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Entropic analysis of non-separability in noisy Dicke states

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Abstract
Characterizing quantum entanglement in mixed states is a longstanding challenge. Among the various methods available, conditional entropies serve as a powerful tool. Notably, the AR q-conditional entropy introduced by Abe and Rajagopal in 2002 has demonstrated significant promise as it often surpasses other entropy-based criteria. The wide-ranging applications of conditional entropy in quantum information underscore the importance of studying and analyzing it for a deeper understanding of quantum correlations and their implications. In this paper, we investigate the non-separability of noisy Dicke states using the AR approach of conditional entropy. Our findings reveal that the entropic criterion is equally effective as the PPT criterion in identifying non-separability across a large subset of N-partite noisy Dicke states with even N and excitation number $k = N/2$. Additionally, for systems with $N > 30$ and $k = 1$, the separability thresholds derived from both criteria converge within 10^{-8}, highlighting their strong agreement in this parameter range. Furthermore, we established a condition based on AR q-conditional entropy for identifying genuine multipartite entanglement (GME) in noisy Dicke states and compared its effectiveness to previous methods. Notably, our condition identifies a broader range of GME, particularly when the number of excitations approaches half the number of qubits (i.e., $N/2$). In contrast, previous methods perform better when the number of excitations is significantly less than $N/2$. We believe these results will pave the way for further advancements in entanglement theory and the development of potential quantum-based applications for conditional entropy.

Keywords Quantum entanglement · Mixed states · Conditional entropy · Dicke states

1 Introduction

Entanglement is a fundamental concept in quantum mechanics, driving groundbreaking advancements in quantum technology. It plays a crucial role as a physical resource in the emerging quantum-based technologies and applications. It enables phenomena

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like quantum teleportation [1], secure quantum communication [2], and ultra-precise metrology [3, 4], as well as applications in quantum information processing [5–10], including quantum dense coding [5], quantum key distribution [7], and quantum cryptography [8]. The potential of entanglement extends to emerging fields like quantum image processing [9] and quantum machine learning [10].

Despite its significance, fully characterizing entanglement remains an open challenge. Determining whether an arbitrary mixed quantum state is entangled is a non-trivial task. This has led to two main directions of research: the first aims to detect or quantify the entanglement of an arbitrary state [11–21], while the second focuses on developing completely entangled subspaces to ensure the entanglement of states acting on them [22, 23]. In the first direction, numerous methods have been developed to detect entanglement, each with its strengths and limitations. The earliest method to detect entanglement was through Bell inequalities [11], which are only violated by entangled states. This established a foundational approach to identifying entangled systems. Subsequently, various entanglement criteria were developed, primarily based on the density matrix. Notable examples include the positive partial transpose (PPT) [12, 13], the computable cross norm (CCNR) [14], and the range criteria [15]. Moreover, entanglement witnesses [16], directly measurable observables, offer necessary and sufficient conditions for detecting entanglement. Beyond detection, quantifying the amount of entanglement in a quantum state is crucial. Measures like entanglement distillation [17], concurrence [18, 19], geometric measure [20], and negativity [21] are used to assess this quantity.

Additionally, entropies, measures of disorder, have proven useful in characterizing entanglement. The negativity of von Neumann conditional entropy [24] for entangled states highlights a unique feature of entanglement: entangled systems can exhibit more local disorder than overall disorder, a phenomenon absent in classical systems. To gain a deeper understanding of mixed states, researchers have introduced more general entropy measures like Tsallis q-entropy [26, 27] and α-Renyi entropy [28, 29], which encompass von Neumann entropy as a special case. These measures offer stricter separability criteria but may not always work with arbitrary entangled states. To address the limitations of existing entropy measures, the Abe-Rajagopal (AR) q-conditional entropy was introduced [30, 31]. Derived from Tsallis entropy, AR q-conditional entropy takes negative values for entangled states, making it a valuable tool for identifying entanglement in mixed states. This powerful tool has been applied to study separability in various scenarios, including single-parameter families of mixed multiqubit states [32] and Gaussian states [33].

AR q-conditional entropy demonstrates greater strength in detecting entanglement compared to Bell inequalities and von Neumann conditional entropy, particularly in the case of the two-qubit Werner state $\rho_W = p|\psi\rangle\langle\psi| + (1-p)\mathbb{1}/4$, where $|\psi\rangle$ is a maximally entangled two qubit state. While Bell inequalities can detect entanglement for $p > \frac{1}{\sqrt{2}} \simeq 0.707$ and conditional entropy for $p > 0.748$, AR q-conditional entropy can detect entanglement for $p > \frac{1}{3}$, effectively identifying all entangled Werner states. Interestingly, it has the same strength as the PPT criterion which can detect entangled Werner states for $p > \frac{1}{3}$ [12, 13]. Moreover, there are some other situations for which the PPT criterion and the AR q-conditional entropy have same strength as a separability measure. For instance, in Ref. [34], it has been shown that the AR q-conditional entropy can detect the full range of separability in case of the set of generalized Werner states of N-partite n-level systems. In addition, the detected range of separability using AR q-conditional entropy for the one parameter symmetric 2 and 3-qubit GHZ states has been found to agree with that detected by PPT criterion [32]. However, the PPT criterion stays much stronger in various situations than AR q-conditional entropy. For example, in the study of one parameter symmetric multiqubit W state and the GHZ state with $N > 3$ the range of separability detected by AR q-conditional entropy has been found to be weaker than that obtained from the PPT criterion [32]. Moreover, it has been shown numerically [35] that PPT is much stronger separability criterion than AR q-conditional one in the limit of $q \to \infty$. In addition, there are situations where this PPT superiority in separability detection has been proven in case of Gaussian states [33].

While entropic measures of entanglement may not be superior to existing methods, they offer a valuable perspective on entanglement and its applications. They illuminate non-additive quantum information scenarios [30, 31]. Conditional entropy, in particular, is pivotal in defining concepts like coherent information [36] and quantum discord [37, 38]. Quantum states with negative conditional entropy are crucial for tasks such as quantum state merging [39, 40] and offer quantum advantages in superdense coding [41, 42]. Additionally, they are essential for one-way entanglement distillation [43], maximizing distributed private randomness distillation rates [44], and reducing uncertainty in incompatible measurement predictions [45]. However, entanglement alone is not a guarantee of success in these tasks, as evidenced by entangled states with non-negative conditional entropy. This highlights the necessity of understanding quantum conditional entropy not just as a theoretical construct but as a practical measure that provides a deeper understanding of quantum correlations and their implications for quantum information processing. Moreover, the connection between quantum conditional entropy and thermodynamics is particularly intriguing. Entanglement can be harnessed to extract work from quantum systems [46], and entanglement entropy is linked to thermodynamic properties, such as the capacity to perform work in quantum heat engines [47]. This bridge between quantum mechanics and classical thermodynamics highlights the significance of quantum conditional entropy in understanding the efficiency of quantum systems in thermodynamic processes and could lead to potential applications in the emerging technologies.

In the current work, we study and characterize the separability of mixed N-qubit Dicke states using the AR q-conditional entropy and compare the detected ranges of separability to that obtained from the PPT criterion. First, in Sect. 2, we delve into the theoretical framework employing conditional entropy as an entanglement measure in quantum systems. Subsequently, in Sect. 3, we present our key findings on characterizing the separability of noisy Dicke states. Finally, in Sect. 4, we discuss the outcomes of detecting genuine entanglement in these noisy Dicke states.

2 Theory of entropic measures of entanglement

By leveraging the concept of von Neumann entropy and the distinction between global and local spectra, this approach provides a deeper insight into the fundamental properties of entangled states. The von Neumann Entropy [24] of a state $\rho = \sum_i \lambda_i |i\rangle\langle i|$ with $\langle i|j\rangle = \delta_{ij}$ is given by

$$S(\rho) = -\mathrm{tr}(\rho \log(\rho)) = -\sum_i \lambda_i \log(\lambda_i), \tag{1}$$

where $S(\rho)$ equals 0 for pure states and $\log(d)$ for the maximally mixed state $\rho = \frac{\mathbb{1}}{d}$. For pure composite systems, two scenarios arise: if the state is separable, then the subsystems are pure states, resulting in $S(\rho_i) = 0$ for subsystem i. Conversely, if the system is entangled, the subsystems are in mixed states, leading to $S(\rho_i) \geq 0$. This suggests that the entropic measure $E_i = S(\rho_i)$ could serve as a quantifier of entanglement, where $0 \leq E_i \leq \log(d)$ and $E_i = \log(d)$ for maximally entangled states. However, in the general case of multipartite mixed states, $E_i > 0$ for separable states, since subsystems are already in a mixed state. Consequently, von Neumann entropy is not a reliable quantifier of entanglement in such scenarios.

Entropy can be understood as a measure of randomness in the systems. Moreover, it is known that for classical systems the local randomness cannot be greater than global randomness [25]. Interestingly, this feature extends to separable quantum states. However, if a state is entangled, then its local disorder can be greater than its global disorder. This feature has been used to develop a measure of entanglement in quantum systems, named von Neumann conditional entropy [24, 25]. This measure is similar to Shannon entropy for classical systems and is given as

$$S(A|B) = S(\rho_{AB}) - S(\rho_B), \tag{2}$$

where $S(A|B)$ is the quantum entropy of subsystem A conditioned on B, and ρ_{AB} and ρ_B represent the density matrices of the entire system and the subsystem B, respectively. Notably, this quantity can be negative for entangled states. While $S(A|B)$ is positive for separable states, the converse does not necessarily hold; entangled systems can also exhibit $S(A|B) > 0$. Thus, positivity of the conditional entropy serves as a weak criterion for separability.

To deepen our understanding of mixed states, researchers have introduced Tsallis q-entropy and α-Renyi entropy. However, they may not always effectively characterize entangled systems. To address this limitation, Abe and Rajagopal [30, 31] proposed a conditional version of Tsallis entropy, known as the AR q-conditional entropy, given by

$$S_q(A|B) = \frac{S_q^T(\rho_{AB}) - S_q^T(\rho_B)}{1 + (1-q) S_q^T(\rho_B)}. \tag{3}$$

Moreover, the Tsallis q-entropy $S_q^T(X)$ is defined by

$$S_q^T(X) = \frac{\mathrm{tr}(X^q) - 1}{1-q}, \tag{4}$$

with tr(.) denoting the trace of the relevant density matrix. For any bipartition $A : B$ of the multipartite system we can rewrite Eq. (3) as

$$S_q(A|B) = \frac{1}{q-1}\left[1 - \frac{\mathrm{tr}(\rho_{AB}^q)}{\mathrm{tr}(\rho_B^q)}\right], \tag{5}$$

and in terms of eigenvalues of the density matrices it can be expressed as

$$S_q(A|B) = \frac{1}{q-1}\left[1 - \frac{\sum_j \lambda_j^q(\rho_{AB})}{\sum_k \lambda_k^q(\rho_B)}\right], \tag{6}$$

where $\lambda_j(\rho_{AB})$ and $\lambda_k(\rho_B)$ are the eigenvalues of the whole system ρ_{AB} and its subsystem ρ_B, respectively.

3 Non-separability of noisy Dicke states

Dicke states [48] are a specific class of quantum states that characterize the collective behavior of systems composed of identical particles. These states find applications across various fields, including quantum optics, condensed matter physics, and quantum information processing. Experimental realizations of Dicke states have evolved from six-photon configurations [49] to scalable multi-atom states in atomic vapors [50]. Recent advancements include the on-chip generation and coherent control of four-photon Dicke states using integrated quantum photonics [51]. Efforts to achieve deterministic preparation of these states have led to reliable methods for generating them for experimental applications [52]. Dicke states facilitate protocols such as open-destination teleportation and quantum secret sharing [49]. Additionally, the development of universal gates for transforming multipartite entangled Dicke states has significant implications for cavity quantum electrodynamics, where they can be utilized for a range of quantum information processing tasks [53]. These states are essential resources in quantum communication and quantum computing, serving as initial states for algorithms like Grover's [54]. Moreover, Dicke states can be leveraged as starting points for combinatorial optimization algorithms, thereby enhancing the efficiency of quantum computations [52]. A Dicke state with a total number of particles N and a number of excitations k can be written as,

$$\left|D_N^{\{k\}}\right\rangle = \binom{N}{k}^{-\frac{1}{2}} \sum_j P_j \left\{|1\rangle^{\otimes k} \otimes |0\rangle^{\otimes (N-k)}\right\}, \tag{7}$$

where $\sum_j P_j\{.....\}$ means the sum over all possible permutations. These states are eigenstates of the collective angular momentum operators J^2 and $J_z{=}\frac{1}{2}\sum_k \sigma_z^{\{k\}}$, simultaneously. In the current study, we focus on the Dicke states mixed with white noise and are defined as follows,

$$\rho^{N,k} = y|D_N^{\{k\}}\rangle\langle D_N^{\{k\}}| + (1-y)\frac{\mathbb{1}}{2^N}, \tag{8}$$

where $\mathbb{1}$ is the identity operator, y represents the percent of pure Dicke state in the mixture and has values $y \in [0, 1]$ and $|.\rangle\langle.|$ is the density matrix of the given state. In addition,

$$\rho_r^{N,k} = \mathrm{tr}_r(\rho^{N,k}), \tag{9}$$

is the reduced density matrix obtained by the partial trace of the first r parties from the noisy Dicke state $\rho^{N,k}$.

In the following sections, we are going to study Dicke states with arbitrary N and k. However, it will be helpful to define a maximum number of excitations $k_{\max}$ for every N-partite Dicke state as,

$$k_{\max} = \begin{cases} N/2, & \text{if } N \text{ is even} \\ (N-1)/2, & \text{if } N \text{ is odd,} \end{cases} \tag{10}$$

because the states with certain N which may permit higher number of excitations than $k_{\max}$ would have similar entanglement properties to those with $k < k_{\max}$ which is a result of the symmetry of Dicke states. For that reason we will present calculations and results for the Dicke states with $1 \le k \le k_{\max}$ as would be clear in the following.

3.1 Peres PPT non-separability measure

In this section, we will study the separability of the states defined by Eq. (8) using the Peres PPT criterion. To obtain the ranges in terms of the parameters y, N and k for which the states (8) are separable we will start by studying few special cases first and then deriving the general form for the separability condition.

3.1.1 Three-qubit state

The three-qubit pure and mixed Dicke states with one excitation ($k = 1$) are given, respectively, by

$$\left|D_3^{\{1\}}\right\rangle = \frac{1}{\sqrt{3}}\left(|100\rangle + |010\rangle + |001\rangle\right), \tag{11}$$

$$\rho^{3,1} = y|D_3^{\{1\}}\rangle\langle D_3^{\{1\}}| + (1-y)\frac{\mathbb{1}}{8}.$$

For a bipartite system, the Peres–Horodecki positive partial transposition (PPT) criterion is evaluated by performing a partial transposition of the state's density matrix with respect to one of the two subsystems, followed by computing the eigenvalues of the resulting matrix. If the partially transposed matrix exhibits at least one negative eigenvalue, the state is entangled and referred to as NPT (negative under partial transposition). Conversely, if all eigenvalues are non-negative, the state is termed PPT. However, PPT positivity does not necessarily imply separability, since there exist bound entangled states that remain PPT [12, 13].

To apply this criterion to the tripartite mixed state $\rho^{3,1}$ defined in Eq. (11), we perform a partial transposition with respect to the first qubit. Due to the permutation symmetry of the Dicke basis, the partially transposed matrix acquires a block-diagonal structure. The eigenvalues of this matrix can then be computed analytically. Since the resulting expressions are symbolic, it is not immediately obvious which eigenvalue may become negative without numerical inspection. A practical approach is therefore to identify the eigenvalue that attains the smallest numerical value.

The eigenvalues of the partial transposition of the density matrix $\rho^{3,1}$ are found to be

$$\lambda^{1:4} = \frac{1-y}{8}, \quad \lambda^5 = \frac{3+5y}{24}, \quad \lambda^6 = \frac{3+13y}{24}, \\ \lambda^7 = \frac{1-y}{8} - \frac{\sqrt{2}\,y}{3}, \quad \lambda^8 = \frac{1-y}{8} + \frac{\sqrt{2}\,y}{3}. \tag{12}$$

These are the eight eigenvalues of the partially transposed state, with the first four being degenerate. For $y \in [0, 1]$, it is straightforward to verify that λ^7 attains the smallest numerical value and can become negative for certain values of y. For consistency of notation, we denote this eigenvalue as

$$\lambda^{3,1}_{\min} = \frac{1-y}{8} - \frac{\sqrt{2}\,y}{3}, \tag{13}$$

which remains non-negative in the range

$$0 \leq y \leq \frac{3}{8\sqrt{2}+3}. \tag{14}$$

Hence, the state $\rho^{3,1}$ becomes non-separable for

$$y > \frac{3}{8\sqrt{2}+3}. \tag{15}$$

It is also worth noting that the three-qubit Dicke state with two excitations, $\rho^{3,2}$, yields identical results to $\rho^{3,1}$, as discussed in Eq. (10).

3.1.2 Arbitrary *N* and *k*

To generalize the non-separability condition of the noisy Dicke states defined in Eq. (8), as detected by the PPT criterion, we analyzed the results obtained for several representative cases with different numbers of qubits ($N = 3-6$) and excitation numbers k, as presented in Sects. 3.1.1, A.0.1, and A.0.2.

A clear and systematic dependence of the minimum eigenvalue of the partially transposed density matrix on both N and k was observed. In all cases, the smallest eigenvalue of the partial transpose, which determines the boundary between separability and non-separability, followed a consistent pattern characterized by the symmetry of Dicke states and the structure of the noise term.

The key observation is that the partial transposition of the Dicke density matrix preserves a block structure whose dimensionality and degeneracies depend only on the number of excitations and the total number of qubits. Specifically, the smallest eigenvalue is associated with a 2×2 sub-block involving the coherences between neighboring Dicke subspaces $|D_N^k\rangle$ and $|D_N^{k\pm1}\rangle$. This sub-block structure leads to an eigenvalue expression of the generic form

$$\lambda_{\min}(N, k) = \frac{1-y}{2^N} - f(N, k)\, y, \tag{16}$$

where $f(N, k)$ is a positive function determined by the coupling between symmetric Dicke subspaces. Empirically, from explicit evaluation for $N = 3$–6, we found that $f(N, k)$ scales as $\sqrt{k(N-k)}/N$. Substituting this relation yields the general separability boundary condition

$$\frac{1-y}{2^N} - \frac{y\sqrt{k(N-k)}}{N} = 0, \tag{17}$$

which leads to the compact analytical expression for the threshold value of y:

$$y > \frac{N}{2^N\sqrt{k(N-k)} + N}. \tag{18}$$

This equation defines the range of non-separability predicted by the PPT criterion for any N-qubit Dicke state with excitation number k, where $1 \leq k \leq k_{\max}$ and $k_{\max}$ are defined in Eq. (10). Accordingly, the critical (minimum) value of y above which the state in Eq. (8) becomes entangled can be written as

$$y_{\min}^P = \frac{N}{2^N\sqrt{k(N-k)} + N}. \tag{19}$$

To validate the robustness of this generalized form, we have verified Eq. (19) analytically for 8-qubit and numerically for several systems with larger sizes up to $N = 10$. The analytical prediction exhibited exact agreement with the numerically computed thresholds across all tested configurations. These verifications are summarized in appendix C and Table 3. The complete agreement between the analytical and

Table 1 The results of the calculated non-separability bounds for noisy Dicke states with several values of N and k compared to that obtained from the PPT criterion

$\rho^{N,k}$	AR q-conditional entropy					PPT
	$S_q(A_1\|A_2\dots A_N)$	$S_q(A_1A_2\|A_3\dots A_N)$	$S_q(A_1A_2A_3\|A_4\dots A_N)$	$S_q(A_1\dots A_4\|A_5\dots A_N)$	$S_q(A_1\dots A_5\|A_6\dots A_N)$	
N,k	y^E_{min}	y^E_{min}	y^E_{min}	y^E_{min}	y^E_{min}	y^P_{min}
3, 1	$\frac{3}{11}$	$\frac{9}{17}$	—	—	—	$\frac{3}{8\sqrt{2}+3}$
4, 1	$\frac{1}{5}$	$\frac{3}{11}$	$\frac{7}{11}$	—	—	$\frac{1}{4\sqrt{3}+1}$
4, 2	$\frac{1}{9}$	$\frac{9}{25}$	$\frac{7}{15}$	—	—	$\frac{1}{9}$
5, 1	$\frac{5}{37}$	$\frac{15}{79}$	$\frac{35}{99}$	$\frac{75}{107}$	—	$\frac{5}{69}$
5, 2	$\frac{5}{69}$	$\frac{15}{79}$	$\frac{35}{99}$	$\frac{75}{139}$	—	$\frac{5}{32\sqrt{6}+5}$
6, 1	$\frac{3}{35}$	$\frac{9}{73}$	$\frac{7}{39}$	$\frac{45}{109}$	$\frac{93}{125}$	$\frac{6}{64\sqrt{5}+6}$
6, 2	$\frac{3}{67}$	$\frac{45}{493}$	$\frac{35}{163}$	$\frac{225}{673}$	$\frac{93}{157}$	$\frac{6}{64\sqrt{8}+6}$
6, 3	$\frac{1}{33}$	$\frac{15}{143}$	$\frac{35}{211}$	$\frac{75}{203}$	$\frac{31}{63}$	$\frac{1}{33}$

numerical results strongly supports Eq. (19) as an accurate and general expression for the PPT separability threshold of noisy Dicke states.

It should be emphasized that this relation was not obtained by simple extrapolation, but rather by identifying the structural dependence of the smallest eigenvalue on N and k within the block-diagonal form of the partially transposed state. The observed scaling is consistent with the combinatorial symmetry of Dicke states, where the coupling between symmetric subspaces naturally introduces the $\sqrt{k(N-k)}$ dependence. Therefore, Eq. (19) provides a compact, physically motivated, and numerically verified general formula that captures the onset of non-separability in noisy Dicke states as detected by the PPT criterion.

3.2 Entropic measure of entanglement

In this subsection, we investigate the separability of the noisy Dicke states defined in Eq. (8) using the Abe–Rajagopal (AR) q-conditional entropy, as given in Eqs. (3), (5), and (6). Following the same strategy used for the PPT criterion, we begin by analyzing a few illustrative cases to make the derivation transparent before presenting the general form of the entropic separability condition.

3.2.1 Three-qubit Dicke states

To compute the AR q-conditional entropy in Eq. (6), we first evaluate the eigenvalues of the full state $\rho^{3,1}$ and its reduced forms $\rho_1^{3,1}$ and $\rho_2^{3,1}$. Because Dicke states are completely symmetric under particle exchange, each reduced density matrix obtained by tracing out a subset of qubits is identical, which substantially simplifies the eigenvalue calculations. The permutation symmetry implies that many basis vectors belong to the same symmetric subspace and therefore the corresponding eigenvalues of the density matrices are degenerate. In practice, this means that sums over all eigenvalues reduce to sums over a small set of distinct eigenvalues weighted by their multiplicities. Exploiting this degeneracy enables an analytical evaluation of the sums appearing in the AR conditional entropy without diagonalizing the full 8×8 matrix numerically.

The eigenvalues of the three-qubit mixed Dicke state and its reductions are therefore obtained in closed form as

$$\lambda^{1:7} = \frac{1-y}{8}, \quad \lambda^{8} = \frac{1+7y}{8}, \tag{20}$$

$$\lambda_1^{1:2} = \frac{1-y}{4}, \quad \lambda_1^{3} = \frac{3+y}{12}, \quad \lambda_1^{4} = \frac{3+5y}{12}, \tag{21}$$

$$\lambda_2^{1} = \frac{3-y}{6}, \quad \lambda_2^{2} = \frac{3+y}{6}. \tag{22}$$

Here, λ_r^i denotes the ith eigenvalue of the reduced state $\rho_r^{N,k}$; for instance, $i \equiv \{1:7\}$ indicates seven degenerate eigenvalues, while $r \equiv \{2\}$ labels the eigenvalues of $\rho_2^{3,1}$ obtained by tracing out the first two qubits from $\rho^{3,1}$.

For this case, there are two distinct conditional entropies of the form in Eq. (6), namely $S_q(A_1|A_2A_3)$ and $S_q(A_1A_2|A_3)$. We focus on the first one, $S_q(A_1|A_2A_3)$,

Table 2 This table presents selected results from Table 1, formatted for improved comparison

N	k	y^P_{min}	y^E_{min}
3	1	$\frac{3}{8\sqrt{2}+3} \approx 0.20959$	$\frac{3}{11} \approx 0.27273$
4	1	$\frac{1}{4\sqrt{3}+1} \approx 0.12613$	$\frac{1}{5} \approx 0.2$
4	2	$\frac{1}{9} \approx 0.11111$	$\frac{1}{9} \approx 0.11111$
5	1	$\frac{5}{69} \approx 0.07246$	$\frac{5}{37} \approx 0.13513$
5	2	$\frac{5}{32\sqrt{6}+5} \approx 0.05996$	$\frac{5}{69} \approx 0.07246$
6	1	$\frac{6}{64\sqrt{5}+6} \approx 0.04024$	$\frac{3}{35} \approx 0.08571$
6	2	$\frac{6}{64\sqrt{8}+6} \approx 0.03208$	$\frac{3}{67} \approx 0.04478$
6	3	$\frac{1}{33} \approx 0.03030$	$\frac{1}{33} \approx 0.03030$

Specifically, it presents the non-separability bounds utilizing the PPT criterion and the entropic measure $S_q(A_1|A_2 \ldots A_N)$ for various values of N and k

since it provides a stronger indicator of non-separability (see Table 1). Substituting the eigenvalues from Eqs. (20) and (21) into Eq. (6), we obtain

$$S_q(A_1|A_2A_3) = \frac{1}{q-1}\left[1 - \frac{\sum_j (\lambda^j)^q}{\sum_k (\lambda_1^k)^q}\right], \tag{23}$$

where the numerator represents the sum of the eigenvalues of $\rho^{3,1}$ each raised to the power q, while the denominator contains the corresponding sum for the reduced state $\rho_1^{3,1}$. Substituting the explicit forms of the eigenvalues from Eqs. (20) and (21) and simplifying gives

$$S_q(A_1|A_2A_3) = \frac{1}{q-1}\left[1 - \frac{\left(\frac{3}{2}\right)^q \left[7(1-y)^q + (1+7y)^q\right]}{2(3-3y)^q + (3+y)^q + (3+5y)^q}\right]. \tag{24}$$

The derivation above relies on the spectral degeneracies produced by permutation symmetry: rather than summing over eight distinct eigenvalues, the sums in Eq. (23) collapse to combinations of a few distinct eigenvalues multiplied by their multiplicities (for example, the factor 7 multiplying $(1-y)^q$ in the numerator of Eq. (24)). This reduction is the concrete reason the conditional entropy admits a closed-form expression in this symmetric setting.

It is worth noting that the three-qubit Dicke state with two excitations ($k = 2$) yields identical results to the case with one excitation ($k = 1$), reflecting the exchange symmetry $\left|D_3^{\{2\}}\right\rangle \leftrightarrow \left|D_3^{\{1\}}\right\rangle$. In the limit $q \to \infty$, $S_q(A_1|A_2A_3)$ remains positive for $0 < y < \frac{3}{11}$ (see the solid magenta line in Fig. 1). Thus, the AR q-conditional entropy in Eq. (24) becomes negative and the state $\rho^{3,1}$ is entangled for

$$y > \frac{3}{11}. \tag{25}$$

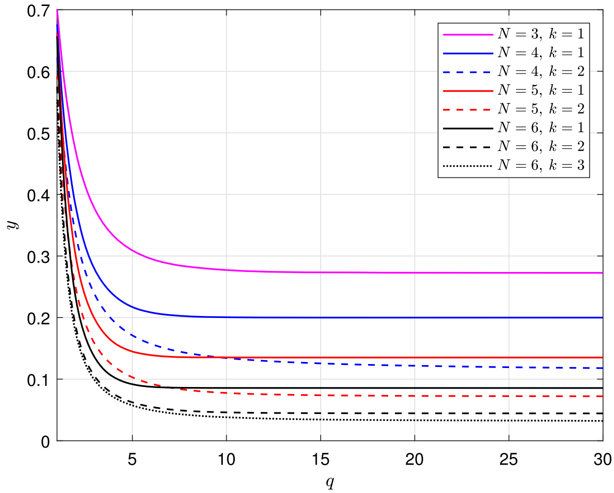

Fig. 1 Minimum value of y for which $S_q(A_1|A_2 \ldots A_N)$ remains positive (i.e., $S_q(A_1|A_2 \ldots A_N) \geq 0$) as a function of the parameter q, shown for different numbers of qubits N and excitations k

3.2.2 States with arbitrary *N* and *k*

Building on the analytical results presented in Sect. 3.2.1 and Appendix B, where explicit derivations were carried out for systems with different numbers of qubits, we now generalize the AR q-conditional entropy separability criterion to arbitrary N and excitation number k. The key observation enabling this generalization is that Dicke states belong to the fully symmetric subspace of the N-qubit Hilbert space. This symmetry implies that all basis states with the same excitation number are equivalent under particle exchange, and thus the density matrix $\rho^{N,k}$ and its reduced forms decompose into a small number of invariant symmetric blocks. Consequently, their eigenvalue spectra display strong degeneracies, allowing the conditional entropy to be expressed in a compact analytical form without performing a full diagonalization for each N.

By exploiting this structural property and extending the patterns observed in the explicitly solved cases for $N = 3$–6, the eigenvalues of the general noisy Dicke state $\rho^{N,k}$ take the form

$$\lambda^{1:(2^N-1)} = \frac{1-y}{2^N}, \qquad \lambda^{2^N} = \frac{1+(2^N-1)y}{2^N}. \tag{26}$$

Here, the $(2^N - 1)$ degenerate eigenvalues correspond to the uniformly mixed component of the state, while the non-degenerate eigenvalue represents the symmetric Dicke component weighted by y. This simple spectral structure is a direct manifestation of the permutation invariance of Dicke states and the isotropic nature of the depolarizing noise.

Among all possible AR conditional entropies, $S_q(A_1|A_2 \ldots A_N)$ was found to provide the strongest separability criterion. To evaluate it, we calculate the eigenvalues of the single-qubit reduced density matrix $\rho_1^{N,k}$, which is identical for any traced subsystem owing to the full symmetry of $\rho^{N,k}$. The reduced matrix exhibits three distinct eigenvalue classes: two are associated with populations of the excited level (arising from the cases where the traced qubit is in the excited or ground state, respectively), and one corresponds to the ground-level population of the traced qubit. The general spectral form can thus be expressed as

$$\lambda_1^{1:(2^{N-1}-2)} = \frac{1-y}{2^{(N-1)}},$$
$$\lambda_1^{(2^{N-1}-1)} = \frac{N + (k\, 2^{(N-1)} - N)y}{N\, 2^{(N-1)}},$$
$$\lambda_1^{2^{N-1}} = \frac{N + ((N-k)\, 2^{(N-1)} - N)y}{N\, 2^{(N-1)}}. \tag{27}$$

This structure highlights how the relative populations of excited and ground levels, determined by k and $(N-k)$, are reflected in the reduced density matrix through these three eigenvalue classes.

Substituting Eqs. (26) and (27) into Eq. (6), we obtain the general closed-form expression for the AR q-conditional entropy:

$$S_q(A_1|A_2 \ldots A_N) = \frac{1}{q-1} \times \left[1 - \frac{(\frac{N}{2})^q((2^N-1)(1-y)^q + (1+(2^N-1)y)^q)}{(2^{(N-1)}-2)(N-Ny)^q + (N+(k\,2^{(N-1)}-N)y)^q + (N+((N-k)\,2^{(N-1)}-N)y)^q}\right]. \tag{28}$$

In this expression, the multiplicative factors such as $(2^N - 1)$ and $(2^{(N-1)} - 2)$ encode the degeneracies induced by the underlying permutation symmetry of the Dicke state and its reductions. These degeneracies effectively compress the exponential dimensionality of the Hilbert space into a concise algebraic representation, making Eq. (28) both computationally and conceptually tractable.

Figure 1 illustrates the dependence of the smallest y value for which $S_q(A_1|A_2 \ldots A_N)$ remains non-negative as a function of q for different values of N and k. This threshold marks the boundary between separable and entangled regions according to the AR criterion.

From Eq. (28), the range over which $S_q(A_1|A_2 \ldots A_N)$ remains positive in the limit $q \to \infty$ can be evaluated analytically as

$$0 \leq y < \frac{N}{k\, 2^N + N}. \tag{29}$$

Table 3 Comparison between analytically predicted and numerically computed thresholds for higher-dimensional noisy Dicke states ($N = 7$–10)

N	k	y^P_{min}		y^E_{min}	
		Numerical	Using Eq. (19)	Numerical	Using Eq. (30)
7	2	0.016999	0.016999	0.026615	0.026615
8	3	0.008004	0.008004	0.010309	0.010309
8	4	0.0077519	0.0077519	0.0077519	0.0077519
9	3	0.004126	0.004126	0.005825	0.005825
9	4	0.003915	0.003915	0.004375	0.004375
10	3	0.002126	0.002126	0.003244	0.003244
10	5	0.001949	0.001949	0.001949	0.001949

For all cases, the minimum y values obtained numerically coincide exactly with those given by the analytical expressions in Eqs. (19) and (30), confirming the correctness and general applicability of the derived formulas

Accordingly, the minimum value of y above which the state $\rho^{N,k}$ becomes entangled according to the AR q-conditional entropy criterion is given by

$$y^E_{min} = \frac{N}{k\,2^N + N}. \tag{30}$$

The entropic separability bound, derived via the approach of Abe and Rajagopal, differs quantitatively from the bound obtained using the Positive Partial Transpose (PPT) criterion (Eq. (19)). Crucially, however, both bounds share the same overall scaling with respect to N and k. Table 1 summarizes a full comparison of the two methods. The general applicability of the entropic bound (Eq. (30)) is further validated by several results. These include additional 3, 4, 5 and 6-qubit case studies presented in Fig. 1, selected quantitative results in Table 2, and high-dimensional system findings in Table 3.

3.3 Entangled states completely detectable by the entropic measure

As can be seen from the results of the previous sections, the PPT criterion is always stronger as a measure of separability. However, we noted that for some mixed Dicke states, with certain values for N and k, both the PPT and entropic measures show the same power as a separability measure. Although this is not always the case for all values of N and k but it represent an important result of the current work and it shows an interesting behavior for the Dicke states. Now the important question of when the separability of a mixed Dicke state of the form (8) can be detected completely by the AR q-conditional entropy (6) with the same detection strength as the famous PPT criterion. In fact from the above presented results and in particular those presented in Tables 1 and 2 we noted that this happens in case of states fulfilling two conditions: firstly, it has even N and secondly has number of excitations $k = N/2$. Only, in this case the AR-q-conditional entropy gives the same results as Peres criterion as

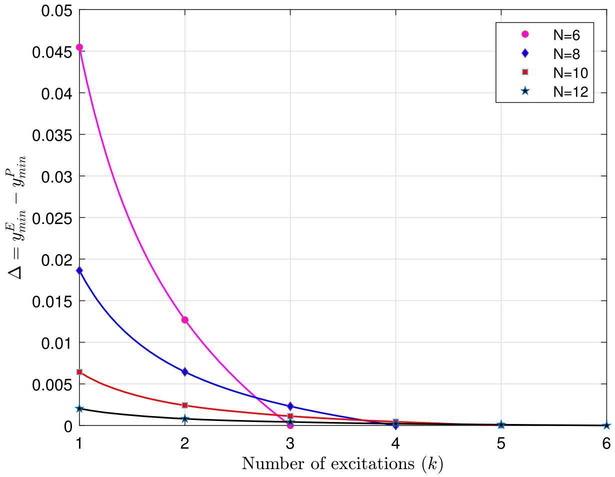

Fig. 2 Difference between the AR q-conditional entropy and Peres PPT non-separability bounds, Δ, plotted as a function of the excitation number k for several values of N. The markers denote the actual computed data points, while the smooth connecting lines are included solely to enhance the visual clarity of the trends, as k takes only integer values

a measure of separability in which case the bound for non-separability from both measures is

$$y > \frac{1}{2^{(N-1)} + 1} \tag{31}$$

It is also interesting to check how the form of Eq. (31) can be obtained from both Eqns. (19) and (30). By putting $k = N/2$ in both equations, it reach the same form as Eq. (31). This is an interesting result, as the entanglement of any mixed Dicke state of the form (8) can be detected completely using entropic measure of entanglement if it fulfills $N = 2k$. Moreover, for the states of the form (8), the difference between the PPT and entropic non-separability bounds reads

$$\Delta = y^E_{min} - y^P_{min}, \tag{32}$$

will reach zero for the cases in which $N = 2k$.

So by plotting the difference between AR-q-conditional entropy and Peres non-separability bounds Δ as a function of the number of excitations k we can note that it is always decreasing by increasing k and it does not reach zero unless the condition $N = 2k$ is fulfilled as can be seen clearly in Fig. 2. Moreover, we present in Fig. 3 the plot of the difference Δ of Eq. (32) as a function of the number of qubits N. We can easily see that Δ is decreasing fast with increasing the number of qubits. This means that for large N both criteria will have almost the same strength as a separability measure which can be seen from Fig. 3. Moreover, by calculating the limit of Δ for infinitely large N we found it $\lim_{N\to\infty} \Delta = 0$ for any value of excitation number k. Even more interesting we found that $\lim_{N\to 30} \Delta \approx 10^{-8}$ for $k = 1$. This means that for larger number of excitations it will, even, be smaller. These results indicate that for systems containing 30 qubits or higher the two criteria of Peres and AR q-conditional entropy

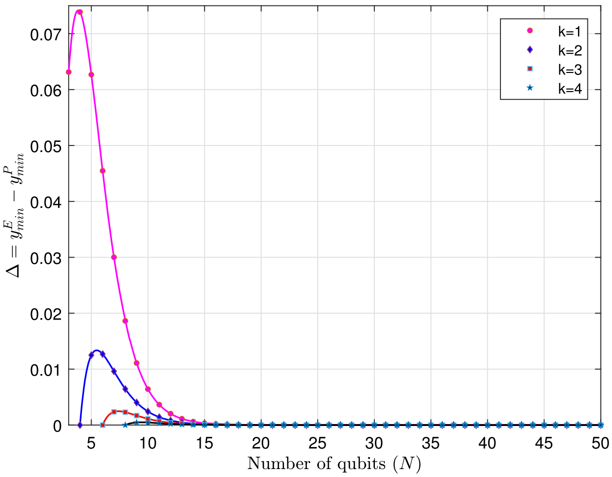

Fig. 3 Difference between the AR q-conditional entropy and Peres PPT non-separability bounds, Δ, plotted as a function of the number of qubits N for various excitation numbers k. The markers indicate the actual data points, whereas the smooth curves are drawn to aid visualization of the overall behavior, since N is discrete and integer-valued

will have approximately the same power to detect entanglement. This is particularly interesting because conditional entropy is crucial for various applications [36–47]. Thus, a robust entanglement criterion based on conditional entropy could significantly benefit practical uses. Moreover, entropic criteria are often easier to implement through measurements compared to other criteria. This makes our results particularly valuable, as we present an entanglement criterion that is as powerful as the PPT criterion for the above-mentioned states while remaining user-friendly in practice.

4 Entropic-based genuine multipartite entanglement criteria for Dicke states

Genuine multipartite entanglement refers to a specific type of quantum systems that involves three or more parties in such a way that the entanglement cannot be reduced to entanglement between any subset of those parties. The detection of genuine multipartite entanglement (GME) in noisy Dicke states is a crucial challenge in quantum information processing. Several criteria [55–60] have been developed to address this issue, including fidelity-based and collective spin-based entanglement witnesses [55]. For example, one approach [57, 58] proposed criteria for detecting GME in n-qubit Dicke states that are more resilient to noise and only require polynomially scaling local observables. Another, method [59] uses semi-definite programming and numerical optimization to identify genuine entanglement, although this is hardly possible for systems with more than seven qubits. Our theory given above indicates that the AR q-conditional entropy in case of multipartite systems is not unique; instead, there are several distinct q-conditional entropies based on the number of subsystems. For instance, in a three-qubit system, there are two distinct AR q-conditional entropies: $S_q(A_1|A_2A_3)$ and $S_q(A_1A_2|A_3)$. For N-partite Dicke states, we generally have $N-1$ distinct conditional entropies. Notably, for Dicke states, the conditional entropy $S_q(s_r|s_{N-r})$, where s_r is any subset of parties with size r and s_{N-r} is the remaining subset, has the same value for any fixed r, regardless of the specific elements in s_r. Thus, in three-qubit Dicke states, e.g., the values of $S_q(A_1|A_2A_3)$, $S_q(A_2|A_1A_3)$ and $S_q(A_3|A_1A_2)$ are all equal. Additionally, it has been observed [32, 34] for several states that $S_q(A_1|A_2 \ldots A_N)$ imposes a stronger condition for separability, although it does not by itself guarantee genuine entanglement. These findings apply to Dicke states as well. Moreover, $S_q(A_1 \ldots A_{N-1}|A_N)$ has been shown to be the weakest condition for separability among all distinct AR q-conditional entropies. Consequently, if $S_q(A_1 \ldots A_{N-1}|A_N)$ is negative, it ensures the negativity of all other distinct AR q-conditional entropies, confirming entanglement across all possible bipartitions of the state and indicating that the state is genuinely entangled. In this case, the range where $S_q(A_1 \ldots A_{N-1}|A_N)$ is positive has been determined using results from Table 1 which is

$$0 \leq y < \frac{N(2^{N-1}-1)}{(N+2k)\,2^{N-1}-N}. \tag{33}$$

Eventually, we found that the state (8) is genuinely entangled for,

$$y > \frac{N(2^{N-1}-1)}{(N+2k)\,2^{N-1}-N}, \tag{34}$$

This developed criterion significantly expands the detection range for genuine multipartite entanglement in the noisy Dicke states $\rho^{4,2}$ compared to the methods in references [56–58]. Specifically, the criterion from Ref. [56] identifies genuine entanglement for $y > \frac{13}{21} \approx 0.619$, while Ref. [57, 58] finds it for $y > \frac{9}{17} \approx 0.529$. In contrast, our entropic-based criterion (34) detects genuine entanglement for $y > \frac{7}{15} \approx 0.467$, which is notably larger than both previous results. Although the numerical method in [59] detects genuine entanglement for $\rho^{4,2}$ with a threshold of $y > 0.461$, which is slightly better than our criterion, it is also more computationally demanding compared to our approach, which relies on direct calculation.

To our knowledge, only a few studies [57, 58, 60] have attempted to establish general criteria for detecting genuine entanglement in noisy Dicke states (8) with arbitrary N and k. The inequality derived in Ref. [57, 58] takes the following form:

$$y > 1 - \frac{2^N}{2^N + (2N-2k-1)\binom{N}{k}}. \tag{35}$$

On the other hand, Ref. [60] provides an analytical condition based on the Positive Partial Transpose (PPT) criterion, which indicates when the state (8) is genuinely entangled. This condition is formulated as follows:

$$y > 1 - \frac{1}{1 + \frac{1}{2^N}\left[\frac{N-k}{k}\right]\sum_{j=0}^{2k}\binom{N}{j} - \frac{N}{k}}. \tag{36}$$

It is intriguing to compare our inequality (34) with those in Eqs. (35) and (36). For the three-qubit state $\rho^{3,1}$, our condition (34) and the one from Eq. (35) share the same detection range of genuine entanglement, $y > \frac{9}{17} \approx 0.529$ which is higher than the range provided by the PPT-based condition (36), where $y > 0.579$. However, as N increases, the scenario changes. For example, when $k = 1$ and $N \geq 5$, both the criteria from (35) and (36) exhibit a broader detection range for genuine entanglement compared to our criterion (34). Conversely, when the number of excitations approaches $N/2$, the criterion from (35) detects a smaller range than both our condition and the PPT-based one. In this regime, our condition (34) performs significantly better, detecting a larger range of genuine entanglement that reaches $y > \frac{1}{2}$ for $N \geq 10$, which matches the range detectable by the PPT-based condition (36) for $k = N/2$. These findings are noteworthy and warrant further investigation, as they could reveal interesting properties of genuinely entangled Dicke states . In summary, our entropic-based condition for detecting genuine entanglement in the states described by (8), as given in (34), is stronger than the other two conditions and identifies broader ranges of genuine entanglement as the number of excitations approaches $N/2$. In contrast, the other two conditions perform better when the number of excitations is significantly less than $N/2$.

5 Possible applications and future work

The present study demonstrates that the Abe–Rajagopal (AR) q-conditional entropy offers a powerful entropic framework for detecting non-separability and genuine multipartite entanglement in noisy Dicke states. While our analysis has focused on depolarizing (white) noise, the developed methodology can be naturally extended to more general and experimentally relevant decoherence models. This section outlines the main directions in which the current results may be further developed and applied.

1. Extension to non-white noise models

A primary avenue for future investigation is the application of the AR entropic criterion to noise models beyond the depolarizing channel:

- **Amplitude-damping noise:** This channel models energy relaxation, where excitations irreversibly decay into the environment. It introduces population asymmetry between the excited and ground states, thereby breaking the uniform permutation symmetry of the Dicke basis.
- **Phase-damping noise:** This process represents pure dephasing without energy exchange. It suppresses quantum coherences while preserving populations, leading to anisotropic decay of off-diagonal elements in the Dicke density matrix.

For such asymmetric noise models, the simple block-diagonal structure of $\rho^{N,k}$ that underpins our analytical derivations no longer holds. The loss of permutation symmetry modifies the coupling between excitation subspaces and alters the eigenvalue spectra of both the global and reduced density matrices. Consequently, the separability thresholds predicted by the AR q-conditional entropy and the PPT criterion are expected to deviate, revealing potentially richer entanglement dynamics. Because the AR framework depends explicitly on the spectral properties of these matrices, it provides a natural and sensitive tool for quantifying how such symmetry-breaking decoherence mechanisms affect multipartite entanglement.

2. Robustness of entanglement in experimental platforms

Dicke-type states arise naturally in various quantum technologies such as atomic ensembles, trapped ions, superconducting qubits, and photonic systems. In all these settings, environmental noise and operational imperfections are inevitable. Future work could use the AR entropic framework to:

- Quantify entanglement robustness under realistic experimental noise conditions;
- Identify parameter regimes where multipartite entanglement remains operationally useful;
- Benchmark AR-based separability thresholds against conventional criteria such as PPT and concurrence.

Such studies would not only validate the analytical results in practical contexts but also help guide the design of entanglement-preserving quantum architectures.

3. Applications in quantum metrology and networking

Because Dicke states are a key resource for achieving sub-shot-noise sensitivity, understanding their entanglement degradation under various noise models is essential. The AR entropic measure can be used to:

- Assess the usability of noisy Dicke states as probes in quantum metrology schemes;
- Analyze distributed entanglement in quantum networks subject to correlated or asymmetric noise;
- Explore how entanglement decay affects collective measurement precision and information transfer.

4. Broader implications and theoretical extensions

Beyond the direct applications to noise characterization, the present work suggests several theoretical directions:

- The parameter q in the AR entropy provides a tunable sensitivity to the spectral structure of the state, enabling selective probing of different correlation regimes;
- Comparative studies with other generalized entropies (e.g., Rényi or Tsallis forms) could reveal universal behaviors in multipartite entanglement detection;
- The AR framework may serve as a diagnostic tool for entropic phase transitions and critical phenomena in complex quantum systems.

The methodology developed here thus establishes a solid foundation for entropic analysis of multipartite entanglement in noisy quantum systems. Extending this framework to non-white noise models, benchmarking it across experimental platforms, and exploiting the tunable nature of the AR entropy are promising directions for future research. Together, these efforts aim to deepen our understanding of how information-theoretic quantities characterize quantum correlations in realistic, decoherence-prone environments and to guide the development of more robust entanglement-based quantum technologies.

6 Conclusions

This work utilized the AR q-conditional entropy to investigate non-separability in noisy Dicke states, successfully deriving analytical formulas that define their entanglement ranges. We found that for a substantial subset of these states which have even N and number of excitations $k = N/2$, the AR q-conditional entropy matches the entanglement detection power of the PPT criterion. Notably, for systems with $N \geq 30$ and a single excitation ($k = 1$), the separability thresholds predicted by the entropic and PPT criteria converge to within 10^{-8}, indicating that both methods exhibit comparable detection capability for large systems. Furthermore, we developed an entropy-based condition for identifying genuine multipartite entanglement (GME) in these states, which outperforms previous methods, especially as the excitation count nears half the qubit number (though older methods may be better for low excitations). These results advance the characterization of entanglement in mixed Dicke states and highlight future research directions concerning both noisy Dicke state properties and the application of conditional entropy for quantifying genuine quantum correlations.

Appendix A Peres PPT measure for more Dicke states

A.0.1 Four-qubit states

The second scenario we examine is the four-qubit state, which, based on condition (10), encompasses two distinct cases. These cases correspond to the number of excitations $k = 1$ and 2. Notably, the case of $k = 3$ yields results identical to those of $k = 1$.

$$\left|D_4^{\{1\}}\right\rangle = \frac{1}{\sqrt{4}}(|1000\rangle + |0100\rangle + |0010\rangle + |0001\rangle), \tag{A1}$$
$$\left|D_4^{\{2\}}\right\rangle = \frac{1}{\sqrt{6}}(|1100\rangle + |0110\rangle + |0011\rangle + |1001\rangle + |1010\rangle + |0101\rangle).$$

By following the same method as in subsection 3.1.1 we found the minimum eigenvalues for the cases of $\rho^{4,1}$ and $\rho^{4,2}$, respectively, as:

$$\lambda_{min}^{4,1} = \frac{1-y}{16} - \frac{\sqrt{3}\,y}{4}, \tag{A2}$$
$$\lambda_{min}^{4,2} = \frac{1-9y}{16},$$

and they are positive in the ranges $0 \leq y \leq \frac{1}{4\sqrt{3}+1}$ and $0 \leq y \leq \frac{1}{9}$, respectively. Clearly, these results prove the entanglement of $\rho^{4,1}$ and $\rho^{4,2}$ in the ranges $y > \frac{1}{4\sqrt{3}+1}$ and $y > \frac{1}{9}$, respectively.

A.0.2 Five and six-qubit states

To reach a general form for the non-separability condition based on PPT criterion we need to study few more cases. For instance, when we have one excitation $k = 1$ the ranges of entanglement detectable by PPT criterion are $y > \frac{5}{69}$ and $y > \frac{6}{64\sqrt{5}+6}$ for the cases of $N = 5$ and 6, respectively. In addition, when $k = 2$ the non-separability ranges detectable by Peres criterion are $y > \frac{5}{32\sqrt{6}+5}$ and $y > \frac{6}{64\sqrt{8}+6}$ for $N = 5$ and 6, respectively. Also, for $k = 3$ the PPT measure can detect non-separability in the range $y > \frac{1}{33}$ when $N = 6$. And as clear from condition (10) we have two nontrivial cases of $k = 1, 2$ for five-qubit and three different excitation numbers $k = 1, 2, 3$ for six-qubit case.

Appendix B Entropic measure for more Dicke states

B.0.1 Four-qubit Dicke states

The calculated eigenvalues of the mixed states $\rho^{4,1}$, $\rho_1^{4,1}$, $\rho_2^{4,1}$ and $\rho_3^{4,1}$ are given, respectively, by

$$\begin{aligned}
&\lambda^{1:15} = \frac{1-y}{16},\ \lambda^{16} = \frac{1+15y}{16}, \\
&\lambda_1^{1:6} = \frac{1-y}{8},\ \lambda_1^{7} = \frac{1+y}{8},\ \lambda_1^{8} = \frac{1+5y}{8}, \\
&\lambda_2^{1:2} = \frac{1-y}{4},\ \lambda_2^{3:4} = \frac{1+y}{4}, \\
&\lambda_3^{1} = \frac{2-y}{4},\ \lambda_3^{2} = \frac{2+y}{4}.
\end{aligned} \tag{B3}$$

Although there are three distinct AR q-conditional entropies, $S_q(A_1|A_2A_3A_4)$, $S_q(A_1A_2|A_3A_4)$ and $S_q(A_1A_2A_3|A_4)$, the first one provides the strongest criterion for separability. This one can be calculated using Eq. (B3) as following

$$S_q(A_1|A_2A_3A_4) = \frac{1}{q-1}\left[1 - \frac{15(1-y)^q + (1+15y)^q}{6(2-2y)^q + (2+2y)^q + (2+10y)^q}\right]. \tag{B4}$$

In the limit $q \to \infty$, we calculated $S_q(A_1|A_2A_3A_4)$ and found it to be negative for $y > \frac{1}{5}$, indicating the non-separability of the state $\rho^{4,1}$ (see the blue solid line of Fig. 1). In addition, the other two AR entropies have led to weaker separability criterion as can be seen from Tables 1 and 2.

Similarly, the eigenvalues of the states $\rho^{4,2}$, $\rho_1^{4,2}$, $\rho_2^{4,2}$ and $\rho_3^{4,2}$ are given, respectively, as following

$$\lambda^{1:15} = \frac{1-y}{16},\ \lambda^{16} = \frac{1+15y}{16},$$
$$\lambda_1^{1:6} = \frac{1-y}{8},\ \lambda_1^{7:8} = \frac{1+3y}{8},$$
$$\lambda_2^{1} = \frac{1-y}{4},\ \lambda_2^{2:3} = \frac{3-y}{12},\ \lambda_2^{4} = \frac{3+5y}{12},$$
$$\lambda_3^{1:2} = \frac{1}{2}. \tag{B5}$$

Similar to the case of one excitation we have three different entropies. By substituting from Eq. (B5) we obtained all the three and found the one which give the strongest separability criterion to be

$$S_q(A_1|A_2A_3A_4) = \frac{1}{q-1}\left[1 - \frac{15(1-y)^q + (1+15y)^q}{6(2-2y)^q + 2(2+6y)^q}\right], \tag{B6}$$

which, succeeded to detect non-separability in the range $y > \frac{1}{9}$ as can be seen from the dashed blue line of Fig. 1 and Table 2. Moreover, the other two entropies have been calculated and are presented in Table 1 and show weaker detectability of entanglement of the state.

B.0.2 Five-qubit Dicke states

Here, we could have two different cases of the number of excitations $k = 1$ and 2 (see Eq. (10)). We study these two cases only as the cases of $k = 3$ and 4 give identical results to the cases of $k = 2$ and 1, respectively. The eigenvalues of mixed state $\rho^{5,1}$ and its reduced forms $\rho_r^{5,1}$ with one excitation are

$$\lambda^{1:31} = \frac{1-y}{32},\ \lambda^{32} = \frac{1+31y}{32},$$
$$\lambda_1^{1:14} = \frac{1-y}{16},\ \lambda_1^{15} = \frac{5+11y}{80},\ \lambda_1^{16} = \frac{5+59y}{80},$$
$$\lambda_2^{1:6} = \frac{1-y}{8},\ \lambda_2^{7} = \frac{5+11y}{40},\ \lambda_2^{8} = \frac{5+19y}{40},$$
$$\lambda_3^{1:2} = \frac{1-y}{4},\ \lambda_3^{3} = \frac{5+3y}{20},\ \lambda_3^{4} = \frac{5+7y}{20},$$
$$\lambda_4^{1} = \frac{5-3y}{10},\ \lambda_4^{2} = \frac{5+3y}{10}. \tag{B7}$$

From which we can obtain the AR q-conditional entropy as

$$S_q(A_1|A_2\ldots A_5) = \frac{1}{q-1}\left[1 - \frac{(\frac{5}{2})^q(31(1-y)^q + (1+31y)^q)}{14(5-5y)^q + (5+11y)^q + (5+59y)^q}\right]. \tag{B8}$$

And the eigenvalues of mixed state $\rho^{5,2}$ and its reduced forms $\rho_r^{5,2}$ with two excitations are

$$\begin{aligned}
&\lambda^{1:31} = \frac{1-y}{32},\ \lambda^{32} = \frac{1+31y}{32}, \\
&\lambda_1^{1:14} = \frac{1-y}{16},\ \lambda_1^{15} = \frac{5+27y}{80},\ \lambda_1^{16} = \frac{5+43y}{80}, \\
&\lambda_2^{1:5} = \frac{1-y}{8},\ \lambda_2^{6} = \frac{5-y}{40},\ \lambda_2^{7} = \frac{5+7y}{40},\ \lambda_2^{8} = \frac{5+19y}{40}, \\
&\lambda_3^{1} = \frac{1-y}{4},\ \lambda_3^{2} = \frac{5+y}{20},\ \lambda_3^{3} = \frac{5-3y}{20},\ \lambda_3^{4} = \frac{5+7y}{20}, \\
&\lambda_4^{1} = \frac{5-y}{10},\ \lambda_4^{2} = \frac{5+y}{10},
\end{aligned} \tag{B9}$$

which leads to the following AR q-conditional entropy

$$S_q(A_1|A_2 \dots A_5) = \frac{1}{q-1}\left[1 - \frac{(\frac{5}{2})^q(31(1-y)^q + (1+31y)^q)}{14(5-5y)^q + (5+27y)^q + (5+43y)^q}\right]. \tag{B10}$$

B.0.3 Six-qubit Dicke states

The eigenvalues for 6-qubit Dicke state $\rho^{6,1}$ and its reduced forms $\rho_r^{6,1}$ with one excitation are given by

$$\begin{aligned}
&\lambda^{1:63} = \frac{1-y}{64},\ \lambda^{64} = \frac{1+63y}{64}, \\
&\lambda_1^{1:30} = \frac{1-y}{32},\ \lambda_1^{31} = \frac{6+26y}{192},\ \lambda_1^{32} = \frac{6+154y}{192}, \\
&\lambda_2^{1:14} = \frac{1-y}{16},\ \lambda_2^{15} = \frac{6+26y}{96},\ \lambda_2^{16} = \frac{6+58y}{96}, \\
&\lambda_3^{1:6} = \frac{1-y}{8},\ \lambda_3^{7:8} = \frac{6+18y}{48}, \\
&\lambda_4^{1:2} = \frac{1-y}{4},\ \lambda_4^{3} = \frac{6+2y}{24},\ \lambda_4^{4} = \frac{6+10y}{24}, \\
&\lambda_5^{1} = \frac{6-4y}{12},\ \lambda_5^{2} = \frac{6+4y}{12},
\end{aligned} \tag{B11}$$

and the corresponding AR q-conditional entropy is

$$S_q(A_1|A_2 \dots A_6) = \frac{1}{q-1}\left[1 - \frac{(\frac{6}{2})^q(63(1-y)^q + (1+63y)^q)}{30(6-6y)^q + (6+26y)^q + (6+154y)^q}\right]. \tag{B12}$$

Moreover, the eigenvalues of the 6-qubit Dicke states $\rho^{6,2}$ and its reduced forms $\rho_r^{6,2}$ with two excitations are given by

$$
\begin{aligned}
&\lambda^{1:63} = \frac{1-y}{64},\ \lambda^{64} = \frac{1+63y}{64},\\
&\lambda_1^{1:30} = \frac{1-y}{32},\ \lambda_1^{31} = \frac{6+58y}{192},\ \lambda_1^{32} = \frac{6+122y}{192},\\
&\lambda_2^{1:13} = \frac{1-y}{16},\ \lambda_2^{14} = \frac{5+27y}{80},\ \lambda_2^{15} = \frac{15+y}{240},\\
&\lambda_2^{16} = \frac{15+113y}{240},\\
&\lambda_3^{1:5} = \frac{1-y}{8},\ \lambda_3^{6:7} = \frac{5+3y}{40},\ \lambda_3^{8} = \frac{5+19y}{40},\\
&\lambda_4^{1} = \frac{1-y}{4},\ \lambda_4^{2} = \frac{5+3y}{20},\ \lambda_4^{3} = \frac{15-11y}{60},\\
&\lambda_4^{4} = \frac{15+17y}{60},\\
&\lambda_5^{1} = \frac{6-2y}{12},\ \lambda_5^{2} = \frac{6+2y}{12},
\end{aligned}
\tag{B13}
$$

from which the AR q-conditional entropy is

$$
S_q(A_1|A_2\ldots A_6) = \frac{1}{q-1}\left[1 - \frac{(\frac{6}{2})^q(63(1-y)^q + (1+63y)^q)}{30(6-6y)^q + (6+58y)^q + (6+122y)^q}\right]. \tag{B14}
$$

While eigenvalues for the 6-qubit Dicke state $\rho^{6,3}$ and its reduced forms $\rho_r^{6,3}$ with 3 excitations are obtained as:

$$
\begin{aligned}
&\lambda^{1:63} = \frac{1-y}{64},\ \lambda^{64} = \frac{1+63y}{64},\\
&\lambda_1^{1:30} = \frac{1-y}{32},\ \lambda_1^{31:32} = \frac{6+90y}{192},\\
&\lambda_2^{1:13} = \frac{1-y}{16},\ \lambda_2^{14:15} = \frac{5+11y}{80},\ \lambda_2^{16} = \frac{5+43y}{80},\\
&\lambda_3^{1:4} = \frac{1-y}{8},\ \lambda_3^{5:6} = \frac{5-3y}{40},\ \lambda_3^{7:8} = \frac{5+13y}{40},\\
&\lambda_4^{1} = \frac{1-y}{4},\ \lambda_4^{2:3} = \frac{5-y}{20},\ \lambda_4^{4} = \frac{5+7y}{20},\\
&\lambda_5^{1:2} = \frac{1}{2},
\end{aligned}
\tag{B15}
$$

and the corresponding AR conditional entropy is

$$S_q(A_1|A_2\dots A_6) = \frac{1}{q-1}\left[1 - \frac{(\frac{6}{2})^q(63(1-y)^q + (1+63y)^q)}{30(6-6y)^q + 2(6+90y)^q}\right]. \tag{B16}$$

Appendix C Verification for higher-dimensional cases: additional examples

In this appendix, we explicitly verify the validity of the analytical expressions derived in the main text for higher-dimensional noisy Dicke states. We perform both analytical and numerical calculations for systems up to $N = 10$ qubits to confirm the applicability and accuracy of the relations obtained for the PPT criterion and the AR conditional entropy.

Specifically, we derive the non-separability range for the eight-qubit Dicke state analytically using both approaches and then implement numerical evaluations for several additional cases ($N = 7$–10). The results presented here demonstrate that the relationships (19) and (30) hold exactly for larger systems, thereby confirming the general validity of the derived formulas.

C.1 Eight-qubit case: analytical verification

We begin with the explicit case of the eight-qubit Dicke state with four excitations ($k = 4$), defined as

$$\rho^{8,4} = y|D_8^{\{4\}}\rangle\langle D_8^{\{4\}}| + (1-y)\frac{\mathbb{1}}{256}. \tag{C17}$$

The minimum eigenvalue of the partially transposed state is found to be

$$\lambda_{min}^{8,4} = \frac{1-y}{256} - \frac{y}{2}, \tag{C18}$$

from which the state in Eq. (C17) becomes non-separable for

$$y > \frac{1}{129}. \tag{C19}$$

This threshold value coincides exactly with the prediction from Eq. (19), thereby confirming the correctness of the general PPT expression.

To compute the AR conditional entropy for this system, we evaluate the eigenvalues of the full density matrix $\rho^{8,4}$ and of its single-qubit reduced form $\rho_1^{8,4}$, obtained by tracing out the first subsystem. The eigenvalues of $\rho^{8,4}$ are

$$\lambda^{1:(255)} = \frac{1-y}{256}, \tag{C20}$$

$$\lambda^{256} = \frac{1+255y}{256},$$

while those of the reduced state $\rho_1^{8,4}$ are

$$\lambda_1^{1:126} = \frac{1-y}{128}, \qquad \lambda_1^{127:128} = \frac{1+63y}{128}. \tag{C21}$$

Substituting Eqs. (C20) and (C21) into Eq. (6) gives the AR q-conditional entropy:

$$S_q(A_1|A_2\dots A_8) = \frac{1}{q-1}\left[1 - \frac{255(1-y)^q + (1+255y)^q}{2^q\,(126(1-y)^q + 2(1+63y)^q)}\right]. \tag{C22}$$

Following the procedure discussed in Section 3.2, we obtain the corresponding non-separability bound

$$y^E_{min} = \frac{1}{129} \approx 0.0077519, \tag{C23}$$

which again agrees exactly with Eq. (30). This analytic agreement between the PPT and AR criteria further reinforces the general validity of the derived relations for higher-dimensional systems.

C.2 Numerical validation for higher-order cases

To further substantiate these analytical findings, we performed numerical optimization to determine the minimum value of y for which the noisy Dicke states defined by Eq. (8) become entangled according to both the PPT and AR criteria. Although computationally demanding, we successfully evaluated y^P_{min} and y^E_{min} for several higher-dimensional states with $N = 7$–10 qubits, in addition to the analytic eight-qubit case discussed above.

The results are summarized in Table 3. For all cases examined, the numerically computed thresholds coincide exactly with the values predicted by Eqs. (19) and (30), confirming the precision and broad applicability of the derived analytical formulas to higher-dimensional systems.

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Data Availability No datasets were generated or analyzed during the current study.

Materials Availability Not applicable.

Code Availability Not applicable.

Declarations

Conflict of interest The authors declare no conflict of interest.

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