



The Dirac equation in general relativity and the 3+1 formalism

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Abstract

I present a review of the Dirac equation in general relativity. Although the generalization of the Dirac equation to a curved spacetime is well known, it is not usually part of the standard toolkit of techniques known to people working on classical general relativity. Recently, there has been some renewed interest in studying solutions of the Einstein–Dirac system of equations, particularly in the context of the so-called “Dirac stars”. Motivated by this, here I present a review of the Dirac equation in general relativity, starting from Minkowski spacetime, and then considering the Lorentz group and the tetrad formalism in order to generalize this equation to the case of a curved spacetime. I also derive the form of the Dirac equation and its associated stress–energy tensor for the case of the 3+1 formalism of general relativity, which can be useful for the study of the evolution of the Dirac field in a dynamical spacetime.

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1 Introduction

The relativistic description of spin $1/2$ particles is given in terms of the Dirac equation. This equation, proposed by Dirac in 1928 [1], is a first order equation in both time and space that is fully Lorentz covariant and does not suffer the problem of having negative probability densities as in the case of the Klein–Gordon equation. The price to pay is the need to introduce a new type of geometric object different from vectors and tensors: a four-component spinor that transforms according to its own special set of rules with respect to a general Lorentz transformation.

The original form of the Dirac equation is perfectly consistent in special relativity, but since Einstein’s work in 1915 we know that in the presence of gravity our Universe is not correctly described by Minkowski spacetime, and one must use instead the curved spacetime formalism of general relativity. A generalization of the Dirac equation to the case of curved spacetimes was quickly found by Fock and Ivanenko in 1929 [2, 3], and later studied by Bargmann [4] and even Schroedinger [5]. However, due to the fact that the gravitational field can be safely ignored when studying atomic physics, this generalization was regarded for a long time as an academic exercise with little practical applications.

Interest in the study of Dirac equation in curved spacetimes increased in the 1970’s with the work of Hawking on quantum field theory on curved spacetimes, and since then one can find some discussion (usually quite short) of the general relativistic version of the Dirac equation in modern textbooks (see e.g. [6, 7]). Still, the formulation of the Dirac equation in a curved spacetime remains, even today, as something that most researchers working in the field of general relativity never study.

More recently, a revived interest in this subject has arisen related to the work on exotic compact objects (ECO), in particular self-gravitating stationary solutions of the Einstein equations coupled to some matter field. The case of ECO’s formed by scalar fields corresponds to the well-known boson stars initially introduced by Kaup

and Ruffini in the late 1960's [8, 9] (a recent review can be found in [10]), while ECO's formed from vector fields correspond to the more recently introduced Proca stars of Brito et al. [11]. The so-called "Dirac stars", that is self-gravitating stationary solutions of the Einstein–Dirac system, have also been considered, originally by Finster in 1998 [12], and more recently by Herdeiro et al. [13] (see also [14–16]).

Motivated by these developments, here I present a review of the Dirac equation in general relativity, starting from Minkowski spacetime, and then considering the Lorentz group and the tetrad formalism in order to generalize this equation to the case of a curved spacetime. Other reviews on this subject already exist in the literature (see e.g. [17–20]), but in my opinion none are fully comprehensive. Though most of the material presented here is known, I try to present it in a pedagogical way starting from first principles. I also derive the form of the Dirac equation and its associated stress–energy tensor for the particular case of the 3+1 formalism of general relativity. To my knowledge, these last sections include new material which can be very useful for the study of the evolution of the Dirac field in a dynamical spacetime. Finally, I consider the particular example of spherical symmetry: First the general case of the Dirac equation in a spherically symmetric spacetime, and later the case of self-consistent spherically symmetric solutions of the Einstein–Dirac system, and the special configurations corresponding to Dirac stars.

A word about my conventions, throughout this paper I use the metric signature $(-, +, +, +)$ and Planck units such that $c = \hbar = G = 1$.

2 The Dirac equation in special relativity

2.1 Dirac equation

The Dirac equation is a relativistic generalization of the Schroedinger equation that describes the behaviour of spin 1/2 particles (the material presented in this section is well known, and can be found in any standard text book on quantum field theory, see e.g. [21–23]). Before Dirac's work in 1928 [1], there was already a relativistic generalization of Schroedinger's equation, namely the Klein–Gordon equation, which takes the form:

$$\square\psi - m^2\psi = 0, \quad (\text{II.1})$$

where \square is the standard d'Alambertian operator in special relativity (though the equation takes exactly the same form in general relativity (GR) with the curved d'Alambertian), and m is a mass parameter which corresponds to the mass of the associated particle when the theory is quantized. For a complex wave function ψ one can show that there is a conserved current given by:

$$j_\mu := -i(\psi^*\partial_\mu\psi - \psi\partial_\mu\psi^*), \quad (\text{II.2})$$

where ψ^* denotes the complex conjugate of ψ , such that:

$$\nabla_\mu j^\mu = 0. \quad (\text{II.3})$$

The main problem with interpreting the Klein–Gordon equation as a quantum equation comes from the fact that the density associated with the above conserved current is given by:

$$\rho \equiv j^0 = i (\psi^* \partial_t \psi - \psi \partial_t \psi^*) . \quad (\text{II.4})$$

This expression is clearly not positive definite, so it can not be associated with a probability density. This problem can be traced back to the fact that, in contrast with the Schroedinger equation, the Klein–Gordon equation is of second order in time. Because of this the Klein–Gordon equation was initially rejected as a valid quantum equation, which motivated Dirac to look for a relativistic equation that was first order in time. The problem with the Klein–Gordon equation was later solved in quantum field theory by associating the density ρ above not with a probability density, but rather with a charge density, allowing it to describe particles and antiparticles of opposite charge. However, this equation does not include the effects of the spin of the particles, so today it is considered to describe only scalar (spin 0) particles such as for example the Higgs boson. But we will not go any deeper into quantum field theory here and we will instead regard both the Klein–Gordon equation and the Dirac equation below as purely “classical” field equations.

Since having an equation that is first order in time and second order in space, such as the Schroedinger equation, clearly violates Lorentz invariance, Dirac proposed a purely first order expression for the Hamiltonian operator of the form:

$$\hat{H} \psi = (\boldsymbol{\alpha} \cdot \hat{\mathbf{p}} + \beta m) \psi . \quad (\text{II.5})$$

where bold letters indicate three-dimensional objects, with \hat{p}_i the usual momentum operator, and where the quantities α_i and β are constant coefficients to be determined. We can immediately see that, if the α_i were the components of a simple three-dimensional vector, the above equation would give preference to a specific direction in space, in clear violation of relativistic invariance, so they must be other type of objects.

If we now want our Hamiltonian operator to satisfy the relativistic energy–momentum relation we must ask for:

$$\hat{H}^2 \psi = (\hat{\mathbf{p}}^2 + m^2) \psi . \quad (\text{II.6})$$

Taking the square of Eq. (II.5) we now find (notice that here we are not assuming that the α_i and β commute):

$$\hat{H}^2 = \sum_i \alpha_i^2 \hat{p}_i^2 + \sum_{i>j} (\alpha_i \alpha_j + \alpha_j \alpha_i) \hat{p}_i \hat{p}_j + \sum_i m (\alpha_i \beta + \beta \alpha_i) \hat{p}_i + \beta^2 m^2 . \quad (\text{II.7})$$

Comparing this with (II.6) we find that we must have $\alpha_i^2 = \beta^2 = 1$, plus the objects $(\alpha_1, \alpha_2, \alpha_3, \beta)$ must all anti-commute with each other. Given these anti-commutation relations we must conclude that these objects are not simple numbers, but must be matrices of some dimension. One can also show that in order to satisfy all these

relationships such matrices must be at least of dimension 4×4 . One such set of matrices are known as the *Dirac–Pauli* matrices, and are given by:

$$\alpha_i = \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}, \tag{II.8}$$

where I_2 is the 2×2 identity matrix, and where the σ_i are the usual 2×2 Pauli matrices:

$$\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{II.9}$$

Given the fact that the Pauli matrices are hermitian (i.e. equal to their conjugate transpose), the α_i and β are also hermitian. It is important to remember at this point that the Pauli matrices anti-commute with each other and are such that $\sigma_i^2 = 1$. Both these properties can be combined into the expression:

$$\{\sigma_i, \sigma_j\} = 2\delta_{ij}I_2, \tag{II.10}$$

where $\{, \}$ denotes the anticommutator defined as $\{\sigma_i, \sigma_j\} := \sigma_i\sigma_j + \sigma_j\sigma_i$.

One should mention the fact that the above choice for the matrices α_i and β is clearly not unique. In fact, any set of matrices related to the above choice by a transformation of the form:

$$\alpha'_k = U\alpha_kU^{-1}, \quad \beta' = U\beta U^{-1}, \tag{II.11}$$

with U a unitary matrix, would satisfy the same relations. Another common representation is the *Weyl* or *quiral* representation, and is given by:

$$\alpha_i = \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \quad \beta = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix}. \tag{II.12}$$

Now, given the fact that the Dirac Eq. (II.5) involves matrix operators of dimension 4×4 , we conclude that the “wave function” ψ must in fact be a complex column vector with four components known as a *Dirac spinor*. The Dirac equation then represents a set of four coupled equations for the four complex components of ψ . Notice that even if the Dirac spinor has four components, it is not a 4-vector in the usual spacetime sense, but rather a collection of four complex numbers that transform in a special way under rotations, as we will see below.

2.2 Covariant form

The Dirac Eq. (II.5) can easily be rewritten in a manifestly covariant form. In order to do this we first write the standard energy and momentum operators in the usual form (remember that we are taking $\hbar = 1$):

$$\hat{H} = i\partial_t, \quad \hat{P}_i = -i\partial_i. \tag{II.13}$$

The equation now takes the form:

$$i \partial_t \psi = -i \alpha^i \partial_i \psi + \beta m \psi , \quad (\text{II.14})$$

where we have defined $\alpha^i \equiv \alpha_i$. From here on we will adopt the convention that Greek indices take values from 0 to 3, with 0 corresponding to the time coordinate, while Latin indices only take values from 1 to 3. We also adopt the Einstein summation convention: repeated indices in the same term, once covariant and once contravariant, are to be summed over all their allowed values. Multiplying the above equation with $-i$ we obtain:

$$\partial_t \psi = -\alpha^i \partial_i \psi - i \beta m \psi . \quad (\text{II.15})$$

Written in this form the Dirac equation can be interpreted as an evolution equation in time for ψ . This is important, for example, if one is interested in dynamical simulations of solutions of the Dirac equation. Multiplying now (II.14) with β from the left, and rearranging terms we find:

$$i \beta \partial_t \psi + i \beta \alpha^i \partial_i \psi - m \psi = 0 , \quad (\text{II.16})$$

where we used the fact that $\beta^2 = 1$. The above equation can now be written in covariant form as:

$$i \gamma^\mu \partial_\mu \psi - m \psi = 0 . \quad (\text{II.17})$$

where we defined the γ^μ matrices as:

$$\gamma^0 := \beta , \quad \gamma^k := \beta \alpha^k . \quad (\text{II.18})$$

The above matrices are the so-called *Dirac matrices*. In the standard representation they take the form:

$$\gamma^0 = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} , \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} , \quad (\text{II.19})$$

while in the Weyl representation they are instead:

$$\gamma^0 = \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} , \quad \gamma^k = \begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix} . \quad (\text{II.20})$$

It is usual to define the operator $\not{\partial} := \gamma^\mu \partial_\mu$, so that the Dirac equation takes the more compact form:

$$(i \not{\partial} - m) \psi = 0 . \quad (\text{II.21})$$

From the definition of the γ^μ matrices one can show that:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2 \eta^{\mu\nu} I_4 , \quad (\text{II.22})$$

with $\eta^{\mu\nu}$ the Minkowski tensor, and where here I_4 denotes the 4×4 identity matrix. Here it is important to mention the fact that the above relation is obtained when using a

signature $(-, +, +, +)$ for the metric. In quantum field theory text books it is common to use the opposite signature, so that the term on the right hand side above changes sign. The previous relation defines what is known as a *Clifford algebra*. In particular we have:

$$(\gamma^0)^2 = I_4, \quad (\gamma^k)^2 = -I_4, \tag{II.23}$$

and:

$$\gamma^0 \gamma^k + \gamma^k \gamma^0 = 0, \tag{II.24}$$

that is, γ^0 and γ^k anti-commute with each other. One can also show that:

$$\gamma^{0\dagger} = \gamma^0, \quad \gamma^{k\dagger} = -\gamma^k, \tag{II.25}$$

where the symbol \dagger denotes the transpose conjugate. That is, γ^0 is hermitian while the γ^k are anti-hermitian. These last relations can be summarized as:

$$\gamma^0 \gamma^\mu \gamma^0 = \gamma^{\mu\dagger}. \tag{II.26}$$

It is common to also define the matrix γ^5 as follows:

$$\gamma^5 := i\gamma^0 \gamma^1 \gamma^2 \gamma^3. \tag{II.27}$$

The use of the number 5 comes from the fact that many older texts take the spacetime coordinates to run from 1 to 4 instead of from 0 to 3 as we do here. This matrix is useful for many calculations in quantum field theory, but we will not consider it further here.

There is another useful relation that can be obtained from the Clifford algebra that allows us to commute two pairs of Dirac matrices. From (II.22) one can show, after some algebra:

$$\gamma^\alpha \gamma^\beta \gamma^\mu \gamma^\nu = \gamma^\mu \gamma^\nu \gamma^\alpha \gamma^\beta + 2(\eta^{\alpha\mu} \gamma^\beta \gamma^\nu - \eta^{\beta\mu} \gamma^\alpha \gamma^\nu + \eta^{\alpha\nu} \gamma^\mu \gamma^\beta - \eta^{\beta\nu} \gamma^\mu \gamma^\alpha). \tag{II.28}$$

2.3 Adjoint equation, conserved current and the Klein–Gordon equation

In order to find a conserved current associated with the Dirac equation we start from considering its hermitian conjugate. We first write the Dirac equation in extended form as:

$$i\gamma^0 \partial_t \psi + i\gamma^k \partial_k \psi - m\psi = 0, \tag{II.29}$$

so that its hermitian conjugate takes the form:

$$-i(\partial_t \psi^\dagger) \gamma^0 - i(\partial_k \psi^\dagger) (-\gamma^k) - m\psi^\dagger = 0. \tag{II.30}$$

Notice that in the above equation ψ^\dagger is now a row vector, while ψ is a column vector. Multiplying the last equation with γ^0 from the right, and defining the *adjoint spinor*

as $\bar{\psi} := \psi^\dagger \gamma^0$, we find:

$$i (\partial_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} = 0 , \tag{II.31}$$

where we used the fact that γ^0 and γ^k anti-commute. The last equation is known as the adjoint Dirac equation.

We can now multiply the Dirac Eq. (II.17) with $\bar{\psi}$ on the left, and the adjoint Eq. (II.31) with ψ on the right, and add the resulting equations together (remembering that the γ^μ matrices are constant). We then find:

$$\bar{\psi} (\gamma^\mu \partial_\mu \psi) + (\partial_\mu \bar{\psi}) \gamma^\mu \psi = \partial_\mu (\bar{\psi} \gamma^\mu \psi) = 0 . \tag{II.32}$$

This result implies that we have a conserved current of the form:

$$j^\mu = \bar{\psi} \gamma^\mu \psi . \tag{II.33}$$

In particular, the associated density is now given by:

$$\rho = \bar{\psi} \gamma^0 \psi = \psi^\dagger \psi = \sum_{i=1}^4 |\psi_i|^2 , \tag{II.34}$$

where here (and in similar expressions below) the index i labels spinor components. Clearly ρ is now positive definite and can be interpreted as a probability density, which was Dirac’s main motivation.

Let us now go back to Eq. (II.17). Applying the operator $-i\gamma^\nu \partial_\nu$ from the left we obtain:

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + im\gamma^\nu \partial_\nu \psi = 0 , \tag{II.35}$$

where we again used the fact that the γ^μ matrices are constant. Using again (II.17) in the second term this reduces to:

$$\gamma^\nu \gamma^\mu \partial_\nu \partial_\mu \psi + m^2 \psi = 0 . \tag{II.36}$$

On the other hand, since partial derivatives commute we can write:

$$(\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu) \partial_\mu \partial_\nu \psi = 2\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi . \tag{II.37}$$

Using now the Clifford algebra, relation (II.22), this implies that:

$$\gamma^\mu \gamma^\nu \partial_\mu \partial_\nu \psi = -\eta^{\mu\nu} \partial_\nu \psi = -\square \psi , \tag{II.38}$$

so that Eq. (II.36) finally reduces to:

$$\square \psi - m^2 \psi = 0 , \tag{II.39}$$

which is nothing more than the Klein–Gordon equation. That is, each of the individual components of the spinor ψ obey the Klein–Gordon equation separately. However,

will see in the following sections that this is strictly true only in a flat spacetime and in Cartesian coordinates.

3 The Lorentz group

3.1 Tensor representation

In order to study the behaviour of the Dirac equation under Lorentz transformations we must first understand in some detail the *Lorentz group*, which includes the proper Lorentz transformations as well as the three-dimensional rotations in space. A general Lorentz transformation is defined as a linear (and real) coordinate transformation that leaves the Minkowski interval invariant. Such a transformation can be represented in the general form:

$$x'^{\alpha} = \Lambda^{\alpha}_{\beta} x^{\beta} , \tag{III.1}$$

where $\{x^{\alpha}\}$ are the original coordinates, $\{x'^{\alpha}\}$ are the new coordinates, and where $\Lambda^{\alpha}_{\beta} := \partial x'^{\alpha} / \partial x^{\beta}$ is the jacobian matrix that must be constant for a linear transformation. Notice that in the above expression the order of the indices in the matrix Λ^{α}_{β} matters, as we will see in a moment.

It is important to mention the fact that in general we are not assuming any symmetry properties for the matrix Λ^{α}_{β} . For example, for as Lorentz “boost” along the x direction the matrix turns out to be symmetric, while for a three-dimensional rotation in space around the x axis it is antisymmetric. We will return to this point below.

We can now raise and lower indices of the jacobian matrix using the Minkowski tensor $\eta_{\alpha\beta}$ to construct, for example:

$$\Lambda_{\alpha\beta} = \eta_{\alpha\mu} \Lambda^{\mu}_{\beta} , \quad \Lambda^{\alpha\beta} = \eta^{\beta\mu} \Lambda^{\alpha}_{\mu} , \quad \Lambda_{\alpha}^{\beta} = \eta_{\alpha\mu} \eta^{\beta\nu} \Lambda^{\mu}_{\nu} . \tag{III.2}$$

In particular, since the Minkowski tensor is invariant under Lorentz transformations by definition, we must have:

$$\eta^{\alpha\beta} = \Lambda^{\alpha}_{\mu} \Lambda^{\beta}_{\nu} \eta^{\mu\nu} = \Lambda^{\alpha\nu} \Lambda^{\beta}_{\nu} , \tag{III.3}$$

which implies:

$$\Lambda^{\alpha\nu} \Lambda_{\beta\nu} = \Lambda^{\alpha}_{\nu} \Lambda_{\beta}^{\nu} = \delta^{\alpha}_{\beta} . \tag{III.4}$$

On the other hand, for the inverse transformation we have:

$$x^{\alpha} = (\Lambda^{-1})^{\alpha}_{\beta} x'^{\beta} , \tag{III.5}$$

so that:

$$(\Lambda^{-1})^{\alpha}_{\mu} \Lambda^{\mu}_{\beta} = \delta^{\alpha}_{\beta} , \quad \Lambda^{\alpha}_{\mu} (\Lambda^{-1})^{\mu}_{\beta} = \delta^{\alpha}_{\beta} . \tag{III.6}$$

Comparing this with (III.4) we find:

$$(\Lambda^{-1})^{\alpha}_{\beta} = \Lambda_{\beta}^{\alpha} , \tag{III.7}$$

so the inverse transformation takes the form:

$$x^\alpha = \Lambda_\beta^\alpha x'^\beta . \tag{III.8}$$

That is, in order to obtain the inverse of the jacobian matrix we must lower the first index, raise the second index, and transpose the matrix. In matrix notation this result can be written as $\Lambda^{-1} = \eta \Lambda^T \eta$. In particular, for a rotation in space raising and lowering indices has no effect, so the inverse is simply the transpose, and since the jacobian matrix is real we see that rotations in space correspond to orthogonal matrices (with inverse equal to the transpose). In contrast, for a Lorentz boost raising one index and lowering the other changes the sign of the first column and first row (keeping the 00 component unchanged), and leaves us again with a symmetric matrix, so now taking the transpose has no effect. A Lorentz boost therefore does not correspond to an orthogonal matrix.

Equation (III.7) also implies the following relations (compare this with (III.4)):

$$\Lambda^{\mu\alpha} \Lambda_{\mu\beta} = \Lambda_\mu^\alpha \Lambda^\mu_\beta = \delta_\beta^\alpha . \tag{III.9}$$

Given the previous results, the Lorentz transformations of vectors and 1-forms take the form:

$$v'^\alpha = \Lambda^\alpha_\beta v^\beta , \quad q'_\alpha = \Lambda_\alpha^\beta q_\beta , \tag{III.10}$$

which can be generalized directly to tensors of arbitrary range. In particular, the coordinate basis vectors (\vec{e}_α) transform in the same way as the components of a 1-form:

$$\vec{e}'_\alpha = \Lambda_\alpha^\beta \vec{e}_\beta . \tag{III.11}$$

On the other hand, since the determinant of the Minkowski tensor is -1 , we also find that:

$$[\det (\Lambda^\alpha_\beta)]^2 = 1 \implies [\det (\Lambda^\alpha_\beta)] = \pm 1 . \tag{III.12}$$

The group of Lorentz transformations is known as $O(3, 1)$, which is the general group that leaves the Minkowski interval invariant. If we restrict ourselves to those transformations that have determinant equal to $+1$ we obtain the “special” or “proper” Lorentz group $SO(3, 1)$. If, moreover, we ask for the direction of time to remain the same, that is we ask for $\Lambda^0_0 \geq 1$, we obtain the orthochronous Lorentz group.

Let us now consider an infinitesimal transformation of the form:

$$\Lambda^\alpha_\beta = \delta^\alpha_\beta + \lambda^\alpha_\beta , \tag{III.13}$$

with $|\lambda^\alpha_\beta| \ll 1$. Raising and lowering indices we find:

$$\Lambda_{\alpha\beta} = \eta_{\alpha\beta} + \lambda_{\alpha\beta} , \quad \Lambda^{\alpha\beta} = \eta^{\alpha\beta} + \lambda^{\alpha\beta} . \tag{III.14}$$

These results imply that, to first order in small quantities, we must have:

$$\lambda^{\alpha\beta} + \lambda^{\beta\alpha} = 0 , \tag{III.15}$$

that is, $\lambda^{\alpha\beta}$ must be antisymmetric. In four dimensions such a matrix has only 6 independent components which correspond to the three spatial rotations and the three possible boosts.

The next step is to introduce a basis for the space of the 4×4 antisymmetric matrices. This basis must be clearly formed from 6 antisymmetric matrices M^A that are linearly independent from each other, with $A = 1, \dots, 6$. In fact, it turns out to be very convenient to replace the index A with a new pair of antisymmetric indices, so that our 6 basis matrices will now be $M^{\rho\sigma}$, with elements given by $(M^{\rho\sigma})^{\alpha\beta}$. This notation can seem somewhat cumbersome at first, but notice that we can now find an explicit expression for our basis matrices as:

$$(M^{\rho\sigma})^{\alpha\beta} = -\eta^{\rho\alpha}\eta^{\sigma\beta} + \eta^{\sigma\alpha}\eta^{\rho\beta} . \tag{III.16}$$

In the above expression (α, β) denote the different elements of a given matrix, while (ρ, σ) denote which particular matrix we are considering. By direct computation it is not difficult to show that the basis introduced above corresponds to taking, for the matrices M^{0i} with $i = 1, 2, 3$:

$$(M^{0i})^{0i} = -(M^{0i})^{i0} = +1 , \tag{III.17}$$

with all other components equal to zero, and for the matrices M^{ij} with $i, j = 1, 2, 3$:

$$(M^{ij})^{ij} = -(M^{ij})^{ji} = -1 , \tag{III.18}$$

again with all other components equal to zero. That is, each basis matrix takes one of the six independent components of a general 4×4 antisymmetric matrix equal to ± 1 , with all other other independent components equal to zero.

The matrix $\lambda^{\alpha\beta}$ associated with an infinitesimal Lorentz transformation can now be written as a linear combination of our basis matrices in the form:

$$\lambda^{\alpha\beta} = \frac{1}{2} C_{\rho\sigma} (M^{\rho\sigma})^{\alpha\beta} , \tag{III.19}$$

where the coefficients $C_{\rho\sigma}$ are six small parameters (antisymmetric in ρ and σ) that identify the type of transformation we are doing, that is, which precise combination of rotations and boosts. The factor $1/2$ is there to compensate for the fact that the sum over (ρ, σ) counts each independent term twice. In practice, in order to apply a Lorentz transformation we need to lower one index and use λ^α_β , where one should remember that the matrices λ^α_β are no longer necessarily antisymmetric (what we actually have is $\lambda^\alpha_\beta + \lambda^\beta_\alpha = 0$). It is interesting to note that, given the form of the matrices M in (III.16), the infinitesimal Lorentz transformation simply reduces to:

$$\lambda^{\alpha\beta} = -C^{\alpha\beta} . \tag{III.20}$$

The matrices $M^{\rho\sigma}$ are known as the *generators* of the Lorentz group, and satisfy the Lie algebra:

$$[M^{\rho\sigma}, M^{\mu\nu}] = \eta^{\rho\mu} M^{\sigma\nu} - \eta^{\rho\nu} M^{\sigma\mu} + \eta^{\sigma\nu} M^{\rho\mu} - \eta^{\sigma\mu} M^{\rho\nu} . \tag{III.21}$$

We can now express a finite Lorentz transformation by using the standard exponential mapping given by:

$$\Lambda^\alpha_\beta = \exp\left(\frac{1}{2} C_{\rho\sigma} (M^{\rho\sigma})^\alpha_\beta\right) . \tag{III.22}$$

Is it common to give alternative names to the matrices M as follows:

$$(B_i)^\alpha_\beta := (M^{0i})^\alpha_\beta , \quad (R_i)^\alpha_\beta = \frac{1}{2} \epsilon_{ijk} (M^{jk})^\alpha_\beta , \tag{III.23}$$

where the Latin indices (i, j, k) only take values from 1 to 3, and with ϵ^{ijk} the Levi-Civita symbol in three dimensions. The names of these new matrices are clearly associated with spatial rotations, R , and Lorentz boosts, B . We find explicitly:

$$B_1 = \begin{pmatrix} 0 & +1 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_2 = \begin{pmatrix} 0 & 0 & +1 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, \quad B_3 = \begin{pmatrix} 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ +1 & 0 & 0 & 0 \end{pmatrix}, \tag{III.24}$$

and:

$$R_1 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & +1 & 0 \end{pmatrix}, \quad R_2 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & +1 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix}, \quad R_3 = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & +1 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}. \tag{III.25}$$

We can now define the complex matrices $J_i := i R_i$ and $K_i := i B_i$, which obey the following algebra:

$$[J_i, J_j] = i \epsilon_{ij}{}^k J_k , \quad [J_i, K_j] = i \epsilon_{ij}{}^k K_k , \quad [K_i, K_j] = -i \epsilon_{ij}{}^k J_k . \tag{III.26}$$

These relations are equivalent to (III.21) and define the algebra of the Lorentz group. The matrices J generate spatial rotations, while the matrices K generate Lorentz boosts. Notice here that the J matrices are purely imaginary and antisymmetric, so they are hermitian, while the matrices K are purely imaginary and symmetric, and as such are anti-hermitian.

With the previous definitions, a general Lorentz transformation can be expressed as:

$$\Lambda = \exp\left(\vec{\theta} \cdot \vec{R}_l - \vec{\varphi} \cdot \vec{B}_l\right) = \exp\left(-i\vec{\theta} \cdot \vec{J} + i\vec{\varphi} \cdot \vec{K}\right) , \tag{III.27}$$

with θ_i and φ_i parameters associated with spatial rotations and boosts, respectively, and where the dot product is the usual one in three dimensions (the negative sign on the term with the φ_i is necessary in order to recover the Lorentz boosts in their usual form). The θ_i represent directly rotation angles, while the φ_i are in fact velocity parameters as will be clear below.

To continue we will now define 6 “partial identity matrices” $I_{\rho\sigma}$ as diagonal matrices such that their only non-zero components are the (ρ, ρ) and (σ, σ) components which are equal to 1. A little algebra now allows us to show that the square of the B and R matrices is given by:

$$(B_1)^2 = +I_{01} , \quad (B_2)^2 = +I_{02} , \quad (B_3)^2 = +I_{03} , \quad \text{(III.28)}$$

$$(R_1)^2 = -I_{23} , \quad (R_2)^2 = -I_{13} , \quad (R_3)^2 = -I_{12} . \quad \text{(III.29)}$$

These relationships allow us to find recurrent formulas for any power of the B and R matrices. In particular, even powers of any B_i are just the respective partial identity matrix, while odd powers are the same B_i again. For the R_i matrices the situation is similar but with alternating signs.

We can then expand the exponential mapping in a Taylor series, so that for the B_i matrices we get only positive signs that can be grouped into hyperbolic sines and cosines. For example, for a boost along the x direction ($\varphi_1 = \varphi, \varphi_2 = \varphi_3 = 0, \theta_i = 0$) we find:

$$\Lambda = \exp(-\varphi B_1) = \begin{pmatrix} \cosh(\varphi) & -\sinh(\varphi) & 0 & 0 \\ -\sinh(\varphi) & \cosh(\varphi) & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} , \quad \text{(III.30)}$$

and similarly for the y and z directions. This can be immediately recognized as a usual Lorentz boost written in terms of the velocity parameter φ defined as $v = \tanh(\varphi)$, with v the speed associated with the boost.

On the other hand, for the R_i matrices the alternating signs result in the Taylor expansion for the standard trigonometric sines and cosines. For example, for a rotation around the x axis ($\varphi_i = 0, \theta_1 = \theta, \theta_2 = \theta_3 = 0$) we now find:

$$\Lambda = \exp(\theta R_1) = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \cos(\theta) & -\sin(\theta) \\ 0 & 0 & \sin(\theta) & \cos(\theta) \end{pmatrix} , \quad \text{(III.31)}$$

which is the usual rotation matrix around the x axis.

3.2 Spinor representation

The algebra of the Lorentz group is defined by the commutation relations (III.21), but the specific form of the matrices, and even their rank, can change when we pass from one representation of the algebra to another. In order to see how this is related to the

Dirac equation, let us define a new set of matrices $\sigma^{\mu\nu}$ of the form:

$$\sigma^{\mu\nu} := \frac{1}{4} [\gamma^\mu, \gamma^\nu] = \frac{1}{2} (\gamma^\mu \gamma^\nu + \eta^{\mu\nu} I_4) , \tag{III.32}$$

with γ^μ the Dirac matrices we defined before in (II.19), and where $[\gamma^\mu, \gamma^\nu] := \gamma^\mu \gamma^\nu - \gamma^\nu \gamma^\mu$ now denotes the commutator, and in the second equality we used the Clifford algebra (II.22). Notice in particular that we have $\sigma^{\mu\mu} = 0$. From this definition it is not difficult to show that:

$$[\sigma^{\mu\nu}, \gamma^\rho] = \eta^{\mu\rho} \gamma^\nu - \eta^{\nu\rho} \gamma^\mu . \tag{III.33}$$

And using this last result we find:

$$[\sigma^{\rho\sigma}, \sigma^{\mu\nu}] = \eta^{\rho\mu} \sigma^{\sigma\nu} - \eta^{\rho\nu} \sigma^{\sigma\mu} + \eta^{\sigma\nu} \sigma^{\rho\mu} - \eta^{\sigma\mu} \sigma^{\rho\nu} . \tag{III.34}$$

But these are precisely the same commutation relations that we had before for the M matrices in (III.21). We then conclude that the σ matrices are a different representation of the Lorentz group (notice in particular that the M matrices are real, while the σ matrices are complex).

Just as we did before, we can now introduce an infinitesimal Lorentz transformation as:

$$s = \frac{1}{2} C_{\rho\sigma} \sigma^{\rho\sigma} , \tag{III.35}$$

where the coefficients $C_{\rho\sigma}$ are the same as before, and the corresponding exponential map as:

$$S = \exp\left(\frac{1}{2} C_{\rho\sigma} \sigma^{\rho\sigma}\right) , \tag{III.36}$$

where now S represents a finite Lorentz transformation. Here one must remember that both s and S are 4×4 matrices. Just as the original Lorentz transformation in the representation Λ acts on vectors v^α (and tensors) as:

$$v'^\alpha = \Lambda^\alpha_\beta v^\beta , \tag{III.37}$$

the Lorentz transformation in the representation S acts on 4-component Dirac spinors in the following way (as will be shown in Sect. 4 below):

$$\psi'^i = S^i_j \psi^j , \tag{III.38}$$

where in the last equation the indices (i, j) are not spacetime indices, but rather spinor indices.

In order to find the explicit form of the $\sigma^{\alpha\beta}$ matrices, let us first consider the purely spatial components associated with spatial rotations. From the definition we find, for

$i \neq j$:

$$\sigma^{ij} = \frac{1}{4} [\gamma^i, \gamma^j] = \frac{1}{2} \gamma^i \gamma^j = \frac{1}{2} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_j \\ -\sigma_j & 0 \end{pmatrix} = -\frac{i}{2} \epsilon^{ijk} \begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix}, \tag{III.39}$$

where we used the fact that the γ^i and γ^j anticommute, and in the last steps we used the form of the γ^i matrices in Dirac’s representation (though in this case one in fact finds the same result in the Weyl representation). If we now define the angles θ^k such that $C_{ij} = -\epsilon_{ijk}\theta^k$, the rotation matrix can be written as:

$$S = \begin{pmatrix} e^{i\vec{\theta}\cdot\vec{\sigma}/2} & 0 \\ 0 & e^{i\vec{\theta}\cdot\vec{\sigma}/2} \end{pmatrix}, \tag{III.40}$$

where the dot product is again the standard in three dimensions. Let us now consider, for example, a rotation around the x axis by an angle θ , in that case we will have:

$$S = \begin{pmatrix} e^{+i(\theta/2)\sigma_1} & 0 \\ 0 & e^{+i(\theta/2)\sigma_1} \end{pmatrix} = \cos(\theta/2) \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} + i \sin(\theta/2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}. \tag{III.41}$$

where in the last step we use the fact that $(\sigma_1)^2 = I_2$ in order to expand the exponential in a Taylor series and regroup terms into sines and cosines. Notice now that the form of this matrix is quite different from the matrix λ associated to the same rotation given by (III.31). In particular, if we take $\theta = 2\pi$ we find $S = -I_4$, so that ψ now changes sign after a full rotation, in contrast with what happens with vectors and tensors that are invariant under a full rotation. This is a well known characteristic of spinors.

On the other hand, for a Lorentz boost we find, in the Dirac representation:

$$\sigma^{0i} = \frac{1}{2} \gamma^0 \gamma^i = \frac{1}{2} \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}. \tag{III.42}$$

Defining now the velocity parameter as $\varphi := -C_{0i}$ we find, for an arbitrary boost:

$$S = \begin{pmatrix} 0 & e^{-\vec{\varphi}\cdot\vec{\sigma}/2} \\ e^{-\vec{\varphi}\cdot\vec{\sigma}/2} & 0 \end{pmatrix}. \tag{III.43}$$

Using now the fact that:

$$\begin{pmatrix} 0 & \sigma_i \\ \sigma_i & 0 \end{pmatrix}^2 = I_4, \tag{III.44}$$

one can show that a boost along the direction x takes the form:

$$S = \begin{pmatrix} 0 & e^{-(\varphi/2)\sigma_1} \\ e^{-(\varphi/2)\sigma_1} & 0 \end{pmatrix} = \cosh(\varphi/2) \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} - \sinh(\varphi/2) \begin{pmatrix} 0 & \sigma_1 \\ \sigma_1 & 0 \end{pmatrix}. \tag{III.45}$$

One should be very careful when applying a Lorentz boost (also a rotation) using the exponential map. The exponential of a matrix is really defined in terms of the

Taylor expansion. This implies, for example, that even if the matrix S given in (III.43) apparently has no elements in the diagonal, once we do the Taylor expansion such diagonal elements do appear (because of Eq. (III.44)). That is the reason why it is easier to use the Lorentz transformation along a specific direction as shown in (III.45).

It turns out that for the case of a Lorentz boost it is more convenient to work in the Weyl representation, in which case we have:

$$\sigma^{0i} = \frac{1}{2} \gamma^0 \gamma^i = \frac{1}{2} \begin{pmatrix} 0 & I_2 \\ I_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma_i \\ -\sigma_i & 0 \end{pmatrix} = \frac{1}{2} \begin{pmatrix} -\sigma_i & 0 \\ 0 & \sigma_i \end{pmatrix}, \tag{III.46}$$

and taking again $\varphi := -C_{0i}$ we find:

$$S = \begin{pmatrix} e^{+\vec{\varphi} \cdot \vec{\sigma} / 2} & 0 \\ 0 & e^{-\vec{\varphi} \cdot \vec{\sigma} / 2} \end{pmatrix}. \tag{III.47}$$

For a boost along the x direction we now have:

$$S = \begin{pmatrix} e^{(\varphi/2)\sigma_1} & 0 \\ 0 & e^{-(\varphi/2)\sigma_1} \end{pmatrix} = \cosh(\varphi/2) \begin{pmatrix} I_2 & 0 \\ 0 & I_2 \end{pmatrix} + \sinh(\varphi/2) \begin{pmatrix} \sigma_1 & 0 \\ 0 & -\sigma_1 \end{pmatrix}. \tag{III.48}$$

The reason for which the Weyl representation is better in this case is that in the Dirac representation the components (1, 2) of the spinor are mixed with the components (3, 4) for a Lorentz boost, but this does not happen in the Weyl representation since the matrix is now block diagonal. Here it is important to notice that the S transformations in general are *not* unitary (i.e. with inverse equal to their transpose conjugate). A three-dimensional rotation is unitary as can be easily seen from (III.40) and the fact that the Pauli matrices σ_i are hermitian, but a Lorentz boost is not unitary.

To finish this section we will show a very important relation between the matrices S and Λ associated with the same Lorentz transformation. It turns out that in general one has:

$$S^{-1} \gamma^\mu S = \Lambda^\mu_\nu \gamma^\nu. \tag{III.49}$$

The previous expression has to be understood with some care. On the left hand side we have the product of three matrices, $S^{-1} \gamma^\mu S$, while on the right hand side we have a linear combination of matrices γ^ν with coefficients given by the Λ^μ_ν . In order to prove this relation we shall work in the limit of infinitesimal transformations. In that case we have, from (III.22) and (III.36) (here and in what follows \simeq denotes approximately equal to):

$$\Lambda \simeq I_4 + \frac{1}{2} C_{\rho\sigma} M^{\rho\sigma}, \quad S \simeq I_4 + \frac{1}{2} C_{\rho\sigma} \sigma^{\rho\sigma}. \tag{III.50}$$

This implies:

$$\Lambda^\mu_\nu \gamma^\nu \simeq \gamma^\mu + \frac{1}{2} C_{\rho\sigma} (M^{\rho\sigma})^\mu_\nu \gamma^\nu, \tag{III.51}$$

and:

$$S^{-1} \gamma^\mu S \simeq \gamma^\mu - \frac{1}{2} C_{\rho\sigma} (\sigma^{\rho\sigma} \gamma^\mu - \gamma^\mu \sigma^{\rho\sigma}), \tag{III.52}$$

where in the last expression we kept only linear terms in the coefficients $C_{\rho\sigma}$, which we assume are small. In order to prove (III.49) we must then ask for:

$$[\sigma^{\rho\sigma}, \gamma^\mu] = -(M^{\rho\sigma})^\mu{}_\nu \gamma^\nu . \tag{III.53}$$

But, from (III.16) we have:

$$(M^{\rho\sigma})^\mu{}_\nu \gamma^\nu = -\eta^{\rho\mu} \gamma^\sigma + \eta^{\sigma\mu} \gamma^\rho , \tag{III.54}$$

and using now (III.33) we obtain precisely (III.53). Equation (III.49) can be easily inverted to find the equivalent expression:

$$S\gamma^\mu S^{-1} = \Lambda_\nu{}^\mu \gamma^\nu . \tag{III.55}$$

4 Lorentz invariance of the Dirac equation

We are now in a position to show the invariance of Dirac’s equation under a Lorentz transformation. We start from Dirac’s equation written in covariant form:

$$i\gamma^\mu \partial_\mu \psi - m\psi = 0 . \tag{IV.1}$$

Under a Lorentz transformation $x'^\mu = \Lambda^\mu{}_\nu x^\nu$ this equation must keep the exact same form, with the same γ^μ matrices, so we must have:

$$i\gamma^\mu \partial'_\mu \psi' - m\psi' = 0 , \tag{IV.2}$$

with $\partial'_\mu = \Lambda_\mu{}^\nu \partial_\nu$. We now propose that the Dirac spinor transforms with a transformation matrix S as:

$$\psi' = S\psi , \tag{IV.3}$$

where at the moment we are not assuming anything about the form of S , except for the fact that it must be a constant matrix (because of the homogeneity and isotropy of spacetime). Substituting this into Eq. (IV.2) we find:

$$i\Lambda_\mu{}^\nu \gamma^\mu S \partial_\nu \psi - mS\psi = 0 , \tag{IV.4}$$

where we used the fact that S is constant and the $\Lambda_\mu{}^\nu$ are real numbers so they commute with the matrices. Multiplying now from the left with S^{-1} we find:

$$i\Lambda_\mu{}^\nu \left(S^{-1} \gamma^\mu S \right) \partial_\nu \psi - m\psi = 0 , \tag{IV.5}$$

and comparing with the original Dirac equation we conclude that we must have:

$$\Lambda_\mu{}^\nu \left(S^{-1} \gamma^\mu S \right) = \gamma^\nu . \tag{IV.6}$$

Using now the fact that $\Lambda_{\mu}^{\nu} = (\Lambda^{-1})^{\nu}_{\mu}$, we can see that the previous expression is equivalent to:

$$S^{-1}\gamma^{\mu}S = \Lambda^{\mu}_{\nu}\gamma^{\nu} . \tag{IV.7}$$

But this is nothing more than Eq. (III.49) from the previous section, so we conclude that the Dirac spinor ψ transforms precisely with the S matrices we discussed before.

From the transformation law for ψ one can easily show that the adjunct spinor $\bar{\psi}$ transforms as:

$$\bar{\psi}' = \psi'^{\dagger}\gamma^0 = \psi^{\dagger}S^{\dagger}\gamma^0 . \tag{IV.8}$$

Using now the relations (II.26) it is not difficult to show that the matrices $\sigma^{\mu\nu}$ defined in (III.32) satisfy:

$$(\sigma^{\mu\nu})^{\dagger} = -\gamma^0\sigma^{\mu\nu}\gamma^0 , \tag{IV.9}$$

which in turn implies:

$$S^{\dagger} = \gamma^0S^{-1}\gamma^0 . \tag{IV.10}$$

The transformation of $\bar{\psi}$ then takes the final form:

$$\bar{\psi}' = \psi^{\dagger}\gamma^0S^{-1} = \bar{\psi}S^{-1} , \tag{IV.11}$$

where we used the fact that $(\gamma^0)^2 = 1$. The result implies, in particular, that the object $(\bar{\psi}\psi)$ is a Lorentz scalar, while the conserved current $(\bar{\psi}\gamma^{\mu}\psi)$ transforms as a vector. Moreover, $(\bar{\psi}\gamma^{\mu}\gamma^{\nu}\psi)$ transforms as a rank 2 tensor, or strictly speaking its antisymmetric part transforms as a tensor (actually a 2-form), while its symmetric part is Lorentz invariant since it is proportional to the Minkowski tensor (see Eq. (II.22)).

5 Dirac equation in general relativity

5.1 The tetrad formalism

In order to write the Dirac equation in the case of a general curved spacetime we first need to introduce the formalism of tetrads. We will not describe here that formalism in great detail as it is quite standard in advanced general relativity text books, and we will limit ourselves to discuss some of its more important properties.

The basic idea behind the tetrad formalism is to choose, at each point of spacetime, a set of four orthonormal vectors $\{\vec{e}_A\}$ that define a local inertial frame, where the index A identifies each of the four vectors ($A = 0, 1, 2, 3$). It is important to emphasize that this orthonormal basis is in principle completely independent from our coordinate system. The basis $\{\vec{e}_A\}$ is known as a *tetrad* (also frequently called a *vierbein* from the German word for “four legs”, or even a *vielbein* meaning “many legs”).

Assume now that we have a general coordinate system $\{x^{\mu}\}$, the corresponding coordinate basis components of the tetrad will then be e^{μ}_A , and since the tetrad basis is orthonormal by definition we will have:

$$\vec{e}_A \cdot \vec{e}_B = g_{\mu\nu}e^{\mu}_A e^{\nu}_B = \eta_{AB} , \tag{V.1}$$

with $g_{\mu\nu}$ the components of the metric tensor and η_{AB} the usual Minkowski tensor. Notice that now we have two different types of indices: indices associated with the spacetime coordinates, denoted by Greek letters, and indices associated to the tetrad basis vectors denoted by uppercase Latin letters and usually called ‘‘Lorentz indices’’. From here on we also take the convention that Greek indices are raised and lowered with $g_{\mu\nu}$, while uppercase Latin indices are raised and lowered with η_{AB} . For example we will have:

$$e_{\mu A} = g_{\mu\nu} e_A^\nu, \quad e^{\mu A} = \eta^{AB} e_B^\mu. \tag{V.2}$$

In particular, this implies:

$$e_{\mu A} e_B^\mu = \eta_{AB}. \tag{V.3}$$

Somewhat more formally, we define a set of four 1-forms $\{\tilde{e}^A\}$ associated with our tetrad such that:

$$\tilde{e}^A (\tilde{e}_B) = e_\mu^A e_B^\mu = \delta_B^A.$$

The 1-forms $\{\tilde{e}^A\}$ are known as the ‘‘co-tetrad’’. However, it is easier just to think of raising and lowering indices with η_{AB} and $g_{\mu\nu}$.

We can now project arbitrary vectors onto the tetrad. For example, for a vector \vec{v} we will have:

$$v^A = v^\mu e_\mu^A, \quad v_A = v^\mu e_{\mu A}, \tag{V.4}$$

and the original vector can be written as:

$$\vec{v} = v^A \tilde{e}_A. \tag{V.5}$$

This relation implies then that:

$$v^\mu = v^A e_A^\mu. \tag{V.6}$$

The dot product of two vectors can then be written as:

$$\vec{v} \cdot \vec{u} = (v^A \tilde{e}_A) \cdot (u^B \tilde{e}_B) = v^A u^B \eta_{AB} = v^A u_A. \tag{V.7}$$

That is, we have $\vec{v} \cdot \vec{u} = v^\mu u_\mu = v^A u_A$.

Let now $\{\vec{z}_\mu\}$ denote the coordinate basis vectors. We can then express this basis in terms of the tetrad as:

$$\vec{z}_\mu = z_\mu^A \tilde{e}_A. \tag{V.8}$$

The dot product of two coordinate basis vectors will then be:

$$\vec{z}_\mu \cdot \vec{z}_\nu = z_{\mu A} z_\nu^A = (e_{\lambda A} z_\mu^\lambda) (e_\sigma^A z_\nu^\sigma) = (e_{\lambda A} \delta_\mu^\lambda) (e_\sigma^A \delta_\nu^\sigma) = e_{\mu A} e_\nu^A, \tag{V.9}$$

where we used the fact that the spacetime components of the coordinate basis are just the Kronecker delta. But the dot product of coordinate basis vectors is precisely the definition of the components of the metric tensor, so we finally have (compare this with (V.3)):

$$e_{\mu A} e_\nu^A = g_{\mu\nu}. \tag{V.10}$$

This is a very important result, and shows that in some sense the tetrad can be understood as a “square root” of the metric tensor. In particular, the last expression implies that the determinant of the metric can be written in terms of the tetrad as:

$$g = \det \left(e_{\mu}^A e_{\nu}^B \eta_{AB} \right) = \left[\det(e_{\mu}^A) \right]^2 \det(\eta_{AB}) = - \left[\det(e_{\mu}^A) \right]^2, \tag{V.11}$$

so the volume element takes the form:

$$|g|^{1/2} = \det(e_{\mu}^A). \tag{V.12}$$

It is interesting to notice that the components of the tetrad e_{μ}^A can also be interpreted as the jacobian of the transformation from the coordinates $\{x^{\mu}\}$ to a new set of locally flat coordinates $\{X^A\}$. If we define $e_{\mu}^A := \partial x^{\mu} / \partial X^A$ we will have, for an arbitrary vector:

$$v^{\mu} = \left(\frac{\partial x^{\mu}}{\partial X^A} \right) v^A = e_{\mu}^A v^A, \tag{V.13}$$

which corresponds to the rule (V.6) that we saw above.

We can also project tensors of arbitrary rank onto the tetrad. For example, for a rank 2 tensor we will have:

$$T_{AB} := e_{\mu}^A e_{\nu}^B T_{\mu\nu}. \tag{V.14}$$

In particular, if we take T as the metric tensor we find immediately $g_{AB} = \eta_{AB}$, as should be expected. The operation that takes a spacetime index into a Lorentz (tetrad) index is usually known as “strangulation” ($v^A = e_{\mu}^A v^{\mu}$), while the opposite operation that takes a Lorentz index into a spacetime index is known as “resurrection” ($v^{\mu} = e_{\mu}^A v^A$).

The importance of the tetrad formalism comes from the fact that, once projected onto the tetrad, the components of vectors and tensors behave as scalar functions under changes of coordinates. In the tetrad formalism we then have two different types of transformations:

1. General coordinate transformations for which the tetrad basis vectors behave in the usual manner, but the components of different geometric objects (vectors, tensors and spinors) projected onto the tetrad behave as scalars.
2. Local Lorentz transformations that take an initial tetrad onto a new tetrad, for which the components of geometric objects transform as tensors in special relativity (or spinors, see next sections).

The next step is to consider the derivatives of vectors and tensors in the tetrad formalism. For this we need to define the so-called *spin connection coefficients*, also known as the *Ricci rotation coefficients*, as:

$$\omega_{\mu B \nu} := \nabla_{\nu} e_{\mu B} \equiv e_{\mu B ; \nu}. \tag{V.15}$$

The order of the indices for the ω ’s is chosen here such that it coincides with the natural order when using the “;” notation for covariant derivatives (this convention is common, but not universal, so care must be taken when reading different references).

It is usual to work with these coefficients with one or both of the spacetime indices projected onto the tetrad (strangled):

$$\omega_{AB\nu} = e_A^\lambda \omega_{\lambda B\nu}, \quad \omega_{ABC} = e_A^\lambda e_C^\sigma \omega_{\lambda B\sigma}. \tag{V.16}$$

One often finds that the name ‘‘spin connection coefficients’’ is reserved for the fully strangled version ω_{ABC} . Similarly, one can work with all three indices of spacetime type (resurrected):

$$\omega_{\mu\lambda\nu} = e_\lambda^B \omega_{\mu B\nu}. \tag{V.17}$$

Notice that with our notation the third index always corresponds to the derivative (but many references put the derivative index as the first one), either directly as in $\omega_{\mu A\nu}$ and $\omega_{AB\nu}$, or strangled as in ω_{ABC} .

From the above definition it is not difficult to show that in general we have:

$$\partial_\mu \vec{e}_A \equiv (e^{\nu A;\mu}) \vec{z}_\nu = \omega^{\nu A\mu} \vec{z}_\nu = \omega^B{}_{A\mu} \vec{e}_B. \tag{V.18}$$

The spin connection that we have just defined turns out to be antisymmetric in the first two indices when they are of the same type. In order to see this one must remember that $\eta_{AB} = g_{\mu\nu} e_A^\mu e_B^\nu$, which implies:

$$\begin{aligned} 0 &= \eta_{AB;\lambda} = g_{\mu\nu} \left(e_A^\mu e_{B;\lambda}^\nu + e_B^\nu e_{A;\lambda}^\mu \right) \\ &= e_{\nu A} e_{B;\lambda}^\nu + e_{\mu B} e_{A;\lambda}^\mu = \omega_{AB\lambda} + \omega_{BA\lambda}, \end{aligned} \tag{V.19}$$

and finally:

$$\omega_{AB\alpha} = -\omega_{BA\alpha}. \tag{V.20}$$

From this we immediately also find $\omega_{\alpha\beta\lambda} = -\omega_{\beta\alpha\lambda}$. This property ensures that when we express the covariant derivative of the metric tensor in terms of the tetrad we will have $\nabla_\mu g_{\alpha\beta} = 0$. To see this notice that from (V.10) we have:

$$\begin{aligned} \nabla_\mu g_{\alpha\beta} &= \left(e_{\alpha A} e_{\beta}^A \right)_{;\mu} = e_{\alpha A} e_{\beta;\mu}^A + e_{\beta}^A e_{\alpha A;\mu} \\ &= e_{\alpha A} \omega_{\beta}^A{}_\mu + e_{\beta}^A \omega_{\alpha A}{}^\mu = \omega_{\beta\alpha\mu} + \omega_{\alpha\beta\mu} = 0. \end{aligned} \tag{V.21}$$

Notice that, from Eq. (V.18) above, the Ricci rotation coefficients play a similar role to the Christoffel symbols when working with the tetrad instead of a coordinate basis. But crucially, while the Christoffel symbols are symmetric in two indices, $\Gamma_{\mu\nu}^\alpha = \Gamma_{\nu\mu}^\alpha$, the Ricci rotation coefficients are antisymmetric in two indices, $\omega_{AB\alpha} = -\omega_{BA\alpha}$. This implies that there are fewer independent $\omega_{AB\alpha}$ than there are $\Gamma_{\mu\nu}^\alpha$. For example, in 4 dimensions there are 40 independent $\Gamma_{\mu\nu}^\alpha$, while there are only 24 independent $\omega_{AB\alpha}$. This is one of the advantages of the tetrad formalism: one has to compute fewer quantities.

The name “rotation coefficients” comes from considering the change of the tetrad for an infinitesimal displacement δx^μ . In that case we have:

$$\delta e^\mu_A = e^\mu_{A;\nu} \delta x^\nu = \omega^\mu_{A\nu} \delta x^\nu = \left(\omega^B_{A\nu} \delta x^\nu \right) e^\mu_B. \tag{V.22}$$

If we now define $\Lambda^B_A := \omega^B_{A\nu} \delta x^\nu$ we find:

$$\delta e^\mu_A = \Lambda^B_A e^\mu_B. \tag{V.23}$$

But this is just a rotation of the tetrad in 4 dimensions, that is a general Lorentz transformation such as those we discussed above.

One can in fact define different concepts of derivatives of geometric quantities in the tetrad formalism. The first one is the standard covariant derivative for which an object completely projected onto the tetrad, i.e. completely strangled, behaves as a scalar. To denote this derivative we will continue to use the ∇ (or ;) symbol.

We can also define two different types of derivatives for geometric objects that have indices of mixed type. The *intrinsic derivative* is defined by first strangling all the spacetime indices, then taking the covariant derivative of the resulting scalar, and finally resurrecting back the indices that had been strangled. To denote this derivative we use a vertical bar instead of as semicolon, for example:

$$T^\mu_{A|v} = \left(T^\lambda_A e^\lambda_B \right)_{;v} e^\mu_B = T^B_{A;\nu} e^\mu_B = T^B_{A,\nu} e^\mu_B, \tag{V.24}$$

where in the last step we used the fact that fully strangled components behave as scalars for the covariant derivative. The intrinsic derivative then corresponds to the change of the tensor components with respect to the tetrad basis. That is, if the tetrad changes from one point to another and the tensor also changes but in such a way that its tetrad components are the same, then its intrinsic derivative vanishes. In particular, from the previous definition it is easy to see that:

$$e^\mu_{A|v} = \left(e^\lambda_A e^\lambda_B \right)_{;v} e^\mu_B = \left(\delta^B_A \right)_{;v} e^\mu_B = 0, \tag{V.25}$$

that is, the intrinsic derivative of the tetrad is always zero, so that such derivative commutes with the strangulation and resurrection operations, for example:

$$e^B_\mu T^\mu_{A|v} = T^B_{A|v}, \quad e^A_\lambda T^\mu_{A|v} = T^\mu_{\lambda|v}. \tag{V.26}$$

Equation (V.24) can also be written explicitly in terms of the spin connection as follows:

$$T^\mu_{A|v} = T^\mu_{A;\nu} + \omega_\sigma{}^\mu{}_\nu T^\sigma_A. \tag{V.27}$$

The first term in the previous expression is the usual covariant derivative, for which strangled indices behave as scalars (so that T^μ_A behaves as a vector). In the second term the spin connection plays a similar role as the usual Christoffel symbols, but

remember that they have different symmetries. It is also possible to strangle the index in the derivative to define a directional derivative along the tetrad, for example:

$$T^{\mu}_{A|C} = e^{\nu}_C T^{\mu}_{A;\nu} = T^{\mu}_{A;C} + \omega_{\sigma}{}^{\mu}{}_C T^{\sigma}_A, \tag{V.28}$$

where we have defined $T^{\mu}_{A;C} := e^{\nu}_C T^{\mu}_{A;\nu}$. Similarly, we also can strangle the μ index to find:

$$T^B_{A|C} = e^B_{\mu} T^{\mu}_{A;C} + \omega_{\sigma}{}^B{}_C T^{\sigma}_A = e^B_{\mu} T^{\mu}_{A;C} + \omega_D{}^B{}_C T^D_A. \tag{V.29}$$

Notice that for the first term we can not simply write $T^B_{A;C}$, since the covariant derivative of the tetrad in general does not vanish.

The intrinsic derivative can be generalized to tensors with multiple indices in the same way as the covariant derivative. For example, for an object with fully strangled indices we have:

$$T^A_{B|v} = T^A_{B,v}, \tag{V.30}$$

while for an object with one covariant spacetime index we will have:

$$T_{\mu A|v} = T_{\mu A;v} - \omega_{\mu}{}^{\sigma}{}_v T_{\sigma A}, \tag{V.31}$$

and for objects with mixed spacetime indices we find:

$$T^{\mu}_{\lambda A|v} = T^{\mu}_{\lambda A;v} + \omega_{\sigma}{}^{\mu}{}_v T^{\sigma}_{\lambda A} - \omega_{\lambda}{}^{\sigma}{}_v T^{\mu}_{\sigma A}. \tag{V.32}$$

A different type of derivative is known as the *invariant derivative*. In this case the definition is the opposite: we first resurrect all tetrad indices, we then take the covariant derivative, and finally we strangle again. We denote this derivative by a dot, for example:

$$T^{\mu}_{A\cdot v} = \left(T^{\mu}_B e^B_{\lambda} \right)_{;v} e^{\lambda}_A = T^{\mu}_{\lambda;v} e^{\lambda}_A. \tag{V.33}$$

In the same way as with the intrinsic derivative, it turns out that the invariant derivative of the tetrad also vanishes:

$$e^{\mu}_{A\cdot v} = \left(e^{\mu}_B e^B_{\lambda} \right)_{;v} e^{\lambda}_A = \left(\delta^{\mu}_{\lambda} \right)_{;v} e^{\lambda}_A = 0. \tag{V.34}$$

The invariant derivative then also commutes with strangulation and resurrection, for example:

$$e^A_{\lambda} T^{\mu}_{A\cdot v} = T^{\mu}_{\lambda\cdot v}, \quad e^B_{\mu} T^{\mu}_{A\cdot v} = T^B_{A\cdot v}. \tag{V.35}$$

In particular, this implies that when a tensor is expressed with purely spacetime indices, its invariant derivative is simply equal to its covariant derivative:

$$T^{\mu}_{\lambda\cdot v} = T^{\mu}_{\lambda;v}. \tag{V.36}$$

In other words, the invariant derivative is just the projection of the usual covariant derivative of a tensor with purely spacetime indices onto the tetrad, in one or several

of its indices. In particular we have, for example:

$$T_{B \cdot C}^A = e_\alpha^A e_B^\beta e_C^\nu T_{\beta; \nu}^\alpha . \tag{V.37}$$

The previous expression can in fact be inverted to find:

$$T_{\beta; \nu}^\alpha = e_A^\alpha e_\beta^B e_\nu^C T_{B \cdot C}^A . \tag{V.38}$$

For a tensor with mixed indices, the invariant derivative gives us the change of the tensor as an abstract geometric object, that is already reconstructed in terms of the corresponding basis. For example, if we have a tensor $\mathbf{T} \equiv T_A^\mu \vec{z}_\mu \vec{e}^A$, then its derivative will be:

$$\partial_\mu (\mathbf{T}) = \partial_\mu \left(T_A^\nu \vec{z}_\nu \vec{e}^A \right) = T_{A \cdot \mu}^\nu \vec{z}_\nu \vec{e}^A . \tag{V.39}$$

It is because of this property that the invariant derivative is the most natural generalization of the covariant derivative in the tetrad formalism.

The invariant derivative can also be written in terms of the spin connection. For example we have:

$$T^{\mu A}{}_{\cdot \nu} = T^{\mu A}{}_{; \nu} + \omega^A{}_{B \nu} T^{\mu B} , \tag{V.40}$$

$$T^\mu{}_{A \cdot \nu} = T^\mu{}_{A; \nu} - \omega^B{}_{A \nu} T^\mu{}_B . \tag{V.41}$$

In a similar way, for an object with mixed covariant and contravariant Lorentz indices we will have:

$$T^{\mu A}{}_{B \cdot \nu} = T^{\mu A}{}_{B; \nu} + \omega^A{}_{C \nu} T^{\mu C}{}_B - \omega^C{}_{B \nu} T^{\mu A}{}_C . \tag{V.42}$$

It is interesting to notice that, while in the case of the intrinsic derivative the spin connection “takes” the spacetime indices of the original tensor, in the case of the invariant derivative it takes the Lorentz indices. In particular, if the original tensor had no spacetime indices the intrinsic derivative has no extra terms and just reduces to the covariant derivative (in fact the partial derivative), while the opposite happens for the case of the invariant derivative where for a tensor with no Lorentz indices it reduces to the usual covariant derivative. Notice that for a tensor with no spacetime indices the invariant derivative simplifies and the first term reduces to a partial derivative, so that we have for example:

$$v^A{}_{\cdot B} = e_B^\mu v_{\cdot \mu}^A = e_B^\mu \left(\partial_\mu v^A + \omega^A{}_{C \mu} v^C \right) = \partial_B v^A + \omega^A{}_{CB} v^C , \tag{V.43}$$

where we defined $\partial_B := e_B^\mu \partial_\mu$. Similarly:

$$v_{A \cdot B} = \partial_B v_A - \omega^C{}_{AB} v_C , \tag{V.44}$$

$$T_{B \cdot C}^A = \partial_C T_B^A + \omega^A{}_{DC} T_B^D - \omega^D{}_{BC} T^A{}_D . \tag{V.45}$$

In particular if we apply the rule (V.41) to the tetrad vectors we find:

$$e_{\mu A \cdot \alpha} = e_{\mu A; \alpha} - \omega^C{}_{A \alpha} e_{\mu C} = \omega_{\mu A \alpha} - \omega_{\mu A \alpha} = 0 , \tag{V.46}$$

so that we recover (V.34). This result is usually called the “tetrad postulate”, and can sometimes be a source of some confusion. But notice that in this case the invariant derivative does not correspond directly with the covariant derivative of the 1-form \tilde{e}_A , which is not zero in general, but rather with the covariant derivative of the “tensor” $e_{\mu\nu}$ projected onto the tetrad, and we have $e_{\mu\nu} = e_{\mu A} e_{\nu}^A = g_{\mu\nu}$. That is, having the invariant derivative of the tetrad vanish simply represents the fact that the covariant derivative of the metric tensor is always zero.

We can now use the expression for the invariant derivative of a vector to find a relation between the spin connection and the Christoffel symbols. If we calculate $v^\lambda_{;\alpha}$ directly we have:

$$v^\lambda_{;\mu} = \partial_\mu v^\lambda + \Gamma^\lambda_{\nu\mu} v^\nu, \tag{V.47}$$

while if we calculate it from the projection of the invariant derivative we find:

$$\begin{aligned} v^\lambda_{;\mu} &= e_B^\lambda e_\mu^A (v^{.B}) = e_B^\lambda e_\mu^A (\partial_A v^B + \omega^B_{CA} v^C) \\ &= e_B^\lambda \partial_\mu v^B + \omega^\lambda_{\nu\mu} v^\nu = \partial_\mu (e_B^\lambda v^B) - v^B \partial_\mu e_B^\lambda + \omega^\lambda_{\nu\mu} v^\nu \\ &= \partial_\mu v^\lambda + (\omega^\lambda_{\nu\mu} - e_\nu^B \partial_\mu e_B^\lambda) v^\nu. \end{aligned} \tag{V.48}$$

Equating both expressions, and using the fact that this must be valid for an arbitrary vector \vec{v} , we find:

$$\Gamma^\lambda_{\nu\mu} = \omega^\lambda_{\nu\mu} - e_\nu^B \partial_\mu e_B^\lambda, \tag{V.49}$$

where we used the fact that the Christoffel symbols are symmetric in the lower indices. Solving for the spin coefficients we finally find:

$$\omega^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu} + e_\nu^B \partial_\mu e_B^\lambda. \tag{V.50}$$

In a similar way, by considering now the derivative of a 1-form we also find:

$$\omega^\lambda_{\nu\mu} = \Gamma^\lambda_{\nu\mu} - e_B^\lambda \partial_\mu e_\nu^B. \tag{V.51}$$

Notice that the last expression can be obtained immediately from the fact that $e_\nu^B e_B^\lambda = \delta^\lambda_\nu$. Finally, projecting the first two indices onto the tetrad we obtain:

$$\omega_{AB\mu} = e_B^\nu (e_{\lambda A} \Gamma^\lambda_{\mu\nu} - \partial_\mu e_{\nu A}). \tag{V.52}$$

Even though not obvious, the expression above is in fact antisymmetric in the indices A and B as will become apparent below. This expression is particularly useful in order to calculate the coefficients $\omega_{AB\mu}$ which, as we well see in the next sections, are necessary in order to write the Dirac equation in general relativity.

If we now substitute the definition of the Christoffel symbols in terms of the metric tensor, and the expression of the metric in terms of the tetrad $g_{\mu\nu} = e_{\mu A} e_{\nu}^A$, a straightforward (but somewhat long) algebra allows us to find:

$$\omega_{ABC} = -\frac{1}{2} [(f_{ABC} + f_{ACB} + f_{CAB}) - A \leftrightarrow B], \tag{V.53}$$

where we have projected the third index onto the tetrad, and where we have defined the quantities:

$$f_{ABC} := (\partial_A e_{\alpha B}) e_C^\alpha . \tag{V.54}$$

The antisymmetry in the first two indices of ω_{ABC} is now evident.

To finish this section we will now write the Riemann curvature tensor in terms of the Ricci rotation coefficients. One finds:

$$R_{AB\mu\nu} = \partial_\mu \omega_{AB\nu} - \partial_\nu \omega_{AB\mu} + \omega_{AC\mu} \omega^C_{B\nu} - \omega_{AC\nu} \omega^C_{B\mu} . \tag{V.55}$$

Notice that in this expression the Riemann tensor has the first two indices projected onto the tetrad, so that it behaves as a rank 2 tensor (in fact a 2-form) with respect to coordinate changes. The previous expression can be proved by direct calculation (the algebra is rather long) by substituting the ω 's in terms of the Christoffel symbols from (V.49), and using the usual definition of the Riemann tensor:

$$R^\alpha_{\beta\mu\nu} := \partial_\mu \Gamma^\alpha_{\beta\nu} - \partial_\nu \Gamma^\alpha_{\beta\mu} + \Gamma^\alpha_{\sigma\mu} \Gamma^\sigma_{\beta\nu} - \Gamma^\alpha_{\sigma\nu} \Gamma^\sigma_{\beta\mu} . \tag{V.56}$$

5.2 Generally covariant form of the Dirac equation

In order to generalize the Dirac equation to the case of a curved spacetime we must first go back to Eq. (II.22) that defines the Clifford algebra, which in the language of tetrads now takes the form:

$$\{\gamma^A, \gamma^B\} = -2\eta^{AB} I_4 , \tag{V.57}$$

with $\{, \}$ the anticommutator, and where the γ^A matrices are the same constant matrices we defined in the case of special relativity. We will now define new γ^μ matrices that depend on the tetrad as:

$$\gamma^\mu := \gamma^A e_A^\mu . \tag{V.58}$$

Notice that these new γ^μ are in general not constant anymore and can change from one point to another. From this definition we find immediately:

$$\{\gamma^\mu, \gamma^\nu\} = \{\gamma^A, \gamma^B\} e_A^\mu e_B^\nu = -2\eta^{AB} e_A^\mu e_B^\nu , \tag{V.59}$$

and using now (V.10) we finally obtain:

$$\{\gamma^\mu, \gamma^\nu\} = -2g^{\mu\nu} I_4 . \tag{V.60}$$

This is the form of the Clifford algebra in general relativity. In particular, notice that the new γ^μ matrices with different indices in general do not anti-commute anymore, and only do so for the special case of orthogonal coordinates. From the previous result it is easy to show that:

$$\gamma_\mu \gamma^\mu = \gamma_A \gamma^A = -4I_4 . \tag{V.61}$$

The next step is to consider the transformation rule for a spinor. For a 4D rotation of the tetrad we will have:

$$\psi' = S(x) \psi , \tag{V.62}$$

where now $S(x)$ is a general Lorentz spinor transformation as those we saw before, but which can now depend on position. The derivative of a spinor, however, does not transform as a spinor anymore since we will have:

$$\partial_\mu \psi' = \partial_\mu (S \psi) = S (\partial_\mu \psi) + (\partial_\mu S) \psi . \tag{V.63}$$

In order to take this into account we will now define a *spinor covariant derivative* as:

$$\mathcal{D}_\mu \psi := \partial_\mu \psi + \Gamma_\mu \psi , \tag{V.64}$$

where the Γ_μ are some matrices to be determined, and are known as the *spinor affine connection* (not to be confused with the spin connection coefficients we saw above), or the *Fock–Ivanenko coefficients* [2, 3]. We will find the explicit form of these coefficients in the next Section.

Consider now the adjoint of Eq. (V.64) which takes the form:

$$\mathcal{D}_\mu \bar{\psi} := \partial_\mu \bar{\psi} + \bar{\psi} \bar{\Gamma}_\mu , \tag{V.65}$$

In order to find the relation between $\bar{\Gamma}_\mu$ and Γ_μ we now ask for our spinor covariant derivative to obey the Leibniz rule, and also for it to be compatible with the usual covariant derivative. Consider then the covariant derivative of $(\bar{\psi} \psi)$, we have:

$$\mathcal{D}_\mu (\bar{\psi} \psi) = (\mathcal{D}_\mu \bar{\psi}) \psi + \bar{\psi} (\mathcal{D}_\mu \psi) = (\partial_\mu \bar{\psi}) \psi + \bar{\psi} (\partial_\mu \psi) + \bar{\psi} (\bar{\Gamma}_\mu + \Gamma_\mu) \psi . \tag{V.66}$$

On the other hand, since we know that $\bar{\psi} \psi$ behaves as a scalar we must also have:

$$\mathcal{D}_\mu (\bar{\psi} \psi) = \partial_\mu (\bar{\psi} \psi) = (\partial_\mu \bar{\psi}) \psi + \bar{\psi} (\partial_\mu \psi) . \tag{V.67}$$

Comparing now both expressions we find immediately:

$$\bar{\Gamma}_\mu = -\Gamma_\mu , \tag{V.68}$$

so that the spinor covariant derivative of $\bar{\psi}$ takes the final form:

$$\mathcal{D}_\mu \bar{\psi} := \partial_\mu \bar{\psi} - \bar{\psi} \Gamma_\mu . \tag{V.69}$$

With the above definitions, the Dirac equation in a curved spacetime can be written as:

$$i \gamma^\mu \mathcal{D}_\mu \psi - m \psi = 0 , \tag{V.70}$$

with $\mathcal{D}_\mu \psi = \partial_\mu \psi + \Gamma_\mu \psi$. Similarly, the adjunct equation takes the form:

$$i (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} = 0 , \tag{V.71}$$

with $\mathcal{D}_\mu \bar{\psi} = \partial_\mu \bar{\psi} - \bar{\psi} \Gamma_\mu$.

5.3 The Fock–Ivanenko coefficients

We still need to find the explicit form of the Fock–Ivanenko coefficients Γ_μ . In order to do this, let us first assume that we have a matrix operator with spacetime indices Q^α , such that $(\bar{\psi} Q^\alpha \psi)$ transforms as a tensor with respect to a general change of coordinates, and where α can represent any combination of covariant and contravariant indices. For the spinor derivative of $(\bar{\psi} Q^\alpha \psi)$ we will have:

$$\begin{aligned} \mathcal{D}_\mu (\bar{\psi} Q^\alpha \psi) &= (\mathcal{D}_\mu \bar{\psi}) Q^\alpha \psi + \bar{\psi} (\mathcal{D}_\mu Q^\alpha) \psi + \bar{\psi} Q^\alpha (\mathcal{D}_\mu \psi) \\ &= (\partial_\mu \bar{\psi}) Q^\alpha \psi + \bar{\psi} Q^\alpha (\partial_\mu \psi) + \bar{\psi} (\mathcal{D}_\mu Q^\alpha - \Gamma_\mu Q^\alpha + Q^\alpha \Gamma_\mu) \psi . \end{aligned} \tag{V.72}$$

On the other hand, since $(\bar{\psi} Q^\alpha \psi)$ behaves as a tensor, and spinors should behave as scalars for the usual covariant derivative (they only have Lorentz indices), we have:

$$\mathcal{D}_\mu (\bar{\psi} Q^\alpha \psi) = \nabla_\mu (\bar{\psi} Q^\alpha \psi) = (\partial_\mu \bar{\psi}) Q^\alpha \psi + \bar{\psi} Q^\alpha (\partial_\mu \psi) + \bar{\psi} (\nabla_\mu Q^\alpha) \psi . \tag{V.73}$$

Equating both expressions we now find:

$$\bar{\psi} (\mathcal{D}_\mu Q^\alpha - \Gamma_\mu Q^\alpha + Q^\alpha \Gamma_\mu) \psi = \bar{\psi} (\nabla_\mu Q^\alpha) \psi , \tag{V.74}$$

and since this must hold for any ψ we finally obtain:

$$\mathcal{D}_\mu Q^\alpha = \nabla_\mu Q^\alpha + [\Gamma_\mu, Q^\alpha] . \tag{V.75}$$

Clearly, if we take $Q = I_4$ we find $\mathcal{D}_\mu I_4 = 0$, as expected. On the other hand, if we take our matrix operator as $Q^{\alpha\beta} = g^{\alpha\beta} I_4$, we immediately find $\mathcal{D}_\mu (g^{\alpha\beta} I_4) = 0$, which indicates that the spinor derivative must be compatible with the metric. Going back to the Clifford algebra, Eq. (V.60), it is easy to see that a sufficient condition for this to be satisfied is to ask for the spinor derivative of the γ^μ matrices to vanish, that is:

$$\mathcal{D}_\mu \gamma^\nu = 0 , \tag{V.76}$$

or more explicitly:

$$\nabla_\mu \gamma^\nu + [\Gamma_\mu, \gamma^\nu] = 0 . \tag{V.77}$$

A somewhat long algebraic calculation shows that the previous equation will be satisfied if we take:

$$\Gamma_\mu = -\frac{1}{4} \omega_{AB\mu} \gamma^A \gamma^B = -\frac{1}{2} \omega_{AB\mu} \sigma^{AB} , \tag{V.78}$$

where the σ^{AB} matrices are the same we had previously defined in (III.32), but now expressed in terms of the tetrad:

$$\sigma^{AB} := \frac{1}{4} [\gamma^A, \gamma^B] = \frac{1}{2} (\gamma^A \gamma^B + \eta^{AB} I_4) . \tag{V.79}$$

Equation (V.78) gives us the explicit form of the Fock–Ivanenko coefficients Γ_μ .

Let us now return to Eq. (V.76). Since by definition we have $\gamma^\mu = \gamma^A e^\mu_A$, and the γ^A are constant, the condition that the spinor derivative of γ^μ should vanish implies that we must ask for:

$$\mathcal{D}_\mu e^v_A = 0 . \tag{V.80}$$

The more natural way to accomplish this is to ask for the spinor derivative of tensor objects with mixed indices (spacetime and Lorentz) to reduce to the invariant derivative that we defined in the previous section.

5.4 Geometric derivation of the Fock–Ivanenko coefficients

In the previous Section we arrived at an explicit form for the Fock–Ivanenko coefficients through a series of algebraic requirements that might seem somewhat obscure. Here we will show an alternative geometric derivation that arrives at the same result (the discussion here is based on that of [24]).

We start by considering the fact that the geometrically meaningful derivative of a spinor ψ can not simply be given by the difference between its values at neighboring points, since the tetrad with respect to which ψ is defined will in general not be parallelly transported between these two points. The first step is then to consider how the tetrad changes. Let $\vec{e}_A(x + dx)$ be the value of the tetrad at point $(x + dx)$, and $\vec{e}_A^\parallel(x + dx)$ the value of the corresponding tetrad that has been parallelly transported from x to $x + dx$. Since both these tetrads are now evaluated at the same point, the difference between them must be an infinitesimal Lorentz transformation, that is:

$$\vec{e}_A(x + dx) - \vec{e}_A^\parallel(x + dx) = \lambda_A^B \vec{e}_B(x + dx) \simeq \lambda_A^B \vec{e}_B(x) , \tag{V.81}$$

with λ_{AB} the infinitesimal Lorentz transformation that we introduced in Section 3 above. But this difference is precisely the definition of the usual covariant derivative, so we must have:

$$e^\mu_{A;v} dx^v = \lambda_A^B e^\mu_B . \tag{V.82}$$

Contracting both sides of this equation with $e_{\mu C}$ we immediately find:

$$e_{\mu C} e^\mu_{A;v} dx^v = \lambda_A^B e^\mu_B e_{\mu C} \implies \lambda_{AC} = e_{\mu C} e^\mu_{A;v} dx^v , \tag{V.83}$$

or:

$$\lambda_{AC} = e_{\mu C} \omega^\mu_{Av} dx^v = \omega_{CAv} dx^v = -\omega_{ACv} dx^v . \tag{V.84}$$

From Eq. (III.35), the change of the spinor ψ under such a Lorentz transformation will then be:

$$\delta\psi = -\frac{1}{2} \lambda_{AB} \sigma^{AB} \psi , \tag{V.85}$$

where we already used the fact that the coefficients for the transformation are given by $-\lambda_{AB}$ (see Eq. (III.20)).

Now, just as before, the geometrically meaningful derivative of ψ must be given by the difference between the value of ψ at point $x + dx$ and the value of ψ^\parallel that has

been parallelly transported from x to $x + dx$:

$$\begin{aligned} \mathcal{D}\psi &= \psi(x + dx) - \psi^\parallel(x + dx) = \psi(x + dx) - (\psi(x) + \delta\psi) \\ &= \partial_\nu \psi dx^\nu - \delta\psi = \partial_\nu \psi dx^\nu + \frac{1}{2} \lambda_{AB} \sigma^{AB} \psi \\ &= \left(\partial_\nu \psi dx^\nu - \frac{1}{2} \omega_{AB\nu} \sigma^{AB} \psi \right) dx^\nu \equiv \mathcal{D}_\nu \psi dx^\nu . \end{aligned} \tag{V.86}$$

Comparing this to our definition for the spinor covariant derivative (V.64) we find:

$$\Gamma_\mu = -\frac{1}{2} \omega_{AB\mu} \sigma^{AB} , \tag{V.87}$$

which is the same as (V.78).

5.5 Spinor Ricci identity

In the same way as the usual covariant derivatives, the commutator of the spinor derivative can also be written in terms of the Riemann tensor. Using the expression for the Fock–Ivanenko coefficients that we found above, as well as the expression for the Riemann tensor in terms of the Ricci rotation coefficients (V.55) together with Eq. (II.28), it is not difficult to show that:

$$[\mathcal{D}_\mu, \mathcal{D}_\nu] \psi = -\frac{1}{2} R_{AB\mu\nu} \sigma^{AB} \psi . \tag{V.88}$$

This expression is a generalization of the Ricci identity for the case of spinor covariant derivatives. The previous result can also be written in terms of the Γ_μ matrices as:

$$\partial_\mu \Gamma_\nu - \partial_\nu \Gamma_\mu + \Gamma_\mu \Gamma_\nu - \Gamma_\nu \Gamma_\mu = -\frac{1}{2} R_{AB\mu\nu} \sigma^{AB} , \tag{V.89}$$

which bears an obvious resemblance to the form that the Riemann tensor takes in terms of the usual Christoffel symbols. In order to prove the previous expressions one must remember that $\mathcal{D}_\mu \psi$ is both a spinor and a 1-form, so that:

$$\begin{aligned} \mathcal{D}_\mu \mathcal{D}_\nu \psi &= \partial_\mu (\mathcal{D}_\nu \psi) + \Gamma_\mu (\mathcal{D}_\nu \psi) - \Gamma_{\mu\nu}^\alpha (\mathcal{D}_\alpha \psi) \\ &= \partial_\mu (\partial_\nu \psi + \Gamma_\nu \psi) + \Gamma_\mu (\partial_\nu \psi + \Gamma_\nu \psi) - \Gamma_{\mu\nu}^\alpha (\partial_\alpha \psi + \Gamma_\alpha \psi) \\ &= \left(\partial_\mu \partial_\nu \psi - \Gamma_{\mu\nu}^\alpha \partial_\alpha \psi \right) + \Gamma_\mu \partial_\nu \psi + \Gamma_\nu \partial_\mu \psi + \left(\partial_\mu \Gamma_\nu - \Gamma_{\mu\nu}^\alpha \Gamma_\alpha + \Gamma_\mu \Gamma_\nu \right) \psi , \end{aligned} \tag{V.90}$$

and finally:

$$\mathcal{D}_\mu \mathcal{D}_\nu \psi = \nabla_\mu \nabla_\nu \psi + \Gamma_\mu \partial_\nu \psi + \Gamma_\nu \partial_\mu \psi + (\nabla_\mu \Gamma_\nu + \Gamma_\mu \Gamma_\nu) \psi , \tag{V.91}$$

where ∇_μ is the usual covariant derivative that acts on ψ as a scalar, and on Γ_μ as a 1-form. Notice that, even if the first three terms are clearly symmetric on (μ, ν) , the last term breaks this symmetry since there is no reason to assume that $\nabla_\mu \Gamma_\nu$ is symmetric, and $\Gamma_\mu \Gamma_\nu$ also isn't symmetric since the Γ_μ in general do not commute.

5.6 Invariance of the spinor affine connection

In the previous Sections we found the final form of the Fock–Ivanenko coefficients Γ_μ in terms of the spin connection given by Eq. (V.78). We still need to show that the spinor covariant derivative that we defined above does indeed transform as a spinor. That is, we want to show that:

$$\mathcal{D}'_\mu \psi' := S \mathcal{D}_\mu \psi, \tag{V.92}$$

where:

$$\mathcal{D}'_\mu \psi' = \partial_\mu \psi' + \Gamma'_\mu \psi'. \tag{V.93}$$

It is not difficult to show that Eq. (V.92) will be satisfied if we ask for the Γ_μ matrices to transform according to the rule:

$$\Gamma'_\mu = S \Gamma_\mu S^{-1} - (\partial_\mu S) S^{-1}. \tag{V.94}$$

We will now show that the Γ_μ given by (V.78) do in fact satisfy this transformation rule. In order to do this, we consider an infinitesimal Lorentz transformation of the form (see Eq. (III.19)):

$$\Lambda^A_B \simeq \delta^A_B + \lambda^A_B, \quad \lambda^A_B := \frac{1}{2} C_{CD} (M^{CD})^A_B, \tag{V.95}$$

with $C_{CD} = C_{CD}(x)$ now functions of position, and $|C_{CD}| \ll 1$. The associated spinor transformation will be:

$$S^A_B \simeq \delta^A_B + s^A_B, \quad s^A_B := \frac{1}{2} C_{CD} (\sigma^{CD})^A_B. \tag{V.96}$$

The inverse transformations will then have the form, to first order in small quantities:

$$(\Lambda^{-1})^A_B \simeq \delta^A_B - \lambda^A_B, \quad (S^{-1})^A_B \simeq \delta^A_B - s^A_B. \tag{V.97}$$

Substituting now the expressions for S and S^{-1} in the transformation rule (V.94), and keeping to first order in C_{AB} we find, after some algebra:

$$\Gamma'_\mu = \Gamma_\mu + [s, \Gamma_\mu] - \partial_\mu s, \tag{V.98}$$

and explicitly substituting the form of s :

$$\Gamma'_\mu = \Gamma_\mu + \frac{1}{2} C_{CD} [\sigma^{CD}, \Gamma_\mu] - \frac{1}{2} (\partial_\mu C_{CD}) \sigma^{CD}. \tag{V.99}$$

Let us now write the Γ_μ in the form (V.78), but instead of using the $\omega_{AB\mu}$ as coefficients of the σ matrices we consider some arbitrary coefficients $B_{AB\mu}$:

$$\Gamma_\mu = -\frac{1}{2} B_{AB\mu} \sigma^{AB}, \tag{V.100}$$

where at the moment we are not assuming anything about the coefficients $B_{AB\mu}$, except for the fact that they must be antisymmetric in their Lorentz indices. Substituting into the transformation law we find:

$$\begin{aligned} B'_{AB\mu} \sigma^{AB} &= B_{AB\mu} \sigma^{AB} + \frac{1}{2} C_{AB} \left[\sigma^{AB}, B_{CD\mu} \sigma^{CD} \right] + \frac{1}{2} (\partial_\mu C_{AB}) \sigma^{AB} \\ &= (B_{AB\mu} + \partial_\mu C_{AB}) \sigma^{AB} + \frac{1}{2} C_{AB} B_{CD\mu} \left[\sigma^{AB}, \sigma^{CD} \right], \end{aligned} \tag{V.101}$$

where we used the fact that the σ^{CD} are constant and as such are invariant, and also that the C 's and B 's are real numbers. Using now the commutation relations of the σ matrices given in (III.34) we find, after some algebra:

$$B'_{AB\mu} \sigma^{AB} = \left[B_{AB\mu} + \partial_\mu C_{AB} - (B_{AC\mu} C_B^C + B_{CB\mu} C_A^C) \right] \sigma^{AB}, \tag{V.102}$$

which implies that the B coefficients must transform as:

$$B'_{AB\mu} = B_{AB\mu} + \partial_\mu C_{AB} - (B_{AC\mu} C_B^C + B_{CB\mu} C_A^C). \tag{V.103}$$

On the other hand, the transformation rule for the spin connection is:

$$\begin{aligned} \omega'_{AB\mu} &= e'^\lambda_A \omega'_{\lambda B\mu} = e'^\lambda_A e'_{\lambda B;\mu} = \Lambda_A^C e^\lambda_C \left(\Lambda_B^D e_{\lambda D} \right)_{;\mu} \\ &= \Lambda_A^C \Lambda_B^D e^\lambda_C e_{\lambda D;\mu} + \Lambda_A^C e^\lambda_C e_{\lambda D} \partial_\mu \Lambda_B^D. \end{aligned} \tag{V.104}$$

Using now the fact that $e^\lambda_C e_{\lambda D} = \eta_{CD}$ we find:

$$\omega'_{AB\mu} = \Lambda_A^C \Lambda_B^D \omega_{CD\mu} + \Lambda_A^C \partial_\mu \Lambda_{BC}. \tag{V.105}$$

Assuming an infinitesimal transformation, the last expression reduces to first order to:

$$\omega'_{AB\mu} = \omega_{AB\mu} + \omega_{AC\mu} \lambda_B^C + \omega_{CB\mu} \lambda_A^C - \partial_\mu \lambda_{AB}, \tag{V.106}$$

where we used the fact that λ_{AB} is antisymmetric. Remembering now that for an infinitesimal transformation we have $\lambda_{AB} = -C_{AB}$ (see Eq. (III.20)), we finally find:

$$\omega'_{AB\mu} = \omega_{AB\mu} + \partial_\mu C_{AB} - (\omega_{AC\mu} C_B^C + \omega_{CB\mu} C_A^C). \tag{V.107}$$

But this is precisely the transformation rule for the B coefficients we found above, so we conclude that the Fock–Ivanenko coefficients given by (V.78) do transform in the correct way.

6 The Schroedinger–Dirac equation

For the case of Minkowski spacetime we have already shown that Dirac’s equation takes us directly to the Klein–Gordon equation for each of the spinor components. In the case of a curved spacetime this is no longer true, and what we find is a generalization of the Klein–Gordon equation known as the Schroedinger–Dirac equation [5] (see also [25] and references therein).

The first step in finding this equation is to calculate the quantity $\mathcal{D}^2\psi := \gamma^\mu \mathcal{D}_\mu (\gamma^\nu \mathcal{D}_\nu)\psi = \gamma^\mu \gamma^\nu \mathcal{D}_\mu \mathcal{D}_\nu \psi$. We have:

$$\begin{aligned} \mathcal{D}^2\psi &= \gamma^\mu \gamma^\nu \mathcal{D}_\mu \mathcal{D}_\nu \psi = \frac{1}{2} (\{\gamma^\mu, \gamma^\nu\} + [\gamma^\mu, \gamma^\nu]) \mathcal{D}_\mu \mathcal{D}_\nu \psi \\ &= -g^{\mu\nu} \mathcal{D}_\mu \mathcal{D}_\nu \psi + \frac{1}{2} [\gamma^\mu, \gamma^\nu] \mathcal{D}_\mu \mathcal{D}_\nu \psi \\ &= -\mathcal{D}^\mu \mathcal{D}_\mu \psi + \frac{1}{4} [\gamma^\mu, \gamma^\nu] [\mathcal{D}_\mu, \mathcal{D}_\nu] \psi \\ &= -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{2} \sigma^{\mu\nu} R_{CD\mu\nu} \sigma^{CD} \psi \\ &= -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{2} R_{ABCD} \sigma^{AB} \sigma^{CD} \psi, \end{aligned} \tag{VI.1}$$

where we used the definition of the $\sigma^{\mu\nu}$ matrices and the expression for the commutator of the spinor derivatives given by (V.88), and the fact that the Riemann tensor is symmetric when we exchange the first and second pairs of indices. Using now the antisymmetry of the Riemann tensor in the two pairs of indices, and substituting the definition of the σ matrices, we can rewrite the previous result as:

$$\mathcal{D}^2\psi = -\mathcal{D}^\mu \mathcal{D}_\mu \psi - \frac{1}{8} R_{ABCD} \gamma^A \gamma^B \gamma^C \gamma^D \psi. \tag{VI.2}$$

The second term of the above result in fact turns out to be proportional to the scalar curvature. In fact we have:

$$R = -\frac{1}{2} R_{ABCD} \gamma^A \gamma^B \gamma^C \gamma^D. \tag{VI.3}$$

We will leave the proof of this result to the end of this section. The quantity $\mathcal{D}^2\psi$ then takes the final form:

$$\mathcal{D}^2\psi = \left(-\mathcal{D}^\mu \mathcal{D}_\mu + \frac{R}{4} \right) \psi. \tag{VI.4}$$

This last expression is the natural form of the Laplace operator when applied to spinors in a curved spacetime. Having a contribution from the curvature scalar is not surprising,

and comes from the fact that the covariant derivatives of spinors in general do not commute (something similar happens in the case of vectors and 1-forms, and the natural Laplace operator in that case is the so-called “de Rham Laplacian” which also has contributions from the curvature tensor).

Having found this result, we can now go back to the Dirac equation. Applying the operator $i\gamma^\mu \mathcal{D}_\mu$ from the left we find:

$$\begin{aligned} i\gamma^\mu \mathcal{D}_\mu (i\gamma^\mu \mathcal{D}_\mu \psi - m\psi) &= 0 \\ \implies -\mathcal{D}^2 \psi - im\gamma^\mu \mathcal{D}_\mu \psi &= 0 \\ \implies \mathcal{D}^2 \psi + m^2 \psi &= 0, \end{aligned} \tag{VI.5}$$

where in the second term of the third row we used again Dirac’s equation. Substituting now (VI.4) we finally find:

$$\left(\mathcal{D}^\mu \mathcal{D}_\mu - \frac{R}{4} - m^2 \right) \psi = 0. \tag{VI.6}$$

This is the Schroedinger–Dirac equation, and is the generalization of the Klein–Gordon equation for spinors in a curved spacetime. In the previous equation one should remember that the operator $\mathcal{D}^\mu \mathcal{D}_\mu$ must be calculated as (see Eq. (V.91)):

$$\mathcal{D}^\mu \mathcal{D}_\mu \psi = \square \psi + 2\Gamma^\mu \partial_\mu \psi + (\nabla_\mu \Gamma^\mu + \Gamma_\mu \Gamma^\mu) \psi, \tag{VI.7}$$

where \square is the usual d’Alambertian applied to scalars, and $\nabla_\mu \Gamma^\mu = \partial_\mu (|g|^{1/2} \Gamma^\mu) / |g|^{1/2}$. Notice that, since the operator $\mathcal{D}^\mu \mathcal{D}_\mu$ involves the Γ_μ matrices, in the Schroedinger–Dirac equation the different components of ψ are in fact coupled, something that does not happen with the Klein–Gordon equation.

We will now prove Eq. (VI.3) that we used in order to derive the Schroedinger–Dirac equation. The first step is to consider the contraction of the Ricci tensor $R_{\mu\nu}$ with two γ matrices. using the symmetry of the Ricci tensor we find:

$$R_{\mu\nu} \gamma^\mu \gamma^\nu = R_{AB} \gamma^A \gamma^B = \frac{1}{2} R_{AB} (\gamma^A \gamma^B + \gamma^B \gamma^A) = -R_{AB} \eta^{AB} = -R. \tag{VI.8}$$

Next we must express $R_{AB} \gamma^A \gamma^B$ in terms of the Riemann tensor:

$$\begin{aligned} R_{AB} \gamma^A \gamma^B &= R_{ACBD} \eta^{CD} \gamma^A \gamma^B = R_{ACBD} \gamma^A \eta^{CD} \gamma^B \\ &= -\frac{1}{2} R_{ACBD} \gamma^A (\gamma^C \gamma^D + \gamma^D \gamma^C) \gamma^B \\ &= \frac{1}{2} (R_{ACDB} \gamma^A \gamma^C \gamma^D \gamma^B - R_{ACBD} \gamma^A \gamma^D \gamma^C \gamma^B) \\ &= \frac{1}{2} (R_{ABCD} \gamma^A \gamma^B \gamma^C \gamma^D - R_{ACBD} \gamma^A \gamma^D \gamma^C \gamma^B), \end{aligned} \tag{VI.9}$$

where we used the antisymmetry of the Riemann tensor in the second pair of indices, and in the last step we renamed indices on the first term. In the first term above we

can recognize already the contraction $R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D$ that we need in order to prove (VI.3). For the second term we use the cyclic symmetry of Riemann so that:

$$\begin{aligned} R_{AB}\gamma^A\gamma^B &= \frac{1}{2} \left[R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D + (R_{ADCB} + R_{ABDC})\gamma^A\gamma^D\gamma^C\gamma^B \right] \\ &= R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D + \frac{1}{2} R_{ABDC}\gamma^A\gamma^D\gamma^C\gamma^B, \end{aligned} \tag{VI.10}$$

where we again renamed indices to show that two of the terms are identical. To calculate the second term of the last expression we use the fact that the Clifford algebra implies:

$$\gamma^D\gamma^C\gamma^B = \gamma^B\gamma^D\gamma^C + 2\eta^{BD}\gamma^C - 2\eta^{BC}\gamma^D. \tag{VI.11}$$

Using this result we can show that:

$$\begin{aligned} R_{ABDC}\gamma^A\gamma^D\gamma^C\gamma^B &= R_{ABDC}\gamma^A \left(\gamma^B\gamma^D\gamma^C + 2\eta^{BD}\gamma^C - 2\eta^{BC}\gamma^D \right) \\ &= R_{ABDC}\gamma^A\gamma^B\gamma^D\gamma^C + 2R_{ABDC}\eta^{BD}\gamma^A\gamma^C - 2R_{ABDC}\eta^{BC}\gamma^A\gamma^D \\ &= R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - 4R_{AB}\gamma^A\gamma^B, \end{aligned} \tag{VI.12}$$

where once more we renamed indices and grouped terms. Collecting our results we find:

$$R_{AB}\gamma^A\gamma^B = \frac{3}{2} R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D - 2R_{AB}\gamma^A\gamma^B, \tag{VI.13}$$

and solving for $R_{AB}\gamma^A\gamma^B$:

$$R_{AB}\gamma^A\gamma^B = \frac{1}{2} R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D. \tag{VI.14}$$

Finally, using Eq. (VI.8) we obtain the desired result:

$$R = -\frac{1}{2} R_{ABCD}\gamma^A\gamma^B\gamma^C\gamma^D. \tag{VI.15}$$

7 Lagrangian and stress–energy tensor of the Dirac field

7.1 Lagrangian, first version

The Lagrangian associated with the Dirac field should be a scalar function that depends on the spinor ψ and its derivatives. Furthermore, since the Dirac equation is of first order, the Lagrangian should also be of first order. One possible expression that satisfies all the previous conditions is:

$$L = \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi, \tag{VII.1}$$

with $\bar{\psi} = \psi^\dagger \gamma^T$, and where here γ^T corresponds to the constant matrix γ^0 used in Minkowski spacetime. The change in notation from γ^0 to γ^T is introduced here in order to avoid a possible confusion between γ^T and coordinate component $\gamma^t = \gamma^A e_A^t$ when considering the case of a curved spacetime. From here on the constant γ matrices will be associated with the tetrad basis and not the coordinate basis (we will discuss this further in Sect. 8.4.)

The Lagrangian (VII.1) is the one found in many text books on quantum field theory in Minkowski spacetime (with $\partial_\mu \psi$ instead of $\mathcal{D}_\mu \psi$). The Lagrangian density then takes the form:

$$\mathcal{L} = \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi |g|^{1/2}, \tag{VII.2}$$

with g the determinant of the metric, and the action integral becomes:

$$S = \int \mathcal{L} d^4x = \int \bar{\psi} (i\gamma^\mu \mathcal{D}_\mu - m) \psi |g|^{1/2} d^4x. \tag{VII.3}$$

For the variation of the action above one must take ψ and $\bar{\psi}$ as independent fields. The Euler–Lagrange equations are then:

$$\frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) - \frac{\partial \mathcal{L}}{\partial \psi} = 0, \tag{VII.4}$$

with an analogous equation for $\bar{\psi}$. In fact, in this case it turns out to be far easier to work with $\bar{\psi}$ since the Lagrangian (VII.1) does not depend on its derivatives. We find:

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}} = (i\gamma^\mu \mathcal{D}_\mu - m) \psi |g|^{1/2}, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi})} = 0, \tag{VII.5}$$

so the Euler–Lagrange equation for $\bar{\psi}$ gives us immediately Dirac’s equation:

$$i\gamma^\mu \mathcal{D}_\mu \psi - m\psi = 0. \tag{VII.6}$$

The Dirac equation for ψ is a bit more subtle. In this case it is necessary to write the spinor covariant derivative explicitly in the Lagrangian density:

$$\mathcal{L} = [i\bar{\psi} \gamma^\mu (\partial_\mu \psi + \Gamma_\mu \psi) - m\psi] |g|^{1/2}. \tag{VII.7}$$

From here we then find:

$$\frac{\partial \mathcal{L}}{\partial \psi} = \bar{\psi} (i\gamma^\mu \Gamma_\mu - m) |g|^{1/2}, \quad \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} = i\bar{\psi} \gamma^\mu |g|^{1/2}, \tag{VII.8}$$

which implies:

$$\begin{aligned} \frac{\partial}{\partial x^\mu} \left(\frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi)} \right) &= i \partial_\mu \left(\bar{\psi} \gamma^\mu |g|^{1/2} \right) \\ &= i \left[(\partial_\mu \bar{\psi}) \gamma^\mu + \bar{\psi} \partial_\mu \gamma^\mu + \frac{1}{2|g|} \bar{\psi} \gamma^\mu \partial_\mu |g| \right] |g|^{1/2}. \end{aligned} \tag{VII.9}$$

The Euler–Lagrange equation then takes the form:

$$i \left[(\partial_\mu \bar{\psi}) \gamma^\mu + \bar{\psi} \partial_\mu \gamma^\mu + \frac{1}{2|g|} \bar{\psi} \gamma^\mu \partial_\mu |g| \right] - \bar{\psi} (i \gamma^\mu \Gamma_\mu - m) = 0, \tag{VII.10}$$

and regrouping terms:

$$i (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} + i \bar{\psi} \left[\partial_\mu \gamma^\mu + \frac{1}{2} \gamma^\mu \partial_\mu \ln |g| + [\Gamma_\mu, \gamma^\mu] \right] = 0. \tag{VII.11}$$

Using now the fact that $\partial_\mu \ln |g| = 2\Gamma_{\nu\mu}^\nu$, we can recognize that the term in square brackets is just the spinor divergence of the γ^μ matrices, but this divergence vanishes since the spinor derivative of the γ^μ is zero. We then finally obtain:

$$i (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu + m \bar{\psi} = 0, \tag{VII.12}$$

which is precisely the adjunct Dirac equation.

7.2 Lagrangian, second version

Even though the Lagrangian (VII.1) results in the correct equations of motion, it has the serious disadvantage of not being symmetric in ψ and $\bar{\psi}$. This can be easily fixed if we define an alternative form of the Lagrangian as:

$$L = \frac{i}{2} [\bar{\psi} \gamma^\mu (\mathcal{D}_\mu \psi) - (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi. \tag{VII.13}$$

This is the form of the Lagrangian that we will use from now on (this form for the Lagrangian is also well known, see for example [6, 7, 26]). The Lagrangian density is now:

$$\mathcal{L} = \left\{ \frac{i}{2} [\bar{\psi} \gamma^\mu (\mathcal{D}_\mu \psi) - (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi \right\} |g|^{1/2}. \tag{VII.14}$$

A similar procedure to the one presented above shows that this Lagrangian density results on precisely the same Dirac equations (one should mention the fact that the i factor is frequently absorbed in the definition of the γ matrices, so that it does not appear in the Lagrangian, or indeed in the Dirac equation). It is interesting to note that both expressions for the Lagrangian in fact become zero when we substitute the Dirac equation, that is when we evaluate them “on shell”. This represents no problem, since

what one is interested in is the functional form of the Lagrangian in terms of ψ and its derivatives, and not its specific numerical value on shell.

The Lagrangian (VII.14) is clearly invariant under a transformation of the form:

$$\psi \rightarrow e^{-i\theta} \psi, \quad \bar{\psi} \rightarrow e^{+i\theta} \bar{\psi}, \tag{VII.15}$$

with θ an arbitrary constant. This implies the existence of a conserved Noether current given by:

$$\begin{aligned} j^\mu &= \left(\frac{\partial L}{\partial(\partial_\mu \psi)} \right) (-i\psi) + (i\bar{\psi}) \left(\frac{\partial L}{\partial(\partial_\mu \bar{\psi})} \right) \\ &= \frac{i}{2} \bar{\psi} \gamma^\mu (-i\psi) - \frac{i}{2} (i\bar{\psi}) \gamma^\mu \psi \\ &= \bar{\psi} \gamma^\mu \psi, \end{aligned} \tag{VII.16}$$

such that $\nabla_\mu j^\mu = 0$. We can now see that this is precisely the same conserved current that we had initially found in Eq. (II.33).

Even if the expression for the Lagrangian (VII.14) is correct, it is interesting to rewrite it in a more illustrative form. We can write the Lagrangian as:

$$L = K - V, \tag{VII.17}$$

where here K is the so-called *kinetic* term given by:

$$\begin{aligned} K &= \frac{i}{2} [\bar{\psi} \gamma^\mu (\mathcal{D}_\mu \psi) - (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu \psi] \\ &= \frac{i}{2} [\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi + \bar{\psi} (\gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu) \psi], \end{aligned} \tag{VII.18}$$

while V is the *potential* term that is simply:

$$V = m \bar{\psi} \psi. \tag{VII.19}$$

The kinetic term can in turn be written in several different forms. If we use the expression for the Γ_μ in terms of the Ricci rotation coefficients given in (V.78) we find:

$$\Gamma_\mu = -\frac{1}{2} \omega_{AB\mu} \sigma^{AB} = -\frac{1}{2} e_\mu^C \omega_{ABC} \sigma^{AB}. \tag{VII.20}$$

Using this relation, it is not difficult to show that:

$$\gamma^\mu \Gamma_\mu + \Gamma_\mu \gamma^\mu = -\frac{1}{2} \omega_{ABC} \gamma^{CAB}, \tag{VII.21}$$

where we have defined $\gamma^{CAB} := \{\gamma^C, \sigma^{AB}\}$. For what follows it is important to notice that γ^{CAB} is totally antisymmetric:

$$\gamma^{CAB} = -\gamma^{CBA} = -\gamma^{ACB} = -\gamma^{BAC} . \tag{VII.22}$$

The kinetic term then reduces to:

$$K = \frac{i}{2} [\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] - \frac{i}{4} \bar{\psi} (\omega_{ABC} \gamma^{CAB}) \psi . \tag{VII.23}$$

Finally, if we substitute the ω 's using Eq. (V.53), and use the anticommutation property of the γ^A matrices, a little algebra allows us to show that the kinetic term takes the final form:

$$K = \frac{i}{2} [\bar{\psi} \gamma^\mu (\partial_\mu \psi) - (\partial_\mu \bar{\psi}) \gamma^\mu \psi] + \frac{i}{4} \bar{\psi} (f_{ABC} \gamma^{CAB}) \psi , \tag{VII.24}$$

where we must remember that the f 's are defined as $f_{ABC} := (\partial_A e_{\nu B}) e_C^\nu = e_A^\mu e_C^\nu \partial_\mu e_{\nu B}$.

7.3 Stress–energy tensor

In order to find the form of the stress–energy tensor associated with the Dirac field we start from the action integral:

$$S = \int L |g|^{1/2} d^4x , \tag{VII.25}$$

with L the Lagrangian that we found above:

$$L = \frac{i}{2} [\bar{\psi} \gamma^\mu (\mathcal{D}_\mu \psi) - (\mathcal{D}_\mu \bar{\psi}) \gamma^\mu \psi] - m \bar{\psi} \psi . \tag{VII.26}$$

At this point one could think of using the standard definition for the Hilbert stress–energy tensor in terms of the Lagrangian L given by:

$$T_{\mu\nu} = -2 \frac{\partial L}{\partial g^{\mu\nu}} + g_{\mu\nu} L . \tag{VII.27}$$

However, in the case of a Dirac field this definition fails since the Lagrangian has terms that depend directly on the tetrad and not just on the metric. Instead, we must now define the stress–energy tensor by considering the variation of the action with respect to the tetrad itself:

$$T_{\mu\nu} = -\frac{1}{2} \left(e_{\mu D} \frac{\delta L}{\delta e_D^\nu} + e_{\nu D} \frac{\delta L}{\delta e_D^\mu} \right) + g_{\mu\nu} L , \tag{VII.28}$$

where we must remember that the metric is given in terms of the tetrad as $g_{\mu\nu} = e_{\mu A}e_{\nu}^A$, and where we have included two symmetrized terms since the final stress–energy tensor must be symmetric. It is not difficult to see that for an action that depends only on the metric both definitions for $T_{\mu\nu}$ are in fact equivalent.

Using the expression above one arrives, after a somewhat lengthy algebra, at:

$$T_{\mu\nu} = \frac{i}{2} [(\mathcal{D}_{(\mu}\bar{\psi})\gamma_{\nu)}\psi - \bar{\psi}\gamma_{(\mu}(\mathcal{D}_{\nu)}\psi)] . \tag{VII.29}$$

This is the stress–energy tensor for the Dirac equation (this expression is also well known, see e.g. [7, 20, 27]). Details of the derivation of this stress–energy tensor, as well as a proof that it satisfies the conservation equations $\nabla^\mu T_{\mu\nu} = 0$, can be found in Appendix A.

There is an property of the stress–energy tensor (VII.29) that is interesting to mention. If we take its trace, and use the Dirac equation, one can easily show that:

$$T^\mu{}_\mu = -m\bar{\psi}\psi = -m(\psi^\dagger\gamma^T\psi) = -m(|\psi_1|^2 + |\psi_2|^2 - |\psi_3|^2 - |\psi_4|^2) . \tag{VII.30}$$

We then see that the trace in fact vanishes for $m = 0$.

8 Dirac equation in the 3+1 formalism

In the previous sections we found the general form of the Dirac equation for a curved spacetime. For numerical applications, or in case one is interested in the Hamiltonian formalism, it is interesting to find the form that the Dirac equation takes in the 3+1 formalism of general relativity.

We assume that the spacetime is globally hyperbolic, so it can be foliated by Cauchy hypersurfaces Σ_t parametrized by a global time function t . The metric can then be written in the general form (see for example [28]):

$$ds^2 = (-\alpha^2 + \beta_i\beta^i) dt^2 + 2\beta_idtdx^i + \gamma_{ij}dx^i dx^j , \tag{VIII.1}$$

where x^i are spatial coordinates, α is the lapse function, β^i is the shift vector, γ_{ij} is the spatial metric induced on the spacelike hypersurfaces of constant t , and where $\beta_i = \gamma_{ij}\beta^j$ (in general the indices of purely spatial tensors are raised and lowered with γ_{ij}).

The next step is to choose a tetrad adapted to our spacetime foliation. In particular we will take:

$$e_T^\mu = n^\mu , \quad e_I^\mu = E_I^\mu , \tag{VIII.2}$$

with n^μ the unit vector orthogonal to the spatial hypersurfaces, and where E_I^μ with $I \in \{1, 2, 3\}$ are three purely spatial vectors orthogonal to each other that from now on we will call the “spatial triad”.

Do notice that we now have four different types of indices that require special notation: 1) Spacetime coordinate indices that take values from 0 to 3, for which we

continue to use Greek letters $\{\alpha, \beta, \dots\}$; 2) tetrad (Lorentz) indices that also take values from 0 to 3, for which we continue to use upper case Latin letters starting from $\{A, B, \dots\}$; 3) purely spatial coordinate indices that only take values from 1 to 3, for which we use lower case Latin letters starting from $\{i, j, \dots\}$; 4) purely spatial triad indices that also only take values from 1 to 3, and for which we will use upper case Latin indices starting from $\{I, J, \dots\}$.

The timelike vector $e_T^\mu = n^\mu$ can now be expressed in terms of the lapse and shift as:

$$e_T^\mu = \left(1/\alpha, -\beta^i/\alpha\right), \quad e_{\mu T} = (-\alpha, 0), \tag{VIII.3}$$

so that we clearly have $e_T^\mu e_{\mu T} = -1$. On the other hand, since the vectors E_I^μ are purely spatial we must have $e_T^\mu E_{\mu I} = e_{\mu T} E_I^\mu = 0$, so that:

$$E_I^0 = 0, \quad E_{0I} = \beta^m E_{mI}. \tag{VIII.4}$$

Notice in particular that the purely spatial indices of E_I^m are raised and lowered with the spatial metric γ_{mn} , that is: $E_{mI} = \gamma_{mn} E_I^n$, $E_I^m = \gamma^{mn} E_{nI}$. We also have:

$$E_{mI} E_I^m = \delta_{IJ}, \quad E_m^I E_{nI} = \gamma_{mn}. \tag{VIII.5}$$

At this point it is convenient to introduce the projection operator onto the spatial hypersurfaces defined as:

$$P_\nu^\mu := \delta_\nu^\mu + n^\mu n_\nu. \tag{VIII.6}$$

Notice that, so defined, this operator corresponds directly with the induced metric on the spatial hypersurfaces, $\gamma_{\mu\nu} \equiv P_{\mu\nu}$. In particular the spatial metric is $\gamma_{ij} = P_{ij}$, as can be seen directly from the above definition.

8.1 Ricci rotation coefficients in 3+1 form

In order to find the components of the Ricci rotation coefficients in the 3+1 formalism we start from Eq. (V.52) which we rewrite here:

$$\omega_{AB\mu} = e_B^\nu \left(e_{\lambda A} \Gamma_{\nu\mu}^\lambda - \partial_\mu e_{\nu A} \right). \tag{VIII.7}$$

The expressions for the Christoffel symbols $\Gamma_{\mu\nu}^\lambda$ in terms of the 3+1 quantities are well known and can be found, for example, in Appendix B of reference [28]. We write them again here for completeness:

$$\Gamma_{00}^0 = (\partial_t \alpha + \beta^m \partial_m \alpha - \beta^m \beta^n K_{mn}) / \alpha, \tag{VIII.8}$$

$$\Gamma_{0i}^0 = (\partial_i \alpha - \beta^m K_{im}) / \alpha, \tag{VIII.9}$$

$$\Gamma_{ij}^0 = -K_{ij} / \alpha, \tag{VIII.10}$$

$$\Gamma_{00}^l = \alpha \partial^l \alpha - 2\alpha \beta^m K_m^l - \beta^l (\partial_t \alpha + \beta^m \partial_m \alpha - \beta^m \beta^n K_{mn}) / \alpha + \partial_t \beta^l + \beta^m D_m \beta^l, \tag{VIII.11}$$

$$\Gamma_{m0}^l = -\beta^l (\partial_m \alpha - \beta^n K_{mn}) / \alpha - \alpha K_m^l + D_m \beta^l, \tag{VIII.12}$$

$$\Gamma_{ij}^l = {}^{(3)}\Gamma_{ij}^l + \beta^l K_{ij} / \alpha, \tag{VIII.13}$$

with D_i the covariant derivative associated with the spatial metric γ_{ij} , ${}^{(3)}\Gamma_{ij}^l$ the corresponding three-dimensional Christoffel symbols, and where K_{ij} refers to the extrinsic curvature tensor of the spatial hypersurfaces of constant time (also known as the second fundamental form), defined as the spatial projection of the covariant derivative of the timelike unit normal vector to the spatial hypersurfaces, that is $K_{\mu\nu} := -P_{\mu}^{\alpha} \nabla_{\alpha} n_{\nu}$. The purely spatial components of the extrinsic curvature can be shown to be related to the time derivative of the spatial metric γ_{ij} as:

$$\partial_t \gamma_{ij} = -2\alpha K_{ij} + D_i \beta_j + D_j \beta_i. \tag{VIII.14}$$

Equations (VIII.8)–(VIII.13) are used for deriving the results that follow.

Below I show the final results for the different components of the Ricci rotation coefficients $\omega_{AB\mu}$. The calculations are straightforward and will not be presented here in detail. Also, remember that the ω 's are antisymmetric in the first two indices, so that in a four-dimensional spacetime they only have 24 independent components. In what follows the index T refers specifically to the Lorentz time component, while the index I refers to Lorentz purely spatial components.

1. Coefficients $\omega_{T10} = -\omega_{IT0}$ (3 coefficients):

$$\omega_{T10} = -\omega_{IT0} = -E_I^m (\partial_m \alpha - \beta^n K_{mn}). \tag{VIII.15}$$

2. Coefficients $\omega_{TIm} = -\omega_{ITm}$ (9 coefficients):

$$\omega_{TIm} = -\omega_{ITm} = E_I^n K_{nm}. \tag{VIII.16}$$

3. Coefficients $\omega_{IJ0} = -\omega_{JI0}$ (3 coefficients):

$$\omega_{IJ0} = -\omega_{JI0} = -E_J^n [\partial_t E_{mI} - E_I^n (-\alpha K_{mn} + D_m \beta_n)]. \tag{VIII.17}$$

Here it is important to mention that, even if it is not immediately evident, when using Eq. (VIII.14) for the time derivative of the spatial metric, plus the fact that $\gamma_{mn} = E_{mI} E_n^I$, a short calculation allows one to show that the previous result is in fact antisymmetric in (I, J) .

4. Coefficients $\omega_{IJm} = -\omega_{JIm}$ (9 coefficients):

$$\omega_{IJm} = -E_J^n D_m E_{nI} . \tag{VIII.18}$$

Again, using the fact that the three-dimensional covariant derivative of the spatial metric is zero, it is not difficult to see that the previous expression is also antisymmetric in (I, J) .

If we now define the three-dimensional Ricci rotation coefficients as:

$$\omega_{IJm}^{(3)} := E_I^n D_m E_{nJ} , \tag{VIII.19}$$

our result reduces simply to:

$$\omega_{IJm} = -\omega_{JIm} = E_I^n D_m E_{nJ} \equiv \omega_{IJm}^{(3)} . \tag{VIII.20}$$

The above results provide us with 3+1 expressions for the 24 independent components of the four-dimensional Ricci rotation coefficients $\omega_{AB\mu}$. Using these results we can now also project the third index of the $\omega_{AB\mu}$ onto the tetrad to obtain ω_{ABC} :

1. Coefficients $\omega_{TIT} = -\omega_{ITT}$ (3 coefficients):

$$\omega_{ITT} = -\omega_{TIT} = E_I^m \partial_m \alpha / \alpha \equiv \partial_I \alpha / \alpha . \tag{VIII.21}$$

2. Coefficients $\omega_{TIJ} = -\omega_{ITJ}$ (9 coefficients):

$$\omega_{TIJ} = -\omega_{ITJ} = E_I^m E_J^n K_{mn} \equiv K_{IJ} . \tag{VIII.22}$$

Notice in particular that the coefficients ω_{TIJ} turn out to be symmetric in (I, J) . On the other hand, from (VIII.14) we have:

$$K_{mn} = -\frac{1}{2\alpha} (\partial_t \gamma_{mn} - D_m \beta_n - D_n \beta_m) . \tag{VIII.23}$$

Equation (VIII.22) can then also be written as:

$$\omega_{TIJ} = -\omega_{ITJ} = -\frac{E_I^m E_J^n}{2\alpha} (\partial_t \gamma_{mn} - D_m \beta_n - D_n \beta_m) . \tag{VIII.24}$$

3. Coefficients $\omega_{IJT} = -\omega_{JIT}$ (3 coefficients):

$$\omega_{IJT} = -\omega_{JIT} = -\frac{1}{\alpha} \left[E_J^m \left(\partial_t E_{mI} - \mathfrak{L}_{\vec{\beta}} E_{mI} \right) + \alpha K_{IJ} \right] , \tag{VIII.25}$$

where $\mathfrak{L}_{\vec{\beta}} E_{mI}$ denotes the Lie derivative of the 1-form E_{mI} with respect to the shift vector β^i .

Again, even if it is not evident, one can show that the previous coefficients are antisymmetric in (I, J) . Notice that this antisymmetry implies that only three of these coefficients are independent from each other.

4. Coefficients $\omega_{IJK} = -\omega_{JIK}$ (9 coefficients):

$$\omega_{IJK} = -\omega_{JIK} = E_I^n E_K^m D_m E_{nJ} \equiv \omega_{IJK}^{(3)} . \tag{VIII.26}$$

There are some interesting facts that we can notice about our results. First, as already mentioned, there are only 24 independent coefficients. We can see that once we have defined a spatial triad E_I^m , nine of these coefficients are given simply by the projection of the extrinsic curvature tensor on the triad, $\omega_{TIJ} = K_{IJ}$. Another nine coefficients are just projections onto the triad of the 3-covariant derivatives of the triad itself, $\omega_{IJK} = E_I^n E_K^m D_m E_{nJ} = \omega_{IJK}^{(3)}$. This means that 18 of the coefficients depend only on information at a given hypersurface. Of the six remaining coefficients, three depend directly on our choice of the lapse function α , in other words on our slicing condition, $\omega_{ITT} = \partial_I \alpha / \alpha$.

The last three coefficients correspond to ω_{IJT} , and they depend on the form in which the spatial triad propagates through time, as can be seen from Eq. (VIII.25). Just as the choice of lapse and shift is free, the choice of the propagation of the spatial triad in time is also free, so it represents a new gauge degree of freedom. Now, since the triad must by definition be orthonormal, the only thing it can do as it evolves is rotate rigidly in space, and this rotation can be parametrized by the usual three Euler angles. This explains why these coefficients have only three degrees of freedom.

8.2 Triad evolution and Fermi–Walker transport

In principle there are many different forms to choose the evolution of the spatial triad, but there is one particular choice that is quite natural and can be useful in many cases. Such a choice consists on asking for the triad *not* to rotate as it propagates along the normal direction to the spatial hypersurfaces, or in other words, asking for the triad to evolve in such a way that it always corresponds to the natural local inertial frame associated with the normal (Eulerian) observers as they move through time. The condition we need to impose to achieve this is known as *Fermi–Walker* transport [29].

Consider the worldline of an arbitrary observer with 4-velocity u^μ , such that $u_\mu u^\mu = -1$. In the general case we say that a vector v^μ is transported without rotation along the curve with tangent u^μ , with v^μ not necessarily normal to u^μ , if we have:

$$u^\mu \nabla_\mu v_\nu = v^\mu (a_\mu u_\nu - a_\nu u_\mu) , \tag{VIII.27}$$

where $a^\mu := u^\nu \nabla_\nu u^\mu$ is the 4-acceleration associated with our observer (if the observer moves on a geodesic we will clearly have $a^\mu = 0$ and Fermi–Walker transport becomes equivalent to parallel transport). The previous condition defines what is known as the Fermi–Walker transport of the vector v^μ along the integral lines of the vector field u^μ .

In the case when we consider the transport of the spatial triad along the worldline of the Eulerian observers we will have $u^\mu = n^\mu$ and $v^\mu = E_I^\mu$, with n^μ the unit timelike vector to the spatial hypersurfaces. Since by construction we have $n^\mu E_{\mu I} = 0$, the second term inside the parenthesis in equation (VIII.27) vanishes and our condition reduces to:

$$n^\mu \nabla_\mu E_{vI} = (E_I^\mu a_\mu) n_\nu , \tag{VIII.28}$$

where $a^\mu := n^\nu \nabla_\nu n^\mu$ is now the 4-acceleration of the Eulerian observers. Let us now see the form that this condition takes in 3+1 language. The first step is to calculate the term $E_I^\mu a_\mu$. We have:

$$\begin{aligned}
 E_I^\mu a_\mu &= E_I^\mu n^\nu \nabla_\nu n_\mu = E_I^\mu n^\nu (\partial_\nu n_\mu - \Gamma_{\mu\nu}^\lambda n_\lambda) \\
 &= E_I^\mu \left[\frac{1}{\alpha} (\partial_t n_\mu - \Gamma_{\mu 0}^\lambda n_\lambda) - \frac{\beta^m}{\alpha} (\partial_m n_\mu - \Gamma_{\mu m}^\lambda n_\lambda) \right] \\
 &= -\frac{E_I^0}{\alpha} (\partial_t \alpha - \beta^m \partial_m \alpha) + E_I^\mu (\Gamma_{\mu 0}^0 - \beta^m \Gamma_{\mu m}^0) = E_I^n (\Gamma_{n 0}^0 - \beta^m \Gamma_{nm}^0) \\
 &= E_I^n \left[\frac{1}{\alpha} (\partial_n \alpha - \beta^m K_{mn}) + \frac{\beta^m}{\alpha} K_{nm} \right] = \frac{\partial_I \alpha}{\alpha}. \tag{VIII.29}
 \end{aligned}$$

The Fermi–Walker condition then becomes:

$$n^\mu \nabla_\mu E_{\nu I} = n_\nu (\partial_I \ln \alpha) . \tag{VIII.30}$$

To proceed, consider first the spatial components of the previous condition by taking $\nu = m$. Since we know that $n_m = 0$, the condition now reduces to simply $n^\mu \nabla_\mu E_{mI} = 0$. Calculating the left hand side we find, after some algebra:

$$n^\mu \nabla_\mu E_{mI} = \frac{1}{\alpha} \left\{ \partial_t E_{mI} - \xi_{\bar{\beta}} E_{mI} + \alpha K_{mI} \right\} . \tag{VIII.31}$$

The Fermi–Walker condition then takes the final form:

$$\partial_t E_{mI} - \xi_{\bar{\beta}} E_{mI} + \alpha K_{mI} = 0 , \tag{VIII.32}$$

or:

$$\partial_t E_{mI} = \xi_{\bar{\beta}} E_{mI} - \alpha K_{mI} . \tag{VIII.33}$$

We still need to consider the time component of equation (VIII.30), that is taking $\nu = 0$. We won't do the calculation here, but it is not too difficult to show that this component in fact adds no new information and reduces again to (VIII.33). This equation is then the full expression that guarantees that the spatial triad propagates without rotation as seen by the Eulerian observers.

Let us now return to equation (VIII.25) for the coefficients ω_{IJT} . This equation can be trivially rewritten as:

$$\omega_{IJT} = -\omega_{JIT} = -\frac{E_J^m}{\alpha} \left[\partial_t E_{mI} - \xi_{\bar{\beta}} E_{mI} + \alpha K_{mI} \right] . \tag{VIII.34}$$

We then see that if we impose the condition that the spatial triad should evolve under Fermi–Walker transport along the normal direction to the spatial hypersurfaces, we will clearly have:

$$\omega_{IJT} = 0 . \tag{VIII.35}$$

We should emphasize here that this result is just a gauge choice, and does not need to apply in the general case. This choice simplifies the equations, and as such can be useful in some cases. On the other hand, it is not difficult to show that if we impose condition (VIII.33) on the spatial triad we recover the evolution equation for the metric (VIII.14), so that imposing the condition is a perfectly consistent gauge choice.

Now, even though condition (VIII.33) is not general, one can always write in the general case:

$$\partial_t E_{mI} = \xi_{\bar{\beta}} E_{mI} - \alpha K_{mI} + \alpha Q_{mI} , \tag{VIII.36}$$

with the Q_{mI} quantities to be chosen. Notice, however, that not all the Q_{mI} can be chosen freely since we need to guarantee that equation (VIII.14) holds. A little algebra allows us to show that this requirement implies that Q_{mn} with purely space indices, or equivalently Q_{IJ} with purely triad indices, must be antisymmetric. This is to be expected since, as mentioned above, the only freedom we really have is a rigid rotation of the triad, which can always be described by a 3×3 antisymmetric matrix. Equation (VIII.36) then allows us to reduce all the gauge freedom associated with the evolution of the spatial triad to the choice of the three-dimensional tensor Q_{mn} . Given equation (VIII.36), the coefficients ω_{IJT} are then given in general by:

$$\omega_{IJT} = -Q_{JI} = +Q_{IJ} , \tag{VIII.37}$$

with $Q_{IJ} = 0$ corresponding to the choice of a triad that is Fermi–Walker transported along the normal direction to the spatial hypersurfaces.

As mentioned above, Fermi–Walker transport can be a good choice in many cases. As we will see below, it is the natural choice in spherical symmetry. However, there are many situations when such a choice might not be adequate. For example, in situations when there is angular momentum asking for the triad not to rotate might be a very bad choice. In fact, it is not difficult to show that in the case of the Kerr spacetime the most natural choice of triad, that is the one associated with the already orthogonal spatial coordinates in the standard Boyer–Lindquist form of the metric, does not satisfy (VIII.33). Another possible choice for the triad evolution is to simply take:

$$\partial_t E_{mI} = \frac{1}{2} E_I^n \partial_t \gamma_{mn} , \tag{VIII.38}$$

or equivalently:

$$\partial_t E_I^m = -\frac{1}{2} E_I^l \gamma^{mn} \partial_t \gamma_{ln} . \tag{VIII.39}$$

This last condition has the advantage of guaranteeing that for a stationary spacetime, such as Kerr for example, the spatial triad is also time independent. We shall call it the *metric driven* triad gauge choice. It is not difficult to show that this condition is

also perfectly consistent in the sense that we find:

$$\begin{aligned} \partial_t (\delta_{IJ}) &= \partial_t (E_I^m E_{mJ}) = E_I^m \partial_t E_{mJ} + E_{mJ} \partial_t E_I^m \\ &= \frac{1}{2} \left(E_I^m E_J^n \partial_t \gamma_{mn} - E_{mJ} E_I^l \gamma^{mn} \partial_t \gamma_{ln} \right) = 0, \end{aligned} \tag{VIII.40}$$

and:

$$\begin{aligned} \partial_t (E_m^I E_{nI}) &= \delta^{IJ} \partial_t (E_{mJ} E_{nI}) = \delta^{IJ} (E_{mJ} \partial_t E_{nI} + E_{nI} \partial_t E_{mJ}) \\ &= \frac{1}{2} \left(E_m^I E_n^a \partial_t \gamma_{an} + E_n^I E_J^a \partial_t \gamma_{am} \right) = \frac{1}{2} (\delta_m^a \partial_t \gamma_{an} + \delta_n^a \partial_t \gamma_{am}) \\ &= \partial_t \gamma_{mn}. \end{aligned} \tag{VIII.41}$$

We can now substitute condition (VIII.38) in the general equation (VIII.36) in order to find the value of Q_{mn} for this gauge choice. We find, after some algebra:

$$Q_{mn} = \frac{1}{\alpha} \left[\frac{1}{2} (D_n \beta_m - D_m \beta_n) - \beta^l (E_n^I D_l E_{mI}) \right]. \tag{VIII.42}$$

The first term in the last expression is clearly antisymmetric. On the other hand, using the fact that $E_n^I E_{mI} = \gamma_{nm}$ it is not difficult to show that the second term is also antisymmetric, so that we can rewrite the expression as:

$$Q_{mn} = -\frac{1}{2\alpha} \left[(\partial_m \beta_n - \partial_n \beta_m) - \beta^l (E_m^K D_l E_{nK} - E_n^K D_l E_{mK}) \right], \tag{VIII.43}$$

or equivalently:

$$Q_{IJ} = -\frac{1}{2\alpha} \left[E_I^m E_J^n (\partial_m \beta_n - \partial_n \beta_m) + \beta^l (E_I^m D_l E_{mJ} - E_J^m D_l E_{mI}) \right]. \tag{VIII.44}$$

where we used the symmetry of the Christoffel symbols on the two lower indices to change covariant derivatives for partial derivatives in the first term. This is the form of the tensor Q when we use the metric driven triad evolution (VIII.38). Notice that this form of Q vanishes for the case when we have no shift vector, $\beta^i = 0$.

Of course, other gauge choices for $\partial_t E_{mI}$ might be useful/interesting, but we will not discuss this issue further here.

8.3 Fock–Ivanenko coefficients in 3+1 form

The next step is to find the form of the Fock–Ivanenko coefficients Γ_μ in terms of 3+1 quantities. In order to do this we start from equation (VIII.45), which we repeat here for completeness:

$$\Gamma_\mu = -\frac{1}{4} \omega_{AB\mu} \gamma^A \gamma^B. \tag{VIII.45}$$

It turns out to be more convenient to work with the coefficients projected onto the tetrad, which now take the form:

$$\Gamma_C = -\frac{1}{4} \omega_{ABC} \gamma^A \gamma^B, \tag{VIII.46}$$

where here we must remember that the indices (A, B, C) take values from 0 to 3, while the indices (I, J, K) will only take values from 1 to 3.

Consider first the time component Γ_T , we have:

$$\Gamma_T = -\frac{1}{4} \omega_{ABT} \gamma^A \gamma^B = -\frac{1}{4} \left[2 \omega_{TIT} \gamma^T \gamma^I + \omega_{IJT} \gamma^I \gamma^J \right], \tag{VIII.47}$$

where we used the fact that the ω_{ABC} are antisymmetric in the first two indices, and $\gamma^A \gamma^B = -\gamma^B \gamma^A$ for $A \neq B$ (from Clifford’s algebra). Substituting now the values of ω_{ITT} and ω_{IJT} from Eqs. (VIII.21) and (VIII.25) we find:

$$\begin{aligned} \Gamma_T &= \frac{1}{4} \left[2 \left(\frac{\partial_I \alpha}{\alpha} \right) \gamma^T \gamma^I + \frac{E^m_J}{\alpha} \left(\partial_I E_{mI} - \epsilon_{\bar{\beta}} E_{mI} + \alpha K_{mI} \right) \gamma^I \gamma^J \right] \\ &= \left(\frac{\partial_I \alpha}{2\alpha} \right) \gamma^T \gamma^I - \frac{1}{4} Q_{IJ} \gamma^I \gamma^J, \end{aligned} \tag{VIII.48}$$

where in the last step we used equation (VIII.36). Remember that in the last expression the antisymmetric matrix Q_{IJ} is a free gauge choice that vanishes when the triad evolves under Fermi–Walker transport.

Before considering the spatial components of Γ_A , let us remember that in Dirac’s equation (V.70) we have the contraction $\gamma^\mu \Gamma_\mu$ coming from the operator $\gamma^\mu \mathcal{D}_\mu$, and we have in general $\gamma^\mu \Gamma_\mu = \gamma^A \Gamma_A = \gamma^T \Gamma_T + \gamma^I \Gamma_I$. At this point we can already calculate the product $\gamma^T \Gamma_T$:

$$\gamma^T \Gamma_T = \left(\frac{\partial_I \alpha}{2\alpha} \right) \left(\gamma^T \right)^2 \gamma^I - \frac{1}{4} Q_{IJ} \gamma^T \gamma^I \gamma^J = \left(\frac{\partial_I \alpha}{2\alpha} \right) \gamma^I - \frac{1}{4} Q_{IJ} \gamma^T \gamma^I \gamma^J. \tag{VIII.49}$$

Here one should emphasize the fact that in the first term above we have a sum over I , while in the second we have sums over I and J . Notice that the term $\gamma^T \Gamma_T$ clearly only depends on our gauge choices, in particular on the choice of the lapse function α and the triad rotation matrix Q_{IJ} . For the particular case when $Q_{IJ} = 0$ we simply have:

$$\gamma^T \Gamma_T = \left(\frac{\partial_I \alpha}{2\alpha} \right) \gamma^I. \tag{VIII.50}$$

Let us now consider the spatial components of the Fock–Ivanenko coefficients Γ_I . We have:

$$\Gamma_I = -\frac{1}{4} \omega_{ABI} \gamma^A \gamma^B = -\frac{1}{4} \left[2 \omega_{TJI} \gamma^T \gamma^J + \omega_{JKI} \gamma^J \gamma^K \right]. \tag{VIII.51}$$

Substituting the values of ω_{TJI} and ω_{JKI} from Eqs. (VIII.22) and (VIII.26) we find:

$$\begin{aligned} \Gamma_I &= -\frac{1}{2} K_{IJ} \gamma^T \gamma^J - \frac{1}{4} (E_J^n E_I^m D_m E_{nK}) \gamma^J \gamma^K \\ &= -\frac{1}{2} K_{IJ} \gamma^T \gamma^J - \frac{1}{4} \omega_{JKI}^{(3)} \gamma^J \gamma^K = -\frac{1}{2} K_{IJ} \gamma^T \gamma^J + \Gamma_I^{(3)}, \end{aligned} \tag{VIII.52}$$

where in the last step we defined the purely three-dimensional Ricci rotation and Fock–Ivanenko coefficients as:

$$\omega_{JKI}^{(3)} := E_J^n E_I^m D_m E_{nK}, \tag{VIII.53}$$

$$\Gamma_I^{(3)} := -\frac{1}{4} \omega_{JKI}^{(3)} \gamma^J \gamma^K. \tag{VIII.54}$$

Let us now calculate the contraction $\gamma^I \Gamma_I$:

$$\gamma^I \Gamma_I = -\frac{1}{2} K_{IJ} \gamma^I \gamma^T \gamma^J + \gamma^I \Gamma_I^{(3)}. \tag{VIII.55}$$

For the first term above we find:

$$-\frac{1}{2} K_{IJ} \gamma^I \gamma^T \gamma^J = \frac{1}{2} (K_{IJ} \gamma^I \gamma^J) \gamma^T = \frac{1}{2} \left(\sum_I K_{II} (\gamma^I)^2 + \sum_{I \neq J} K_{IJ} \gamma^I \gamma^J \right) \gamma^T. \tag{VIII.56}$$

Now, since K_{IJ} is symmetric, and from Clifford’s algebra we know that $\gamma^I \gamma^J = -\gamma^J \gamma^I$ for $I \neq J$, the second term in the last equation cancels. On the other hand, we also have $(\gamma^I)^2 = -1$, so that we finally find:

$$K_{IJ} \gamma^I \gamma^J = -\sum_I K_{II} = -K, \tag{VIII.57}$$

with K the trace of K_{mn} . We can then rewrite the contraction $\gamma^I \Gamma_I$ as:

$$\gamma^I \Gamma_I = -\left(\frac{K}{2}\right) \gamma^T + \gamma^I \Gamma_I^{(3)}. \tag{VIII.58}$$

To finish, we can add both contributions to $\gamma^A \Gamma_A$ to obtain:

$$\gamma^A \Gamma_A = \gamma^\mu \Gamma_\mu = \left[\left(\frac{\partial_I \alpha}{2\alpha}\right) \gamma^I - \frac{1}{4} Q_{IJ} \gamma^T \gamma^I \gamma^J - \left(\frac{K}{2}\right) \gamma^T + \gamma^I \Gamma_I^{(3)} \right]. \tag{VIII.59}$$

Notice that we can also write the spacetime components of the Fock–Ivanenko coefficients as $\Gamma_\mu = e_\mu^A \Gamma_A$, from which we find:

$$\begin{aligned} \Gamma_t &= e_t^A \Gamma_A = \alpha \Gamma_T + \beta^I \Gamma_I \\ &= \left(\frac{\partial_I \alpha}{2} \right) \gamma^T \gamma^I - \frac{\alpha}{4} Q_{IJ} \gamma^I \gamma^J - \beta^I \left(\frac{K_{IJ}}{2} \gamma^T \gamma^J - \Gamma_I^{(3)} \right), \end{aligned} \tag{VIII.60}$$

$$\Gamma_m = e_m^A \Gamma_A = E_m^I \Gamma_I = -\frac{K_{mJ}}{2} \gamma^T \gamma^J + \Gamma_m^{(3)}. \tag{VIII.61}$$

From the above expression one can now easily verify find that:

$$\Gamma_t - \beta^m \Gamma_m = \left(\frac{\partial_I \alpha}{2} \right) \gamma^T \gamma^I - \frac{\alpha}{4} Q_{IJ} \gamma^I \gamma^J = \alpha \Gamma_T. \tag{VIII.62}$$

8.4 Dirac equation in 3+1 form

We are now ready to write the Dirac equation in 3+1 language. Remember that the Dirac equation in general relativity takes the form (V.70):

$$i \gamma^\mu \mathcal{D}_\mu \psi - m \psi = 0, \tag{VIII.63}$$

or equivalently:

$$(\gamma^\mu \partial_\mu + \gamma^\mu \Gamma_\mu + im) \psi = 0, \tag{VIII.64}$$

The last equation can also be written as:

$$(\gamma^t \partial_t + \gamma^m \partial_m) \psi = -(\gamma^\mu \Gamma_\mu + im) \psi. \tag{VIII.65}$$

Notice now that (remember that I takes values from 1 to 3):

$$\gamma^I = e_A^I \gamma^A = e_T^I \gamma^T + \cancel{e_I^I \gamma^I}^0 = \left(\frac{1}{\alpha} \right) \gamma^T, \tag{VIII.66}$$

$$\gamma^m = e_A^m \gamma^A = e_T^m \gamma^T + e_I^m \gamma^I = -\left(\frac{\beta^m}{\alpha} \right) \gamma^T + E_I^m \gamma^I. \tag{VIII.67}$$

Substituting these results into the Dirac equation we find:

$$\gamma^T (\partial_t - \beta^m \partial_m) \psi = -\alpha (\lambda^m \partial_m + \gamma^\mu \Gamma_\mu + im) \psi, \tag{VIII.68}$$

where we have defined the purely spatial Dirac matrices as $\lambda^m := E_I^m \gamma^I$. Notice that, so defined, λ^m is different from $\gamma^m = e_A^m \gamma^A$. In fact we have:

$$\gamma^m = e_A^m \gamma^A = e_T^m \gamma^T + e_I^m \gamma^I = -\left(\frac{\beta^m}{\alpha} \right) \gamma^T + E_I^m \gamma^I = -\left(\frac{\beta^m}{\alpha} \right) \gamma^T + \lambda^m. \tag{VIII.69}$$

The λ^m can also be defined in a completely equivalent way by simply projecting the γ^μ onto the spatial hypersurfaces:

$$\lambda^\mu := P_\nu^\mu \gamma^\nu, \tag{VIII.70}$$

with P_ν^μ the projection operator defined above in (VIII.6). From this definition we find immediately $\lambda^t = 0$, as expected. Notice also that even if $\lambda^m \neq \gamma^m$, if we now lower the indices of λ^m using the spatial metric, that is if we define $\lambda_m := \gamma_{mn} \lambda^n$, then we do find that $\lambda_m = \gamma_m$.

The purely spatial Dirac matrices satisfy the three-dimensional Clifford algebra:

$$\lambda^m \lambda^n + \lambda^n \lambda^m = -2\gamma^{mn} I_3, \tag{VIII.71}$$

with γ^{mn} the inverse spatial metric and I_3 the 3×3 identity matrix. There is one important comment to make here with respect to the spatial Dirac matrices λ^m . When we project these matrices back onto the triad we find $\lambda^I = E_m^I \lambda^m = E_m^I E_J^m \gamma^J = \delta_J^I \gamma^J = \gamma^I$. So that we have $\lambda^I = \gamma^I$, but crucially $\lambda^m \neq \gamma^m$ whenever $\beta^m \neq 0$. The reason for this is that we have $\lambda^m := E_I^m \gamma^I$, while $\gamma^m := e_A^m \gamma^A$. In particular, for any three-dimensional tensor T_m we will have:

$$\gamma^m T_m = \left(\lambda^m - \frac{\beta^m}{\alpha} \right) T_m \neq \lambda^m T_m = \lambda^I T_I = \gamma^I T_I, \tag{VIII.72}$$

so that in general we have $\gamma^m T_m \neq \gamma^I T_I$, while $\lambda^m T_m = \lambda^I T_I$. Because of this, in order to avoid possible confusions, it is best to always try to use the λ 's instead of the γ 's when considering purely spatial contractions of indices, be them coordinate or triad indices.

Multiplying now equation (VIII.68) with γ^T from the left, and using the fact that $(\gamma^T)^2 = 1$, we find:

$$(\partial_t - \beta^m \partial_m) \psi = -\alpha \gamma^T (\lambda^m \partial_m + \gamma^\mu \Gamma_\mu + im) \psi. \tag{VIII.73}$$

We can now use our result for $\gamma^\mu \Gamma_\mu$, equation (VIII.59), to obtain:

$$\begin{aligned} (\partial_t - \beta^m \partial_m) \psi = & -\alpha \gamma^T \left\{ \lambda^m \partial_m + \left[\left(\frac{\partial_I \alpha}{2\alpha} \right) \gamma^I \right. \right. \\ & \left. \left. - \frac{1}{4} Q_{IJ} \gamma^T \gamma^I \gamma^J - \left(\frac{K}{2} \right) \gamma^T + \gamma^I \Gamma_I^{(3)} \right] + im \right\} \psi. \end{aligned} \tag{VIII.74}$$

The above equation can be further simplified by noticing first that:

$$\gamma^I \partial_I = \gamma^I (e_I^\mu \partial_\mu) = \gamma^I E_I^m \partial_m = \lambda^m \partial_m. \tag{VIII.75}$$

Here one must remember again that in general $\lambda^m \neq \gamma^m$, so that $\gamma^I \partial_I = \lambda^m \partial_m \neq \gamma^m \partial_m$. Similarly we find $\gamma^I \Gamma_I^{(3)} = \lambda^m \Gamma_m^{(3)}$. The Dirac equation then takes the form:

$$(\partial_t - \beta^m \partial_m) \psi = \alpha \left[-\gamma^T \lambda^m \left(\partial_m + {}^{(3)}\Gamma_m + \frac{\partial_m \alpha}{2\alpha} \right) + \left(\frac{K}{2} - im\gamma^T \right) + \frac{1}{4} Q_{IJ} \lambda^I \lambda^J \right] \psi. \tag{VIII.76}$$

Finally, if we define the three-dimensional spinorial covariant derivative as:

$$D_m \psi := \partial_m \psi + \Gamma_m^{(3)} \psi, \tag{VIII.77}$$

the Dirac equation then becomes:

$$(\partial_t - \beta^m \partial_m) \psi = -\alpha \gamma^T \left[\lambda^m \left(D_m + \frac{\partial_m \alpha}{2\alpha} \right) + im \right] \psi + \alpha \left(\frac{K}{2} + \frac{1}{4} Q_{mn} \lambda^m \lambda^n \right) \psi. \tag{VIII.78}$$

The above equation can be written in a somewhat more compact form as:

$$(\partial_t - \beta^m \partial_m) \psi = -\alpha \gamma^T (\lambda^m D_m \psi + im \psi) + \alpha \left(\frac{K}{2} - \Gamma_T \right) \psi, \tag{VIII.79}$$

where Γ_T is given by (confront equation (VIII.48)):

$$\Gamma_T = \gamma^T \lambda^m \left(\frac{\partial_m \alpha}{2\alpha} \right) - \frac{1}{4} Q_{mn} \lambda^m \lambda^n. \tag{VIII.80}$$

Equation (VIII.79) is the final form of the Dirac equation in the 3+1 formalism.

The expression for Γ_T given above is in principle valid for any arbitrary choice of the triad evolution gauge represented by the matrix Q_{mn} . In the particular case when we choose a triad that evolves via Fermi–Walker transport we have $Q_{mn} = 0$, so the Γ_T reduces simply to:

$$\Gamma_T = \gamma^T \lambda^m \left(\frac{\partial_m \alpha}{2\alpha} \right). \tag{VIII.81}$$

Notice also that in the Dirac equation there is an explicit dependence on the Dirac matrix γ^T . Since γ^T is associated to the time direction in a local inertial frame, from now on we will simply take $\gamma^T = \gamma^0$, with γ^0 the usual Dirac matrix from Minkowski spacetime, so that:

$$\gamma^T = \begin{pmatrix} I_2 & 0 \\ 0 & -I_2 \end{pmatrix}. \tag{VIII.82}$$

However, when we resurrect the spacetime index we will now have:

$$\gamma^t = e^t_A \gamma^A = n^t_T \gamma_T + \cancel{E^t_I}^0 \gamma_I = \frac{\gamma^T}{\alpha}. \tag{VIII.83}$$

We then see that the matrix γ^t is not just the γ^0 used in Minkowski spacetime, so we must be somewhat careful with the notation. Therefore, from here on we will always write γ^T instead of γ^0 when we refer to the time component of the Dirac matrices in

a local inertial frame. In particular, the above result implies that when we project the gamma matrices onto the normal direction to the spatial hypersurfaces we find:

$$n_\mu \gamma^\mu = -\alpha \gamma^t = -\gamma^T. \tag{VIII.84}$$

From the expression for Γ_T above it is not difficult to show that $\gamma^T \Gamma_T^\dagger \gamma^T = -\Gamma_T$. Using this result, a little algebra allows us to find the adjunct Dirac equation in 3+1 form:

$$(\partial_t - \beta^m \partial_m) \bar{\psi} = -\alpha \left((D_m \bar{\psi}) \lambda^m - im \bar{\psi} \right) \gamma^T + \alpha \bar{\psi} \left(\frac{K}{2} + \Gamma_T \right). \tag{VIII.85}$$

8.5 Conserved current and stress–energy tensor in 3+1 form

Let us now consider the conserved current in 3+1 form. To do this we must first find the adjoint spinor $\bar{\psi} = \psi^\dagger \gamma^T$:

$$\bar{\psi} = (\psi_1^*, \psi_2^*, -\psi_3^*, -\psi_4^*). \tag{VIII.86}$$

As we have shown before, the conserved current will now be given by $j^\mu = \bar{\psi} \gamma^\mu \psi$. We now define the particle density measured by the Eulerian observers as $\rho_p := -n_\mu j^\mu$. We find:

$$\rho_p := -n_\mu j^\mu = \alpha j^t = \alpha \bar{\psi} \gamma^t \psi = \bar{\psi} \gamma^T \psi = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2, \tag{VIII.87}$$

which is, of course, what we would have expected.

On the other hand, the particle flux measured by the Eulerian observers is defined as $f^i := P_\nu^i j^\nu$, with P_ν^μ the projection operator onto the spatial hypersurfaces. We now find:

$$f^i = P_\nu^i j^\nu = P_\nu^i (\bar{\psi} \gamma^\nu \psi) = \bar{\psi} \left(P_\nu^i \lambda^\nu \right) \psi = \bar{\psi} \lambda^i \psi, \tag{VIII.88}$$

where we used the definition of the purely spatial Dirac matrices (VIII.70).

Let us now consider the stress–energy tensor expressed in 3+1 terms. For this, it turns out to be convenient to define $\Pi := n^\mu \mathcal{D}_\mu \psi$. A little algebra then allows us to show that:

$$\Pi = \frac{1}{\alpha} \left(\partial_t \psi - \beta^i \partial_i \psi \right) + \Gamma_T \psi. \tag{VIII.89}$$

From this definition one can see that Π represents the geometric change of the spinor along the normal direction to the spatial hypersurfaces. In an analogous way we also define $\bar{\Pi} := n^\mu \mathcal{D}_\mu \bar{\psi}$, so that:

$$\bar{\Pi} = \frac{1}{\alpha} \left(\partial_t \bar{\psi} - \beta^i \partial_i \bar{\psi} \right) - \bar{\psi} \Gamma_T. \tag{VIII.90}$$

The energy density measured by the Eulerian observers is now defined as $\rho_E := n^\mu n^\nu T_{\mu\nu}$. Using the expression for the stress–energy tensor of the Dirac field, equation (VII.29), we find:

$$\rho_E = \frac{i}{2} n^\mu n^\mu [(\mathcal{D}_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\mathcal{D}_{\nu)} \psi)] = \frac{i}{2} [\bar{\psi} \gamma^T \Pi - \bar{\Pi} \gamma^T \psi], \tag{VIII.91}$$

where we used the fact that $n^\mu \gamma_\mu = -\gamma^T$. We can simplify this further by noticing first that $\bar{\psi} \gamma^T = \psi^\dagger$. Also, from the transpose of the definition of Π we clearly have:

$$\Pi^\dagger = \frac{1}{\alpha} (\partial_t \psi^\dagger - \beta^i \partial_i \psi^\dagger) + \psi^\dagger \Gamma_T^\dagger. \tag{VIII.92}$$

Multiplying this from the right with γ^T we find:

$$\Pi^\dagger \gamma^T = \frac{1}{\alpha} (\partial_t \bar{\psi} - \beta^i \partial_i \bar{\psi}) + \psi^\dagger \Gamma_T^\dagger \gamma^T = \frac{1}{\alpha} (\partial_t \bar{\psi} - \beta^i \partial_i \bar{\psi}) - \bar{\psi} \Gamma_T, \tag{VIII.93}$$

which implies that $\Pi^\dagger \gamma^T = \bar{\Pi}$, or equivalently $\bar{\Pi} \gamma^T = \Pi^\dagger$. The energy density then reduces to:

$$\rho_E = \frac{i}{2} (\psi^\dagger \Pi - \Pi^\dagger \psi). \tag{VIII.94}$$

It is clear from this expression that the energy density is not positive definite, as already mentioned. In terms of the components of the spinor we will have:

$$\rho_E = \frac{i}{2} [(\psi_1^* \Pi_1 + \psi_2^* \Pi_2 + \psi_3^* \Pi_3 + \psi_4^* \Pi_4) - c.c.], \tag{VIII.95}$$

where *c.c.* denotes the complex conjugate of the previous expression.

There is an interesting observation to be made with respect to our final expression for the energy density, equation (VIII.94). If we now define:

$$\tilde{\Pi} := n^\mu \partial_\mu \psi = \frac{1}{\alpha} (\partial_t \psi - \beta^i \partial_i \psi), \tag{VIII.96}$$

we will clearly have $\Pi = \tilde{\Pi} + \Gamma_T \psi$. In the same way, if we define $\tilde{\Pi}^\dagger := n^\mu \partial_\mu \psi^\dagger$ we find $\Pi^\dagger = \tilde{\Pi}^\dagger + \psi^\dagger \Gamma_T^\dagger$. The energy density then becomes:

$$\rho_E = \frac{i}{2} [(\tilde{\psi}^\dagger \Pi - \tilde{\Pi}^\dagger \psi) + \psi^\dagger (\Gamma_T - \Gamma_T^\dagger) \psi]. \tag{VIII.97}$$

Using now the expression for Γ_T given by equation (VIII.80) it is not difficult to show that:

$$\Gamma_T - \Gamma_T^\dagger = -\frac{1}{2} Q_{mn} \lambda^m \lambda^n. \tag{VIII.98}$$

The energy density then becomes:

$$\rho_E = \frac{i}{2} \left[(\psi^\dagger \tilde{\Pi} - \tilde{\Pi}^\dagger \psi) - \frac{1}{2} \psi^\dagger (Q_{mn} \lambda^m \lambda^n) \psi \right]. \tag{VIII.99}$$

The interesting fact about this last expression is that all dependencies coming from the gradient of the lapse that appear in Γ_T have cancelled, and we are only left with a dependency on the Q_{mn} . For a triad that evolves via Fermi–Walker transport we have $Q_{mn} = 0$, and the energy density reduces simply to:

$$\rho_E = \frac{i}{2} (\psi^\dagger \tilde{\Pi} - \tilde{\Pi}^\dagger \psi). \tag{VIII.100}$$

Let us now consider the momentum density measured by the Eulerian observers, which is defined as $J_i := -n^\mu P_i^\nu T_{\mu\nu}$. We will then have:

$$\begin{aligned} J_i &= -\frac{i}{2} n^\mu P_i^\nu [(\mathcal{D}_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\mathcal{D}_{\nu)} \psi)] \\ &= -\frac{i}{4} n^\mu P_i^\nu [\{(\mathcal{D}_\mu \bar{\psi}) \gamma_\nu + (\mathcal{D}_\nu \bar{\psi}) \gamma_\mu\} \psi - \bar{\psi} \{\gamma_\mu (\mathcal{D}_\nu \psi) + \gamma_\nu (\mathcal{D}_\mu \psi)\}] \\ &= -\frac{i}{4} \left[\tilde{\Pi} \lambda_i \psi - \bar{\psi} \lambda_i \Pi + \bar{\psi} \gamma^T (P_i^\nu \mathcal{D}_\nu \psi) - (P_i^\nu \mathcal{D}_\nu \bar{\psi}) \gamma^T \psi \right], \end{aligned} \tag{VIII.101}$$

where we used the fact that $n^\mu \gamma_\mu = -\gamma^T$ and $P_i^\mu \gamma_\mu = \lambda_i$. We now need to calculate the projections of the spinor derivatives $P_i^\nu \mathcal{D}_\nu \psi$ and $P_i^\nu \mathcal{D}_\nu \bar{\psi}$. In order to do this notice first that, since the covariant components of the normal vector vanish, we will have $P_i^\mu = \delta_i^\mu$ (confront (VIII.6)). This implies:

$$P_i^\nu \mathcal{D}_\nu \psi = \mathcal{D}_i \psi = \partial_i \psi + \Gamma_i \psi = \partial_i \psi + \left(\Gamma_i^{(3)} - \frac{K_{im}}{2} \gamma^T \lambda^m \right) \psi = D_i \psi - \frac{K_{im}}{2} \gamma^T \lambda^m \psi, \tag{VIII.102}$$

with D_i the three-dimensional covariant derivative we define above, and where we used equation (VIII.61). Similarly:

$$P_i^\nu \mathcal{D}_\nu \bar{\psi} = \mathcal{D}_i \bar{\psi} = D_i \bar{\psi} + \frac{K_{im}}{2} \bar{\psi} \gamma^T \lambda^m. \tag{VIII.103}$$

We then find:

$$\begin{aligned} &\bar{\psi} \gamma^T (P_i^\nu \mathcal{D}_\nu \psi) - (P_i^\nu \mathcal{D}_\nu \bar{\psi}) \gamma^T \psi \\ &= \bar{\psi} \gamma^T (D_i \psi) - (D_i \bar{\psi}) \gamma^T \psi - \frac{K_{im}}{2} \bar{\psi} (\lambda^m + \gamma^T \lambda^m \gamma^T) \psi \\ &= \psi^\dagger (D_i \psi) - (D_i \psi^\dagger) \psi, \end{aligned} \tag{VIII.104}$$

where we used the fact that $(\gamma^T)^2 = 1$ and $\gamma^T \lambda^m \gamma^T = E_I^m \gamma^T \gamma^I \gamma^T = -E_I^m \gamma^I (\gamma^T)^2 = -\lambda^m$. The momentum density then takes the final form:

$$J_i = -\frac{i}{4} \left[\bar{\Pi} \lambda_i \psi - \bar{\psi} \lambda_i \Pi + \psi^\dagger (D_i \psi) - (D_i \psi^\dagger) \psi \right]. \tag{VIII.105}$$

Finally, the spatial stress tensor is defined as defined as $S_{ij} := P_i^\mu P_j^\nu T_{\mu\nu} = \delta_i^\mu \delta_j^\nu T_{\mu\nu} = T_{ij}$, where we used the fact that $P_i^\mu = \delta_i^\mu$ since the covariant spatial components of the normal vector vanish, $n_i = 0$ (confront (VIII.6)). We now find:

$$\begin{aligned} S_{ij} &= \frac{i}{2} \left[(D_{(i} \bar{\psi}) \lambda_{j)} \psi - \bar{\psi} \lambda_{(i} (D_{j)} \psi) \right] \\ &= \frac{i}{2} \left[(D_{(i} \bar{\psi}) \lambda_{j)} \psi - \bar{\psi} \lambda_{(i} (D_{j)} \psi) + \frac{1}{2} \bar{\psi} \left(K_{m(i} \gamma^T \lambda^m \lambda_{j)} + \lambda_{(i} K_{j)m} \gamma^T \lambda^m \right) \psi \right] \\ &= \frac{i}{2} \left[(D_{(i} \bar{\psi}) \lambda_{j)} \psi - \bar{\psi} \lambda_{(i} (D_{j)} \psi) + \frac{1}{2} \bar{\psi} K_{m(i} \left(\gamma^T \lambda^m \lambda_{j)} + \lambda_{j)} \gamma^T \lambda^m \right) \psi \right] \\ &= \frac{i}{2} \left[(D_{(i} \bar{\psi}) \lambda_{j)} \psi - \bar{\psi} \lambda_{(i} (D_{j)} \psi) + \frac{1}{2} (\bar{\psi} \gamma^T) K_{m(i} (\lambda^m \lambda_{j)} - \lambda_{j)} \lambda^m) \psi \right]. \end{aligned} \tag{VIII.106}$$

The last expression can be further simplified noticing first that $\bar{\psi} \gamma^T = \psi^\dagger$, and:

$$\begin{aligned} K_{mi} (\lambda^m \lambda_j - \lambda_j \lambda^m) &= K_{mi} \gamma_{jn} (\lambda^m \lambda^n - \lambda^n \lambda^m) = -2K_{mi} \gamma_{jn} (\lambda^n \lambda^m + \gamma^{nm}) \\ &= -2(K_{ji} + K_{mi} \lambda_j \lambda^m), \end{aligned} \tag{VIII.107}$$

where we used the spatial Clifford algebra. We finally find for the spatial stress tensor:

$$S_{ij} = \frac{i}{2} \left[(D_{(i} \bar{\psi}) \lambda_{j)} \psi - \bar{\psi} \lambda_{(i} (D_{j)} \psi) - \psi^\dagger (K_{ij} + K_{m(i} \lambda_{j)} \lambda^m) \psi \right]. \tag{VIII.108}$$

It is interesting at this point to calculate the trace of this spatial stress tensor. We have:

$$S \equiv S^m_m = \frac{i}{2} \left[(D_m \bar{\psi}) \lambda^m \psi - \bar{\psi} \lambda^m (D_m \psi) - \psi^\dagger (K + K_{mn} \lambda^m \lambda^n) \psi \right]. \tag{VIII.109}$$

But, from equation (VIII.57), we know that $K_{mn} \lambda^m \lambda^n = -K$, so that S reduces to:

$$S = \frac{i}{2} \left[(D_m \bar{\psi}) \lambda^m \psi - \bar{\psi} \lambda^m (D_m \psi) \right]. \tag{VIII.110}$$

The previous expression can be further simplified by using the Dirac equation (VIII.79) and its adjunct (VIII.85), which can be written in terms of Π and $\bar{\Pi}$ as:

$$\Pi = -\gamma^T (\lambda^m (D_m \psi) + im\psi) + \frac{K}{2} \psi, \tag{VIII.111}$$

$$\bar{\Pi} = -((D_m \bar{\psi})\lambda^m - im\bar{\psi})\gamma^T + \frac{K}{2} \bar{\psi}. \tag{VIII.112}$$

Solving for $\lambda^m (D_m \psi)$ and $(D_m \bar{\psi})\lambda^m$ from the above equations, and substituting into S we find, after some algebra:

$$S = \frac{i}{2} [\psi^\dagger \Pi - \Pi^\dagger \psi] - m\bar{\psi}\psi. \tag{VIII.113}$$

By comparing this with the expression for the energy density above it is easy to see that:

$$S = \rho_E - m\bar{\psi}\psi. \tag{VIII.114}$$

But this result is to be expected since the trace of the full stress–energy tensor can be written in 3+1 terms as $T^\mu{}_\mu = S - \rho_E$, so that we have $T^\mu{}_\mu = -m\bar{\psi}\psi$, which is precisely the result we found before in equation (VII.30).

9 Example: Spherically symmetric spacetime

As an example of the Dirac equation in a curved spacetime, we will consider the particular case of a spacetime with spherical symmetry. We will first only consider the Dirac field as a test field in a background spherically-symmetric spacetime, and only later we will consider the self-gravitating case.

We start by writing the metric of a spherically-symmetric spacetime in spherical coordinates (t, r, θ, φ) as:

$$ds^2 = (-\alpha^2 + \beta_r \beta^r) dt^2 + 2\beta_r dr dt + a^2 dr^2 + b^2 r^2 d\Omega^2, \tag{IX.1}$$

where $d\Omega^2 = d\theta^2 + \sin^2(\theta) d\varphi^2$ is the standard solid angle element, $\alpha = \alpha(r, t)$ is the lapse function, $\beta^r = \beta^r(r, t)$ the shift vector which in this case only has a non-zero radial component, and where $a = a(r, t)$ and $b = b(r, t)$ are the spatial metric components. In particular we have $\beta_r := \gamma_{rr} \beta^r = a^2 \beta^r$. Notice that we recover the Minkowski metric in spherical coordinates if we take $\alpha = a = b = 1, \beta^r = 0$.

9.1 Dirac equation

The first step in order to write the Dirac equation is to choose our tetrad. As we have already mentioned when discussing the Dirac equation in the 3+1 formalism, for the timelike vector we take the unit normal vector to the spatial hypersurfaces:

$$e_T^\mu = n^\mu = (1/\alpha, -\beta^r/\alpha, 0, 0). \tag{IX.2}$$

Notice that for $\beta^r \neq 0$ this vector has a non-trivial radial component. For the spatial vectors we take as a natural choice the unit vectors along the radial and angular directions, which now take the form:

$$e_R^\mu = \left(0, \frac{1}{a}, 0, 0\right), \quad e_\Theta^\mu = \left(0, 0, \frac{1}{rb}, 0\right), \quad e_\Phi^\mu = \left(0, 0, 0, \frac{1}{rb \sin \theta}\right). \quad (\text{IX.3})$$

It is clear that these three vectors are already orthogonal to each other.

In order to avoid confusions, from here on we will always denote the spacetime indices by (t, r, θ, φ) , and their associated Lorentz indices by (T, R, Θ, Φ) . Using now the metric (IX.1) one can now show that the associated 1-forms (the co-tetrad) are:

$$e_{\mu T} = (-\alpha, 0, 0, 0), \quad e_{\mu R} = (a\beta^r, a, 0, 0), \quad e_{\mu \Theta} = (0, 0, rb, 0), \quad e_{\mu \Phi} = (0, 0, 0, rb \sin \theta). \quad (\text{IX.4})$$

Notice again that for $\beta^r \neq 0$ the radial 1-form has a non-zero time component. When thinking only of the spatial triad we will have:

$$E_R^i = \left(\frac{1}{a}, 0, 0\right), \quad E_\Theta^i = \left(0, \frac{1}{rb}, 0\right), \quad E_\Phi^i = \left(0, 0, \frac{1}{rb \sin \theta}\right), \quad (\text{IX.5})$$

and:

$$E_{iR} = (a, 0, 0), \quad E_{i\Theta} = (0, rb, 0), \quad E_{i\Phi} = (0, 0, rb \sin \theta), \quad (\text{IX.6})$$

Next, we need to construct the Dirac matrices. Since spherical coordinates are already orthogonal, the natural choice is to associate the γ matrices directly to the coordinate directions. We will start by defining the γ^A matrices with Lorentz indices since they correspond to a local inertial frame, and can therefore be constructed directly from the usual Dirac matrices in Minkowski spacetime. As already mentioned, we will take $\gamma^T = \gamma^0$ along the time direction, but we now have to ask ourselves in what order should be associate the γ^i to the spatial coordinates (r, θ, φ) . An obvious choice (used frequently) is to associate γ^1 to r , γ^2 to θ and γ^3 to φ . However, it turns out to be more convenient when separating variables (see below) to make a different choice and associate instead γ^3 to the radial coordinate r , and γ^2 and γ^1 to the angular coordinates θ and φ respectively. This is the choice we will follow here. Notice that the different choices only correspond to changing the order of the coordinates and should be completely equivalent physically. We will then take:

$$\gamma^T = \gamma^0, \quad \gamma^R = \gamma^3, \quad \gamma^\Theta = \gamma^2, \quad \gamma^\Phi = \gamma^1. \quad (\text{IX.7})$$

The spacetime components of the γ matrices are then defined as $\gamma^\mu = e_A^\mu \gamma^A$. Using our choice of tetrad above we now find:

$$\gamma^t = \frac{\gamma^T}{\alpha}, \quad \gamma^r = \frac{\gamma^R}{a} - \frac{\beta^r \gamma^T}{\alpha}, \quad \gamma^\theta = \frac{\gamma^\Theta}{rb}, \quad \gamma^\varphi = \frac{\gamma^\Phi}{rb \sin \theta}. \quad (\text{IX.8})$$

Notice that when $\beta^r \neq 0$ the matrix γ^r now has a contribution from γ^T . It is not difficult to verify that the above matrices satisfy the Clifford algebra:

$$\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = -2g^{\mu\nu} . \tag{IX.9}$$

On the other hand, for the purely spatial Dirac matrices λ we have:

$$\lambda^R = \gamma^3 , \quad \lambda^\ominus = \gamma^2 , \quad \lambda^\Phi = \gamma^1 , \tag{IX.10}$$

and:

$$\lambda^r = \frac{\gamma^R}{a} , \quad \lambda^\theta = \frac{\gamma^\ominus}{rb} , \quad \lambda^\varphi = \frac{\gamma^\Phi}{rb \sin \theta} . \tag{IX.11}$$

Notice in particular that when β^r is not zero we clearly have $\lambda^r \neq \gamma^r$.

The λ^m now clearly satisfy the spatial Clifford algebra:

$$\lambda^m \lambda^n + \lambda^n \lambda^m = -2\gamma^{mn} I_3 , \tag{IX.12}$$

with γ^{mn} .

We now need to calculate the three-dimensional Ricci rotation coefficients, since they are necessary in order to obtain the Fock–Ivanenko coefficients. In order to do this we first need to find the 3D Christoffel symbols ${}^{(3)}\Gamma^i_{jk}$ and use (VIII.53). The calculation is not particularly illuminating so we will not write it in detail here. In particular, for the $\omega_{ABC}^{(3)}$ we find that the only non-zero components are:

$$\omega_{R\ominus\ominus}^{(3)} = \omega_{R\Phi\Phi}^{(3)} = -\frac{1}{a} \left(\frac{1}{r} + \frac{\partial_r b}{b} \right) , \quad \omega_{\ominus\Phi\Phi}^{(3)} = -\frac{\cot \theta}{rb} . \tag{IX.13}$$

The next step is to find the matrices $\sigma^{IJ} = [\gamma^I, \gamma^J]/4$. Given equation (V.78), plus the fact that we only have three non-zero Ricci rotation coefficients, it is not difficult to see that we only need the three matrices $\sigma^{R\ominus}$, $\sigma^{R\Phi}$ and $\sigma^{\ominus\Phi}$. Using now equation (III.39) we find:

$$\sigma^{R\ominus} = -\sigma^{23} = +\frac{i}{2} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix} , \tag{IX.14}$$

$$\sigma^{R\Phi} = -\sigma^{13} = -\frac{i}{2} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} , \tag{IX.15}$$

$$\sigma^{\ominus\Phi} = -\sigma^{12} = +\frac{i}{2} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix} , \tag{IX.16}$$

with σ_i the usual Pauli matrices (see Eq. (II.9)). We can now use the previous results to calculate the three-dimensional Fock–Ivanenko coefficients from equation (VIII.54).

We find:

$$\Gamma_R^{(3)} = 0, \tag{IX.17}$$

$$\Gamma_\Theta^{(3)} = -\frac{i}{2} \omega_{R\Theta\Theta} \begin{pmatrix} \sigma_1 & 0 \\ 0 & \sigma_1 \end{pmatrix}, \tag{IX.18}$$

$$\Gamma_\Phi^{(3)} = +\frac{i}{2} \omega_{R\Phi\Phi} \begin{pmatrix} \sigma_2 & 0 \\ 0 & \sigma_2 \end{pmatrix} - \frac{i}{2} \omega_{\Theta\Phi\Phi} \begin{pmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{pmatrix}, \tag{IX.19}$$

and for the coordinate components $\Gamma_r^{(3)} = 0, \Gamma_\theta^{(3)} = rb \Gamma_\Theta^{(3)}, \Gamma_\varphi^{(3)} = rb \sin \theta \Gamma_\Phi^{(3)}$.

We can now calculate the contraction $\lambda^I \Gamma_I^{(3)}$ that appears in Dirac’s equation. A little algebra yields:

$$\lambda^I \Gamma_I^{(3)} = \lambda^m \Gamma_m^{(3)} = \frac{1}{2} \begin{pmatrix} 0 & 0 & +M_1 & +M_2 \\ 0 & 0 & -M_2 & -M_1 \\ -M_1 & -M_2 & 0 & 0 \\ +M_2 & +M_1 & 0 & 0 \end{pmatrix}, \tag{IX.20}$$

where:

$$M_1 := \frac{2}{a} \left(\frac{1}{r} + \frac{\partial_r b}{b} \right), \quad M_2 := -i \frac{\cot \theta}{rb}. \tag{IX.21}$$

We now have all the necessary ingredients to write down the Dirac equation. Notice first that in spherical symmetry our spatial triad does not rotate, so we have $Q_{IJ} = 0$ and the Dirac equation reduces to:

$$(\partial_t - \beta^r \partial_r) \psi = -\alpha \gamma^T \left[\lambda^r \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} \right) + \lambda^m \Gamma_m^{(3)} + im \right] \psi + \left(\frac{\alpha K}{2} \right) \psi, \tag{IX.22}$$

or explicitly for each of the spinor components:

$$\partial_t \psi_1 - \beta^r \partial_r \psi_1 = \alpha \left[-\frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_3 + \frac{i}{rb} \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_4 + \left(\frac{K}{2} - im \right) \psi_1 \right], \tag{IX.23}$$

$$\partial_t \psi_2 - \beta^r \partial_r \psi_2 = \alpha \left[+\frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_4 - \frac{i}{rb} \left(\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_3 + \left(\frac{K}{2} - im \right) \psi_2 \right], \tag{IX.24}$$

$$\partial_t \psi_3 - \beta^r \partial_r \psi_3 = \alpha \left[-\frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_1 + \frac{i}{rb} \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_2 + \left(\frac{K}{2} + im \right) \psi_3 \right], \tag{IX.25}$$

$$\partial_t \psi_4 - \beta^r \partial_r \psi_4 = \alpha \left[+\frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_2 - \frac{i}{rb} \left(\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_1 + \left(\frac{K}{2} + im \right) \psi_4 \right]. \tag{IX.26}$$

These are the Dirac equations in a general spherically symmetric spacetime.

In particular, by taking $\beta^r = K = 0$ and $\alpha = a = b = 1$ we will have the explicit form of the Dirac equation for the case of Minkowski spacetime in spherical coordinates. If, on the other hand, we take $\beta^r = K = 0$, $\alpha = (1 - 2M/r)^{1/2}$, $a = 1/(1 - 2M/r)$ and $b = 1$ we will have the Dirac equation in a Schwarzschild spacetime in the standard coordinates. Alternatively, by taking $\alpha = 1/(1 + 2M/r)^{1/2}$, $\beta^r = 2M/(r + 2M)$, $a = (1 + 2M/r)$, $b = 1$, and $K = (2M/r^2)(1 + 3M/r)/(1 + 2M/r)^{3/2}$ we will have the Dirac equation in a Schwarzschild spacetime in horizon penetrating coordinates of Kerr–Schild type.

At this point is it important to mention the fact that, even though we arrived at the previous equations using the 3+1 form of the Dirac equation, we would have obtained precisely the same result starting from the fully covariant four-dimensional version.

The equations we just found can be written in a more compact form if we define:

$$\psi_\pm^I := \psi_1 \pm \psi_3, \quad \psi_\pm^{II} := \psi_4 \mp \psi_2. \tag{IX.27}$$

The Dirac equations then reduce to:

$$\partial_t \psi_\pm^I - \beta^r \partial_r \psi_\pm^I = \alpha \left[\mp \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_\pm^I + \frac{i}{rb} \left(\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_\mp^{II} + \frac{K}{2} \psi_\pm^I - im \psi_\mp^I \right], \tag{IX.28}$$

$$\partial_t \psi_\pm^{II} - \beta^r \partial_r \psi_\pm^{II} = \alpha \left[\mp \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) \psi_\pm^{II} - \frac{i}{rb} \left(\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) \psi_\mp^I + \frac{K}{2} \psi_\pm^{II} - im \psi_\mp^{II} \right],$$

$$-\frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \Big) \psi_{\mp}^I + \frac{K}{2} \psi_{\pm}^{II} + im\psi_{\mp}^{II} \Big]. \tag{IX.29}$$

An interesting property of the previous system of equations is the fact that we cannot have a spinor with spherical symmetry, which makes perfect sense since spinors represent spin 1/2 particles. To see this notice that if even if we start with initial data such that all the different spinor components are functions only of the radial coordinate r , the terms with $\cot \theta$ above will immediately introduce a dependence on the angular coordinate θ during evolution.

9.2 Conserved current and stress–energy tensor

We can now calculate the particle density and its associated flux. For the particle density we simply have, from equation (VIII.87):

$$\rho_p = |\psi_1|^2 + |\psi_2|^2 + |\psi_3|^2 + |\psi_4|^2. \tag{IX.30}$$

On the other hand, the spatial components of the current which give us the particle flux, can be found using equation (VIII.88). For the radial flux we find:

$$f_r = \bar{\psi} \lambda_r \psi = a \left(\psi^\dagger \gamma^T \lambda^R \right) \psi = a [(\psi_1 \psi_3^* - \psi_2 \psi_4^*) + c.c.] , \tag{IX.31}$$

and for the angular components:

$$f_\theta = \bar{\psi} \lambda_\theta \psi = rb \left(\psi^\dagger \gamma^T \lambda^\Theta \psi \right) = irb [(\psi_1 \psi_4^* - \psi_2 \psi_3^*) - c.c.] , \tag{IX.32}$$

$$f_\varphi = \bar{\psi} \lambda_\varphi \psi = rb \sin \theta \left(\psi^\dagger \gamma^T \lambda^\Phi \psi \right) = rb \sin \theta [(\psi_1 \psi_4^* + \psi_2 \psi_3^*) + c.c.] . \tag{IX.33}$$

The next step is to find the 3+1 components of the stress–energy tensor. These components involve the quantity Π defined in (VIII.89), so it is convenient at this point to find an expression for the Fock–Ivanenko coefficient Γ_T that appears in the definition of Π for the case of spherical symmetry. First, notice that in spherical symmetry our spatial triad does not rotate so we clearly have $Q_{mn} = 0$. Also, the lapse function only depends on the radial coordinate r , so the coefficient Γ_T reduces in this case to:

$$\Gamma_T = \gamma^T \lambda^r \left(\frac{\partial_r \alpha}{2\alpha} \right) . \tag{IX.34}$$

Consider now the energy density. Since, as we just mentioned, in spherical symmetry we have $Q_{mn} = 0$, we can use equation (VIII.100) for the energy density in terms of $\tilde{\Pi}$ instead of Π . We find:

$$\rho_E = \frac{i}{2} \left[\psi^\dagger \tilde{\Pi} - \tilde{\Pi}^\dagger \psi \right] = \frac{i}{2} \left[\left(\psi_1^* \tilde{\Pi}_1 + \psi_2^* \tilde{\Pi}_2 + \psi_3^* \tilde{\Pi}_3 + \psi_4^* \tilde{\Pi}_4 \right) - c.c. \right] , \tag{IX.35}$$

where now:

$$\tilde{\Pi}_i = \frac{1}{\alpha} (\partial_r \psi_i - \beta^r \partial_r \psi_i) . \tag{IX.36}$$

The momentum density J_i is given by equation (VIII.105). In particular, for the radial component we find, using the fact that $\Gamma_r^{(3)} = 0$:

$$J_r = -\frac{i}{4} \left[\tilde{\Pi} \lambda_r \psi - \bar{\psi} \lambda_r \Pi + \psi^\dagger (\partial_r \psi) - (\partial_r \psi^\dagger) \psi \right] . \tag{IX.37}$$

On the other hand, substituting $\Pi = \tilde{\Pi} + \Gamma_T \psi$, we have:

$$\begin{aligned} \tilde{\Pi} \lambda_r \psi - \bar{\psi} \lambda_r \Pi &= \Pi^\dagger \gamma^T \lambda_r \psi - \psi^\dagger \gamma^T \lambda_r \Pi = \tilde{\Pi}^\dagger \gamma^T \lambda_r \psi - \psi^\dagger \gamma^T \lambda_r \tilde{\Pi} \\ &+ \psi^\dagger \left(\Gamma_T^\dagger \gamma^T \lambda_r - \gamma^T \lambda_r \Gamma_T \right) \psi . \end{aligned} \tag{IX.38}$$

Using now our expression for Γ_T above, plus the fact that $(\gamma^T)^\dagger = \gamma^T$, $(\lambda^m)^\dagger = -\lambda^m$, $(\gamma^T)^2 = 1$, and the spatial Clifford algebra, one can show that in the case of spherical symmetry we have:

$$\Gamma_T^\dagger \gamma^T \lambda_i - \gamma^T \lambda_i \Gamma_T = \frac{\partial_r \alpha}{\alpha} (\lambda_i \lambda^r + \delta_i^r) . \tag{IX.39}$$

For $i = r$ the Clifford algebra implies that $\lambda_r \lambda^r = -1$, so the above term vanishes. The radial component of the momentum density can then be written using $\tilde{\Pi}$ instead of Π as:

$$J_r = -\frac{i}{4} \left[\tilde{\Pi}^\dagger \gamma^T \lambda_r \psi - \psi^\dagger \gamma^T \lambda_r \tilde{\Pi} + \psi^\dagger (\partial_r \psi) - (\partial_r \psi^\dagger) \psi \right] , \tag{IX.40}$$

or in terms of the spinor components:

$$\begin{aligned} J_r &= -\frac{i}{4} \left[a \left(\psi_1 \tilde{\Pi}_3^* - \psi_2 \tilde{\Pi}_4^* + \psi_3 \tilde{\Pi}_1^* - \psi_4 \tilde{\Pi}_2^* \right) \right. \\ &\quad \left. + (\psi_1^* \partial_r \psi_1 + \psi_2^* \partial_r \psi_2 + \psi_3^* \partial_r \psi_3 + \psi_4^* \partial_r \psi_4) - c.c. \right] . \end{aligned} \tag{IX.41}$$

The angular components of the momentum density can also be calculated in a straightforward way, but they turn out to be rather long expressions that are not particularly interesting and I will not write them here. But it is important to stress the fact that these angular components do not vanish in general since, even if here we are considering a spherically symmetric spacetime, we are not asking for a spherically symmetric ψ (but see Sect. 9.4 below).

Finally, for the spatial stress tensor S_{ij} we use equation (VIII.108). Let us consider the diagonal components S_{ii} . Since in spherical symmetry the metric is diagonal, the Clifford algebra implies that (no sum) $\lambda_i \lambda^i = -1$. Moreover, in spherical symmetry the extrinsic curvature tensor K_{ij} is also diagonal. One can then easily see that the contributions from the extrinsic curvature in equation (VIII.108) cancel out for $i = j$. The diagonal components of the spatial stress tensor then become:

$$\begin{aligned}
 S_{ii} &= \frac{i}{2} [(D_i \bar{\psi}) \lambda_i \psi - \bar{\psi} \lambda_i (D_i \psi)] \\
 &= \frac{i}{2} [(\partial_i \bar{\psi}) \lambda_i \psi - \bar{\psi} \lambda_i (\partial_i \psi) - \bar{\psi} (\Gamma_i^{(3)} \lambda_i + \lambda_i \Gamma_i^{(3)}) \psi] . \quad (\text{IX.42})
 \end{aligned}$$

We can simplify this even further. Using now the expressions for the $\Gamma_i^{(3)}$ and λ_i found above, plus the fact that the Pauli matrices anti-commute with each other, it is not difficult to show that in our case we have $\Gamma_i^{(3)} \lambda_i + \lambda_i \Gamma_i^{(3)} = 0$ for all three possible values of i . The diagonal components of the spatial stress tensor then reduce simply to:

$$S_{ii} = \frac{i}{2} [(\partial_i \bar{\psi}) \lambda_i \psi - \bar{\psi} \lambda_i (\partial_i \psi)] , \quad (\text{IX.43})$$

or more explicitly:

$$\begin{aligned}
 S_{rr} &= \frac{i}{2} [(\partial_r \bar{\psi}) \lambda_r \psi - \bar{\psi} \lambda_r (\partial_r \psi)] \\
 &= \frac{ia}{2} [(\psi_1 \partial_r \psi_3^* - \psi_2 \partial_r \psi_4^* + \psi_3 \partial_r \psi_3^* - \psi_4 \partial_r \psi_2^*) - c.c.] , \quad (\text{IX.44})
 \end{aligned}$$

$$\begin{aligned}
 S_{\theta\theta} &= \frac{i}{2} [(\partial_\theta \bar{\psi}) \lambda_\theta \psi - \bar{\psi} \lambda_\theta (\partial_\theta \psi)] \\
 &= \frac{rb}{2} [(\psi_2 \partial_\theta \psi_3^* + \psi_4 \partial_\theta \psi_1^* - \psi_1 \partial_\theta \psi_4^* - \psi_3 \partial_\theta \psi_2^*) + c.c.] , \quad (\text{IX.45})
 \end{aligned}$$

$$\begin{aligned}
 S_{\varphi\varphi} &= \frac{i}{2} [(\partial_\varphi \bar{\psi}) \lambda_\varphi \psi - \bar{\psi} \lambda_\varphi (\partial_\varphi \psi)] \\
 &= \frac{irb \sin \theta}{2} [(\psi_1 \partial_\varphi \psi_4^* + \psi_2 \partial_\varphi \psi_3^* + \psi_3 \partial_\varphi \psi_2^* + \psi_4 \partial_\varphi \psi_1^*) - c.c.] . \quad (\text{IX.46})
 \end{aligned}$$

The mixed components of S_{ij} for $i \neq j$ again turn out to be rather long expressions that I will not write here.

9.3 Separation of variables

We are interested in finding solutions to Dirac’s equation that are compatible with a spherically symmetric spacetime. For the moment we will still consider the Dirac field as a test field on a fixed background spacetime, and only later consider the self-gravitating case.

The first step in looking for solutions is to use the method of separation of variables. We then propose an ansatz of the form (the discussion here is based in part in that of [14]):

$$\psi_i = R_i(t, r) T_i(\theta, \varphi) , \quad (\text{IX.47})$$

where here $i = 1, 2, 3, 4$ denotes spinor components, and R_i and T_i are complex functions to be determined. Substituting this in Eqs. (IX.23)–(IX.26), and regrouping

terms we find, after some algebra:

$$\begin{aligned} & \frac{rb}{R_4} \left[\frac{T_1}{T_3} \left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 \right] \\ & = + \frac{i}{T_3} \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_4, \end{aligned} \tag{IX.48}$$

$$\begin{aligned} & \frac{rb}{R_3} \left[\frac{T_2}{T_4} \left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_2 - \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_4 \right] \\ & = - \frac{i}{T_4} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_3, \end{aligned} \tag{IX.49}$$

$$\begin{aligned} & \frac{rb}{R_2} \left[\frac{T_3}{T_1} \left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 \right] \\ & = + \frac{i}{T_1} \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_2, \end{aligned} \tag{IX.50}$$

$$\begin{aligned} & \frac{rb}{R_1} \left[\frac{T_4}{T_2} \left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_4 - \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_2 \right] \\ & = - \frac{i}{T_2} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_1. \end{aligned} \tag{IX.51}$$

The right hand side of the previous equations is now only a function of the angular coordinates (θ, φ) , but the separation of variables is not complete since we still have angular functions on the left hand side. This can be fixed if we ask for $T_3 = aT_1$ and $T_4 = bT_2$, with (a, b) some proportionality constants. In that case the left hand side of the above equations will now be only a function of (t, r) .

The constants (a, b) are in principle arbitrary, but a convenient choice is $a = 1$ and $b = -1$. With this choice we can see that, except for a sign, the right hand side of Eqs. (IX.48) and (IX.50) is now the same, and also the right hand side of Eqs. (IX.49) and (IX.51). This implies that we must now have:

$$\begin{aligned} & \frac{rb}{R_4} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 \right] \\ & = - \frac{rb}{R_2} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 \right], \end{aligned} \tag{IX.52}$$

$$\begin{aligned} & \frac{rb}{R_3} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_2 - \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_4 \right] \\ & = - \frac{rb}{R_1} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_4 - \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_2 \right]. \end{aligned} \tag{IX.53}$$

We can now reduce these two equations to just one if we take $R_2 = cR_1$ and $R_4 = cR_3$, with c a new constant. Again, the value of c is arbitrary, but a convenient choice is

$c = i$. With these choices we now have:

$$R_2 = iR_1, \quad R_4 = iR_3, \quad T_3 = T_1, \quad T_4 = -T_2, \tag{IX.54}$$

and our system of equation reduces to:

$$\begin{aligned} & \frac{rb}{R_3} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 \right] \\ & = + \frac{1}{T_1} \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_2, \end{aligned} \tag{IX.55}$$

$$\begin{aligned} & \frac{rb}{R_3} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 \right] \\ & = - \frac{1}{T_2} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_1, \end{aligned} \tag{IX.56}$$

$$\begin{aligned} & \frac{rb}{R_1} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 \right] \\ & = - \frac{1}{T_1} \left[\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_2, \end{aligned} \tag{IX.57}$$

$$\begin{aligned} & \frac{rb}{R_1} \left[\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 \right] \\ & = + \frac{1}{T_2} \left[\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right] T_1. \end{aligned} \tag{IX.58}$$

The previous equations now have the following structure:

$$f_1(t, r) = +g_1(\theta, \varphi), \tag{IX.59}$$

$$f_1(t, r) = -g_2(\theta, \varphi), \tag{IX.60}$$

$$f_2(t, r) = -g_1(\theta, \varphi), \tag{IX.61}$$

$$f_2(t, r) = +g_2(\theta, \varphi). \tag{IX.62}$$

Since one side of these equations depends only on (r, t) , and the other only on (θ, φ) , we conclude that both sides must be equal to the same constant. Also, given the above structure we must have $f_1 = g_1 = -f_2 = -g_2 = k$, with k a separation constant to be determined. We will then have the following two radial equations:

$$\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 = + \frac{kR_3}{rb}, \tag{IX.63}$$

$$\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 = - \frac{kR_1}{rb}, \tag{IX.64}$$

and two angular equations:

$$\left(\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) T_1 = -k T_2, \tag{IX.65}$$

$$\left(\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi + \frac{\cot \theta}{2} \right) T_2 = +k T_1. \tag{IX.66}$$

The angular equations are particularly interesting. In order to see this let us first define the operators:

$$\not\partial_s^+ := -\partial_\theta - \frac{i}{\sin \theta} \partial_\varphi + s \cot \theta, \tag{IX.67}$$

$$\not\partial_s^- := -\partial_\theta + \frac{i}{\sin \theta} \partial_\varphi - s \cot \theta, \tag{IX.68}$$

with s an integer or half-integer constant. The above operators are known as the raising and lowering spin operators respectively, and are associated with the spherical harmonics with spin weight s first introduced by Newman and Penrose in [30] (see also Appendix D of reference [28]). For an integer s such that $|s| < l$, the spin weighted spherical harmonics are defined in terms of the usual spherical harmonics $Y^{l,m}(\theta, \varphi)$ as:

$${}_s Y^{l,m} := \begin{cases} \left[\frac{(l-s)!}{(l+s)!} \right]^{1/2} \not\partial_{s-1}^+ \cdots \not\partial_0^+ (Y^{l,m}), & +l \geq s \geq 0, \\ (-1)^s \left[\frac{(l-s)!}{(l+|s|)!} \right]^{1/2} \not\partial_{s-1}^- \cdots \not\partial_0^- (Y^{l,m}), & -l \leq s \leq 0, \\ 0, & |s| > l. \end{cases} \tag{IX.69}$$

We also define ${}_0 Y^{l,m} := Y^{l,m}$. For example, we find:

$$\begin{aligned} \pm_1 Y^{l,m} &= \pm \left[\frac{(l-1)!}{(l+1)!} \right]^{1/2} \not\partial_0^\pm Y^{l,m} \\ &= \mp \left[\frac{(l-1)!}{(l+1)!} \right]^{1/2} \left(\partial_\theta \pm \frac{i}{\sin \theta} \partial_\varphi \right) Y^{l,m}, \end{aligned} \tag{IX.70}$$

$$\begin{aligned} \pm_2 Y^{l,m} &= \left[\frac{(l-2)!}{(l+2)!} \right]^{1/2} \not\partial_1^\pm \not\partial_0^\pm Y^{l,m}, \\ &= \left[\frac{(l-2)!}{(l+2)!} \right]^{1/2} \left(\partial_\theta^2 - \cot \theta \partial_\theta \pm \frac{2i}{\sin \theta} (\partial_\theta - \cot \theta) \partial_\varphi - \frac{1}{\sin^2 \theta} \partial_\varphi^2 \right) Y^{l,m}. \end{aligned} \tag{IX.71}$$

On the other hand, the above definitions imply that:

$$\mathcal{J}_s^+ \left({}_s Y^{l,m} \right) = + [(l - s)(l + s + 1)]^{1/2} {}_{s+1} Y^{l,m} , \tag{IX.72}$$

$$\mathcal{J}_s^- \left({}_s Y^{l,m} \right) = - [(l + s)(l - s + 1)]^{1/2} {}_{s-1} Y^{l,m} , \tag{IX.73}$$

which explains why they are called spin raising and lowering operators. These relations also allow us to show that:

$$\mathcal{J}_{s+1}^- \mathcal{J}_s^+ \left({}_s Y^{l,m} \right) = - [l(l + 1) - s(s + 1)] {}_s Y^{l,m} , \tag{IX.74}$$

$$\mathcal{J}_{s-1}^+ \mathcal{J}_s^- \left({}_s Y^{l,m} \right) = - [l(l + 1) - s(s - 1)] {}_s Y^{l,m} , \tag{IX.75}$$

that is, the ${}_s Y^{l,m}$ are eigenfunctions of the operators $\mathcal{J}_{s+1}^- \mathcal{J}_s^+$ and $\mathcal{J}_{s-1}^+ \mathcal{J}_s^-$. In particular, for a function f with spin weight $s = 0$ we have $\mathcal{J}_1^- \mathcal{J}_0^+ = \mathcal{J}_{-1}^+ \mathcal{J}_0^- = L^2$, with L^2 the usual angular Laplacian operator:

$$L^2 f = \frac{1}{\sin \theta} \partial_\theta (\sin \theta \partial_\theta f) + \frac{1}{\sin^2 \theta} \partial_\varphi^2 f = \partial_\theta^2 f + \cot \theta \partial_\theta f + \frac{1}{\sin^2 \theta} \partial_\varphi^2 f . \tag{IX.76}$$

Notice now that in our equations (IX.65) and (IX.66) we in fact have the raising and lowering operators with spin $s = \pm 1/2$. In that case the definition (IX.69) can not be used since we have a half-integer value for s . However, we can define the functions ${}_{\pm 1/2} Y^{l,m}$ simply as the eigenfunctions of the corresponding operators $\mathcal{J}_{s+1}^- \mathcal{J}_s^+$ and $\mathcal{J}_{s-1}^+ \mathcal{J}_s^-$ with $s = \pm 1/2$. Notice that in that case we must also have l and m as half-integers, with $m = -l, \dots, l$. For $s = +1/2$ we find:

$$\mathcal{J}_{3/2}^- \mathcal{J}_{1/2}^+ f = \partial_\theta^2 f + \cot \theta \partial_\theta f + \frac{1}{\sin^2 \theta} \left(\partial_\varphi^2 f + i \cos \theta \partial_\varphi f \right) - \frac{1}{4} \left(\frac{1}{\sin^2 \theta} - 3 \right) f , \tag{IX.77}$$

$$\mathcal{J}_{-1/2}^+ \mathcal{J}_{1/2}^- = \partial_\theta^2 f + \cot \theta \partial_\theta f + \frac{1}{\sin^2 \theta} \left(\partial_\varphi^2 f + i \cos \theta \partial_\varphi f \right) - \frac{1}{4} \left(\frac{1}{\sin^2 \theta} + 1 \right) f , \tag{IX.78}$$

while for $s = -1/2$ we find:

$$\mathcal{J}_{1/2}^- \mathcal{J}_{-1/2}^+ f = \partial_\theta^2 f + \cot \theta \partial_\theta f + \frac{1}{\sin^2 \theta} \left(\partial_\varphi^2 f - i \cos \theta \partial_\varphi f \right) - \frac{1}{4} \left(\frac{1}{\sin^2 \theta} + 1 \right) f , \tag{IX.79}$$

$$\not\partial_{-3/2}^+ \not\partial_{-1/2}^- = \partial_\theta^2 f + \cot \theta \partial_\theta f + \frac{1}{\sin^2 \theta} \left(\partial_\varphi^2 f - i \cos \theta \partial_\varphi f \right) - \frac{1}{4} \left(\frac{1}{\sin^2 \theta} - 3 \right) f . \tag{IX.80}$$

In terms of our raising and lowering operators for spin 1/2 the angular equations (IX.65) and (IX.66) can now be written as:

$$\not\partial_{+1/2}^- T_1 = k T_2 , \quad \not\partial_{-1/2}^+ T_2 = -k T_1 . \tag{IX.81}$$

Comparing this with equations (IX.72) and (IX.73) it is clear that we can take as solutions:

$$T_1 = {}_{+1/2} Y^{l,m} , \quad T_2 = {}_{-1/2} Y^{l,m} , \tag{IX.82}$$

with the separation constant given by $k = -(l + 1/2)$.

General expressions for the ${}_s Y^{l,m}$ with both integer and half-integer indices are well known, and can be found in terms of the Wigner rotation matrices commonly used in quantum mechanics. Here we will just consider the cases with $l = 1/2$. For $l = 1/2, s = \pm 1/2, m = 1/2$ we have, in the standard normalization:

$${}_{\pm 1/2} Y^{1/2,1/2} = \left(\frac{1}{\sqrt{4\pi}} \right) e^{+i\varphi/2} y_\pm(\theta) , \tag{IX.83}$$

where $y_+(\theta) = \sin(\theta/2)$ and $y_- = \cos(\theta/2)$. On the other hand, for $l = 1/2, s = \pm 1/2, m = -1/2$ we have:

$${}_{\pm 1/2} Y^{1/2,-1/2} = \pm \left(\frac{1}{\sqrt{4\pi}} \right) e^{-i\varphi/2} y_\mp(\theta) . \tag{IX.84}$$

Notice that the functions ${}_{\pm 1/2} Y^{1/2,\pm 1/2}$ are discontinuous at $\varphi = 0$. This is an indication of the fact that spinors change sign under a full rotation, as mentioned when we discussed the spinor representation of the Lorentz group in Sect. 3.2.

It is now not difficult to show that taking either $T_1 = {}_{+1/2} Y^{1/2,1/2}$ and $T_2 = {}_{-1/2} Y^{1/2,1/2}$, or alternatively $T_1 = {}_{+1/2} Y^{1/2,-1/2}$ and $T_2 = {}_{-1/2} Y^{1/2,-1/2}$, we will have two independent solutions for our angular equations with $k = -1$. Of course, we can take higher half-integer values of (l, m) while keeping $s = 1/2$ in order to find more solutions, but here we will only consider these two cases. In that case the radial Eqs. (IX.63) and (IX.63) become:

$$\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) + im - \frac{K}{2} \right) R_1 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_3 = -\frac{R_3}{rb} , \tag{IX.85}$$

$$\left(\frac{1}{\alpha} (\partial_t - \beta^r \partial_r) - im - \frac{K}{2} \right) R_3 + \frac{1}{a} \left(\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \right) R_1 = +\frac{R_1}{rb} , \tag{IX.86}$$

If we now take $R_1(t, r) \equiv F(t, r)$ and $R_3(t, r) \equiv G(t, r)$ these equations can be written as:

$$\partial_t F = \beta^r \partial_r F - \frac{\alpha}{a} \left[\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \left(1 + \frac{a}{b} \right) \right] G + \alpha \left(\frac{K}{2} - im \right) F, \tag{IX.87}$$

$$\partial_t G = \beta^r \partial_r G - \frac{\alpha}{a} \left[\partial_r + \frac{\partial_r \alpha}{2\alpha} + \frac{\partial_r b}{b} + \frac{1}{r} \left(1 - \frac{a}{b} \right) \right] F + \alpha \left(\frac{K}{2} + im \right) G, \tag{IX.88}$$

These are now evolution equations for the functions F and G , and can be solved either numerically given some adequate initial data, or considering some particular ansatz. The spinors associated with our two solutions will then be given by:

$$\psi_{\pm} = \frac{e^{\pm i\varphi/2}}{(4\pi)^{1/2}} \begin{pmatrix} F(t, r) y_{\pm}(\theta) \\ \pm i F(t, r) y_{\mp}(\theta) \\ G(t, r) y_{\pm}(\theta) \\ \mp i G(t, r) y_{\mp}(\theta) \end{pmatrix}, \tag{IX.89}$$

with $y_+(\theta) = \sin(\theta/2)$ and $y_- = \cos(\theta/2)$. Equations (IX.87) and (IX.87) can be used, for example, to study the evolution of our spinors in a general curved spherically symmetric spacetime, such as Schwarzschild for example, given some adequate initial data for both F and G .

To finish this section, it is important to find how the functions F and G behave near the origin $r = 0$. In order to do this we take $r \ll 1$, and expand our functions as powers of r :

$$F = \sum_{n=0}^{\infty} F_n(t)r^n, \quad G = \sum_{n=0}^{\infty} G_n(t)r^n. \tag{IX.90}$$

On the other hand, regularity of the metric at the origin implies that a, b and α must be even functions of r , while β^r must be odd:

$$a \simeq a_0(t) + a_2(t)r^2 + \mathcal{O}(r^4), \tag{IX.91}$$

$$b \simeq b_0(t) + b_2(t)r^2 + \mathcal{O}(r^4), \tag{IX.92}$$

$$\alpha \simeq \alpha_0(t) + \alpha_2(t)r^2 + \mathcal{O}(r^4), \tag{IX.93}$$

$$\beta^r \simeq \beta_1(t)r + \mathcal{O}(r^3). \tag{IX.94}$$

Furthermore, in order for the metric to be locally flat at $r = 0$ we must ask for $b_0(t) = a_0(t)$. Finally, from the definition of the extrinsic curvature we find that we must also have $K \simeq K_0(t) + K_2(t)r^2 + \mathcal{O}(r^4)$. When looking at equations (IX.87) and (IX.88) there are two terms that appear to be singular at $r = 0$. Their behavior

close to the origin is:

$$\frac{G}{r} \left(1 + \frac{a}{b}\right) \simeq \frac{2G_0}{r}, \quad \frac{F}{r} \left(1 - \frac{a}{b}\right) = \frac{F}{rb} (b - a) \simeq \frac{rF}{b_0} (b_2 - a_2), \tag{IX.95}$$

where we used the fact that $b_0 = a_0$. The second term is now clearly regular at $r = 0$. In order for the first term to be also regular we must now ask for $G_0 = 0$, so the function $G(r)$ must vanish at the origin.

We can go further in the analysis, but it is easier to work in the case of Minkowski spacetime for which we take $\alpha = a = b = 1, \beta^f = K = 0$ (the general case is more complicated but the conclusions are the same). Substituting the expansions for F and G into equations (IX.87) y (IX.88) we now find:

$$\sum_{n=0}^{\infty} (\dot{F}_n + imF_n) r^n + \sum_{n=0}^{\infty} (n + 2) G_n r^{n-1} = 0, \tag{IX.96}$$

$$\sum_{n=0}^{\infty} (\dot{G}_n - imG_n) r^n + \sum_{n=0}^{\infty} nF_n r^{n-1} = 0. \tag{IX.97}$$

Again, from the first equation it is clear that we must have $G_0 = 0$. Moreover, in the second term of the second equation the sum can be taken from $n = 1$ since the $n = 0$ term vanishes. We then have:

$$\sum_{n=0}^{\infty} (\dot{F}_n + imF_n) r^n + \sum_{n=1}^{\infty} (n + 2) G_n r^{n-1} = 0, \tag{IX.98}$$

$$\sum_{n=0}^{\infty} (\dot{G}_n - imG_n) r^n + \sum_{n=1}^{\infty} nF_n r^{n-1} = 0. \tag{IX.99}$$

Taking now $n \rightarrow n + 1$ in the second term of both equations we can rewrite them as:

$$\sum_{n=0}^{\infty} (\dot{F}_n + imF_n + (n + 3)G_{n+1}) r^n = 0, \tag{IX.100}$$

$$\sum_{n=0}^{\infty} (\dot{G}_n - imG_n + (n + 1)F_{n+1}) r^n = 0. \tag{IX.101}$$

Finally, cancelling each power of r separately we find:

$$G_{n+1} = -\frac{\dot{F}_n + imF_n}{n + 3}, \quad F_{n+1} = -\frac{\dot{G}_n - imG_n}{n + 1}. \tag{IX.102}$$

Since we must have $G_0 = 0$, the above result implies that $F_1 = 0$, which in turn implies $G_2 = 0$, which now implies $F_3 = 0$, etc. We finally find that F must be an

even function of r , while G must be odd:

$$F = F_0(t) + F_2(t)r^2 + \dots, \quad G = G_1(t)r + G_3(t)r^3 + \dots \tag{IX.103}$$

This behavior must be taken into account if one wishes to construct initial data for F and G .

9.4 Spherically symmetric solutions

As we have already mentioned, it is not possible to have spherically symmetric solutions with the Dirac equation. However, we will now show that one can have solutions that are compatible with spherical symmetry, in the sense of having both a conserved current and a stress–energy tensor that are spherically symmetric, if we add the two particular solutions that we found in the last section with the same amplitude, but considering them as independent fields. We start from the conserved current, which is given by:

$$j_\mu^{\text{Tot}} = (j_\mu)_+ + (j_\mu)_- = \bar{\psi}_+ \gamma_\mu \psi_+ + \bar{\psi}_- \gamma_\mu \psi_- , \tag{IX.104}$$

with ψ_\pm the spinors given by (IX.89). For the particle density we find, using (IX.30):

$$\rho_p^{\text{Tot}} = \frac{1}{2\pi} (|F|^2 + |G|^2) . \tag{IX.105}$$

On the other hand, the flux of particles in the radial direction now takes the form, from (IX.31):

$$f_r^{\text{Tot}} = \frac{a}{2\pi} (FG^* + GF^*) . \tag{IX.106}$$

Since both F and G depend only on (t, r) we see that ρ_p^{Tot} and f_r^{Tot} are clearly spherically symmetric. Notice in particular that from the expansions for small r we found above for F and G , it is clear that we will also have $f_r \sim r$ close to the origin.

Consider now the particle flux in the θ direction. It is not difficult to see that equation (IX.32) immediately implies $j_{\theta+} = j_{\theta-} = 0$, so that we clearly have $f_\theta^{\text{Tot}} = j_\theta^{\text{Tot}} = 0$. Finally, for the flux in the φ direction we use equation (IX.33). We now find that $j_{\varphi+} \neq 0$ and $j_{\varphi-} \neq 0$, but crucially $j_{\varphi-} = -j_{\varphi+}$ so that we have $f_\varphi^{\text{Tot}} = j_\varphi^{\text{Tot}} = 0$. We then see that the conserved current has both angular components equal to zero, so that it is indeed compatible with spherical symmetry.

Let us now consider the total stress–energy tensor:

$$T_{\mu\nu}^{\text{Tot}} = T_{\mu\nu+} + T_{\mu\nu-} . \tag{IX.107}$$

For the energy density we find, using (IX.35):

$$\rho_E^{\text{Tot}} = \frac{i}{4\pi} (F^* \tilde{\Pi}_F + G^* \tilde{\Pi}_G - c.c.) , \tag{IX.108}$$

with:

$$\tilde{\Pi}_F := \frac{1}{\alpha} (\partial_t F - \beta^r \partial_r F) , \quad \tilde{\Pi}_G := \frac{1}{\alpha} (\partial_t G - \beta^r \partial_r G) . \tag{IX.109}$$

The previous result for the total energy density is written in a very compact form, but it is convenient to rewrite it using the definitions of $\tilde{\Pi}_F$ and $\tilde{\Pi}_G$ and the evolution equations (IX.87) y (IX.88). Doing this we obtain an equivalent, though somewhat larger expression, that does not involve time derivatives:

$$\rho_E^{\text{Tot}} = \frac{1}{2\pi} \left[\text{Im} \left(\frac{1}{a} (F^* \partial_r G + G^* \partial_r F) + \frac{2}{rb} F^* G \right) + m (|F|^2 - |G|^2) \right] , \tag{IX.110}$$

where here $\text{Im}(q)$ indicates the imaginary part of q in the sense that, if $q = a + ib$ with both a and b real, then $\text{Im}(q) \equiv b$. The energy density is then purely real, as expected.

On the other hand, for the total momentum density in the radial direction we find, from (IX.41):

$$\begin{aligned} J_r^{\text{Tot}} &= \frac{1}{4\pi} \text{Im} \left[F^* \partial_r F + G^* \partial_r G - a (F^* \tilde{\Pi}_G + G^* \tilde{\Pi}_F) \right] \\ &= \frac{1}{2\pi} \text{Im} [F^* \partial_r F + G^* \partial_r G] , \end{aligned} \tag{IX.111}$$

where in the last step we substituted the definitions of $\tilde{\Pi}_F$ and $\tilde{\Pi}_G$, and used again the evolutions Eqs. (IX.87) and (IX.88). The calculation for the angular components of the momentum density is longer, but after some algebra one finds that both vanish, $J_\theta^{\text{Tot}} = J_\varphi^{\text{Tot}} = 0$. Again, it interesting to notice that J_θ in fact vanishes for both individual solutions, $J_{\theta+} = J_{\theta-} = 0$, while J_φ is non-zero for each individual solution but the sum vanishes.

Finally, for the diagonal components of the spatial stress tensor S_{ij} we find, from Eqs. (IX.44)–(IX.46):

$$S_{rr}^{\text{Tot}} = \frac{a}{2\pi} \text{Im} [F^* \partial_r G + G^* \partial_r F] , \tag{IX.112}$$

$$S_{\theta\theta}^{\text{Tot}} = \frac{rb}{2\pi} \text{Im} [F^* G] , \tag{IX.113}$$

$$S_{\varphi\varphi}^{\text{Tot}} = (\sin^2 \theta) S_{\theta\theta}^{\text{Tot}} . \tag{IX.114}$$

All off-diagonal components of S_{ij}^{Tot} now vanish. We then conclude that the total stress-energy tensor is indeed compatible with spherical symmetry.

In particular, the total trace of $T_{\mu\nu}$ turns out to be:

$$(T^\mu{}_\mu)^{\text{Tot}} = (S^i{}_i)^{\text{Tot}} - \rho_E^{\text{Tot}} = (S^r{}_r)^{\text{Tot}} + 2 (S^\theta{}_\theta)^{\text{Tot}} - \rho_E^{\text{Tot}} = -\frac{m}{2\pi} (|F|^2 - |G|^2) , \tag{IX.115}$$

in complete agreement with equation (VII.30). The extra factor of $1/2\pi$ comes from our normalization of the spinors (see equation (IX.89)), and from the fact that we now have two spinors with equal amplitude.

9.5 Dirac stars

As a particular example of a Dirac field in spherical symmetry we will consider the so-called Dirac stars, which are self-gravitating stationary solutions of the Einstein–Dirac equations, analogous to the usual boson stars for the case of the Klein–Gordon field (see reference [10] for a very complete review of boson stars and their relatives). Dirac stars have been previously studied in some detail for example in [13, 15, 16]. Because of this, here we will only consider the basic equations describing the system and will not discuss any particular family of solutions.

We start from a spacetime metric in spherical symmetry of the form:

$$ds^2 = -\alpha^2 dt^2 + a^2 dr^2 + r^2 d\Omega^2, \tag{IX.116}$$

where now α and a are only functions of the radial coordinate r . In terms of our general spherically symmetric metric (IX.1) we are taking $\beta^r = 0$ and $b = 1$, so that we assume that our radial coordinate is the areal radius.

For the Dirac field we will use the spherically symmetric formalism we developed in the previous sections, so that we take a solution of the form $\psi = \psi_+ + \psi_-$, with the spinors ψ_{\pm} given by equation (IX.89):

$$\psi_{\pm} = \frac{e^{\pm i\varphi/2}}{(4\pi)^{1/2}} \begin{pmatrix} F(t, r) y_{\pm}(\theta) \\ \pm i F(t, r) y_{\mp}(\theta) \\ G(t, r) y_{\pm}(\theta) \\ \mp i G(t, r) y_{\mp}(\theta) \end{pmatrix}. \tag{IX.117}$$

We have already shown that the total spinor ψ is compatible with spherical symmetry in the sense that both the total conserved current j^{μ} and the total stress-energy tensor $T_{\mu\nu}$ maintain that symmetry. But if we now want to have a static solution we must also ask for j^{μ} and $T_{\mu\nu}$ to be time independent, and for the associated flux of particles and momentum density to vanish. In order to achieve this we introduce an ansatz with a harmonic time dependence for the functions F and G that define our spinors:

$$F(r, t) = f(r)e^{-i\omega t}, \quad G(r, t) = ig(r)e^{-i\omega t}, \tag{IX.118}$$

where now both $f(r)$ and $g(r)$ are purely real functions. It is now easy to see that with this ansatz both the conserved current and the stress-energy tensor are time independent. The minus sign in the exponential comes from the fact that the energy operator is given by $\hat{E} = i\partial_t$ (remember that we are working in Planck units), so the sign guarantees that we will have positive energy solutions for $\omega > 0$ (see below).

For the particle density and flux we find, from Eqs. (IX.105) and (IX.106):

$$\rho_p = \frac{1}{2\pi} (f^2 + g^2), \quad f_r = 0. \tag{IX.119}$$

Notice that the particle flux vanishes, as expected for a static solution.

On the other hand for the different components of the stress-energy tensor we find, from Eqs. (IX.110), (IX.111), (IX.112) and (IX.113):

$$\rho_E = \frac{1}{2\pi} \left[\frac{1}{a} (fg' - f'g) + \frac{2fg}{r} + m(f^2 - g^2) \right], \tag{IX.120}$$

$$J_r = 0, \tag{IX.121}$$

$$S^r_r = \frac{1}{2\pi a} (fg' - f'g), \tag{IX.122}$$

$$S^\theta_\theta = S^\varphi_\varphi = \frac{fg}{2\pi r}, \tag{IX.123}$$

where the prime denotes derivatives with respect to r . Notice again how the momentum density J_r vanishes, as expected for a static solution. We also find for the trace pf $T_{\mu\nu}$:

$$T^\mu_\mu = -\rho_E + S^r_r + 2S^\theta_\theta = -\frac{m}{2\pi} (f^2 - g^2), \tag{IX.124}$$

consistent with equation (VII.30).

Here one should notice that, although the expression for ρ_E above is correct, we can in fact find an equivalent more compact expression using equation (IX.108), where in this case we have from our ansatz $\Pi_F = \partial_t F/\alpha$ and $\Pi_G = \partial_t G/\alpha$. We then find:

$$\rho_E = \frac{\omega}{2\pi\alpha} (f^2 + g^2). \tag{IX.125}$$

Notice that we will clearly have $\rho_E > 0$ for $\omega > 0$.

The next step is to find the equations that must be satisfied by the stationary solution. Substituting our ansatz for the metric and the functions F and G into Eqs. (IX.87) and (IX.88) we find:

$$\omega f = +\frac{\alpha}{a} \left[g' + g \left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1+a) \right) \right] + \alpha m f, \tag{IX.126}$$

$$\omega g = -\frac{\alpha}{a} \left[f' + f \left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1-a) \right) \right] - \alpha m g, \tag{IX.127}$$

where we also used the fact that for a static spacetime the extrinsic curvature vanishes, so that $K = 0$. Solving for f' and g' we obtain:

$$f' = -f \left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1-a) \right) - ag \left(m + \frac{\omega}{\alpha} \right), \tag{IX.128}$$

$$g' = -g \left(\frac{\alpha'}{2\alpha} + \frac{1}{r}(1+a) \right) - af \left(m - \frac{\omega}{\alpha} \right). \tag{IX.129}$$

We also need equations for the metric functions a and α . The equation for the radial metric a is obtained directly from the Hamiltonian constraint. On the other hand, the

equation for the lapse function α is obtained from the so-called polar-areal gauge, which corresponds to asking for the time derivative of the angular component of the extrinsic curvature $K_{\theta\theta}$ to vanish. We will not write down the general expressions for the Hamiltonian constraint and the polar-areal gauge condition here since they are well known and can be found in text books (see e.g. [28]). In our case these two conditions reduce to:

$$a' = \frac{a}{2} \left(\frac{1 - a^2}{r} + 8\pi r a^2 \rho_E \right). \tag{IX.130}$$

and:

$$\alpha' = \alpha \left(\frac{a^2 - 1}{2r} + 4\pi r a^2 S^r_r \right). \tag{IX.131}$$

The final system of equations to be solved for the functions (a, α, f, g) is then:

$$\partial_r a = \frac{a}{2} \left(\frac{1 - a^2}{r} + 8\pi r a^2 \rho_E \right), \tag{IX.132}$$

$$\partial_r \alpha = \alpha \left(\frac{a^2 - 1}{2r} + 4\pi r a^2 S^r_r \right), \tag{IX.133}$$

$$\partial_r f = -f \left(\frac{\partial_r \alpha}{2\alpha} + \frac{1}{r}(1 - a) \right) - ag \left(m + \frac{\omega}{\alpha} \right), \tag{IX.134}$$

$$\partial_r g = -g \left(\frac{\partial_r \alpha}{2\alpha} + \frac{1}{r}(1 + a) \right) - af \left(m - \frac{\omega}{\alpha} \right), \tag{IX.135}$$

with ρ_E given by (IX.125), and S^r_r given by:

$$S^r_r = \frac{1}{2\pi a} (fg' - f'g) = \rho_E - \frac{1}{\pi} \left(\frac{fg}{r} + \frac{m}{2} (f^2 - g^2) \right). \tag{IX.136}$$

Notice that in the equations for $\partial_r f$ and $\partial_r g$ above there are derivatives of the lapse on the right hand side, but these can be eliminated using (IX.133).

It is important to consider the behavior of solutions of our system of equations both at infinity and at the origin. Consider first the limit $r \rightarrow \infty$. For asymptotically flat solutions we can assume our spacetime is Minkowski far away, so that we must have $a \simeq 1$, $\alpha \simeq 1$ and $1/r \rightarrow 0$. The equations for f and g then reduce to:

$$\partial_r f \simeq -g(m + \omega), \quad \partial_r g \simeq -f(m - \omega). \tag{IX.137}$$

Taking a second derivative of the first equation, and substituting the result in the second, we find:

$$\partial_r^2 f \simeq f(m^2 - \omega^2). \tag{IX.138}$$

It is now clear that if we want to have exponentially decaying solutions at infinity we must have $m^2 > \omega^2$. Of course, in principle we will also have solutions that grow exponentially, which is incompatible with having an asymptotically flat spacetime.

We will only have decaying solutions for specific values of ω , so that we must solve an eigenvalue problem.

Consider now the behaviour of the solutions near the origin $r = 0$. Since spacetime must be locally flat there we must ask for the radial metric component a to behave as:

$$a \simeq 1 + \mathcal{O}(r^2) . \tag{IX.139}$$

Similarly, for the lapse function α we will have:

$$\alpha \simeq \alpha_0 + \mathcal{O}(r^2) , \tag{IX.140}$$

with α_0 some constant. In principle we don't know the value of α_0 , but notice that our system of Eqs. (IX.132)–(IX.135) is invariant under the rescaling:

$$\alpha \rightarrow k\alpha , \quad \omega \rightarrow k\omega , \tag{IX.141}$$

with k an arbitrary constant. This means that we can simply take $\alpha_0 = 1$, solve the system, and then rescale α and ω so that we have $\alpha \rightarrow 1$ at infinity.

On the other hand, we have already shown above that f must be an even function of r , while g must be odd, so that we will have:

$$f \simeq f_0 + \mathcal{O}(r^2) , \quad g \simeq g_1 r + \mathcal{O}(r^3) , \tag{IX.142}$$

with f_0 and g_1 some constants. Substituting our expansions into the system of equations we find that at the origin we must have:

$$\partial_r a |_{r=0} = 0 , \quad \partial_r \alpha |_{r=0} = 0 , \quad \partial_r f |_{r=0} = 0 . \tag{IX.143}$$

The condition for g is more interesting due to the presence of the term $(1 + a)/r$ in its equation, which might seem to be singular at $r = 0$. The equation, however, is in fact regular since this factor is multiplied with g which goes as $\sim r$ close to the origin. When we substitute our expansions for small r we now find:

$$\partial_r g |_{r=0} = g_1 , \tag{IX.144}$$

with $g_1 = f_0(\omega/\alpha_0 - m)/3$, so that g_1 is not independent of f_0 .

To solve the full system of equations (IX.132)–(IX.135) one can then choose f_0 as our only free parameter (taking $\alpha_0 = 1$), and look for solutions for which f and g decay exponentially at infinity in order to find the eigenvalue ω , using a variety of numerical techniques. For example, for a given value of f_0 one can choose a trial value of ω and integrate outward from the origin with some standard ODE integrator (for example fourth order Runge–Kutta), and use a shooting algorithm to modify the value of ω until one finds exponentially decaying solutions at infinity.

By changing the value of f_0 one can construct a whole family of solutions for the Dirac stars. As mentioned above, we will not discuss the family of solutions here, as this has already been done before in some detail in references [13, 15, 16].

10 Final remarks

The Dirac equation is one of the most fundamental equations in physics. It describes the behavior of fermions such as leptons and quarks, and is at the heart of the standard model of particles and fields. On the other hand, general relativity is our modern theory of gravity, and describes with great success astrophysical phenomena that go from the structure of neutron stars, to the formation of black holes, the emission of gravitational waves, and the evolution of the Universe as a whole.

Though currently we do not have a successful theory of quantum gravity (though we certainly have candidates in the form of string theory, loop quantum gravity, dynamical causal triangulations, etc.), it is nevertheless very important to be able to study the evolution of quantum fields in a curved spacetime. This was, for example, what led Hawking to the discovery that black holes in fact radiate energy.

For the case of scalar or tensor fields, such as the Higgs or electromagnetic fields, the generalization to a curved spacetime is rather straightforward and follows directly from the equivalence principle. However, in the case of spinor fields this generalization is not that simple, and requires the introduction of the Lorentz group and the tetrad formalism.

Here, I have presented a pedagogical review of the Dirac equation in the case of general relativity, starting from first principles. Even though I have ignored the quantization of the Dirac field and have treated it as a purely “classical” field, I believe that this review can be useful to researchers in general relativity who might not be used to working with spinor fields. In the last sections I have also derived expressions for the Dirac equation and its associated stress–energy tensor in the 3+1 formalism, and shown how this can be applied to the special case of spherical symmetry. To my knowledge, these last sections include new material which can be very useful for the study of the evolution of the Dirac field in a dynamical spacetime.

11 Derivation of the stress–energy tensor for the Dirac equation

We start by considering the variation of the action with respect to the spacetime metric $g_{\mu\nu}$:

$$\delta_g S = \delta \int L |g|^{1/2} d^4x . \quad (\text{A.1})$$

As we know, the Dirac Lagrangian has terms that depend directly on the tetrad and not on the metric, so we now need to consider variations of the tetrad itself. Notice first that from the expression for the metric in terms of the tetrad, $g_{\mu\nu} = e_{\mu A} e_{\nu}^A$, we immediately find:

$$\delta g_{\mu\nu} = \eta_{AB} \left(e_{\mu}^A \delta e_{\nu}^B + e_{\nu}^B \delta e_{\mu}^A \right) . \quad (\text{A.2})$$

The variation of the tetrad $\delta e_{\mu A}$ can be naturally decomposed into two parts, a “symmetric part” $\delta^+ e_{\mu A}$ that induces variations of the metric, and an “antisymmetric part”

$\delta^- e_{\mu A}$ that leaves the metric unchanged [31]:

$$\delta^\pm e_\mu^A = \frac{1}{2} \left(\delta e_\mu^A \mp \eta^{AB} g_{\mu\nu} \delta e_B^\nu \right), \quad \delta e_\mu^A = \delta^+ e_\mu^A + \delta^- e_\mu^A. \tag{A.3}$$

From these definitions we find:

$$\begin{aligned} \delta^\pm g_{\mu\nu} &= \eta_{AB} \left(e_\mu^A \delta^\pm e_\nu^B + e_\nu^B \delta^\pm e_\mu^A \right) = \frac{\eta_{AB}}{2} \left[e_\mu^A \left(\delta e_\nu^B \mp \eta^{BC} g_{\nu\lambda} \delta e_C^\lambda \right) \right. \\ &\quad \left. + e_\nu^B \left(\delta e_\mu^A \mp \eta^{AC} g_{\mu\lambda} \delta e_C^\lambda \right) \right] \\ &= \frac{\eta_{AB}}{2} \left[e_\mu^A \delta e_\nu^B + e_\nu^B \delta e_\mu^A \right] \mp \frac{1}{2} \left[g_{\nu\lambda} e_\mu^A \delta e_A^\lambda + g_{\mu\lambda} e_\nu^A \delta e_A^\lambda \right] \\ &= \frac{\delta g_{\mu\nu}}{2} \pm \frac{1}{2} \left[g_{\nu\lambda} e_A^\lambda \delta e_\mu^A + g_{\mu\lambda} e_A^\lambda \delta e_\nu^A \right] \\ &= \frac{\delta g_{\mu\nu}}{2} \pm \frac{1}{2} \left[e_{\nu A} \delta e_\mu^A + e_{\mu A} \delta e_\nu^A \right] = \frac{\delta g_{\mu\nu}}{2} \pm \frac{\delta g_{\mu\nu}}{2}, \end{aligned} \tag{A.4}$$

where we used the fact that $e_\mu^A e_A^\nu = \delta_\mu^\nu$ implies $e_\mu^A \delta e_A^\nu = -e_A^\nu \delta e_\mu^A$. We then find:

$$\delta^+ g_{\mu\nu} = \delta g_{\mu\nu}, \quad \delta^- g_{\mu\nu} = 0. \tag{A.5}$$

We can also show that:

$$\begin{aligned} \eta^{AB} e_B^\nu \delta g_{\mu\nu} &= \eta^{AB} e_B^\nu \left(e_\nu^C \delta e_{\mu C} + e_{\mu C} \delta e_\nu^C \right) = \delta e_\mu^A + \eta^{AB} e_{\mu C} e_B^\nu \delta e_\nu^C \\ &= \delta e_\mu^A - \eta^{AB} e_{\mu C} e_\nu^C \delta e_B^\nu = \delta e_\mu^A - \eta^{AB} g_{\mu\nu} \delta e_B^\nu = 2\delta^+ e_\mu^A, \end{aligned} \tag{A.6}$$

so that we can express $\delta^+ e_\mu^A$ entirely in terms of $\delta g_{\mu\nu}$ as:

$$\delta^+ e_\mu^A = \frac{1}{2} \eta^{AB} e_B^\nu \delta g_{\mu\nu} = \frac{1}{2} e^{\nu A} \delta g_{\mu\nu}. \tag{A.7}$$

And similarly:

$$\delta^+ e_A^\mu = \frac{1}{2} e_{\nu A} \delta g^{\mu\nu} = -\frac{1}{2} g^{\mu\alpha} e_A^\beta \delta g_{\alpha\beta}. \tag{A.8}$$

Now, the stress-energy tensor is defined in terms of the variation of the action integral with respect to changes in the metric as:

$$\delta_g S = \frac{1}{2} \int T^{\alpha\beta} \delta g_{\alpha\beta} |g|^{1/2} d^4x. \tag{A.9}$$

This means that if the Lagrangian is expressed in terms of the tetrad, as is the case of the Dirac Lagrangian, we must only consider the changes in the tetrad that modify the metric, that is we should only consider $\delta^+ e_A^\mu$. The variation in the action will then

take the form:

$$\delta_g S = \int \left[|g|^{1/2} \left(\frac{\delta L}{\delta e_A^\mu} \right) \delta^+ e_A^\mu + L \delta |g|^{1/2} \right] d^4 x . \quad (\text{A.10})$$

For the second term we have:

$$L \delta |g|^{1/2} = \frac{L}{2|g|^{1/2}} \delta |g| = \frac{L}{2|g|^{1/2}} |g| g^{\alpha\beta} \delta g_{\alpha\beta} = \frac{L}{2} |g|^{1/2} g^{\alpha\beta} \delta g_{\alpha\beta} . \quad (\text{A.11})$$

On the other hand, for the first term we find:

$$\begin{aligned} |g|^{1/2} \left(\frac{\delta L}{\delta e_A^\mu} \right) \delta^+ e_A^\mu &= -\frac{1}{2} |g|^{1/2} \left(\frac{\delta L}{\delta e_A^\mu} \right) g^{\mu\alpha} e_A^\beta \delta g_{\alpha\beta} \\ &= -\frac{1}{4} |g|^{1/2} \left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu} \right) \delta g_{\alpha\beta} . \end{aligned} \quad (\text{A.12})$$

The variation of the action then becomes:

$$\delta_g S = \frac{1}{2} \int \left[-\frac{1}{2} \left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu} \right) + g^{\alpha\beta} L \right] \delta g_{\alpha\beta} |g|^{1/2} d^4 x . \quad (\text{A.13})$$

Comparing this result with (A.9) we find for the stress–energy tensor:

$$T^{\alpha\beta} = -\frac{1}{2} \left(g^{\mu\alpha} e_A^\beta \frac{\delta L}{\delta e_A^\mu} + g^{\mu\beta} e_A^\alpha \frac{\delta L}{\delta e_A^\mu} \right) + g^{\alpha\beta} L , \quad (\text{A.14})$$

and lowering the indices we recover the expression for the stress-energy tensor given in (VII.28):

$$T_{\mu\nu} = -\frac{1}{2} \left(e_{\mu D} \frac{\delta L}{\delta e_D^\nu} + e_{\nu D} \frac{\delta L}{\delta e_D^\mu} \right) + g_{\mu\nu} L . \quad (\text{A.15})$$

One should stress the fact that at this point we are still not replacing the variations with partial derivatives since the Lagrangian can depend also on derivatives of the tetrad (see below).

Notice now that, since the Dirac Lagrangian (VII.13) vanishes on shell, for case of the Dirac field we can in fact ignore the last term in the above expression. Moreover, the mass term that appears in the Dirac Lagrangian is independent of the tetrad, so the stress-energy tensor reduces to:

$$T_{\mu\nu} = -\frac{1}{2} \left(e_{\mu D} \frac{\delta K}{\delta e_D^\nu} + e_{\nu D} \frac{\delta K}{\delta e_D^\mu} \right) , \quad (\text{A.16})$$

with K the kinetic term in the Lagrangian. The kinetic term can further be split into two parts, $K = K_1 + K_2$, given by:

$$K_1 = \frac{i}{2} \left[e_A^\beta \bar{\psi} \gamma^A (\partial_\beta \psi) - e_A^\alpha (\partial_\alpha \bar{\psi}) \gamma^A \psi \right], \tag{A.17}$$

$$K_2 = \frac{i}{4} \bar{\psi} \left(e_A^\alpha e_C^\beta \partial_\alpha e_{\beta B} \right) \gamma^{CAB} \psi. \tag{A.18}$$

Now, the variations that appear in (A.16) can be substituted with partial derivatives when K depends only on the tetrad and not its derivatives. This is clearly the case for K_1 , so its contribution is not difficult to find and turns out to be:

$$(T_{\mu\nu})_1 = \frac{i}{2} \left[(\partial_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\partial_{\nu)} \psi) \right]. \tag{A.19}$$

The contribution from K_2 is somewhat more difficult to find since it depends not only on the tetrad, but also on its derivatives. The variation is then given by the so-called Euler derivative (this can be shown by following the same procedure used to find the Euler–Lagrange equations through integration by parts starting from the action integral):

$$\frac{\delta K_2}{\delta e_D^\mu} = \frac{\partial K_2}{\partial e_D^\mu} - \partial_\lambda \left(\frac{\partial K_2}{\partial (\partial_\lambda e_D^\mu)} \right). \tag{A.20}$$

From the definition of K_2 we can see that the term we are interested in is:

$$f_{ABC} = e_A^\alpha e_C^\beta \partial_\alpha e_{\beta B}. \tag{A.21}$$

Using now the fact that $\partial_\alpha (e_A^\beta e_{\beta B}) = \partial_\alpha \eta_{AB} = 0$, one can show that we have $\partial_\alpha e_{\beta B} = -e_B^D e_{\sigma B} \partial_\alpha e_D^\sigma$, so that we can rewrite f_{ABC} as:

$$f_{ABC} = -e_A^\alpha e_C^\beta e_B^D e_{\sigma B} \partial_\alpha e_D^\sigma = -e_{\sigma B} e_A^\alpha \partial_\alpha e_C^\sigma. \tag{A.22}$$

Consider first the derivative of f_{ABC} with respect to the tetrad e_D^ν :

$$\begin{aligned} \frac{\partial f_{ABC}}{\partial e_D^\nu} &= -\frac{\partial (e_{\sigma B} e_A^\alpha)}{\partial e_D^\nu} \partial_\alpha e_C^\sigma = -\left(e_{\sigma B} \frac{\partial e_A^\alpha}{\partial e_D^\nu} + e_A^\alpha \frac{\partial e_{\sigma B}}{\partial e_D^\nu} \right) \partial_\alpha e_C^\sigma \\ &= -\left(e_{\sigma B} \delta_\nu^\alpha \delta_A^D - e_A^\alpha e_\sigma^D e_{\nu B} \right) \partial_\alpha e_C^\sigma = -e_{\sigma B} \delta_A^D \partial_\nu e_C^\sigma + e_\sigma^D e_{\nu B} \partial_\alpha e_C^\sigma, \end{aligned} \tag{A.23}$$

where we used the fact that $\partial e_{\sigma B} / \partial e_D^\nu = -e_\sigma^D e_{\nu B}$, which can be easily shown from $\partial (e_A^\beta e_{\beta B}) / \partial e_D^\nu = \partial \eta_{AB} / \partial e_D^\nu = 0$. The contribution to the stress-energy tensor coming

from this term will then be:

$$\begin{aligned}
 P_{\mu\nu} &:= \frac{i}{8} \bar{\psi} \left[e_{\mu D} \left(e_{\sigma B} \delta_A^D \partial_\nu e_C^\sigma - e_{\sigma}^D e_{\nu B} \partial_A e_C^\sigma \right) + \mu \leftrightarrow \nu \right] \gamma^{CAB} \psi \\
 &= \frac{i}{8} \bar{\psi} \left[e_{\mu A} e_{\sigma B} \partial_\nu e_C^\sigma - g_{\mu\sigma} e_{\nu B} \partial_A e_C^\sigma + \mu \leftrightarrow \nu \right] \gamma^{CAB} \psi .
 \end{aligned} \tag{A.24}$$

In fact, it is more convenient to project this result onto the tetrad to find:

$$\begin{aligned}
 P_{IJ} = e_I^\mu e_J^\nu P_{\mu\nu} &= \frac{i}{8} \bar{\psi} \left[\eta_{IA} e_{\sigma B} \partial_J e_C^\sigma - \eta_{JB} e_{\sigma I} \partial_A e_C^\sigma + I \leftrightarrow J \right] \gamma^{CAB} \psi \\
 &= \frac{i}{8} \bar{\psi} \left[-\eta_{IA} f_{JBC} + \eta_{JB} f_{AIC} + I \leftrightarrow J \right] \gamma^{CAB} \psi \\
 &= -\frac{i}{8} \bar{\psi} \left[(f_{JAB} + f_{AJB}) \gamma_I^{AB} + (f_{IAB} + f_{AIB}) \gamma_J^{AB} \right] \psi ,
 \end{aligned} \tag{A.25}$$

where in the last step we used the symmetry properties of the γ^{ABC} and renamed indices.

Consider now the term associated with the derivatives of f_{ABC} with respect to the derivatives of the tetrad:

$$\begin{aligned}
 -\partial_\lambda \left(\frac{\partial f_{ABC}}{\partial(\partial_\lambda e_D^\nu)} \right) &= -\partial_\lambda \left(\frac{\partial(e_{\sigma B} e_A^\alpha \partial_\alpha e_C^\sigma)}{\partial(\partial_\lambda e_D^\nu)} \right) = -\partial_\lambda \left(e_{\sigma B} e_A^\alpha \delta_\alpha^\lambda \delta_\nu^\sigma \delta_C^D \right) \\
 &= -\delta_C^D \partial_\lambda (e_A^\lambda e_{\nu B}) = -\delta_C^D (e_A^\lambda \partial_\lambda e_{\nu B} + e_{\nu B} \partial_\lambda e_A^\lambda) .
 \end{aligned} \tag{A.26}$$

The contribution to the stress-energy tensor coming from this term is then:

$$Q_{\mu\nu} := -\frac{i}{8} \bar{\psi} \left[e_{\mu C} (\partial_A e_{\nu B} + e_{\nu B} \partial_\lambda e_A^\lambda) + \mu \leftrightarrow \nu \right] \gamma^{CAB} \psi . \tag{A.27}$$

Again, it is convenient to project this result onto the tetrad to find:

$$\begin{aligned}
 Q_{IJ} &= -e_I^\mu e_J^\nu Q_{\mu\nu} \\
 &= -\frac{i}{8} \bar{\psi} \left[(\eta_{IC} e_J^\nu + \eta_{JC} e_I^\nu) \partial_A e_{\nu B} + (\eta_{IC} \eta_{JB} + \eta_{JC} \eta_{IB}) \partial_\lambda e_A^\lambda \right] \gamma^{CAB} \psi .
 \end{aligned} \tag{A.28}$$

Notice now that the term proportional to the divergence $\partial_\lambda e_A^\lambda$ is symmetric in C and B , but is contracted with γ^{CAB} which is antisymmetric, so it cancels. We finally find:

$$\begin{aligned}
 Q_{IJ} &= -\frac{i}{8} \bar{\psi} \left[\eta_{IC} e_J^\nu \partial_A e_{\nu B} + \eta_{JC} e_I^\nu \partial_A e_{\nu B} \right] \gamma^{CAB} \psi \\
 &= -\frac{i}{8} \bar{\psi} \left[e_J^\nu \partial_A e_{\nu B} \gamma_I^{AB} + e_I^\nu \partial_A e_{\nu B} \gamma_J^{AB} \right] \psi \\
 &= -\frac{i}{8} \bar{\psi} \left[f_{ABJ} \gamma_I^{AB} + f_{ABI} \gamma_J^{AB} \right] \psi .
 \end{aligned} \tag{A.29}$$

Adding both contributions coming from K_2 we obtain:

$$\begin{aligned}
 P_{IJ} + Q_{IJ} &= -\frac{i}{8} \bar{\psi} \left[(f_{ABJ} + f_{JAB} + f_{AJB}) \gamma_I^{AB} + I \leftrightarrow J \right] \psi \\
 &= \frac{i}{8} \bar{\psi} \left[\omega_{ABJ} \gamma_I^{AB} + \omega_{ABI} \gamma_J^{AB} \right] \psi .
 \end{aligned}
 \tag{A.30}$$

And going back to spacetime indices:

$$\begin{aligned}
 (T_{\mu\nu})_2 &= \frac{i}{8} \bar{\psi} \left[\omega_{AB\mu} \gamma_\nu^{AB} + \omega_{AB\nu} \gamma_\mu^{AB} \right] \psi \\
 &= \frac{i}{8} \bar{\psi} \left[\omega_{AB\mu} \left\{ \gamma_\nu, \sigma^{AB} \right\} + \omega_{AB\nu} \left\{ \gamma_\mu, \sigma^{AB} \right\} \right] \psi \\
 &= \frac{i}{8} \bar{\psi} \left[\left\{ \gamma_\nu, \omega_{AB\mu} \sigma^{AB} \right\} + \left\{ \gamma_\mu, \omega_{AB\nu} \sigma^{AB} \right\} \right] \psi \\
 &= -\frac{i}{4} \bar{\psi} \left[\left\{ \gamma_\nu, \Gamma_\mu \right\} + \left\{ \gamma_\mu, \Gamma_\nu \right\} \right] \psi = -\frac{i}{2} \bar{\psi} \left\{ \gamma_{(\mu}, \Gamma_{\nu)} \right\} \psi .
 \end{aligned}
 \tag{A.31}$$

Adding now the contributions to the stress-energy tensor from K_1 and K_2 we find:

$$\begin{aligned}
 T_{\mu\nu} = (T_{\mu\nu})_1 + (T_{\mu\nu})_2 &= \frac{i}{2} \left[(\partial_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\partial_{\nu)} \psi) - \bar{\psi} \left\{ \gamma_{(\mu}, \Gamma_{\nu)} \right\} \psi \right] \\
 &= \frac{i}{2} \left[(\partial_{(\mu} \bar{\psi} - \bar{\psi} \Gamma_{(\mu}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\partial_{\nu)} \psi + \Gamma_{\nu)} \psi) \right] ,
 \end{aligned}
 \tag{A.32}$$

and finally:

$$T_{\mu\nu} = \frac{i}{2} \left[(\mathcal{D}_{(\mu} \bar{\psi}) \gamma_{\nu)} \psi - \bar{\psi} \gamma_{(\mu} (\mathcal{D}_{\nu)} \psi) \right] .
 \tag{A.33}$$

This is the final form of the stress-energy tensor for the Dirac field. Notice that there is no explicit contribution from the mass term in this tensor, which might seem strange at first glance, but such a contribution is implicitly there since ψ must satisfy Dirac’s equation.

We still need to show that the stress-energy tensor we just found does in fact satisfy the conservation equations $\nabla^\mu T_{\mu\nu} = 0$. Now, since this tensor involves spinors, and for tensors the spinor derivative reduces to the covariant derivative, what we must show is that we have $\mathcal{D}^\mu T_{\mu\nu} = 0$. Substituting the expression for $T_{\mu\nu}$ given in (VII.29), and ignoring constant factors, we have:

$$\begin{aligned}
 \mathcal{D}^\mu T_{\mu\nu} &\propto \mathcal{D}^\mu \left[(\mathcal{D}_{\mu} \bar{\psi}) \gamma_\nu \psi + (\mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi - \bar{\psi} \gamma_\mu (\mathcal{D}_\nu \psi) - \bar{\psi} \gamma_\nu (\mathcal{D}_\mu \psi) \right] \\
 &= (\mathcal{D}^\mu \mathcal{D}_\mu \bar{\psi}) \gamma_\nu \psi + (\mathcal{D}_\mu \bar{\psi}) \gamma_\nu (\mathcal{D}^\mu \psi) + (\mathcal{D}^\mu \mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi + (\mathcal{D}_\nu \bar{\psi}) \gamma_\mu (\mathcal{D}^\mu \psi) \\
 &\quad - (\mathcal{D}^\mu \bar{\psi}) \gamma_\mu (\mathcal{D}_\nu \psi) - \bar{\psi} \gamma_\mu (\mathcal{D}^\mu \mathcal{D}_\nu \psi) - (\mathcal{D}^\mu \bar{\psi}) \gamma_\nu (\mathcal{D}_\mu \psi) - \bar{\psi} \gamma_\nu (\mathcal{D}^\mu \mathcal{D}_\mu \psi) \\
 &= (\mathcal{D}^\mu \mathcal{D}_\mu \bar{\psi}) \gamma_\nu \psi + (\mathcal{D}^\mu \mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi + (\mathcal{D}_\nu \bar{\psi}) \gamma_\mu (\mathcal{D}^\mu \psi) \\
 &\quad - (\mathcal{D}^\mu \bar{\psi}) \gamma_\mu (\mathcal{D}_\nu \psi) - \bar{\psi} \gamma_\mu (\mathcal{D}^\mu \mathcal{D}_\nu \psi) - \bar{\psi} \gamma_\nu (\mathcal{D}^\mu \mathcal{D}_\mu \psi) ,
 \end{aligned}
 \tag{A.34}$$

where we used the fact that $\mathcal{D}_\mu \gamma^\nu = 0$, and in the last step we cancelled two terms that were clearly equal. In the first and last terms of the previous expression we can now use the Schroedinger–Dirac equation (VI.6) for ψ and $\bar{\psi}$ to show that those two terms again cancel. On the other hand, the third and fourth terms can be simplified using the Dirac equation to find:

$$\begin{aligned} (\mathcal{D}_\nu \bar{\psi}) \gamma_\mu (\mathcal{D}^\mu \psi) - (\mathcal{D}^\mu \bar{\psi}) \gamma_\mu (\mathcal{D}_\nu \psi) &= -im [(\mathcal{D}_\nu \bar{\psi}) \psi + \bar{\psi} (\mathcal{D}_\nu \psi)] \\ &= -im \mathcal{D}_\nu (\bar{\psi} \psi). \end{aligned} \tag{A.35}$$

We then have:

$$\mathcal{D}^\mu T_{\mu\nu} \propto (\mathcal{D}^\mu \mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi - \bar{\psi} \gamma_\mu (\mathcal{D}^\mu \mathcal{D}_\nu \psi) - im \mathcal{D}_\nu (\bar{\psi} \psi). \tag{A.36}$$

For the first two terms in the previous expression we can now use the commutation relation for the spinor derivatives (V.88):

$$\begin{aligned} (\mathcal{D}^\mu \mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi - \bar{\psi} \gamma_\mu (\mathcal{D}^\mu \mathcal{D}_\nu \psi) &= g^{\mu\lambda} [(\mathcal{D}_\lambda \mathcal{D}_\nu \bar{\psi}) \gamma_\mu \psi - \bar{\psi} \gamma_\mu (\mathcal{D}_\lambda \mathcal{D}_\nu \psi)] \\ &= \left(\mathcal{D}_\nu \mathcal{D}_\lambda \bar{\psi} - \frac{1}{2} R_{AB\lambda\nu} \sigma^{AB} \bar{\psi} \right) \gamma^\lambda \psi - \bar{\psi} \gamma^\lambda \left(\mathcal{D}_\nu \mathcal{D}_\lambda \psi - \frac{1}{2} R_{AB\lambda\nu} \sigma^{AB} \psi \right) \\ &= \mathcal{D}_\nu [(\mathcal{D}_\lambda \bar{\psi}) \gamma^\lambda] \psi - \bar{\psi} \mathcal{D}_\nu (\gamma^\lambda \mathcal{D}_\lambda \psi) \\ &= im [(\mathcal{D}_\nu \bar{\psi}) \psi + \bar{\psi} (\mathcal{D}_\nu \psi)] \\ &= im \mathcal{D}_\nu (\bar{\psi} \psi), \end{aligned} \tag{A.37}$$

where we again used Dirac’s equation. We finally find:

$$\mathcal{D}^\mu T_{\mu\nu} = 0. \tag{A.38}$$

We then see that the stress-energy tensor (VII.29) does indeed satisfy the conservation laws.

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Declarations

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