

Review

# Quantum Gravity Phenomenology from the Thermodynamics of Spacetime

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**Abstract:** This work is based on the formalism developed in the study of the thermodynamics of spacetime used to derive Einstein equations from the proportionality of entropy within an area. When low-energy quantum gravity effects are considered, an extra logarithmic term in the area is added to the entropy expression. Here, we present the derivation of the quantum modified gravitational dynamics from this modified entropy expression and discuss its main features. Furthermore, we outline the application of the modified dynamics to cosmology, suggesting the replacement of the Big Bang singularity with a regular bounce.

**Keywords:** models of quantum gravity; spacetime singularities; black holes; classical theories of gravity



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## 1. Introduction

In the absence of a final theory of quantum gravity, the study of its phenomenological effects has obtained relevance in recent years. The aim of these approaches is to shed some light on the effective, dynamical low-energy quantum gravity effects near singularities [1–6]. Phenomenological models are naturally constrained to a particular candidate theory of quantum gravity and usually also to simple particular models that prevent the extrapolation of results. In order to improve this situation we aim at finding the general phenomenological effects of quantum gravity, allowing us to extract general features of and possible constraints on the final theory. Thus, we suggest the derivation of the phenomenology from general thermodynamics tools.

The use of thermodynamics to understand gravitational mechanics was pointed out first in the context of black hole thermodynamics and subsequently extended to general spacetimes, giving rise to the derivation of Einstein equations from thermodynamics [7–9]. Specifically, the starting point of the derivation is the equilibrium condition for maximal entropy on the horizon. This relationship between thermodynamics and gravitational dynamics has been seen to not be just a particular characteristic of General Relativity, but relevant for many modified theories of gravity [10–16] and for the introduction of quantum fields as gravitational sources [8,16]. In this framework of understanding the interface between gravitational dynamics and thermodynamics we propose a further step. By introducing quantum gravity effects to thermodynamics we obtain a set of modified gravitational dynamics, which encode the low-energy effects of quantum gravity.

As it is a suitable modification of thermodynamics using quantum gravity effects, we make use of the leading order modification of Bekenstein entropy. Different approaches to quantum gravity predict the same qualitative logarithmic correction to this entropy, such as loop quantum gravity (LQG) [17,18], string theory [19,20], and AdS/CFT correspondence [21,22]; this also appears in some model-independent thought experiments and phenomenological approaches such as the generalized uncertainty principle (GUP) [2,4].

Due to the fact that we are considering modifications on the entropy associated with a horizon, we note that the studied modification of entanglement entropy also gives rise to the same correction term [23–25]. In fact, some proposals interpret Bekenstein entropy as an entanglement entropy [23,26,27]. The convergence of approaches leading to the same qualitative modification of entropy allows for a universal interpretation of it as a general feature of quantum gravity effects, additionally providing different quantitative results that could be used to constrain the models.

This article presents a review of our work that covers the contribution presented in the workshop "The Quantum and The Gravity" and is organised as follows. In Section 2, we briefly review the classical results of the derivation of Einstein equations that we will extend later into the quantum realm. In Section 3, we first introduce the quantum gravity modification of entropy. Then, we use methods from the thermodynamics of spacetime to derive effective quantum gravitational equations of motion and discuss the results. Finally we present a sketch of the application of our equations to a simple cosmological model, where our scheme provides an effective avoidance of the singularity. We conclude in Section 5 by summarizing our results and outlining possible future work perspectives. Throughout the paper, we express equations in SI units.

## 2. Einstein Equations of Motion from Thermodynamics

In general, derivations in the thermodynamics of spacetime are based on the idea that gravitational dynamics are encoded in the equilibrium condition for the maximal entropy on the horizon,  $\delta S = 0$ . Thus, all the technical challenges lie in the definitions of the relevant entropies and of the horizon itself. In this work we will follow two recent derivations performed from different definitions for the entropy of the matter fields, being the entanglement entropy of the matter present there [8] or the Clausius entropy flux across the horizon [9,28]. The consideration of both cases allow us to analyze the possible equivalence of these definitions that has been already established at the semi-classical level (see the detailed analysis by the authors in [9,28]). Regarding the definition of the horizon, we develop both derivations associated with the horizon of geodesic local causal diamonds (GLCD) that we will introduce in this paper<sup>1</sup>. We would like to note that the derivations in the semi-classical case seem to point to the emergence of Unimodular Gravity instead of General Relativity. These two gravitational theories are dynamically equivalent in the semi-classical regime, so to reliably determine which theory is likelier, one would need to introduce quantum gravity effects.

### 2.1. Geodesic Local Causal Diamonds

Let us consider an arbitrary point  $P$  of spacetime and any of unit time-like vector  $n^\mu$  emerging from it. If we choose Riemann normal coordinates (RNC), holding  $n = \partial/\partial t + O(l)$ , and expand around  $P$ , we obtain [32]:

$$g_{\mu\nu}(x) = \eta_{\mu\nu} - \frac{1}{3}R_{\mu\alpha\nu\beta}(P)x^\alpha x^\beta + O(x^3). \tag{1}$$

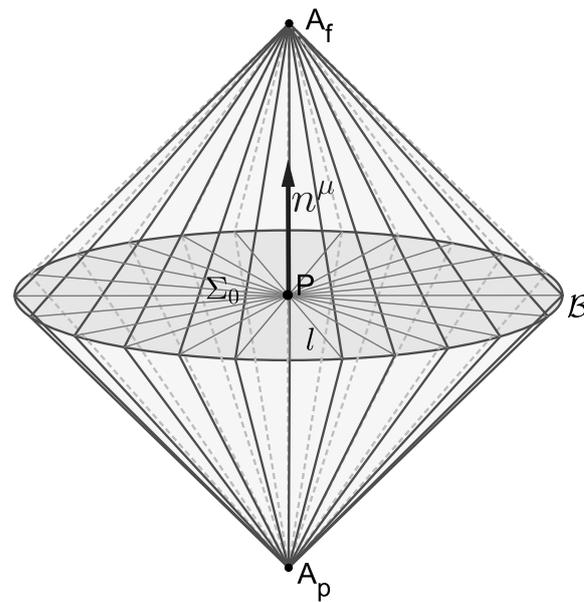
When sending a family of geodesics orthogonal to  $n^\mu$  with parameter length  $l$  out from  $P$ , we obtain a three-dimensional geodesic ball,  $\Sigma_0$ . The causal region determined by this geodesic ball is what defines a geodesic local causal diamond (see Figure 1).

One of the relevant quantities for our derivation will be the area of the 2-sphere defined by the boundary  $\mathcal{B}$  of  $\Sigma_0$  (for a value of  $l$  much smaller than the local curvature length) [8]

$$\mathcal{A} = 4\pi l^2 - \frac{4\pi}{9}l^4 G_{00}(P) + O(l^5), \tag{2}$$

where  $G_{00} \equiv G_{\mu\nu}n^\mu n^\nu$ .

The other relevant quantity is the conformal Killing vector, which generates a conformal Killing horizon on the null boundary of the GLCD, which will allow us to define the variation of matter entanglement entropy inside the geodesic ball.



**Figure 1.** A representation of a GLCD having originated from point  $P$  (where the angular coordinate is suppressed) and with a unit time-like vector  $n^\mu$ . The geodesics of length  $l$  form the spatial geodesic ball  $\Sigma_0$ , whose boundary  $B$  is an approximate 2-sphere. The boundary of the diamond given by the null geodesic generators runs from the past apex  $A_p$  ( $t = -l/c$ ) to the future apex  $A_f$  ( $t = l/c$ ).

### 2.2. Classical Derivation of Einstein Equations

In basic terms, the gravitational dynamics are encoded in the equilibrium condition for maximal entropy on the horizon, that is  $\delta S = 0$ . The entropy on the horizon is composed of the sum of the entropy of quantum correlations across the horizon and the entropy of the matter–energy crossing it.

On one side, the entropy given by the correlations of the vacuum fluctuations of quantum fields across the horizon is calculated as von Neumann (entanglement) entropy. This entropy comes from having a region of the spacetime inaccessible to an observer and it is associated with the horizon defining that boundary. Then, as this entropy describes any causal horizon, the observer-dependent horizons are of interest in this framework [12,26,27]. The entanglement entropy was found to be proportional to the area of the horizon  $S = \eta A$  [23,26,27,33], where  $\eta$  depends on a UV cut-off and, in principle, can also depend on the position in spacetime [12]. When we perform a variation of this entropy, it corresponds to a variation of the area (for constant  $\eta$ ). The variation of the area in this model is determined by the previous equation in the diamond, performing a variation of it from a maximally symmetric spacetime [34].

Let us remark that, in order to recover the Einstein equations of motion from this derivation, it is necessary to assume an equivalence of entanglement entropy with the Bekenstein entropy of black holes [35–37]:

$$S_{BH} = \frac{k_B \mathcal{A}}{4l_P^2}, \tag{3}$$

where  $\mathcal{A}$  is the area of the black hole’s event horizon,  $l_P = \sqrt{G\hbar/c^3}$  is the Planck length, and  $k_B$  is the Boltzmann constant. This assumption is not something new or exclusive to thermodynamic derivations. Among the many proposals for microscopic interpretations of this entropy [38], one of the ideas classifies the appearance of Bekenstein entropy as a product of the quantum entanglement between two causally separated regions [26]<sup>2</sup>. Under this identification method, the proportionality constant takes a universal value of  $\eta = k_B/4l_P^2$ . Note also that setting the equivalence with Bekenstein entropy in this

context of thermodynamic derivation implies the assumption of the Strong Equivalence Principle [40].

On the other hand, we need to consider the variation of matter–energy entropy crossing the horizon. The standard method used is to calculate the thermodynamic entropy of the horizon, as was the case in the seminal work of Jacobson [7]. The issue there was whether thermodynamic entropy can be summed directly with the entanglement entropy to result in a general equilibrium condition, because of the different characters of both. Later, a computation for the entropy of matter was developed in terms of entanglement entropy for GLDC explicitly evaluated for small perturbations from a vacuum [34]. This entropy can be combined with the entanglement entropy of the geometry to generate a total equilibrium condition, and this will be the derivation on which we will base our study. The expression of this entropy comes from realizing that the vacuum state of the field can be written as an expression of a thermal density matrix at the Unruh temperature. Thus, in the semi-classical framework,  $\delta S_m$  is given by (for a detailed derivation see [34]):

$$\delta S_m = \frac{2\pi k_B}{\hbar c} \frac{4\pi l^4}{15} (\delta \langle T_{00} \rangle + \delta X), \tag{4}$$

where  $X$  is a spacetime scalar depending on  $l$ , which reflects the presence of non-conformal fields [34]. This equation is valid as long as there is a fixed UV point for each field.

When one sums up these two contributions to calculate the total entropy and demands the equilibrium condition, after some calculations (and fixing the value of  $\eta$  as described earlier) and assuming the Einstein Equivalence Principle [40] (in order to generalize the local equation to all the components of the Einstein tensor in the whole spacetime [7]), the traceless equations of motion emerge in the derivation:

$$R_{\mu\nu} - \frac{1}{4} R g_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \right). \tag{5}$$

Imposing the local conservation of the energy–momentum tensor results in the Einstein equations, with the cosmological constant  $\Lambda$  appearing as an integration constant:

$$G_{\mu\nu} + \Lambda g_{\mu\nu} = \frac{8\pi G}{c^4} T_{\mu\nu}. \tag{6}$$

In order to both understand a possible equivalence between thermodynamic and entanglement entropies, and to check the results via a different method, we developed another procedure. In this derivation [9], the matter–energy entropy expression comes from the explicit definition of a Clausius flux,  $S_C$ , crossing the horizon [28]. One starts by defining a class of time-like observers traveling inside the GLCD with constant acceleration  $a$ . Then, from the velocity of these observers,  $V^\mu$ , the normal,  $N^\mu$ , used to time the hyperbolic sheet of these observers’ sweep out,  $\Sigma$ , and the energy momentum tensor, it can be defined the heat (matter–energy) crossing a segment of  $\Sigma$  as:

$$\delta Q = -\frac{1}{c} \int_{\Sigma} T_{\mu\nu} V^\mu N^\nu d^3\Sigma. \tag{7}$$

From this expression, one can directly obtain the Clausius entropy by invoking the Unruh temperature measured by these uniformly accelerated observers  $dS_C = \delta Q/T$ . The general expression for the Clausius entropy for a bifurcate null horizon is then found by taking a limit,  $a \rightarrow \infty$ , yielding [28]:

$$S_C(\lambda) = S(\mathcal{B}) + \frac{2\pi k_{BC}}{\hbar} \int_0^\lambda \int_{S(\lambda)} \tilde{\lambda} T_{\mu\nu} k_\pm^\mu k_\pm^\nu d^2\mathcal{A} d\tilde{\lambda} + O(\lambda^3), \tag{8}$$

where  $\lambda$  is the affine parameter along the geodesic generators of the null surface,  $k_\pm^\mu$  are null vectors tangent to the surface (for positive and negative values of  $\lambda$ , respectively),

$d^2\mathcal{A}$  is the area element of the null surface’s spatial cross-section  $\mathcal{S}(\lambda)$ , and  $S(\mathcal{B})$  is an unspecified Clausius entropy referred to the 2-surface  $\mathcal{B}$  (given at  $\lambda = 0$ ).

From the definition of the Clausius entropy, we can integrate this entropy into the diamond to obtain the total entropy flux across the horizon, and then add this contribution of the matter–energy flux to the entanglement entropy, demanding the equilibrium condition for maximal entropy. Using this procedure, it was shown that the same derivation for the Einstein equations could be completed under the same assumptions as the previous procedure [9].

There are few remarks concerning these classical derivations that we consider relevant for our next analysis. The first is the found equivalence of the Clausius and entanglement entropies at the semi-classical level (for a detailed analysis see [9]). It is not clear to what extent this equivalence hold when we introduce quantum effects, as the very definition of Clausius entropy is no longer valid in this instance. The second remark comes from noting the traceless form of the derived equations of motion, with the local conservation of the energy–momentum tensor appearing as an additional assumption. This is characteristic of Unimodular Gravity theories. From this, it could be argued that General Relativity does not emerge from the thermodynamics of spacetime, rather Unimodular Gravity does. However, these two theories are dynamically equivalent at the semi-classical level, preventing us from reaching a clear conclusion<sup>3</sup>. In contrast, these two theories might differ when one tries to quantize them. So, we expect that the introduction of quantum effects in the thermodynamic derivation can shed light on which of the two theories emerges from thermodynamics.

### 3. Modified Equations of Motion

Now, we introduce low-energy effects of quantum gravity on thermodynamics and analyze the effective equations of motion that emerge from it. For this purpose, we first studied the modified entropy of the horizon as a general prediction from the inclusion of quantum gravity effects. Once we obtained the modified thermodynamic tools, we proceeded to derive the equations of motion by following an extension of the two classical procedures we mentioned in previous section.

#### 3.1. Modified Entropy of the Horizon

Modification of the horizon entropy is characterized by the appearance of a logarithmic correction. This modification emerges from the quantum gravity effects on both the Bekenstein entropy associated with the black hole horizon and on the entanglement entropy associated with observer-dependent causal horizons.

The modification of Bekenstein entropy by quantum gravity effects has been argued in very different contexts, e.g., in LQG [17,18], GUP phenomenology [4,6], entanglement entropy calculations [23,45], AdS/CFT duality [21,22], string theory [19,20], and in the analysis of statistical fluctuations around equilibrium [46]. We can express the modified entropy in a general form:

$$S_{BH,q} = \frac{k_B \mathcal{A}}{4l_P^2} + Ck_B \ln \left( \frac{\mathcal{A}}{\mathcal{A}_0} \right) + O \left( \frac{k_B l_P^2}{\mathcal{A}} \right), \tag{9}$$

where  $C$  is a real dimensionless constant and  $\mathcal{A}_0$  is another constant with dimensions of area. The specific values of these constants are characteristic of the different theories and models.

The entanglement entropy calculations of the logarithmic modifications are particularly appropriate in our case (for a detailed description of these methods see, e.g., [23–25]). On one hand, these calculations are appropriate, because they provide an expression for the logarithmic modification for virtual observer-dependent horizons. On the other hand, these calculations are appropriate, because they show how the corrections depend on the horizon’s topology, such that they do not appear for a plane (such as the Rindler horizon)

but they emerge for 2-spheres [23]. So, it is more appropriate to use these calculations for the derivation of a horizon with closed spatial cross-sections such as the one of the GLCD, in contrast with the Rindler horizon used in Jacobson’s original paper [7].

Let us briefly remark that only the sign of the constant  $C$  is relevant to determine the character of the effective equations of motion. Some of the mentioned models have proposed opposite signs for it, which could be used in the future as a constraint on the proposal. In general, one could classify the modifications to entropy in two categories regarding the sign of  $C$ .  $C < 0$  is associated with microcanonical modifications, providing more accuracy on the microstates at fixed horizon areas. As expected, those modifications reduce the uncertainty, being reflected in a negative correction term to the entropy [6,47,48]. Conversely,  $C > 0$  is associated with canonical modifications, coming from the thermal fluctuations of the horizon area at a fixed temperature, providing an additional source of uncertainty [6,47,48]. Throughout this work we will keep a general value of  $C$  in Equation (9), which can later be set to any particular value, in order to study the general features of the theory (as we will see in the simple cosmological example).

### 3.2. Modified Equations of Motion

We now incorporate the modified entanglement entropy into the semi-classical methods developed in the thermodynamics of spacetime. By doing this, we aim to obtain a general expression for modified equations of motion reflecting the phenomenological low-energy effects of quantum gravity dynamics. Note that we still assume in this regime that spacetime exists as a four-dimensional Lorentzian manifold, i.e., we are restricting the study to length scales still significantly larger than those of the Planck scale.

From a technical point of view, in order to develop our derivation, we will use causal diamonds, as they provide a closed region of spacetime with a naturally defined boundary, which is especially well suited to local calculations. Moreover, as we have previously mentioned in the subsection which defined closed causal horizons, the use of causal diamonds is necessary for the logarithmic modifications to entanglement entropy<sup>4</sup>. In the following sections, we explain the results of the derivation in terms of the two independent methods mentioned in previous section (for a detailed computation of the results, see [49]).

#### 3.2.1. Derivation from MVEH

This derivation is based on the maximal vacuum entanglement hypothesis (MVEH) [8] concerning GLCD, which establishes that a first order variation of the modified total entropy (i.e., summing the entropies for the geometry and quantum fields) from a vacuum of maximally symmetric spacetime vanishes in a small geodesic ball at a fixed volume. We can express this as  $\delta S_{e,q} + \delta S_m = 0$ , where  $S_{e,q}$  is the modified vacuum entanglement entropy and  $S_m$  the entanglement entropy of the matter.

On one hand, in order to evaluate  $\delta S_{e,q}$ , we consider the entropy as given by:

$$S_{e,q} = \eta \mathcal{A} + k_B C \ln \frac{\mathcal{A}}{\mathcal{A}_0} + O\left(\frac{k_B l_P^2}{\mathcal{A}}\right), \tag{10}$$

where  $C$  is a dimensionless constant, and the area of  $\mathcal{B}$ , using the expression for the area (corresponding to  $\mathcal{B}$ ), is determined by Equation (2). Note that we have kept  $\eta$  as a general constant for the moment, instead of fixing it to the Bekenstein value. In order to find the equations of motion, we need to consider a variation of the metric that leaves fixed the volume of  $\Sigma_0$  [8,15] and compute the corresponding change  $\delta S_{e,q}$  as:

$$\begin{aligned} \delta S_{e,q} &= S_{e,q} - S_{e,q}^{MSS} \\ &= \eta \delta \mathcal{A}|_V + k_B C \frac{\delta \mathcal{A}|_V}{\mathcal{A}_{MSS}} - k_B \frac{C}{2} \left(\frac{\delta \mathcal{A}|_V}{\mathcal{A}_{MSS}}\right)^2 + O\left((\delta \mathcal{A}|_V)^3\right), \end{aligned} \tag{11}$$

where the superindex MSS accounts for the maximally symmetric spacetime from where we perform the variation.

On the other hand, we also need to evaluate  $\delta S_m$ , expressing it in terms of the variation of the energy–momentum tensor expectation value,  $\delta \langle T_{\mu\nu} \rangle$ . In order to perform that evaluation, we express the vacuum state of the field in terms of a thermal density matrix at the Unruh temperature [7,49]. The consideration of the Unruh effect implies assuming the ground state of the quantum fields, approximated locally as the Minkowski vacuum, which implies assuming the Einstein Equivalence Principle [12,40], as we have already seen in the semi-classical case. However the inclusion of the quantum gravity effect could lead to the violation of the EEP (as it has been studied, for example, in GUP phenomenology [50–53], with very different conclusions). Nevertheless, in this modified picture, it has been argued that the Unruh effect is also modified due to quantum gravity effects. In that case, such a modification already accounts for changes in the local quantum field theory induced by a possible EEP violation. This allows us to consider the (modified) Unruh temperature regardless of the validity of the EEP. Some modifications of this kind have been suggested in the framework of GUP [54,55], following previous discussions of Hawking temperature modifications [4]. Even though the proposed models modifying the Unruh temperature differ, all of them show a leading order correction term proportional to  $T^3 \propto a^3$ , where  $T$  is the standard Unruh temperature. Thus, it can be written generally as:

$$T_{GUP} = \frac{\hbar a \left( 1 + \psi \frac{l_P^2}{c^4} a^2 \right)}{2\pi k_{BC}} + O\left( \frac{l_P^4 a^5}{c^8} \right), \tag{12}$$

where  $\psi$  is a real constant (with an estimation of being of the order of unity) [54,55].

It is noteworthy that a complete rigorous analysis of possible modifications of Unruh temperature has not been developed yet, and, therefore, we lack conclusive arguments for or against them. Nevertheless, due to the general basis of the validity of GUP approaches [2], it is worth considering this possibility and showing that, even if some modifications appear, our construction is consistent. Hence, we will keep the more general expression, using the standard Unruh temperature when  $\psi = 0$  as a particular case. We will see that resulting gravitational dynamics has no dependence on the exact expression for the Unruh temperature (as one would expect from the fact that both Hawking and Unruh effects are kinematic results [56]).

The variation of  $S_m$  can be expressed as the semi-classical variation plus an extra term from the temperature modification:

$$\delta S_m = \frac{2\pi k_{BC}}{\hbar} \frac{4\pi}{15} l^4 (\delta \langle T_{00} \rangle + \delta X) - 4\psi l^2 C^2 \frac{2\pi k_B l_P^2}{\hbar c^3} \frac{4\pi}{15} l^4 (\delta \langle T_{00} \rangle + \delta X) + O(l^5), \tag{13}$$

where  $C$  is a constant from an arbitrary normalization of the conformal Killing vector [8,34].

When bringing together both variations of entropy, one can check that the necessary condition implied by MVEH reads [49]:

$$S_{\mu\nu}(P)n^\mu n^\nu + \frac{Cl_{Pl}^2}{30\pi} S_{\alpha\beta}(P)n^\alpha n^\beta S_{\mu\nu}(P)n^\mu n^\nu - \Phi(P) = \frac{8\pi G}{c^4} \delta \langle T_{\mu\nu}(P) \rangle (P)n^\mu n^\nu, \tag{14}$$

where  $S_{\mu\nu} = R_{\mu\nu} - Rg_{\mu\nu}/4$  denotes the traceless part of the Ricci tensor, and  $\Phi$  is a scalar independent of  $l$ . In order to recover Einstein equations in the semi-classical limit, when  $C \rightarrow 0$ , we set  $\eta = k_B/4l_P^2$ . This assumption implies the consideration of  $G$  as a universal constant. The equation is valid for any unit of the time-like vector  $n^\mu$  in  $P$  (note that time indices used previously were the result of contractions with the time-like vector  $n^\mu$ ), as there is no preferred time direction. In order to determine  $\Phi$ , we can obtain a system of conditions by differentiating the equation with respect to  $n^\mu$  and thus finding that  $\Phi$  is

uniquely reduced to a single undetermined scalar function,  ${}^{(0)}\Phi^5$ , leading the equations to be expressed as:

$$\left( S_{\mu\nu}(P) - \frac{Cl_{Pl}^2}{30\pi} S_{\mu\lambda}(P) S^\lambda{}_\nu(P) - \frac{8\pi G}{c^4} T_{\mu\nu}(P) \right) n^\mu n^\nu = {}^{(0)}\Phi(P). \tag{15}$$

Similar to the semi-classical case [9], the contractions containing  $n^\mu$  and a dependence on  $P$  can be disregarded when considering the validity of the Einstein Equivalence principle. This already-mentioned assumption allows for a generalization of the equations throughout the whole spacetime, thus:

$$S_{\mu\nu} - \frac{Cl_{Pl}^2}{30\pi} S_{\mu\lambda} S^\lambda{}_\nu - \frac{8\pi G}{c^4} \delta\langle T_{\mu\nu} \rangle = -{}^{(0)}\Phi g_{\mu\nu}. \tag{16}$$

Finally, taking the trace of these equation, we fully determine  ${}^{(0)}\Phi$  and obtain the following expression for the traceless equations of gravitational dynamics:

$$S_{\mu\nu} - \frac{Cl_{Pl}^2}{30\pi} S_{\mu\lambda} S^\lambda{}_\nu + \frac{Cl_{Pl}^2}{120\pi} \left( R_{\kappa\lambda} R^{\kappa\lambda} - \frac{1}{4} R^2 \right) g_{\mu\nu} = \frac{8\pi G}{c^4} \left( \delta\langle T_{\mu\nu} \rangle - \frac{1}{4} \delta\langle T \rangle g_{\mu\nu} \right).$$

These represent completely general effective equations of motion that reduce to (traceless) Einstein equations within an appropriate limit. In order to obtain a deeper understanding of them, and before we analyze them in detail, we performed an alternative derivation via the expression for the Clausius entropy flux.

### 3.2.2. Derivation from the Clausius Entropy Flux

In this derivation, as we have seen in the semi-classical case, we use the flux of Clausius entropy crossing the null boundary of a GLCD to quantify the contribution of matter–energy entropy. However, the standard semi-classical prescription for this derivation lacks validity here. Hence, we first need to modify the definition of Clausius entropy.

The modified Unruh temperature is measured as the standard temperature, by uniformly accelerating observers so entropy is defined in a similar way to the semi-classical case as  $dS_C = \delta Q/T_{GUP}$ . One relevant change in this framework is that, in the semi-classical set up, entropy is calculated in the limit  $a \rightarrow \infty$  [28], when  $\Sigma$  approaching the bifurcate null surface; however, when the Unruh temperature is modified according to Equation (12), the term proportional to  $\psi a^2$  becomes dominant for  $a$  going to infinity. Therefore, we instead need to consider an  $a$  value that is smaller than  $1/\sqrt{\psi}$  but still very large<sup>6</sup>. By introducing these modifications, one can finally find that the time-derivative Clausius entropy takes the form [49]:

$$\frac{dS_C(t)}{dt} = \frac{2\pi k_B c}{\hbar} t \int_{S(t)} T_{\mu\nu}(x(t, \theta, \phi)) k^\mu_\pm k^\nu_\pm d^2\mathcal{A} + O(l^4) + O\left(\psi \frac{l_P^2 a^2}{c^4}\right) + O\left(\frac{1}{a^2}\right), \tag{17}$$

which recovers semi-classical results when  $\psi = 0$ . From this expression, we can directly obtain the total flux of Clausius entropy across the GLCD horizon during its lifetime by integrating it from the bifurcation surface,  $\mathcal{B}$ , at  $t = 0$  to the diamond’s future apex  $A_f$ , at  $t = l/c$  [9,49] obtaining:

$$\Delta S_{Clausius} = -\frac{8\pi^2 k_B l^4}{9\hbar c} \left( T_{00}(P) + \frac{1}{4} T(P) \right) + O(l^5) + O\left(\psi \frac{l_P^2 a^2}{c^4}\right) + O\left(\frac{1}{a^2}\right). \tag{18}$$

Once we obtained this expression, we demanded, as before, thermodynamic equilibrium in addition to the change of entanglement entropy  $S_{e,q}$  associated with the GLCD horizon (this will have the same expression as the other derivation as well as the same origin). In this case, to perform this addition inside the thermodynamic equilibrium,

we assume that, at a leading order in  $l$ , the Clausius entropy is equivalent to the matter entanglement entropy, and this will allow us to check the validity of that equivalence.

We can proceed with the calculations in a similar way as for the other derivation, i.e., finding that the modified gravitational equations of motion result from that process, thus:

$$S_{\mu\nu} - \frac{Cl_p^2}{18\pi} S_{\mu\lambda} S^\lambda{}_\nu + \frac{Cl_p^2}{72\pi} \left( R_{\kappa\lambda} R^{\kappa\lambda} - \frac{1}{4} R^2 \right) g_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \right). \tag{19}$$

We can clearly see that these equations are nearly identical to the ones obtained from the previous derivation. The only variation between them is a numerical difference in the proportionality constant. In this case, the modification term carries a factor of  $-\mathcal{C}/18\pi$ , in contrast to the factor of  $-\mathcal{C}/30\pi$  found in previous derivations. This minor discrepancy could show some modifications in the equivalence between Clausius and matter entanglement entropy when quantum effects become relevant [9,49]. This discrepancy is expected from the classical character of Clausius entropy but it can be interesting to analyze it in detail to understand the relationship among different entropies (this issue will be addressed in a future work)<sup>7</sup>. In any case, the 5/3 difference factor will have no relevant effects on the obtained physics, and its value will also be encompassed in the indeterminacy of the coefficient  $\mathcal{C}$ .

#### 4. Interpretation of the Modified Dynamics

In view of the similarity of both derivations of the modified dynamics, we can write a general expression for the modified equations of motion as:

$$S_{\mu\nu} - D l_p^2 S_{\mu\lambda} S^\lambda{}_\nu + \frac{D l_p^2}{4} \left( R_{\kappa\lambda} R^{\kappa\lambda} - \frac{1}{4} R^2 \right) g_{\mu\nu} = \frac{8\pi G}{c^4} \left( T_{\mu\nu} - \frac{1}{4} T g_{\mu\nu} \right), \tag{20}$$

where  $D = \mathcal{C}/30\pi$  or  $D = \mathcal{C}/18\pi$  for the equations derived using the MVEH or Clausius entropy approaches, respectively.

As we mentioned before, the introduction of quantum effects tips the scale in favor of the emergence of Unimodular Gravity instead of General Relativity. This occurs in part because of the traceless character of the equations, but even more so in that case, where the additional assumption of the local energy–momentum conservation gives rise to:

$$\frac{1}{4} R_{;\mu} - D l_p^2 \left( S^{\lambda\nu} S_{\mu\lambda} \right)_{;\nu} + \frac{D l_p^2}{2} \left( R^{\kappa\lambda} R_{\kappa\lambda;\mu} - \frac{1}{4} R R_{;\mu} \right) = -\frac{2\pi G}{c^4} T_{;\mu}, \tag{21}$$

which, in contrast to the semi-classical dynamics, cannot be generally solved for  $T$ , preventing us from expressing them as modified Einstein equations. This shows in a clear way the tendency of the thermodynamics of spacetime towards Unimodular Gravity rather than General Relativity when introducing phenomenological effects of quantum gravity<sup>8</sup>.

On another note, both derivations are not affected by any possible modifications of the Hawking and Unruh temperatures. This is expected and can be considered as a consistency check of the process by realizing that both effects are kinematic and independent of gravitational dynamics [56].

In order to understand the generality of the equations and their possible predictive power, we need to remember that the modification is completely governed by the extra logarithmic term in the modified horizon entanglement entropy model. Due to the fact that the emergence of this term from quantum gravity effects is predicted by many different methods of calculating entropy, we can conclude that our results are considerable and robust and can point, in the future, towards constraints on the proportionality factor. The extension of the modifications in any model is controlled by the squared Planck length, so, in any case, they become relevant only in a window where the curvature length scale approaches the Planck scale. In our model, the curvature length is significantly larger than

the Planck scale, allowing the consideration of spacetime as a Lorentzian manifold to be retained<sup>9</sup>.

*Application to a Simple Cosmological Model*

In order to better understand the possible physical effects that these modified equations of motion produce, we briefly analyzed a simple flat Friedmann–Lemaître–Robertson–Walker (FLRW) model, whose metric is given by:

$$ds^2 = -c^2 dt^2 + a(t)^2 (dr^2 + r^2 d\Omega^2), \tag{22}$$

where  $a(t)$  is the scale factor. For the sake of simplicity we also assume a universe filled with dust such that  $T_{\mu\nu} = \rho \delta_\mu^0 \delta_\nu^0$ . Solving the phenomenological equations of motion in this model gives rises to a modified Raychaudhuri equation that can be written as:

$$\dot{H} - D \frac{l_p^2 \dot{H}^2}{c^2} = -4\pi G \rho, \tag{23}$$

where  $H = \dot{a}/a$  is the Hubble parameter, and the dot denotes the coordinate time derivative.

In the same way that the modifications of entropy came from an expansion in powers of  $l_p^2$  around the semi-classical value,  $S = k_B A/4l_p^2$ , in this case, the modified Hubble parameter comes from an expansion around the classical one,  $H_0$ , as:

$$H = H_0 + l_p^2 H_1 + O(l_p^4). \tag{24}$$

Taking into account that  $\dot{H}_0$  must satisfy the standard Raychaudhuri equation, we determine:

$$\dot{H} = -4\pi G \rho \left( 1 - 4\pi D \frac{\rho}{\rho_P} \right). \tag{25}$$

In this case, local energy–momentum conservation can be imposed (mainly because of the vanishing of Weyl tensor), resulting in  $\rho = \rho_0/a^3$ , where  $\rho_0$  is an arbitrary constant with the dimensions of energy density. When we substitute this value into the previous equation and integrate it, we obtain a modified Friedmann equation of the form [49]:

$$H^2 = \frac{8\pi G \rho}{3} \left( 1 - \frac{2\pi D \rho}{\rho_P} \right) + \tilde{\Lambda}, \tag{26}$$

where there is an arbitrary integration constant,  $\tilde{\Lambda}$ , which corresponds to the cosmological term ( $\tilde{\Lambda} = \Lambda c^2/3$ ), as it is characterized in Unimodular Gravity. From this equation, we can show how positive modifications to the GLCD entanglement entropy (implying  $D > 0$ ) modify the gravitational dynamics close to the singularity in a way that allows the avoidance of this cosmological singularity. This is also in agreement with the resulting quantum bounce in the effective dynamics of loop quantum cosmology [59]. In that approach, similar equations are obtained, e.g.,  $H^2 = \frac{8\pi G}{3} \rho \left( 1 - \frac{\rho}{\rho_{sup}} \right)$ , which, when are compared to our results, fix  $\rho_{sup} = \rho_P/2\pi D$ <sup>10</sup>. In contrast, the equations indicate that the appearance of corrections with  $D < 0$  would not only not avoid the singularity but even strengthen it. This feature deserves a deeper analysis in future research.

**5. Discussion**

We have reviewed and described general quantum phenomenological gravitational dynamics that emerge from the description of thermodynamics. The low-energy quantum gravity effects are codified in the modification of entropy via a logarithmic extra term. The generality of this framework comes from the universality of the logarithmic modification of entropy in very different approaches to quantum gravity, such as LQG, string theory,

AdS/CFT correspondence, path integral quantum gravity, and some phenomenological approaches, such as GUP.

Our results show how the dynamics are modified when quantum gravity effects become relevant (but we can still consider spacetime as a smooth Lorentzian manifold) via a completely general expression. These equations then allow us to investigate the dynamics in particular models of interest. As a first approach, we studied a simple FLRW cosmological model, where the modified dynamics give rise to the avoidance of the singularity and its replacement through a bounce (in close analogy with the bounce found in LQC). We also expect to find relevant results and connections with other particular effective approaches by analyzing other more complex solutions.

Further analysis of these phenomenological equations in the future could allow us to set some constraints on the approaches to phenomenological quantum gravity effects and the underlying theories, via the analysis of parameter  $D$  and the finding of the action implied by the equations.

In connection with the standard thermodynamics of spacetime, there are two effects that will deserve more analysis in the future. The first is the finding of the expected breakdown of the equivalence of General Relativity and Unimodular Gravity when quantum effects are introduced. This breakdown also shows that the thermodynamics of spacetime appear to imply the emergence of Unimodular Gravity, as opposed to General Relativity. The second idea relates to strengthening the equivalence of Clausius and entanglement entropy, even beyond the semi-classical approach. We have seen that the introduction of quantum gravity effects, despite breaking the complete equivalence, shows the persistence of a strong relationship between both entropies that will be investigated further in future works.

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## Notes

<sup>1</sup> for a deeper analysis of GLCDs, see, e.g., [29–31].

<sup>2</sup> We remember that this interpretation, as are all interpretations, is not free of controversy. It has been argued that this interpretation states that entropy depends on the number of fields and their coupling to gravity [23]. However, some scenarios have been also proposed to solve this issue [39].

<sup>3</sup> More precisely, the behavior of the thermodynamics of local causal horizons under Weyl transformations actually suggests that the derived gravitational values correspond to Weyl Transverse Gravity [41,42]. This is a theory of gravity invariant under both metric determinant preserving diffeomorphisms and Weyl transformations (in fact, Unimodular Gravity can then be understood as a gauge fixed form of Weyl Transverse Gravity). While it has been argued that Unimodular Gravity is not physically distinguishable from General Relativity [43], these arguments do not apply to Weyl Transverse Gravity [44]. Therefore, Weyl Transverse Gravity represents a distinct alternative to General Relativity, which offers a new perspective on some of the problems associated with the value of the cosmological constant [44]. We plan to address the possibility of the emergence of Weyl Transverse Gravity from thermodynamics in a future study.

<sup>4</sup> Note that, by following these requirements, we could also have used light cones [14], and we expect that the result would be equivalent.

<sup>5</sup> For a detailed derivation of this argument, see the discussion in [49].

<sup>6</sup> See that this mathematical consideration agrees with the proposal that quantum gravity establishes a maximal attainable acceleration [57], and that this correction would be consistent with our modification by taking the particular case  $\psi = 2$  [55].

- 7 Some studies have already pointed out some mechanisms for the emergence of a fundamental concept of entropy in the quantum regime, e.g., [58].
- 8 Note that, as we will see in the simple example of a cosmological model, in some particular cases there exists a solution to the previous condition. In that case, we will see that the cosmological constant  $\Lambda$  would appear as an arbitrary integration constant, as it is characteristic in Unimodular Gravity.
- 9 For a complete discussion on the possible features and consistency checks of these equations, see the discussion in [49].
- 10 It is also worth noting that our results are also in agreement with particular GUP-induced modifications of FLRW universes [5].

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