

## Article

# Sasaki–Ricci Flow and Deformations of Contact Action–Angle Coordinates on Spaces $T^{1,1}$ and $Y^{p,q}$

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**Abstract:** In this paper, we are concerned with completely integrable Hamiltonian systems and generalized action–angle coordinates in the setting of contact geometry. We investigate the deformations of the Sasaki–Einstein structures, keeping the Reeb vector field fixed, but changing the contact form. We examine the modifications of the action–angle coordinates by the Sasaki–Ricci flow. We then pass to the particular cases of the contact structures of the five-dimensional Sasaki–Einstein manifolds  $T^{1,1}$  and  $Y^{p,q}$ .

**Keywords:** contact geometry; Sasaki–Einstein space; Sasaki–Ricci flow; contact Hamiltonian systems

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## 1. Introduction

Over the past four decades, contact geometry has undergone a rapid development in pure mathematics [1] and in applied areas as mechanics, dissipative systems, optics, thermodynamics, or control theory [2].

As it is well-known, the description of Hamiltonian mechanics is developed on symplectic manifolds. Contact geometry has been applied to give a Hamiltonian-type description of mechanical systems with dissipation [3], field theories, and gravitation in an odd number of dimensions, Sasaki–Einstein geometries [4,5]. An analogous theory to complete integrability in symplectic geometry was constructed in contact geometry [6,7].

A well-known method for generating Einstein metrics on manifolds is the Ricci flow introduced by Hamilton in [8] and extended to Kähler manifolds in [9]. Recently, the method was applied to Sasaki manifolds in [10] to generate new Sasaki structures, the authors providing the well-posedness.

In this paper, we concentrate on the special class of toric contact structures on  $S^2 \times S^3$  denoted by  $Y^{p,q}$  [11] which contain the homogeneous space  $T^{1,1}$  as a special case [12].

We introduce local holomorphic coordinates and construct the Sasakian local potential, analogous to the Kähler potential. We investigate local deformations of Sasakian structures exploiting the transverse structure of Sasakian manifolds. On the analogy of Kähler–Ricci flow, Sasaki–Ricci flow preserves the Sasaki condition in the sense that the evolved metrics remain Sasaki.

We consider local deformations of Sasakian structures using particular basic functions which satisfy the Sasaki–Ricci flow equations. As in the case of standard symplectic dynamics, we introduce the action–angle variables and evaluate the frequencies of the flow of toric action.

The paper is organized as follows: we start by recalling some background in Sasaki geometry, deformations of Sasaki metrics, and Sasaki–Ricci flow. In Section 3, we present the Hamiltonian dynamics in the setting of contact geometry and introduce the generalized contact action–angle variables. In Section 4, the general results are applied to the five-dimensional Sasaki–Einstein spaces  $T^{1,1}$  and  $Y^{p,q}$ . In the final section, we provide some closing remarks.

## 2. Preliminaries

In this section, we review basic definitions and results concerning the geometry of Sasaki manifolds and their deformations by the Sasaki–Ricci flow.

### 2.1. Sasaki Manifolds

Let  $(M, g)$  be a Riemannian manifold. The cone manifold  $C(M)$  of  $M$  is a Riemannian manifold diffeomorphic to  $(0, \infty) \times M$  equipped with the metric

$$\bar{g} = dr^2 + r^2 g$$

where  $r$  is a coordinate on  $\mathbb{R}^+ = (0, +\infty)$ .

$M$  is said to be a Sasaki manifold if the cone manifold  $C(M)$  has a Kähler structure  $(J, \bar{g})$ . Notice that any Sasaki manifold  $M$  is of odd dimension  $2n + 1$  where  $n + 1$  is the complex dimension of the Kähler cone  $C(M)$ . If the Sasaki space is Einstein ( $Ric_g = 2ng$ ), then the Kähler metric cone is Ricci flat ( $Ric_{\bar{g}} = 0$ ), i.e., a Calabi–Yau manifold.

On  $C(M)$ , we have a vector field  $\bar{\xi}$  and a 1-form  $\bar{\eta}$  defined by

$$\bar{\xi} = J r \frac{\partial}{\partial r} \text{ and } \bar{\eta}(\cdot) = \frac{1}{r^2} \bar{g}(\bar{\xi}, \cdot)$$

respectively. The vector field  $\bar{\xi}$  restricted to  $M$  is called the *characteristic vector field* or the *Reeb vector field* (let us note it by  $\xi$ ). Let now  $\mathcal{D} = \ker \eta$ , where  $\eta$  is the restriction of  $\bar{\eta}$  to  $M$ . We have the  $g$ -splitting of the tangent bundle  $TM$  of  $M$ :

$$TM = \mathcal{D} \oplus L_{\xi}$$

where  $L_{\xi}$  is the trivial line bundle generated by  $\xi$ .

Restrict  $J$  to  $\mathcal{D}$  and extend it to an endomorphism  $\Phi \in \text{End}(TM)$  by setting  $\Phi \xi = 0$ .  $\Phi$  satisfies

$$\Phi^2 = -1 + \eta \otimes \xi$$

and

$$g(\Phi(X), \Phi(Y)) = g(X, Y) - \eta(X)\eta(Y)$$

for any smooth vector fields  $X, Y$  on  $M$ .

We have a global 2-form  $\Omega^T$  on  $M$  coming from the contact 1-form  $\eta$

$$\Omega^T = \frac{1}{2} d\eta.$$

We get that  $(\mathcal{D}, \Phi|_{\mathcal{D}}, d\eta)$  gives  $M$  a transverse Kähler structure with Kähler form  $\Omega^T$  and transverse metric  $g^T$  given by

$$g^T(X, Y) = d\eta(X, \Phi Y)$$

for any smooth vector fields  $X, Y$  on  $M$  and related to the Sasakian metric  $g$  on  $M$  by

$$g = g^T + \eta \otimes \eta.$$

Using the transverse metric  $g^T$ , it is possible to define a connection  $\nabla^T$  on  $\mathcal{D}$  which is torsion free such that  $\nabla^T g^T = 0$ . Moreover, the Sasaki–Einstein manifold is transverse Kähler–Einstein [1,4].

One can choose local coordinates  $(x, z^1, \dots, z^n)$  on a small neighborhood  $U = I \times V$  of  $M$  with  $I \in \mathbb{R}$  and  $V \in \mathbb{C}^n$ . In the chart  $U$ , we may write [13]

$$\begin{aligned}
\xi &= \frac{\partial}{\partial x} \\
\eta &= dx + i \sum_{j=1}^n (K_{j\bar{j}} dz^j) - i \sum_{\bar{j}=1}^n (K_{j\bar{j}} d\bar{z}^j) \\
d\eta &= -2i \sum_{j,\bar{k}=1}^n K_{j\bar{k}} dz^j \wedge d\bar{z}^k \\
g &= \eta \otimes \eta + g^T = \eta \otimes \eta + 2 \sum_{j,\bar{k}=1}^n K_{j\bar{k}} dz^j d\bar{z}^k \\
\Phi &= -i \sum_{j=1}^n [(\partial_j - iK_{j\bar{j}} \partial_x) \otimes dz^j] + i \sum_{\bar{j}=1}^n (\partial_{\bar{j}} + iK_{j\bar{j}} \partial_x) \otimes d\bar{z}^j
\end{aligned}$$

where  $K : U \rightarrow \mathbb{R}$  is a local *basic function*, i.e.,  $\frac{\partial K}{\partial x} = 0$  and  $K_{j\bar{j}} = \frac{\partial K}{\partial z^j}$  and  $K_{j\bar{k}} = \frac{\partial^2 K}{\partial z^j \partial \bar{z}^k}$ . Every Sasakian manifold is locally generated by a real function  $K$ , called the *Sasaki potential*, which is the analogue of the Kähler potential.

## 2.2. Sasaki–Ricci Flow

There are various ways to deform Sasakian structures. We shall consider deformations keeping the Reeb field  $\xi$  fixed and varying the contact form  $\eta$  by perturbing it with a basic function  $\varphi$ :

$$\tilde{\eta} = \eta + d_B^c \varphi \quad (1)$$

where  $d_B^c = \frac{i}{2}(\bar{\partial}_B - \partial_B)$  with

$$\partial_B = \sum_{j=1}^n dz^j \frac{\partial}{\partial z^j} \quad \bar{\partial}_B = \sum_{j=1}^n d\bar{z}^j \frac{\partial}{\partial \bar{z}^j}.$$

To introduce the transverse Kähler–Ricci flow, also called Sasaki–Ricci flow, we consider the flow  $(\xi, \eta(t), \Phi(t), g(t))$  with initial data  $(\xi, \eta(0), \Phi(0), g(0)) = (\xi, \eta, \Phi, g)$  generated by a basic function  $\varphi(t)$ . The Sasaki–Ricci flow equation is [10,14]

$$\frac{\partial g^T}{\partial t} = -Ric_{g(t)}^T + (2n+2)g^T(t)$$

where  $Ric^T$  is the transverse Ricci curvature. In the case of the deformation (1) with a basic function  $\varphi$ , in local coordinates, the Sasaki–Ricci flow can be expressed as a parabolic Monge–Ampère equation [10,14]

$$\frac{\partial \varphi}{\partial t} = \ln \det(g_{j\bar{k}}^T + \varphi_{j\bar{k}}) - \ln(\det g_{j\bar{k}}^T) + (2n+2)\varphi. \quad (2)$$

It is shown in [14] that the flow is well-posed and preserves the Sasakian structure of the manifold. It was proved the existence of transverse Kähler–Ricci solitons (or Sasaki–Ricci solitons) on compact toric Sasakian manifolds, of which the basic first Chern form of the normal bundle of the Reeb foliation is positive and the first Chern class of the contact bundle is trivial. More details on stability and convergence of the Sasaki–Ricci flow can be found in [15].

## 3. Contact Hamiltonian Systems

A vector field  $X$  on the contact manifold  $(M, \mathcal{D})$  is called an infinitesimal automorphism of the contact structure if its flow preserves the contact structure  $\mathcal{D}$ .

The condition for  $X$  to be an infinitesimal automorphism can be written as [16]:

$$\mathcal{L}_X \eta = \rho \eta \quad (3)$$

for some function  $\rho : M \rightarrow \mathbb{R}$ .

The local flow of  $X$  preserves the contact form  $\eta$  if and only if  $\rho = 0$ , i.e.,  $\mathcal{L}_X\eta = 0$ .

Fixing a contact form  $\eta$ , a function  $h \in C^\infty(M)$  gives a unique Hamiltonian vector field  $X_h$  that satisfies

$$h = \eta(X_h). \quad (4)$$

The function  $h$  will be called the contact Hamiltonian associated with the vector field  $X_h$ .

The mapping (4) establishes a one-to-one correspondence between the vector space of infinitesimal contact automorphisms and smooth functions  $h : M \rightarrow \mathbb{R}$ .

Let  $X_h$  be an infinitesimal automorphism. From (3), we get

$$\mathcal{L}_{X_h} = \iota_{X_h} d\eta + d(\eta(X_h)) = \iota_{X_h} d\eta + dh = \rho\eta.$$

Applying this equation to the Reeb vector field  $\xi$ , we have

$$\rho = \xi(h).$$

Note that  $X_h$  preserves the contact form ( $\rho = 0$ ) if and only if the Hamiltonian function  $h$  is basic.

In connection with the isomorphism (4), the Lie algebra of  $C^\infty(M)$  functions is given by the Jacobi bracket [16,17]

$$[f, g]_\eta = \eta([X_f, X_g]). \quad (5)$$

Sometimes it is considered the function  $1 = \eta(\xi)$  as the Hamiltonian making the Reeb vector field the Hamiltonian vector field.

A smooth function  $f \in C^\infty(M)$  is a first integral of the contact Hamiltonian structure  $(M, \mathcal{D}, \eta, h)$  if  $f$  is constant along the flow of the Hamiltonian vector field  $X_h$ , i.e.,  $X_h f = 0$ . The subset of first integrals  $\{h, f_1, \dots, f_j\}$  is independent if the corresponding set  $\{X_h, X_{f_1}, \dots, X_{f_j}\}$  of Hamiltonian vector fields is pointwise linearly independent on a dense open set.

A Hamiltonian contact structure is completely integrable if there exists  $(n+1)$  first integrals  $\{h, f_1, \dots, f_n\}$  that are independent and in involution with respect to the Jacobi bracket (5). In addition, a completely integrable Hamiltonian contact structure is said to be of toric type if the corresponding vector fields  $\mathcal{X} = \{X_h, X_{f_1}, \dots, X_{f_n}\}$  form the Lie algebra of a torus  $\mathbb{T}^{n+1}$ . The action of a torus  $\mathbb{T}^{n+1}$  on a contact  $(2n+1)$ -dimensional manifold  $(M, \eta)$  is completely integrable if it is effective and preserves the contact structure  $\eta$  [18]. The Reeb vector field is an element of this algebra. In this case, we have a regular completely integrable contact structure studied in [6]. It is possible to find the local coordinates  $(\theta_0, \theta_1, \dots, \theta_n, y_1, \dots, y_n)$  such that the contact form has the following canonical form:

$$\eta_0 = y_0 d\theta_0 + y_1 d\theta_1 + \dots + y_n d\theta_n$$

where  $y_0$  is a smooth function of  $(y_1, \dots, y_n)$ . We refer the set of local coordinates  $(\theta_0, \theta_1, \dots, \theta_n, y_1, \dots, y_n)$  as generalized contact action-angle coordinates [7].

The flow of  $\mathcal{X}$  on invariant tori is quasi-periodic

$$(\theta_0, \theta_1, \dots, \theta_n) \longrightarrow (\theta_0 + t\omega_0, \theta_1 + t\omega_1, \dots, \theta_n + t\omega_n) \quad , \quad t \in \mathbb{R}$$

where frequencies  $\omega_0, \dots, \omega_n$  depend only on  $y_i$  [6,7].

Consider now the vector field  $Y = -f\xi$ , where  $f$  is a basic function, a first integral of the Reeb vector field  $\xi$ . The flow  $\phi_t$  of  $Y$  is a complete flow and preserves the toric fibration. Let us define the family of 1-forms

$$\eta_t = \eta_0 + tdf \quad (6)$$

which is also a contact form having the Reeb vector field  $\xi$ . Using the Moser's deformation [6,19], we have

$$\mathcal{L}(Y)\eta_t = -df = -\frac{\partial\eta_t}{\partial t}$$

which imply

$$\frac{d}{dt}(\phi_t^*\eta_t) = \phi^*\left(\mathcal{L}(Y)\eta_t + \frac{\partial\eta_t}{\partial t}\right) = 0.$$

Therefore,  $\phi_1^*\eta_1 = \eta_0$  and we can obtain the coordinates in which the 1-form (6) has the canonical form. Choosing in turn the first integrals  $f_i$  of the completely integrable Hamiltonian contact structure, a change of variables  $\phi = \phi_{-1}$  permits to extract the frequencies  $\omega_i$ .

#### 4. Action–Angle Coordinates and Sasaki–Ricci Flow on Spaces $T^{1,1}$ and $Y^{p,q}$

In this section we consider the Sasaki–Ricci flow on five-dimensional Sasaki–Einstein spaces  $T^{1,1}$  and  $Y^{p,q}$ . We evaluate the action–angle coordinates for these spaces and produce some explicit solutions of the Sasaki–Ricci flow equation.

##### 4.1. Sasaki–Einstein Space $T^{1,1}$

We recall that  $T^{1,1} = S^2 \times S^3$  is one of the most renowned example of homogeneous Sasaki–Einstein space in five-dimensions.

The standard metric on this manifold is [12,20]

$$ds^2 = \frac{1}{6}(d\theta_1^2 + \sin^2\theta_1 d\phi_1^2 + d\theta_2^2 + \sin^2\theta_2 d\phi_2^2) + \frac{1}{9}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2)^2$$

where  $\theta_i \in [0, \pi)$ ,  $\phi_i \in [0, 2\pi)$ ,  $i = 1, 2$  and  $\psi \in [0, 4\pi)$ . The contact 1-form  $\eta$  is

$$\eta = \frac{1}{3}(d\psi + \cos\theta_1 d\phi_1 + \cos\theta_2 d\phi_2) \quad (7)$$

and the Reeb vector field has the form

$$\xi = 3\frac{\partial}{\partial\psi}. \quad (8)$$

In what follows, we introduce  $\nu = \frac{1}{2}\psi$  so that  $\nu$  has canonical period  $2\pi$ .

The Hamiltonian contact structure of Reeb type  $(T^{1,1}, \eta, \xi)$  is completely integrable. To describe the effectively acting  $\mathbb{T}^3$  action, we employ the basis [12]

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial}{\partial\phi_1} + \frac{1}{2}\frac{\partial}{\partial\nu} \\ \mathbf{e}_2 &= \frac{\partial}{\partial\phi_2} + \frac{1}{2}\frac{\partial}{\partial\nu} \\ \mathbf{e}_3 &= \frac{\partial}{\partial\nu} \end{aligned}$$

which preserves the contact structure  $\eta$ .

As on  $T^{1,1}$ , the transverse structure is locally isomorphic to a product  $S^2 \times S^2$ , for each  $S^2$  sphere the complex coordinates  $z^j$  are related to the spherical coordinates as

$$z^j = \tan\frac{\theta_j}{2}e^{i\phi_j} \quad j = 1, 2. \quad (9)$$

The Sasaki potential of the transverse metric  $g^T$  is

$$K = \frac{1}{3} \sum_{j=1}^2 \ln(1 + z^j \bar{z}^j) - \frac{1}{6} \sum_{j=1}^2 \ln(z^j \bar{z}^j).$$

Let  $\mathcal{F} = (f_0, f_1, f_2)$  the set of independent first integrals in involution and  $\mathcal{X} = (R_\eta, X_{f_1}, X_{f_2})$  the corresponding set of infinitesimal automorphisms of  $\eta$ . Let  $T$  be a compact connected component of the level set  $\{f_1 = c_1, f_2 = c_2\}$  and  $df_1 \wedge df_2 \neq 0$  on  $T$ .  $T$  is diffeomorphic to a  $T^3$  torus and there exist a neighborhood  $U$  of  $T$  and a diffeomorphism  $\phi : U \rightarrow T^3 \times D$

$$\phi(x) = (\vartheta_0, \vartheta_1, \vartheta_2, y_1, y_2)$$

where  $D \in \mathbb{R}^2$ , such that the contact form has the following canonical expression [6,21]:

$$\eta_0 = (\phi^{-1})^* \eta = y_0 d\vartheta_0 + y_1 d\vartheta_1 + y_2 d\vartheta_2.$$

Note that  $\eta_0(\frac{\partial}{\partial \vartheta_i}) = y_i$  are the contact Hamiltonians of the independent set of vector fields  $\mathcal{X}$ . Let us remark that the action of the torus  $T^3$  is given by translations of the angles  $\vartheta_i$ .

Taking into account the 1-form  $\eta$  (7), it is convenient to choose

$$\vartheta_0 = \frac{2}{3}\nu \quad \vartheta_1 = \phi_1 \quad \vartheta_2 = \phi_2$$

and accordingly we have

$$y_0 = 1 \quad y_1 = \frac{1}{3} \cos \theta_1 \quad y_2 = \frac{1}{3} \cos \theta_2.$$

These functions are first integrals of the Hamiltonian contact structure

$$f_0 = y_0 = 1, \quad f_i = y_i = \frac{1}{3} \cos \theta_i \quad , \quad i = 1, 2 \quad (10)$$

which are independent and in involution

$$[1, f_i]_\eta = [f_i, f_j]_\eta = 0 \quad , \quad i, j = 1, 2$$

as can be seen through a direct evaluation of the respective Jacobi brackets (5).

The flows of the set  $\mathcal{X}$  on invariant tori is quasi-periodic

$$(\vartheta_0, \vartheta_1, \vartheta_2) \rightarrow (\vartheta_0 + t\omega_0, \vartheta_1 + t\omega_1, \vartheta_2 + t\omega_2) \quad (11)$$

where the frequencies  $\omega_i$  depend only on  $y_i$ .

As it was shown in Section 3, we consider the vector field  $X = -f\xi$ , where  $f$  is a basic function, a first integral of the Reeb vector field  $\xi$ . Choosing the first integrals  $f_i = y_i$  as in (10), a simple calculation permits us to extract the frequencies

$$\omega_i = \ln \cos \theta_i \quad , \quad i = 1, 2.$$

Finally, we evaluate the modifications of the action-angle coordinates by the Sasaki-Ricci flow. We consider deformations of the contact form (7) with a basic function  $\varphi$  solution of the Sasaki-Ricci flow Equation (2). For a concrete realization of the deformation of the contact structure, we need an explicit analytical solution of the Sasaki-Ricci flow equation.

For the Sasaki–Einstein space  $T^{1,1}$ , a distinguished class of solutions of the Sasaki–Ricci flow equation is represented by the following families of basic functions [22]

$$\varphi(t) = (e^{6t} - 1) \sum_{j=1,2} \left[ c_j (\ln z^j + \ln \bar{z}^j) + d_j (\ln^2 z^j + \ln^2 \bar{z}^j) \right] \quad (12)$$

with  $c_j, d_j$  arbitrary constants and the complex coordinates  $z^j$  are given in (9).

In terms of angular coordinates, we have

**Proposition 1.** *The families of contact forms*

$$\tilde{\eta} = \eta + \frac{e^{6t} - 1}{2} \sum_j \left[ -c_j d\phi_j + d_j \frac{\phi_j}{\sin \theta_j} d\theta_j + d_j \ln \tan \frac{\theta_j}{2} d\phi_j \right] \quad (13)$$

with arbitrary real constants  $c_j, d_j$   $j = 1, 2$ , represent deformations of the canonical contact structure of  $T^{1,1}$ .

We remark that, if the constants  $d_j$  in (13) are not zero, the angles  $\phi_j$  interfere in the deformed metric and the Reeb vector field (8) remains the only Killing vector. Therefore, the primary toric symmetry of  $T^{1,1}$  is broken for this class of deformations. However, if the constants  $d_j = 0$ , the toric  $\mathbb{T}^3$  symmetry is preserved with the same angle coordinates. For the action coordinates, instead of the first integrals (10), we get the modified ones

$$\tilde{y}_i = \tilde{f}_i = \frac{1}{3} \cos \theta_i - c_j \frac{e^{6t} - 1}{2} \quad , \quad i = 1, 2 .$$

Regarding the frequencies, they are modified accordingly.

#### 4.2. Sasaki–Ricci Space $Y^{p,q}$

In the framework of AdS/CFT correspondences, spaces  $Y^{p,q}$  have been employed to provide an infinite class of dualities [23].

The metric of the Sasaki–Einstein space  $Y^{p,q}$  is given by the line element [12]

$$\begin{aligned} ds^2 = & \frac{1-y}{6} (d\theta^2 + \sin^2 \theta d\phi^2) + \frac{1}{w(y)q(y)} dy^2 + \frac{w(y)q(y)}{36} (d\beta - \cos \theta d\phi)^2 \\ & + \frac{1}{9} [d\psi + \cos \theta d\phi + y(d\beta - \cos \theta d\phi)]^2 \end{aligned}$$

where

$$\begin{aligned} w(y) &= \frac{2(a - y^2)}{1 - y} \\ q(y) &= \frac{a - 3y^2 + 2y^3}{a - y^2} \end{aligned}$$

and the constant  $a$  is chosen in the range  $0 < a < 1$ .

The contact 1-form  $\eta$  is [12]

$$\eta = \frac{1}{3} d\psi + \frac{1}{3} y d\beta + \frac{1-y}{3} \cos \theta d\phi \quad (14)$$

and the Reeb vector field is

$$\xi = 3 \frac{\partial}{\partial \psi} . \quad (15)$$

The angular coordinates span the ranges  $0 \leq \theta \leq \pi$ ,  $0 \leq \phi \leq 2\pi$ ,  $0 \leq \psi \leq 2\pi$ . In order to specify the range of the variable  $\beta$ , we note that it is connected with another variable  $\alpha$

$$\beta = -(6\alpha + \psi).$$

The range of  $\alpha$  is

$$0 \leq \alpha \leq 2\pi\ell$$

where

$$\ell = \frac{q}{3q^2 - 2p^2 + p(4p^2 - 3q^2)^{1/2}}.$$

We introduce a local set of transverse complex coordinates appropriate for the transverse Kähler structure of  $Y^{p,q}$  [24,25]

$$\begin{aligned} z^1 &= \tan \frac{\theta}{2} e^{i\phi} \\ z^2 &= \frac{\sin \theta}{f_1(y)} e^{i\beta} \end{aligned} \quad (16)$$

where

$$f_1(y) = \exp \left( \int \frac{3(1-y)}{a - 3y^2 + 2y^3} dy \right).$$

The Sasaki–Kähler potential of the transverse metric is [24]

$$K = \frac{1}{3} \left[ \left( 1 + \frac{1}{z^1 \bar{z}^1} \right) f_2(y) \right] + \frac{1}{6} \ln(z^1 \bar{z}^1) \quad (17)$$

where

$$f_2(y) = \exp \left( \int \frac{3y(1-y)}{a - 3y^2 + 2y^3} dy \right) = \frac{1}{\sqrt{a - 3y^2 + 2y^3}}.$$

As in the case of the Sasaki–Ricci flow equation for the space  $T^{1,1}$ , an explicit analytical solution can be found. Quite interestingly, a class of explicit solutions has the same form as in the case of the space  $T^{1,1}$  (12), but, of course, with the complex coordinates  $z^j$  given by Equation (16).

**Proposition 2.** *The families of contact forms*

$$\begin{aligned} \tilde{\eta} = \eta + \frac{e^{6t} - 1}{2} \left[ \frac{c_1 \phi}{\sin \theta} d\theta + \left( -d_1 + c_1 \ln \tan \frac{\theta}{2} \right) d\phi \right. \\ \left. + \frac{c_2 \beta}{\rho} d\rho + (-d_2 + c_2 \ln \rho) d\beta \right] \end{aligned} \quad (18)$$

with real arbitrary constants  $c_j, d_j$ ,  $j = 1, 2$  represent deformations of the canonical contact structure of  $Y^{p,q}$ .

To find the action–angle coordinates, we choose the following basis of an effectively acting  $\mathbb{T}^3$  action [12,26]

$$\begin{aligned} \mathbf{e}_1 &= \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \psi} \\ \mathbf{e}_2 &= \frac{\partial}{\partial \phi} - \frac{(p-q)\ell}{2} \frac{\partial}{\partial \alpha} \\ \mathbf{e}_3 &= \ell \frac{\partial}{\partial \alpha}. \end{aligned}$$

To write the contact form (14) and the Reeb vector field (15) in the canonical forms, we introduce the angle variables

$$\vartheta_0 = \frac{\psi}{3} \quad \vartheta_1 = -6\alpha - \psi \quad \vartheta_2 = \phi$$

and the generalized action variables

$$y_0 \equiv 1 \quad y_1 = \frac{y}{3} \quad y_2 = \frac{y-1}{3} \cos \theta.$$

Concerning the frequencies, their evaluation proceeds as in the case of the space  $T^{1,1}$ .

**Remark 1.** We observe that the presence of the constants  $c_j, d_j$  in the deformed contact form (18) implies that the Reeb vector field (15) remains the only Killing vector of the deformed metric. Therefore, the initial toric symmetry is broken during the Sasaki–Ricci flow deformation.

## 5. Discussion

An important point of interest in physics is to find the conserved quantities and investigate the integrability of the systems. Having in mind that Sasaki–Einstein manifolds have become of significant interest in many areas of physics, we investigate the integrability in the frame of contact geometry.

Unlike the symplectic case, the contact structures are automatically Hamiltonian. Moreover, for the manifolds  $T^{1,1}$  and  $Y^{p,q}$ , the toric action  $\mathbb{T}^3$  is effective and preserves the contact structures implying the complete integrability. We introduce generalized action–angle variables which are similar to the ones in Hamilton dynamics and evaluate the frequencies of the flow of toric action.

We examine the Kähler structure of the transverse Kähler geometry and consider deformations of the contact structure perturbing the contact form with a basic function. In the case of the five-dimensional spaces  $T^{1,1}$  and  $Y^{p,q}$ , we have explicit solutions of the Sasaki–Ricci flow equation. Finally, we investigate the modifications of the action–angle variables by the Sasaki–Ricci flow.

It would be interesting to study the Sasaki–Ricci flow on higher-dimensional Sasaki–Einstein spaces as well as other contact spaces with 3-Sasaki structures [27] or mixed 3-structures [28].

It is worth extending the study of deformations of the metric using other kind of deformations. For instance, the so-called  $\mathcal{D}$ -homotetic deformation is defined

$$\eta' = a\eta \quad \xi' = \frac{1}{a}\xi \quad g' = ag + a(a-1)\eta \otimes \eta$$

for a positive constant  $a$ . Other deformations of interest in Sasaki geometry are obtained by defining a new Sasakian structure  $(M, \eta', \xi')$  with  $\eta' = f\eta$  for a positive function  $f \neq \text{constant}$  and  $\xi'$  is the corresponding Reeb vector field [29].

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