

Conformal field theory generalizations of cohomology methods

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Abstract. In this paper we review recent applications of cohomology methods associated with vertex operator algebras to problems in conformal field theory and differential geometry.

1 Introduction

The natural question of calculating cohomology of holomorphic structures on complex manifolds [2, 4, 22, 11, 10] belongs to fundamental problems. The continuous cohomology of Lie algebras of C^∞ -vector fields [6] turned to be a subject of great geometrical interest. Computational methods involving recurrent relations for n -point correlation functions [23, 14, 15, 18, 30] description proved their effectiveness in conformal field theory with some applications in other areas.

There exist several approaches to cohomology of vertex algebras [9, 8, 13, 1]. In [8] the notion of a cohomology of grading-restricted vertex algebras was introduced. The main construction of coboundary operator was given using considerations of rational functions obtained as matrix elements for such vertex algebras. As we know [23] from the genus zero theory of correlation functions for vertex algebras, matrix elements correspond to the choice of formal parameters for vertex operators to be local coordinates on the complex sphere. In this review we discuss the reduction cohomology [24, 25] and product-type cohomology [26, 28].

2 Vertex algebras

A vertex algebra [9] $(V, Y_V, \mathbf{1}_V)$, consists of a \mathbb{Z} -graded complex vector space $V = \bigoplus_{n \in \mathbb{Z}} V_{(n)}$, where $\dim V_{(n)} < \infty$ for each $n \in \mathbb{Z}$, a linear map $Y_V : V \rightarrow \text{End}(V)[[z, z^{-1}]]$, for a formal parameter z and a distinguished vector $\mathbf{1}_V$. For each $v \in V$, the image under the map Y_V is the vertex operator $Y_V(v, z) = \sum_{n \in \mathbb{Z}} v(n)z^{-n-1}$, with modes $(Y_V)_n = v(n) \in \text{End}(V)$, where $Y_V(v, z)\mathbf{1} = v + O(z)$. In addition to that we have: *the grading-restriction condition*: for $n \in \mathbb{Z}$, $\dim V_{(n)} < \infty$, and when n is sufficiently negative, $V_{(n)} = 0$; *the lower-truncation condition for vertex operators*: for $u, v \in V$, $Y_V(u, z)v$ contain only finitely many negative power terms, $Y_V(u, z)v \in V((z))$ (the space of formal Laurent series in z with coefficients in V and finitely



many negative power terms); *the identity property*: Let 1_V be the identity operator on V . Then $Y_V(\mathbf{1}, z) = 1_V$; *the creation property*: For $u \in V$, $Y_V(u, z)\mathbf{1} \in V[[z]]$ and $\lim_{z \rightarrow 0} Y_V(u, z)\mathbf{1} = u$; *the duality*: for $u_1, u_2, v \in V$, $v' \in V' = \prod_{n \in \mathbb{Z}} V_n^*$, the series

$$\begin{aligned} \langle v', Y_V(u_1, z_1)Y_V(u_2, z_2)v \rangle, \quad \langle v', Y_V(u_2, z_2)Y_V(u_1, z_1)v \rangle, \\ \langle v', Y_V(Y_V(u_1, z_1 - z_2)u_2, z_2)v \rangle, \end{aligned}$$

are absolutely convergent in the regions $|z_1| > |z_2| > 0$, $|z_2| > |z_1| > 0$, $|z_2| > |z_1 - z_2| > 0$, respectively, to a common rational function in z_1 and z_2 with the only possible poles at $z_1, z_2 = 0$ and $z_1 = z_2$; *the $L_V(0)$ -bracket formula*: let $L_V(0) : V \rightarrow V$ be defined by $L_V(0)v = nv$ for $v \in V_{(n)}$. Then

$$[L_V(0), Y_V(v, z)] = Y_V(L_V(0)v, z) + z \frac{d}{dz} Y_V(v, z)$$

, for $v \in V$; *the $L_V(-1)$ -derivative property*: let $L_V(-1) : V \rightarrow V$ be the operator given by

$$L_V(-1)v = \text{Res}_z z^{-2} Y_V(v, z)\mathbf{1} = Y_{(-2)}(v)\mathbf{1}$$

, for $v \in V$. Then for $v \in V$,

$$\frac{d}{dz} Y_V(u, z) = Y_V(L_V(-1)u, z) = [L_V(-1), Y_V(u, z)].$$

One also has

$$Y_V(\mathbf{1}_V, z) = 1, \quad \lim_{z \rightarrow 0} Y(u, z)\mathbf{1}_V = u.$$

The conformal vector is defined by ω is

$$Y(\omega, z) = \sum_{n \in \mathbb{Z}} L(n)z^{-n-2},$$

where $L(n)$ forms a Virasoro algebra for central charge c

$$[L_V(m), L_V(n)] = (m - n)L_V(m + n) + \frac{c}{12}(m^3 - m)\delta_{m, -n}\text{Id}_V.$$

$L_V(0)$ describes the \mathbb{Z} -grading with

$$L_V(0)a = \text{wt}(a)a,$$

for weight $\text{wt}(a) \in \mathbb{Z}$ and

$$V_r = \{a \in V | \text{wt}(a) = r\}.$$

We quote the standard commutator property of vertex operator superalgebra, for $x_1 = (a, z_1)$, $x = (b, z_2)$

$$[a(m), Y(x)] = \sum_{j \geq 0} \binom{m}{j} Y(a(j).x)z_1^{m-j}.$$

Taking $a = \omega$ this implies for b of weight $\text{wt}(b)$ that

$$[L_V(0), b(n)] = (\text{wt}(b) - n - 1)b(n),$$

so that

$$b(n)V_r \subset V_{r+\text{wt}(b)-n-1}.$$

In particular, we define for a of weight $\text{wt}(a)$ the zero mode $o(a) = a(\text{wt}(a) - 1)$, which is then extended by linearity to all $a \in V$. Vertex operator algebras are very important in completely solvable models (see, e.g., [16]).

2.1 The square bracket formalism

Define [23] the square bracket operators for V by

$$Y[x] = Y\left(q_z^{L_V(0)}v, e^z - 1\right) = \sum_{n \in \mathbb{Z}} v[n]z^{-n-1}.$$

For v of weight $\text{wt}(v)$ and $k \in \mathbb{Z}$, we have

$$\sum_{j \geq 0} \binom{k + \text{wt}(v) - 1}{j} v(j) = \sum_{m \geq 0} \frac{k^m}{m!} v(m).$$

The square bracket operators form an isomorphic vertex operator algebra with Virasoro vector

$$\tilde{\omega} = \omega - \frac{c}{24} \mathbb{1}_V.$$

Let us now introduce the shifted Virasoro vector

$$\omega_h = \omega + h(-2) \mathbb{1}_V,$$

where

$$h = -\frac{\lambda}{\alpha} J,$$

for $\lambda \in \mathbb{Z}$, and $J \in V_1$, such that $J(0)$ acts semisimply on V .

Then the shifted grading operator is

$$L_h(0) = L(0) - h(0) = L(0) + \frac{\lambda}{\alpha} J(0).$$

Denote the square bracket vertex operator for the shifted Virasoro vector by

$$Y[x]_h = Y\left(e^{z L_h(0)}v, e^z - 1\right) = \sum_{n \in \mathbb{Z}} v[n]_h z^{-n-1},$$

Therefore,

$$Y[a, z]_h = e^{z\lambda} Y[a, z],$$

or equivalently,

$$a[n]_h = \sum_{m \geq 0} \frac{\lambda^m}{m!} a[n + m].$$

2.2 Elliptic functions

Let $q_z = e^{2\pi iz}$. Define for $m \in \mathbb{N}$ the elliptic Weierstrass functions [17]

$$P_{m+1}(z, \tau) = \frac{(-1)^{m+1}}{m!} \sum_{n \in \mathbb{Z} \setminus \{0\}} \frac{n^m q_z^n}{1 - q^n}.$$

The modular Eisenstein series $E_k(\tau)$ are defined by $E_k = 0$ for k for odd and $k \geq 2$ even

$$E_k(\tau) = -\frac{B_k}{k!} + \frac{2}{(k-1)!} \sum_{n \geq 1} \frac{n^{k-1} q^n}{1 - q^n},$$

where B_k is the k -th Bernoulli number defined by

$$(e^z - 1)^{-1} = \sum_{k \geq 0} \frac{B_k}{k!} z^{k-1}.$$

It is convenient to define $E_0 = -1$. E_k is a modular form for $k > 2$ and a quasi-modular form for $k = 2$.

3 Reduction cohomology

In most cases, even for higher genus Riemann surfaces [23, 18] there exist algebraic formulas relating n -point functions with $n - 1$ -point functions in a linear way. Coefficients in the reduction formulas are expressed in terms of quasi-modular forms. The reduction cohomology is defined via such recursion formulas. Let us introduce a complex of n -point correlation functions. We denote by $\mathbf{v}_n = (v_1, \dots, v_n)$, $v_i \in V$ vertex algebra elements. Denote by $\mathbf{z}_n = (z_1, \dots, z_n)$ local coordinates around n points on the torus. Let us introduce the notation: $\mathbf{x}_n = (\mathbf{v}_n, \mathbf{z}_n)$. For a vertex operator algebra V with Virasoro vector ω of central charge c we have the following definition of the torus correlation functions. The genus one $n \geq 0$ -point function is defined by

$$\mathcal{Z}_V(\mathbf{x}_n) = \text{Tr}_V \left(Y_V(q_{z_1}^{L_V(0)} v_1, q_{z_1}) \dots Y_V(q_{z_n}^{L_V(0)} v_n, q_{z_n}) q^{L_V(0) - c/24} \right),$$

with $z \in \mathbb{C}$, and τ being the modular parameter of the torus. For a vertex operator algebra V , we consider the spaces of n -point correlation functions

$$C^n = \{ \mathcal{Z}_V(\mathbf{x}_n), n \geq 0, \}.$$

The coboundary operator $\delta^n(\mathbf{x}_{n+1})$ acting on C^n -space is defined according to the reduction formula [23, 15]. For $n \geq 0$, and any $\mathbf{x}_{n+1} \in V \times \mathbb{C}$, define

$$\delta^n(\mathbf{x}_{n+1}) : C^n \rightarrow C^{n+1},$$

with operators $T_j(v[m].)$, $j \geq 0$, given by the reduction formulas

$$\delta^n(\mathbf{x}_{n+1}) \mathcal{Z}_V(\mathbf{x}_n) = \sum_{\substack{k=0 \\ m \geq 0}}^n f_{k,m}(\mathbf{x}_{n+1}) T_k(v_{n+1}[m].) \mathcal{Z}_V(\mathbf{x}_n).$$

The operators $T_k(v[m].)$ are insertion operators of vertex algebra modes $v[m].$, $m \geq 0$, into $\mathcal{Z}_V(\mathbf{x}_n; B)$ at the k -th entry:

$$T_k(v[m].) \mathcal{Z}_V(\mathbf{x}_n) = \mathcal{Z}_V(T_k(v[m].) \mathbf{x}_n),$$

where we use the notation

$$(\Gamma)_k \mathbf{x}_n = (x_1, \dots, \Gamma x_k, \dots, x_n).$$

For $n \geq 0$, let us denote by \mathcal{B}_n the subsets of all \mathbf{x}_n , such that the chain condition

$$\delta^{n+1}(\mathbf{x}_{n+1}) \delta^n(\mathbf{x}_n) \mathcal{Z}_V(\mathbf{x}_n) = 0,$$

for the coboundary operators is satisfied. Explicitly, the chain condition leads to an infinite $n \geq 0$ set of equations involving functions $f_{k,m}(\mathbf{x}_{n+1})$ and $\mathcal{Z}_V(\mathbf{x}_n)$:

$$\sum_{\substack{k', k=0 \\ m', m \geq 0}}^{n+1, n} f_{k', m'}(\mathbf{x}_{n+1}) f_{k, m}(\mathbf{x}_n) T_{k'}(v_{n+2}[m'].) T_k(v_{n+1}[m].) \mathcal{Z}_V(\mathbf{x}_n) = 0.$$

The C^n -spaces constitute a semi-infinite chain complex

$$0 \longrightarrow C^0 \xrightarrow{\delta^0(\mathbf{x}_1)} C^1 \xrightarrow{\delta^1(\mathbf{x}_2)} \dots \xrightarrow{\delta^{n-2}(\mathbf{x}_{n-1})} C^{n-1} \xrightarrow{\delta^{n-1}(\mathbf{x}_n)} C^n \xrightarrow{\delta^n(\mathbf{x}_{n+1})} \dots$$

For $n \geq 1$, we call corresponding cohomology

$$H^n(V) = \text{Ker } \delta^n(\mathbf{x}_{n+1}) / \text{Im } \delta^{n-1}(\mathbf{x}_n),$$

the n -th reduction cohomology of vertex operator algebra V correlation functions on the torus. The genus one coboundary operator is given by

$$\begin{aligned} f_0(\mathbf{x}_{n+1}; \tau) T_0(v_{n+1}[m]) &= T_0(o(v_{n+1})), \\ f_{k,m}(\mathbf{x}_{n+1}; \tau) &= P_{m+1}(z_{n+1} - z_k, \tau) T_k(v_{n+1}[m]). \end{aligned}$$

Here $o(v_1) = v_1(\text{wt } v_1 - 1)$, $\text{wt } v = \alpha(v)$, $L_V(0)v = \alpha v$.

3.1 Cohomology

In some cases of V , we are able to compute the reduction cohomology of V -correlation functions.

Proposition 1. *The n -th reduction cohomology of the space of correlation functions for V is given by the space of solutions $\mathcal{Z}_V(\mathbf{x}_n)$ to the equations*

$$\sum_{k=0}^n \sum_{m \geq 0} f_{k,m}(\mathbf{x}_n) T_k(v_{n+1}[m].) \mathcal{Z}_V(\mathbf{x}_n) = 0,$$

with $x_i \notin \mathcal{B}_i$, for $1 \leq i \leq n$. These are given by the space of quasi-modular forms in terms of series of Weierstrass functions, recursively generated by reduction/recursion formulas.

In certain cases of coboundary operators, we are able to compute the n -th cohomology even more explicitly by using recursion formulas in terms of generalized elliptic functions [15]. In particular, for n -point correlation functions associated to a vertex operator superalgebra we obtain using [15]

Corollary 1. For $\mathbf{v}_n \notin \mathcal{B}_n$, the n -th cohomology is given by the space of determinants of $n \times n$ -matrices containing deformed elliptic functions depending on $z_i - z_j$, $1 \leq i, j \leq n$, for all possible combinations of \mathbf{v}_n -modes.

4 Product-type cohomology

One defines the configuration space $F_n\mathbb{C} = \{(z_1, \dots, z_n) \in \mathbb{C}^n \mid z_i \neq z_j, i \neq j\}$, for $n \in \mathbb{Z}_+$. Let V be a grading-restricted vertex algebra, and \bar{V} its algebraic completion. A \bar{V} -valued rational function Φ in (z_1, \dots, z_n) with the only possible poles at $z_i = z_j, i \neq j$, is a map

$$\begin{aligned} \mathcal{F} : F_n\mathbb{C} &\rightarrow \mathbb{C}, \\ (z_1, \dots, z_n) &\mapsto \mathcal{F}(z_1, \dots, z_n), \end{aligned}$$

such that for any $w \in V'$ (the dual space), $\mathcal{F} = \langle w, \Phi(z_1, \dots, z_n) \rangle$, is a rational function in (z_1, \dots, z_n) with the only possible poles at $z_i = z_j, i \neq j$. One defines the action of S_n on Φ :

$$\sigma(\Phi)(v_1, z_1; \dots; v_n, z_n) = \Phi(v_{\sigma(1)}, z_{\sigma(1)}; \dots; v_{\sigma(n)}, z_{\sigma(n)}),$$

for $\sigma \in S_n$ and $v_1, \dots, v_n \in V$. For $n \in \mathbb{Z}_+$, a linear map

$$\mathcal{F}(v_1, z_1; \dots; v_n, z_n) = V^{\otimes n} \rightarrow \mathbb{C},$$

is said to have the $L_V(-1)$ -derivative property if

$$\partial_{z_i} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}(v_1, z_1; \dots; L_V(-1)v_i, z_i; \dots; v_n, z_n),$$

for $i = 1, \dots, n, v_1, \dots, v_n \in V$, and

$$\sum_{i=1}^n \partial_{z_i} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = L_V(-1).\mathcal{F}(v_1, z_1; \dots; v_n, z_n),$$

with some action " ." of $L_V(-1)$ on $\mathcal{F}(v_1, z_1; \dots; v_n, z_n)$. A linear map

$$\mathcal{F} : V^{\otimes n} \rightarrow \mathbb{C},$$

has the $L_V(0)$ -conjugation property if for $v_1, \dots, v_n \in V, (z_1, \dots, z_n) \in F_n\mathbb{C}$, and $z \in \mathbb{C}^\times$, such that $(zz_1, \dots, zz_n) \in F_n\mathbb{C}$,

$$z^{L_V(0)} \mathcal{F}(v_1, z_1; \dots; v_n, z_n) = \mathcal{F}(z^{L_V(0)}v_1, zz_1; \dots; z^{L_V(0)}v_n, zz_n).$$

To provide convergence, one adds also the notion of maps composable with vertex operators: in order to multiply maps by vertex operators and form coboundary operators, one has to apply extra conditions [8]. The space of \bar{V} -valued rational function Φ in n variables satisfying the conditions above, and composable with m vertex operators is denoted by C_m^n . We also find

Lemma 1. Let $C_m^0 = V$. Then we have $C_m^n \subset C_{m-1}^n$, for $m \in \mathbb{Z}_+$.

The co-boundary operator for the spaces C_m^n is defined by

$$\delta_m^n : C_m^n \rightarrow C_{m-1}^{n+1}.$$

For $\Phi \in C_m^n$, $v_1, \dots, v_{n+1} \in V$, $w \in V'$, and $(z_1, \dots, z_{n+1}) \in F_{n+1}\mathbb{C}$, it is given by

$$\begin{aligned} \delta_m^n \mathcal{F} &= \langle w, \delta_m^n \cdot \Phi(v_1, z_1; \dots; v_{n+1}, z_{n+1}) \rangle \\ &= \langle w, Y_V(v_1, z_1) \Phi(v_2, z_2; \dots; v_{n+1}, z_{n+1}) \rangle \\ &\quad + \sum_{i=1}^n (-1)^i \langle w, \Phi(v_1, z_1; \dots; v_{i-1}, z_{i-1}; Y_V(v_i, z_i - z_{i+1})v_{i+1}; \\ &\quad \quad \quad \dots; v_{n+1}, z_{n+1}) \rangle \\ &\quad + (-1)^{n+1} \langle w, Y_V(v_{n+1}, z_{n+1}) \Phi(v_1, z_1; \dots; v_n, z_n) \rangle. \end{aligned}$$

Proposition 2. For $n \in \mathbb{N}$ and $m \in \mathbb{Z}_+ + 1$, the co-boundary operator δ_m^n satisfies the chain condition, i.e.,

$$\delta_{m-1}^{n+1} \circ \delta_m^n = 0,$$

One introduces the n -th cohomology $H_m^n(V)$ of a grading-restricted vertex algebra V composable with m vertex operators:

$$H_m^n(V) = \ker \delta_m^n / \text{im } \delta_{m+1}^{n-1}.$$

4.1 Čech-de Rham cohomology of foliations

Let us recall [3] the formulation of Čech-de Rham complex for a foliation on a smooth manifold. For $n \geq 0$, and $1 \leq m \leq k$, we define the space

$$C_m^n(F, \mathcal{U}) = \bigcap_{\substack{U_1 \xrightarrow{h_1} \dots \xrightarrow{h_{m-1}} U_m \\ 1 \leq j \leq m}} C^m(F)(U_j),$$

where the intersection ranges over all possible $(m - 1)$ -tuples of holonomy embeddings h_j , $j \in \{1, \dots, m - 1\}$, between transversal sections of a basis \mathcal{U} for a foliation F . Then a construction [12] of cohomology classes for foliations (starting from an integrability condition of differential forms) motivates us to construct cohomology classes for vertex operator algebra if we introduce a product of elements of our complex.

4.2 Products of elements for the complex

In analogy with the foliation theory [3, 12], in order to determine cohomology classes of a complex, one requires to define a product of a number of elements of subspaces with the image in another complex subspace coherent with respect to the original coboundary operator satisfying some symmetry conditions, $L_V(0)$ -conjugation and $L_V(-1)$ -derivative properties. Let us give an example of product. For $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k$, and $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n$ let us give an example of product

$$\begin{aligned} &\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \\ &\mapsto \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n), \end{aligned}$$

is given by the rational function

$$\begin{aligned} &\langle w, \Phi(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n) \rangle \\ &= \sum_{u \in V} \langle w, \Phi(v_1, x_1; \dots; v_k, x_k, u, \zeta_1) \rangle \\ &\quad \langle w', \Phi(v'_1, y_1; \dots; v'_n, y_n, \bar{u}, \zeta_2) \rangle, \end{aligned}$$

parametrized by $\zeta_1, \zeta_2 \in \mathbb{C}$. The sum is taken over any V_l -basis $\{u\}$, where \bar{u} is the dual of u with respect to a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ over V .

Convergence of a product of that type can be proven by methods of [7].

Proposition 3. For $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k$ and $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n$, the product $\mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n)$ belongs to the space $C_{m+m'}^{k+n}$, i.e.,

$$\cdot : C_m^k \times C_{m'}^n \rightarrow C_{m+m'}^{k+n}.$$

Let us give an example of product. For $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k$, and $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n$, an example of product

$$\begin{aligned} & \mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \\ & \mapsto \mathcal{F}(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n), \end{aligned}$$

is given by the rational function

$$\begin{aligned} & \langle w, \Phi(v_1, x_1; \dots; v_k, x_k; v'_1, y_1; \dots; v'_n, y_n) \rangle \\ & = \sum_{u \in V} \langle w, \Phi(v_1, x_1; \dots; v_k, x_k, u, \zeta_1) \rangle \\ & \quad \langle w', \Phi(v'_1, y_1; \dots; v'_n, y_n, \bar{u}, \zeta_2) \rangle, \end{aligned}$$

parametrized by $\zeta_1, \zeta_2 \in \mathbb{C}$. The sum is taken over any V_l -basis $\{u\}$, where \bar{u} is the dual of u with respect to a non-degenerate bilinear form $\langle \cdot, \cdot \rangle$ over V . The form of the product defined above is natural in terms of the theory of higher genus correlation functions for vertex operator algebras [5, 23, 14, 19, 20, 21]. The product admits the action of the coboundary operator $\delta_{m+m'}^{k+n}$. The co-boundary operator has a variation of Leibniz law with respect to the product.

Proposition 4. For $\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \in C_m^k$ and $\mathcal{F}(v'_1, y_1; \dots; v'_n, y_n) \in C_{m'}^n$, the action of $\delta_{m+m'}^{k+n}$ on their product is given by

$$\begin{aligned} & \delta_{m+m'}^{k+n} (\mathcal{F}(v_1, x_1; \dots; v_k, x_k) \cdot \mathcal{F}(v'_1, y_1; \dots; v'_n, y_n)) \\ & = (\delta_m^k \mathcal{F}(v_1, z_1; \dots; v_k, z_k)) \cdot \mathcal{F}(v_{k+1}, z_{k+1}; \dots; v_{k+n}, z_{k+n}) \\ & \quad + (-1)^k \mathcal{F}(v_1, z_1; \dots; v_k, z_k) \cdot (\delta_{m'}^n \mathcal{F}(v_1, z_{k+1}; \dots; v_{k+n}, z_{k+n})). \end{aligned}$$

Finally, we see that the product extends the chain complex to all products $C_m^k \times C_{m'}^n$, $k, n \geq 0, m, m' \geq 0$.

4.3 Product-type cohomology classes

Let us define an extra product the product of Φ and Ψ , $\Phi * \Psi : V^{\otimes(k+n)} \rightarrow V$, $\Phi * \Psi = [\Phi, \Psi] = \Phi \cdot \Psi - \Psi \cdot \Phi$, where brackets denote ordinary commutator in \bar{V} . Due to the properties of the maps $\Phi \in C_m^k$ and $\Psi \in C_{m'}^n$, the map $\Phi * \Psi$ belongs to the space $C_{m+m'}^{k+n}$. In analogy with differential forms, it is possible to introduce cohomology classes of the form $[\delta\eta]$ which are counterparts of the Godbillon class for codimension one foliations. We call a map $\Phi \in C_k^n$, closed if $\delta_k^n \Phi = 0$. For $k \geq 1$, we call it exact if there exists $\Psi \in C_{k-1}^{n+1}$, such that $\Psi = \delta_k^n \Phi$. For $\Phi \in C_k^n$ we call the cohomology class of mappings $[\Phi]$ the set of all closed elements that differ from Φ by an exact mapping, i.e., for $\Lambda \in C_{k+1}^{n-1}$, $[\Phi] = \Phi + \delta_{k+1}^{n-1} \Lambda$. In analogy with the notion of integrability for differential forms, we use here the notion of the orthogonality for the spaces of a complex. Let us require that for a pair of elements $\Phi \in C_m^k$ and $\Psi \in C_{m'}^n$, $\Phi * \delta_{m'}^n \Psi = 0$. We call this the orthogonality condition for mappings. As for Čech-de Rham complex due to [3], we derive product-type invariants in the vertex operator algebra terms.

Proposition 5. *The product, coboundary operator, and the orthogonality condition applied to elements of the complex subspaces generate non-vanishing cohomology classes $[(\delta_m^n \Phi) * \Phi]$ independent on the choice of $\Phi \in C_m^n$ for certain values of (n, m) -pairs.*

The cohomology classes we obtain are vertex algebra cohomology counterparts of the Godbillon class which non-triviality can be proven by using methods of [12] for codimension one foliations [29].

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