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Genuine multipartite entanglement in three-qubit XYZ Heisenberg systems

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Abstract. Some measures of genuine multipartite entanglement in the three-qubit XYZ Heisenberg model are reported for a system that is embedded in a uniform magnetic field. A quantitative characterization of these measures is accomplished to determine the optimal parameters that provide maximal entanglement.

1. Introduction

Multipartite entangled systems composed of n qubits are a fundamental resource for quantum information protocols [1, 2]. In particular, solid-state systems seem more than suitable for producing these systems in the laboratory [3–8]. Among a diversity of models, the idea of having a chain of spins whose most relevant interaction is that of the first neighbors has gained special interest in recent years [7–16]. This simple system, known as the Heisenberg model [17], admits analytical solution in different scenarios [18], including interactions with uniform external magnetic fields [19–21]. Different investigations have been oriented to the characterization of entanglement for these systems [9–16, 21–23], other works focus on the control of entanglement in terms of either the internal or external degrees of freedom [14,16,21], particularly the coupling constants and external magnetic fields [21].

This work is addressed to study genuine multipartite entanglement (GME) in a three-qubit XYZ Heisenberg system that is embedded in a uniform magnetic field. Many measures have been introduced to quantify GME [24–28]; any of these must to satisfy at least the requirements (a) the measure vanishes for separable and biseparable states and (b) the measure must be positive for states that cannot be factorized as a product of states [26, 27]. In our case, as measure of entanglement, the GME-concurrence [27] and the generalized geometric measure (GGM) of entanglement [28, 29] are appropriate. The GME-concurrence is an extension of the concurrence defined originally for bipartite systems [30], and satisfies the minimal requirements for a measure of genuine entanglement. In turn, the GGM is a generalization of the geometric measure of entanglement proposed in [31, 32], where the optimization process takes into account every separable state. The interest in these measures is because they only require the smallest eigenvalues of the reduced one-qubit density matrices. Thus, no additional correlation measurements are required to get information about GME in the system we are going to study.

2. The XYZ Heisenberg spin chain

Consider a chain of three spin-$\frac{1}{2}$ particles that interact according to the Heisenberg model [17]. The Hamiltonian for the XYZ case is written as follows

$$H_{xyz} = \sum_{i=1}^{3} \left(J_1 \sigma_i^x \sigma_{i+1}^x + J_2 \sigma_i^y \sigma_{i+1}^y + J_3 \sigma_i^z \sigma_{i+1}^z \right), \tag{1}$$

with

$$\sigma_1^k = \sigma^k \otimes \mathbb{1} \otimes \mathbb{1}, \quad \sigma_2^k = \mathbb{1} \otimes \sigma^k \otimes \mathbb{1}, \quad \sigma_3^k = \mathbb{1} \otimes \mathbb{1} \otimes \sigma^k, \quad k = x, y, z. \tag{2}$$

Here $\{\sigma_i^x, \sigma_i^y, \sigma_i^z\}$ is the set of Pauli matrices for the ith interaction, and $\{J_i\}_{i=1}^3$ denotes the corresponding strength. For a general treatment of the Heisenberg model, including the XYZ-Hamiltonian (1) see the review paper [18]. Hereafter we adopt the periodic boundary condition $\sigma_4^k = \sigma_1^k$, with $k = x, y, z$. If the interaction with a uniform external magnetic field $\vec{B} = B\hat{e}_z$ is included, the Hamiltonian (1) acquires the form

$$H = H_{xyz} + B \sum_{i=1}^{3} \sigma_i^z. \tag{3}$$

The above Hamiltonian has been widely studied in the context of solid-state theory [19, 20], where two important cases can be distinguished [21]: isotropic ferromagnetic ($J_i = J < 0$) and isotropic antiferromagnetic ($J_i = J > 0$). In quantum information theory, the systems modeled by this Hamiltonian represent a convenient resource of entanglement [7–16, 21].

Following [18], the diagonalization of (3) provides the eigenvalues

$$E_0 = J_1 + J_2 + J_3 + B + \sqrt{[2(J_3 + B) - (J_1 + J_2)]^2 + 3(J_1 - J_2)^2}, \tag{4}$$

$$E_7 = J_1 + J_2 + J_3 - B + \sqrt{[2(J_3 - B) - (J_1 + J_2)]^2 + 3(J_1 - J_2)^2}, \tag{5}$$

$$E_1 = J_1 + J_2 + J_3 - B - \sqrt{[2(J_3 - B) - (J_1 + J_2)]^2 + 3(J_1 - J_2)^2}, \tag{6}$$

$$E_3 = J_1 + J_2 + J_3 + B - \sqrt{[2(J_3 + B) - (J_1 + J_2)]^2 + 3(J_1 - J_2)^2}, \tag{7}$$

$$E_2 = E_4 = \quad - (J_1 + J_2 + J_3) + B, \tag{8}$$

$$E_5 = E_6 = \quad - (J_1 + J_2 + J_3) - B. \tag{9}$$

Observe that the pairs E_2, E_4, and E_5, E_6, are degenerate. On the other hand, if $B = 0$ then also the pairs E_0, E_7, and E_1, E_3, are degenerate. The eigenvectors of the above system are given by the expressions

$$|E_0\rangle = \cos\alpha_0 \, |000\rangle + \sin\alpha_0 |\widetilde{W}\rangle, \tag{10}$$

$$|E_1\rangle = -\sin\alpha_1 \, |111\rangle + \cos\alpha_1 |W\rangle, \tag{11}$$

$$|E_3\rangle = -\sin\alpha_0 \, |000\rangle + \cos\alpha_0 |\widetilde{W}\rangle, \tag{12}$$

$$|E_7\rangle = \cos\alpha_1 \, |111\rangle + \sin\alpha_1 |W\rangle, \tag{13}$$

$$|E_2\rangle = -\tfrac{1}{\sqrt{6}} \, |001\rangle + \sqrt{\tfrac{2}{3}} \, |010\rangle - \tfrac{1}{\sqrt{6}} \, |100\rangle \,, \tag{14}$$

$$|E_4\rangle = \tfrac{1}{\sqrt{2}} \, |100\rangle - \tfrac{1}{\sqrt{2}} \, |001\rangle \,, \tag{15}$$

$$|E_5\rangle = -\tfrac{1}{\sqrt{6}} \, |011\rangle + \sqrt{\tfrac{2}{3}} \, |101\rangle - \tfrac{1}{\sqrt{6}} \, |110\rangle \,, \tag{16}$$

$$|E_6\rangle = \tfrac{1}{\sqrt{2}} \, |110\rangle - \tfrac{1}{\sqrt{2}} \, |011\rangle \,, \tag{17}$$

where

$$|\widetilde{W}\rangle = \tfrac{1}{\sqrt{3}}\left(|011\rangle + |101\rangle + |110\rangle\right). \tag{18}$$

The parameters α_0 and α_1 satisfy the equations

$$\tan(2\alpha_0) = \frac{\sqrt{3}(J_1 - J_2)}{2(J_3 + B) - (J_1 + J_2)}, \quad \tan(2\alpha_1) = \frac{\sqrt{3}(J_1 - J_2)}{2(J_3 - B) - (J_1 + J_2)}. \tag{19}$$

Remark that vectors $|E_2\rangle$, $|E_4\rangle$, $|E_5\rangle$ and $|E_6\rangle$, are not parameterized by the system variables. They belong to the degenerate eigenvalues (8)-(9). Furthermore, these vectors are paired under local unitary (LU) operations, satisfying $|E_5\rangle = U_x|E_2\rangle$ and $|E_6\rangle = e^{i\pi}U_x|E_4\rangle$, with $U_x = \sigma_x \otimes \sigma_x \otimes \sigma_x$.

In turn, vectors $|E_0\rangle$ and $|E_7\rangle$ must exhibit similar entanglement properties since $U_x|E_0\rangle$ and $|E_7\rangle$, as well as $|E_0\rangle$ and $U_x|E_7\rangle$, are defined by a similar linear combination. The same can be said about the pair $|E_1\rangle$ and $|E_3\rangle$. This symmetry is clear by noticing that $|E_3\rangle$ is a $\frac{\pi}{2}$-shifted version of $|E_0\rangle$ while $|E_7\rangle$ is (minus) $\frac{\pi}{2}$-shifted of $|E_1\rangle$. That is, $|E_3\rangle = |E_0\rangle|_{\alpha_0+\pi/2}$ and $|E_7\rangle = |E_1\rangle|_{\alpha_1-\pi/2}$.

On the other hand, in absence of the field ($B = 0$), from (19) we find $\alpha_1 + m\pi/2$, $m \in \mathbb{Z}$. The eigenvectors of the degenerate pairs E_0, E_7, and E_1, E_3 are LU-correlated as follows:

$$|E_0\rangle = \cos\left(\tfrac{m\pi}{2}\right)U_x|E_7\rangle, \quad |E_3\rangle = \cos\left(\tfrac{m\pi}{2}\right)U_x|E_1\rangle, \quad \mathbb{Z} \ni m = \text{even},$$

$$|E_0\rangle = \sin\left(\tfrac{m\pi}{2}\right)U_x|E_1\rangle, \quad |E_3\rangle = -\sin\left(\tfrac{m\pi}{2}\right)U_x|E_7\rangle, \quad \mathbb{Z} \ni m = \text{odd}.$$

2.1. Measuring genuine entanglement

To measure entanglement we use the genuine multipartite entanglement (GME)-concurrence [16] which, adapted to the case under study, is given by the expression

$$\mathcal{C}(\psi) = \min_{i\in\{1,2,3\}} \sqrt{2\left[1 - Tr\rho_i^2\right]} = \min_{i\in\{1,2,3\}} 2\sqrt{\lambda_i\left(1 - \lambda_i\right)}, \tag{20}$$

where ρ_i is the reduced density matrix of the ith qubit and λ_i is its smallest eigenvalue. Note that $0 \le \mathcal{C} \le 1$, since $0 \le \lambda_i \le \frac{1}{2}$. We also use the generalized geometric measure (GGM) [28,29], written in the form [33]:

$$\mathcal{G}(\psi) = 1 - \max_{i\in\{1,2,3\}}\{1 - \lambda_i\} = \min_{i\in\{1,2,3\}} \lambda_i. \tag{21}$$

Similar to the previous case, $0 \le \mathcal{G} \le 1/2$. In addition, it is worth noting that the above expressions satisfy the relation [33]:

$$\left[2\mathcal{G}(\psi) - 1\right]^2 + \mathcal{C}(\psi)^2 = 1. \tag{22}$$

Compared with other measures of entanglement, where the whole information about the system is required, the expressions (20) and (21) have the advantage that they are determined by the smallest eigenvalues only.

The smallest eigenvalues λ_i are invariant under LU operations [34], so the pair $|E_6\rangle$ and $|E_4\rangle$ share the same eigenvalue set $\{1/2, 0, 1/2\}$. Therefore $\mathcal{C}(E_4) = \mathcal{C}(E_6) = \mathcal{G}(E_4) = \mathcal{G}(E_6) = 0$. Similarly, for $|E_5\rangle$ and $|E_2\rangle$, the eigenvalues $\{1/6, 1/3, 1/6\}$ yield $\mathcal{C}(E_2) = \mathcal{C}(E_5) = \sqrt{5}/3$ and $\mathcal{G}(E_2) = \mathcal{G}(E_5) = 1/6$, so the latter states are genuinely entangled.

For vectors $|E_\ell\rangle$, $\ell = 0, 1, 3, 7$, the smallest eigenvalues satisfy $\lambda_1 = \lambda_2 = \lambda_3 = \kappa_\ell$, with

$$\kappa_\ell = \begin{cases} \frac{1}{2} - \frac{1}{2}\left|1 - \frac{4}{3}\sin^2\alpha_\ell\right|, & \ell = 0, 7 \\ \frac{1}{2} - \frac{1}{2}\left|1 - \frac{4}{3}\cos^2\alpha_\ell\right|, & \ell = 1, 3 \end{cases}, \quad \alpha_7 = \alpha_1, \alpha_3 = \alpha_0. \tag{23}$$

Then we have

$$\mathcal{C}(E_\ell) = \begin{cases} \sqrt{1-\left(1-\frac{4}{3}\sin^2\alpha_\ell\right)^2}, & \ell = 0,7 \\ \sqrt{1-\left(1-\frac{4}{3}\cos^2\alpha_\ell\right)^2}, & \ell = 1,3 \end{cases}, \tag{24}$$

and

$$\mathcal{G}(E_\ell) = \kappa_\ell, \quad \ell = 0,1,3,7. \tag{25}$$

For both measures, (24) and (25), the pair $|E_0\rangle$, $|E_7\rangle$, is maximally entangled at the following values of the parameters

$$\alpha_\ell = m\pi \pm (-1)^m \tfrac{\pi}{3}, \quad m \in \mathbb{Z}, \quad \ell = 0,7. \tag{26}$$

In turn, $|E_1\rangle$, $|E_3\rangle$, is maximally entangled at

$$\alpha_\ell = 2m\pi \pm \tfrac{\pi}{6}, \quad 2m\pi \pm \tfrac{5\pi}{6}, \quad m \in \mathbb{Z}, \quad \ell = 1,3. \tag{27}$$

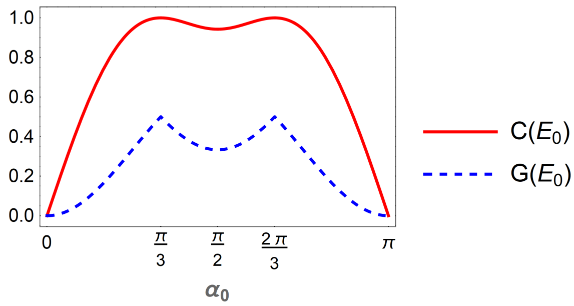

Figure 1: For the eigenvector $|E_0\rangle$ of the XYZ-Heisenberg Hamiltonian (3) the entanglement measures $\mathcal{C}(E_0)$ (solid red curve) and $\mathcal{G}(E_0)$ (dashed blue curve) are π-periodic functions of the parameter α_0. The maximum in the period is reached at the points $\alpha_0 = \pi/3, 2\pi/3$, and the minimum at $\alpha_0 = \pi/2$, where $\mathcal{C} = 2\sqrt{2}/3$ and $\mathcal{G} = 1/3$ and $|E_0\rangle = |\widetilde{W}\rangle$. At $\alpha_0 = 0, \pi$, it is found $\mathcal{C} = \mathcal{G} = 0$ and $|E_0\rangle = \pm|000\rangle$.

Figure 1 shows the results for measuring entanglement in the state $|E_0\rangle$. It is separable at $\alpha_0 = m\pi$, for which $\mathcal{C} = \mathcal{G} = 0$. Local minima are reached at $\alpha_0 = (2m \pm 1/2)\pi$, for which $\mathcal{C} = 2\sqrt{2}/3$ and $\mathcal{G} = 1/3$ and $|E_0\rangle = \pm|\widetilde{W}\rangle$. The description of $|E_7\rangle$ in terms of $\alpha_7 = \alpha_1$ is completely equivalent. That is, except at $\alpha_0 = m\pi$ and $\alpha_1 = n\pi$, the states $|E_0\rangle$ and $|E_7\rangle$ are genuinely entangled.

The results for state $|E_1\rangle$ are just a $\frac{\pi}{2}$-shifted version of the above configurations, see Figure 2. A similar description is valid for state $|E_3\rangle$. Thus, $|E_1\rangle$ and $|E_3\rangle$ are genuinely entangled except if the corresponding angles acquire the values $(2m \pm 1/2)\pi$.

3. Conclusions

We have studied the genuine three-qubit entanglement associated with the eigenvectors of the XYZ Heisenberg Hamiltonian (3). The system has been diagonalized, with eight real eigenvalues, four of which are paired in two separated degenerate subsystems. To measure entanglement, we have used the genuine multipartite entanglement concurrence $\mathcal{C}$ and the generalized geometric measure $\mathcal{G}$. The analytic expressions for these measures, in terms of the smallest eigenvalues of the reduced one-qubit density matrices, have been provided in Eqs. (20) and (21), respectively. We have shown that only four of the eight eigenvectors are parameterized by the system variables (the four belonging to the degenerate subspaces are not parameterized

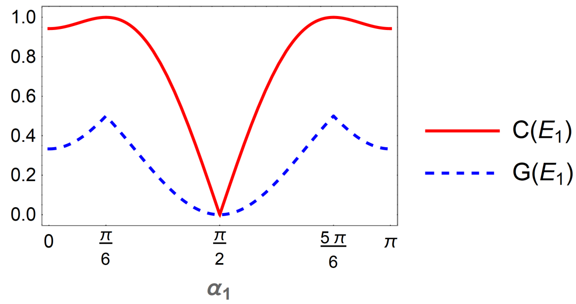

Figure 2: For eigenvector $|E_1\rangle$, the entanglement measures $\mathcal{C}(E_1)$ (solid red curve) and $\mathcal{G}(E_1)$ (dashed blue curve) are $\frac{\pi}{2}$-shifted versions of the functions shown in Figure 1. In this case, the maximum is reached at $\alpha_1 = \pi/6, 5\pi/6$. At the global minimum $\alpha_1 = \pi/2$, the state is reduced to $|E_1\rangle = -|111\rangle$. The local minima $\alpha_1 = 0, \pi$, yield $|E_1\rangle = \pm|W\rangle$.

in any form), they exhibit genuine tripartite entanglement. We have obtained the conditions of maximal entanglement for these eigenvectors.

All the results reported in this work have been achieved assuming an implicit dependence of the parameters with the dynamical variables of the system. A detailed discussion of the topic is beyond the scope of the present work and will therefore be provided elsewhere.

We would like to emphasize that, although the three-qubit framework representes the simplest version of multipartite entanglement, it has been relevant in the understanding of quantum mechanics foundations as well as for several applications in the area of quantum information [1, 35–37]. Then, we hope that our work sheds some light on the understanding of the XYZ Heisenberg model.

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