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THE ANALYSIS OF SUBHARMONIC RESONANT BEAM KNOCKOUT SYSTEMS

In some experiments time-of-flight techniques are used for the identification of particles.[1] To make this feasible, a beam consisting of widely separated single-bunch pulses is desirable.[2] In Ref. 2, a subharmonic resonant system has been suggested, which produces single transmitted bunches followed by the interception of the next 35 bunches.* Thereby a separation of about 12.5 nsec is provided, adequate to differentiate between particles originating from different beam pulses.

The purpose of this note is to compare various possible arrangements for the subharmonic resonant system. Since no computer program exists at this time which would trace the electron trajectories for the geometries considered, some simplifying assumptions were made. These were the following:

(1) The incoming electrons form a parallel beam with a diameter of 0.63 cm. The beam cross section remains unchanged during deflection.

(2) The beam energy is homogeneous, equals 100 keV.

(3) The beam is bunched to 60° rf.

(4) The effects of field fringing are neglected. It can be shown that fringing in the y-direction (Fig. 1) aids, fringing in the x-direction hinders deflection; it will be assumed that these effects approximately cancel.

(5) The symmetry of the geometry in the x- and y-direction is perfect.

It is hoped that the error due to these approximations is not more than ±2 dB; simple experiments will be performed to test their effect, and if necessary, the theoretical treatment will be augmented taking them into account. In particular, a deflection experiment using dc deflection voltage will be used to find the actual deflection sensitivity and the voltage at which the (shortened) plates intercept the beam. A bench experiment, using a synchronized current generator to simulate the beam loading, may help to verify the theoretical circuit element values as well as the calculated voltages and currents.

The views of the writers of this note were, on some of the topics discussed below, strongly influenced (and sometimes reversed) by stimulating arguments with E. L. Chu, R. Helm, R. Koontz, G. Loew, R. Miller, C. Olson and B. Richter.

*For some other suggested systems, see Refs. 3 and 4.

I. DEFLECTION VOLTAGE, LONG DEFLECTION PLATES

Assume that the physical arrangement for the deflection system is that of Fig. 1; the tentative dimensions are indicated in brackets.

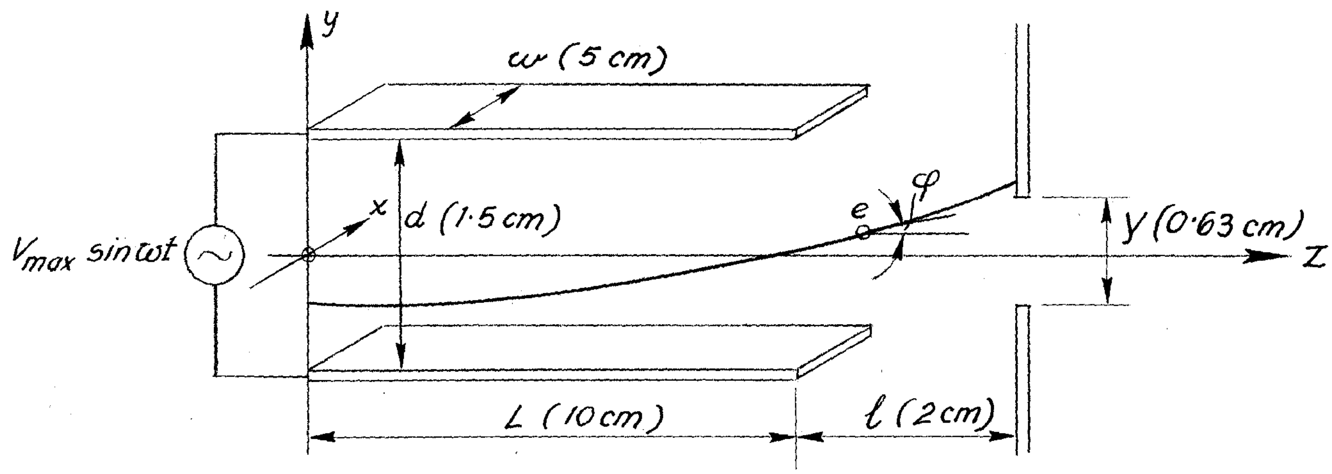

FIG. 1

The design criteria are:

(1) The first electron bunch B_o, arriving at time t_{A_o}, should reach the aperture A undeflected; i.e., $y_o = 0$.

(2) The second bunch B_1, arriving at t_{A_1}, should be intercepted; i.e., assuming a beam-diameter equal to Y, at least $|y_1| = Y$ (Fig. 1).

The transversal momentum gained by the electron is

$$p_y = \int_{t_A}^{t_A+\tau} eE\,dt = -\frac{eV_{max}}{\omega d}\left[\cos\omega(t_A+\tau) - \cos\omega t_A\right], \qquad (1)$$

assuming a homogeneous field. For B_o and B_1, the arrival time is

$$|t_A| < \frac{1}{f_{RF}} \cong 0.35 \text{ ns} \qquad (2)$$

and with the accelerating voltage $V_{acc} = 100$ kV, the total time spent between the electrodes is

$$T \cong \frac{L}{v_z} \cong 0.61 \text{ ns} \tag{3}$$

hence,

$$\omega |t_A + \tau| < 2\pi f_{defl} \left(\frac{1}{f_{RF}} + \frac{L}{v_z} \right) \sim 0.077 \pi \tag{4}$$

and thus the approximation

$$\cos \omega(t_A + \tau) - \cos \omega t_A = - \omega^2 (t_A + \frac{T}{2}) \tau \tag{5}$$

is valid, with an error smaller than

$$\frac{\omega^4 (t_A + T)^4}{24} \sim 3.4 \times 10^{-3} \tag{6.a}$$

i.e., a relative error smaller than

$$\frac{\omega^2 (t_A + T)^3}{24T} \sim 0.382 \% \quad . \tag{6.b}$$

The deflection at the aperture A, at $z = \ell + L$, is (Fig. 1)

$$y = \int_0^L \mathrm{tg}\, \varphi(z)\, dz + \mathrm{tg}\, \varphi(L)\, \ell$$

$$= \int_0^L \frac{p_y(z)}{p_z(z)} dz + \ell\, \mathrm{tg}\, \varphi(L) = \frac{1}{p_z} \int_{t_A}^{t_A+T} p_y\, v_z\, dt + \ell \frac{p_y(t_A + T)}{p_z} \quad . \tag{7}$$

It will now be assumed that $m(z) \cong$ cons., for $0 \leq z \leq L$. Anticipating the result obtained below [in Eq. (8)] for V_{A_0}, V_{D_0}, V_{A_1} and V_{D_1}, it can be shown that for B_o and B_1 the relative mass-increase is

$$\frac{\Delta m}{m} < 1\% \tag{8}$$

hence, the assumption is permissible. Thus

$$\begin{aligned} y &\cong \frac{1}{m} \int_{\tau=0}^{T} p_y(\tau)\, d\tau + \frac{\ell}{p_z} (p_y)_{\tau=T} \\ &= \frac{e}{m} \frac{V_{max}\, \omega T^2}{2d} \left(t_A + \frac{T}{3}\right) + \frac{\ell}{mv_z} \frac{eV_{max}\, \omega}{d} \left(t_A + \frac{T}{2}\right) T \\ &= \frac{e}{m} \frac{V_{max}\, \omega T^2}{2d} \left[t_A \left(1 + \frac{2\ell}{L}\right) + T\left(\frac{1}{3} + \frac{\ell}{L}\right) \right] \end{aligned} \tag{9}$$

Now for B_o, t_{A_o} must be so chosen that

$$y_o\,(t_{A_o}) = 0 \quad , \tag{10}$$

i.e.,

$$t_{A_o} = -\, T \frac{\frac{1}{3} + \frac{\ell}{L}}{1 + \frac{2\ell}{L}} \cong -\, 0.232 \text{ ns}^* \quad . \tag{11}$$

For B_1 then

$$t_{A_1} = -\, \frac{1}{f_{RF}} + t_{A_o} \cong +\, 0.12 \text{ ns} \tag{12}$$

$$y_1 = \frac{e}{m} \frac{V_{max}\, \omega T^2}{2d} \frac{1}{f_{RF}} \left(1 + \frac{2\ell}{L}\right) \tag{13}$$

and the required deflection Y is obtained if

$$V_{max} = \frac{dY}{\frac{e}{m} T^2 \pi \frac{f_{def\ell}}{f_{RF}} \left(1 + \frac{2\ell}{L}\right)} \tag{14}$$

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*The $t = 0$ corresponds to $V = 0$.

Considering an electron energy of 100 keV; i.e.,

$$\frac{m}{m_o} \cong 1 + \frac{eV_{acc}}{m_o c^2} \cong 1 + 1.95 \times 10^{-6}\, V_{acc} = 1.195 \tag{15}$$

$$\frac{v_z}{c} \cong \sqrt{1 - \left(\frac{m_o}{m}\right)^2} \cong 0.548 \tag{17}$$

the required peak voltage is

$$\left| V_{max} \right| \cong 28.4 \text{ kV} . \tag{17}$$

The voltages at the arrival and departure of the two bunches are

(a) $V_{A_o} = 28.4 \sin \omega t_{A_o} = 28.4 \sin \; 2\pi \times 40 \times 10^6 \times (-\,0.232 \times 10^{-9}) \;\; \cong 1.66$ kV

(b) $V_{D_o} = 28.4 \sin \omega t_{D_o} = 28.4 \sin (\omega \times 0.377 \times 10^{-9}) \cong +\,2.7$ kV

(c) $V_{A_1} = 28.4 \sin \omega t_{A_1} = 28.4 \sin (\omega \times 0.12 \times 10^{-9}) \cong +\,0.856$ kV (18)

(d) $V_{D_1} = 28.4 \sin \omega t_{D_1} = 28.4 \sin (\omega \times 0.729 \times 10^{-9}) \cong +\,5.17$ kV

The conditions are illustrated in Fig. 2.

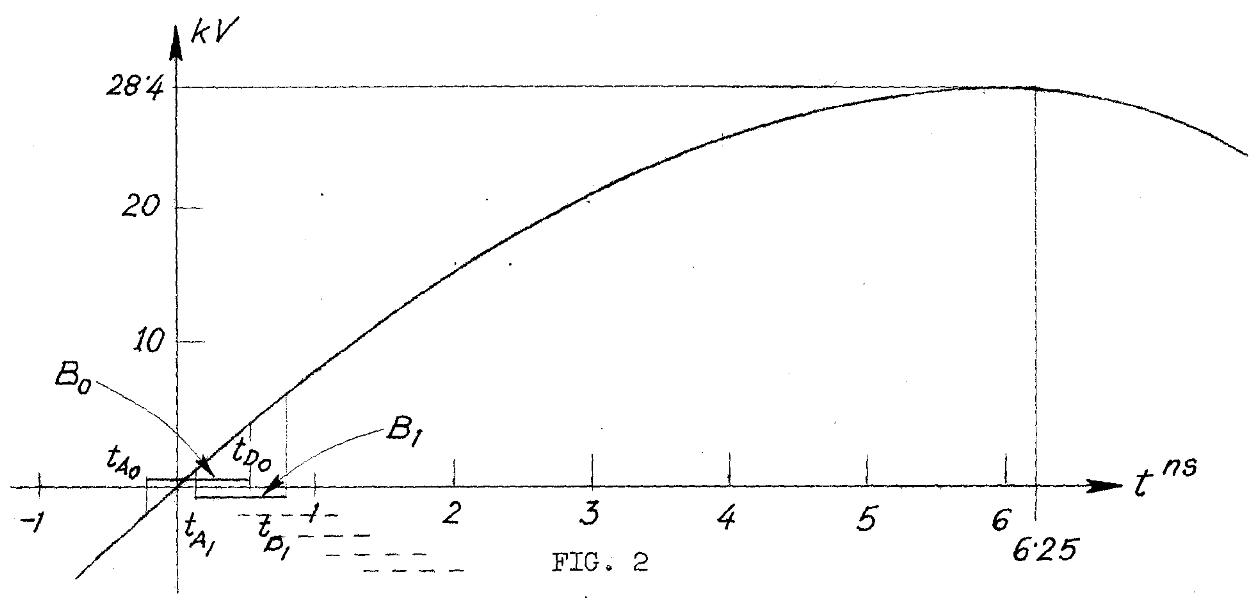


FIG. 2

The angle φ_o of B_o when passing through A can be calculated from (1) and (5):

$$tg\ \varphi_o = \frac{p_y}{p_z} \cong \frac{eV_{max}\ \omega(t_{A_o} + \frac{T}{2})\ T}{md\ v_z} \quad . \tag{19}$$

Using (11) and (14):

$$tg\ \varphi_o \cong \frac{e}{m}\frac{V_{max}}{d}\ \omega\ \frac{T^2}{6(1 + \frac{2L}{L})}\ \frac{1}{v_z} = \frac{f_{RF}\ Y}{3(1 + \frac{2\ell}{L})^2\ v_z} \quad . \tag{20}$$

With the given numerical values

$$tg\ \varphi_o \cong 0.0186 \cong 1.07^o \quad . \tag{21}$$

The peak deflection of B_o is for $v_y = 0$, i.e., by (1) and (2), for $\tau = -2t_{A_o}$. Then

$$y_{o_{max}} = \int_0^{-2t_{A_o}} v_y\ d\tau = \frac{2}{3}\frac{eV_{max}\ \omega}{md}\ t_{A_o} \tag{21}$$

or, by (14)

$$y_{o_{max}} = \frac{4}{3}\ f_{RF}\ T\ \frac{(\frac{1}{3} + \frac{\ell}{L})^3}{(1 + \frac{2\ell}{L})^4}\ Y \cong 0.0576\ cm \tag{23}$$

From (9), (11) and (14), the deflection at $z = \ell + L$ (at the aperture) is simply

$$y = Y\ f_{RF}\ (t_A - t_{A_o}) \tag{24}$$

for electrons with approximately constant mass.

The analysis can also be performed without the simplifying assumption (5); the results are

$$y = \frac{eV_{max}\ T}{m\ \omega d} \left\{ \sin \omega t_A \left[\frac{1 - \cos \omega T}{\omega T} + \frac{\ell}{L} \sin \omega T \right] + \cos \omega t_A \left[\frac{\omega T - \sin \omega T}{\omega T} + \frac{\ell}{L} (1 - \cos \omega T) \right] \right\} \tag{9.a}$$

$$\tan \omega t_{A_o} = - \frac{\omega T - \sin \omega T + \frac{\ell}{L} \omega T (1 - \cos \omega T)}{1 - \cos \omega T + \frac{\ell}{L} \omega T \sin \omega T} \tag{11.a}$$

$$V_{max} = \frac{Yd\ \omega^2 \cos \omega t_{A_o}}{\frac{e}{m} \sin \left(\frac{\omega}{f_{RF}} \right) \left[(1 - \cos \omega T) + \frac{\ell}{L} \omega T \sin \omega T \right]} \tag{14.a}$$

$$y = Y \frac{\sin \omega (t_A - t_{A_o})}{\sin \left(\frac{\omega}{f_{RF}} \right)} \tag{24.a}$$

Using series expansion on functions of $\omega(t_A - t_{A_o})$, ωt_{A_o}, $\frac{\omega}{f_{RF}}$ and ωT, the approximating equations are recovered.

The necessary voltage V_{max} and the corresponding φ_o can be cut to half if in the injector deflection system the 36th subharmonic (~80 mc) is used, rather than the 72nd (~40 Mc). The unwanted beam pulses (every second) will encounter peak deflection voltage in the second stage of the knockout system, which is still driven with 40 Mc, and will thus be removed.

If this means too much dissipated energy and radiation at the Sector 1 end, the order of the two deflection systems can be reversed, as illustrated on Fig. 3.

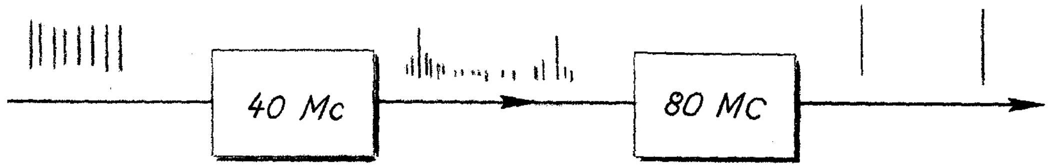


FIG. 3

II. POWER

For power considerations, the current actually reaching the plates has to be considered. It can be shown that the tail end of B_1 already is intercepted by the plates; hence, for beam loading calculations the full beam current I_{beam} can be considered as the loading current. The average electron passes through a voltage drop $\frac{V(t)}{2}$; therefore, the average power needed for the deflection can be obtained from

$$P_{defl} = \frac{1}{\frac{1}{2f_d}} \int_0^{\frac{1}{2f_d}} I_{beam} \frac{V_{max}}{2} \sin \omega t \, dt$$

$$= \frac{I_{beam} V_{max}}{\pi} = \frac{2^A \times 30^{kV}}{\pi} \cong 19.1 \text{ kW} \qquad (25)$$

Here it was assumed that a 30 kV deflection voltage is used, somewhat higher than the approximate value derived in (17); also, I_{beam} is taken as 2A.

Using the circuit of Fig. 4, $P_{def\ell}$ cossesponds to an admittance

$$g_b = \frac{2}{V_{max}^2} P_{def\ell} = \frac{2}{\pi} \frac{I_{beam}}{V_{max}} = 0.0424 \text{ m}\Omega \,, \qquad (a)$$

(26)

$$r_b = 23.6 \text{ k}\Omega \qquad (b)$$

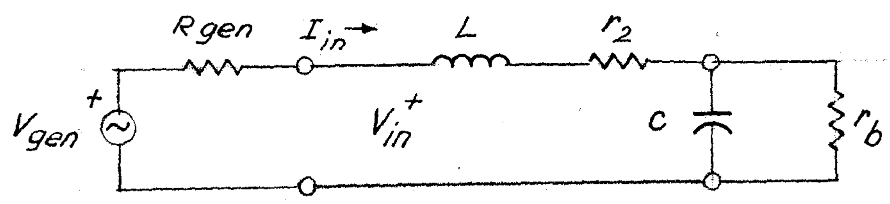


FIG. 4

It must be remembered that r_b is the average value of a nonlinear load, valid for power calculations.

r_b can also be obtained from the equivalent circuit of Fig. 5. a. The waveform of I(t) is shown

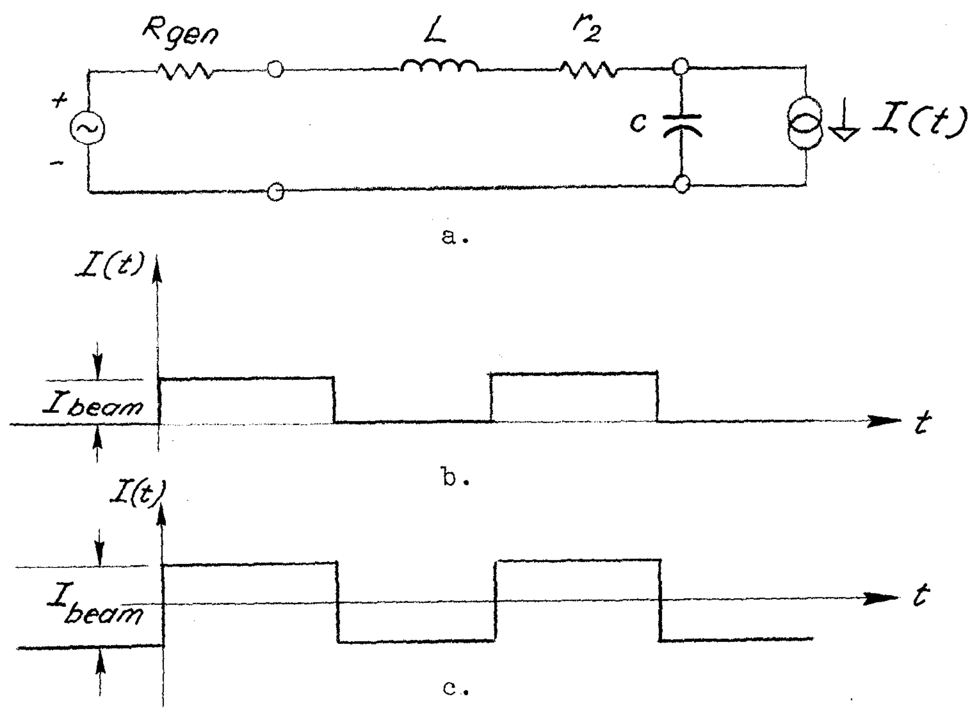
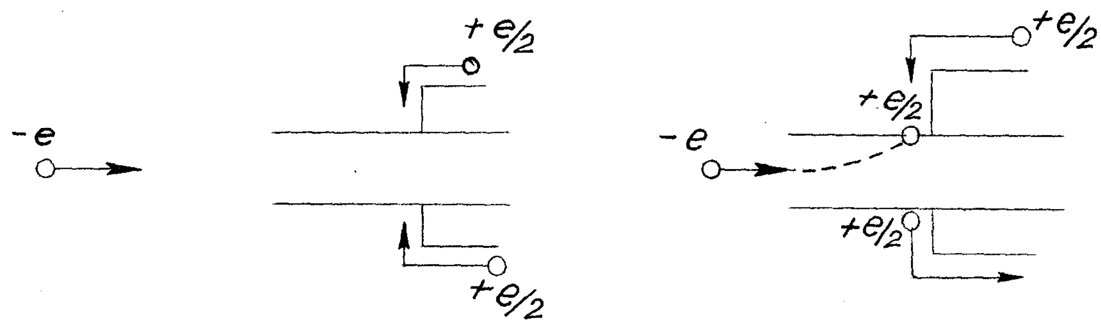


FIG. 5

in Fig. 5.b for unbalanced, 5.c for balanced drive. Fig. 5.b is obvious; 5.c follows from the process illustrated in Fig. 6. The on-axis approach of an electron



a. b.

FIG. 6

induces identical e/2 charges in both plates, a process which affects only the beam's energy slightly (Fig. 6.a). The interception, on the other hand, causes the flow of e/2 charge through the circuit and the generator for every intercepted e (Fig. 6.b). Hence, the peak value of the I(t) square wave is $I_{beam}/2$, as in Fig. 5.c.

In both cases, the fundamental sine-wave component has a peak value $\frac{2}{\pi} I_{beam}$. The other current components have a smaller amplitude and, because of the selective effect of the antiresonant circuit, produce a much smaller voltage (cf. Eqs. (36) - (37), below). Finally, the higher harmonics and the dc component dissipate very little power, since their product with the deflecting voltage has a zero average value. Hence, to a good approximation, they can be neglected and only the fundamental current wave be used in calculations. The fundamental current, in turn, can be represented at a given fixed deflection voltage by a conductance (since I_1 is in phase with $V_{def\ell}$) of the value given by (26).

The design of the circuit of Fig. 4 can now be carried out. The capacitance of the deflector plates is

$$C_d^{(pF)} \cong 8.86 \frac{A^{(m^2)}}{d^{(m)}} \cong 2.95 \text{ pF} . \tag{28}$$

Because of inevitable strays and fringing effects, this is at least doubled in the actual circuit

$$C \geq 6 \text{ pF}, \qquad L \leq \frac{1}{\omega^2 C} = 2.63 \text{ nH} , \tag{29}$$

and

$$Q_C = \omega C \, r_b \geq 35.5 \tag{30}$$

Assuming that $Q_L = 50$, regardless of the size of $\underline{L}$ within a reasonable value range, it is easy to derive from Fig. 4

$$Z_{res} = \frac{V_{in}}{I_{in}} \cong \frac{1}{\omega C} \left(Q_L^{-1} + Q_C^{-1}\right) \tag{31}$$

$$V_{in} \cong jV_{max} \left(Q_L^{-1} + Q_C^{-1}\right) \qquad \text{(a)}$$

$$\text{(32)}$$

$$I_{in} = (j\omega C + g_b) \, V_{max} \cong j\omega C \, V_{max} \qquad \text{(b)}$$

$$P_{in} = \frac{1}{2} V_{in} \, I_{in} = \frac{V_{max}^2}{2 \, r_b} \left(1 + \frac{Q_C}{Q_L}\right) . \tag{33}$$

Hence, for given V_{max}, r_b and Q_L, the necessary power is minimized if Q (i.e., C) is kept as small as possible. With the values given in (29) - (30),

$$P_{in}^{(k\omega)} \geq \frac{(30^{kV})^2}{2 \times 23.6^{k\Omega}} \left(1 + \frac{35.5}{50}\right) \cong 32.7 \text{ kW} , \tag{34}$$

$$|I_{in}| \cong \omega C \, V_{max} = 45.2 \text{ A}, \tag{35}$$

$$|V_{in}| \cong 30^{kV} \left(\frac{1}{50} + \frac{1}{35.5}\right) \cong 1.45 \text{ kV} ,$$

$$Z_{res} \cong 32 \, \Omega .$$

It must be remembered that P_{in} is only the power required in the tank circuit proper; the total required generator power also includes the dissipation in R_{gen}. This loss will be minimized if the mismatch between R_{gen} and Z_{res} is as large as possible. If matching is required, e.g., to avoid reflections in the cable connecting power amplifier and tank circuit, P_{gen} is twice the value of P_{in}.

Another important design consideration is the amount of nonlinearity introduced by the harmonics of the beam current; e.g., the third harmonic,

$$I_3 = \frac{I_{beam}}{2} \frac{4}{3\pi} \sin 3\,\omega t \;, \tag{36}$$

gives rise to a voltage

$$V_3 \cong j \frac{3}{8} \sqrt{\frac{L}{C}}\, I_3 = -\, j \frac{I_{beam}}{4\pi} \frac{1}{\omega C} \sin 3\,\omega t \tag{37}$$

(where $3\omega L \gg R + R_{gen}$ was assumed). Using the value of C in (29),

$$V_{3_{max}} = \frac{I_{beam}}{4\pi\, \omega C} \cong 105 \text{ V} \;. \tag{38}$$

This voltage is 90° out of phase with V_{defl}.

The disadvantages of the above design are the following:

(1) The strays form an appreciable part of C.

(2) The deflecting voltage is quite strongly dependent on I_{beam} because of the low value of Q_C.

(3) V_3 is appreciable; hence, the phase stability may be too low.

The single advantage is the low power consumption.

It may be preferable to "pad" C, by connecting an external capacitor C_e in parallel (C_e may be variable, to facilitate tuning). E.g., if C is thus increased to 20 pF,

$$V'_{in} = V_{in} \frac{Q_L^{-1} + Q_C^{-1}}{Q_L^{-1} + Q_C^{-1}} = 856 \text{ V} \qquad \text{(a)}$$

(39)

$$P'_{in} = P_{in} \frac{1 + \frac{\omega\, r_b}{Q_L} C'}{1 + \frac{\omega\, r_b}{Q_L} C} \cong 64.5 \text{ kW} \qquad \text{(b)}$$

but now

$$Q'_C = 118.5 \tag{40}$$

and

$$V'_{3_{max}} \cong 30.9 \text{ V} \cong 0.1\% \text{ of } V_{max} \tag{41}$$

The natural frequencies of the network of Fig. 4 are

$$p_{1,2} = - d \pm \sqrt{\delta^2 - \omega^2} \cong - d + j\omega \tag{42}$$

where

$$d = \frac{1}{2}\left(\frac{r_L + R_{gen}}{L} + \frac{1}{r_b C}\right) = \frac{1}{2}\left(\frac{R_{gen}}{L} + \frac{\omega}{Q_L} + \frac{\omega}{Q_C}\right) \tag{43}$$

and

$$\delta = \frac{1}{2}\left(\frac{R_{gen}}{L} + \frac{\omega}{Q_L} - \frac{\omega}{Q_C}\right) \tag{44}$$

Hence, the "time-constant" is, by (31)

$$\tau = \frac{1}{d} = \frac{2L}{R_{gen} + Z_{res}} \tag{45}$$

For the 6 pF circuit

$$\tau^{(\mu s)} = \frac{5.26}{R_{gen} + 32} < 0.165\ \mu s\ . \tag{46}$$

For the 20 pF circuit

$$Z'_{res} = 5.65\ \Omega \tag{47}$$

$$L' = 0.79\ \mu H \tag{48}$$

hence

$$\tau' = \frac{1.58}{R_{gen} + 5.65} < 0.28\ \mu s\ . \tag{49}$$

The stability of the deflecting voltage for beam current changes can be approximately calculated from (26) and (32.b). Assuming <u>I_{in} = const.,</u> i.e., $R_{gen} \gg Z_{res}$), from (32.b)

$$\frac{\partial\, V_{max}}{\partial\, I_{beam}} = I_{in} \frac{\partial \left(\frac{1}{j\omega c + g_b} \right)}{\partial I_{beam}} \cong \frac{I_{in}}{(\omega c)^2} \frac{\partial g_b}{\partial I_{beam}} \tag{50}$$

from (26),

$$\frac{\partial g_b}{\partial\, I_{beam}} = \frac{2}{\pi} \frac{V_{max} - I_{beam} \frac{\partial\, g_b}{\partial\, I_{beam}}}{V_{max}^2}\ . \tag{51}$$

Equations (50) - (51) give, with (32.b),

$$\frac{\partial V_{max}}{\partial I_{beam}} \cong -\frac{2}{\pi} \frac{V_{max}}{I_{in}} \cong -\frac{2}{\pi} \frac{1}{j\omega C}\ . \tag{52}$$

Hence, the 40 Mc voltage phase is changed by

$$\Delta\beta \cong \frac{\Delta V_{max}}{V_{max}} = -\frac{2}{\pi}\frac{\Delta I_{beam}}{I_{in}} = -\frac{2}{\pi}\frac{I_{beam}}{I_{in}}\frac{\Delta I_{beam}}{I_{beam}} ; \tag{53}$$

e.g., for 10% beam current change, in the 6 pF curcuit

$$\Delta\beta \cong -\frac{2}{\pi}\frac{2}{45.2}\ 0.1 = -0.0028 = -0.16^{\circ} \tag{54}$$

in the 20 pF circuit

$$\Delta\beta' \cong -0.00084 = -0.048^{\circ} . \tag{55}$$

For larger changes in I_{beam} there is, of course, appreciable amplitude, as well as phase modulation in V_{max}, with the speed defined by τ or τ'.

If, on the other hand, $\underline{V_{in} = \text{const.}}$ (i.e., $R_{gen} \ll Z_{res}$) is assumed

$$\frac{\partial V_{max}}{\partial I_{beam}} = V_{in}\frac{\partial}{\partial g_b}\left[\frac{\frac{1}{pc + g_b}}{r_L + pL + \frac{1}{pC + g_b}}\right]\frac{\partial g_b}{\partial I_{beam}} . \tag{56}$$

Straightforward arithmetic gives from (56) and (51)

$$\frac{\partial V_{max}}{\partial I_{beam}} \cong -\frac{2}{\pi} Q_L\ \omega L = -\frac{2}{\pi}\frac{Q_L}{\omega C} \cong -\frac{2Q_L}{\pi}\frac{V_{max}}{I_{in}} . \tag{57}$$

(Note that Eqs. (52) and (57) could have been obtained directly from the model of Fig. 5, with only the fundamental wave

$$I_1(t) = \frac{2}{\pi} I_{beam} \tag{58}$$

used for I(t). Then, simply,

$$\frac{\partial V_{max}}{\partial I_{beam}} = -\frac{2}{\pi} Z \tag{59}$$

where $\underline{Z}$ is the impedance seen by the current generator, with V_{gen} shorted. Z is illustrated in Fig. 5.)

The value obtained in (57) is jQ_L times that given in (52); hence, the value of the change in V_{max} will be about 50 times larger, but real. For 10% beam current change, therefore,

$$\frac{\Delta V_{max}}{V_{max}} \cong -0.14 = -14\% \tag{60}$$

in the case of the 6 pF circuit; and

$$\frac{\Delta V_{max}}{V_{max}} \cong -4.2\% \tag{61}$$

for the 20 pF network.

Comparison of (54) with (55), or (60) with (61), illustrates the increased stability of the "padded" circuit.

III. ALTERNATIVE GEOMETRIES

If the effects of the beam-loading on the phase and amplitude cannot be tolerated, the geometry may be modified to prevent the beam from reaching the plates. The simplest change is to reduce L, while keeping the total length ($L + \ell = \lambda$) and all other dimensions in Fig. 1 unaltered.

First, the new plate length L_s will be found. Since it is anticipated that the deflection voltage must be significantly increased, the calculations will take into account the relativistic effects.

The transversal momentum of the electrons arriving at

$$\omega t_A = \frac{\pi}{2}, \qquad t_A = \frac{1}{4f}, \tag{62}$$

i.e., at the time of peak deflecting voltage V_{max_s}, increases approximately linearly with time:

$$p_y = \int_{t_A}^{t_A+\tau} eE\, d\tau \cong \frac{eV_{max_s}\,\tau}{d} \tag{63}$$

The mass when entering the plates is

$$m_e = m_o + \frac{eV_{acc}}{c^2} = m_o \left(1 + \frac{V_{acc}}{V_o}\right) \tag{64}$$

and the axial velocity is initially

$$v_{z_o} = c\, \frac{\sqrt{1 + 2\, V_o/V_{acc}}}{1 + V_o/V_{acc}} \;; \tag{65}$$

hence, the (constant) axial momentum is

$$p_z = m_o c \frac{V_{acc}}{V_o} \sqrt{1 + 2 \frac{V_o}{V_{acc}}} \tag{66}$$

where

$$V_o = \frac{m_o c^2}{e} \cong 0.512 \times 10^6 \text{ volts} \tag{67}$$

is the equivalent energy (in eV) of the electron mass at rest. Between the plates, the mass increases:

$$m = \sqrt{m_e^2 + \frac{p_y^2}{c^2}} = \frac{m_e}{\delta} \sqrt{\delta^2 + \tau^2} , \tag{68}$$

where

$$\delta = \frac{m_e cd}{eV_{max}} . \tag{69}$$

The transit time, T, of the electron between the plates can be obtained from

$$L_s = \int_0^T v_z(\tau)\, d\tau = p_z \int_0^T \frac{d\tau}{m(\tau)} \tag{70}$$

Using (68) – (70) ,

$$T = \delta \sinh\left(\frac{L_s}{v_{z_o}\delta}\right) . \tag{71}$$

The deflection at $z = L_s$ $(t = T)$

$$y_T = \int_0^T v_y(\tau)\, d\tau = \int_0^T \frac{p_y(\tau)}{m(\tau)}\, d\tau = 2c\delta \sinh^2\left(\frac{L_s}{2v_{z_o}\delta}\right) \tag{72}$$

where (63), (68) and (71) have been used. For no interception

$$y_T \leq \frac{d}{2} \tag{73}$$

Substituting from (14) the value of V_{max} (with $m \rightarrow m_e$, $L \rightarrow L_s$, $T \rightarrow L_s/v_{z_o}$), we get

$$0 < \left[\frac{c}{v_{z_o}} \frac{2\lambda}{d} x - \frac{y}{d} \frac{f_{RF}}{\pi f}\right] \left[\frac{\sinh \frac{x}{2}}{\frac{x}{2}}\right]^2 \leq 1 \tag{74}$$

where

$$x = \frac{L_s}{v_{z_o} \delta} \quad . \tag{75}$$

Substitution shows that

$$0.33 < x < 0.3761 \tag{76}$$

and hence

$$\left(\frac{\sinh \frac{x}{2}}{\frac{x}{2}} \right)^2 \cong 1 \tag{77}$$

with an error less than 1.2%. Therefore, using (74) as an equality to obtain the maximum permissible plate length,

$$L_s = v_{z_o} \delta \, x = \frac{2\lambda}{1 + \frac{Y}{d} \frac{f_{RF}}{\pi f}} \tag{78}$$

where (14) has again been used. (Note that (14), although obtained from nonrelativistic calculations, remains valid since it refers to B_o and B_1, which are affected by only small deflecting voltages.)

(78), (17) and (14) now give

$$L_s \cong 2.26 \text{ cm} \tag{79}$$

and

$$V_{max\ s} = V_{max} \frac{L(2\lambda - L)}{L_s(2\lambda - L_s)} \cong 80.9 \text{ kV} \ . \tag{80}$$

It will now be shown that it is not expedient to try to decrease V_{max} by permitting part of I_{beam} to reach the plate. The intercepted current waveform obtained by making the length of the plates L', where

$$L_s < L' < L \quad , \tag{81}$$

is illustrated in Fig. 7.

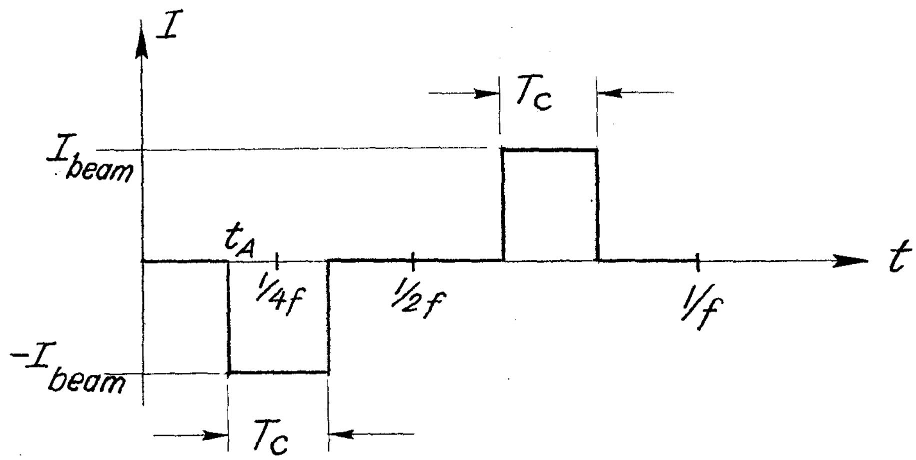


FIG. 7

The fundamental wave of $I(t)$ has an amplitude

$$I_1' = \frac{4I_b}{\pi} \sin \frac{\omega T_c}{2} \quad . \tag{82}$$

The electrons first intercepted enter at $t = t_A$ (Fig. 7) when the deflection voltage is

$$V = V_{max} \sin \omega t_A = V_{max} \sin \left[\omega \left(\frac{1}{4f} - \frac{T_c}{2} \right) \right] = V_{max} \cos \frac{\omega T_c}{2} \quad . \tag{83}$$

The permissible plate-length is now

$$L_s' \leq \frac{2\lambda}{1 + \frac{Y}{d} \frac{f_{RF}}{\pi f} \cos \frac{\omega T_c}{2}} = \frac{L_s}{\frac{L_s}{2\lambda} + \left(1 - \frac{L_s}{2\lambda} \right) \cos \frac{\omega T_c}{2}} \quad . \tag{84}$$

If all electrons reach the plates (square-wave, $T_c = \frac{1}{2f}$), $I_1 - \frac{4I_b}{\pi}$; hence, the reduction due to the shorter plates is $I_1'/I_1 = \sin(\omega T_c/2)$. Thus, for a 20% fundamental beam current interception $(I_1'/I_1 = 0.2)$,

$$L_s' = \frac{L_s}{\frac{L_s}{2\lambda} + \left(1 - \frac{L_s}{2\lambda}\right)\sqrt{1 - \left(\frac{I_1'}{I_1}\right)^2}} \cong 2.3 \text{ cm} \tag{85}$$

while the corresponding deflection voltage is 79.6 kV, or only about 1.6% lower than before. Hence, this is a very inefficient way to decrease V_{max_s}.

The above derivation also indicates that most of the beam current just barely misses the plates if the L_s of (79) is used. Because of the approximations used, it is therefore advisable to make L_s less; e.g.,

$$L_s = 2 \text{ cm} \tag{86}$$

may be suitable. This gives, by (80)

$$V_{max_s} = 90.4 \text{ kV} \tag{87}$$

The power necessary for deflection will next be estimated. For an electron reaching the plates at time t , Eqs. (63) - (72) remain valid if V_{max_s} is replaced by $V(t) = V_{max_s} \sin \omega t$. Hence, by (72) and (69) the voltage drop for this electron is

$$\Delta V \cong \frac{V(t)}{d} y(t) = \frac{2m_e c^2}{e} \sinh^2 \left[\frac{x}{2} \sin \omega t\right] \tag{88}$$

Using (77),

$$\Delta V \cong \frac{m_e c^2 x^2}{2e} \sin^2 \omega t = \frac{eV_{max}^2 L_s^2}{2d^2 m_e v_{z_o}^2} \sin^2 \omega t . \tag{89}$$

The total work during a half-cycle can be found from Fig. 8. The instantaneous power is $I_{beam}\,\Delta V(t)$ and hence

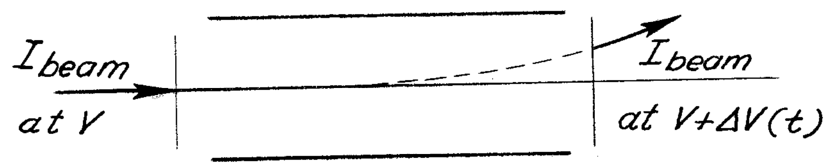


FIG. 8

$$W = \int_0^{\frac{1}{2f}} I_{beam} \; \Delta V(t) \; dt = \frac{e V_{max}^2 \; L_s^2 \; I_{beam}}{8d^2 \; m_e \; v_{z_o}^2 \; f} \tag{90}$$

and the average power is

$$P_{def\ell} = \frac{e V_{max}^2 \; L_s^2 \; I_{beam}}{4 m_e \; v_{z_o}^2 \; d^2} \cong 40 \text{ kW} \tag{91}$$

where (86) - (87) have been used.

The electrons re-radiate most of this power, and thus return it to the field, if the aperture A is kept at ground potential. Hence, only a fraction of P has to be provided in steady state. There will, however, be a small phase-shift in returning the power, corresponding to a small reactive shunting of the plates.

It is estimated that probably less than 5% of the 40 kW power, or 2 kW is lost in the process. Using this value and (87),

$$r_b = \frac{1}{2} \frac{V^2_{max_s}}{P_{diss}} \cong 2.04 \text{ M}\Omega \tag{92}$$

(Fig. 4). Assuming

$$C \cong 5 \text{ pF} \tag{93}$$

for the plates, wiring, etc., and $Q_L = 50$,

$$L = \frac{1}{\omega^2 C} \cong 3.16 \ \mu\text{H} \tag{94}$$

$$Q_C = \omega C r_b \cong 2570 \tag{95}$$

$$I_{max} = \omega C \ V_{max} \cong 113.5 \text{ A} \tag{96}$$

$$Z_{res} = r_L \left(1 + \frac{Q_L}{Q_C}\right) \cong r_L \cong 16 \ \Omega \tag{97}$$

so that

$$V_{in} \cong 1.82 \text{ kV} \tag{98}$$

(Fig. 4) and

$$P_{in} \cong 103 \text{ kW.} \tag{99}$$

For greater stability, C should be "padded," e.g., for $C = 20$ pF,

$$L \cong 0.792\ \mu H \tag{100}$$

$$I_{max} \cong 454\ A \tag{101}$$

$$Z_{res} \cong r_L \cong 4\Omega \tag{102}$$

$$V_{in} \cong 1.82\ kV \tag{103}$$

as before, and

$$P_{in} \cong 413\ kW\ . \tag{104}$$

For both circuits, in a first-order approximation,

$$\frac{\partial V_{max}}{\partial I_{beam}} \cong 0\ , \tag{105}$$

since the beam-loading is negligible.

For the angle φ_o, (20) is applicable; it gives now 0.0003 ra or 0.0173°.

Another possible solution for decreasing the beam-loading is replacing each deflection plate by a pair of rods, as illustrated in Fig. 9. This will be approximated by the idealized arrangement of Fig. 10.

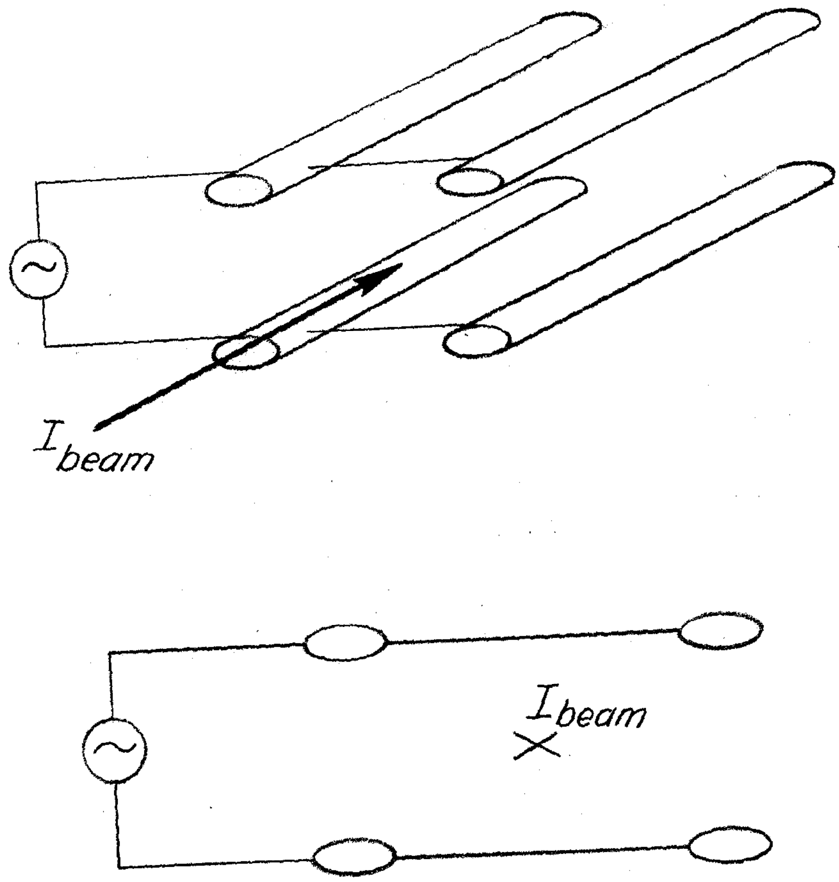


FIG. 9

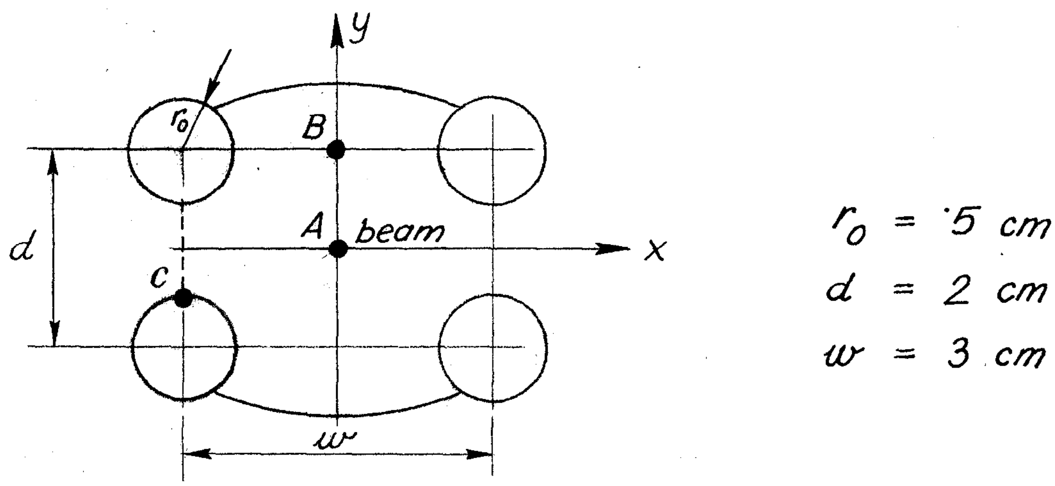


FIG. 10

The idealized system will now be analyzed. Along the y-axis the electric field is

$$E = E_y = \frac{Q}{\pi\epsilon_o \ell}\left[\frac{\frac{d}{2} - y}{\left(\frac{d}{2} - y\right)^2 + \left(\frac{w}{2}\right)^2} + \frac{\frac{d}{2} + y}{\left(\frac{d}{2} + y\right)^2 + \left(\frac{w}{2}\right)^2}\right]$$

$$= \frac{Qd}{\pi\epsilon_o \ell} \; \frac{\left(\frac{d}{2}\right)^2 + \left(\frac{w}{2}\right)^2 - y^2}{\left[\left(\frac{d}{2}\right)^2 + \left(\frac{w}{2}\right)^2 - y^2\right]^2 + w^2 y^2} \; . \tag{106}$$

In particular, at the origin

$$E_A = \frac{4Qd}{\pi\epsilon_o \ell \,(d^2 + w^2)} \tag{107}$$

and at $y = \frac{d}{2}$

$$E_B = \frac{4Qd}{\pi\epsilon_o \ell \,(4d^2 + w^2)} \; . \tag{108}$$

The voltage can be found by integrating E_y along the $x = -\frac{w}{2}$ line (dotted on Fig. 10):

$$V = \frac{Q}{2\pi\epsilon_o \ell} \int_{-(\frac{d}{2}-r_o)}^{+(\frac{d}{2}-r_o)} \left[\frac{1}{y + \frac{d}{2}} + \frac{1}{\frac{d}{2} - y} + \frac{\frac{d}{2} + y}{w^2 + (d - y)^2} + \frac{\frac{d}{2} - y}{w^2 + (d - y)^2} \right] dy$$

$$= \frac{Q}{\pi\epsilon_o \ell} \ln \left[\frac{d - r_o}{r_o} \sqrt{\frac{w^2 + (d - r_o)^2}{w^2 + r_o^2}} \right] . \tag{109}$$

Requiring E_A to equal the field-strength of the long parallel plate arrangement,

$$E_A = \frac{4Qd}{\pi\epsilon_o \ell \, (d^2 + w^2)} = \frac{V_{max_o}}{d_o} = 20 \, \frac{kV}{cm} \tag{110}$$

$$\frac{Q}{\pi\epsilon_o \ell} = 32.5 \text{ kV} \tag{111}$$

and from (109)

$$V_{max} = 32.5 \times 1.197 \cong 39 \text{ kV} . \tag{112}$$

The field at the surface of a wire

$$E_c \cong E_{yc} = \frac{Q}{2\pi\epsilon_o \ell} \left[\frac{1}{r_o} + \frac{1}{d - r_o} + \frac{r_o}{w^2 + r_o^2} + \frac{d - r_o}{w^2 + (d - r_o)^2} \right]$$

$$\cong \frac{32.5}{2} \, 2.854 = 46.4 \text{ kV/cm} \tag{113}$$

The capacitance is

$$C = \frac{Q}{V} = \frac{\pi\epsilon_o \ell}{\ell n\left[\frac{d - r_o}{r_o}\sqrt{\frac{w^2 + (d - r_o)^2}{w^2 + r_o^2}}\right]} = 2.32 \text{ pF} \tag{114}$$

The field-strength at B is only

$$E_B = E_A \frac{w^2 + d^2}{w^2 + 4d^2} = 0.52\ E_A = 10.4 \text{ kV/cm} \tag{115}$$

but the critical deflection of B_1 takes place near the origin, so this is of no consequence.

The design of the circuit is similar to that of the short-plate arrangement [Eqs. (92)-(95), (97), (100)-(102)]. However, because of the lower voltage required:

$$V_{max} \cong 40 \text{ kV}, \tag{116}$$

the power requirements are much lower. For the 5 pF circuit

$$V_{in} \cong 870 \text{ V} \tag{117}$$

$$P_{in} \cong 21.9 \text{ kW}\,. \tag{118}$$

For the 20 pF circuit, V_{in} remains as in (117) and

$$P_{in} \cong 87.6 \text{ kW}. \tag{119}$$

The reduction in power is about 80%.

Equation (105) is again valid if the symmetry of the circuit is good and no electron reaches the wires.

Estimating again that 5% of the deflection power is lost, and assuming that (25) represents a good approximation, r_b is about twenty times that given in (26) or about 0.5 MΩ.

In addition to the geometries already discussed, several other possibilities are open. For example, the current interception can be decreased by using a grid mesh rather than solid plates, which preserves most of the electric field. However, this may lead to extremely high local fields and breakdown.

An interesting possibility is the combination of different arrangements. A promising geometry is shown in Fig. 11. This is a mixture of the short

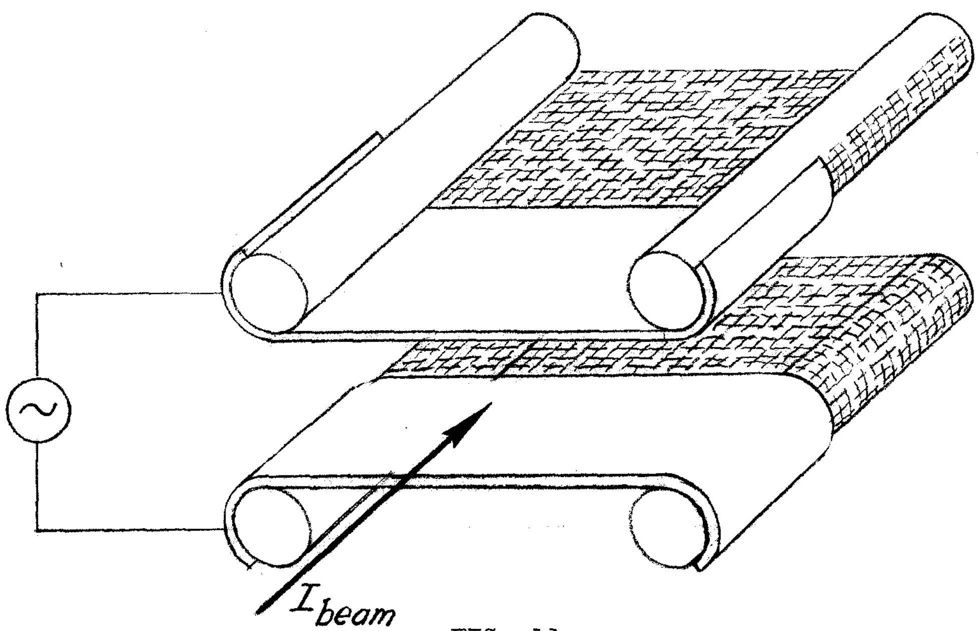


FIG. 11

plates and the rods. Assuming the voltage to be the arithmetic mean of those required for the long plates and for the rods, i.e., 35 kV, (69), (72), and (73) can be used to find the permissible length of solid plates:

$$\delta = \frac{m_e}{e} \frac{cd}{V_{max}} \cong 0.873 \text{ ns} \tag{120}$$

$$L_s \leq 2v_{z_o} \delta \sinh^{-1} \sqrt{\frac{d}{4\delta c}} \cong 3.39 \text{ cm} , \quad (121)$$

The rest of the space between the rods may be filled with a wire mesh (Fig. 11).

The circuit design is idential to that of the rod circuit. The voltage and power requirements, however, are lower:

$$V_{in} \cong 760 \text{ V} \quad (122)$$

$$P_{in} \cong 16.8 \text{ kW} \quad (123)$$

for 5 pF; and

$$P_{in} = 67.3 \text{ kW} \quad (124)$$

for the 20 pF circuit.

IV. STABILITY

Ideally, the deflection voltage source should be rigidly synchronized with the rf. If, however, an independent source is used, or a large phase-shift is required, the frequency stability of the source (or, the stability of the phase-shifter) should be specified. If a drift of bunches through the gate at t_{A_o} is to be avoided, extreme stability is needed.

As a design criterion, it is suggested that the effect of the instabilities should be less than the inevitable effect due to the nonzero beam pulse length. Assuming 60^o bunching, the bunches pass through any cross section in

$$T_b = \frac{1}{6 f_{RF}} \cong 0.0583 \text{ ns} . \quad (125)$$

TABLE I

SYSTEM	L^{cm}	C^{pF}	$L^{\mu H}$	r_L^{Ω}	$r_b^{k\Omega}$	Z_{res}^{Ω}	I_{in}^{A}	V_{max}^{kV}	V_{in}^{kV}	P_{in}^{kW}	V_3^{v}	$\tau_{max}^{\mu s}$	$(\Delta\beta)^{*}_{\Delta I_{beam}=10\%}$	$(\Delta V)^{**}_{\Delta I_{beam}=10\%}$
Long plates, unpadded	10	6	2.63	13.2	23.6	32	45.2	30	1.45	32.7	105	0.165	- 0.16°	- 14%
Long plates, padded	10	20	0.79	3.96	23.6	5.6	151	30	0.856	64.5	31	0.28	- 0.048°	- 4.2%
Short plates, unpadded	2	5	3.16	15.9	2040	16	113.5	90	1.82	103	~0	0.395	~0	~0
Short plates, padded	2	20	0.79	3.96	2040	4	454	90	1.82	413	~0	0.395	~0	~0
Rods, unpadded	10	5	3.16	15.9	500	17.3	50.4	40	0.87	21.9	~0	0.365	~0	~0
Rods, padded	10	20	0.79	3.96	500	4.3	202	40	0.87	87.6	~0	0.365	~0	~0
Rods and short plates, unpadded	10, 3.4	5	3.16	15.9	500	17.3	44.1	35	0.76	16.8	~0	0.365	~0	~0
Rods and short plates, padded	10, 3.4	20	0.79	3.96	500	4.3	177	35	0.76	67.3	~0	0.365	~0	~0

*For current generator feed

**For voltage source feed

Hence, from (24.a), there is a vertical spread at the aperture:

$$\Delta y \cong \pm \frac{\partial y}{\partial t_A} \frac{T_b}{2} = \pm \frac{Y}{12} \frac{\cos \omega (t_A - t_{A_O})}{\left(\sin \frac{\omega}{f_{RF}}\right) / \frac{\omega}{f_{RF}}} \cong \pm \frac{Y}{12} \tag{126}$$

which can be tolerated.

The frequency (or phase) stability should be such that the cumulative effect of frequency and phase variation be less than Δy. This means that if the experiment lasts for T_e seconds, the total change in zero-crossing time

$$|\Delta T| = T_e f \quad \Delta \left(\frac{1}{f}\right) = T_e \frac{|\Delta f|}{f} \leq \frac{T_b}{2} \tag{127}$$

or the relative frequency-change

$$\frac{|\Delta f|}{f} \leq \frac{T_b}{2T_e} = \frac{1}{12 \, f_{RF} \, T_e} \cong \frac{2.92 \times 10^{-11}}{T_e \, (\mathrm{sec})} \tag{128}$$

Such stability is probably impractical without synchronization.

The amplitude-stability is much less critical; as long as the voltage is higher than necessary for the interception of B_1, only φ_o, the angle of B_o at the aperture, will be slightly affected. From (20),

$$\frac{\Delta \varphi_o}{\varphi_o} = \frac{\Delta V_{max}}{V_{max}} \, . \tag{129}$$

Since magnetic focusing follows, this effect is negligible.

φ_o is also affected by ω changes and nonzero T_b. Since, by (20), the effect of T_b is

$$\frac{\Delta\varphi_o}{\varphi_o} = \pm \frac{1}{\varphi}\frac{\partial\varphi}{\partial t_A}\frac{T_b}{2} = \pm \frac{\frac{\ell}{L} + \frac{1}{2}}{f_{RF}\, T} \quad , \tag{130}$$

for the long plates

$$\frac{\Delta\varphi_o}{\varphi_o} \simeq \pm\, 0.4 \quad ; \tag{131}$$

and for the short plates

$$\frac{\Delta\varphi_o}{\varphi_o} \sim \pm\, 11.7 \quad . \tag{132}$$

The spread is less than 0.5° in both cases.

For a synchronized system, the phase-difference between electron bunches and deflection voltage must be kept less than $\pm 30^\circ$ RF angle; i.e.,

$$|\Delta\beta| < \frac{30^\circ}{72} \cong 0.42^\circ \tag{133}$$

of 40 Mc. Even this may be difficult to maintain for a long time, especially in the (Class C) power amplifier stage.

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