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Twisted rational r-matrices and algebraic Bethe ansatz: Application to generalized Gaudin and Richardson models

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Abstract

In the present paper we develop the algebraic Bethe ansatz approach to the case of non-skew-symmetric $gl(2) \otimes gl(2)$-valued Cartan-non-invariant classical r-matrices with spectral parameters. We consider the two families of these r-matrices, namely, the two non-standard rational r-matrices twisted with the help of second order automorphisms and realize the algebraic Bethe ansatz method for them. We study physically important examples of the Gaudin-type and BCS-type systems associated with these r-matrices and obtain explicitly the Bethe vectors and the spectrum for the corresponding quantum hamiltonians in terms of solutions of Bethe equations.

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1. Introduction

1.1. Classical r-matrices and quantum integrable systems

Quantum integrable systems are of utmost importance for the quantum mathematical physics. In some sense, the most interesting quantum solvable systems are the ones admitting Lax representation, i.e. the so-called Lax-integrable models. These systems can be further divided into the two classes: the ones for which the relevant Lax algebra is linear and the ones for which the Lax algebra is quadratic. The commutation relations of linear and quadratic Lax algebras are determined by classical r-matrices and quantum R-matrices, respectively.

For a long while it was commonly believed that integrable systems associated with linear Lax algebras are just the artefacts of the integrable systems associated with quadratic algebras i.e. quantum groups. That is to say that for quantum integrability are pertinent only the *skew-symmetric* classical r-matrices $r_{12}(u_1,u_2) \in \mathfrak{g} \otimes \mathfrak{g}$, such that $r_{12}(u_1,u_2) = -r_{21}(u_2,u_1)$, and the corresponding linear Lax algebras. Here $\mathfrak{g}$ is (semi)simple Lie algebra or reductive Lie algebra $gl(n)$.

In the papers of the first author [17,18,21,20] it was shown that one can associate quantum integrable systems also with more general, non-skew-symmetric, classical r-matrices $r_{12}(u_1,u_2)$ satisfying instead of the ordinary classical Yang-Baxter equation the generalized classical Yang-Baxter equation [7–9]:

$$[r_{12}(u_1,u_2), r_{13}(u_1,u_3)] = [r_{23}(u_2,u_3), r_{12}(u_1,u_2)] - [r_{32}(u_3,u_2), r_{13}(u_1,u_3)]. \tag{1.1}$$

Notice that the variety of solutions to the equation (1.1) is wider than the variety of solutions to the ordinary classical Yang-Baxter equation [3,4], in particular, it contains the set of skew-symmetric classical r-matrices as a subset. It also contains the "twisted" non-skew-symmetric classical r-matrices of the following form:

$$r^{\sigma}_{12}(u_1,u_2) = r_{12}(u_1,u_2) - \sigma_2 r_{12}(u_1,u_2^{\sigma}), \tag{1.2}$$

where $r_{12}(u_1,u_2)$ is a skew-symmetric classical r-matrix such that

$$r_{12}(u_1,u_2) = -\sigma_1\sigma_2 r_{12}(u_1^{\sigma},u_2^{\sigma}),$$

here σ is an involutive automorphism of $\mathfrak{g}$ and $u \to u^{\sigma}$ is an involution in $\mathbb{C}$. The variety of solutions to the equation (1.1) contains also many other non-skew-symmetric classical r-matrices [16,17,20].

Any solution to the equation (1.1) yields different quantum integrable models, in particular, the generalized Gaudin models [16–18] and generalized Gaudin models in an external magnetic field [21], where the role of the "integrable" magnetic field is played by the shift element $c(u)$ satisfying the following linear equation:

$$[r_{12}(u_1,u_2), c_1(u_1)] - [r_{21}(u_2,u_1), c_2(u_2)] = 0. \tag{1.3}$$

In the case of the Lie algebra $\mathfrak{g} = gl(2)$, considered in the present paper, the corresponding mutually commuting hamiltonians have the following explicit form:

$$\hat{H}_n = \sum_{m=1, m\neq n}^{N} \sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m,\nu_n)\hat{S}^{(m)}_{ij}\hat{S}^{(n)}_{kl} +$$

$$+\frac{1}{2}\sum_{i,j,k,l=1}^{2} r_0^{ij,kl}(\nu_n,\nu_n)(\hat{S}^{(n)}_{ij}\hat{S}^{(n)}_{kl}+\hat{S}^{(n)}_{kl}\hat{S}^{(n)}_{ij})+\sum_{i,j=1}^{2} c_{ij}(\nu_n)\hat{S}^{(n)}_{ij}, \quad (1.4)$$

where $r_{12}(u,v) = \sum_{i,j,k,l=1}^{2} r^{ij,kl}(u,v) X_{ij}\otimes X_{kl}$ is a classical r-matrix, $r^{(0)}_{12}(u,v) = \sum_{i,j,k,l=1}^{2} r_0^{ij,kl}(u,v) X_{ij}\otimes X_{kl}$ its regular part, $c(u)=\sum_{i,j=1}^{2} c_{ij}(u)X_{ij}$ is a shift element, X_{ij}, $i,j\in 1,2$ is a standard basis of $gl(2)$: $(X_{ij})_{\alpha\beta}=\delta_{i\alpha}\delta_{j\beta}$, $\hat{S}^{(n)}_{kl}=\pi^{(n)}(X_{ij})$ and $\pi^{(n)}$, $n\in\overline{1,N}$ are representations of the Lie algebra $gl(2)$.

The physical importance of the generalized Gaudin hamiltonians (1.4) is based on the fact that upon fermionization of the spin operators $\hat{S}^{(n)}_{kl}$ they produce integrable fermion hamiltonians of the BCS-Richardson-type used in nuclear physics [2], in the theory of quantum dots and small metallic grains [11–13] etc. The generalized Gaudin models and non-skew-symmetric classical r-matrices have attracted considerable interest [33,35,37–40,32,34,36,41].

1.2. Aims of the paper

The aim of this paper is two-fold. The first aim is to continue the study of the generalized Gaudin models based on the non-skew-symmetric classical r-matrices, their generalized Richardson's counterparts and to specify physically interesting ones. This study was initiated by the first author in [16,17] for the case of the Gaudin-type models and in [22] for the case of the Richardson-type models. Moreover, the cases of r-matrices which are diagonal in some natural basis, have been studied in the papers [22–24,27], including their Richardson-type systems and corresponding spectral problem. With the present paper we begin the study of the general non-diagonal, non-Cartan-invariant non-skew-symmetric r-matrices, their Richardson-type models and their spectra. For this purpose we consider two non-standard rational *skew-symmetric* r-matrices written as follows [4,10]:

$$r_{12}(u,v)=\frac{\sum_{i,j=1}^{2} X_{ij}\otimes X_{ji}}{u-v}+c\big(v(X_{11}-X_{22})\otimes X_{21}-uX_{21}\otimes(X_{11}-X_{22})\big), \quad (1.5a)$$

$$r_{12}(u,v)=\frac{\sum_{i,j=1}^{2} X_{ij}\otimes X_{ji}}{u-v}+c\big((X_{11}-X_{22})\otimes X_{21}-X_{21}\otimes(X_{11}-X_{22})\big), \quad (1.5b)$$

where c is an arbitrary parameter. Furthermore, we construct their twisted counterparts (1.2) having, up to an equivalence, the following explicit form:

$$r^{\sigma}_{12}(u,v)=\frac{\sum_{i,j=1}^{2} X_{ij}\otimes X_{ji}}{u-v}+c(X_{11}-X_{22})\otimes X_{21}, \quad (1.6a)$$

$$r^{\sigma}_{12}(u,v)=\frac{v^2}{u^2-v^2}(X_{11}\otimes X_{11}+X_{22}\otimes X_{22})+\frac{uv}{u^2-v^2}(X_{12}\otimes X_{21}+X_{21}\otimes X_{12})+ \\ +cv(X_{11}-X_{22})\otimes X_{21}, \quad (1.6b)$$

here, again, c is an arbitrary parameter. These r-matrices may also be viewed as two one-parametric families of non-skew-symmetric, non-Cartan-invariant deformations of standard skew-symmetric rational and non-skew-symmetric trigonometric r-matrix [24]. The corresponding Richardson-type models will be one-parametric deformations of the Richardson model of the s-type [2] and $p_x + ip_y$-type [23,24], [28]. In particular, the r-matrix (1.6b) yields the following integrable Richardson-type fermion hamiltonian:

$$\hat{H} = \sum_{n=1}^{N} \epsilon_n (c^{\dagger}_{n,\epsilon'} c_{n,\epsilon'} + c^{\dagger}_{n,\epsilon} c_{n,\epsilon}) + g \sum_{m,n=1}^{N} \sqrt{\epsilon_n \epsilon_m} c^{\dagger}_{m,\epsilon'} c^{\dagger}_{m,\epsilon} c_{n,\epsilon} c_{n,\epsilon'} + \\ + cg \sum_{m=1}^{N} (c^{\dagger}_{m,\epsilon'} c_{m,\epsilon'} + c^{\dagger}_{m,\epsilon} c_{m,\epsilon}) \sum_{n=1}^{N} \sqrt{\epsilon_n} c_{n,\epsilon} c_{n,\epsilon'}, \quad (1.7)$$

constituting a one-parametric family of deformations of the $p_x + ip_y$ reduced BCS hamiltonian [23,24]. Here $c^{\dagger}_{n,\epsilon}$, $c_{m,\epsilon'}$, $m, n \in \overline{1, N}$, $\epsilon, \epsilon' \in \{+, -\}$ are standard fermion creation-anihilation operators, ϵ_n is a free energy of the n-th fermion and $g_{nm} = g\sqrt{\epsilon_n \epsilon_m}$ is a pairing interaction strength.

Notice that non-standard rational skew-symmetric r-matrices (1.5a)-(1.5b) have no diagonal shift elements satisfying the equation (1.3). Therefore one can not associate with them Richardson-type hamiltonians possessing the kinetic terms. This confirms the observation of the first author [27] that non-skew-symmetric classical r-matrices are more pertinent to the constructions of the integrable Richardson-type hamiltonians than the skew-symmetric ones.

Now we come to the second aim of our paper, which is to the develop methods for complete integrability of the quantum systems related to the classical r-matrices.

1.3. Algebraic Bethe ansatz

The exact solvability of the quantum integrable models, i.e. the diagonalization of the corresponding Hamiltonian and related integrals of motion, can be obtained by several methods. For the Lax integrable models the most important method is the algebraic Bethe ansatz. For the quadratic Lax algebras and Cartan-invariant quantum R-matrices the Bethe ansatz technique was proposed and developed in the papers of Leningrad school of mathematical physics (see [14,15] for the reviews), for the linear Lax algebra case and the Cartan-invariant *skew-symmetric* classical r-matrices it has been proposed in [1] and developed in [5,6].

In the papers of the first author the algebraic Bethe ansatz approach was applied on the Lax-integrable systems governed by Cartan-invariant non-skew-symmetric classical r-matrices: for the case of Lie algebras $\mathfrak{g} = sl(2), gl(2)$ in [19,23] and for the case of Lie algebras $\mathfrak{g} = gl(n)$ in [25,26]. The Cartan-non-invariant case is investigated much less [32,34]. Thus the second aim of this paper is to develop the algebraic Bethe ansatz for the Lax-integrable systems with linear Lax algebras governed by *Cartan-non-invariant non-skew-symmetric* classical r-matrices $r_{12}(u, v)$.

For the Cartan-invariant case and Lie algebras $\mathfrak{g} = sl(2), gl(2)$ the algebraic Bethe ansatz is implemented to the non-skew-symmetric case just in the same manner as in the skew-symmetric case [5]. In more details, the eigenvectors of the quantum hamiltonians are given by:

$$|v_1, v_2, ..., v_M\rangle = \hat{B}(v_1)\hat{B}(v_2)....\hat{B}(v_M)|0\rangle,$$

where $|0\rangle$ is the vacuum vector such that

$$\hat{C}(u)|0\rangle = 0, \quad \hat{A}(u)|0>= \Lambda_1(u)|0\rangle, \quad \hat{D}(u)|0>= \Lambda_2(u)|0\rangle, \tag{1.8}$$

where

$$\hat{A}(u) = \hat{L}^{11}(u), \quad \hat{D}(u) = \hat{L}^{22}(u), \quad \hat{B}(u) = \hat{L}^{21}(u), \quad \hat{C}(u) = \hat{L}^{12}(u),$$

$\hat{L}(u) = \sum\limits_{i,j=1}^{2} \hat{L}^{ij}(u) X_{ij}$ is the Lax matrix and rapidities v_i satisfy a set of Bethe-type equations.

In order to apply the algebraic Bethe ansatz for the considered Cartan-non-invariant r-matrices (1.6a)-(1.6b) we use the approach of the second author to the skew-symmetric classical r-matrix (1.5a)-(1.5b) in [29,30] with the assumption that non-skew-symmetric case is organized analogously. Namely, for the Bethe states we take the vectors of the following from

$$|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle,$$

where the vacuum vector satisfies the conditions (1.8) and the operators $\hat{B}_k(v_k)$ are defined as follows:

$$\hat{B}_k(v_k) = \hat{L}^{21}(v_k) + (2k-1) f(v_k)\mathrm{Id},$$

here $f(v_k) = c$ in the case of the r-matrix (1.6a) and $f(v_k) = cv_k$ in the case of the r-matrix (1.6b).

Moreover, the form of the Bethe equations that guarantee the diagaonalization of the generating functions of the quantum Hamiltonians for the r-matrices (1.6a)-(1.6b) — as the for the skew-symmetric r-matrices (1.5a)-(1.5b) — is the same as in the undeformed $c = 0$ case. Thus we have that the one-parametric families of the obtained deformed BCS-Richardson's Hamiltonians of s- and $p_x + ip_y$-type have the same spectrum as the non-deformed Hamiltonians of the same type, but their eigenvectors are different and depend on the deformation parameter c.

1.4. The structure of the paper

The structure of the paper is the following: in the Section 2 we outline some general facts about the quantum integrable systems and non-skew-symmetric classical r-matrices. In the Sections 3 and 4 we consider the non-skew-symmetric r-matrices (1.6a)-(1.6b) correspondingly, their Gaudin-type and Richardson-type models and the implementation of the algebraic Bethe ansatz for them.

2. Quantum integrable systems and classical r-matrices

In this section we will briefly review the relation of the theory of general non-skew-symmetric classical r-matrices with spectral parameters with the theory of quantum integrable systems [16–19]. Although the constructions presented in this section hold true for any simple (reductive) Lie algebra, we will state them in the case of the reductive Lie algebra $\mathfrak{g} = gl(2)$.

2.1. Definition and notations

Let $\mathfrak{g} = gl(2)$ be the Lie algebra of the general linear group over the field of complex numbers. Let $X_{ij}, i, j = 1, 2$ be a standard basis in $gl(2)$ with the commutation relations:

$$[X_{ij}, X_{kl}] = \delta_{kj} X_{il} - \delta_{il} X_{kj}. \tag{2.1}$$

Definition 1. A function of two complex variables $r(u_1, u_2)$ with values in the tensor square of the algebra $\mathfrak{g} = gl(2)$ is called a classical r-matrix if it satisfies the following generalized classical Yang-Baxter equation [7,9,8]

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] - [r_{32}(u_3, u_2), r_{13}(u_1, u_3)], \quad (2.2)$$

where $r_{12}(u_1, u_2) \equiv \sum_{i,j,k,l=1}^{2} r^{ij,kl}(u_1, u_2) X_{ij} \otimes X_{kl} \otimes 1$, $r_{13}(u_1, u_3) \equiv \sum_{i,j,k,l=1}^{2} r^{ij,kl}(u_1, u_3) X_{ij} \otimes 1 \otimes X_{kl}$, etc. and $r^{ij,kl}(u, v)$ are matrix elements of the r-matrix $r(u, v)$.

Remark 1. In the case of skew-symmetric r-matrices, i.e. when $r_{12}(u_1, u_2) = -r_{21}(u_2, u_1)$ the generalized classical Yang-Baxter equation reduces to the proper classical Yang-Baxter equation [3,4]:

$$[r_{12}(u_1, u_2), r_{13}(u_1, u_3)] = [r_{23}(u_2, u_3), r_{12}(u_1, u_2)] + [r_{23}(u_2, u_3), r_{13}(u_1, u_3)]. \quad (2.3)$$

In the present paper we are interested in the meromorphic r-matrices that possess the decomposition:

$$r(u_1, u_2) = \frac{\Omega}{u_1 - u_2} + r_0(u_1, u_2), \quad (2.4)$$

where $r_0(u_1, u_2)$ is a regular function with values in $gl(2) \otimes gl(2)$, $\Omega = \sum_{i,j=1}^{2} X_{ij} \otimes X_{ji}$ is tensor Casimir.

For the subsequent we will also need the following definition:

Definition 2. A $gl(2)$-valued function $c(u) = \sum_{i,j=1}^{2} c^{ij}(u) X_{ij}$ of one complex variable is called generalized shift element if it satisfies the following equation:

$$[r_{12}(u_1, u_2), c_1(u_1)] - [r_{21}(u_2, u_1), c_2(u_2)] = 0. \quad (2.5)$$

2.2. *The σ-twisted classical r-matrices with spectral parameters*

Let $r_{12}(u_1, u_2)$ be a skew-symmetric r-matrix i.e. a non-degenerate meromorphic solution of the proper classical Yang-Baxter equation (2.3). Let σ be an automorphism of $\mathfrak{g} = gl(2)$ of the second order: $\sigma^2 = 1$.

Let us assume that the following anti-invariance condition is satisfied:

$$r_{12}(u_1, u_2) = -\tilde{\sigma}_1 \tilde{\sigma}_2 r_{12}(u_1, u_2) = -\sigma_1 \sigma_2 r_{12}(u_1^{\sigma}, u_2^{\sigma}), \quad (2.6)$$

where $\tilde{\sigma}$ is an extension of σ onto the algebra of meromorphic functions, i.e. $\tilde{\sigma}(X(u)) = \sigma(X(u^{\sigma}))$, $\sigma : u \to u^{\sigma}$ is a certain involution in $\mathbb{C}$, e.g. $u^{\sigma} = -u$ or $u^{\sigma} = u^{-1}$. The index $i \in 1, 2$ in σ_i refers to the component of the tensor product in which the automorphism σ acts.

Taking into account the definitions above, it is possible to show [18,19] that the following tensor

$$r^{\sigma}_{12}(u_1, u_2) = r_{12}(u_1, u_2) - \sigma_2 r_{12}(u_1, u_2^{\sigma}) \quad (2.7)$$

is a solution of the generalized classical Yang-Baxter equation (2.2), i.e. is a non-skew-symmetric classical r-matrix with spectral parameters. A particular case of such a non-skew-symmetric classical r-matrices will be the main interest of the present article.

2.3. Lie algebra of Lax operators

In the space of certain $gl(2)$ valued functions of the complex parameter u, using the classical r-matrix $r(u_1, u_2)$, it is possible define the tensor Lie bracket [7,9,8]

$$[\hat{L}_1(u_1), \hat{L}_2(u_2)] = [r_{12}(u_1, u_2), \hat{L}_1(u_1)] - [r_{21}(u_2, u_1), \hat{L}_2(u_2)], \tag{2.8}$$

where $\hat{L}_1(u_1) = L(u_1) \otimes 1$, $\hat{L}_2(u_2) = 1 \otimes L(u_2)$, $\hat{L}(u) = \sum\limits_{i,j=1}^{3} \hat{L}^{ij}(u) X_{ij}$, $r_{21}(u_2, u_1) = P_{12} r_{12}(u_2, u_1) P_{12}$, and P_{12} is the permutation operator which interchanges the first and second spaces in the tensor product.

Tensor bracket (2.8) between the Lax matrices $\hat{L}_1(u_1)$ and $\hat{L}_2(u_2)$ is a symbolic notation for the Lie brackets between their matrix entries. Explicitly, in terms of matrix entries the bracket (2.8) takes the following form

$$[\hat{L}^{ij}(u), \hat{L}^{kl}(v)] = \sum_{s=1}^{2} (r^{is,kl}(u,v) \hat{L}^{sj}(u) - r^{sj,kl}(u,v) \hat{L}^{is}(u)) - \\ - \sum_{s=1}^{2} (r^{ks,ij}(v,u) \hat{L}^{sl}(v) - r^{sl,ij}(v,u) \hat{L}^{ks}(v)). \tag{2.9}$$

Remark 2. From the explicit form of the Lie bracket (2.9) it follows that the operators $(\hat{L}^{11}(u) - \hat{L}^{22}(u))$, $\hat{L}^{12}(u)$, $\hat{L}^{21}(u)$ span a subalgebra and an ideal in the Lax algebra spanned over the elements $\hat{L}^{11}(u)$, $\hat{L}^{22}(u)$, $\hat{L}^{12}(u)$, $\hat{L}^{21}(u)$, independently on the form of the classical r-matrix.

2.4. Generating functions of quantum integrals

Let us now consider the following linear function on the Lax algebra

$$\hat{\tau}^{(1)}(u) = \hat{L}^{11}(u) + \hat{L}^{22}(u). \tag{2.10}$$

From the tensor form of the Poisson brackets (2.8) it follows that it generates commutative subalgebra. Moreover, the following Proposition is valid

Proposition 2.1. *The function $\hat{\tau}^{(1)}(u)$ is a generating function of the center of the Lax algebra (2.8) if and only if the following conditions on the components of the r-matrix hold true*

$$\sum_{k=1}^{2} r^{ii,kk}(u,v) = \sum_{k=1}^{2} r^{jj,kk}(u,v), \qquad i, j \in 1, 2. \tag{2.11a}$$

$$\sum_{k=1}^{2} r^{ij,kk}(u,v) = 0, \qquad i, j \in 1, 2, \ i \neq j. \tag{2.11b}$$

Proof. The statement follows from the brackets (2.9).

By the direct calculation one can also prove the following Proposition

Proposition 2.2. *The $gl(2)\otimes gl(2)$-valued r-matrices $r_{12}(u,v)$ that satisfy the conditions (2.11) can be brought to the $sl(2)\otimes sl(2)$-valued form by the equivalence transformation*[1]*:*

$$r_{12}(u,v)\to r_{12}(u,v)+1\otimes X(u,v), \tag{2.12}$$

where $X(u,v)$ takes the value in $gl(2)$.

Let us now consider the following quadratic function on the Lax algebra

$$\hat{\tau}^{(2)}(u)=\frac{1}{2}\big(\hat{L}^{11}(u)\hat{L}^{11}(u)+\hat{L}^{22}(u)\hat{L}^{22}(u)+\hat{L}^{12}(u)\hat{L}^{21}(u)+\hat{L}^{21}(u)\hat{L}^{12}(u)\big). \tag{2.13}$$

We can state the following Theorem

Theorem 2.1. *Let the classical r-matrix $r(u,v)$ satisfy the conditions (2.4) and (2.11). Then*

$$[\hat{\tau}^{(2)}(u),\hat{\tau}^{(2)}(v)]=0. \tag{2.14}$$

Proof. In order to prove the theorem, we note that in the new notations $\hat{H}(u)=\frac{1}{2}(\hat{L}^{11}(u)-\hat{L}^{22}(u))$, $\hat{C}(u)=\hat{L}^{12}(u)$, $\hat{B}(u)=\hat{L}^{21}(u)$ corresponding to the $sl(2)$ basis the generating function $\hat{\tau}^{(2)}(u)$ is re-written in the following way

$$\hat{\tau}^{(2)}(u)=\frac{1}{2}\big(\frac{1}{2}\hat{H}^2(u)+\hat{C}(u)\hat{B}(u)+\hat{B}(u)\hat{C}(u)\big)+\frac{1}{4}(\hat{\tau}^{(1)}(u))^2.$$

Now we note that under the conditions of the Theorem $\hat{\tau}^{(1)}(u)$ is a central element, hence it commutes with everything and its presence or absence does not influence the relation (2.14). Moreover, under the conditions of the Theorem by the virtue of the Proposition 2.1 and the arguments of the previous subsection, the Lax algebra (2.9) is a direct sum of $sl(2)$-valued Lax algebra with the generating functions of the basis being $\hat{H}(u)$, $\hat{B}(u)$ and $\hat{C}(u)$ and a center with the generating function being $\hat{\tau}^{(1)}(u)$. Finally, by the virtue of the Proposition 2.2 the r-matrix $r_{12}(u,v)$ is equivalent to the $sl(2)\otimes sl(2)$-valued one. That is why the proof of the commutativity of the generating functions $\hat{\tau}^{(2)}(u)$ and $\hat{\tau}^{(2)}(v)$ is reduced to the proof of the commutativity of the generating functions $\hat{\tau}(u)$ and $\hat{\tau}(v)$, where

$$\hat{\tau}(u)=\big(\frac{1}{2}\hat{H}^2(u)+\hat{C}(u)\hat{B}(u)+\hat{B}(u)\hat{C}(u)\big)$$

is a quadratic generating function on $sl(2)$-valued Lax algebra with the $sl(2)\otimes sl(2)$-valued r-matrix. Finally, to prove the theorem we recall that the proof of the commutativity of $\hat{\tau}(u)$ and $\hat{\tau}(v)$ under the condition (2.4) was done in the paper [19].

[1] The transformation that preserves the Lie brackets (2.8) is called the equivalence transformation.

2.5. Generalized Gaudin models in external magnetic field

2.5.1. General case

Let $\hat{S}_{ij}^{(m)}, i, j = 1, 2, m = 1, 2, ..., N$ be linear operators in some Hilbert space that span Lie algebra isomorphic to $gl(2)^{\oplus N}$ with the commutation relations

$$[\hat{S}_{ij}^{(m)}, \hat{S}_{kl}^{(n)}] = \delta^{nm}(\delta_{kj}\hat{S}_{il}^{(m)} - \delta_{il}\hat{S}_{kj}^{(m)}). \tag{2.15}$$

Let us fix N distinct points of the complex plain ν_m, $m \in \overline{1, N}$. It is possible to introduce the following quantum Lax operator [16,17,21]

$$\hat{L}(u) = \sum_{i,j=1}^{2} \hat{L}^{ij}(u)X_{ij} \equiv \sum_{m=1}^{N}\sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m, u)\hat{S}_{ij}^{(m)}X_{kl} + \sum_{i,j=1}^{2} c_{ij}(u)X_{ij}, \tag{2.16}$$

where $c(u) = \sum\limits_{i,j=1}^{2} c_{ij}(u)X_{ij}$ is a shift element satisfying the equation (2.5). Using generalized classical Yang-Baxter equation one can show that it satisfies a linear r-matrix algebra (2.8). This quantum Lax operator corresponds to the generalized $gl(2)$-Gaudin system in the external magnetic field.

As our next step, we apply the results of the previous subsection to the Lax operators of the generalized Gaudin systems. A direct calculation yields the explicit form of the corresponding generating functions:

$$\hat{\tau}^{(1)}(u) = \sum_{l=1}^{N}\sum_{i,j,k=1}^{2} r^{ij,kk}(\nu_l, u)\hat{S}_{ij}^{(l)} + \sum_{i=1}^{2} c_{ii}(u). \tag{2.17}$$

Under the conditions (2.11) on the r-matrix the generating function (2.17) is further reduced to the following form:

$$\hat{\tau}^{(1)}(u) = \sum_{l=1}^{N}(\sum_{k=1}^{2} r^{11,kk}(\nu_l, u))(\hat{S}_{11}^{(l)} + \hat{S}_{22}^{(l)}) + \sum_{i=1}^{2} c_{ii}(u),$$

i.e. it becomes the linear combinations of the linear Casimirs of $gl(2)$.

The second order generating function is given by

$$\begin{aligned}\hat{\tau}^{(2)}(u) = \frac{1}{2}\Big(&\sum_{n,m=1}^{N}\sum_{i,j,k,l,p,q=1}^{2} r^{ij,kl}(\nu_m, u)r^{pq,lk}(\nu_n, u)\hat{S}_{ij}^{(m)}\hat{S}_{pq}^{(n)} \\ &+ \sum_{m=1}^{N}\sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m, u)\hat{S}_{ij}^{(m)}c_{lk}(u) + \\ &+ \sum_{n=1}^{N}\sum_{k,l,p,q=1}^{2} r^{pq,lk}(\nu_n, u)\hat{S}_{pq}^{(n)}c_{kl}(u) + \sum_{k,l=1}^{2} c_{kl}(u)c_{lk}(u)\Big).\end{aligned}$$

Maybe somewhat more transparent are the residues of $\hat{\tau}^{(2)}(u)$ at $u = \nu_n$:

$$\hat{H}_n = -\operatorname*{Res}_{u=\nu_n} \hat{\tau}^{(2)}(u), \quad n \in \overline{1, N}.$$

They have the following form

$$\hat{H}_n = \sum_{m=1, m\neq n}^{N} \sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m, \nu_n)\hat{S}^{(m)}_{ij}\hat{S}^{(n)}_{kl} + \\ + \frac{1}{2}\sum_{i,j,k,l=1}^{2} r_0^{ij,kl}(\nu_n, \nu_n)(\hat{S}^{(n)}_{ij}\hat{S}^{(n)}_{kl} + \hat{S}^{(n)}_{kl}\hat{S}^{(n)}_{ij}) + \sum_{i,j=1}^{2} c_{ij}(\nu_n)\hat{S}^{(n)}_{ij}, \quad (2.18)$$

where $r_0^{ij,kl}(\nu_n, \nu_n)$ are the matrix elements of the regular part of the classical r-matrix $r(u, v)$ at the point $u = v = \nu_n$. The Hamiltonians (2.18) are the generalized Gaudin Hamiltonians corresponding to the $gl(2) \otimes gl(2)$-valued r-matrix [16,17]. By the virtue of the results of the previous subsection as well as of the general results of [17,21] they mutually commute

$$[\hat{H}_m, \hat{H}_n] = 0, \quad \forall m, n \in \overline{1, N}.$$

2.5.2. The σ-twisted case

The non-skew-symmetric classical r-matrices we will consider in the present paper are the σ-twisted classical r-matrices (2.7). The corresponding Lax matrix of the Gaudin-type models in the external magnetic field can be written as follows

$$\hat{L}(u) = \sum_{m=1}^{N} \sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m, u)\hat{S}^{(m)}_{ij} X_{kl} - \sum_{m=1, m\neq n}^{N} \sum_{i,j,k,l=1}^{2} (\sigma_2 \cdot r)^{ij,kl}(\nu_m, u^\sigma)\hat{S}^{(m)}_{ij} X_{kl} + \\ + \sum_{i,j=1}^{2} c_{ij}(u) X_{ij}. \quad (2.19)$$

In this case, the Gaudin-type Hamiltonians (2.18) have the following form

$$\hat{H}_n = \sum_{m=1, m\neq n}^{N} \sum_{i,j,k,l=1}^{2} r^{ij,kl}(\nu_m, \nu_n)\hat{S}^{(m)}_{ij}\hat{S}^{(n)}_{kl} - \sum_{m=1, m\neq n}^{N} \sum_{i,j,k,l=1}^{2} (\sigma_2 \cdot r)^{ij,kl}(\nu_m, \nu_n^\sigma)\hat{S}^{(m)}_{ij}\hat{S}^{(n)}_{kl} \\ - \frac{1}{2}\sum_{i,j,k,l=1}^{2} (\sigma_2 \cdot r)_0^{ij,kl}(\nu_n, \nu_n^\sigma)(\hat{S}^{(n)}_{ij}\hat{S}^{(n)}_{kl} + \hat{S}^{(n)}_{kl}\hat{S}^{(n)}_{ij}) + \sum_{i,j=1}^{2} c_{ij}(\nu_n)\hat{S}^{(n)}_{ij}, \quad (2.20)$$

where $r^{ij,kl}(\nu_m, u)$ in the formula (2.20) are the components of the initial skew-symmetric classical r-matrix from which the r-matrix (2.7) is obtained. By the virtue of all we have stated above as well as the results of [17,18,21], the Hamiltonians (2.20) mutually commute

$$[\hat{H}_m, \hat{H}_n] = 0, \quad \forall m, n \in \overline{1, N}.$$

Remark 3. In all the applications below we will consider only the case where the involution is given by $\nu_n^\sigma = -\nu_n$.

2.6. Integrable fermion models

Base on integrable quantum spin chains it is possible to derive integrable fermion systems. To this end it is necessary to consider the realization of the corresponding spin operators in terms of fermion creation-anihilation operators. In other words, it is necessary to obtain the fermionization of the underlying Lie algebra $gl(2)^{\oplus N}$, where N is the length of the chain.

2.6.1. Fermionization

Here we will consider only the simplest fermionization of the Lie algebra $gl(2)^{\oplus N}$ corresponding to the case when all the Lie algebras $gl(2)$ in the direct sum have the representation with the lowest weight $\lambda = (0, 1)$.

More explicitly, let $c_{j,\epsilon'}$, $c^{\dagger}_{i,\epsilon}$, $i, j \in \overline{1, N}$, $\epsilon, \epsilon' \in \{+, -\}$ be fermion creation-anihilation operators, then we have

$$c^{\dagger}_{i,\epsilon} c_{j,\epsilon'} + c_{j,\epsilon'} c^{\dagger}_{i,\epsilon} = \delta_{\epsilon\epsilon'}\delta_{ij}, \quad c^{\dagger}_{i,\epsilon} c^{\dagger}_{j,\epsilon'} + c^{\dagger}_{j,\epsilon'} c^{\dagger}_{i,\epsilon} = 0, \quad c_{i,\epsilon} c_{j,\epsilon'} + c_{j,\epsilon'} c_{i,\epsilon} = 0. \tag{2.21}$$

By direct calculation it is possible to show that the following formulae

$$\hat{S}^{(j)}_{12} = c^{\dagger}_{j,\epsilon'} c^{\dagger}_{j,\epsilon}, \ \hat{S}^{(j)}_{21} = c_{j,\epsilon} c_{j,\epsilon'}, \ \hat{S}^{(j)}_{11} = c^{\dagger}_{j,\epsilon'} c_{j,\epsilon'}, \ \hat{S}^{(j)}_{22} = c_{j,\epsilon} c^{\dagger}_{j,\epsilon}, \ i, j \in 1, 2, ...N, \ \epsilon \neq \epsilon', \tag{2.22}$$

provide realization of the Lie algebra $gl(2)^{\oplus N}$ with the lowest weight $\lambda_j = (0, 1)$, $j \in \overline{1, N}$. Here operators $c_{j,\epsilon}$ are chosen to annihilate, and operators $c^{\dagger}_{j,\epsilon}$ are chosen to create fermion in the state j, ϵ.

Remark 4. Note, that after the restriction to the subalgebra $sl(2)$ and after the identification $\hat{S}^{(j)}_{+} = \hat{S}^{(j)}_{12}$, $\hat{S}^{(j)}_{-} = \hat{S}^{(j)}_{21}$, $2\hat{S}^{(j)}_{3} = \hat{S}^{(j)}_{11} - \hat{S}^{(j)}_{22}$ we obtain the well-known fermionization of the Lie algebra $sl(2)$ [11,12].

2.6.2. BCS-type Hamiltonians

In order to construct some interesting integrable fermion Hamiltonian $\hat{H}$ of the BCS-type it is necessary to apply the above fermionization formulae to the certain combination of the generalized Gaudin Hamiltonians, Casimir functions and, possibly, some linear integrals. The concrete form of the Hamiltonian $\hat{H}$ will depend on the underlying classical r-matrix. We will consider some explicit examples in the next sections.

3. The rational r-matrix

3.1. Twisted non-standard rational r-matrix

Let us consider skew-symmetric rational r-matrix of the following form [10]

$$r_{12}(u, v) = \frac{\sum\limits_{i,j=1}^{2} X_{ij} \otimes X_{ji}}{u - v} + c\big(v(X_{11} - X_{22}) \otimes X_{21} - u X_{21} \otimes (X_{11} - X_{22})\big).$$

Furthermore, let σ be a trivial automorphism of $gl(2)$: $\sigma(X) = X$ and let us also consider the following involution in $\mathbb{C}$: $u^{\sigma} = -u$.

Due to the fact that $r_{12}(-u, -v) = -r_{12}(u, v)$, it immediately follows that

$$\sigma_1\sigma_2 \cdot r_{12}(-u, -v) = -r_{12}(u, v).$$

Thus, we can define the following twisted non-skew-symmetric classical r-matrix of the type (2.7)

$$r^{\sigma}_{12}(u,v) = r_{12}(u,v) - \sigma_2 \cdot r_{12}(u,-v) = \frac{\sum\limits_{i,j=1}^{2} 2v X_{ij} \otimes X_{ji}}{u^2 - v^2} + 2cv(X_{11} - X_{22}) \otimes X_{21}.$$

Making the equivalence transformation, namely, dividing this r-matrix by $2v$, and changing the parametrization: $u^2 \to u$, $v^2 \to v$ we come to the following shifted non-skew-symmetric rational r-matrix

$$r^{\sigma}_{12}(u,v) = \frac{\sum\limits_{i,j=1}^{2} X_{ij} \otimes X_{ji}}{u - v} + c(X_{11} - X_{22}) \otimes X_{21}, \tag{3.1}$$

which we will consider in this section. In order to simplify our notation in the following we will denote it simply by $r_{12}(u,v)$.

Remark 5. Observe that the r-matrix (3.1) can be viewed as a shifted standard rational r-matrix

$$r_{12}(u,v) = r^{rat}_{12}(u-v) + c_{12},$$

with the shift tensor being constant and non-skew-symmetric: $c_{12} = c(X_{11} - X_{22}) \otimes X_{21}$.

The r-matrix (3.1) evidently satisfies the condition (2.4) as well as the conditions (2.11).

It is straightforward to show that the r-matrix (3.1) possess the following constant shift element

$$c(u) = c_{11} X_{11} + c_{22} X_{22}. \tag{3.2}$$

3.2. Linear Lax algebra and generating functions of the integrals of motion

With the help of the classical r-matrix (3.1) one can define the linear Lax algebra (2.8)

$$[\hat{L}_1(u), \hat{L}_2(v)] = [r_{12}(u,v), \hat{L}_1(u)] - [r_{21}(v,u), \hat{L}_2(v)],$$

where $\hat{L}_1(u) = \hat{L}(u) \otimes 1$, $\hat{L}_2(v) = 1 \otimes \hat{L}(v)$, and $\hat{L}(u) = \sum\limits_{i,j=1}^{2} \hat{L}^{ij}(u) X_{ij}$.

As a consequence of the fact that the r-matrix (3.1) satisfies the conditions (2.11) the function

$$\hat{\tau}^{(1)}(u) = \hat{L}^{11}(u) + \hat{L}^{22}(u)$$

generates a center of the linear Lax algebra (2.8). Also, it follows from the Theorem 2.1 the function

$$\hat{\tau}^{(2)}(u) = \frac{1}{2}\left(\hat{L}^{11}(u)\hat{L}^{11}(u) + \hat{L}^{22}(u)\hat{L}^{22}(u) + \hat{L}^{12}(u)\hat{L}^{21}(u) + \hat{L}^{21}(u)\hat{L}^{12}(u)\right)$$

is a generating function of the commuting quantum integrals

$$[\hat{\tau}^{(2)}(u), \hat{\tau}^{(2)}(v)] = 0.$$

The main task of the subsequent subsections will be to diagonalize this operator function.

For the subsequent it will be convenient to introduce the following notations:

$$\hat{A}(u) = \hat{L}^{11}(u), \quad \hat{B}(u) = \hat{L}^{21}(u), \quad \hat{C}(u) = \hat{L}^{12}(u), \quad \hat{D}(u) = \hat{L}^{22}(u).$$

In terms of these operators the commutation relations of the considered Lax algebra (2.9) become

$$[\hat{A}(u), \hat{A}(v)] = [\hat{A}(u), \hat{D}(v)] = [\hat{D}(u), \hat{D}(v)] = 0, \tag{3.3a}$$

$$[\hat{A}(u), \hat{B}(v)] = \frac{1}{u-v}(\hat{B}(u) - \hat{B}(v)), \tag{3.3b}$$

$$[\hat{D}(u), \hat{B}(v)] = -\frac{1}{u-v}(\hat{B}(u) - \hat{B}(v)), \tag{3.3c}$$

$$[\hat{B}(u), \hat{B}(v)] = -2c(\hat{B}(u) - \hat{B}(v)), \tag{3.3d}$$

$$[\hat{A}(u), \hat{C}(v)] = -\frac{1}{u-v}(\hat{C}(u) - \hat{C}(v)), \tag{3.3e}$$

$$[\hat{D}(u), \hat{C}(v)] = \frac{1}{u-v}(\hat{C}(u) - \hat{C}(v)), \tag{3.3f}$$

$$[\hat{C}(u), \hat{C}(v)] = 0, \tag{3.3g}$$

$$[\hat{B}(u), \hat{C}(v)] = \frac{1}{u-v}\big((\hat{A}(u) - \hat{D}(u)) - (\hat{A}(v) - \hat{D}(v))\big) - 2c\hat{C}(v). \tag{3.3h}$$

Also, the generating functions $\hat{\tau}^{(1)}(u)$, $\hat{\tau}(u) \equiv \hat{\tau}^{(2)}(u)$ are given by

$$\hat{\tau}^{(1)}(u) = A(u) + D(u), \quad \hat{\tau}(u) = \frac{1}{2}\big(\hat{A}^2(u) + \hat{D}^2(u) + \hat{C}(u)\hat{B}(u) + \hat{B}(u)\hat{C}(u)\big).$$

3.3. Algebraic Bethe ansatz

3.3.1. Vacuum vector and Bethe vectors

Let us assume that in a space **V** of representation of the Lax algebra (3.3) there exists a vacuum vector $|0\rangle$ such that

$$\hat{C}(u)|0\rangle = 0, \quad \hat{A}(u)|0>= \Lambda_1(u)|0\rangle, \quad \hat{D}(u)|0>= \Lambda_2(u)|0\rangle. \tag{3.4}$$

Let us, following the ideas of [29–31], construct the following vectors $|v_1, v_2, ..., v_M\rangle$ in the space **V**

$$|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle, \tag{3.5}$$

where the operators $\hat{B}_k(u)$ are defined by the following formula

$$\hat{B}_k(u) = \hat{B}(u) + (2k-1)c\mathrm{Id}. \tag{3.6}$$

From the commutation relation (3.3d) we have that:

$$\hat{B}_k(u)\hat{B}_{k+1}(v) = \hat{B}_k(v)\hat{B}_{k+1}(u). \tag{3.7}$$

Therefore the vector $|v_1, v_2, ..., v_M\rangle$ is a symmetric function of its arguments.

3.3.2. The spectrum of the generating functions

In this subsection we seek the spectrum of the generating functions $\hat{\tau}^{(1)}(u)$ and $\hat{\tau}^{(2)}(u)$ corresponding to the Bethe vectors $|v_1, v_2, ..., v_M\rangle$ (3.5).

Notice that due to the fact that $\hat{\tau}^{(1)}(u)$ is a Casimir function, the spectrum of $\hat{\tau}^{(1)}(u)$ is

$$\hat{\tau}^{(1)}(u)|v_1, v_2, ..., v_M\rangle = \big(\Lambda_1(u) + \Lambda_2(u)\big)\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle.$$

Below we will simplify our notation by $\hat{\tau}(u) \equiv \hat{\tau}^{(2)}(u)$. To calculate the spectrum of $\hat{\tau}(u)$ we will need the following Proposition.

Proposition 3.1. *The following commutation relations hold*[2]

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)] =$$
$$= -\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\big((\hat{A}(u)-\hat{D}(u))\sum_{i=1}^{M}\frac{1}{u-v_i} - \sum_{i=1,j=2,i<j}^{M}\frac{2}{(u-v_i)(u-v_j)}\mathrm{Id}\big)+$$
$$+\sum_{k=1}^{M}\hat{B}_1(v_1)\hat{B}_2(v_2)....\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\frac{1}{u-v_k}\big((\hat{A}(v_k)-\hat{D}(v_k)) - \sum_{s=1,s\neq k}^{M}\frac{2}{v_k-v_s}\mathrm{Id}\big)+$$
$$+ 2cM\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{C}(u). \quad (3.8)$$

Proof. The proof of the proposition is by the mathematical induction. To prove the first step we calculate the following commutator using the commutation relations (3.3) and the direct calculation. We obtain that

$$[\hat{\tau}(u), \hat{B}(v)] = -(\hat{B}(v) + c\mathrm{Id})(\hat{A}(u)-\hat{D}(u))\frac{1}{u-v}+$$
$$+ (\hat{B}(u)+c\mathrm{Id})\frac{1}{u-v}(\hat{A}(v)-\hat{D}(v)) + 2c(\hat{B}(v)+c\mathrm{Id})\hat{C}(u).$$

Using the definition of the operator $\hat{B}_1(v)$ (3.6) the $M=1$ case follows directly from the formula above

$$[\hat{\tau}(u), \hat{B}_1(v)] = -\hat{B}_1(v)(\hat{A}(u)-\hat{D}(u))\frac{1}{u-v}+\hat{B}_1(u)\frac{1}{u-v}(\hat{A}(v)-\hat{D}(v))+2c\hat{B}_1(v)\hat{C}(u). \quad (3.9)$$

Now let us assume that the formula (3.8) is valid for M. We have to prove that it is also valid for $M+1$. Using the Leibnitz rule for the commutator we have

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_{M+1}(v_{M+1})] = [\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]\hat{B}_{M+1}(v_{M+1})+$$
$$+ \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)[\hat{\tau}(u), \hat{B}_{M+1}(v_{M+1})]. \quad (3.10)$$

Using further the formulae (3.8) and (3.9) we obtain

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_{M+1}(v_{M+1})] =$$
$$= -\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(v_{M+1})(\hat{A}(u)-\hat{D}(u))\frac{1}{u-v_{M+1}}$$
$$+ \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(u)\frac{1}{u-v_{M+1}}(\hat{A}(v_{M+1})-\hat{D}(v_{M+1}))+$$
$$+ 2c\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(v_{M+1})\hat{C}(u)$$
$$- \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\big((\hat{A}(u)-\hat{D}(u))\sum_{i=1}^{M}\frac{1}{u-v_i}-$$

[2] Check over the operator $\hat{B}_k(v_k)$ means that it is omitted in the corresponding product.

$$- \sum_{i=1, j=2, i<j}^{M} \frac{2}{(u-v_i)(u-v_j)}) \hat{B}_{M+1}(v_{M+1}) +$$

$$+ \sum_{k=1}^{M} \hat{B}_1(v_1)\hat{B}_2(v_2)....\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\frac{1}{u-v_k}\big((\hat{A}(v_k)-\hat{D}(v_k))-$$

$$- \sum_{s=1, s\neq k}^{M} \frac{2}{v_k - v_s}) \hat{B}_{M+1}(v_{M+1}) +$$

$$+ 2cM\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{C}(u)\hat{B}_{M+1}(v_{M+1}). \quad (3.11)$$

In order to obtain the desired formula we have to pass the operator $\hat{B}_{M+1}(v_{M+1})$ to the left of the operators $\hat{A}(u)$, $\hat{D}(u)$, $\hat{C}(u)$, $\hat{A}(v_k)$, $\hat{D}(v_k)$ on the rhs of the formula (3.11). To this end we use

$$[\hat{A}(u) - \hat{D}(u), \hat{B}_{M+1}(v_{M+1})] = \frac{2}{u - v_{M+1}}(\hat{B}_{M+1}(u) - \hat{B}_{M+1}(v_{M+1})), \quad (3.12a)$$

$$[\hat{A}(v_k) - \hat{D}(v_k), \hat{B}_{M+1}(v_{M+1})] = \frac{2}{v_k - v_{M+1}}(\hat{B}_{M+1}(v_k) - \hat{B}_{M+1}(v_{M+1})), \quad (3.12b)$$

$$[\hat{C}(u), \hat{B}_{M+1}(v_{M+1})] = \frac{1}{u - v_{M+1}}\big((\hat{A}(v_{M+1}) - \hat{D}(v_{M+1})) - (\hat{A}(u) - \hat{D}(u))\big) + 2c\hat{C}(u). \quad (3.12c)$$

The careful analysis shows that the additional terms obtained from the commutators (3.12) transform the right hand side of (3.11) to the form (3.8) with $M \to M+1$. To show this we have used the operator identity

$$\hat{B}_1(v_1)...\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\hat{B}_{M+1}(v_k) = \hat{B}_1(v_1)...\hat{B}_k(v_k)...\hat{B}_M(v_M)\hat{B}_{M+1}(u),$$

the definition of $\hat{B}_{M+1}(v)$ and the following identity:

$$\frac{1}{u-v_k}\left(\frac{1}{u-v_{M+1}} - \frac{1}{v_k - v_{M+1}}\right) = -\frac{1}{u-v_{M+1}}\frac{1}{v_k - v_{M+1}}.$$

Therefore the proposition is shown.

Now we can formulate the following Theorem

Theorem 3.1. *Let the rapidities v_k, $k \in \overline{1, M}$ satisfy the following set of Bethe equations*

$$\Lambda_1(v_k) - \Lambda_2(v_k) - \sum_{s=1, s\neq k}^{M} \frac{2}{(v_k - v_s)} = 0. \quad (3.13)$$

Then the Bethe vectors $|v_1, v_2, ..., v_M\rangle$ are eigenvectors of the generating functions of the integrals

$$\hat{\tau}(u)|v_1, v_2, ..., v_M\rangle = \Lambda(u, v_1, v_2, ..., v_M)|v_1, v_2, ..., v_M\rangle$$

with the following eigenvalues $\Lambda(u, v_1, v_2, ..., v_M)$

$$\Lambda(u, v_1, v_2, ..., v_M) = \frac{1}{2}(\Lambda_1^2(u) + \Lambda_2^2(u)) - \frac{1}{2}(\partial_u \Lambda_1(u) - \partial_u \Lambda_2(u)) + \\ + (\Lambda_1(u) - \Lambda_2(u)) \sum_{i=1}^{M} \frac{1}{u - v_i} - \sum_{i=1, j=2, i<j}^{M} \frac{2}{(u - v_i)(u - v_j)}. \quad (3.14)$$

Proof. The statement of the theorem follows from the previous Proposition. Indeed, we have

$$\hat{\tau}(u)|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{\tau}(u)|0\rangle + \\ + [\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]|0\rangle.$$

On the other hand we have that

$$\hat{\tau}(u) = \frac{1}{2}\big(\hat{A}^2(u) + \hat{D}^2(u) - (\partial_u \hat{A}(u) - \partial_u \hat{D}(u)) + 2\hat{B}(u)\hat{C}(u)\big),$$

where we have used the commutation relations (3.3h) in the limit $v \to u$. Then, using the relations (3.4) we obtain

$$\hat{\tau}(u)|0\rangle = \big(\frac{1}{2}(\Lambda_1^2(u) + \Lambda_2^2(u)) - \frac{1}{2}(\partial_u \Lambda_1(u) - \partial_u \Lambda_2(u))\big)|0\rangle.$$

Now, making use of the Proposition 3.1 we obtain that, by the virtue of the conditions (3.4), for the rapidities v_i that satisfy Bethe equations (3.13) the following equality holds true:

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]|0\rangle = \big((\Lambda_1(u) - \Lambda_2(u)) \sum_{i=1}^{M} \frac{1}{u - v_i} - \\ - \sum_{i=1, j=2, i<j}^{M} \frac{2}{(u - v_i)(u - v_j)}\big)|v_1, v_2, ..., v_M\rangle$$

This completes the proof of the Theorem.

Remark 6. It is of interest to notice that the spectrum and the Bethe equations of the models associated with the r-matrix (3.1) coincide with that in the standard rational case, when $c = 0$ [5]. But, as we have shown, when $c \neq 0$, the Bethe vectors have different form.

3.4. Corresponding generalized Gaudin models

3.4.1. Lax matrix and Gaudin-type Hamiltonians

Let us now consider the Lax algebra of the Gaudin-type models. In the case when r-matrix is (3.1) the corresponding Lax matrix (2.16) has the following form

$$\hat{L}(u) = \sum_{m=1}^{N} (\sum_{i,j=1}^{2} \frac{\hat{S}_{ij}^{(m)} X_{ji}}{v_m - u} + c(\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)}) X_{21}) + c_{11} X_{11} + c_{22} X_{22}. \quad (3.15)$$

The mutually commuting quantities it produces with the help of the generating functions $\hat{\tau}^{(1)}(u)$, $\hat{\tau}^{(2)}(u)$ are linear and quadratic Casimirs of the direct sum $gl(2)^{\oplus N}$

$$\hat{C}_m^{(1)} = \hat{S}_{11}^{(m)} + \hat{S}_{22}^{(m)}, \quad \hat{C}_m^{(2)} = \frac{1}{2}(\hat{S}_{11}^{(m)}\hat{S}_{11}^{(m)} + \hat{S}_{22}^{(m)}\hat{S}_{22}^{(m)} + \hat{S}_{12}^{(m)}\hat{S}_{21}^{(m)} + \hat{S}_{21}^{(m)}\hat{S}_{12}^{(m)}),$$

as well as the following Gaudin-type Hamiltonians (2.18) in an external magnetic field

$$\hat{H}_n = \sum_{m=1,m\neq n}^{N} (\sum_{i,j=1}^{2} \frac{\hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(n)}}{\nu_m - \nu_n} + c(\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)})\hat{S}_{21}^{(n)}) + \frac{c}{2}((\hat{S}_{11}^{(n)} - \hat{S}_{22}^{(n)})\hat{S}_{21}^{(n)} + + \hat{S}_{21}^{(n)}(\hat{S}_{11}^{(n)} - \hat{S}_{22}^{(n)})) + c_{11}\hat{S}_{11}^{(n)} + c_{22}\hat{S}_{11}^{(n)}. \quad (3.16)$$

3.4.2. Algebraic Bethe ansatz

Let us apply the construction of the previous subsection to the case of the Lax operators of the generalized Gaudin models. Let us consider a finite-dimensional irreducible representation of the algebra $gl(2)^{\oplus N}$ in some space V. Due to the fact that any irreducible representation of the direct sum of the Lie algebras is a tensor product of irreducible representations of their components, we will have $V = V^{\lambda_1} \otimes V^{\lambda_2} \otimes \cdots \otimes V^{\lambda_N}$, where V^{λ_k} is an irreducible finite-dimensional representation of the k-th copy of $gl(2)$ with the lowest weight $\lambda_k = (\lambda_1^{(k)}, \lambda_2^{(k)})$, with $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{N}$. Each representation V^{λ_k} contains the lowest weight vector v_{λ_k} such that

$$\hat{S}_{11}^{(k)} \mathrm{v}_{\lambda_k} = \lambda_1^{(k)} \mathrm{v}_{\lambda_k}, \ \hat{S}_{22}^{(k)} \mathrm{v}_{\lambda_k} = \lambda_2^{(k)} \mathrm{v}_{\lambda_k}, \quad (3.17a)$$

$$\hat{S}_{21}^{(k)} \mathrm{v}_{\lambda_k} = 0. \quad (3.17b)$$

Therefore, the whole space V^{λ_k} is spanned by the vectors $\mathrm{v}_{\lambda_k}^m = (\hat{S}_{12}^{(k)})^m \mathrm{v}_{\lambda_k}$, $m \in \overline{0, (\lambda_2^{(k)} - \lambda_1^{(k)})}$. The Casimir function $\hat{C}_k^{(2)}$

$$\hat{C}_k^{(2)} = \frac{1}{2}(\hat{S}_{11}^{(k)}\hat{S}_{11}^{(k)} + \hat{S}_{22}^{(k)}\hat{S}_{22}^{(k)} + \hat{S}_{21}^{(k)}\hat{S}_{12}^{(k)} + \hat{S}_{12}^{(k)}\hat{S}_{21}^{(k)}),$$

acts on each vector $\mathrm{v}_{\lambda_k}^m \in V^{\lambda_k}$ in the usual way $\hat{C}_k^{(2)} \mathrm{v}_{\lambda_k}^m = \frac{1}{2}\big((\lambda_k^{(1)})^2 + (\lambda_k^{(2)})^2 + (\lambda_k^{(2)} - \lambda_k^{(1)})\big)\mathrm{v}_{\lambda_k}^m$.

Let us consider the following vacuum vector in the space V

$$|0\rangle = \mathrm{v}_{\lambda_1} \otimes \mathrm{v}_{\lambda_2} \otimes \cdots \otimes \mathrm{v}_{\lambda_N}. \quad (3.18)$$

From the definition of the Lax matrix (3.15) it follows that

$$\hat{C}(u) = \hat{L}^{12}(u) = \sum_{m=1}^{N} \frac{\hat{S}_{21}^{(m)}}{\nu_m - u}, \quad \hat{A}(u) = \hat{L}^{11}(u) = \sum_{m=1}^{N} \frac{\hat{S}_{11}^{(m)}}{\nu_m - u} + c_{11},$$

$$\hat{D}(u) = \hat{L}^{22}(u) = \sum_{m=1}^{N} \frac{\hat{S}_{22}^{(m)}}{\nu_m - u} + c_{22}. \quad (3.19)$$

Thus, we have the following action of the entries of the Lax matrix on the vacuum vector

$$\hat{C}(u)|0\rangle = 0, \quad \hat{A}(u)|0\rangle = \Lambda_{11}(u)|0\rangle, \quad \hat{D}(u)|0\rangle = \Lambda_{22}(u)|0\rangle,$$

where, due to (3.19) and (3.17a)-(3.17b), the eigenvalues are given by

$$\Lambda_{ii}(u) \equiv \sum_{k=1}^{N} \frac{\lambda_i^{(k)}}{\nu_k - u} + c_{ii}, \qquad i \in 1, 2 .$$

In this case, the creation operators $\hat{B}_k(u)$ have the following form

$$\hat{B}_k(u) = \sum_{m=1}^{N} \frac{\hat{S}_{12}^{(m)}}{\nu_m - u} + c \sum_{m=1}^{N} (\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)}) + (2k-1)c\mathrm{Id}. \quad (3.20)$$

Due to the results of the previous subsection we know that the Bethe vectors are given by

$$|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle,$$

where the rapidities v_i should satisfy the Bethe equations (3.13), which now read

$$\sum_{s=1}^{N} \frac{\lambda_s^{(1)} - \lambda_s^{(2)}}{\nu_s - v_k} - \sum_{s=1, s\neq k}^{M} \frac{2}{(v_k - v_s)} = c_{22} - c_{11}, \qquad k \in \overline{1, M}. \tag{3.21}$$

The spectrum of the generating function $\hat{\tau}(u)$ is given by

$$\Lambda(u, v_1, v_2, ..., v_M) = \frac{1}{2}((\sum_{k=1}^{N} \frac{\lambda_1^{(k)}}{\nu_k - u} + c_{11})^2 + (\sum_{k=1}^{N} \frac{\lambda_2^{(k)}}{\nu_k - u} + c_{22})^2) - \\ - \frac{1}{2}(\sum_{k=1}^{N} \frac{\lambda_1^{(k)}}{(\nu_k - u)^2} - \frac{\lambda_2^{(k)}}{(\nu_k - u)^2}) + (\sum_{k=1}^{N} \frac{\lambda_1^{(k)} - \lambda_2^{(k)}}{\nu_k - u} + c_{11} - c_{22}) \sum_{i=1}^{M} \frac{1}{u - v_i} - \\ - \sum_{i=1, j=2, i<j}^{M} \frac{2}{(u - v_i)(u - v_j)}. \tag{3.22}$$

Also, the spectrum of the generalized Gaudin Hamiltonians is

$$h_n = \sum_{i=1}^{2} \sum_{k=1, k\neq n}^{N} \frac{\lambda_i^{(k)} \lambda_i^{(n)}}{\nu_k - \nu_n} + \sum_{i=1}^{2} c_{ii} \lambda_{ii}^{(n)} - (\lambda_1^{(n)} - \lambda_2^{(n)}) \sum_{i=1}^{M} \frac{1}{v_i - \nu_n}. \tag{3.23}$$

Remark 7. The spectra of the generalized Gaudin Hamiltonians and the Bethe equations are the same as in the standard rational case, when $c = 0$. However, for $c \neq 0$ the Bethe states of the model are different.

3.5. Corresponding BCS-type models

3.5.1. The BCS-type Hamiltonian in the spin and fermion form

Let us consider the following combination of the generalized Gaudin Hamiltonians (3.16) and the second order Casimir operators:

$$\hat{H} = \sum_{n=1}^{N} \nu_n \hat{H}_n - \sum_{n=1}^{N} \hat{C}_n^{(2)}. \tag{3.24}$$

More explicitly we have

$$\hat{H} = c_{11} \sum_{n=1}^{N} \nu_n \hat{S}_{11}^{(n)} + c_{22} \sum_{n=1}^{N} \nu_n \hat{S}_{22}^{(n)} - \frac{1}{2} \sum_{i,j=1}^{2} \sum_{m,n=1}^{N} \hat{S}_{ij}^{(m)} \hat{S}_{ji}^{(n)} + \\ + c \sum_{m=1}^{N} (\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)} + \mathrm{Id}) \sum_{n=1}^{N} \nu_n \hat{S}_{21}^{(n)}. \tag{3.25}$$

Applying the fermionization formulae (2.22) we obtain the following Hamiltonians of the BCS-type

$$\hat{H} = c_{11} \sum_{n=1}^{N} \nu_n c^{\dagger}_{n,\epsilon'} c_{n,\epsilon'} + c_{22} \sum_{n=1}^{N} \nu_n c_{n,\epsilon} c^{\dagger}_{n,\epsilon} -$$
$$- \frac{1}{2} \sum_{m,n=1}^{N} (c^{\dagger}_{m,\epsilon'} c^{\dagger}_{m,\epsilon} c_{n,\epsilon} c_{n,\epsilon'} + c_{m,\epsilon} c_{m,\epsilon'} c^{\dagger}_{n,\epsilon'} c^{\dagger}_{n,\epsilon}) -$$
$$- \frac{1}{2} \sum_{m,n=1}^{N} (c^{\dagger}_{m,\epsilon'} c_{m,\epsilon'} c^{\dagger}_{n,\epsilon'} c_{n,\epsilon'} + c_{m,\epsilon} c^{\dagger}_{m,\epsilon} c_{n,\epsilon} c^{\dagger}_{n,\epsilon}) +$$
$$+ c \sum_{m=1}^{N} (c^{\dagger}_{m,\epsilon'} c_{m,\epsilon'} - c_{m,\epsilon} c^{\dagger}_{m,\epsilon} + 1) \sum_{n=1}^{N} \nu_n c_{n,\epsilon} c_{n,\epsilon'}, \quad (3.26)$$

where $\epsilon, \epsilon' \in 1, 2$ and $\epsilon \neq \epsilon'$.

Remark 8. It maybe of interest to discuss briefly the terms of the Hamiltonian (3.26). The first two terms are the kinetic ones. They acquire the standard form upon setting $c_{22} = -c_{11}$. The third term is the s-type pairing interaction. The fifth term is new, non-standard and it is due to the additional summand in the considered r-matrix. Finally, the fourth term corresponds to the negative half sum of $\hat{S}^2_{11} = (\sum_{m=1}^{N} \hat{S}^{(m)}_{11})^2$ and $\hat{S}^2_{22} = (\sum_{m=1}^{N} \hat{S}^{(m)}_{22})^2$. This term is absent in the standard BCS-Richardson's Hamiltonian since in the standard rational case $\hat{S}_{11} = \sum_{m=1}^{N} \hat{S}^{(m)}_{11}$ and $\hat{S}_{22} = \sum_{m=1}^{N} \hat{S}^{(m)}_{22}$ are the integrals of motion and one can add $\frac{1}{2}(\hat{S}^2_{11} + \hat{S}^2_{22})$ to the Hamiltonian (3.25) without spoiling its integrability. In the $c \neq 0$ case, the operators $\hat{S}_{11} = \sum_{m=1}^{N} \hat{S}^{(m)}_{11}$ and $\hat{S}_{22} = \sum_{m=1}^{N} \hat{S}^{(m)}_{22}$ are not the integrals of motion and therefore their functions can not be added to the Hamiltonian (3.25).

3.5.2. The spectra and Bethe equations

In particular case when $\lambda^{(k)}_2 = 1$, $\lambda^{(k)}_1 = 0$, $k \in \overline{1, N}$ the Bethe equations (3.21) have the following simple form

$$\sum_{s=1}^{N} \frac{1}{\nu_s - v_k} - \sum_{s=1, s\neq k}^{M} \frac{2}{v_k - v_s} = c_{22} - c_{11}, \qquad k \in \overline{1, M}. \quad (3.27)$$

The spectra of the generalized Gaudin Hamiltonians is also simple in this case

$$h_n = \sum_{k=1, k\neq n}^{N} \frac{1}{\nu_k - \nu_n} + c_{22} + \sum_{i=1}^{M} \frac{1}{v_i - \nu_n}, \quad (3.28)$$

where the rapidities v_i satisfy the Bethe equations (3.27).

The spectrum of the BCS-like Hamiltonian (3.26), up to the constant, has the following form

$$h = \sum_{n=1}^{N} \sum_{i=1}^{M} \frac{\nu_n}{v_i - \nu_n} = -MN + (c_{22} - c_{11}) \sum_{i=1}^{M} v_i, \tag{3.29}$$

here we have used the definition of $\hat{H}$ and the Bethe equations (3.27). It is important to notice that the spectrum of the Hamiltonian $\hat{H}$ (3.26) is the same as in the standard Richardon's case [2,11], but the Bethe vectors are different.

4. The trigonometric r-matrix

4.1. Shifted twisted trigonometric r-matrix

Let us consider another non-standard skew-symmetric rational r-matrix of the following form [4]:

$$r_{12}(u-v) = \frac{\sum\limits_{i,j=1}^{2} X_{ij} \otimes X_{ji}}{u-v} + c\big((X_{11} - X_{22}) \otimes X_{21} - X_{21} \otimes (X_{11} - X_{22})\big).$$

It is straightforward to check that

$$\sigma_1 \sigma_2 r_{12}(u^\sigma - v^\sigma) = -r_{12}(u-v),$$

where $u^\sigma = -u$, $v^\sigma = -v$ and the automorphism σ on $\mathfrak{g} = gl(2)$ is defined by

$$\sigma(X_{ij}) = (-1)^{i+j} X_{ij}.$$

Thus we can define the following non-skew-symmetric classical r-matrix

$$r^\sigma_{12}(u,v) = r_{12}(u-v) - \sigma_2 r_{12}(u+v).$$

By making the equivalence transformation, namely, multiplying this r-matrix by $\frac{v}{2}$ we come to the following non-skew-symmetric r-matrix of the type (2.7)

$$r^\sigma_{12}(u,v) = \frac{v^2}{u^2-v^2}(X_{11} \otimes X_{11} + X_{22} \otimes X_{22}) + \frac{uv}{u^2-v^2}(X_{12} \otimes X_{21} + X_{21} \otimes X_{12}) + \\ + cv(X_{11} - X_{22}) \otimes X_{21}. \tag{4.1}$$

In this section we will focus on this r-matrix and it will be denoted simply by $r_{12}(u,v)$.

Remark 9. Notice that the r-matrix (4.1) may be also viewed as a shifted trigonometric r-matrix:

$$r_{12}(u,v) = r^{trig}_{12}(u,v) + c_{12}(v),$$

where

$$r^{trig}_{12}(u,v) = \frac{1}{2}\frac{u^2+v^2}{u^2-v^2}(X_{11} \otimes X_{11} + X_{22} \otimes X_{22}) + \frac{uv}{u^2-v^2}(X_{12} \otimes X_{21} + X_{21} \otimes X_{12})$$

with the shift tensor $c_{12}(v)$ defined as follows $c_{12}(v) = -\frac{1}{2}(X_{11} \otimes X_{11} + X_{22} \otimes X_{22}) + cv(X_{11} - X_{22}) \otimes X_{21}$.

The trigonometric parametrization is obtained by the following substitution $u = \exp\left(\frac{i\phi}{2}\right)$, $v = \exp\left(\frac{i\psi}{2}\right)$. The r-matrix (4.1) satisfies the condition (2.4) in the trigonometric parametrization, since the r-matrix $r_{12}^{trig}(u, v)$ satisfies it in this parametrization. Furthermore, is straightforward to show that the r-matrix (4.1) satisfies the conditions (2.11).

4.2. Linear Lax algebra and generating functions of the integrals of motion

In the standard way the classical r-matrix (4.1) defines the linear Lax algebra (2.8):

$$[\hat{L}_1(u), \hat{L}_2(v)] = [r_{12}(u, v), \hat{L}_1(u)] - [r_{21}(v, u), \hat{L}_2(v)],$$

where $\hat{L}_1(u) = \hat{L}(u) \otimes 1$, $\hat{L}_2(v) = 1 \otimes \hat{L}(v)$, and $\hat{L}(u) = \sum_{i,j=1}^{2} \hat{L}^{ij}(u) X_{ij}$.

Using the fact that the r-matrix (4.1) satisfies the conditions (2.11) we have that the function

$$\hat{\tau}^{(1)}(u) = \hat{L}^{11}(u) + \hat{L}^{22}(u)$$

generates a center of the linear Lax algebra (2.8). Also, it follows from the Theorem 2.1, that the function

$$\hat{\tau}^{(2)}(u) = \frac{1}{2}\big(\hat{L}^{11}(u)\hat{L}^{11}(u) + \hat{L}^{22}(u)\hat{L}^{22}(u) + \hat{L}^{12}(u)\hat{L}^{21}(u) + \hat{L}^{21}(u)\hat{L}^{12}(u)\big)$$

is a generating function of the commuting quantum integrals of the second order

$$[\hat{\tau}^{(2)}(u), \hat{\tau}^{(2)}(v)] = 0.$$

The spectral decomposition of this function will be the main topic of the subsequent subsections. As in the previous section, it will be convenient to use the following notations

$$\hat{A}(u) = \hat{L}^{11}(u), \quad \hat{B}(u) = \hat{L}^{21}(u), \quad \hat{C}(u) = \hat{L}^{12}(u), \quad \hat{D}(u) = \hat{L}^{22}(u).$$

In these terms the commutation relations of the Lax algebra (2.8) acquire the following form:

$$[\hat{A}(u), \hat{A}(v)] = [\hat{A}(u), \hat{D}(v)] = [\hat{D}(u), \hat{D}(v)] = 0, \tag{4.2a}$$

$$[\hat{A}(u), \hat{B}(v)] = \frac{uv}{u^2 - v^2}\hat{B}(u) - \frac{u^2}{u^2 - v^2}\hat{B}(v), \tag{4.2b}$$

$$[\hat{D}(u), \hat{B}(v)] = -\frac{uv}{u^2 - v^2}\hat{B}(u) + \frac{u^2}{u^2 - v^2}\hat{B}(v), \tag{4.2c}$$

$$[\hat{B}(u), \hat{B}(v)] = -2c(v\hat{B}(u) - u\hat{B}(v)), \tag{4.2d}$$

$$[\hat{A}(u), \hat{C}(v)] = -\frac{uv}{u^2 - v^2}\hat{C}(u) + \frac{u^2}{u^2 - v^2}\hat{C}(v), \tag{4.2e}$$

$$[\hat{D}(u), \hat{C}(v)] = \frac{uv}{u^2 - v^2}\hat{C}(u) - \frac{u^2}{u^2 - v^2}\hat{C}(v), \tag{4.2f}$$

$$[\hat{C}(u), \hat{C}(v)] = 0, \tag{4.2g}$$

$$[\hat{C}(u), \hat{B}(v)] = -\frac{uv}{u^2 - v^2}\big((\hat{A}(u) - \hat{D}(u)) - (\hat{A}(v) - \hat{D}(v))\big) + 2cv\hat{C}(u). \tag{4.2h}$$

In terms this notation, the generating functions $\hat{\tau}^{(1)}(u)$, $\hat{\tau}(u) \equiv \hat{\tau}^{(2)}(u)$ have the following form:

$$\hat{\tau}^{(1)}(u) = A(u) + D(u), \quad \hat{\tau}(u) = \frac{1}{2}\big(\hat{A}^2(u) + \hat{D}^2(u) + \hat{C}(u)\hat{B}(u) + \hat{B}(u)\hat{C}(u)\big).$$

4.3. Algebraic Bethe ansatz

4.3.1. Vacuum vector and Bethe vectors

As usual, let us assume that in the representation space **V** of the Lax algebra (4.2) there exists a vacuum vector $|0\rangle$ such that:

$$\hat{C}(u)|0\rangle = 0, \quad \hat{A}(u)|0>= \Lambda_1(u)|0\rangle, \quad \hat{D}(u)|0>= \Lambda_2(u)|0\rangle. \tag{4.3}$$

Following the ideas of [29–31], we consider the following vectors $|v_1, v_2, ..., v_M\rangle$ in the space **V**

$$|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle, \tag{4.4}$$

where the operators $\hat{B}_k(u)$ are defined by

$$\hat{B}_k(u) = \hat{B}(u) + (2k-1)uc\mathrm{Id}. \tag{4.5}$$

From the commutation relation (4.2d) we have that:

$$\hat{B}_k(u)\hat{B}_{k+1}(v) = \hat{B}_k(v)\hat{B}_{k+1}(u). \tag{4.6}$$

Therefore the vector $|v_1, v_2, ..., v_M\rangle$ is a symmetric function of its arguments.

4.3.2. The spectrum of the generating functions and Bethe equations

In this subsection we will study the spectra of the generating functions $\hat{\tau}^{(1)}(u)$ and $\hat{\tau}(u)$ relative to the Bethe vectors $|v_1, v_2, ..., v_M\rangle$ obtained in the previous subsection.

Since $\hat{\tau}^{(1)}(u)$ is a Casimir function, its spectrum on the Bethe vectors (4.4) is

$$\hat{\tau}^{(1)}(u)|v_1, v_2, ..., v_M\rangle = \big(\Lambda_1(u) + \Lambda_2(u)\big)\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle.$$

In order to calculate the spectrum of $\hat{\tau}(u)$ we will need the following Proposition.

Proposition 4.1. *The following commutation relation holds*[3]*:*

$$\begin{aligned}
&[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)] = \\
&\quad = -\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\Big((\hat{A}(u) - \hat{D}(u) - \mathrm{Id})\sum_{i=1}^{M}\frac{u^2}{u^2 - v_i^2} - \\
&\quad - \sum_{i=1, j=2, i<j}^{M}\frac{2u^4}{(u^2 - v_i^2)(u^2 - v_j^2)}\mathrm{Id}\Big) + \\
&\quad + \sum_{k=1}^{M}\hat{B}_1(v_1)\hat{B}_2(v_2)....\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\frac{uv_k}{u^2 - v_k^2}\big(\hat{A}(v_k) - \hat{D}(v_k) - \mathrm{Id} - \\
&\quad - \sum_{s=1, s\neq k}^{M}\frac{2v_k^2}{v_k^2 - v_s^2}\mathrm{Id}\big) + \\
&\quad + 2cuM\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{C}(u).
\end{aligned} \tag{4.7}$$

[3] Check over the operator $\hat{B}_k(v_k)$ means that it is omitted in the corresponding product.

Proof. The proof of the Proposition is by an induction procedure. To prove the first step we use the commutation relations (4.2) to obtain

$$[\hat{\tau}(u), \hat{B}(v)] = -(\hat{B}(v) + cv\mathrm{Id})(\hat{A}(u) - \hat{D}(u) - \mathrm{Id})\frac{uv}{u^2 - v^2} + \\ + \frac{u^2}{u^2 - v^2}(\hat{B}(u) + cu\mathrm{Id})(\hat{A}(v) - \hat{D}(v) - \mathrm{Id}) \\ + 2cu(\hat{B}(v) + cv\mathrm{Id})\hat{C}(u).$$

Using the definition of the creation operator $\hat{B}_1(v)$ (4.5) we obtain the $M = 1$ case

$$[\hat{\tau}(u), \hat{B}_1(v)] = -\frac{uv}{u^2 - v^2}\hat{B}_1(v)(\hat{A}(u) - \hat{D}(u) - \mathrm{Id}) + \\ + \hat{B}_1(u)\frac{u^2}{u^2 - v^2}(\hat{A}(v) - \hat{D}(v) - \mathrm{Id}) + 2cu\hat{B}_1(v)\hat{C}(u). \quad (4.8)$$

Now let us assume that the formula (4.7) is valid for M. We have to show that it is also valid for $M + 1$. Using the Leibnitz rule for the commutator we have

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_{M+1}(v_{M+1})] = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)[\hat{\tau}(u), \hat{B}_{M+1}(v_{M+1})] + \\ + [\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]\hat{B}_{M+1}(v_{M+1}). \quad (4.9)$$

Using further the formulae (4.7) and (4.8) we obtain:

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_{M+1}(v_{M+1})] = \\ = -\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(v_{M+1})(\hat{A}(u) - \hat{D}(u) - \mathrm{Id})\frac{u^2}{u^2 - v_{M+1}^2} + \\ \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(u)\frac{uv_{M+1}}{u^2 - v_{M+1}^2}(\hat{A}(v_{M+1}) - \hat{D}(v_{M+1}) - \mathrm{Id}) + \\ + 2cu\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{B}_1(v_{M+1})\hat{C}(u) \\ - \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\big((\hat{A}(u) - \hat{D}(u) - \mathrm{Id})\sum_{i=1}^{M}\frac{u^2}{u^2 - v_i^2} - \\ - \sum_{i=1, j=2, i<j}^{M}\frac{2u^4}{(u^2 - v_i^2)(u^2 - v_j^2)}\big)\hat{B}_{M+1}(v_{M+1}) + \\ + \sum_{k=1}^{M}\hat{B}_1(v_1)\hat{B}_2(v_2)....\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\frac{1}{u - v_k}\big((\hat{A}(v_k) - \hat{D}(v_k) - \mathrm{Id}) - \\ - \sum_{s=1, s\neq k}^{M}\frac{2v_k^2}{v_k^2 - v_s^2}\big)\hat{B}_{M+1}(v_{M+1}) + \\ + 2cM\hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{C}(u)\hat{B}_{M+1}(v_{M+1}). \quad (4.10)$$

In order to obtain the desired result we have to pass the operator $\hat{B}_{M+1}(v_{M+1})$ to the left of the operators $\hat{A}(u)$, $\hat{D}(u)$, $\hat{C}(u)$, $\hat{A}(v_k)$, $\hat{D}(v_k)$ on the right hand side of the formula (4.10). For this purpose we use

$$[\hat{A}(u) - \hat{D}(u), \hat{B}_{M+1}(v_{M+1})] = \frac{2uv_{M+1}}{u^2 - v_{M+1}^2}\hat{B}_{M+1}(u) - \frac{2u^2}{u^2 - v_{M+1}^2}\hat{B}_{M+1}(v_{M+1}), \tag{4.11a}$$

$$[\hat{A}(v_k) - \hat{D}(v_k), \hat{B}_{M+1}(v_{M+1})] = \frac{2v_k v_{M+1}}{v_k^2 - v_{M+1}^2}\hat{B}_{M+1}(v_k) - \frac{2v_k^2}{v_k^2 - v_{M+1}^2}\hat{B}_{M+1}(v_{M+1}), \tag{4.11b}$$

$$[\hat{C}(u), \hat{B}_{M+1}(v_{M+1})] = \frac{uv_{M+1}}{u^2 - v_{M+1}^2}\big((\hat{A}(v_{M+1}) - \hat{D}(v_{M+1})) - (\hat{A}(u) - \hat{D}(u))\big) + 2cv\hat{C}(u). \tag{4.11c}$$

It can be seen that the additional terms obtained from the commutators (4.11) transform the right hand side of (4.10) to the form (4.7) with $M \to M+1$. In order to show this one also has to use that

$$\hat{B}_1(v_1)...\check{\hat{B}}_k(v_k)\hat{B}_k(u)...\hat{B}_M(v_M)\hat{B}_{M+1}(v_k) = \hat{B}_1(v_1)...\hat{B}_k(v_k)...\hat{B}_M(v_M)\hat{B}_{M+1}(u),$$

the definition of the creation operator $\hat{B}_{M+1}(v)$ (4.5) and the following identity

$$\frac{u^2}{u^2 - v_k^2}\frac{uv_{M+1}}{u^2 - v_{M+1}^2} - \frac{uv_k}{u^2 - v_k^2}\frac{v_k v_{M+1}}{v_k^2 - v_{M+1}^2} = -\frac{uv_{M+1}}{u^2 - v_{M+1}^2}\frac{v_{M+1}^2}{v_k^2 - v_{M+1}^2}.$$

This completes the proof of the proposition.

Now we can state the following Theorem

Theorem 4.1. *Let the rapidities v_k, $k \in \overline{1, M}$ satisfy the following Bethe equations*

$$\Lambda_1(v_k) - \Lambda_2(v_k) - \sum_{s=1, s\neq k}^{M} \frac{2v_k^2}{v_k^2 - v_s^2} = 1. \tag{4.12}$$

Then the Bethe vectors $|v_1, v_2, ..., v_M\rangle$ are eigenvectors of the generating function of the integrals of motion

$$\hat{\tau}(u)|v_1, v_2, ..., v_M\rangle = \Lambda(u, v_1, v_2, ..., v_M)|v_1, v_2, ..., v_M\rangle$$

with the eigenvalues

$$\Lambda(u, v_1, v_2, ..., v_M) = \frac{1}{2}(\Lambda_1^2(u) + \Lambda_2^2(u)) - \frac{u}{4}(\partial_u \Lambda_1(u) - \partial_u \Lambda_2(u)) + \\ + (\Lambda_1(u) - \Lambda_2(u) - 1)\sum_{i=1}^{M}\frac{u^2}{u^2 - v_i^2} - \sum_{i=1, j=2, i<j}^{M}\frac{2u^4}{(u^2 - v_i^2)(u^2 - v_j^2)}. \tag{4.13}$$

Proof. The statement of the theorem follows from the previous Proposition. Indeed, we have:

$$\hat{\tau}(u)|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)\hat{\tau}(u)|0\rangle + \\ + [\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]|0\rangle.$$

On the other hand we have that:

$$\hat{\tau}(u) = \frac{1}{2}\big(\hat{A}^2(u) + \hat{D}^2(u) - \frac{u}{2}(\partial_u \hat{A}(u) - \partial_u \hat{D}(u)) + 2\hat{B}(u)\hat{C}(u)\big),$$

where we have used the commutation relations (4.2h) in the limit $v \to u$. That is why using the relations (4.3) we obtain that:

$$\hat{\tau}(u)|0\rangle = \big(\frac{1}{2}(\Lambda_1^2(u) + \Lambda_2^2(u)) - \frac{u}{4}(\partial_u \Lambda_1(u) - \partial_u \Lambda_2(u))\big)|0\rangle.$$

Now, making use of the Proposition 4.1 we obtain that, by the virtue of the conditions (4.3), and for the rapidities v_i that satisfy Bethe equations (4.12), the following formula holds true

$$[\hat{\tau}(u), \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)]|0\rangle =$$
$$= \big((\Lambda_1(u) - \Lambda_2(u) - 1)\sum_{i=1}^{M} \frac{u^2}{u^2 - v_i^2} - \sum_{i=1, j=2, i<j}^{M} \frac{2u^4}{(u^2 - v_i^2)(u^2 - v_j^2)}\big)|v_1, v_2, ..., v_M\rangle$$

This completes the proof of the theorem.

Remark 10. The spectrum and Bethe equations of the systems associated with the r-matrix (4.1) coincide with the ones of the non-skew-symmetric trigonometric $c = 0$ case [24]. But the form of the Bethe states is different.

4.4. Corresponding generalized Gaudin models

4.4.1. Lax matrix and Gaudin-type Hamiltonians

Let us consider the universal example of the Lax algebra which yields the Gaudin models. The relevant Lax matrix (2.16) has the form

$$\hat{L}(u) = \sum_{m=1}^{N}(\sum_{i,j=1}^{2} \frac{u^2}{\nu_m^2 - u^2}(\hat{S}_{11}^{(m)} X_{11} + \hat{S}_{22}^{(m)} X_{22}) + \frac{\nu_m u}{\nu_m^2 - u^2}(\hat{S}_{12}^{(m)} X_{21} + \hat{S}_{21}^{(m)} X_{12}) +$$
$$+ cu(\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)})X_{21}) + c_{11}X_{11} + c_{22}X_{22}. \quad (4.14)$$

The commutative quantities it produce with the help of the generating functions $\hat{\tau}^{(1)}(u)$, $\hat{\tau}(u)$ are linear and quadratic Casimirs of the direct sum $gl(2)^{\oplus N}$

$$\hat{C}_m^{(1)} = \hat{S}_{11}^{(m)} + \hat{S}_{22}^{(m)}, \quad \hat{C}_m^{(2)} = \frac{1}{2}(\hat{S}_{11}^{(m)}\hat{S}_{11}^{(m)} + \hat{S}_{22}^{(m)}\hat{S}_{22}^{(m)} + \hat{S}_{12}^{(m)}\hat{S}_{21}^{(m)} + \hat{S}_{21}^{(m)}\hat{S}_{12}^{(m)}),$$

as well as the following Gaudin-type Hamiltonians (2.18)

$$\hat{H}_n = \sum_{m=1, m\neq n}^{N} (\frac{\nu_n^2}{\nu_m^2 - \nu_n^2}(\hat{S}_{11}^{(m)}\hat{S}_{11}^{(n)} + \hat{S}_{22}^{(m)}\hat{S}_{22}^{(n)}) + \frac{\nu_m \nu_n}{\nu_m^2 - \nu_n^2}(\hat{S}_{12}^{(m)}\hat{S}_{21}^{(n)} + \hat{S}_{21}^{(m)}\hat{S}_{12}^{(n)}) +$$
$$+ c\nu_n(\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)})\hat{S}_{21}^{(n)}) - \frac{1}{2}(\hat{S}_{11}^{(n)}\hat{S}_{11}^{(n)} + \hat{S}_{22}^{(n)}\hat{S}_{22}^{(n)}) + \frac{c}{2}\nu_n((\hat{S}_{11}^{(n)} - \hat{S}_{22}^{(n)})\hat{S}_{21}^{(n)} + \hat{S}_{21}^{(n)}(\hat{S}_{11}^{(n)} - \hat{S}_{22}^{(n)})) +$$
$$+ c_{11}\hat{S}_{11}^{(n)} + c_{22}\hat{S}_{22}^{(n)}. \quad (4.15)$$

4.4.2. Algebraic Bethe annsatz

Let us apply the construction of the previous subsection to the case of the Lax operators of the generalized Gaudin models. Let us consider a finite-dimensional irreducible representation of the algebra $gl(2)^{\oplus N}$ in some space V, just as we have done in the subsection 3.4.2. As we have seen already, any irreducible representation of the direct sum of the Lie algebras is a tensor product of irreducible representations of their components. That is to say that $V = V^{\lambda_1} \otimes V^{\lambda_2} \otimes \cdots \otimes V^{\lambda_N}$, where V^{λ_k} is an irreducible finite-dimensional representation of the k-th copy of $gl(2)$ with the lowest weight $\lambda_k = (\lambda_1^{(k)}, \lambda_2^{(k)})$ with $\lambda_1^{(k)}, \lambda_2^{(k)} \in \mathbb{N}$. Each representation V^{λ_k} contains the lowest weight vector v_{λ_k} such that

$$\hat{S}_{11}^{(k)} \mathrm{v}_{\lambda_k} = \lambda_1^{(k)} \mathrm{v}_{\lambda_k}, \ \hat{S}_{22}^{(k)} \mathrm{v}_{\lambda_k} = \lambda_2^{(k)} \mathrm{v}_{\lambda_k}, \tag{4.16a}$$

$$\hat{S}_{21}^{(k)} \mathrm{v}_{\lambda_k} = 0, \tag{4.16b}$$

and the whole space V^{λ_k} is spanned by $\mathrm{v}_{\lambda_k}^m = (\hat{S}_{12}^{(k)})^m \mathrm{v}_{\lambda_k}, m \in \overline{0, (\lambda_2^{(k)} - \lambda_1^{(k)})}$.

The Casimir function $\hat{C}_k^{(2)}$

$$\hat{C}_k^{(2)} = \frac{1}{2}(\hat{S}_{11}^{(k)} \hat{S}_{11}^{(k)} + \hat{S}_{22}^{(k)} \hat{S}_{22}^{(k)} + \hat{S}_{21}^{(k)} \hat{S}_{12}^{(k)} + \hat{S}_{12}^{(k)} \hat{S}_{21}^{(k)}), \tag{4.17}$$

acts on each vector $\mathrm{v}_{\lambda_k}^m \in V^{\lambda_k}$ in the usual way $\hat{C}_k^{(2)} \mathrm{v}_{\lambda_k}^m = \frac{1}{2}\big((\lambda_k^{(1)})^2 + (\lambda_k^{(2)})^2 + (\lambda_k^{(2)} - \lambda_k^{(1)})\big)\mathrm{v}_{\lambda_k}^m$. Also, as we already know, the vacuum vector in the space V is given by

$$|0\rangle = \mathrm{v}_{\lambda_1} \otimes \mathrm{v}_{\lambda_2} \otimes \cdots \otimes \mathrm{v}_{\lambda_N}. \tag{4.18}$$

From the definition of the Lax matrix (4.14) we can readout its entries

$$\hat{C}(u) = \hat{L}^{12}(u) = \sum_{m=1}^{N} \frac{u \nu_m \hat{S}_{21}^{(m)}}{\nu_m^2 - u^2}, \quad \hat{A}(u) = \hat{L}^{11}(u) = \sum_{m=1}^{N} \frac{u^2 \hat{S}_{11}^{(m)}}{\nu_m^2 - u^2} + c_{11},$$

$$\hat{D}(u) = \hat{L}^{22}(u) = \sum_{m=1}^{N} \frac{u^2 \hat{S}_{22}^{(m)}}{\nu_m^2 - u^2} + c_{22}. \tag{4.19}$$

The action of these operators on the vacuum vector follows from the equations above

$$\hat{C}(u)|0\rangle = 0, \quad \hat{A}(u)|0\rangle = \Lambda_{11}(u)|0\rangle, \quad \hat{D}(u)|0\rangle = \Lambda_{22}(u)|0\rangle, \tag{4.20}$$

where

$$\Lambda_{ii}(u) \equiv \sum_{k=1}^{N} \frac{u^2 \lambda_i^{(k)}}{\nu_k^2 - u^2} + c_{ii}, \qquad i \in 1, 2.$$

In this case, the creation operator $\hat{B}_k(u)$ has the following form

$$\hat{B}_k(u) = \sum_{m=1}^{N} \frac{u \nu_m \hat{S}_{12}^{(m)}}{\nu_m^2 - u^2} + cu \sum_{m=1}^{N} (\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)}) + (2k-1)uc\mathrm{Id}. \tag{4.21}$$

It follows from the results of the previous subsection that the Bethe vectors are given by the action of the creation operator on the vacuum vector

$$|v_1, v_2, ..., v_M\rangle = \hat{B}_1(v_1)\hat{B}_2(v_2)....\hat{B}_M(v_M)|0\rangle,$$

where the rapidities v_i satisfy the Bethe equations (4.12), which in this case read

$$\sum_{s=1}^{N} \frac{v_k^2(\lambda_1^{(s)} - \lambda_2^{(s)})}{v_s^2 - v_k^2} - \sum_{s=1, s\neq k}^{M} \frac{2v_k^2}{v_k^2 - v_s^2} = c_{22} - c_{11} + 1, \qquad k \in \overline{1, M}. \tag{4.22}$$

The spectrum of the generating function $\hat{\tau}(u)$ is given by

$$\Lambda(u, v_1, v_2, ..., v_M) = \frac{1}{2}((\sum_{k=1}^{N} \frac{u^2\lambda_1^{(k)}}{v_k^2 - u^2} + c_{11})^2 + (\sum_{k=1}^{N} \frac{u^2\lambda_2^{(k)}}{v_k^2 - u^2} + c_{22})^2) -$$
$$- \frac{1}{2}\sum_{k=1}^{N} \frac{u^2 v_k^2}{(v_k^2 - u^2)^2}(\lambda_1^{(k)} - \lambda_2^{(k)}) + (\sum_{k=1}^{N} \frac{u^2\lambda_1^{(k)}}{v_k^2 - u^2} - \sum_{k=1}^{N} \frac{u^2\lambda_2^{(k)}}{v_k^2 - u^2} + c_{11} - c_{22} - 1)\sum_{i=1}^{M} \frac{u^2}{u^2 - v_i^2} -$$
$$- \sum_{i=1, j=2, i<j}^{M} \frac{2u^4}{(u^2 - v_i^2)(u^2 - v_j^2)}. \tag{4.23}$$

Furthermore, the spectra of the generalized Gaudin Hamiltonians are

$$h_n = \sum_{i=1}^{2} \sum_{k=1, k\neq n}^{N} \frac{v_n^2 \lambda_i^{(k)} \lambda_i^{(n)}}{v_k^2 - v_n^2} + c_{11}\lambda_1^{(n)} + c_{22}\lambda_2^{(n)} - \frac{1}{2}((\lambda_1^{(n)})^2 + (\lambda_2^{(n)})^2) -$$
$$- (\lambda_1^{(n)} - \lambda_2^{(n)}) \sum_{i=1}^{M} \frac{v_n^2}{v_i^2 - v_n^2}. \tag{4.24}$$

Remark 11. Thus we can conclude that the spectra of the generalized Gaudin Hamiltonians (4.24) and the corresponding Bethe equations (4.22) are the same as in the $c = 0$ case [24,23]. However, the Bethe vectors of the system are different.

4.5. Corresponding BCS-type models

4.5.1. The BCS-type Hamiltonian in the spin and fermion form

Here we consider the following combination of the generalized Gaudin Hamiltonians (4.15) and the second order Casimir operators (4.17)

$$\hat{H} = \sum_{n=1}^{N} v_n^{-2} \hat{H}_n + \sum_{n=1}^{N} v_n^{-2} \hat{C}_n^{(2)}. \tag{4.25}$$

In terms of the local $gl(2)$ generators this Hamiltonian is given by

$$\hat{H} = c_{11} \sum_{n=1}^{N} v_n^{-2} \hat{S}_{11}^{(n)} + c_{22} \sum_{n=1}^{N} v_n^{-2} \hat{S}_{22}^{(n)} + \frac{1}{2} \sum_{i,j=1}^{2} \sum_{m,n=1}^{N} v_n^{-1} v_m^{-1} (\hat{S}_{12}^{(m)} \hat{S}_{21}^{(n)} + \hat{S}_{21}^{(m)} \hat{S}_{12}^{(n)}) +$$
$$+ c \sum_{m=1}^{N} (\hat{S}_{11}^{(m)} - \hat{S}_{22}^{(m)} + \mathrm{Id}) \sum_{n=1}^{N} v_n^{-1} \hat{S}_{21}^{(n)}. \tag{4.26}$$

Using the fermionization formulae (2.22) we obtain the following BCS-type Hamiltonian

$$\hat{H} = c_{11} \sum_{n=1}^{N} v_n^{-2} c^\dagger_{n,\epsilon'} c_{n,\epsilon'} + c_{22} \sum_{n=1}^{N} v_n^{-2} c_{n,\epsilon} c^\dagger_{n,\epsilon} +$$
$$+ \frac{1}{2} \sum_{m,n=1}^{N} v_n^{-1} v_m^{-1} (c^\dagger_{m,\epsilon'} c^\dagger_{m,\epsilon} c_{n,\epsilon} c_{n,\epsilon'} + c_{m,\epsilon} c_{m,\epsilon'} c^\dagger_{n,\epsilon'} c^\dagger_{n,\epsilon}) +$$
$$+ c \sum_{m=1}^{N} (c^\dagger_{m,\epsilon'} c_{m,\epsilon'} - c_{m,\epsilon} c^\dagger_{m,\epsilon} + 1) \sum_{n=1}^{N} v_n^{-1} c_{n,\epsilon} c_{n,\epsilon'}, \quad (4.27)$$

where $\epsilon, \epsilon' \in 1, 2$ and $\epsilon \neq \epsilon'$. Moreover, it can be shown that, using the fermion anti-commutation relations, this Hamiltonian takes the form (1.7), up to the term proportional to the identity operator.

The Hamiltonian (4.27) is a one-parametric deformation of the $p + ip$ BCS Hamiltonian [23,24]. Indeed, its first two terms are the kinetic ones. They acquire a standard form upon putting $c_{22} = -c_{11}$. The third term is a $p + ip$ the pairing interaction term. Finally, the fourth term is a new term that is due to the additional term in the considered r-matrix. In the limit $c \to 0$ the Hamiltonian (4.27) coincides with the $p + ip$ BCS Hamiltonian [23,24].

4.5.2. The spectra

In the case when we specify $\lambda_2^{(k)} = 1$, $\lambda_1^{(k)} = 0$, $k \in \overline{1, N}$ the Bethe equations (4.22) become

$$\sum_{s=1}^{N} \frac{v_k^2}{v_k^2 - v_s^2} - \sum_{s=1, s\neq k}^{M} \frac{2v_k^2}{v_k^2 - v_s^2} = c_{22} - c_{11} + 1, \qquad k \in \overline{1, M}, \quad (4.28)$$

and the spectra of the generalized Gaudin Hamiltonians become

$$h_n = \sum_{k=1, k\neq n}^{N} \frac{v_n^2}{v_k^2 - v_n^2} + c_{22} - \frac{1}{2} + \sum_{i=1}^{M} \frac{v_n^2}{v_i^2 - v_n^2}, \quad (4.29)$$

where the rapidities v_i satisfy the Bethe equations (4.28). Furthermore, up to the constant, the spectrum of the BCS-like Hamiltonian (4.27) is given by

$$h = \sum_{n=1}^{N} \sum_{i=1}^{M} \frac{1}{v_i^2 - v_n^2} = (c_{22} - c_{11} + 1) \sum_{i=1}^{M} \frac{1}{v_i^2}, \quad (4.30)$$

where we have used the definition of $\hat{H}$ (4.25) and the Bethe equations (4.28).

It follows that the spectrum of the Hamiltonian $\hat{H}$ is the same as in the $p_x + ip_y$ case [22–24], but that the Bethe vectors are different.

CRediT authorship contribution statement

T. Skrypnyk: Conceptualization, Methodology, Draft preparation. N. Manojlovich: Methodology, Reviewing and Editing.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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References

[1] M. Gaudin, J. Phys. 37 (1976) 1087.
[2] R.W. Richardson, N. Sherman, Nucl. Phys. 52 (1964) 221;
R.W. Richardson, N. Sherman, Nucl. Phys. 52 (1964) 253.
[3] E. Sklyanin, On the integrability of Landau-Lifshitz equation, preprint LOMI E-3, 1979.
[4] A. Belavin, V. Drinfeld, Funct. Anal. Appl. 16 (3) (1982) 1.
[5] E. Sklyanin, Zap. LOMI 164 (1987) 151.
[6] B. Jurco, J. Math. Phys. 30 (1989) 1289.
[7] L. Freidel, J.M. Maillet, Phys. Lett. B 262 (1991) 278.
[8] O. Babelon, C. Viallet, Phys. Lett. B 237 (1990) 411.
[9] J. Avan, M. Talon, Phys. Lett. B 241 (1990) 77.
[10] A. Stolin, Math. Scand. 69 (1991) 57–80.
[11] M.C. Cambiaggio, A.M.F. Rivas, M. Saraceno, Nucl. Phys. A 624 (1997) 157.
[12] L. Amico, A. Di Lorenzo, A. Osterloh, Phys. Rev. Lett. 86 (2001) 5759.
[13] J. Dukelsky, C. Esebbag, P. Schuck, Phys. Rev. Lett. 87 (2001) 066403.
[14] L.D. Faddeev, in: Proceedings of the Les Houches Summer School, North Holland, 1998, pp. 149–219.
[15] E. Sklyanin, Quantum inverse scattering method, Selected topics: arxiv preprint, arXiv:hep-th/9211111.
[16] T. Skrypnyk, Phys. Lett. A 334 (5–6) (2005) 390–399;
T. Skrypnyk, Phys. Lett. A 347 (4) (2005) 266–267.
[17] T. Skrypnyk, J. Geom. Phys. 57 (1) (2006).
[18] T. Skrypnyk, J. Math. Phys. 47 (3) (2006).
[19] T. Skrypnyk, J. Math. Phys. 48 (2) (2007) 023506.
[20] T. Skrypnyk, J. Math. Phys. 48 (11) (2007) 113521.
[21] T. Skrypnyk, J. Phys. A 40 (7) (2007) 1611–1623.
[22] T. Skrypnyk, Nucl. Phys. B 806 (3) (2009) 504–528.
[23] T. Skrypnyk, J. Math. Phys. 50 (3) (2009) 033504.
[24] T. Skrypnyk, J. Phys. A 42 (47) (2009) 472004.
[25] T. Skrypnyk, Nucl. Phys. B 891 (2015) 200–229.
[26] T. Skrypnyk, Nucl. Phys. B 913 (2016) 327–356.
[27] T. Skrypnyk, Nucl. Phys. B 941 (2019) 225–248.
[28] M. Ibañez, J. Links, G. Sierra, S.-Y. Zhao, Phys. Rev. B 79 (2009) 180501(R).
[29] N. Cirilo António, N. Manojlović, J. Math. Phys. 46 (10) (2005) 102701.
[30] P.P. Kulish, N. Manojlović, M. Samsonov, A. Stolin, Proc. Est. Acad. Sci. 59 (2010) 326–331.
[31] N. Cirilo António, N. Manojlović, A. Stolin, J. Math. Phys. 52 (10) (2011) 103501.
[32] N. Cirilo António, N. Manojlović, I. Salom, Nucl. Phys. B 889 (2014) 87–108.
[33] I. Lukyanenko, P.S. Isaac, J. Links, Nucl. Phys. B 886 (2014) 364–398.
[34] N. Cirilo António, N. Manojlović, E. Ragoucy, I. Salom, Nucl. Phys. B 893 (2015) 305–331.
[35] J. Links, Nucl. Phys. B 916 (2017) 117–131.
[36] N. Manojlović, I. Salom, Nucl. Phys. B 923 (2017) 73–106.
[37] Y. Shen, P.S. Isaac, J. Links, Nucl. Phys. B 937 (2018) 28–55.
[38] C. Dimo, A. Faribault, J. Phys. A 51 (32) (2018).
[39] P. Claeys, C. Dimo, S. De Baerdemacker, A. Faribault, J. Phys. A 52 (8) (2019) 09LT01.
[40] V. Caudrelier, N. Crampe, Lett. Math. Phys. 109 (4) (2019) 843–856.
[41] N. Manojlović, I. Salom, Symmetry 12 (3) (2020) 352.