Indian J Phys (March 2023) 97(3):923–931
https://doi.org/10.1007/s12648-022-02474-1

ORIGINAL PAPER

On the time dependence of the photon Compton wavelength

E Fiordilino*

Dipartimento di Fisica e Chimica - Emilio Segré, Università degli Studi di Palermo, Via Archirafi 36, 90123 Palermo, Italy

Received: 07 April 2022 / Accepted: 10 September 2022 / Published online: 23 November 2022

Abstract: The Proca equation is a generalisation of the Maxwell equations that modifies the potential of a constant charge at rest from the Coulomb to the Yukawa form and describes massive photons. In this paper, we assume time-dependent photon mass and solve the Proca equation for a charge at rest. It is found that the vector potential is non-vanishing and that the scalar potential has the form $\Phi(r, x^0) = f(r, x^0)/r$ with $x^0 = ct$ and, thus, is space-time dependent. The solution shows that an oscillating potential pulse leaves the charge at $x^0 = 0$ and propagates with the speed of the light to modify the initial Yukawa potential. Experimentally the change of sign of the potential field at the charge location can be interpreted as a change of the sign of the charge. The picture that emerges from the theory is, then, that of charges that at the initial time emit massive radiation that fills the Universe. The period of the field oscillation is, however, increasing and suggests that after a long time the potential may become stabilised. A noteworthy consequence of the theory is that the present form of the Maxwell electrodynamics is only temporarily true.

Keywords: Proca equation; Time-dependent physical constants; Photon mass; Electromagnetic potential

1. Introduction

1.1. Context

From the very beginning of our study of Physics we meet numerical parameters that can be determined only through experiments: they are the physical constants whose value and role within Physics are intriguing and vividly discussed. In general the parameters are dimensionful with numerical value dependent upon the adopted units but often they can be algebraically combined to form dimensionless quantities that are free of any human choice. Their intrinsic meaning can be extracted from the role played within the theory; in fact, whatever the units might be, the speed of the light c is the natural upper limit of the speed of any mass or field; the elementary charge e and half the Planck constant $\hbar$ are the lower limits for the absolute value of the free electric charges and nonzero angular moment.

Although not often explicitly declared, the value of the physical constants is considered as given a priori by Nature up to any decimal digit and, in principle, measurable with any desired precision. The case of the speed of light c is emblematic: its value is given by definition to be $c = 299,792,458$ m/s with no error bar and thus is known up to any decimal digit (always zero). Granted our freedom in the choice of the units, several assumptions are implicit in the definition of c; for example we must be sure that c is not space-time dependent and, perhaps, that the mass of the photon is zero [1–3]. The physical constants permit us to calculate the evolution of the Universe as a whole or of that small fragments of matter that we call atom. In this picture of Physics the constants play the same role of Plato's ideas: a priori given, they disclose the knowledge of the world to us. Yet, to speculate that the distribution of matter and energy in the Universe and the value of the physical parameters are mutually dependent in a self-consistent way is possible. Other options can be envisaged, for example the parameters could not be known with any desirable precision or be fluctuating [4]. Which of these possibilities is the true one cannot be determined on personal philosophical preferences but on experimental grounds. The full picture of the issue is much richer, though.

Noticing that the ratio of the electric to gravitational force between electron and proton is of the order of the age of the Universe T (expressed in atomic units where $\hbar = e = m_e = 1$, with m_e the electron mass) and that the ratio between the mass of the Universe and that of the proton is of the order of T^2, Dirac was led to hypothesise that G might be time dependent [5] thus setting the basis of the so-called Dirac's cosmology [6].

*Corresponding author, E-mail: emilio.fiordilino@unipa.it

Dirac's suggestion opened several channels of investigations on the possibility that other constants might be space-time dependent and the present paper is devoted to the investigation of the consequences brought about by the hypothesis that some fundamental constants are time dependent; we refer the interested reader to the existing reviews for a view of the topics of variation of the physical constants [7, 8].

From the experimental side, the issue appears to be a conundrum; it has been hinted that the fine structure constant may be function of time or space [9–12] but independent experiments do not confirm the suggestion [13–15]. Other investigations discuss the space-time dependence of $\hbar$ [16, 17], G [18–20], e [21], c [22–26], electron Compton wavelength [27] but a debate is active if the quest for the variation of dimensionful parameters has any meaning at all and, instead, if only dimensionless parameters should be taken into consideration in the first place. Unfortunately the fervency of the discussion is proportional to the lack of clear-cut experimental data. From the theoretical point of view it is important to explore all the options in order to have a picture of what may happen and to suggest new and, may be, maverick methods of investigation [28–36].

1.2. Motivations

In this context, it has been studied [27] the Dirac equation for a free electron with time-dependent Compton wavelength

$$i\partial_0 \underset{..}{\psi}(x^\mu) = -i\sum_1^3 \mathbb{a}^j \partial_j \underset{..}{\psi}(x^\mu) + \varkappa_e(x^0)\mathbb{B}\underset{..}{\psi}(x^\mu) \quad (1)$$

with

$$\mathbb{a}^j = \begin{pmatrix} \mathbb{0} & \sigma^j \\ \sigma^j & \mathbb{0} \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} \mathbb{1} & \mathbb{0} \\ \mathbb{0} & -\mathbb{1} \end{pmatrix} \quad (2)$$

the standard Dirac matrices and $\underset{..}{\psi}$ the four component electron wavefunction whose bispinorial nature is indicated by the two dots under the symbol. Here

$$\varkappa_e = \frac{m_e c}{\hbar} = \frac{1}{\lambda_e} \quad (3)$$

and λ_e is the (reduced) Compton wavelength of the electron assumed to be time dependent. Throughout this paper a Greek index runs from 0 to 3 and a Latin index from 1 to 3; the components of the four vector x^ν are (x^0, x^1, x^2, x^3) (or, concisely, $(x^0, \boldsymbol{r})$) with metric $g^{00} = +1$, $g^{jk} = -\delta^{j,k}$ (the standard Kronecker symbol); however the Einstein sum convention over repeated indices in not adopted; moreover in the following we shall loosely refer to $x^0 = ct$ as *time*. The surprising result is that the analytical solution of Eq. (1) predicts positive–negative charge oscillations. Thus, time dependency of λ_e, which does not depend upon the charge, requires charge non-conservation which in turn requires that the photon has a nonzero mass [37, 38] and at the end of the line also that the Coulomb law is not any more valid [39]. Such a string of implications is disturbing in front of the fact that the experiments set a very long constraint on the charge decay time $\tau_e \geq 6.6 \times 10^{28}$ yr although the value is obtained by assuming a particular decay of an electron into a photon and a neutrino [21]. Also very tight is the constraint in the photon mass; recent experimental limits being $mc^2 \leq 6 \cdot 10^{-17}$ eV [40] and $mc^2 \leq 10^{-15}$ eV [41] and appears to be compatible with the null result. Of course experiments should be designed to set also a limit on m in the far past. However, it is well clear that any variation of the physical constants would change some of our deepest-rooted ideas of natural bases such as the inertial law and the equivalence principle [8, 42].

Since the time dependence of the electron Compton wavelength implies that the photon must have a mass, to study such prospects from a different point of view is essential. As it is well known, Coulomb law and Maxwell equations require that the photon is massless; nevertheless by generalising these fundamental laws, to introduce the concept of massive photon is possible; the task is accomplished by the Proca equation that in Gaussian units is [39, 43]

$$\Box A^\sigma + \varkappa^2 A^\sigma = \frac{4\pi}{c} j^\sigma \quad (4)$$

where

$$\Box = \frac{\partial}{\partial x_0^2} - \nabla, \quad (5)$$

is the d'Alembert operator, A^σ is the four electromagnetic potential, j^σ the four electric current,

$$\varkappa = \frac{mc}{\hbar} = \frac{1}{\lambda}, \quad 6$$

m the mass and λ the reduced Compton wavelength of the photon. The reader interested in the genesis, primeval motivation and modern vision on the Proca equation and the mass of the photon is referred to the existing literature [40, 44, 45].

The presence of the mass term $\varkappa$ in Eq. (4) explicitly breaks the U(1) gauge symmetry of the usual electromagnetic theory but the relativistic covariance is preserved in the Lorenz gauge $\sum_{\nu=0}^{3} \partial_\nu A^\nu = 0$ which is, henceforth assumed. This is an important point as charge conservation and Lorentz gauge are strictly related:

$$4\pi \sum_{\nu=0}^{3} \partial_\nu j^\nu = \varkappa^2 c \sum_{\nu=0}^{3} \partial_\nu A^\nu \tag{7}$$

so that the assumption of the Lorenz gauge requires charge conservation.

2. Theory

We wish to solve Eq. (4) for a time-dependent Compton wavelength of the photon and therefore we set $\varkappa \equiv \varkappa(x^0)$; it implies a time-dependent photon mass. For simplicity sake we consider a *constant* point charge q at rest at the origin of the coordinate system; thus

$$j^\sigma = (cq\delta(\boldsymbol{r}), \boldsymbol{0}) \tag{8}$$

however, even if the particle is at rest, nothing forces the vector potential $A^n(x^\sigma)$ to vanish, therefore we write

$$A^\sigma = (\Phi, \boldsymbol{A}) \tag{9}$$

with $\Phi(x^\sigma)$ the scalar potential. In this way Proca equation splits into two differential equations:

$$\begin{cases} (\partial_0^2 - \boldsymbol{\nabla}^2)\Phi(x^\sigma) + \varkappa^2(x^0)\Phi(x^\sigma) = 4\pi q\delta(\boldsymbol{r}) \\ (\partial_0^2 - \boldsymbol{\nabla}^2)A^n(x^\sigma) + \varkappa^2(x^0)A^n(x^\sigma) = 0 \end{cases}. \tag{10}$$

2.1. The adiabatic approximation

The second of Eq. (10) is easier and can be solved by separation of the variables by setting (no sum over n)

$$A^n(x^\sigma) = a^n(\boldsymbol{r})T^n(x^0). \tag{11}$$

Confining to Appendix (A) the mathematical details, the result is that the equation for $T^n(x^0)$ becomes:

$$\begin{cases} \partial_0^2 T^n_{k^n}(x^0) + (\varpi_{k^n})^2(x^0)T^n_{k^n}(x^0) = 0 \\ (\varpi_{k^n})^2(x^0) \equiv (k^n)^2 + \varkappa^2(x^0) \end{cases} \tag{12}$$

which is the equation of a parametric oscillator with time-dependent angular frequency $\omega_{k^n}(x^0) = c\varpi_{k^n}(x^0)$ and leads us to predict an oscillating potential. Since the equation is second order on x^0 we shall need two integration constants to be determined from the initial conditions. This is a far reaching characteristic as the plane waves are not harmonically oscillating.

At the best of our knowledge, there is no fundamental theory nor cogent reason requiring a time variation of the photon Compton wavelength, thus to proceed with the calculations we need to use some speculation. As far as our present day experiments say, any time dependence of the photon Compton wavelength λ – or of $\varkappa^2(x^0)$ – if real, is very slow or would have been detected, thus we believe that to hypothesise that λ has been slowly varying even in the very far past is reasonable; therefore we make the crucial adiabatic ansatz [46]:

$$T^n_{k^n}(x^0) = \cos\left[\theta_{k^n}(x^0) + \varphi_{k^n}\right] \tag{13}$$

with

$$\theta_{k^n}(x^0) = \int_0^{x^0} \varpi_{k^n}(\xi)\mathrm{d}\xi \tag{14}$$

and φ_{k^n} is one of the two integration constants. We stress that the particular choice of $T^n_{k^n}(x^0)$ is motivated by the (parametric) oscillator form of its equation.

2.2. The vector potential

Always after Appendix (A), the space-time dependence of the components of the vector potential is

$$\begin{aligned} 2\boldsymbol{A}_{\{k^n\}}(\boldsymbol{r}, x^0) = \Big(& a^1_{\boldsymbol{k}^1} \mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}} \cos\left[\theta_{k^1}(x^0) + \varphi_{k^1}\right], \\ & a^2_{\boldsymbol{k}^2} \mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}} \cos\left[\theta_{k^2}(x^0) + \varphi_{k^2}\right], a^3_{\boldsymbol{k}^3} \mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}} \cos \\ & \left[\theta_{k^3}(x^0) + \varphi_{k^3}\right]\Big) + cc \end{aligned} \tag{15}$$

with $a^n_{\boldsymbol{k}^n}$ amplitudes to be determined from the initial condition (we recall that $k^n = |\boldsymbol{k}^n|$). The full solution is given by a linear superposition of these solutions:

$$\begin{aligned} 2\boldsymbol{A}(\boldsymbol{r}, x^0) = \Big(& \int \mathrm{d}^3k^1 a^1_{\boldsymbol{k}^1} \mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}} \cos\left[\theta_{k^1}(x^0) + \varphi_{k^1}\right], \\ & \int \mathrm{d}^3k^2 a^2_{\boldsymbol{k}^2} \mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}} \cos\left[\theta_{k^2}(x^0) + \varphi_{k^2}\right], \int \mathrm{d}^3k^3 a^3_{\boldsymbol{k}^3} \mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}} \\ & \cos\left[\theta_{k^3}(x^0) + \varphi_{k^3}\right]\Big) + cc \end{aligned} \tag{16}$$

2.3. The scalar potential

We are interested in the scalar potential $\Phi(x^\sigma)$ to observe how the Coulomb law is affected by our hypothesis that the photon Compton wavelength is time dependent. However, the direct solution of the first of Eq. (10) is hindered by the presence of the $\delta(\boldsymbol{r})$ and we prefer the use of the Lorenz gauge $\sum_{\sigma=0}^{3} \partial_\sigma A^\sigma = 0 \Rightarrow \partial^0\Phi + \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$. The calculations leading to $\boldsymbol{\nabla}\cdot\boldsymbol{A}$ are long and time consuming and are detailed in Appendix (B). The final result is

$$\Phi(r, x^0) = \frac{2q}{\pi r}\int_0^\infty k\frac{\mathcal{F}_k(x^0)/\mathcal{F}_k(0)}{k^2 + \varkappa^2(0)}\sin(kr)\mathrm{d}k \tag{17}$$

with

$$\mathcal{F}_{k^n}(x^0) = \int \cos\left[\theta_{k^n}(x^0) + \varphi_{k^n}\right] \mathrm{d}x^0. \tag{18}$$

After so many titanic calculations, the final result appears to be quite simple. The expression shows that the potential generated by a charge put at the origin of the coordinates is of the form $\Phi(\boldsymbol{r}, x^0) = f(\boldsymbol{r}, x^0)/r$. $\mathcal{F}_{k^n}(x^0)$ contains the infinite number of phases φ_{k^n} to be determined.

3. A model

Among the many possible models, we adopt a periodic time dependence for $\varkappa(x^0)$; of course it is a toy model, but it allows advanced analytical calculations and, moreover, is based on some ground. In fact hypotheses have been advanced that the physical constants may be dependent upon some privileged space direction and in this case the Earth orbital motion would feel a periodic variation of the physical constants [7, 47]; therefore we set:

$$\varkappa(x^0) = \varkappa_0 + \varkappa_1 \sin(\kappa x^0), \quad (\varkappa_1, \kappa) \ll \varkappa_0 \tag{19}$$

leading to

$$\varpi^2_{k^n}(x^0) \cong (k^n)^2 + \varkappa_0^2 + 2\varkappa_0\varkappa_1 \sin(\kappa x^0), \tag{20}$$

$$\begin{aligned}\theta_{k^n}(x^0) &\cong \int_0^{x^0} \sqrt{(k^n)^2 + \varkappa_0^2 + 2\varkappa_0\varkappa_1 \sin(\kappa\xi)}\, \mathrm{d}\xi \\ &\cong \sqrt{(k^n)^2 + \varkappa_0^2} x^0 - \frac{\varkappa_0\varkappa_1}{\kappa\sqrt{(k^n)^2 + \varkappa_0^2}}\left[\cos(\kappa x^0) - 1\right]\end{aligned} \tag{21}$$

and, after redefinition of φ_{k^n}

$$\mathcal{F}_{k^n}(x^0) = \int \cos\left[\sqrt{(k^n)^2 + \varkappa_0^2} x^0 - \frac{\varkappa_0\varkappa_1}{\kappa\sqrt{(k^n)^2 + \varkappa_0^2}} \cos(\kappa x^0) + \varphi_{k^n}\right] \mathrm{d}x^0. \tag{22}$$

This integral can be manipulated to give for the function $\mathcal{F}_{k^n}(x^0)$ the form:

$$\begin{aligned}\mathcal{F}_{k^n}(x^0) &= \sum_{n=-\infty}^{\infty} (-1)^n J_n\left(\frac{\varkappa_0\varkappa_1}{\kappa\sqrt{(k^n)^2 + \varkappa_0^2}}\right) \\ &\frac{\sin\left[\left(\sqrt{(k^n)^2 + \varkappa_0^2} + n\kappa\right)x^0 + \varphi_{k^n}\right]}{\sqrt{(k^n)^2 + \varkappa_0^2} + n\kappa}\end{aligned} \tag{23}$$

with $J_n(x)$ the Bessel function of integer order n; details on the analytical steps leading to this expression are given in Appendix (C). A glance to Eq. (17) gives the final expression for the scalar potential:

$$\Phi(\boldsymbol{r}, x^0) = \frac{q(0)}{r} f(r, x^0) \tag{24}$$

with

$$\begin{aligned}f(r, x^0) &= -\frac{2}{\pi} \sum_{n=-\infty}^{+\infty} (-1)^n \times \\ &\times \int_0^{\infty} J_n\left(\frac{\varkappa_0\varkappa_1}{\kappa\sqrt{k^2 + \varkappa_0^2}}\right) \frac{k \sin\left[\left(\sqrt{k^2 + \varkappa_0^2} + n\kappa\right)x^0 + \varphi_{k^n}\right]}{\left[k^2 + \varkappa_0^2\right]\left[\sqrt{k^2 + \varkappa_0^2} + n\kappa\right]} \\ &\frac{\sin(kr)}{\mathcal{F}_k(0)} \mathrm{d}k.\end{aligned} \tag{25}$$

The presence of r at the denominator is reassuring as reminiscent of the Coulomb law; in the following we shall discuss the function $f(r, x^0)$ alone.

The Bessel functions $J_n(x)$ peak when $x \approx n$; thus we see that the number of addends to be taken into account in $f(r, x^0)$ depends essentially on the ratio $\varkappa_1/\kappa$. Complications may arise if the quantity $\sqrt{k^2 + \varkappa_0^2} + n\kappa$ vanishes because n may be negative. To look after this sort of *resonance* might be rewarding; since here we present model calculations, in the numerical evaluation of the scalar potential, we use values of the load parameters which avoid the resonance.

4. Results and discussion

We have numerically evaluated the integrals present in Eq. (25) to observe how the Coulomb law is changed by a time variation of the photon Compton wavelength. At this stage of the discussion there is no need to ascribe the variation to any of the constants defining λ but we shall deal with this point later. The integral has the form of a sin Fourier transform; the upper limit and step of integration were selected in order to achieve numerical convergence. We have checked it also by using functions whose integral could be analytically carried out.

The value of the load parameters $\varkappa_0$, $\varkappa_1$, κ in Eq. (19) is tricky since these should comply with the experimental state of the art but, then, calculations would be prohibitively long; thus we choose values apt to unveil the physics of the problem without requiring a too long numerical effort. In the following we use $\varkappa_0 = 1.074\ \mathrm{cm}^{-1}$ (which is very large but permits to discern at a glance exponential decreasing behaviours in the plots), $\varkappa_1 = 1.17 \times 10^{-2}\varkappa_0\ \mathrm{cm}^{-1}$ and $\kappa = 5.13 \times 10^{-3}\varkappa_0\ \mathrm{cm}^{-1}$ ($2\pi/\kappa = 1.14 \times 10^3$ cm). The reason of the unhandy choice of the values is given by the need to avoid the previously discussed resonances and also possible beats between the frequencies. The phases φ_k are, as yet, undetermined but for simplicity sake we assign the value $\varphi_k = \pi/2.05$ determined, after few checks, to give the slowest evolution of the potential.

In Fig. 1 we show few snapshots of $f(r, x^0)$ as a function of r. From the plots a very interesting picture of the scalar potential appears. At $x^0 = 0$ a pulse of potential leaves the charge with speed c and modifies the potential from the initial Yukawa's form in eq. (40). For $r > x^0$ the potential is almost unaffected. The pulse is chirped and shows a time-dependent frequency. This is more evident from two snapshots taken after a long time x^0 and shown in Fig. 2.Now, however, it is evident that the potential at $x^0 = 0$ changes sign; thus a constant and positive charge creates a negative scalar potential; these awkward characteristics in an experiment may be interpreted as a transition from a positive to a negative charge. In Fig. 3 we give the temporal evolution of $f(r, x^0)$ at $r = 5$ cm: we see that almost nothing happens until $x^0 = r$ when the field begins to oscillate.

An interesting feature is that the period of the positive–negative potential oscillation increases with time; thus the sign of the charge tends towards a stabilisation. The inset of Fig. 3 shows the square modulus of the Fourier transform of $f(5, x^0)$ which presents a broad component content. Of course our results are obtained from a toy model, however, for $x^0 \ll 2\pi/\kappa$ we may say that the physical development of $f(r, x^0)$ is practically model independent.

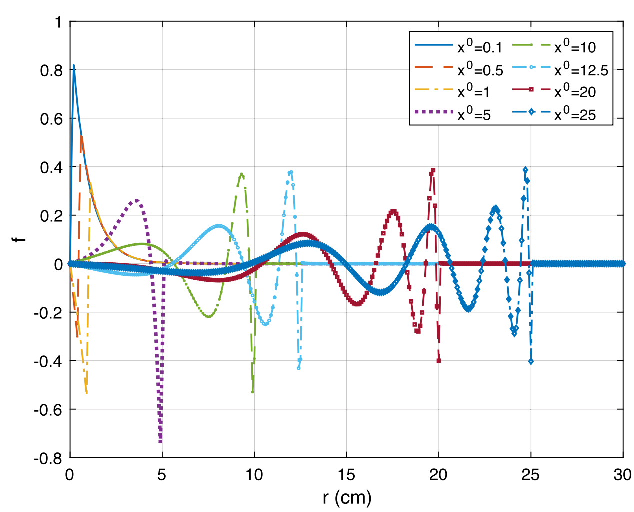

Fig. 1 Colour on line. Snapshots of $f(r, x^0)$ as function of r in cm. The snapshots at small $x^0 = ct$ in cm show that $f(r, x^0)$ exponentially decreases for $r > x^0$. It is clearly visible the change of sign of $f(r, x^0)$

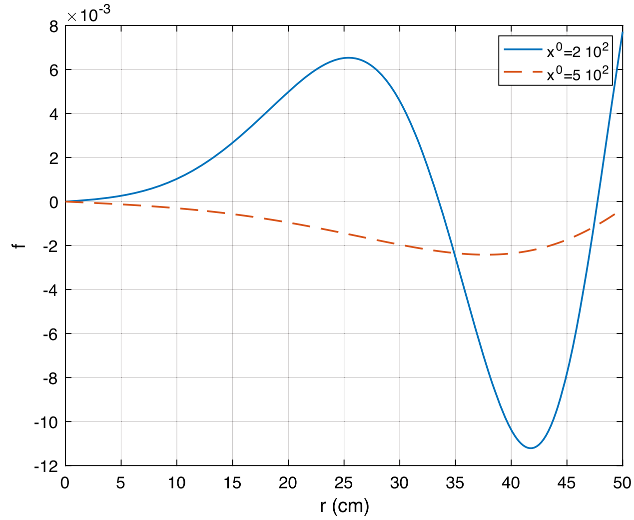

Fig. 2 Colour on line. Two snapshots of $f(r, x^0)$ as function of r in cm; x^0 in cm. The change of sign of $f(r, x^0)$ can be read as a change of the sign of the charge. The period of oscillation of $f(r, x^0)$ increases with x^0

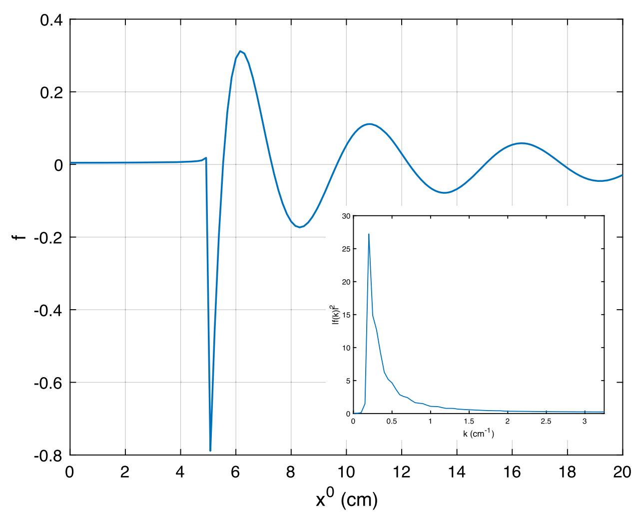

Fig. 3 Plot of $f(r, x^0)$ as a function of x^0 in cm at $r = 5$ cm. The wave arrives at $x^0 = ct$ at $x^0 = 5$ cm; afterwards the field oscillates with increasing period. The inset shows the square modulus (in arbitrary units) of the Fourier transform of $f(5, x^0)$: the spectrum is formed of a broad distribution of components

5. Conclusions

Charge conservation, electron stability and massless photon are amongst the most constrained principles of Physics both from the experimental and the theoretical point of view [21, 37, 38, 45, 48–50]. Proca equation describes massive photons and implies that the potential generated by an electric charge at rest is modified from the pure Coulomb potential into the Yukawa short range potential (Eq. (40)). In this paper, we hypothesise that the Compton wavelength of the photon is time dependent and accordingly modify the Proca equation which now assumes the form of a parametric oscillator with very slow variation of the frequency thus permitting the adiabatic approximation (Eq. (13)). The Yukawa potential is assumed to be the initial potential. The surprising result is that a wave propagates from the field source with speed c and modifies the static potential energy; eventually the potential changes sign even at the charge location and this can be read, from the experimental point of view, as a change of the sign of the charge. The oscillation of the potential is rapid at the initial time but then becomes increasingly slower and suggests a picture of the evolution of the field rather interesting, because at the beginning of time massive photons leave the charges and fill the Universe; as time goes by, the sign of the potential becomes stable. For the sake of argument, by pushing the discussion beyond the borders of the model here presented, we may envisage that we are now living at the particular stage of the evolving Universe when $\varkappa(x^0) = 0$ and the Coulomb law is strictly true.

Three issues should now be shortly addressed. To begin with, we are faced with what can be called unity of Nature. Proca theory violates the gauge invariance of electromagnetism and moreover the present Paper assumes that the ratio of few physical constants changes in time. The consequences reverberate everywhere: the Coulomb law is not any more valid, energy flows from the charges albeit, perhaps, mainly at the far past and an apparent charge non-conservation is obtained. But electromagnetism is the basis and benchmark of all relativistic field theories thus contradictions are to be expected everywhere. As a second point, in our approach the dependence of the potentials upon x^0 is described by an intrinsically time-dependent differential equation with assigned initial conditions thus the evolution affects the potential at any spatial point; the fact that the modification of the scalar potential is essentially represented by a wave leaving the source of the field and, thus, preserving causality, is indication that our approach, albeit approximated, catches the essential physics of the problem.

As a third issue we must address the problem to which of the parameters entering the definition of the photon Compton wavelength λ should the variation be ascribed. The discussion requires careful handling as it is often told that even thinking about variation of dimensionful parameters is meaningless. Nevertheless here we are concerned with the variation of a dimensionful parameter. To some extent our calculations do not force a choice, but it appears to us that if the photon mass m is eternally zero, then all of the presented work would be pointless thus, within our theory, the choice imposes itself.

A The vector potential

In this Appendix, we detail the steps needed to solve for the vector potential in the set of Eq (10). Because of the large amount of calculations, we adopt a rational notation that, albeit unorthodox, is helpful. Accordingly, throughout the manuscript we keep a notation allowing constant control of dimensional consistency. Thus, quantities with L^{-1} dimension are indicated with symbols of the k family, quantities with the dimension T^{-1} with ω and similar symbols. Adopting this strategy to avoid errors is simpler. In equation (35) k is the integration variable.

We start from the separation in Eq. (11) and substitute it into the second equation (10) to obtain

$$\frac{\partial_0^2 T^n}{T^n} - \frac{\nabla^2 a^n}{a^n} + \varkappa^2(x^0) = 0; \tag{26}$$

since a^n and T^n are function of independent variables, we set

$$\nabla^2 a^n(\boldsymbol{r}) = -(K^n)^2 a^n(\boldsymbol{r}) \tag{27}$$

giving for T^n

$$\frac{\partial_0^2 T^n}{T^n} + \varkappa^2(x^0) = -(K^n)^2; \tag{28}$$

with K^n the separation constant.

The spatial part for the generic nth component can be solved, again, by separation of the variables; thus

$$a^n(\boldsymbol{r}) = a^{n;1}(x^1)a^{n;2}(x^2)a^{n;3}(x^3) \tag{29}$$

with the second index after the semicolon flagging the variable the amplitude a^n is depending from. By following standard procedure we arrive at

$$\frac{1}{a^{n;1}}\frac{d^2a^{n;1}}{d(x^1)^2} + \frac{1}{a^{n;2}}\frac{d^2a^{n;2}}{d(x^2)^2} + \frac{1}{a^{n;3}}\frac{d^2a^{n;3}}{d(x^3)^2} = -(K^n)^2 \tag{30}$$

so that

$$\frac{1}{a^{n;1}}\frac{d^2a^{n;1}}{d(x^1)^2} = -(k_1^n)^2 \Rightarrow a^{n;1}(x^1) = a^{n;1}_{k_1^n} \mathrm{e}^{ik_1^n x^1}. \tag{31}$$

Straightforward reiteration for x^2 and x^3 gives

$$a^n_{\boldsymbol{k}^n}(\boldsymbol{r}) = a^{n;1}_{k^n_1} a^{n;2}_{k^n_2} a^{n;3}_{k^n_3} \mathrm{e}^{i(k^n_1 x^1 + k^n_2 x^2 + k^n_3 x^3)} \tag{32}$$

or, concisely:

$$a^n_{\boldsymbol{k}^n}(\boldsymbol{r}) = a^n_{\boldsymbol{k}^n} \mathrm{e}^{i\boldsymbol{k}^n\cdot\boldsymbol{r}}. \tag{33}$$

The notation might seem overburdening but is meant to recall that the amplitudes a^n does indeed depend from $\boldsymbol{k}^n$. From Eq (27)

$$(K^n)^2 = \boldsymbol{k}^n \cdot \boldsymbol{k}^n = (k^n)^2. \tag{34}$$

The vector potential at the initial time is, then, given by

$$2\boldsymbol{A}_{\{k^n\}}(\boldsymbol{r},0) = \left(a^1_{\boldsymbol{k}^1}\mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}}, a^2_{\boldsymbol{k}^2}\mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}}, a^3_{\boldsymbol{k}^3}\mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}}\right) + cc. \tag{35}$$

As a warning, $\boldsymbol{k}^j$ is not the jth component of the vector $\boldsymbol{k}$ but the vector pertaining to the jth coordinate. Thus, speaking about the spatial part, any component of the vector potential is a plane wave.

To obtain the full expression of the vector potential we need to solve for the temporal part. Eq. (26) gives

$$\begin{cases} \partial^2_0 T^n_{k^n}(x^0) + (\varpi_{k^n})^2(x^0) T^n_{k^n}(x^0) = & 0 \\ (\varpi_{k^n})^2(x^0) \equiv (k^n)^2 + \varkappa^2(x^0) & \end{cases}. \tag{36}$$

B The scalar potential

In this Appendix, we detail the steps leading to the expression of the scalar potential. We evaluate the divergence of $\boldsymbol{A}$

$$\begin{aligned} 2\nabla\cdot\boldsymbol{A}(\boldsymbol{r},x^0) = & \\ & + i\int \mathrm{d}^3k^2 a^2_{\boldsymbol{k}^2}(\boldsymbol{k}^2)^2 \mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}} \cos\left[\theta_{k^2}(x^0) + \varphi_{k^2}\right] \\ & + i\int \mathrm{d}^3k^3 a^3_{\boldsymbol{k}^3}(\boldsymbol{k}^3)^3 \mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}} \cos\left[\theta_{k^3}(x^0) + \varphi_{k^3}\right] + cc. \end{aligned} \tag{37}$$

By setting

$$\mathcal{F}_{k^n}(x^0) = \int \cos\left[\theta_{k^n}(x^0) + \varphi_{k^n}\right] \mathrm{d}x^0 \tag{38}$$

we obtain

$$\begin{aligned} -2\Phi(\boldsymbol{r},x^0) &= 2\int \nabla\cdot\boldsymbol{A}(\boldsymbol{r},x^0)\mathrm{d}x^0 \\ &= i\int \mathrm{d}^3k^1\left[a^1_{\boldsymbol{k}^1}\mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}} - (a^1_{\boldsymbol{k}^1})^*\mathrm{e}^{-i\boldsymbol{k}^1\cdot\boldsymbol{r}}\right]\mathcal{F}_{k^1}(x^0)(\boldsymbol{k}^1)^1 \\ &\quad + i\int \mathrm{d}^3k^2\left[a^2_{\boldsymbol{k}^2}\mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}} - (a^2_{\boldsymbol{k}^2})^*\mathrm{e}^{-i\boldsymbol{k}^2\cdot\boldsymbol{r}}\right]\mathcal{F}_{k^2}(x^0)(\boldsymbol{k}^2)^2 \\ &\quad + i\int \mathrm{d}^3k^3\left[a^3_{\boldsymbol{k}^3}\mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}} - (a^3_{\boldsymbol{k}^3})^*\mathrm{e}^{-i\boldsymbol{k}^3\cdot\boldsymbol{r}}\right]\mathcal{F}_{k^3}(x^0)(\boldsymbol{k}^3)^3. \end{aligned} \tag{39}$$

It is well known that the static solution of Proca equation for the scalar potential with constant mass term $\varkappa \Rightarrow \mu$ is the Yukawa potential [39, 51]

$$\Phi(r) = q\frac{e^{-\mu r}}{r} \tag{40}$$

thus we require the initial condition (i.e. at $x^0 = 0$)

$$\Phi(r,0) = q\frac{e^{-\varkappa(0)r}}{r}; \tag{41}$$

by equating the Fourier transform of Eqs (39)and (41) we arrive at:

$$\begin{aligned} -\frac{q}{p^2+\varkappa^2(0)} &= \pi^2 i\Big\{\left[a^1_{\boldsymbol{p}} + (a^1_{-\boldsymbol{p}})^*\right]p^1 \\ &\quad + \left[a^2_{\boldsymbol{p}} + (a^2_{-\boldsymbol{p}})^*\right]p^2 + \left[a^3_{\boldsymbol{p}} + (a^3_{-\boldsymbol{p}})^*\right]p^3\Big\}\mathcal{F}_p(0). \end{aligned} \tag{42}$$

The right-hand side of the equation must be real, thus we set

$$a^n_{\boldsymbol{p}} + (a^n_{-\boldsymbol{p}})^* = i\beta^n_{\boldsymbol{p}} \tag{43}$$

giving

$$\frac{q}{p^2+\varkappa^2(0)} = \pi^2\left\{\beta^1_{\boldsymbol{p}}p^1 + \beta^2_{\boldsymbol{p}}p^2 + \beta^3_{\boldsymbol{p}}p^3\right\}\mathcal{F}_p(0) \tag{44}$$

and

$$\begin{aligned} &\int \mathrm{d}^3k^1 (a^1_{\boldsymbol{k}^1})^*\mathrm{e}^{-i\boldsymbol{k}^1\cdot\boldsymbol{r}}(\boldsymbol{k}^1)^1\mathcal{F}_{k^1}(x^0) \\ &\quad = -\int \mathrm{d}^3k^1 (a^1_{-\boldsymbol{k}^1})^*\mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}}(\boldsymbol{k}^1)^1\mathcal{F}_{k^1}(x^0). \end{aligned} \tag{45}$$

From Eq. (39) we obtain

$$\begin{aligned}
-2\Phi(\boldsymbol{r},x^0) &= i\int \mathrm{d}^3k^1\left[a^1_{\boldsymbol{k}^1}+\left(a^1_{-\boldsymbol{k}^1}\right)^*\right]\mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}}\mathcal{F}_{k^1}(x^0)(\boldsymbol{k}^1)^1\\
&\quad + i\int \mathrm{d}^3k^2\left[a^2_{\boldsymbol{k}^2}+\left(a^2_{-\boldsymbol{k}^2}\right)^*\right]\mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}}\mathcal{F}_{k^2}(x^0)(\boldsymbol{k}^2)^2\\
&\quad + i\int \mathrm{d}^3k^3\left[a^3_{\boldsymbol{k}^3}+\left(a^3_{-\boldsymbol{k}^3}\right)^*\right]\mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}}\mathcal{F}_{k^3}(x^0)(\boldsymbol{k}^3)^3\\
&= -\int \mathrm{d}^3k^1\beta^1_{\boldsymbol{k}^1}\mathrm{e}^{i\boldsymbol{k}^1\cdot\boldsymbol{r}}\mathcal{F}_{k^1}(x^0)(\boldsymbol{k}^1)^1\\
&\quad -\int \mathrm{d}^3k^2\beta^2_{\boldsymbol{k}^2}\mathrm{e}^{i\boldsymbol{k}^2\cdot\boldsymbol{r}}\mathcal{F}_{k^2}(x^0)(\boldsymbol{k}^2)^2\\
&\quad -\int \mathrm{d}^3k^3\beta^3_{\boldsymbol{k}^3}\mathrm{e}^{i\boldsymbol{k}^3\cdot\boldsymbol{r}}\mathcal{F}_{k^3}(x^0)(\boldsymbol{k}^3)^3\\
&= -\int \mathrm{d}^3k(\beta_{\boldsymbol{k}}k^1+\beta_{\boldsymbol{k}}k^2+\beta_{\boldsymbol{k}}k^3)\mathcal{F}_k(x^0)\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}\\
&= -\frac{q}{\pi^2}\int \mathrm{d}^3k\,\frac{\mathcal{F}_k(x^0)/\mathcal{F}_k(0)}{k^2+\varkappa^2(0)}\mathrm{e}^{i\boldsymbol{k}\cdot\boldsymbol{r}}
\end{aligned} \tag{46}$$

or, performing the angular integrations:

$$\Phi\left(r,x^0\right)=\frac{2q}{\pi r}\int_0^\infty k\,\frac{\mathcal{F}_k\left(x^0\right)/\mathcal{F}_k(0)}{k^2+\varkappa^2(0)}\sin(kr)\mathrm{d}k. \tag{47}$$

C An integral involving Bessel functions

By using the generating function of the Bessel functions [52] in the identity

$$\begin{aligned}
&\cos\left[ax-b\cos(\kappa x)+c\right]\\
&\quad=\cos(ax+c)\cos\left[b\sin(\kappa x)\right]+\sin(ax+c)\sin\left[b\sin(\kappa x)\right]\\
&\quad=\cos(ax+c)\left[J_0(b)+2\sum_{n=1}^{\infty}J_{2n}(b)\cos(2n\kappa x)\right]\\
&\quad\quad+2\sin(ax+c)\left[\sum_{n=0}^{\infty}J_{2n+1}(b)\sin\left[(2n+1)\kappa x\right]\right]
\end{aligned} \tag{48}$$

it is possible to obtain a closed form for $\mathcal{F}_{k^n}\left(x^0\right)$. We set

$$a=\sqrt{k^2+\varkappa_0^2};\quad b=\frac{\varkappa_0\varkappa_1}{\kappa\sqrt{k^2+\varkappa_0^2}};\quad c=\varphi_{k^n} \tag{49}$$

and proceed:

$$\begin{aligned}
&\mathcal{F}_{k^n}(x^0)=\\
&\quad+2\sum_{n=1}^{\infty}J_{2n}(b)\left\{\frac{\sin[(a-2n\kappa)x^0+c]}{2(a-2n\kappa)}+\frac{\sin[(a+2n\kappa)x^0+c]}{2(a+2n\kappa)}\right\}\\
&\quad+2\sum_{n=0}^{\infty}J_{2n+1}(b)\left\{\frac{\sin\{[a-(2n+1)\kappa]x^0+c\}}{2[a-(2n+1)\kappa]}\right.\\
&\quad\left.-\frac{\sin\{[(a+(2n+1)\kappa]x^0+c\}}{2[a+(2n+1)\kappa]}\right\}\\
&\quad=\frac{J_0(b)}{a}\sin(ax^0+c)+\sum_{n=1}^{\infty}J_n(b)\frac{\sin[(a-n\kappa)x^0+c]}{a-n\kappa}\\
&\quad+\sum_{n=1}^{\infty}(-1)^nJ_n(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}
\end{aligned} \tag{50}$$

with the substitution $n\to -n$ in the first summation and use of the relation $J_{-n}(x)=(-1)^nJ_n(x)$ to gather the expression under the same summation is possible:

$$\begin{aligned}
\mathcal{F}_{k^n}(x^0)&=\frac{J_0(b)}{a}\sin(ax^0+c)+\sum_{n=-\infty}^{-1}J_{-n}(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}\\
&\quad+\sum_{n=1}^{\infty}(-1)^nJ_n(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}\\
&=\frac{J_0(b)}{a}\sin(ax^0+c)+\sum_{n=-\infty}^{-1}(-1)^nJ_n(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}\\
&\quad+\sum_{n=1}^{\infty}(-1)^nJ_n(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}\\
&=\sum_{n=-\infty}^{\infty}(-1)^nJ_n(b)\frac{\sin[(a+n\kappa)x^0+c]}{a+n\kappa}.
\end{aligned} \tag{51}$$

Acknowledgements The author acknowledges discussions with Antonino Di Piazza.

Funding Open access funding provided by Università degli Studi di Palermo within the CRUI-CARE Agreement.

Declarations

Conflict of interest The author does not have conflict of interest.

References

[1] B W Petley *Nature* **303** 373 (1983)
[2] J L Flowers and B . W . Petley *Rep. Prog. Phys.* **64** 1191 (2001)
[3] R Th Udem and T W Hänsch *Nature* **416** 233 (2002)
[4] G Mangano and F Lizzi *Int. J. Mod. Phys. A* **30** 1550209 (2015)
[5] P A M Dirac *Nature* page 323 (Feb 1937)
[6] P A M Dirac *Proc. R. Soc. London, Series A* **165** 199 (1938)
[7] C J A P Martins *Rep. Prog. Phys.* **80** 126902 (2017)
[8] J-Ph Uzan Living Rev *Relativity* **14** 155 (2011)
[9] J K Webb et al. *Phys. Rev. Lett.* **87** 091301 (2001)
[10] J K Webb, J A King, M T Murphy, V V Flambaum, R F Carswell and M B Bainbridge *Phys. Rev. Lett.* **107** 191101 (2011)
[11] M R Wilczynska et al. *Sci. Adv.* **6** eaay9672 (2020)
[12] T D Le *Heliyon* **6** e05011 (2020)
[13] S M Kotuš, M T Murphy and R F Carswell *MNRAS* **464** 3679 (2017)
[14] J B Whitmore and M T Murphy *MNRAS* **447** 446 (2015)
[15] Planck Collaboration:, P A R Ade, N Aghanim, M Arnaud, M Ashdown, J Aumont, C Baccigalupi, A J Banday, R B Barreiro, E Battaner, K Benabed, A Benoit-Lévy, J-P Bernard, M Bersanelli, P Bielewicz, J R Bond, J Borrill, F R Bouchet, C Burigana, R C Butler, E Calabrese, A Chamballu, H C Chiang, P R Christensen, D L Clements, L P L Colombo, F Couchot, A Curto, F Cuttaia, L Danese, R D Davies, R J Davis, P de Bernardis, A de Rosa, G de Zotti, J Delabrouille, J M Diego, H Dole, O Doré, X Dupac, T A Enßlin, H K Eriksen, O Fabre, F Finelli, O Forni, M Frailis, E Franceschi, S Galeotta, S Galli, K Ganga, M Giard, J González-Nuevo, K M Górski, A Gregorio, A Gruppuso, F K Hansen, D Hanson, D L Harrison, S Henrot-Versillé, C Hernández-Monteagudo, D Herranz, S R Hildebrandt, E Hivon, M Hobson, W A Holmes, A Hornstrup, W Hovest, K M Huffenberger, A H Jaffe, W C Jones, E Keihänen, R Keskitalo, R Kneissl, J Knoche, M Kunz, H Kurki-Suonio, J-M Lamarre, A Lasenby, C R Lawrence, R Leonardi, J Lesgourgues, M Liguori, P B Lilje, M Linden-Vørnle, M López-Caniego, P M Lubin, J F Macías-Pérez, N Mandolesi, M Maris, P G Martin, E Martínez-González, S Masi, S Matarrese, P Mazzotta, P R Meinhold, A Melchiorri, L Mendes, E Menegoni, A Mennella, M Migliaccio, M-A Miville-Deschênes, A Moneti, L Montier, G Morgante, A Moss, D Munshi, J A Murphy, P Naselsky, F Nati, P Natoli, H U Nørgaard-Nielsen, F Noviello, D Novikov, I Novikov, C A Oxborrow, L Pagano, F Pajot, D Paoletti, F Pasian, G Patanchon, O Perdereau, L Perotto, F Perrotta, F Piacentini, M Piat, E Pierpaoli, D Pietrobon, S Plaszczynski, E Pointecouteau, G Polenta, N Ponthieu, L Popa, G W Pratt, S Prunet, J P Rachen, R Rebolo, M Reinecke, M Remazeilles, C Renault, S Ricciardi, L Ristorcelli, G Rocha, G Roudier, B Rusholme, M Sandri, G Savini, D Scott, L D Spencer, V Stolyarov, R Sudiwala, D Sutton, A-S Suur-Uski, J-F Sygnet, J A Tauber, D Tavagnacco, L Terenzi, L Toffolatti, M Tomasi, M Tristram, M Tucci, J-P Uzan, L Valenziano, J Valiviita, B Van Tent, P Vielva, F Villa, L A Wade, D Yvon, A Zacchei, A Zonca *A &A* **580** A22 (2015)
[16] R A Hutchin *Opt. Phot. J.* **6** 124 (2016)
[17] J Kentosh and M Mohageg *Phys. Rev. Lett.* **108** 110801 (2012)
[18] D-C Dai Phys *Rev. D* **103** 064059 (2021)
[19] J D Anderson, G Schubert, V Trimble and M R Feldman *EPL* **110** 10002 (2015)
[20] J Mbelek *Gravit. and Cosmol.* **25** 250 (2019)
[21] M Agostini et al. *G Zuzel Phys. Rev. Lett.* **115** 231802 (2015)
[22] F Selleri *Found. Phys. Lett.* **10** 73 (1997)
[23] J D Barrow *Phys. Rev. D* **59** 043515 (1999)
[24] P C W Davies, T M Davis and C H Lineweaver *Nature* **418** 602 (2002)
[25] J Magueijo *Rep. Prog. Phys.* **66** 2025 (2003)
[26] L Iorio *Gen. Relativ. Gravit.* **42** 199 (2010)
[27] E Fiordilino *Eur. Phys. J. Plus* **136** 1129 (2021)
[28] M A de Gosson *Phys. Lett. A* **381** 3033 (2017)
[29] R Dannenberg *Symmetry* **12** 490 (2020)
[30] E Fiordilino *Eur. Phys. J. Plus* **136** 54 (2021)
[31] E Fiordilino *Eur. Phys. J. Plus* **136** 821 (2021)
[32] H Fritzsch, J Solà and R C Nunes *Eur. Phys. J. C* **77** 193 (2017)
[33] L F Pašteka, Y Hao, A Borschevsky, V V Flambaum and P Schwerdtfeger *Phys. Rev. Lett.* **122** 160801 (2019)
[34] A-K Hansmann and R Berger *J. Phys. Chem. A* **124** 6682 (2020)
[35] K Mori and K Nomoto *Symmetry* **12** 404 (2020)
[36] R P Gupta *Galaxies* **7** 55 (2019)
[37] S Weinberg *Phys. Rev.* **135** B1049 (1964)
[38] L B Okun and Ya B Zeldovich *Phys. Lett.* **78B** 597 (1978)
[39] J D Jackson, *Classical Electrodynamics*, *(USA John Wiley & Sons) (III ed.)* (2001)
[40] L-C Tu, J Luo and G T Gillies *Rep. Prog. Phys.* **68** 77 (2005)
[41] H Wang, X Miao and L Shao *Phys. Lett. B* **820** 136596 (2021)
[42] M S Safronova *Ann. Phys. (Berlin)* **531** 1800364 (2019)
[43] V E Díez, B Gording, J A Méndez-Zavaleta and A Schmidt-May *Phys. Rev. D* **101** 045008 (2020)
[44] L B Okun *Acta Phys. Pol. B* **37** 565 (2006)
[45] A S Goldhaber and M N Nieto *Rev. Mod. Phys.* **82** 939 (2010)
[46] C J Joachain and N J Kylstra (R M Potvliege: Atoms in intense laser fields (UK Cambridge University Press)) (2012)
[47] L Iorio *Class. Quant. Grav.* **33** 045004 (2016)
[48] R N Mohapatra *Phys. Rev. Lett.* **59** 1510 (1987)
[49] J-J Wei and X-F Wu *J. Cosmol. Astropart. Phys.* **2018** 045 (2018)
[50] J Rosato *Eur. Phys. J. D* **73** (2019)
[51] M Maggiore *A modern introduction to quantum field theory (UK Oxford university press)* (2005)
[52] M Abramowitz, I A Stegun *Handbook of Mathematical Functions with Formulas, Graphs, and Mathematical Tables USA Dover* (1974)