Covariant Schwarz maps in $M_2(\mathbb{C})$

Dariusz Chruściński

Institute of Physics, Faculty of Physics, Astronomy and Informatics, Nicolaus Copernicus University, Grudziadzka 5/7, 87-100 Toruń, Poland

E-mail: darch@fizyka.umk.pl

Abstract. We analyze a class of qubit maps displaying diagonal unitary and orthogonal symmetries. For unital maps we characterize all covariant maps satisfying an operator Schwarz inequality. In particular well known Pauli maps are completely characterized. Going beyond the unital case we consider recently proposed generalizations of Schwarz inequality and provide the corresponding necessary and sufficient conditions for the entire class of covariant maps.

1 Introduction

A linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ (with $M_n(\mathbb{C})$ being an algebra of $n \times n$ complex matrices) is covariant w.r.t. a subgroup G of $U(n)$ if

$$U\Phi(X)U^\dagger = \Phi(UXU^\dagger), \tag{1}$$

for all $X \in M_n(\mathbb{C})$ and all elements $U \in G$ (in what follows we identify G with its n-dimensional unitary representation). Recall, that Φ is positive whenever $\Phi(X) \geq 0$ for all $X \geq 0$ and completely positive whenever the extended map $\mathrm{id}_n \otimes \Phi$ is positive (id_n stands for the identity map in $M_n(\mathbb{C})$) [1, 2, 3]. Both positive and completely positive maps play important role in quantum physics. In particular completely positive trace-preserving maps provide mathematical representation for quantum channels – key objects in quantum information theory [4, 5]. On the other hand maps which are positive but not completely positive provide basic tool for entanglement detection [6, 7, 8]. Covariant maps (in particular covariant channels) were analyzed by many researchers (see e.g. [9, 10, 11, 12, 13, 14, 15]).

In this paper we consider maps which interpolate between positive and completely positive maps. A map is called unital if $\Phi(\mathbb{1}_n) = \mathbb{1}_n$ ($\mathbb{1}_n$ is an identity matrix in $M_n(\mathbb{C})$). A unital map Φ satisfies an operator Schwarz inequality if [16, 17, 18] (see also [1, 2, 3])

$$\Phi(X^\dagger X) \geq \Phi(X)^\dagger \Phi(X), \tag{2}$$

for all $X \in M_n(\mathbb{C})$. In what follows we call such Φ a *Schwarz map*. It turns out that any completely positive unital map is necessarily a Schwarz map. However, there are Schwarz maps which are not completely positive [18]. Note, that (2) implies that any Schwarz map is necessarily positive. However, there are unital positive maps which are not Schwarz. A prominent example is provided by a transposition map. Interestingly, any unital positive map satisfies (2) for Hermitian X [16]. Interestingly, it was observed by Choi [17, 18] that any positive unital map satisfies (2) for normal matrix X. Recall, that any positive trace-preserving map is a contraction in the trace norm in $M_n(\mathbb{C})$, i.e.

$$\|\Phi(X)\|_1 \leq \|X\|_1, \tag{3}$$

where $\|X\|_1 = \mathrm{Tr}|X|$. Similarly, any trace-preserving Schwarz map is a contraction in the Hilbert-Schmidt norm

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$$\|\Phi(X)\|_{\rm HS} \leq \|X\|_{\rm HS}, \tag{4}$$

where $\|X\|^2 = {\rm Tr}(X^\dagger X)$. Indeed, one has

$$\|\Phi(X)\|^2_{\rm HS} = {\rm Tr}(\Phi(X)^\dagger \Phi(X)) \leq {\rm Tr}\Phi(X^\dagger X) = {\rm Tr}(X^\dagger X) = \|X\|^2_{\rm HS}. \tag{5}$$

Any completely positive map is characterized by its Kraus representation

$$\Phi(X) = \sum_\ell K_\ell X K_\ell^\dagger, \tag{6}$$

and if the map is unital the Kraus operators K_ℓ satisfy $\sum_\ell K_\ell K_\ell^\dagger = \mathbb{1}_n$. Neither positive nor Schwarz maps admit simple Kraus-like representation which makes the problem of characterization and classification of positive and Schwarz maps very difficult. In $M_2(\mathbb{C})$ all positive maps are decomposable, i.e. can be represented as

$$\Phi = \Phi_1 + \Phi_2 \circ {\rm T}, \tag{7}$$

where Φ_1 and Φ_2 are completely positive and T stands for a transposition. In this paper we analyze linear maps $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ which are covariant w.r.t. a diagonal unitary matrices and diagonal orthogonal matrices [19, 20]. In particular we characterize covariant Schwarz maps which were recently analyzed in [21]. Moreover, we consider recently proposed generalizations of Schwarz maps beyond unital scenario and provide necessary and sufficient conditions for a covariant map to satisfy the generalized Schwarz inequality.

2 Covariant Schwarz maps

The characterization of bistochastic (i.e. unital and trace-reserving) Schwarz maps $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ was initiated in [22, 23] (see also [24, 25, 26] for further development). Recall, that any such map can be represented as follows: for $X = z_0 \mathbb{1}_2 + \boldsymbol{z} \cdot \boldsymbol{\sigma}$ one has

$$\Phi(X) = z_0 \mathbb{1}_2 + (T\boldsymbol{z}) \cdot \boldsymbol{\sigma}, \tag{8}$$

where $T \in M_3(\mathbb{R})$, $\boldsymbol{z} \in \mathbb{C}^3$, $z_0 \in \mathbb{C}$ and $\boldsymbol{\sigma} = (\sigma_1, \sigma_2, \sigma_3)$ is a vector of Pauli matrices. Matrix elements of T are defined via $T_{ij} = {\rm Tr}(\sigma_i \Phi(\sigma_j))$. Hence, essentially all properties of Φ are encoded into the real matrix T_{ij}. Authors of [22, 23] proved the following result

Theorem 1 *A bistochastic hermiticity-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ satisfies operator Schwarz inequality if and only if*

$$\|T\boldsymbol{z}\| \leq \|\boldsymbol{z}\|\ , \tag{9}$$

$$\|T(\boldsymbol{z} \times \overline{\boldsymbol{z}}) - T\boldsymbol{z} \times T\overline{\boldsymbol{z}}\| \leq \|\boldsymbol{z}\|^2 - \|T\boldsymbol{z}\|^2, \tag{10}$$

for all $\boldsymbol{z} \in \mathbb{C}^3$ (where $\boldsymbol{a} \times \boldsymbol{b}$ stands for the standard vector product in $\mathbb{C}^3$ and $\|\boldsymbol{z}\|^2 = \sum_k |z_k|^2$).

Note, that if $\boldsymbol{z} = \boldsymbol{x} \in \mathbb{R}^3$, then (10) reduces to (9) and hence one recovers well known result

Corollary 1 *A unital and trace-preserving map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is positive if and only if $\|T\boldsymbol{x}\| \leq \|\boldsymbol{x}\|$ for all $\boldsymbol{x} \in \mathbb{R}^3$.*

In this paper we relax the condition of bistochasticity, i.e. we do not assume that Φ is trace-preserving, but restrict our analysis to the class of maps satisfying additional symmetry requirements. A linear map $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ is covariant w.r.t. diagonal orthogonal matrices iff [19, 20]

$$\Phi(X) = \left(\begin{array}{cc} a_{11}X_{11} + a_{12}X_{22} & \lambda X_{12} + \overline{\mu} X_{21} \\ \overline{\lambda} X_{21} + \mu X_{12} & a_{21}X_{11} + a_{22}X_{22} \end{array} \right), \tag{11}$$

where X_{ij} are matrix elements of $X \in M_2(\mathbb{C})$. Moreover, Φ is covariant w.r.t. diagonal unitary matrices if additionally $\mu = 0$. Note, that Φ enjoys diagonal orthogonal covariance if

$$\sigma_z \Phi(X) \sigma_z = \Phi(\sigma_z X \sigma_z), \tag{12}$$

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and diagonal unitary covariance if

$$e^{-i\sigma_z\phi}\Phi(X)e^{i\sigma_z\phi} = \Phi(e^{-i\sigma_z\phi}Xe^{i\sigma_z\phi}), \quad \phi \in \mathbb{R}, \tag{13}$$

and in this case it is often called a *phase covariant*.

Proposition 1 *A family of invertible maps (11) defines a noncommutative group and invertible phase-covariant maps (i.e. $\mu = 0$) define its subgroup.*

Identity map corresponds to $a_{11} = a_{22} = 1$, $a_{12} = a_{21} = 0$, $\lambda = 1$, and $\mu = 0$. Now, if Φ' is parameterized by a'_{ij}, λ', and μ', then $\widetilde{\Phi} := \Phi' \circ \Phi$ belongs to (11) and it is parameterized by

$$\tilde{a}_{ij} = \sum_{k=1}^{2} a'_{ik} a_{kj},$$

together with

$$\tilde{\lambda} = \lambda\lambda' + \mu\overline{\mu}' \ , \quad \tilde{\mu} = \overline{\lambda}\overline{\mu}' + \overline{\mu}\lambda'.$$

Finally, the inverse of Φ corresponds to the inverse of a_{ij} matrix, together with parameters ℓ and m defined by

$$\ell = \frac{\overline{\lambda}}{|\lambda|^2 - |\mu|^2} \ , \quad m = \frac{\mu}{|\mu|^2 - |\lambda|^2}.$$

Note, that Φ is Hermiticity preserving, i.e. $\Phi(X^\dagger) = \Phi(X)^\dagger$ if $a_{ij} \in \mathbb{R}$. Moreover, Φ is trace-preserving if $\sum_i a_{ij} = 1$ and unital if $\sum_j a_{ij} = 1$. One proves [8, 19]

Proposition 2 *A Hermiticity preserving map (11) is positive iff $a_{ij} \geq 0$ together with*

$$|\lambda| + |\mu| \leq \sqrt{a_{11}a_{22}} + \sqrt{a_{12}a_{21}}, \tag{14}$$

Moreover, Φ is completely positive iff

$$|\lambda| \leq \sqrt{a_{11}a_{22}} \ , \quad \text{and} \quad |\mu| \leq \sqrt{a_{12}a_{21}}. \tag{15}$$

If the map (11) is unital it is convenient to introduce a new parametrization

$$\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix} = \begin{pmatrix} a & 1-a \\ 1-b & b \end{pmatrix} \ , \qquad a, b \in [0,1]. \tag{16}$$

One proves [21]

Proposition 3 *A positive unital map (11) is Schwarz iff*

$$\frac{|\lambda|^2}{a} + \frac{|\mu|^2}{1-a} \leq 1, \quad \text{and} \quad \frac{|\lambda|^2}{b} + \frac{|\mu|^2}{1-b} \leq 1. \tag{17}$$

If the map is doubly stochastic, i.e. both unital and trace-preserving, then $a = b$ and conditions (17) reduces to a single elliptic condition.

Corollary 2 *A unital phase covariant map (11) is Schwarz iff $a, b \in [0,1]$ and*

$$|\lambda| \leq \min\{\sqrt{a}, \sqrt{b}\}. \tag{18}$$

Example 1 *A first example of a Schwarz map which is not completely positive was proposed by Choi [18]*

$$\Phi(X) = \frac{1}{4}\mathbb{1}_2 \mathrm{Tr} X + \frac{1}{2}X^{\mathrm{T}}, \tag{19}$$

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where X^{T} denotes transposed matrix. One finds that this map belongs to (11) with $a = b = \frac{3}{4}$, $\lambda = 0$, and $\mu = \frac{1}{2}$. Conditions (17) are satisfied and hence this map is Schwarz. However, it is not completely positive since (15) is violated. A well known reduction map [6]

$$\Phi(X) = \mathbb{1}_2 \mathrm{Tr} X - X, \tag{20}$$

corresponds to $a = b = 0$, $\mu = 0$, and $\lambda = -1$. It is positive (condition (14) holds), however, it is not Schwarz since (17) is violated. Similarly, a transposition map corresponds to $a = b = 1$, $\mu = 1$, and $\lambda = 0$. Again, condition (17) is violated.

A prominent example of a map (11) is provided by so called Pauli map, that is a map defined by

$$\Phi(X) = \sum_{\alpha=1}^{3} p_\alpha \sigma_\alpha X \sigma_\alpha, \tag{21}$$

where σ_α are Pauli matrices (as usual $\sigma_0 = \mathbb{1}_2$). This map is Hermiticity preserving if $p_\alpha \in \mathbb{R}$. It is unital (and at the same time trace-preserving) if $\sum_\alpha p_\alpha = 1$. Note, that Pauli maps belongs to (11) with

$$a_{11} = a_{22} = p_0 + p_3 \ , \ a_{12} = a_{21} = p_1 + p_2 \ , \ \lambda = p_0 - p_3 \ , \ \mu = p_1 - p_2 .$$

Corollary 3 *A Hermiticity preserving unital Pauli map is*

- *positive iff $|p_0 - p_3| + |p_1 - p_2| \leq 1$,*
- *completely positive iff $p_\alpha \geq 0$,*
- *Schwarz iff*

$$\frac{(p_0 - p_3)^2}{p_0 + p_3} + \frac{(p_1 - p_2)^2}{p_1 + p_2} \leq 1. \tag{22}$$

It is well known that if a Pauli map is positive but not completely positive then only single p_α is negative. Let $p_0 = -\alpha < 0$. The following unital Pauli map

$$\Phi_\alpha(X) = \frac{1}{3 - \alpha}(\sigma_1 X \sigma_1 + \sigma_2 X \sigma_2 + \sigma_3 X \sigma_3 - \alpha X), \tag{23}$$

is positive iff $\alpha \leq 1$, and Schwarz iff $\alpha \leq \frac{1}{3}$. Note, that it corresponds

$$a = b = \frac{1 - \alpha}{3 - \alpha}, \quad \lambda = -\frac{1 + \alpha}{3 - \alpha}, \quad \mu = 0,$$

and hence for $\alpha = \frac{1}{3}$, it leads to

$$a = b = \frac{1}{4}, \quad \lambda = -\frac{1}{2}, \quad \mu = 0,$$

which is essentially equivalent to the Choi Schwarz map (19). Moreover, if $\alpha = 1$, then it recovers the reduction map. This simple example clearly shows the difference between positive and Schwarz qubit maps.

3 Generalized Schwarz maps

Recently, [27] the following generalization of Schwarz maps was proposed: a linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ is called a generalized Schwarz map if

$$\left(\begin{array}{cc} \Phi(\mathbb{1}_n) & \Phi(X) \\ \Phi(X)^\dagger & \Phi(X^\dagger X) \end{array} \right) \geq 0, \tag{24}$$

for all $X \in M_n(\mathbb{C})$. Let us observe that Φ is generalized Schwarz if and only if

$$\Phi(X^\dagger X) \geq \Phi(X)^\dagger \Phi(\mathbb{1}_n)^+ \Phi(X), \tag{25}$$

where X^+ denotes a generalized Moore-Penrose inverse of A [29]. It is clear that if Φ is unital, then $\Phi(\mathbb{1}_n) = \mathbb{1}_n$ and hence (25) reduces to (2). One proves [21]

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Proposition 4 *A covariant map (11) is generalized Schwarz if and only if $a_{ij} \geq 0$ together with*

$$\frac{|\lambda|^2}{a_{11}} + \frac{|\mu|^2}{a_{12}} \leq a_{21} + a_{22}, \quad \frac{|\lambda|^2}{a_{22}} + \frac{|\mu|^2}{a_{21}} \leq a_{11} + a_{12}. \tag{26}$$

Given a linear map $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ one defines its adjoint $\Phi^\ddagger$ w.r.t. the Hilbert-Schmidt inner product

$$\langle \Phi^\ddagger(X)|Y\rangle_{\rm HS} := \langle X|\Phi(Y)\rangle_{\rm HS}, \tag{27}$$

where $\langle X|Y\rangle_{\rm HS} = {\rm Tr}(X^\dagger Y)$. Recall that Φ is unital iff $\Phi^\ddagger$ is trace-preserving.

Proposition 5 *A class of maps (11) defines a unital C^*-algebra $\mathfrak{A}$.*

The involution is defined by the adjoint w.r.t. the Hilbert-Schmidt inner product. One easily finds that

$$\Phi^\ddagger(X) = \begin{pmatrix} a_{11}X_{11} + a_{21}X_{22} & \overline{\lambda}X_{12} + \overline{\mu}X_{21} \\ \lambda X_{21} + \mu X_{12} & a_{12}X_{11} + a_{22}X_{22} \end{pmatrix}, \tag{28}$$

and hence it does belong to (11).

Proposition 6 $\Phi^\ddagger$ *is (completely) positive iff* Φ *is (completely) positive.*

Such property is no longer true for Schwarz maps. Note, that if Φ is Schwarz then $\Phi^\ddagger$ in general is not unital and hence the operator Schwarz inequality (2) is not even well defined. Let Φ from (28) be a generalized Schwarz map. Is $\Phi^\ddagger$ generalized Schwarz? Interestingly, contrary to (complete) positivity the generalized Schwarz property is not preserved by the adjoint operation. However, one proves [21]

Proposition 7 *If Φ defined by (11) is bistochastic (i.e. unital and trace-preserving), then Φ is Schwarz if and only if $\Phi^\ddagger$ is Schwarz.*

Similarly, one proves

Proposition 8 *If Φ defined by (11) satisfies $\Phi(\mathbb{1}_2) = \Phi^\ddagger(\mathbb{1}_2)$, then Φ is generalized Schwarz if and only if $\Phi^\ddagger$ is generalized Schwarz.*

Indeed, condition $\Phi(\mathbb{1}_2) = \Phi^\ddagger(\mathbb{1}_2)$ is equivalent to $a_{12} = a_{21}$, and then conditions (26) are invariant under swapping $a_{12} \leftrightarrow a_{21}$.

Composition of any two Schwarz maps is again Schwarz due to

$$(\Phi_1 \circ \Phi_2)(X^\dagger X) = \Phi_1(\Phi_2(X^\dagger X)) \geq \Phi_1(\Phi_2(X)^\dagger \Phi_2(X)) \geq (\Phi_1 \circ \Phi_2)(X)^\dagger (\Phi_1 \circ \Phi_2)(X).$$

However, a composition of two generalized maps from (11) is no longer generalized Schwarz [28]

Example 2 *Indeed, consider two maps: Φ_1 corresponding to*

$$a_{11} = a_{22} = 1,\ a_{12} = 2,\ a_{21} = 1,\ \lambda = \sqrt{2},\ \mu = 0,$$

and Φ_2 corresponding to

$$a_{11} = 1,\ a_{22} = 2,\ a_{12} = a_{21} = 0,\ \lambda = \sqrt{2},\ \mu = 0.$$

These two maps are generalized Schwarz. One finds that $\Phi_1 \circ \Phi_2$ corresponds to

$$a_{11} = 1,\ a_{22} = 2,\ a_{12} = 4,\ a_{21} = 1,\ \lambda = 2,\ \mu = 0,$$

and violates (26). Hence $\Phi_1 \circ \Phi_2$ is not generalized Schwarz.

However, one proves

Proposition 9 *If Φ_1 and Φ_2 are phase-covariant generalized Schwarz and satisfy $\Phi_\ell(\mathbb{1}_2) = \Phi_\ell^\ddagger(\mathbb{1}_2)$, then $\Phi_1 \circ \Phi_2$ is generalized Schwarz.*

Note, that in Example 2 one has $\Phi_1(\mathbb{1}_2) \neq \Phi_1^\ddagger(\mathbb{1}_2)$. We conjecture that the above property holds for the entire class of covariant maps (11).

Recall, that any 2-positive map is necessarily generalized Schwarz [27]. Clearly, composition of two 2-positive maps $\Phi_1 \circ \Phi_2$ is 2-positive and hence generalized Schwarz. One has

$$(\Phi_1 \circ \Phi_2)(X^\dagger X) \geq \Phi_1(\Phi_2(X^\dagger)\Phi_2(\mathbb{1}_n)^{-1}\Phi_2(X)) = \Phi_1((Y\Phi_2(X))^\dagger(Y\Phi_2(X)),$$

where $Y = \Phi_2(\mathbb{1}_n)^{-1/2}$. Now, any 2-positive map Φ satisfies [1]

$$\Phi(A^\dagger A) \geq \Phi(A^\dagger B)\Phi(B^\dagger B)^{-1}\Phi(B^\dagger A),$$

and hence

$$(\Phi_1 \circ \Phi_2)(X^\dagger X) \geq \Phi_1((Y\Phi_2(X))^\dagger Y^{-1})\Phi_1(YY)^{-1}\Phi_1(Y^{-1}(Y\Phi_2(X)),$$

which proves that indeed

$$(\Phi_1 \circ \Phi_2)(X^\dagger X) \geq \Phi_1(\Phi_2(X^\dagger))\Phi_1(\Phi_2(\mathbb{1}_n))^{-1}\Phi_1(\Phi_2(X)),$$

that is, $\Phi_1 \circ \Phi_2$ is generalized Schwarz. Note, however, it is no longer true if Φ_1 is not 2-positive.

Recently, [30] another generalization of Schwarz map was considered, namely

$$\|\Phi\|_\infty \Phi(X^\dagger X) \geq \Phi(X)^\dagger \Phi(X), \tag{29}$$

where

$$\|\Phi\|_\infty := \sup_{\|X\|_\infty = 1} \|\Phi(X)\|_\infty,$$

and $\|X\|_\infty$ stands for an operator norm. Recall, that if Φ is a positive map then $\|\Phi\|_\infty = \|\Phi(\mathbb{1}_n)\|_\infty$. If Φ is unital one has $\|\Phi\|_\infty = 1$ and hence (29) reduces to the original Schwarz inequality. Note, that (25) implies (29), that is, one has

$$\Phi(X^\dagger X) \geq \Phi(X)^\dagger \Phi(\mathbb{1}_n)^+ \Phi(X) \geq \frac{1}{\|\Phi\|_\infty} \Phi(X)^\dagger \Phi(X). \tag{30}$$

One has the following

Proposition 10 *A covariant map (11) is generalized Schwarz if and only if* $a_{ij} \geq 0$ *together with*

$$\frac{|\lambda|^2}{a_{11}} + \frac{|\mu|^2}{a_{12}} \leq \|\Phi\|_\infty, \quad \frac{|\lambda|^2}{a_{22}} + \frac{|\mu|^2}{a_{21}} \leq \|\Phi\|_\infty. \tag{31}$$

The above results provide generalization of Proposition 4. Interestingly, the following results holds

Proposition 11 *If* Φ_1 *and unital* Φ_2 *satisfy (29) then the composition* $\Phi_1 \circ \Phi_2$ *satisfies (29) as well.*

4 Conclusions

We provided a detailed analysis of the class of covariant maps (11). In particular we characterized maps satisfying an operator Schwarz inequality (so called Schwarz maps). Bistochastic Schwarz qubit maps we considered recently in [22, 23]. In this paper we go beyond bistochastic maps but restrict our analysis to the class of covariant maps. In particular the complete characterization of (bistochastic) Pauli maps was provided. Presented results provide completion of the recent paper [21]. In what followed we consider recently proposed generalizations of Schwarz inequality and provided necessary and sufficient condition within a class of covariant maps (11). It would be interesting to generalize our analysis to higher dimensional matrix algebras $M_n(\mathbb{C})$.

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