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Closed Orbit Instability for Strong Focusing Lattice *

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Abstract

We extend analysis of Ref. [1] of the closed orbit instability driven by the resistive wall impedance for the case of strong focusing lattice. A numerical estimate for the Low Energy Ring of PEP-II shows that the threshold current is three times higher than the nominal one.

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1 Introduction

It was pointed out in a recent paper [1] that multi-turn accumulation of the transverse resistive wall wake field may lead to a coherent instability of a closed orbit. In contrast to the conventional transverse instabilities, this one is not coupled to the betatron oscillations and is not affected by chromaticity effects or Landau damping.

Assuming that the transverse impedance per unit length is given by $Z_\perp(s,\Omega)$, an equation for the closed orbit trajectory $y(s)$ and its drift eigen-frequencies Ω was found in Ref. [1]:

$$\begin{aligned} y(s) &= \frac{iNr_0}{2\gamma T\sin(\pi\nu)}\oint ds' Z_\perp(s',\Omega) \\ &\times \sqrt{\beta(s)\beta(s')}\cos[\pi\nu-\psi(s,s')](y(s')-y_0(s'))\,. \end{aligned} \tag{1}$$

Here N is the total number of particles in the ring with the classical radius r_0, γ is the relativistic factor, T is the revolution period, ν is the tune, and $\beta(s)$ is the beta-function. The phase advance $\psi(s,s')$ between two points in the ring, s and s', is determined so that $0<\psi(s,s')<2\pi\nu$. The closed orbit coordinate $y(s)$ at azimuth s is measured here relatively its position at zero current, and $y_0(s)$ is an offset of the chamber axis relatively this zero point.

Eq. (1) is valid for arbitrary long-range wake. For the resistive wall impedance of a finite thickness vacuum chamber $Z_\perp(\Omega)$ can be found explicitly [2, 3]

$$Z_\perp(\Omega) = -i\frac{Z_0}{\pi b^2}\begin{cases} 1/(\kappa b), & \text{for } |\delta|\ll d, \\ g/(1-i\Omega/\lambda), & \text{for } |\delta|\gg d. \end{cases} \tag{2}$$

Here $\kappa=(1-i)/\delta$ with $\delta=c/\sqrt{2\pi\sigma\Omega}$ as the skin depth, b and d are the chamber radius and thickness, $Z_0=4\pi/c=377$ Ohm. The numerical factor g and the wake decay rate λ depend on the external surrounding outside of the chamber at $r>b+d$. In case of vacuum, $g=1/2$ and $\lambda=c^2/(2\pi\sigma bd)$; for a vacuum chamber immersed into magnetic material with $\mu\gg 1$, $g=1$ and $\lambda=c^2/(4\pi\sigma bd)$.

In the smooth approximation, solutions of Eq. (1) can be found in terms of Fourier components $y_n=\int y(s)\exp(ins/R)$ and their eigen-frequencies $\Omega_n\equiv i\Gamma_n$, where $R=\Pi/(2\pi)$ is the average radius of the storage ring. As it is discussed below, the resistive impedance has to be taken in the limit of large skin depth, $|\delta|\gg d$. In this case, the eigen-frequency turns out to be equal to

$$\Gamma_n = -\lambda\left(1-\frac{N}{N_s(n)}\right)\,, \tag{3}$$

where $N_s(n)$ is the threshold number of particles for the azimuthal harmonic n,

$$N_s(n) = \frac{2\pi^2 b^2\gamma\nu(\nu-n)}{gr_0\Pi}\,. \tag{4}$$

The most unstable is the mode which number is equal to the tune integer part, $n=[\nu]$; modes with $n>[\nu]$ are always stable. Note that at the threshold Ω vanishes and the skin depth tends to infinity; thus, the thick-skin impedance approximation has to be used in a vicinity of the threshold.

The applicability condition for existence of this modes is the requirement that the skin depth calculated for the frequency Ω_n should be much larger than the wall thickness of the vacuum chamber. Using Eq. (3), it is easy to see that $d/|\delta| \simeq \sqrt{|1 - N/N_s| d/b}$ and the condition $d/|\delta| \ll 1$ is satisfied when $N \ll N_s b/d$. It turns out that the last condition is always satisfied in practice.

2 Strong Focusing Lattice

The smooth approximation may not be accurate enough for real lattices with significant variation of the lattice functions along the orbit. At the same time, variation of the vacuum chamber characteristics in the ring can often be neglected in first approximation. In other words, the resistive impedance can be taken as a constant along the ring. This leads to the following integral equation for the closed orbit motion:

$$\left(1 + \frac{\Gamma}{\lambda}\right) y(s) = \frac{N r_0 Z_0 g}{2\pi b^2 \gamma T \sin(\pi\nu)} \oint ds' \sqrt{\beta(s)\beta(s')} \cos[\pi\nu - \psi(s,s')] y(s') \,. \tag{5}$$

The kernel of this integral equation is real and symmetric, hence all its eigenvalues $1 + \Gamma/\lambda$ are real. This means that all the eigen-frequencies Γ are real, $\mathrm{Im}\Gamma = 0$.

Equation (5) can be simplified if the lattice consists of M identical cells with the phase advance μ_c. Let L be the cell length, $L = \Pi/M$. In what follows, we will also assume that M is a large number, $M \gg 1$, which is typical for rings with strong focusing. The periodicity of the lattice allows the use of the Floquet's theorem — the translation of the eigenfunction $y(s)$ by the period L results in the multiplication by a phase factor: $y(s+L) = \exp(iqL)y(s)$. Since $y(s)$ is also a periodic function with the period Π, the parameter q must satisfy the condition $q = 2\pi n/\Pi$, where n is an arbitrary integer. Introducing a new function $\xi(s)$ such that

$$y(s) = e^{iqs}\xi(s), \tag{6}$$

we conclude that $\xi(s)$ is a periodic function with the period L, $\xi(s) = \xi(s+L)$. Substituting Eq. (6) into Eq. (5) yields

$$\left(1 + \frac{\Gamma}{\lambda}\right) \xi(s) = \frac{N r_0 Z_0 c g}{4\pi^2 b^2 \gamma L(\nu - n)} \int_{s-L}^{s} ds' \sqrt{\beta(s)\beta(s')} \exp(i\chi(s,s'))\xi(s') \,, \tag{7}$$

where $\chi(s,s') = \psi(s,s') - (s-s')\mu_c/L$ is a periodic, with the period L, oscillating part of the phase advance. In the derivation of Eq. (7), we assumed that $\nu, n \gg 1$ and neglected a term of order of $(\nu+n)^{-1}$ in comparison with $(\nu-n)^{-1}$.

Rearranging terms in Eq. (7) we can express the growth rate of the instability as follows

$$\Gamma = \lambda \left(\frac{N}{N_{\mathrm{th}}} - 1\right) , \tag{8}$$

where

$$N_{\rm th} = \frac{N_s}{\Lambda}\,, \tag{9}$$

with N_s as the smooth-approximation threshold given by Eq. (4) and Λ as the largest eigenvalue of the integral equation

$$\Lambda\xi(s) = \frac{\mu_c}{L^2}\int_{s-L}^{s} ds'\sqrt{\beta(s)\beta(s')}\exp(i\chi(s,s'))\xi(s')\,. \tag{10}$$

Note that in a smooth approximation when $\beta(s) = \text{const}$ and $\chi(s,s') = 0$, one finds $\Lambda = 1$. For real lattices the parameter Λ differs from unity and can be calculated by numerical solution of Eq. (10) on the lattice period L.

There is, however, a different way to find the threshold of the instability, which reduces the problem to the solution of a differential equation. To formulate this equation we note that at the threshold, in addition to the design equilibrium orbit in the ring, there exists another static trajectory, given by the solution of Eq. (5) at $\Omega = 0$. The appearance of the new orbit is due to the interaction of the beam with the wall and shifting of the betatron tune to the nearest integer value, $\nu \to \nu - \{\nu\}$. In the case of a periodic lattice, it means that the phase advance per unit cell decreases by $\Delta\mu = 2\pi\{\nu\}/M$; note that since we assume $M \gg 1$, the change of the phase advance is small, $\Delta\mu \ll 1$. Assuming that the interaction with the wall is due to the electrostatic force only (there is no magnetic material outside of the vacuum chamber), it is easy to calculate the image charge force F_y acting on an electron in a round pipe of radius b when the beam offset is equal to y: $F_y = 2e^2Ny/\Pi b^2$. In a more general case, $F_y = 4ge^2Ny/\Pi b^2$, with the factor g introduced in Section 1. With this force, the equation of motion for an electron in a magnetic lattice is

$$\frac{d^2y}{ds^2} + k(s)y = wy\,, \tag{11}$$

where

$$w = \frac{F_y}{mc^2\gamma} = \frac{4Nr_0g}{\gamma\Pi b^2}\,, \tag{12}$$

and $k(s)$ is the focusing strength of the lattice. Using the method of Green's function one can show that the solution of Eq. (11) with the period equal to the ring circumference Π also satisfies Eq. (5) at $\Omega = 0$. Conversely, in case of $\Omega = 0$, the solution of Eq. (5) is also a solution of Eq. (11).

Note that finding a periodic solution of Eq. (11) is an eigenvalue problem — such solutions exist only for special values of the parameter w.

The coherent tune shift in the cell can be found in the thin-lens approximation for a FODO lattice from the Twiss presentation of the cell transfer matrix, where the right hand side of Eq. (11) is considered as a small perturbation. Introducing the drift matrix $\mathbf{M}_d(l)$ for a drift of length l and the focusing quad matrix $\mathbf{M}_q(f)$ with the focal length f,

$$\mathbf{M}_d(l) = \begin{pmatrix} 1 & l \\ 0 & 1 \end{pmatrix}, \qquad \mathbf{M}_q(f) = \begin{pmatrix} 1 & 0 \\ -f^{-1} & 1 \end{pmatrix}, \tag{13}$$

one can calculate that the effect of the weak focusing due to the presence of the right hand side in Eq. (11) results in the change of the drift matrix $\mathbf{M}_d(l) \to \mathbf{M}_d(l) + \mathbf{M}_w(l)$, where

$$\mathbf{M}_w(l) = w \begin{pmatrix} l^2/2 & l^3/6 \\ l & l^2/2 \end{pmatrix} . \tag{14}$$

The full transfer matrix $\mathbf{M}$ for the FODO cell can be written as

$$\begin{aligned} \mathbf{M} \;=\; & \mathbf{M}_q(2f) \cdot (\mathbf{M}_d(L/2) + \mathbf{M}_w(L/2)) \cdot \mathbf{M}_q(-f) \\ & \cdot (\mathbf{M}_d(L/2) + \mathbf{M}_w(L/2)) \cdot \mathbf{M}_q(2f) \, . \end{aligned} \tag{15}$$

The phase advance for this transfer matrix consists of its unperturbed value μ_c and the perturbation due to the image forces $\Delta\mu \propto w$. Keeping only linear terms in w results in the following phase shift $\Delta\mu$

$$\Delta\mu = -wL^2 \frac{5 + \cos(\mu_c)}{12 \sin(\mu_c)} \, . \tag{16}$$

The condition for the instability is that the tune shift $M\Delta\mu/(2\pi)$ for the ring is equal to the fractional part of the nominal tune

$$\Delta\mu = 2\pi \frac{\{\nu\}}{M} . \tag{17}$$

Comparing this condition with Eqs. (9) and (4) gives the parameter Λ for the FODO lattice

$$\Lambda = \frac{\mu_c(5 + \cos(\mu_c))}{6 \sin(\mu_c)} \, . \tag{18}$$

The plot of Λ as a the function the bare phase advance μ_c is shown in Fig. (1). Note that $\Lambda = 1.31$ for a 90 degrees lattice.

3 Estimate for PEP-II

We will estimate the instability condition for the Low Energy Ring (LER) of the PEP-II assuming a lattice with 90 degrees phase advance, $\mu_c = 90°$, and $L = 15.2$ m, $M \approx 150$, $\{\nu\} \approx 0.5$. The ring consists of straight sections, with the round pipe of radius $b = 4.76$ cm, and arcs where the pipe cross section can be approximated by an ellipsoid with the vertical half axis 2.5 cm and the horizontal one 4.8 cm. The length of the straight sections is 517 m, and the total length of the arcs is 1522 m. The ring circumference Π is 2200 m and the magnets occupy only a small fraction of the ring. In the ellipsoidal part of the vacuum chamber we will approximate the pipe by two parallel horizontal plates located at the distance equal to 5 cm.

For parallel plates without magnetic material outside, the impedance factor g in Eq. (2) contains two parts: driving and detuning, see Ref. [4]. The driving impedance describes an electro-magnetic reaction on the beam offset, while a test particle is not deflected. The detuning impedance describes a force felt by the test particle when it is deflected, but the

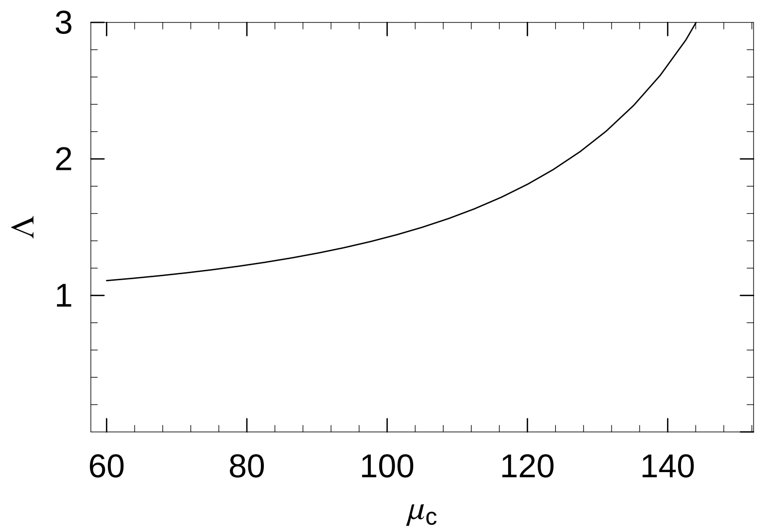

Figure 1: Factor Λ as a function of the bare phase advance μ_c.

beam is on axis. For the problem under study, the test particle has the same offset as the beam; thus, the effective impedance is a sum of the driving and detuning terms. In the low-frequency case, both impedances reduce to a reaction of the image charges and can be calculated by a summation of forces from an infinite series of reflections. This method, presented in Ref. [5] for calculation of the Laslett tune shift (detuning), leads to the detuning contribution to the factor g as $g_{\text{detuning}} = \pi^2/48$, assuming that b is a half-gap between the plates. The same method applied to the driving term gives two times larger value: $g_{\text{driving}} = \pi^2/24$. Both contributions have the same sign (defocusing), so the final result is given by their summation: $g = g_{\text{driving}} + g_{\text{detuning}} = \pi^2/16$.

From Eq. (5) it follows that in case when the pipe properties vary along the ring, the quantity $b^2/(g\Pi)$ should be replaced by the inverse integrated value of g/b^2

$$\frac{b^2}{g\Pi} \to \left(\int ds \frac{g(s)}{b^2(s)} \right)^{-1} .$$

Using this relation and assuming $[\nu] = 36$, $\{\nu\} = 0.5$, $\gamma = 6000$, we find the critical current for the instability in LER of PEPII

$$I_{\text{th}} = 7.2 \text{ A} . \tag{19}$$

This should be compared with the design current $I = 2$ A.

4 Discussion

Even if the beam orbit is stable, the mode can be observed as a slow motion of the beam orbit under the influence of external perturbations. The equation that describes this motion can be obtained in the following way.

In the smooth approximation, the closed orbit equation (1), expressed in terms of the azimuthal Fourier components y_n can be also presented as

$$\frac{1}{\lambda}\frac{dy_n(t)}{dt} + y_n(t) = \frac{N}{N_s(n)}(y_n(t) - y_{0n})\,, \tag{20}$$

where y_{0n} is the Fourier component of the chamber offset relative to the equilibrium position of the beam with a zero current, and a substitution $\Gamma = d/dt$ has been applied. A solution of this equation,

$$y_n(t) = -y_{0n}\frac{N}{N_s(n) - N}(1 - e^{\Gamma_n t}) + y_n(0)e^{\Gamma_n t}\,, \tag{21}$$

shows evolution of the closed orbit from a non-zero initial condition $y_n(0)$. Below the threshold $\Gamma_n < 0$, and the closed orbit returns to its equilibrium position $-y_{0n}N/(N_s(n) - N)$ with time Γ_n^{-1}.

5 Conclusion

In this paper we presented analysis that extends the previous treatment of Ref. [1] of the closed orbit instability driven by the resistive wall impedance. Rather than using an integral equation, we showed that the threshold of the instability can be also found as a solution of a differential equation for the orbit that accounts for the interaction of the beam with image charges in the wall. For large rings, the new approach can be more effective than the direct solution of the integral equation, and it is easily generalizable for non-periodic lattices.

Our estimate for the PEPII LER shows that the critical current for the instability is about 3 times higher than the nominal current in the ring. The instability should be taken into account in the study of future upgrades of PEPII [6] which call for much higher current in the rings. To obtain more accurate estimate for the instability threshold, an additional analysis is needed that takes into account the real lattice of the machine and a possible displacement of the equilibrium orbit from the axis.

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