Exact Solutions to General Relativistic Force-Free Electrodynamics

By

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B.S. Physics & Mathematics, Troy University, 2020

Submitted to the graduate degree program in the Department of Physics and Astronomy and the Graduate Faculty of the University of Kansas in partial fulfillment of the requirements for the degree of Doctor of Philosophy.

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Date Approved: 2 July 2024

Abstract

Force-free electrodynamics (FFE) is a theoretical framework used to describe electromagnetic fields in regions where plasma inertia is negligible compared to the electromagnetic field strength and the field can evolve without contributions from the plasma degrees of freedom.

Force-free fields are extensively studied around accreting black holes as these fields are believed to extract rotational energy from the black hole in the form of relativistic jets. However, the equations of FFE are coupled non-linear partial differential equations, and finding exact solutions has historically been difficult. Similarly, many studies of cosmological magnetic fields assume force-free fields, but despite that, there have never been systematic studies on what type of force-free fields are allowed in an expanding universe described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) metric. In this dissertation, we successfully employ the geometric approach to force-free fields pioneered by Menon that involves the study of foliations of a spacetime manifold into submanifolds that allow for the existence of force-free fields.

As a result, we are able to find several non-null solutions to FFE equations in Kerr spacetime. Similarly, we present several null and non-null solutions in the FLRW spacetime, and we also present, to our knowledge, the first ever exact force-free field that transitions smoothly from magnetically dominated to null and then to an electrically dominated regime.

Acknowledgment

I am deeply grateful to my advisor, Misha, whose unwavering patience, tireless support, and remarkable knack for distilling complex problems into elegant, tangible concepts have been invaluable throughout my graduate school journey. Under his mentorship, navigating academic challenges has been not just manageable but genuinely enjoyable.

Additionally, I extend my heartfelt appreciation to Dr. Menon for his steadfast encouragement and invaluable guidance. His insightful feedback has played a pivotal role in refining my scholarship, and his support has been instrumental in bringing this dissertation to fruition.

I would like to thank Aniket whose companionship over the years was the source of my strength and sanity.

I want to extend my thanks to my friend Ryan who braved the relentless barrage of my, ahem, somewhat unconventional ideas without so much as a whimper or a flinch. I cannot adequately convey the depth of my appreciation for Neno, who, at times, didn't hesitate to pretend not to be unfunny just to humor me. I would like to thank Keita for helping me with all the cosmology work that I did at the beginning of my graduate school and Mike for his help all these years.

I would like to thank Joel, Kristin, Matt, Mohammad, and Kayla for their support all these years. I would also like to thank my committee members for their suggestions.

I would also like to express my gratitude to those who bravely offered 'helpful' suggestions like "Have you tried turning it off and on again?" – while technically not applicable to partial differential equations, the sentiment is appreciated.

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Chapter 1

Introduction

Force-free electrodynamics is a paradigm describing the dynamics of plasma with very strong magnetic fields. Here, we provide a systematic introduction to the formalism of FFE whereas motivations to study these fields will be developed in the next chapter.

Maxwell's equations can be written in covariant notation as follows

$$\nabla_\nu F^{\mu\nu} = J^\mu, \tag{1.1}$$

$$\nabla_{[\mu} F_{\nu\lambda]} = 0. \tag{1.2}$$

F is the Faraday tensor, J is the current density 4-vector and [] refer to the usual anti-symmetrization of the indices. Because $F^{\mu\nu}$ is antisymmetric, 1.1 implies that the current is automatically conserved.

The energy-momentum tensor associated with the electromagnetic field is given by

$$T^{EM}_{\mu\nu} = F_{\mu\alpha} F_\nu{}^\alpha - \frac{1}{4} g_{\mu\nu} F^{\alpha\beta} F_{\alpha\beta}. \tag{1.3}$$

We note that the electromagnetic stress-energy is not conserved in general i.e.

$$\nabla_\nu T^{\mu\nu} = -F^{\mu\nu} J_\nu. \tag{1.4}$$

The electric and magnetic field as measured by an observer with unit time-like vector field v is given by

$$B_\mu = v^\mu * F_{\mu\nu} \tag{1.5}$$

$$E_\mu = v^\mu F_{\mu\nu}. \tag{1.6}$$

Here $*F$ is the Hodge dual given by $*F = *F_{\alpha\beta} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\mu\nu}$ and ε is the volume element compatible with the metric tensor. The two Lorentz invariants of electromagnetic fields are given by

$$F^{\mu\nu}F_{\mu\nu} = 2(B^2 - E^2), \tag{1.7}$$

and

$$*F_{\mu\nu}F^{\mu\nu} = \frac{1}{2}\varepsilon_{\alpha\beta\mu\nu}F^{\alpha\beta}F^{\mu\nu} = -4\vec{B}\cdot\vec{E}. \tag{1.8}$$

While we need to specify an observer (unit time-like vector field) to write the electric and magnetic field, because the above quantities are Lorentz invariant, all observers will agree on their values.

1.1 Degenerate Electrodynamics and Force-Free Electrodynamics

An electromagnetic field $F^{\mu\nu}$ is called degenerate if there exists a vector w such that $F^{\mu\nu}w_\nu = 0$. If the vector field w is time-like, this statement is equivalent to saying that there is no electric field in the frame of the observer moving along w.

Force-free electrodynamics turns out to be a special case of degenerate electrodynamics and is given by the following equation

$$J^{\mu}F_{\mu\nu} = 0. \tag{1.9}$$

This can be written in terms of electric and magnetic fields as follows

$$\vec{E} \cdot \vec{j} = 0 \text{ and} \tag{1.10}$$

$$\rho\,\vec{E} + \vec{j} \times \vec{B} = 0. \tag{1.11}$$

Using 1.1, we can actually eliminate the 4-current density from our equations and write the equations for force-free electrodynamics as

$$\nabla_{[\mu}F_{\nu\lambda]} = 0, \tag{1.12}$$

and,

$$F_{\mu\nu}\nabla_{\alpha}F^{\nu\alpha} = 0. \tag{1.13}$$

1.2 Force-free Limit of Magnetohydrodynamics

Plasmas are quasi-neutral gas of charged particles that exhibit collective behavior. To completely describe such a system we would need to know the position and velocity of all the particles comprised in the system. Obviously, this is very impractical. Therefore, the next best thing to do is to make simplifying assumptions while still accurately describing the physical effects at hand. One such approach is the fluid model of plasma where plasma is described at a macroscopic scale via averaged quantities like density and averaged velocity (bulk velocity).

If we assume the plasma to be a fluid, then we can couple Maxwell's equations with equations of hydrodynamics and the resulting set of equations describe an electrically conducting fluid. This is called the magnetohydrodynamic limit of plasma physics. The stress-energy tensor for a perfect fluid plasma is given by

$$\begin{aligned} T^{\mu\nu} &= T^{\mu\nu}_{fluid} + T^{\mu\nu}_{field} \\ &= \left((\rho + P)u^{\mu}u^{\nu} + P\, g^{\mu\nu}\right) + \left(F^{\mu\alpha}F^{\nu}_{\alpha} - \frac{1}{4}g^{\mu\nu}F_{\alpha\beta}F^{\alpha\beta}\right), \end{aligned} \tag{1.14}$$

where ρ is the plasma density and P is the fluid pressure. The equations of magnetohydrodynamics are:

$$\nabla_{\mu}(n\, u^{\mu}) = 0, \tag{1.15}$$

and,

$$\nabla_{\mu}\, T^{\mu\nu} = 0. \tag{1.16}$$

If we assume that the electric field vanishes in the fluid frame [1], then we arrive at ideal magnetohydrodynamics and have the following additional constraints

$$u^{\mu}F_{\mu\nu} = 0. \tag{1.17}$$

In the Kerr background, the equations of magnetohydrodynamics are extremely complicated and to our knowledge, the only exact solution in the literature is the one given by Komissarov [58].[2]

When the pressure and the energy density of the plasma are neglected, 1.16 reduces to the force-free condition:

$$F_{\mu\nu}\nabla_{\mu}F^{\mu\nu} = F_{\mu\nu}J^{\nu} = 0. \tag{1.18}$$

This is why FFE can be considered the low inertia limit of ideal mhd.

[1] This is also called the Generalized Ohm's Law.

[2] In fact, even numerical simulations of general relativistic magnetohydrodynamics is considered prohibitively expensive.

1.3 Force-Free Electrodynamics in Forms

It turns out that exterior calculus provides a natural setting for degenerate as well as force-free fields and so our work will follow the exterior calculus formalism of Gralla and Jacobson [43] closely. In this formalism, the electromagnetic field is given by a 2-form F which is closed i.e.

$$dF = 0, \tag{1.19}$$

and satisfies the following equation

$$*d*F = J. \tag{1.20}$$

Here, d is the exterior derivative on forms, * is the Hodge dual operator (also called the Hodge star) and J is the current 1-form.

A field F is degenerate if there exists a vector field w such that the inner product of w with the field vanishes, i.e.

$$i_w F \equiv F(w,\cdot) = 0. \tag{1.21}$$

Obviously, the degenerate field is a force-free field if the current is one such vector,

$$i_J F = 0. \tag{1.22}$$

1.4 Euler Potentials

Because the kernel of a force-free field is integrable, there exist (at least locally, two scalar fields ϕ_1 and ϕ_2 such that

$$^{3}F = d\phi_1 \wedge d\phi_2. \tag{1.23}$$

The flux surfaces then are the level sets to these scalar fields i.e. surfaces of constant ϕ_1 and ϕ_2 This was demonstrated in a series of papers in 1997 by Uchida.

[3] Obviously, we can also write F in terms of a vector potential A i.e. $F_{\mu\nu} = \nabla_{[\mu}A_{\nu]}$.

We note that FFE in Euler potential is shown to be ill posed [95].

1.5 Transformation Properties of Force-free Fields

Conformal transformations are position-dependent rescaling of the metric tensor i.e.

$$g_{\mu\nu} \to g'_{\mu\nu} = \Omega(x)^2 g_{\mu\nu}. \tag{1.24}$$

These transformations leave the light cones and causal structure of the spacetime (and 4-vectors) invariant. It is also well known that conformal transformations leave Maxwell's equations invariant i.e. if F solves Maxwell's equations in a given spacetime, F still solves the equations in any spacetime conformally related to the given spacetime. When we look at Maxwell's equations written in forms, we see that the spacetime metric only appears through the Hodge star as the exterior derivative is defined independent of the background spacetime

$$dF = 0 \tag{1.25}$$

and,

$$d * F = *J. \tag{1.26}$$

This means that for vacuum electromagnetic fields (J = 0) if F solves Maxwell's equations in one spacetime, it is also a solution in any spacetime that preserves $*F$.

It turns out that the most relevant class of non-vacuum electromagnetic fields that are preserved by transformations that preserve the Hodge star are force-free fields and what follows is a concise summary of the paper by Harte [49].

A metric for the form

$$\bar{g}_{\mu\nu} = g^{flat}_{\mu\nu} - 2V l_\mu l_\nu, \tag{1.27}$$

is called a Kerr-Schild metric where l_μ is null with respect to the flat metric

$$g^{flat}_{\mu\nu} l^\mu l^\nu = 0,$$

and V is a scalar field [8]. The transformation is called a Kerrr-Schild transformation and inverse transformation also turns out to be linear in V i.e.

$$\bar{g}^{\mu\nu} = g^{\mu\nu}_{flat} + 2V l^\mu l^\nu. \tag{1.28}$$

Transformations of this type are particularly interesting because the Kerr spacetime was discovered with the ansatz of a Kerr-Schild form and enforcing algebraic conditions. Furthermore, a large class of solutions to the Einstein field equations are known to be related to the flat spacetime via Kerr-Schild transformation [103]. Using the definition of the Hodge star

$$*F_{ab} = \frac{1}{2}\varepsilon_{abcd}F^{cd}, \tag{1.29}$$

it can be shown that under a Kerr-Schild type transformation, $*F$ transforms as follows

$$*F_{ab} \to *F_{ab} + \varepsilon_{abcd} V l^c F^d{}_f l^f \tag{1.30}$$

The Hodge star remains unchanged if

$$l_{[a}F_{b]c} g^{cd} l_d = 0. \tag{1.31}$$

This is equivalent to saying that l is an eigenvector of the field F with respect to the spacetime metric g. Because both null vector fields, as well as null eigenvectors, are preserved under an arbitrary rescaling, l^b defines a principal null direction (PND) of the field F with respect to the metric. This means that a force-free field F in a background spacetime g is still a force-free field in a metric given by $\Omega^2(g_{ab} + V l_a l_b)$. This provides us with another avenue for searching force-free fields. If we found a force-free solution in the flat spacetime and the null vector generating the Kerr-Schild transformation belonged to the principal null direction of the electromagnetic field then the field is force-free in Kerr spacetime as well.

Given the limited number of exact force-free solutions in the Kerr background in the literature, this approach could provide a promising avenue for future searches.

Chapter 2

Motivation

A variety of rotating compact objects, such as pulsars, magnetars, and accreting black holes exhibit highly magnetized plasma-filled magnetosphere. While pulsars and magnetars possess their intrinsic magnetic fields generated via the astrophysical dynamo, accreting black holes get their magnetic field from the advection of magnetized plasma around them. When the energy stored in the magnetic field is significantly greater than the inertia of the plasma, the dynamics is believed to be governed by Force-free electrodynamics.

The force-free electrodynamic limit of magnetohydrodynamics is very often a reasonable approximation to model BH and NS magnetospheres. The FFE is the low-inertia limit of MHD. This approximation has several assumptions. First, one assumes that there is plasma that short-circuits large-scale electric field components in the direction of the magnetic field. This is a physically reasonable assumption because large-scale electric fields would accelerate 'seed' particles which are naturally present in astrophysical environments, to set off plasma-creation avalanches. The regions where such a process happens are called 'gaps'(or spark gaps), where FFE may break down. Similarly, it is assumed that plasma inertia is so small that it can be neglected altogether. This assumption means that the structure and dynamics of the BH and NS magnetospheres are controlled by the electromagnetic (EM) fields only. Plasma response time vanishes, so plasma simply follows the field.

2.1 Pulsar Magnetospheres

Pulsars and magnetars are rapidly rotating neutron stars with magnetic field strengths going up to 10^{12} and 10^{16} Gauss respectively [1].Observations show their radiation spans from radio waves at low frequencies to gamma rays at the high-frequency end. Despite their discovery exceeding five decades ago, the precise mechanism responsible for the observed coherent radio emission from magnetars remains elusive. The short-duration, impulsive radio bursts and the high-intensity X-ray flares they produce are likely driven by magnetic reconnection events within their magnetospheres. While magnetars are a prime candidate source for the recently discovered Fast Radio Bursts (FRBs) due to their extreme properties, the specific emission mechanism for FRBs is still under active investigation.

Soon after their discovery [50] in 1968, pulsars were identified with rotating neutron stars, and in 1969 Goldreich and Julian [42] presented their model of a pulsar which comprised of a highly conducting rotating magnetized neutron star.

Their model comprised a magnetic dipole aligned with the rotation axis of the neutron star. They showed that the rotation of the highly conducting magnetized star in a vacuum induces an electric field that is many orders of magnitude stronger than the gravitational field of the neutron star. Such an electromagnetic field would rip charges apart from the surface of the neutron star to populate the vacuum outside the surface, forming a plasma-filled magnetosphere. The charges would then move along the magnetic field lines and screen the component of the electric field along the magnetic field lines, making the magnetosphere essentially force-free. Similarly, an unscreened electric field along the magnetic field outside the star will cause charged particles to be accelerated to such high energy along the curved magnetic field lines that curvature radiation will induce a cascade of electron-positron pair which will then fill the magnetosphere with plasma and eventually render the plasma force-free. The condition that the electric field be screened along the magnetic field implies that charges in the magnetosphere must corotate with the star. However, this means at a large enough radius, called the light cylinder R_{LC}, charges must cease to corotate as the corotation

[1]For comparison, earth's magnetic field at the surface is of the order 0.5 Gauss.

velocity approaches the speed of light. The light cylinder is defined by $R_{LC} = \frac{c}{\Omega}$, where Ω is the angular speed of the star and it effectively forms a boundary condition for the electromagnetic fields.

Under the assumption of axisymmetry and force-free electrodynamics, the fields satisfy one equation, called the pulsar equation, given by

$$\left(1-\frac{r^2\sin^2\theta}{R_{LC}^2}\right)\left[\frac{\partial^2\Psi}{\partial r^2}-\frac{\partial\Psi}{\partial\theta}\frac{\cos\theta}{r^2\sin\theta}+\frac{1}{r^2}\frac{\partial^2\Psi}{\partial\theta^2}\right]-\frac{2r\sin\theta}{R_{LC}^2}\left[\frac{\partial\Psi}{\partial\theta}\frac{\cos\theta}{r}+\frac{\partial\Psi}{\partial r}\sin\theta\right]+II'(\Psi)=0 \tag{2.1}$$

Here, Ψ is the flux function from which the radial and poloidal magnetic field are obtained by

$$B_r = \frac{1}{r^2\sin\theta}\frac{\partial\Psi}{\partial\theta},$$

and

$$B_\theta = -\frac{1}{r\sin\theta}\frac{\partial\Psi}{\partial r}.$$

The function $I(\Psi)$ is called the poloidal current from which the toroidal magnetic field is obtained via

$$B_\varphi = \frac{I}{r\sin\theta}.$$

The first consistent solution to this equation with a dipole magnetic field near the NS was obtained by [26] [2] who were able to obtain a numerical solution by an iterative process. Since then, other authors have solved this equation confirming the validity of the solution (e.g. [109]). More recently, in solving the time-dependent equations of FFE, numerical models of non-aligned magnetospheres of spinning NS were obtained by [100], and since then, other authors [55, 62] have obtained similar solutions.

Over the years, many general relativistic magnetohydrodynamic (GRMHD) codes [28, 3, 76, 29, 35, 115, 119, 81, 64] have been developed to study the magnetospheres of both neutron

[2] See [27] for a recently found solution to the pulsar equation.

stars and accreting magnetized black holes as well as relativistic jet formation. More recently, there have also been works incorporating additional resistive and/or radiative effects on top of magnetohydrodynamics [106, 13, 31, 19].

One of the most recent developments includes a machine learning approach to studying magnetospheres by partitioning the magnetosphere into regions of closed and open field lines and subsequently training physics-informed neural networks to generate solutions within each region [30].

2.2 AGN Jets

Active Galactic Nuclei (AGNs) are highly luminous compact regions at the heart of galaxies and are characterized by intense radiation across the electromagnetic spectrum ranging from radio waves to high-energy gamma rays. They are widely believed to be powered by the accretion of matter into supermassive black holes with masses ranging from millions to billions of solar masses. Even though their sizes are minuscule compared to the host galaxies, they can outshine the host galaxies by several orders of magnitude.

AGN jets, despite originating within a few gravitational radii of the supermassive black hole, can stretch for hundreds of kiloparsecs. The small gyroradius and Debye length compared to the length scale of the jet allows for a magnetohydrodynamic description of the plasma processes. However, owing to the complex and non-linear nature of the magnetohydrodynamic equations, only numerical solutions are feasible and one has to resort to simplified models to describe large-scale processes and force-free electrodynamics appears to be one such leading candidate.

Using FFE, Blandford and Znajek showed in their seminal paper [9] that it is possible to directly extract electromagnetic energy from black hole spin for a slowly rotating black hole. Theirs was a perturbative solution in the spin parameter and the prevailing electromagnetic fields allow for the extraction of rotational energy via pulsar winds in the form of a Poynting flux. Blandford-Znajek mechanism is one of the leading contenders for explaining the formation and collimation of relativistic jets fro AGNs.

Observations of relativistic jets indicate a strong positive correlation between black hole spin and the power of the relativistic jets, thus supporting the rotational energy extraction mechanism [101]. It was found in [107] that, for a fast spinning black hole steady force-free electrodynamic models and successfully explain limb brightened feature observed by Hada et al. [47] in the jet of the giant elliptical galaxy M87. Similarly, the jet collimation profile for a giant radio galaxy NGC 315 studied via Very Long Baseline Interferometry (VBLI) was also found to be in good agreement with steady axisymmetric force-free electrodynamics [87]. It is assumed that when plasma starts to get depleted in the magnetosphere, gap regions are formed where the electric field is not screened. In the gaps, charged particles can be accelerated to high energies which in turn leads to the creation of electron-positron pairs, thus replenishing the plasma. However these processes are assumed to be intermittent [61], so our force-free assumption is fairly reasonable.

Ever since its inception, there has been considerable effort to generalize Blandford and Znajek's solution to a physically relevant exact solution that allows for energy extraction. However, the problem remains intractable owing to the complicated nature of the coupled partial differential equations that arise in force-free electrodynamics in the Kerr background.

By now, however, two classes of exact solutions in Kerr have emerged. A class of exact solutions in Kerr geometry valid at the event horizon was first given by Menon and Dermer [74] nearly thirty years after the publication of [9]. Recently, a generalization to this solution was obtained by Brennan et al. by employing the Newman-Penrose tetrads and assuming the current to be aligned to one of the null tetrads [12]. Other notable works in the theory of force-free electrodynamics include [66, 43, 44, 83, 84, 85, 82]. In the course of this study, we have been able to find four new classes of force-free solutions in Kerr spacetime by studying the specific foliations of the spacetime that allow for the existence of force-free fields.

While exact solutions to FFE have been rare, higher-order perturbative solutions to the equations of FFE have continued to remain an active area of exploration [4, 16]. Since the extraction of rotational energy from the black hole takes place near the horizon, the force-free paradigm can be explored in the simplified metric describing the region near the horizon of an extremal Kerr black hole. Because of the simplified nature of this metric, many classes of exact solutions are possible [45] [65] [14] [24].

While the Blandford-Znajek mechanism is widely considered the leading mechanism for energy extraction from rotating compact objects, there are alternative theories in the literature. For example, there have been studies of magnetic reconnection and how it could affect energy extraction around accreting magnetized black holes [99, 120]. For highly spinning black holes immersed in a magnetic field, magnetic reconnection inside the ergosphere has also been proposed as a viable mechanism for energy extraction [23, 63, 118].

Force-free electrodynamics is also extensively utilized in studying binary compact systems. The studies of electromagnetic counterparts of neutron star mergers are often modeled by force-free electrodynamics which is applicable for highly magnetized closely orbiting binary systems [78, 77].

There is growing evidence that the energy output from black holes through AGN feedback helps shape galaxy morphology, and possibly evolution of clusters, by suppressing star formation rates. Thus, an analytical model of relativistic jets from AGN is also an invaluable tool for studying the dynamics of galaxies in the late universe [98, 36, 21].

Apart from being fascinating astrophysical objects, neutron stars and accreting black holes are excellent laboratories for studying gravity and electromagnetism in the strong sector which puts force-free electrodynamics at the forefront of relativistic astrophysics.

2.3 Blackhole Mimickers and Alternative Theories of Gravity

The Kerr metric is widely believed to describe the exterior of a rotating black hole in general relativity. While observations from telescope arrays as well as gravitational wave detections have helped rule out several alternative theories of gravity as well as black hole models with additional hair [2], they do not completely rule out alternative theories of gravity or alternative candidates for rotating black holes [114, 60]. Tests to determine whether or not the Kerr spacetime describes an astrophysical black hole [3] remain an active area of research [5, 111, 97, 108]. There exist several alternative black hole candidates and determining which of these candidates are consistent with observations is an active area of research.

Space-based gravitational wave observatories are expected to enhance the already revolutionized field of multi-messenger astronomy and with their advent, the observation of electromagnetic counterparts to these binary compact object mergers will give us another window into strong gravity. As the measurements of the power of the relativistic jets as well as black hole spin get more precise in the future, it is believed that the Blandford-Znajek mechanism could help rule out these black hole candidates. As a result, there has been significant effort dedicated to exploring FFE and the BZ mechanism in both modified gravity theories [32] as well as in metrics other than the Kerr metric [89] [60] [15]. For instance, it was shown that relativistic jets are more powerful in stringy black holes [20]. More recently, it was found in GRMHD simulations of magnetized accretion flow into rotating Loop-Quantum black holes that the LQG effect amplifies the frame-dragging effect which then leads to the enhancement of jet power [53].

2.4 Force-free Fields in Cosmology

Magnetic fields are known to permeate the universe from the scales of planets and stars to galaxies and even galaxy clusters [41]. At the scale of galaxies, magnetic fields have a strength of order 10^{-6} G and are coherent over kpc scales. Micro-Gauss magnetic fields have also been observed in galaxy clusters [37, 10]. Recent observations suggest that intergalactic space may harbor magnetic fields of strength 10^{-16} G coherent over Mpc scales. [79].

The currently accepted paradigm explaining the existence of this all-pervading magnetic field states that these magnetic fields originated from the amplification of seed magnetic fields via various astrophysical dynamos. While at smaller scales (at the level of planets and stars), these fields require constant rejuvenation to replenish the loss from dissipation, the time scales for dissipation for large-scale magnetic fields may be of the order of the age of the universe.

[3] This is sometimes called the Kerr hypothesis.

While the amplification of the large-scale magnetic field is attributed to the gravitational collapse of flux-frozen matter during structure formation, the dynamo effect can only amplify a preexisting non-zero ”seed” magnetic field. The origin of such seed magnetic field itself is not well understood and is a subject of extensive study, especially over the last two decades. Electroweak phase transition [54, 11] and inflationary generation of the seed magnetic fields [33, 51, 104] are the most widely studied phenomena for the generation of primordial magnetic fields in the early universe. Other mechanisms for the generation of primordial magnetic fields include Lorentz invariance violation [7], relativistic positron abundance [102], and non-Gaussian perturbations to the baryon density to name a few [38].

A variety of methods exist for detecting and/or constraining magnetic fields in the universe. At low redshifts, these include the observation of Zeeman splitting [48] and Faraday rotation of linearly polarized radio sources [117]. At higher redshifts, the existence of magnetic fields can be inferred from the effects of primordial magnetic fields on the polarization of the Cosmic Microwave Background (CMB) [1, 86, 92, 105], effects of magnetic pressure on the abundance of light elements during big-bang nucleosynthesis [56] as well as (non) detection of inverse-Compton scattered CMB photons from blazar observations [79, 34]. Recently, it was shown that hydrodynamical simulations of structure formation in the universe can also constrain the primordial magnetic fields by studying their ability to reproduce in the simulations the scaling relations observed in dwarf galaxies [96].

When the energy stored in the electromagnetic field is much greater than the plasma pressure, the Lorenz force vanishes. Force-free electrodynamics is the framework that describes such a system. Force-free fields often appear in the study of primordial magnetogenesis [52, 94, 110, 88, 67, 46, 91]. Several studies of primordial magnetic fields assume the vanishing of the Lorentz force [57, 69, 40, 22, 39]. To our knowledge, there has been no significant and systematic attempt to study the types of force-free fields allowed in the Friedmann–Lemaître–Robertson–Walker (FLRW) metric.

Force-free electrodynamics is also widely utilized in studying magnetic fields in the solar corona [116]. In this dissertation, however, we will exclusively discuss force-free fields around accreting rotating black holes and in the early universe.

Chapter 3

The Theory of Force-Free Electrodynamics

3.1 Force Free fields and Foliations

The first attempt at studying the geometrical features of force-free fields in general relativity was taken by Carter in 1979 [18]. MacDonald and Thorne [68] reformulated force-free electrodynamics int 3+1 formalism. Considerable analytical work was done by Uchida [112, 113] who also developed the Euler potential formalism of force-free electrodynamics. More work was done by Komissarov [58, 59] who also demonstrated that force-free electrodynamics is well posed for the magnetically dominated case. In 2014, Gralla and Jacobson [43] reformulated the theory in the language of differential forms demonstrating that exterior calculus provided a natural setting for the study of force-free fields.

Then, in 2020, in a series of papers [71, 72, 73] in 2020, Menon was able to formulate the theory of force-free fields in arbitrary electrically neutral spacetime in completely geometric terms, laying the groundwork for a completely new approach to finding force-free fields. What follows is a summary of the aforementioned results culminating in Menon's theorems that will be utilized in the next chapters to generate new solutions to the equations of FFE in Kerr as well as the FLRW spacetimes.

It is well known that a degenerate electromagnetic field is given by a simple 2-form i.e.

$$F = \alpha \wedge \beta, \tag{3.1}$$

where α and β are 1-forms. The kernel of F is the two-dimensional subspace spanned by forms u and v that annihilate F i.e.

$$ker(F) = \{\text{span}(u,v) : \alpha(v) = \alpha(u) = \beta(v) = \beta(w) = 0\}. \tag{3.2}$$

For force-free electromagnetic fields, the kernel of the field tensor F, denoted by $\ker F$, is a 2-dimensional distribution that is closed under Lie brackets. This means that whenever $v, w \in \ker F$, we have that $[v,w] \in \ker F$. Such subspaces of tangent vectors are referred to as an involutive distribution. Consequently, Frobenius's theorem then implies that when a force-free F exists on $\mathscr{M}$, the spacetime manifold can be foliated by 2-dimensional integral submanifolds of the distribution spanned by $\ker F$. The leaves of the foliation, which are the integral submanifolds of $\ker F$, will be denoted as $\mathscr{F}_a$. A particular submanifold $\mathscr{F}_a$ is referred to as a *field sheet* (or sometimes flux surfaces). Here a belongs to some indexing set A. The key points here are that

$$\mathscr{F}_a \cap \mathscr{F}_b = 0 \text{ whenever } a \neq b \in A\ ,\ \ \cup_{a\in A}\ \mathscr{F}_a = \mathscr{M}\ .$$

The general theory of FFE splits into three categories. Two of them, namely the magnetically and electrically dominated solutions, have very similar properties, and they differ geometrically only in detail. Nonetheless, from the point of view of the initial value problem in partial differential equations, it should be noted that only the magnetically dominated solutions are well posed (for example see [90] and [17]). The general case where non-null solutions reach the null limit can sometimes lead to divergences in the field tensor. These ideas have been explored in [71].

Non-Null Force-free Fields

In this section, we will state all the central results of [72]. Consider any foliation $\mathscr{F} = \{\mathscr{F}_a : a \in A\}$ of spacetime by 2-dimensional Lorentzian manifolds $\mathscr{F}_a$. In the magnetically dominated case, the metric when restricted to $\mathscr{F}_a$ has a Lorentz signature. We will refer to such a foliation as a 2-D *Lorentzian foliation* of $\mathscr{M}$ and denote it by $\mathscr{F}^{2L}$. The leaves of such a 2-D Lorentzian foliation are denoted by $\mathscr{F}_a^{2L}$. For any $p \in \mathscr{M}$ there exists an open set U_p of p and an inertial frame (e_0, e_1, e_2, e_3) on U_p such that the tangent space for each of the leaves of the foliation, when restricted to U_p, is spanned by e_0 and e_1. We will refer to such frames as 2-D *Lorentzian foliation adapted frames* and denote it by $\mathbf{F}_{2L}$. Note that, here, $g(e_\mu, e_\nu) = \eta_{\mu\nu}$, where $\eta_{\mu\nu}$ are the components of the Minkowski metric.

Let V, W be vector fields tangent to any $\mathscr{F}_a^{2L}$. Then the *shape tensor* or the *second fundamental form* Π of $\mathscr{F}_a^{2L}$ is defined by

$$\Pi(V,W) = (\nabla_V W)^\perp \ .$$

Here $\perp$ takes the component of the vector normal to the surface $\mathscr{F}_a^{2L}$. The mean curvature field at any point of $\mathscr{F}_a^{2L}$ is then defined by

$$H = \frac{1}{2}\Big[-\Pi(e_0,e_0) + \Pi(e_1,e_1)\Big]$$

in any $\mathbf{F}_{2L}$. Similarly, we can define a 'dual'mean curvature field. We note, however, that the complimentary orthogonal distribution of the foliation spanned locally by e_2 and e_3 may not necessarity be involutive,

$$\tilde{H} = \frac{1}{2}\Big[\Pi(e_2,e_2) + \Pi(e_3,e_3)\Big] \ .$$

We are now able to state the central theorem that will enable the search for magnetically dominated solutions [72]. The analogous result in the restricted case of stationary, axis-symmetric force-free electrodynamics in a Kerr background was previously developed in

[25].

Theorem 1 *Let $\mathscr{F}^{2L}$ be any 2-D Lorentzian foliation of $\mathscr{M}$ with leaves $\{\mathscr{F}_a^{2L},\ a \in A\}$. Let $\mathbf{F}_{2L}$ be a Lorentzian frame field on a starlike open set U_p about any $p \in \mathscr{M}$. Then, up to a constant factor in u, $F = u\, e_2^\flat \wedge e_3^\flat$ is the unique magnetically dominated force-free electrodynamic field in U_p such that*

$$\ker F|_{U_p \cap \mathscr{F}_a{}^{2L}} = T(\mathscr{F}_a{}^{2L})$$

if and only if

$$dH^\flat = -d\tilde{H}^\flat\ , \tag{3.3}$$

where H and $(\tilde{H})$ are the mean (dual) curvature field associated with the foliation. Moreover, in this case,

$$d(\ln u) = 2(H+\tilde{H})^\flat\ . \tag{3.4}$$

It should be kept in mind that the theorem above generates vacuum solutions as well. Note that the resulting vacuum electromagnetic field continues to be degenerate. This inclusion of vacuum solutions in the methodology stems from the fact that when $j = 0$, the resulting field is trivially "force-free".

An analogous result applies to an electrically dominated field. The only difference is that here (e_2, e_3) spans the kernel of F and hence forms an involutive pair, and the force-free field is given by $F = u\, e_0^\flat \wedge e_1^\flat$. The expression for u, in this case, continues to be given by eq. (3.4).

3.2 Generating Non-Null Solutions from Foliations

Here $\Lambda(x)$ is a spacetime-dependent general homogeneous Lorentz transformation $\mathscr{L}$ satisfying $\eta = \Lambda^T\, \eta\, \Lambda$. It then suffices to pick one tetrad and study its Lorentz transformations that are subject to theorem 1.

This freedom gives us a six-parameter family of functions. For magnetically dominated solutions, without loss of generality, all that we require is that the pair $(\bar{e}_0, \bar{e}_1)$ is involutive and further that the foliation by integral submanifolds of the distribution satisfies eq.(3.3). Similarly, for electrically dominated solutions, it is the pair $(\bar{e}_2, \bar{e}_3)$ that must form an involutive distribution, and once again, that the foliation by integral submanifolds of the distribution satisfies eq.(3.3). Admittedly, this technique is not a recipe, but rather, a systematic method for a search for new solutions. This is currently the best-known technique available to seek benchmark analytic solutions for the non-null solutions describing the Blandford-Znajek mechanism and other force-free configurations.

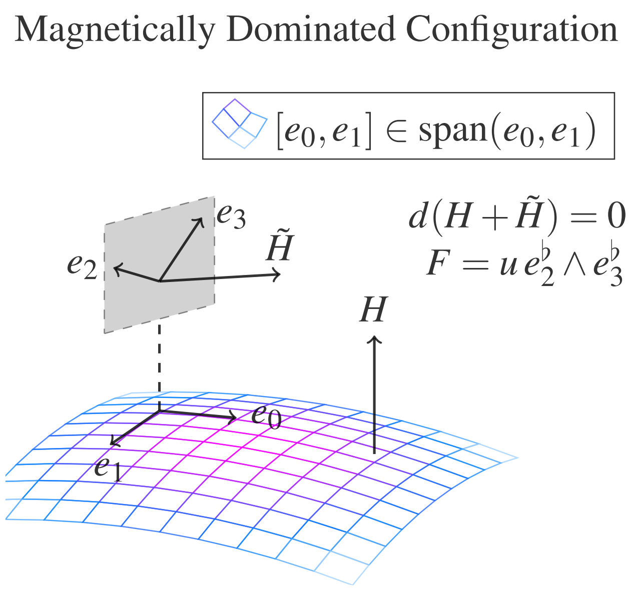

Figure 3.1: Time-like foliation generating a magnetically dominated force-free field

We have the following method for systematically searching for magnetically dominated force-free fields.

Given a set of tetrads e_μ we find a lorentz transformation Λ i.e.

$$\bar{e} = \Lambda e, \tag{3.5}$$

such that a pair of time-like and space-like vectors form an involutive distribution. i.e. $[e_0, e_1] \in \mathrm{span}(e_0, e_1)$[1].

We then calculate the mean curvature field H and its dual $\tilde{H}$ as follows

$$2H = [-g(\nabla_{e_0} e_0, e_2) + g(\nabla_{e_1} e_1, e_2)]\, e_2 + [-g(\nabla_{e_0} e_0, e_3) + g(\nabla_{e_1} e_1, e_3)]\, e_3\ , \tag{3.6}$$

and

$$2\tilde{H} = [-g(\nabla_{e_2} e_2, e_0) - g(\nabla_{e_3} e_3, e_0)]\, e_0 + [g(\nabla_{e_2} e_2, e_1) + g(\nabla_{e_3} e_3, e_1)]\, e_1\ . \tag{3.7}$$

If the condition $d(H + \tilde{H}) = 0$ [2] is satisfied, we are guaranteed a magnetically dominated force-free solution which is given by

[1] This is trivially satisfied if the vector fields commute i.e. $[e_0, e_1] = 0$.
[2] Such one form is called a closed form.

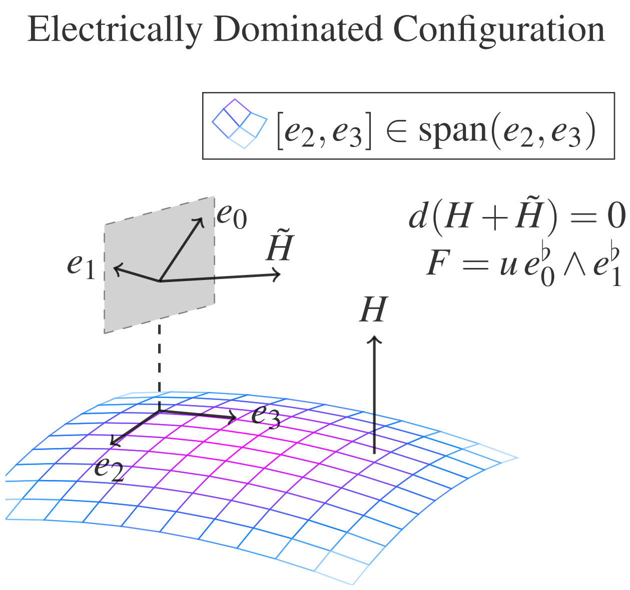

Figure 3.2: Space-like foliation giving rise to an electrically dominated force-free field.

$$F = u\, e_2^\flat \wedge e_3^\flat. \tag{3.8}$$

Here, u is found by solving $d(ln(u)) = 2(H + \tilde{H})$.

The procedure for generating electrically dominated force-free fields is similar to the one for magnetically dominated case with the notable difference being the nature of the vectors spanning the kernel of F.

For an electrically dominated solution, we start with the Lorentz transformation as usual but look for transformations that result in two spacelike vectors being involutive i.e. $[e_2, e_2] \in \text{span}(e_2, 2_3)$.

We calculate H and $\tilde{H}$ as in 3.6 and 3.7 and if their sum is closed, we are guaranteed an electrically dominated force-free field given by

$$F = u\, e_0^\flat \wedge e_1^\flat. \tag{3.9}$$

3.3 The Null Force Free Field

In contrast to the non-null case, the null solutions may be easier to obtain. Indeed, if we find a new solution, as will be made clear, we will have found a class of solutions. Null solutions come with two-parameter freedom that can be used to fine-tune the nature of the Poynting flux. The techniques in this section were derived in [73].

Let F be a null electromagnetic field on $\mathscr{M}$, and let $\mathscr{F}_a$ be the associated field sheets. Then g restricted to the tangent bundle of $\mathscr{F}_a$ denoted by $T(\mathscr{F}_a)$ is degenerate. To see how this happens, in eq.(**??**) consider the plane spanned by α and β. By a judicious reassignment if necessary, pick $\alpha \perp \beta$ in the sense that $g(\alpha, \beta) = 0$. Since,

$$F^2 = 2\alpha^2\beta^2 = 0\,,$$

either α or β must be a null vector. Without loss of generality, set $\beta^2 = 0$, and thus α is spacelike, and $\ker F$ consists of all vectors orthogonal to α and β. In particular $\beta^\sharp \in \ker F$. Rename $\beta^\sharp$ [3] as l to indicate that it is a light-like vector. Clearly, l is orthogonal to every vector in $\ker F$.

Henceforth, foliations by 2-dimensional submanifolds of $\mathscr{M}$, where the restriction of g on the leaves of the foliation is degenerate, will be referred to as a null foliation. Given a null foliation, there may or may not be an associated force-free null field. However, foliation adapted charts, $(U_p, \phi_p = (x^1, \ldots, x^4))$, such that surfaces of constant values for x^3 and x^4 that agree with the leaves of the foliation continue to exist. Such charts are referred to as a null foliation adapted chart.

The ray along l generates all the null vectors in $T(\mathscr{F}_a)$. In this manner, a null force-free field F defines a unique null ray in spacetime that is tangent to the field sheets; i.e., locally, one obtains a smooth lightlike vector field l in $\ker F$. Since $\mathscr{F}_a$ is 2-dimensional, there exists a local spacelike vector field s in $\ker F$. We will normalize s so that $g(s,s) = 1$. Together l and s span $\ker F$. Finally, normalize α so that $g(\alpha, \alpha) = 1$, and to complete the tetrad, let n be a null

[3] $\beta^\sharp$ is the tangent vector that is metrically equivalent to β.

vector field such that

$$n^\flat(l) = -1,\ \ n^\flat(s) = 0 = \alpha(n)\ .$$

To recap, l and s span $\ker F$, the span of $l^\flat$ and α are all the forms that annihilate vectors in $\ker F$, n completes the tetrad, and $(s,l,\alpha^\sharp,n)$ span $T(\mathscr{M})$. We shall refer to $(s,l,\alpha^\sharp,n)$ as a null foliation adapted frame for a null foliation $\mathscr{F}$.

Although we plan to proceed in a completely geometric, coordinate-free formalism, it will be useful to establish a relationship between foliation-adapted charts and frames. Let

$$\begin{pmatrix} \alpha \\ l^\flat \end{pmatrix} = \begin{pmatrix} \alpha_3 & \alpha_4 \\ l^\flat_3 & l^\flat_4 \end{pmatrix} \begin{pmatrix} dx^3 \\ dx^4 \end{pmatrix}\ . \tag{3.10}$$

Then

$$u(x^3,x^4)\ dx^3 \wedge dx^4 = (u\cdot\kappa)\ \alpha\wedge l^\flat\ ,$$

where

$$\kappa = (\alpha_3\ l^\flat_4 - \alpha_4\ l^\flat_3)^{-1}\ . \tag{3.11}$$

Thus

$$F = (u\cdot\kappa)\ \alpha\wedge l^\flat\ . \tag{3.12}$$

The null mean curvature or the null expansion scalar, θ, for the congruence generated by l is given by

$$\theta = \frac{1}{2}\left[\ g(\nabla_s\, l,\ s) + g(\nabla_{\alpha^\sharp}\, l,\ \alpha^\sharp)\ \right]\ . \tag{3.13}$$

Definition 1 *Let $(s,l,\alpha^\sharp,n)$ be a null foliation adapted frame for a null foliation $\mathscr{F}$. If*

$$\theta = g(\nabla_s\, l,\ s)$$

we say that $\mathscr{F}$ admits equipartition of null mean curvature with respect to the null pregeodesic vector field l.

Note that when $\mathscr{F}$ admits an equipartition of null mean curvature with respect to the null

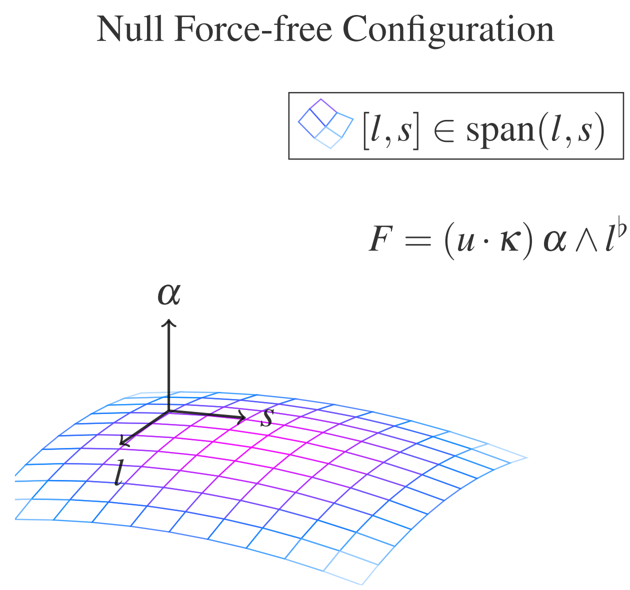

Figure 3.3: Null foliation generating a force-free field

pregeodesic vector field l, the value of θ is equally shared by $g(\nabla_s\, l,\ s)$ and $g(\nabla_{\alpha^\sharp}\, l, \alpha^\sharp\,)$. We finally write down the conditions for a null force-free field in completely geometric terms.

Theorem 2 *Let $\mathscr{F}$ be a null foliation of $\mathscr{M}$ with metric g. Let $(U_p, \phi_p = (x^1,\ldots,x^4))$ be a null foliation adapted chart about any arbitrary point $p \in \mathscr{M}$, and let $(s, l, \alpha^\sharp, n)$ be a null foliation adapted frame for $\mathscr{F}$ in U_p. Then F given by eq.(3.12) for any smooth function u on $\mathscr{F}|_{U_p}$ is a unique class of null, force-free solutions in U_p such that* $\ker F$ *contains exactly the vectors in $T(\mathscr{F}_a)$ if and only if*

1. *l is a pregeodesic vector field,* [4] *and*

2. *$\mathscr{F}$ admits equipartition of null mean curvature with respect to the null pregeodesic vector field l.*

We note that because null fields cannot be normalized, l is not unique.

[4] l can be re-parameterized as a geodesic.

3.4 Existence of the Dual Null Solution

Given a null solution generated by a pregeodesic congruence l as described in the previous section, there exists a generalized dual null solution when the following condition is satisfied.

$$g(\nabla_s l, \alpha^\sharp) + g(\nabla_{\alpha^\sharp} l, s) = 0. \tag{3.14}$$

If the above condition is satisfied, then l is said to admit a uniform equipartition of the null mean curvature.

If the above requirement is satisfied, we can find smooth functions A and B such that
$A^2 + B^2 = 1$, and

$$\hat{s} = A\, s + B\, \alpha^\sharp, \tag{3.15}$$

such that $\hat{s}$ and l form an involutive distribution.

Our generalized solution is then given by

$$F = (u \cdot \kappa)\, \alpha \wedge l^\flat, \tag{3.16}$$

where

$$\begin{bmatrix} \hat{s} \\ \hat{\alpha}^\sharp \end{bmatrix} = \begin{bmatrix} A & B \\ -B & A \end{bmatrix} \begin{bmatrix} s \\ \alpha^\sharp \end{bmatrix}. \tag{3.17}$$

3.5 Adapted Frame Formalism

So far our discussion has separated the foliations into three distinct groups based on the nature of F^2, and as such the nature of the tetrads and the conditions necessary for the existence of null, magnetically dominated, and electrically dominated solutions are distinct. However, there is nothing that precludes a force-free field from smoothly transitioning from a non-null region to a null region. In fact, we will explicitly demonstrate such a solution in the FLRW metric.

Our aforementioned tetrad formalism is obviously not suited for such ”type-changing” solutions so we resort to a frame that is adapted to the force-free field.

Given a manifold $\mathscr{M}$, for any point $p \in \mathscr{M}$, there exists a coordinated chart $(U_p, \phi_p(x^1, .., x^4)$ centered about the point p i.e.

$$\phi_p(p) = (x^1(p), ..., x^4(p)) = (0, ..., 0)$$

, that is adapted to our distribution. We can, without loss of generalization, choose for coordinates such that

$$\text{span}\,\{\partial_{x^1}, \partial_{x^2}\} = \ker F\,|_{U_p}. \tag{3.18}$$

Our field then can be written as

$$F = u\,dx^3 \wedge dx^4, \tag{3.19}$$

for some function $u(x^3, x^4)$ [5].

Imposing the force-free condition $* d * F = 0$ for the aforementioned form of F gives us

$$(J^\sharp)^a = 0, \;\; \text{for } a = 3, 4.$$

For ease of calculation we define the following quantities Despite the index, these are not actually 4-vectors.

$$\begin{aligned} M^r &= g^{r3} g^{r4} - g^{33} g^{r4} \\ N^r &= g^{r3} g^{44} - g^{34} G^{r4}. \end{aligned} \tag{3.20}$$

[5] $dF = 0$ implies that u can only be a function of x^3 and x^4.

The force-free condition reduces to

$$M^4 \frac{\partial}{\partial x^4} \ln|u| = -\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^r}(\sqrt{-g}\, M^r) \tag{3.21}$$

$$N^3 \frac{\partial}{\partial x^3} \ln|u| = -\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^r}(\sqrt{-g}\, N^r) \tag{3.22}$$

We will explicitly construct a frame adapted to a force-free field for a type-changing solution in FLRW spacetime.

Chapter 4

The Schwarzschild and the Kerr Spactime

4.1 The Schwarzschild Spacetime

The Schwarzschild spacetime is a vacuum solution to the Einstein field equations that describes the exterior of a static spherically symmetric mass M. The metric is given by

$$ds^2 = -\left(1 - \frac{2M}{r}\right) dt^2 + \left(1 - \frac{2M}{r}\right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin\theta^2\, d\varphi^2. \tag{4.1}$$

As the solution describes a non-rotating black hole, there cannot be any extraction of rotational energy. However, solutions of FFE in one spacetime can sometimes be promoted to solutions in a different, more general spacetime. Hence, it is worth looking at solutions in this spacetime.

4.2 The Kerr Spacetime

The Kerr-Spacetime is defined by an axisymmetric, stationary (but not static), algebraically special, asymptotically flat, vacuum solution to the Einstein Field Equations. It was shown [93] that when any uncharged, nearly spherical mass distribution collapses into a black hole, the non-spherical part of the fields will be radiated away, and the spacetime will settle down to a form given by the Kerr metric.[1]

The Kerr metric can be written in the Boyer-Lindquist Coordinates as follows.

$$ds^2 = -\left(1 - \frac{2Mr}{\rho^2}\right) dt^2 - \frac{4aMr\sin^2\theta}{\rho^2} dt d\phi + \frac{\rho^2}{\Delta} dr^2 + \rho^2 d\theta^2 + \frac{\sin^2\theta}{\rho^2}\left[(r^2+a^2)d\phi - adt\right]^2 \tag{4.2}$$

where

$$\begin{aligned} \rho^2 &= r^2 + a^2\cos^2\theta \\ \Delta &= r^2 - 2Mr + a^2 \end{aligned} \tag{4.3}$$

We observe that the metric, written in the above form, is singular at two points: $\Delta = 0$ and $\rho^2 = 0$. However, the curvature invariants (for example the Kretschmann scalar, $K = R^{\mu\nu\alpha\beta}R_{\mu\nu\alpha\beta}$), are only singular at $\rho^2 = 0$. This tells us that the singularity at $\Delta = 0$ is not a singularity of the manifold, but of the chosen coordinates that can be transformed away. Because the singularity at the (outer) horizon, $\Delta = 0$, is a coordinate singularity, we require that the electromagnetic field be well-defined at the (outer) horizon. This is called the Znajek regularity condition. Because the Boyer-Lindquist coordinates are singular at the event horizon, we impose the regularity condition by transforming the field to the Kerr-Schild coordinates and then requiring that all the coefficients of the term of the form $\frac{1}{\Delta}$ vanish at the horizon.

The stationary and axisymmetric nature of the spacetime is manifest in the coordinate system i.e. the metric does not explicitly depend on the coordinates t and ϕ. We note that the metric is not stationary i.e. it is not invariant under time reversal $t \longrightarrow -t$. The metric is, however, invariant under simultaneous t and ϕ reversal i.e. $(t, \phi) \longrightarrow (-t, -\phi)$.

The constant a is interpreted as the angular momentum per unit mass and we observe that the Kerr metric reduces to the Schwarzschild metric when $a \longrightarrow 0$ *and* $M \neq 0$. And obviously, the

[1] While the Kerr Spacetime is known to describe the outside of a rotating black hole, to this day, a physically reasonable interior solution has not been found.

Metric is asymptotically flat i.e. it reduces to the Minkowski metric in the limit $r \longrightarrow \infty$.[2]

Finally, we observe that when $a > M$, the spacetime no longer describes a black hole as there is no event horizon to shroud the singularity. Such a system is called a naked singularity.[3]. Therefore, we will always assume that $a < M$[4]i.e. the singularity is always shrouded by an event horizon.

4.2.1 The Horizons

We can rewrite the coordinate singularity at Δ as follows:

$$\begin{aligned}\Delta &= r^2 + a^2 - 2Mr \\ &= (r - (M + \sqrt{M^2 - a^2}))(r - (M - \sqrt{M^2 - a^2})) \\ &= (r - r_+)(r - r_-)\end{aligned}$$

Here,

$$\begin{aligned}r_+ &= M + \sqrt{M^2 - a^2} \\ r_- &= M - \sqrt{M^2 - a^2}\end{aligned}$$

and $r = r_+$ and $r = r_-$ are the surfaces describing the coordinate singularity and are called the outer and inner horizon respectively.[5]
The horizons divide the Kerr Spacetime into three distinct regions.

Region I: $r > r_+$ Surfaces of constant r are timelike in this region so can be crossed in both directions.

Region II: $r_- < r < r_+$ Surfaces of constant r are spacelike in this region and can only be

[2]Interestingly, the Kerr metric also reduces to the Minkowski metric when $M \longrightarrow 0$ *and* $a \neq 0$.
[3]The cosmic censorship conjecture by Roger Penrose states that naked singularities cannot arise in nature from physically reasonable initial conditions
[4]Numerical simulations of collapse of matter into black holes also suggest that $a < M$.
[5]The normal vector to these hyper-surfaces are null ($v_\mu v^\mu = 0$) and hence these surfaces are called null-hyper-surfaces and have the property that they can be crossed only in one direction.

crossed in one direction. An object in this region can only continue falling to a decreasing value of r until it reaches region III.

Region III: $r < r_-$ This region includes the physical singularity and is of no interest to our work.

4.2.2 The Ergosphere

We can observe that $g_{tt} = -1 + \frac{2Mr}{r^2 + a^2\cos^2\theta}$ changes signs at two places $r_{S\pm}$ given by[6]:

$$r_{S\pm} = M \pm \sqrt{M^2 - a^2\cos^2\theta} \tag{4.4}$$

The region outside the outer horizon, where $g_{tt} > 0$ is called the ergoregion the boundary of which is called the ergosphere. The ergoregion lies outside the horizon except at the poles where it meets with the (outer)horizon.
We can consider the usual "stationary" observer with a timelike velocity of the form $u^\mu = (-\alpha, 0, 0, 0)$. From the normalization of 4-velocity

$$u_\mu u^\mu = -1$$

we get that $\alpha = \sqrt{\frac{\rho^2}{\Delta - a^2\sin^2\theta}}$.
Which in turn implies that: $r > M + \sqrt{M^2 - a^2\cos^2\theta} = r_{S+}$

This means that the "stationary" observer cannot exist inside the ergosphere. The ergosphere is therefore also called the stationary limit surface.

The Kerr metric is singular on the surface $r^2 + a^2\cos\theta = 0$ which is the famed ring singularity.

4.3 The Kerr-Schild Coordinates

Since the Boyer-Lindquist coordinates are not well-behaved at the event horizon of the black hole, we need to transform into a new coordinate system. Depending on the choice of infalling

[6]In Schwarzschild metric however, g_{tt} changes sign at the event horizon so the ergosphere and the event horizon coincide.

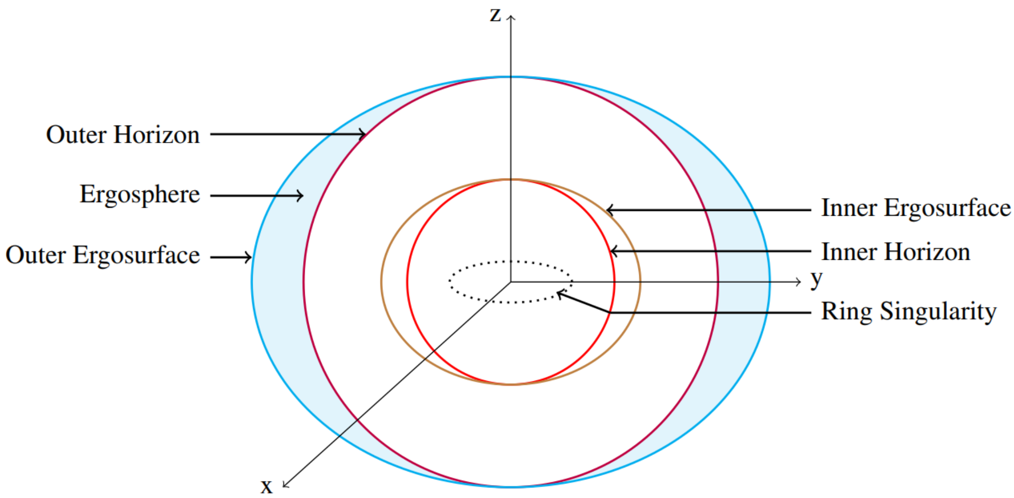

Figure 4.1: Ergosphere, horizons and the ring singularity in the Kerr metric

or outgoing radial null geodesics, we have the black hole and white hole coordinate systems respectively.

The transformation from the Boyer-Lindquist to the Black Hole coordinates $(\bar{t}, r, \theta, \bar{\varphi})$ is given by

$$d\bar{t} = dt + \frac{r^2 + a^2}{\Delta} dr \, , \text{ and } d\bar{\varphi} = d\varphi + \frac{a}{\Delta} dr \, .$$

In this coordinate system, the metric takes the following form

$$ds^2 = (-1+z)\, dt^2 + 2\, dt\, dr + \rho^2 d\theta^2 - za \, - 2a \sin^2(\theta)\, d\varphi\, dr + \frac{\Sigma^2 \sin^2(\theta)}{\rho^2}\, d\varphi^2 \qquad (4.5)$$

where, $z = \dfrac{2Mr}{\rho^2}$.

Similarly, the transformation to White Hole coordinates $(t^*, r, \theta, \bar{\varphi}^*)$ is given by

$$dt^* = dt - \frac{r^2 + a^2}{\Delta} dr \, , \text{ and } d\varphi^* = d\varphi - \frac{a}{\Delta} dr \, .$$

4.4 Black Hole Energy Extraction

The existence of an asymptotically timelike Killing vector field ∂_{t*} in Kerr spacetime allows us to define the notion of energy conservation. Therefore, we can calculate the rate of extraction of energy from the past horizon of the spacetime $\mathscr{H}_-$ by

$$\frac{d\varepsilon}{dt^*}|_{\mathscr{H}_-} = \int_{r^*=r_+} T^{r^*}{}_{t^*}\,\rho^2.\sin\theta d\theta d\varphi^* \tag{4.6}$$

Here, ε is given by

$$\varepsilon = -\int_{r^*=r_+}^{r^*=\infty} T^{r^*}{}_{t^*}\,\rho^2 \sin\theta d\theta d\varphi^*. \tag{4.7}$$

Chapter 5

Existing Solutions

5.1 Solutions in Flat Spacetime

To illustrate our foliation-based approach to finding force-free fields, we present a pedagogical description of our method in flat spacetime.

We have the following canonical set of tetrads

$$[e_0, e_1, e_2, e_3] = [-\partial_t, \partial_x, \partial_y, \partial_z]. \tag{5.1}$$

A new set of transformed tetrads is given by:

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & f & \sqrt{-2f^2+1} & f \\ 0 & -\frac{\sqrt{2}\sqrt{-2f^2+1}}{2} & \sqrt{2}f & -\frac{\sqrt{2}\sqrt{-2f^2+1}}{2} \\ 0 & -\frac{\sqrt{2}}{2} & 0 & \frac{\sqrt{2}}{2} \end{bmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} \tag{5.2}$$

Here, f is an arbitrary function of time and we can easily check that (e_2, e_3) form an involutive distribution. We calculate the following quantities

$$g(\nabla_{e_0}e_0, e_2) = g(\nabla_{e_1}e_1, e_2) = 0, \tag{5.3a}$$

$$g(\nabla_{e_0}e_0, e_3) = g(\nabla_{e_1}e_1, e_3) = 0\,, \tag{5.3b}$$

$$g(\nabla_{e_2}e_2, e_0) = g(\nabla_{e_3}e_3, e_0) = 0\,, \tag{5.3c}$$

$$g(\nabla_{e_2}e_2, e_1) = g(\nabla_{e_3}e_3, e_1) = 0\,. \tag{5.3d}$$

It turns out that the sum of the mean curvature field and its dual is identically zero.

$$2(H + \tilde{H}) = 0. \tag{5.4}$$

Now that all the conditions necessary for an electrically dominated solution are satisfied, we can write the solution as follows

$$\begin{aligned} F &= u\, e_0^\flat \wedge e_1^\flat \\ &= u_0 \left(f\, dt \wedge dx - 2\sqrt{1-2f^2} dt \wedge dy + f\, dt \right) \wedge dz \end{aligned} \tag{5.5}$$

Here $u = u_0$ has been obtained by directly integrating 3.4.

It can be easily shown that the following Euler potentials can be used to generate the solution

$$\begin{aligned} \phi_1 &= g_1(t), \\ \phi_2 &= -\frac{u_0}{g_1'(t)} \left(y\sqrt{1-2f^2} - g_2(t)\, g_1'(t) + (x+z)\, f \right). \end{aligned} \tag{5.6}$$

The current 4-vector is given by

$$J = u_0\, f' \left(\partial_z + \partial_z - \frac{2f}{\sqrt{1-2f^2}}\, \partial_y \right). \tag{5.7}$$

It is clear from $F^2 = -2u_0^2$ that this is indeed an electrically dominated solution.

To clarify the nature of the field, we can find the relevant fields. To that end, we have to make a choice of an observer i.e. unit time-like vector field. We choose the canonical observer with respect to the coordinates given by $V = -\partial_t$ or $v^\mu = (-1, 1, 1, 1)$.

The electric and magnetic fields measured by this observer are given as follows

$$E_\mu = v^\mu F_{\mu\nu} = [E_x, E_y, E_z] = \left[f, 2\sqrt{1-2f^2}, f\right], \text{and}$$
$$B_\mu = v^\mu * F_{\mu\nu} = [0,0,0].$$

Since there is no magnetic field experienced by our observer, it is obvious that the field is force-free. Similarly, since our 4-current density has no time component, there ARE no charges at rest.

Because Maxwell's equations are conformally invariant, the above solution remains a force-free solution in a conformally related spacetime. One such spacetime that is of interest is the FLRW spacetime with no spatial curvature.

In the Cartesian coordinate system, the line element is given by

$$ds^2 = a(t)^2 \left(-dt^2 + dx^2 + dy^2 + dz^2\right). \tag{5.8}$$

It is easy to check that our solution remains force-free after this conformal transformation.

5.2 Solutions in Kerr Spacetime

Since energy extraction from rotating black holes is our major goal, we start our study of force-free fields in the Kerr spacetime. We begin with a set of O'Neill's canonical inertial tetrad [80] given by

$$e_0 = \frac{1}{\sqrt{\rho^2\Delta}}\left[(r^2+a^2)\partial_t + a\partial_\varphi\right] , \tag{5.9a}$$

$$e_1 = \frac{1}{\sqrt{\rho^2}\sin\theta}\left[a\sin^2\theta\partial_t + \partial_\varphi\right] , \tag{5.9b}$$

$$e_2 = \sqrt{\frac{\Delta}{\rho^2}}\,\partial_r , \tag{5.9c}$$

$$e_3 = \frac{1}{\sqrt{\rho^2}}\partial_\theta . \tag{5.9d}$$

We can then easily calculate the following

$$g(\nabla_{e_0}e_0, e_3) = -\frac{a^2\cos\theta\sin\theta}{\rho^2\sqrt{\rho^2}} , \tag{5.10a}$$

$$g(\nabla_{e_0}e_0, e_2) = \frac{a^2\cos^2\theta(r-M)+r(Mr-a^2)}{\rho^2\sqrt{\rho^2\Delta}} , \tag{5.10b}$$

$$g(\nabla_{e_1}e_1, e_3) = -\frac{\cos\theta(r^2+a^2)}{\rho^2\sqrt{\rho^2}\sin\theta} , \tag{5.10c}$$

$$g(\nabla_{e_1}e_1, e_2) = -\frac{r}{\rho^2}\sqrt{\frac{\Delta}{\rho^2}} , \tag{5.10d}$$

$$g(\nabla_{e_2}e_2, e_1) = g(\nabla_{e_3}e_3, e_1) = 0 , \tag{5.10e}$$

$$g(\nabla_{e_2}e_2, e_0) = g(\nabla_{e_3}e_3, e_0) = 0 . \tag{5.10f}$$

It turns out that (e_0, e_1) as given above is not just an involutive pair but actually commutes. We can then consider a Lorentzian foliation of the manifold $\mathscr{M}$ generated by the involutive distribution given by the span of (e_0, e_1). The calculations using equations (5.10) reveals that, in this case

$$\begin{aligned} 2(H^\flat + \tilde{H}^\flat) &= \frac{\cot\theta}{\sqrt{\rho^2}}e_3^\flat - \frac{(r-M)}{\sqrt{\rho^2\Delta}}e_2^\flat \\ &= -\cot\theta\, d\theta - \frac{(r-M)}{\Delta}dr . \end{aligned} \tag{5.11}$$

It is easy to see that the 1-form is exact. The above equation is easily integrated using (3.4) to obtain a final expression for u given by

$$u = \frac{u_0}{\sin\theta\sqrt{\Delta}} ,$$

where u_0 is an arbitrary integration constant. Since all the requirements of theorem (1) are satisfied we get the following vacuum degenerate field [1]

$$F_1 = \frac{u_0}{\sin\theta\sqrt{\Delta}} e_2^\flat \wedge e_3^\flat = u_0 \frac{\rho^2}{\sin\theta\Delta} dr \wedge d\theta . \tag{5.12}$$

Naturally, as expected, here $\ker F_1 = \mathrm{span}\{e_0, e_1\}$. And similarly, the choice of the involutive pair (e_2, e_3) yields another degenerate field given by

$$F_2 = \frac{u_0}{\sin\theta\sqrt{\Delta}} e_0^\flat \wedge e_1^\flat = u_0\, dt \wedge d\varphi . \tag{5.13}$$

The solutions F_1 and F_2 are Hodge-star dual of each other. As shown in [72], vacuum, degenerate, and non-null solutions come in pairs. Our canonical tetrad yields another pair of vacuum solutions given by

$$F_3 = \frac{u_0}{\sin\theta\sqrt{\Delta}} e_0^\flat \wedge e_3^\flat = -\frac{u_0}{\sin\theta}\, d\theta \wedge (-dt + a\sin^2\theta d\varphi) , \tag{5.14}$$

when $\ker F_3 = \mathrm{span}\{e_1, e_2\}$ and

$$F_4 = \frac{u_0}{\sin\theta\sqrt{\Delta}} e_1^\flat \wedge e_2^\flat = \frac{u_0}{\Delta} dr \wedge (a\, dt - (r^2 + a^2)\, d\varphi) , \tag{5.15}$$

when $\ker F_4 = \mathrm{span}\{e_0, e_3\}$. All the above vacuum solutions have been previously presented in [72]. These solutions are not well defined on the event horizon, and F_1, F_2, and F_3 are not defined on the symmetry axis.

[1] We reiterate the fact that our theory does not a priori distinguish between proper force-free solutions ($j \neq 0$) and degenerate vacuum solutions.

5.2.1 Transformed Tetrads

Given a fixed choice of a tetrad (e_0, e_1, e_2, e_3), any other tetrad $(\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ is related to the original one by a simple, spacetime-dependent transformation given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \Lambda(x) \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} . \tag{5.16}$$

5.2.2 Known Solutions in Kerr Space-time

Consider a simple transformation of our canonical tetrad in the (e_1, e_3) plane given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{1}{\sqrt{1+L^2}} & 0 & \frac{L}{\sqrt{1+L^2}} \\ 0 & 0 & 1 & 0 \\ 0 & \frac{L}{\sqrt{1+L^2}} & 0 & \frac{-1}{\sqrt{1+L^2}} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} .$$

Here L is an arbitrary function of the Boyer-Lindquist coordinate r. The pair $(\bar{e}_0, \bar{e}_1)$ is involutive, and here again, the foliation by the integral submanifolds satisfies eq. (3.3). Remarkably, $2(H + \tilde{H})^\flat$ is again given by eq.(5.11), and so the magnetically dominated solution, in this case, is then given by

$$F_5 = \frac{u_0}{\sin\theta\sqrt{\Delta}}\, \bar{e}_2^\flat \wedge \bar{e}_3^\flat . \tag{5.17}$$

In Boyer-Lindquist coordinates, this becomes

$$F_5 = -\frac{u_0}{\Delta}\frac{1}{\sqrt{1+L^2}} dr \wedge \left[\frac{\rho^2}{\sin\theta} d\theta + L(a\,dt - (r^2+a^2)d\varphi)\right] . \tag{5.18}$$

This is precisely the magnetically dominated solution presented in [70] albeit in a $3+1$

formalism of electrodynamics using electric and magnetic fields, and later in [72] as an example of theorem(1). The current density vector j_5 in this case is given by

$$j_5 = \frac{-u_0 L'}{(1+L^2)\sin\theta\sqrt{\rho^2}}\,\bar{e}_1 = \frac{-u_0 L'}{(1+L^2)^{3/2}\rho^2\sin^2\theta}\Big(a\sin^2\theta\partial_t + \partial_\varphi + L\sin\theta\partial_\theta\Big). \tag{5.19}$$

Here $L' = dL/dr$.

5.3 Null Solution in Kerr Geometry

Here we demonstrate how the null foliations can be exploited to generate null and force-free electromagnetic fields and we do so by reformulating a null solution first discovered by Menon and Dermer in [75] and then generalized by Brennan et al. in [12].

We start with the well-known outgoing null geodesic in Kerr spacetime in Boyer-Lindquist coordinates given by

$$l = \frac{r^2+a^2}{\Delta}\,\partial_t + \partial_r + \frac{a}{\Delta}\,\partial_\varphi. \tag{5.20}$$

For ease of calculation, we transform to the white-hole coordinate system (*) where the vector field becomes

$$l = \partial_{r*}. \tag{5.21}$$

We pick the following two unit space-like vector fields

$$\begin{aligned} s &= \frac{1}{\sqrt{\rho^2}}\left(a\,\sin^2\theta\,\partial_{t*} + \partial_{\varphi*}\right) \\ \alpha^\sharp &= \frac{1}{\sqrt{\rho^2}}\,\partial_\theta. \end{aligned} \tag{5.22}$$

We can now calculate the null mean curvature

$$\theta = g(\nabla_s l, s) = g(\nabla_{\alpha^\sharp} l, \alpha^\sharp) = \frac{r}{\rho^2}. \tag{5.23}$$

It is easy to check that

$$[l,s] = -\frac{r}{\rho^2}\, s \in \mathrm{span}(l,s). \tag{5.24}$$

Now that all the conditions required for the existence of a non-null force-free field are satisfied, we define a new set of vector fields

$$X_1 = \partial_{r^*}, \quad X_2 = \sin\theta\sqrt{\rho^2}\, s,$$

$$X_3 = \partial_{\varphi^*} \text{ ,and } X_4 = \partial_\theta + 2t^* \cot\theta\, \partial_{t^*}.$$

We have the following coordinate 1-forms

$$\begin{aligned}
dx^1 &= dr, \\
dx^2 &= \frac{1}{a\sin\theta^2}\left(dt^* - 2t^* \cot\theta\, d\theta\right), \\
dx^3 &= -dx^2 + d\varphi^*, \text{ and} \\
dx^4 &= d\theta
\end{aligned} \tag{5.25}$$

Our null and force-free field is then given by

$$\begin{aligned}
F &= u \cdot \kappa\, dx^3 \wedge dx^4, \\
&= u(x^3,\theta)\left(-\frac{dt^*}{a\sin^2\theta} + d\varphi^*\right) \wedge d\theta
\end{aligned} \tag{5.26}$$

Using

$$\begin{pmatrix} \alpha \\ l \end{pmatrix} = \begin{pmatrix} 0 & \sqrt{\rho^2} \\ a\sin^2\theta & -2t^*\cot\theta \end{pmatrix} \begin{pmatrix} dx^3 \\ dx^4 \end{pmatrix},$$

we find that

$$\kappa = -\frac{1}{a\sin^2\theta\,\sqrt{\rho^2}}.$$

So we can also write the solution as

$$F = u(x^3,\theta)\,\frac{(\rho^2)^{\frac{3}{2}}}{a\sin^2\theta}\,\alpha\wedge l^\flat. \tag{5.27}$$

Now we can try to check if a generalized solution exists. We observe that

$$[l,\alpha] = -\frac{r}{\rho^2}\,\alpha \in \mathrm{span}(l,\alpha). \tag{5.28}$$

We define two new vector fields $\hat{\alpha}$ and $\hat{s}$ that $\hat{s}$ forms an involutive distribution with l as follows

$$\hat{s} = A\,s + B\,\alpha^\sharp, \text{ and,}$$
$$\hat{\alpha} = -A\,s + B\,\alpha^\sharp. \tag{5.29}$$

$\hat{s}$ forms an involutive distribution with l when $A = A(t^*,\theta,\varphi^*)$ and it satisfies

$$a\sin^2(\theta)\,\partial_{t^\star}A + \partial_{\varphi^\star}A - \partial_\theta B = 0\,. \tag{5.30}$$

We can check that l admits a uniform equipartition of the null mean curvature

$$g(\nabla_{\alpha^\sharp}\,\partial_{r^*},s) = g(\nabla_s\,\partial_{r^*},\alpha^\sharp). \tag{5.31}$$

We now have a generalized solution given by

$$\begin{aligned} F &= (u\cdot\,\kappa)(A\,\alpha - B\,s^{\flat})\,\wedge\,l^{\flat} \\ &= (A\,d\theta - B\,\sin\theta\,d\varphi^{*})\wedge l^{\flat}. \end{aligned} \tag{5.32}$$

This solution also has an interesting feature that the current 4-vector is pointed towards the null geodesic ∂_{r^*}.

To describe the solution given above near the future horizon of the black hole [2], consider the infalling (black hole) Kerr-Schild spacetime denoted by $(\bar{t},\bar{r},\bar{\theta},\bar{\varphi})$. They are related to the Boyer-Lindquist coordinates by the following relations:

$$\bar{r} = r\,, \quad \bar{\theta} = \theta\,, \quad d\bar{t} = dt + \frac{r^2+a^2}{\Delta}dr\,, \quad \text{and} \quad d\bar{\varphi} = d\varphi + \frac{a}{\Delta}dr.$$

Kerr metric in the black hole coordinates is given by

$$\bar{g}_{\mu\nu} = \begin{bmatrix} z-1 & 1 & 0 & -za\sin^2(\theta) \\ 1 & 0 & 0 & -a\sin^2(\theta) \\ 0 & 0 & \rho^2 & 0 \\ -za\sin^2(\theta) & -a\sin^2(\theta) & 0 & \Sigma^2\sin^2(\theta)/\rho^2 \end{bmatrix}. \tag{5.33}$$

In the black hole coordinates $F^{\text{ out,Kerr}}$ takes the form

$$F^{\text{ out,Kerr}} = \left(A\,d\theta - \frac{2aB}{\Delta}dr + B\,d\bar{\varphi}\right)\wedge\left(-d\bar{t} + \frac{2\rho^2}{\Delta}dr + a\sin^2\theta d\bar{\varphi}\right). \tag{5.34}$$

Clearly, $F^{\text{ out,Kerr}}$ is well defined at the future horizon if and only if

$$\lim_{\bar{r}\to r_+}\frac{A}{\Delta} = \alpha_H(\bar{t},\theta,\bar{\varphi}) \quad \text{and} \quad \lim_{\bar{r}\to r_+}\frac{B}{\Delta} = \beta_H(\bar{t},\theta,\bar{\varphi}) \tag{5.35}$$

for some well-defined functions α_H and β_H at the future horizon $\mathscr{H}_+$. At first glance, one

[2] The future horizon is described by the surface $r = M + \sqrt{M^2 - a^2}$ for finite values of $\bar{t}$ and $\bar{\varphi}$.

might think this is impossible since A and B do not depend on r (more correctly $r^\star$). However,

$$t^\star = \bar{t} - 2\left(r + \frac{r_+^2 + a^2}{r_+ - r_-}\ \ln|r - r_+| - \frac{r_-^2 + a^2}{r_+ - r_-}\ \ln|r - r_-|\right)$$

and

$$\varphi^\star = \bar{\varphi} - \frac{2a}{r_+ - r_-}\ \ln\left|\frac{r - r_+}{r - r_-}\right| .$$

Therefore, as we approach the future horizon $\bar{r} \to r_+$ for finite $\bar{t}$ and $\bar{\varphi}$ we see that $t^\star$ and $\varphi^\star$ approach $+\infty$. Consequently, the limit in eq.(5.35) is more accurately written as

$$\lim_{t^\star,\varphi^\star \to \infty} \lim_{\bar{r} \to r_+} \frac{A(t^\star, \theta, \varphi^\star)}{\Delta} = \alpha_H(\bar{t}, \theta, \bar{\varphi}) \quad \text{and} \quad \lim_{t^\star,\varphi^\star \to \infty} \lim_{\bar{r} \to r_+} \frac{B(t^\star, \theta, \varphi^\star)}{\Delta} = \beta_H(\bar{t}, \theta, \bar{\varphi}) .$$

The fact that we will need below can now be written in the form

$$\lim_{t^\star,\varphi^\star \to \infty} \lim_{\bar{r} \to r_+} A(t^\star, \theta, \varphi^\star) = 0 = \lim_{t^\star,\varphi^\star \to \infty} \lim_{\bar{r} \to r_+} B(t^\star, \theta, \varphi^\star) . \tag{5.36}$$

The rate of extraction of energy from the past horizon $\mathscr{H}_-$ is given by

$$\left.\frac{d\mathscr{E}}{dt^\star}\right|_{\mathscr{H}_-} = -\int_{r^\star = r_+} T^{r^\star}{}_{t^\star}\, \rho^2 \sin\theta\ d\theta d\varphi^\star , \tag{5.37}$$

where

$$\mathscr{E} = -\int_{r^\star = r_+}^{r^\star = \infty} T^{t^\star}{}_{t^\star}\, \rho^2 \sin\theta\ dr d\theta d\varphi^\star .$$

For our null solution, we have that

$$\left.\frac{d\mathscr{E}}{dt^\star}\right|_{\mathscr{H}_-} = \int_{r^\star = r_+} \left(A^2 + \frac{B^2}{\sin^2\theta}\right) \sin\theta\ d\theta d\varphi^\star . \tag{5.38}$$

In the maximal Kerr geometry, the exterior of the black hole has two distinct outer horizons. The future horizon $\mathscr{H}_+$ in the white hole coordinates is located at $r^\star \to r_+$ as $t^\star, \varphi^\star \to \infty$. In exactly the same way as in the white hole case, the energy extracted from the black hole is given by

$$\frac{d\mathscr{E}}{d\bar{t}}\Big|_{\mathscr{H}_+} = \lim_{t^\star,\varphi^\star\to\infty}\int_{\bar{r}=r_+}\left(A^2+\frac{B^2}{\sin^2\theta}\right)\sin\theta\; d\theta d\bar{\varphi}\to 0\,. \tag{5.39}$$

The last equality is obtained from eq.(5.36). Incidentally, constraints in eqs.(5.30) and (5.36) as easily realized [3]. As attractive as this exact solution is, clearly, it does not allow the extraction of energy from the future horizon of black holes. This is certainly not what we mean by energy extraction from astrophysical black holes. This exact solution while illuminating cannot describe the effect we observe.

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[3]For example, set

$$A = A_0\cos\theta\; e^{-\omega(t^\star)^2}\;, \quad \text{and} \quad B = -\frac{2}{3}A_0\;\omega a\sin^3\theta t^\star\; e^{-\omega(t^\star)^2}\;.$$

Chapter 6

New Solutions

6.1 Non-Null Solutions in Kerr Spacetime

In the previous example, we considered a rotation in the (e_1, e_3) plane. Now consider a Lorentz boost in the (e_0, e_2) plane along with a label change $e_1 \leftrightarrow e_3$ given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \frac{1}{C} \begin{pmatrix} \sqrt{C^2+g^2} & 0 & -g & 0 \\ 0 & 0 & 0 & C \\ -g & 0 & \sqrt{C^2+g^2} & 0 \\ 0 & C & 0 & 0 \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} .$$

Here g is an arbitrary function of θ and $C > 0$ is a constant. It is easy to check that $(\bar{e}_2, \bar{e}_3)$ forms an involutive pair, and here again, the foliations by the integral submanifolds satisfy eq. (3.3). Surprisingly, once again $2(H + \tilde{H})^\flat$ is given by eq.(5.11), and so the electrically dominated solution, in this case, is then given by

$$F_6 = \frac{u_0}{\sin\theta\sqrt{\Delta}}\, \bar{e}_0^\flat \wedge \bar{e}_1^\flat .$$

In Boyer-Lindquist coordinates, this becomes

$$F_6 = \frac{u_0}{C\sin\theta}\, d\theta \wedge \left[\sqrt{C^2+g^2}\,(dt - a\sin^2\theta\, d\varphi) + g\,\frac{\rho^2}{\Delta} dr\right] . \tag{6.1}$$

The current density vector j_6 in this case is given by

$$\begin{aligned} j_6 &= -\frac{u_0 g'}{C\sqrt{\rho^2\Delta(C^2+g^2)}\sin\theta}\,\bar{e}_2 \\ &= \frac{u_0 g'}{C\rho^2\sin\theta}\left[\frac{g}{\Delta\sqrt{C^2+g^2}}\Big((r^2+a^2)\partial_t + a\partial_\varphi\Big) - \partial_r\right] . \end{aligned} \tag{6.2}$$

Here $g' = dg/d\theta$. In the exterior region, F_6 when contracted with itself gives $F_6^2 = -2\,u_0^2/\Delta\,\sin\theta^2$, and further

$$J_6^2 = \frac{u_0^2\, g'^2}{(g^2+C^2)\Delta\rho^2\sin\theta^2} ,$$

showing that the current density vector is spacelike in the exterior geometry.

It is interesting to note that when $C \to 0$, the expression for CF_6 reduces to the null solution presented in [74]. In this sense, the above force-free field can be considered as an electrically dominated perturbation of the null solution presented in [74].

A minor modification of the above solution leads to a magnetically dominated solution given by

$$F_7 = \frac{u_0}{C\sin\theta}\, d\theta \wedge \left[\sqrt{g^2-C^2}\,(dt - a\sin^2\theta\, d\varphi) + g\,\frac{\rho^2}{\Delta} dr\right] . \tag{6.3}$$

Although F_7 is similar in appearance to F_6, as a magnetically dominated solution, it is generated by a completely different foliation. Here

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \frac{1}{C}\begin{pmatrix} -g & 0 & \sqrt{g^2-C^2} & 0 \\ 0 & C & 0 & 0 \\ \sqrt{g^2-C^2} & 0 & -g & 0 \\ 0 & 0 & 0 & C \end{pmatrix}\begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} ,$$

and $(\bar{e}_0, \bar{e}_1)$ is defines the involutive distribution. Both F_6 and F_7 are undefined on the event horizon and along the symmetry axis, and further $F_7^2 = -F_6^2$. The above two solutions can be combined into one and is given by the form

$$\frac{\sqrt{g^2 - C}}{\sin\theta} d\theta \wedge \left(dt - a\sin\theta^2\, d\varphi\right) + \frac{g\rho^2}{\Delta\,\sin\theta} dr \wedge d\theta\ .$$

In this case, C can take on any real value such that $g^2 - C \geq 0$. Unfortunately, this solution is undefined at the event horizon and the symmetry axis given by $\theta = 0$.

II

For constants A, B, α and β, let

$$g(\varphi) = \alpha\cos(A\varphi) - \beta\sin(A\varphi)\ ,$$

and let

$$\begin{pmatrix} f_1 \\ \\ f_2 \end{pmatrix} = \begin{pmatrix} -g & -g'/A \\ \\ g'/A & -g \end{pmatrix} \begin{pmatrix} \cos(Bt/a) \\ \\ \sin(Bt/a) \end{pmatrix}\ .$$

In the above equation, $'$ denotes the derivative with respect to φ. Now consider the pure rotation along with $-e_1 \to \bar{e}_3$ given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & \frac{f_2}{\sqrt{\alpha^2+\beta^2}} & 0 & \frac{f_1}{\sqrt{\alpha^2+\beta^2}} \\ 0 & -\frac{f_1}{\sqrt{\alpha^2+\beta^2}} & 0 & \frac{f_2}{\sqrt{\alpha^2+\beta^2}} \\ 0 & 0 & -1 & 0 \end{bmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}\ .$$

Then $(\bar{e}_2, \bar{e}_3)$ defines an involutive distribution. The mean (/dual) curvature fields associated with the foliation gives

$$2(H^\flat + \tilde{H}^\flat) = \frac{M - r}{\Delta} dr - \frac{A + \cos\theta + B\sin^2\theta}{\sin\theta} d\theta\ .$$

The above expression is an exact 1-form, and from eq.(3.4) we get that

$$u = \frac{u_0 \exp(B\cos\theta)}{\sin\theta\sqrt{\Delta}(\csc\theta - \cot\theta)^A}.$$

The electrically dominated force-free field in this case is given by

$$\begin{aligned} F_8 &= \frac{u_0 \exp(B\cos\theta)}{\sin\theta\sqrt{\Delta}(\csc\theta - \cot\theta)^A}\, \bar{e}_0^\flat \wedge \bar{e}_1^\flat \\ &= \frac{u_0 \exp(B\cos\theta)}{\sqrt{\alpha^2+\beta^2}\, \sin\theta\,(\csc\theta - \cot\theta)^A} \left[f_1 d\theta \wedge (dt - a\sin^2\theta d\varphi) - f_2 \sin\theta dt \wedge d\varphi \right]. \end{aligned} \tag{6.4}$$

The current density is given by

$$j_8 = -\frac{u}{\rho^2\sqrt{\Delta}} \left(\frac{(A+B)a^2 + Br^2}{\sqrt{\alpha^2+\beta^2}} \right) \left[f_1 \sin\theta \left(\partial_t + \frac{1}{a\sin^2\theta}\partial_\varphi \right) - \frac{f_2}{a}\partial_\theta \right]. \tag{6.5}$$

For completeness, here, $F_8^2 = -2u^2$ and,

$$j_8^2 = \frac{4 f_1 f_2 (c_1 a^2 \cos^2\theta + c_3)}{a^2 \Delta \rho^2 \sin^4\theta}.$$

Once again, this solution is undefined at the event horizon and the symmetry axis given by $\theta = 0$.

III

Now consider the following new tetrad $(\bar{e}_0, \bar{e}_1, \bar{e}_2, \bar{e}_3)$ generated by

$$\Lambda = \begin{bmatrix} \frac{(f_2+f_1)^2}{2\sqrt{f_2 f_1}(f_1-f_2)} & -\frac{2\sqrt{f_2 f_1}}{f_1-f_2} & \frac{f_2+f_1}{2\sqrt{f_2 f_1}} & 0 \\ \frac{f_2+f_1}{f_1-f_2} & \frac{f_2+f_1}{f_2-f_1} & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{f_1-f_2}{2\sqrt{f_2 f_1}} & 0 & \frac{f_2+f_1}{2\sqrt{f_2 f_1}} & 0 \end{bmatrix}$$

in eq.(5.16). Here,

$$f_1 = \alpha \exp\left(\frac{ac_1t^* - a^2c_1\varphi^* - c_2\varphi^*}{a}\right)$$

and

$$f_2 = \beta \exp\left(\frac{c_1a^2\bar{\varphi} + c_2\bar{\varphi} - c_1a\bar{t}}{a}\right) .$$

Here star $(*)$ and bar $(-)$ labels represent the white hole and black hole coordinates respectively and c_1, c_2, c_3 and c_4 are real constants. It is easily verified that $(\bar{e}_2, \bar{e}_3)$ defines an involutive distribution, and further

$$2(H^\flat + \tilde{H}^\flat) = \frac{(c_1r^2 + M - r - c_3)}{\Delta} dr - \cot\theta d\theta .$$

Therefore, eq.(3.4) gives that

$$u = u_0 \frac{2\sqrt{f_2 f_1}}{\sin\theta\sqrt{\Delta}} .$$

So, the electrically dominated solution is given by

$$\begin{aligned} F_9 &= u\,\bar{e}_0^\flat \wedge \bar{e}_1^\flat \\ &= f_1\left(-\frac{a}{\Delta}dt\wedge dr + dt\wedge d\varphi - \frac{r^2+a^2}{\Delta}dr\wedge d\varphi\right) + f_2\left(\frac{a}{\Delta}dt\wedge dr + dt\wedge d\varphi + \frac{r^2+a^2}{\Delta}dr\wedge d\varphi\right) . \end{aligned} \tag{6.6}$$

The above solution can be written in a compact form in the mixed black hole and white hole coordinates as

$$F_9 = f_1\, dt^* \wedge d\varphi^* + f_2\, d\bar{t} \wedge d\bar{\varphi} .$$

For completeness, the current density is given by

$$j_9 = \frac{(c_3 + a^2c_1\cos^2\theta)}{\rho^2\sin^2\theta}\left[\frac{(f_1 - f_2)}{a}\frac{r^2+a^2}{\Delta}\partial_t + \frac{(f_1+f_2)}{a}\partial_r + \frac{(f_1+f_2)}{\Delta}\partial_\varphi\right] , \tag{6.7}$$

and

$$F_9^2 = -\frac{8f_1f_2}{\Delta\sin^2\theta} .$$

Depending on the sign of c_1 and c_2, f_1, f_2 can become unbounded in the remote future and past. Additionally, f_1, f_2 are not well-defined as φ sweeps a complete circle. While F_9 is undefined along the symmetry axis of the Kerr black hole, modulo the discontinuity along φ, it is well defined at the future and past event horizon under a mild restriction. To see this, we write the solution in black hole coordinates. Then

$$F_9 = (f_1 + f_2)\, d\bar{t} \wedge d\bar{\varphi} - \frac{2 f_1}{\Delta} \left(a\, d\bar{t} \wedge dr + (r^2 + a^2)\, dr \wedge d\bar{\varphi} \right) \, . \tag{6.8}$$

I.e., if f_1/Δ is well defined at the future horizon, then so is F_9. Since

$$\begin{aligned} f_1 &= \alpha \exp\left(c_1 t^* - a c_1 \varphi^* - \frac{c_2}{a} \varphi^* \right) \\ &\propto \alpha \exp\left(c_1\, \bar{t} - \bar{\varphi}(a\, c_1 + c_2) \right) (r - r_+)^{\dfrac{2(a\, c_2 - r_+\, c_1)}{r_+ - r_-}} \, . \end{aligned}$$

I.e., F_9 is well defined at the future horizon when

$$\frac{2(a\, c_2 - r_+^2\, c_1)}{r_+ - r_-} \geq 1. \tag{6.9}$$

A similar argument confirms regularity at the past horizon. Since

$$\frac{f_1 f_2}{\Delta} = \left(\frac{f_1 f_2}{\Delta^2} \right) \Delta \, ,$$

when the regularity condition is met we see that F_9 becomes null at the horizons. This is in contrast to F_1 as explained in [71].

We can attempt to calculate the rate of energy extraction and to that end, we calculate the following component of the electromagnetic stress-energy tensor

$$T^{r*}{}_{t*} = \frac{f_2^2 - f_1^2}{\rho^2 \sin^2\theta}. \tag{6.10}$$

The rate of extraction of energy from the past horizon is then given by 4.6

$$\frac{d\varepsilon}{dt^*}|_{\mathscr{H}_-} = \int_{r*=r_+} T^{r*}{}_{t*}\, \rho^2 \sin\theta d\theta d\varphi^* \tag{6.11}$$

Because f_1 and f_2 are independent of θ, unfortunately, the integral diverges at the poles where the denominator goes to zero.

IV

For any constant k, consider the Lorentz transformation for the remainder of this subsection.

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{bmatrix} \cosh\Theta & 0 & \sinh\Theta & 0 \\ 0 & \cos k & 0 & \sin k \\ 0 & \sin k & 0 & -\cos k \\ \sinh\Theta & 0 & \cosh\Theta & 0 \end{bmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix} .$$

IV A

When $\sin k \neq 0$, let $\Theta = \beta + \alpha \left(\frac{a\cos\theta\cos k}{\sin k} + t \right)$, $(\bar{e}_2, \bar{e}_3)$ form an involutive pair. In this case,

$$2(H^\flat + \tilde{H}^\flat) = \frac{M - r - \alpha(r^2 + a^2)}{\Delta} dr - \cot\theta d\theta \ ,$$

in which case u can be integrated to give

$$u = \frac{u_0}{\sin\theta} \exp\left(\int \frac{M - r - \alpha(r^2 + a^2)}{\Delta} dr \right).$$

The electrically dominated solution in this case is given by

$$\begin{aligned} F_{10A} &= u\, e_0^\flat \wedge e_1^\flat \\ &= \frac{u}{\sqrt{\Delta}} \Big[a\sin\theta\sin k\sinh\Theta\, dt \wedge dr + \Delta\cos k\cosh\Theta dt \wedge d\theta - \Delta\sin k\sin\theta\cosh\Theta\, dt \wedge d\varphi \\ &\quad - \rho^2\cos k\sinh\Theta\, dr \wedge d\theta + (r^2 + a^2)\sin k\sin\theta\sinh\Theta\, dr \wedge d\varphi + a\Delta\cos k\sin^2\theta\cosh\Theta d\theta \wedge d\varphi \Big]. \end{aligned} \tag{6.12}$$

The current density vector j_{10A} is given by

$$j_{10A} = \frac{au\alpha\sin\theta}{\sin k\sqrt{\Delta}\rho^2} \left[(r^2 + a^2)\sinh\Theta\, \partial_t + \Delta\cosh\Theta\partial_r + a\, \sinh\Theta\partial_\varphi \right]. \tag{6.13}$$

Also, $F_{10A}^2 = -2u^2$, and

$$j_{10A}^2 = \frac{\alpha^2 a^2 \sin^2\theta}{u^2\rho^2\sin^2 k}.$$

IV B

Similarly, when $\cos k \neq 0$, let $\Theta = \beta + \alpha\left(t - \frac{a\cos\theta\sin k}{\cos k}\right)$, $(\bar{e}_0, \bar{e}_1)$ form an involutive pair. The exact same expression for u as above leads to the magnetically dominated solution given by

$$\begin{aligned}
F_{10B} &= u\, e_2^\flat \wedge e_3^\flat \\
&\frac{u}{\sqrt{\Delta}}\Big[-a\cos k\sin\theta\cosh\Theta dt\wedge dr + \Delta\sin k\sinh\Theta dt\wedge d\theta + \Delta\cos k\sin\theta\sinh\Theta dt\wedge d\varphi \\
&-\rho^2\sin k\cosh\Theta dr\wedge d\theta - (r^2+a^2)\cos k\sin\theta\cosh\Theta dr\wedge d\varphi + a\Delta\sin k\sin^2\theta\sinh\Theta d\theta\wedge d\varphi\Big].
\end{aligned} \tag{6.14}$$

The current density vector j_{10B} is given by

$$j_{10B} = -\frac{a\alpha u\sin\theta}{\rho^2\sqrt{\Delta}\cos\theta}\left[(r^2+a^2)\cosh\Theta\partial_t + \Delta\sinh\Theta\partial_r + a\cosh\Theta\partial_\varphi\right]. \tag{6.15}$$

Here $F_{10B}^2 = -F_{10A}^2$, and $J_{10B}^2 = -J_{10A}^2$.

IV C

When

$$\Theta = \beta + \alpha\left(\varphi - ln(\csc\theta - \cot\theta)\cot k\right),$$

we find that $\bar{e}_2$ and $\bar{e}_3$ are involutive. Here,

$$2(H^\flat + \tilde{H}^\flat) = \frac{M - r - a\alpha}{\Delta}dr - \cot\theta\, d\theta$$

and

$$u = \frac{u_0}{\sin\theta}\exp\left(\int \frac{M-r-a\alpha}{\Delta}dr\right).$$

The electrically dominated solution is then given by

$$\begin{aligned}F_{10C} &= u\, e_0^\flat \wedge e_1^\flat \\ &= \frac{u}{\sqrt{\Delta}}\Big[a\sin\theta\sin k\sinh\Theta\, dt\wedge dr + \Delta\cos k\cosh\Theta dt\wedge d\theta - \Delta\sin k\sin\theta\cosh\Theta\, dt\wedge d\varphi \\ &\quad - \rho^2\cos k\sinh\Theta\, dr\wedge d\theta + (r^2+a^2)\sin k\sin\theta\sinh\Theta\, dr\wedge d\varphi + a\Delta\cos k\sin^2\theta\cosh\Theta d\theta\wedge d\varphi\Big].\end{aligned} \tag{6.16}$$

The current density vector is given by

$$j_{10C} = \frac{\alpha\, u}{\rho^2\sqrt{\Delta}\sin k\sin\theta}\Big[(r^2+a^2)\sinh\Theta\partial_t + \Delta\cosh\Theta\partial_r + a\sinh\Theta\partial_\varphi\Big]. \tag{6.17}$$

Here $F_{10C}^2 = -2\,u^2$, and

$$j_{10C}^2 = \frac{\alpha^2}{u^2\rho^2\sin^2 k\sin^2\theta}.$$

IV D

Finally when

$$\Theta = \beta + \alpha\left(\varphi + ln(\csc\theta - \cot\theta)\tan k\right)\ ,$$

we find that $\bar{e}_0$ and $\bar{e}_1$ are involutive. Once again, the exact same expression for u as above leads to the magnetically dominated solution given by

$$\begin{aligned}F_{10D} &= u\, e_2^\flat \wedge e_3^\flat \\ &\frac{u}{\sqrt{\Delta}}\Big[-a\cos k\sin\theta\cosh\Theta dt\wedge dr + \Delta\sin k\sinh\Theta dt\wedge d\theta + \Delta\cos k\sin\theta\sinh\Theta dt\wedge d\varphi \\ &\quad - \rho^2\sin k\cosh\Theta dr\wedge d\theta - (r^2+a^2)\cos k\sin\theta\cosh\Theta dr\wedge d\varphi + a\Delta\sin k\sin^2\theta\sinh\Theta d\theta\wedge d\varphi\Big].\end{aligned} \tag{6.18}$$

Here, the current density is given by

$$j_{10D} = \frac{u\alpha}{\rho^2\sqrt{\Delta}\cos k\sin\theta}\Big[(r^2+a^2)\cosh\Theta\,\partial_t + \Delta\sinh\Theta\,\partial_r + a\cosh\Theta\,\partial_\varphi\Big]. \tag{6.19}$$

$$J^2_{10D} = -J^2_{10C}$$

Chapter 7

Force Free Electrodynamics in Cosmology

7.1 The FLRW Spacetime

The Friedmann–Lemaître–Robertson–Walker metric describes an expanding homogenous and isotropic universe. In comoving coordinates is given by the following line element.

$$ds^2 = \left(-dt^2 + a^2(t)\frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2\right), \tag{7.1}$$

where $a(t)$ is the scale factor and $k = -1, 0, 1$ describes a space with negative, zero, or positive curvature respectively.

Because this metric describes a spacetime that is not charged[1], the formalism we have used for analyzing FFE in the Kerr spacetime can be directly applied to here as well. Furthermore, if we assume a flat universe (k=0) as observations strongly suggest[6], the metric is simplified as are the equations of FFE.

It turns out that the choice of conformal time greatly simplifies the calculations and hence, we will be using conformal time η throughout our calculations.

The transformation into conformal time is given by

[1] We note that, unlike the Kerr spacetime, this spacetime is not a vacuum spacetime i.e the stress-energy tensor is not identically zero.

$$\eta = \int \frac{1}{a(t)} dt. \tag{7.2}$$

The metric now takes the following form

$$ds^2 = \eta(t)^2 \left(-dt^2 + \frac{dr^2}{1-kr^2} + r^2 d\theta^2 + r^2 \sin^2\theta \, d\phi^2 \right). \tag{7.3}$$

7.2 Solutions in FLRW Spacetime

Since we will rely primarily on the tetrad formalism described above, we begin by listing a set of canonical orthonormal tetrad for the metric given above:

$$e_0 = \frac{1}{a(\eta)} \partial_\eta, \tag{7.4a}$$

$$e_1 = \frac{\sqrt{1-Kr^2}}{a(\eta)} \partial_r, \tag{7.4b}$$

$$e_2 = \frac{1}{a(\eta)\, r} \partial_\theta \,, \tag{7.4c}$$

$$e_3 = \frac{}{a(\eta)\, r \sin\theta} \partial_\varphi \,. \tag{7.4d}$$

Solution I

Consider a Lorentz transformation of the canonical tetrads given by

$$\begin{pmatrix} \bar{e}_0 \\ \bar{e}_1 \\ \bar{e}_2 \\ \bar{e}_3 \end{pmatrix} = \begin{pmatrix} \alpha \sin(\theta) f_2 & 0 & r\, f_1 f_2 & r\, f f_2 \\ 0 & 1 & 0 & 0 \\ \sqrt{\beta}\, r f_2 & 0 & \frac{f_1 \alpha \sin(\theta)\, f_2}{\sqrt{\beta}} & \frac{\alpha \sin(\theta)\, f\, f_2}{\sqrt{\beta}} \\ 0 & 0 & \frac{f}{\sqrt{\beta}} & -\frac{f_1}{\sqrt{\beta}} \end{pmatrix} \begin{pmatrix} e_0 \\ e_1 \\ e_2 \\ e_3 \end{pmatrix}, \tag{7.5}$$

where f is any function of r, α and β are real constants, and

$$f_1 = \sqrt{\beta - f^2} \quad \text{and} \quad f_2 = \frac{1}{\sqrt{\alpha^2 \sin^2\theta - \beta\, r^2}}.$$

Then the pair of vector fields $(\bar{e}_2, \bar{e}_3)$ are involutive and further

$$2(H+\tilde{H}) = \frac{2\dot{a}(\eta)}{a(\eta)}\, d\eta + \frac{2\,\alpha^2 \sin^2\theta - \beta\, r^2}{r(\beta\, r^2 - \alpha^2 \sin^2\theta)}\, dr + \frac{\beta^2\, r^2 \cot\theta}{\alpha^2 \sin^2\theta - \beta\, r^2} d\theta\ . \tag{7.6}$$

It is easy to verify that $d(H+\tilde{H}) = 0$. Then from eqs.(3.3), (3.8) and (3.4),

$$u = \frac{u_0 \sqrt{\alpha^2 \sin^2\theta - \beta\, r^2}}{a(\eta)^2\, r^2 \sin\theta}, \tag{7.7}$$

and our electrically dominated solution is given by

$$\begin{aligned} F_1 &= u\, e_0^\flat \wedge e_1^\flat \\ &= \frac{u_0\, \alpha}{r^2\sqrt{1-Kr^2}} d\eta \wedge dr + \frac{u_0 \sqrt{\beta - f^2}}{\sin\theta \sqrt{1-Kr^2}} dr \wedge d\theta + \frac{u_0\, f}{\sqrt{1-Kr^2}} dr \wedge d\varphi\ . \end{aligned} \tag{7.8}$$

Here, the current density is given by

$$j_1 = \frac{\sqrt{1-K\,r^2} f'}{a(\eta)^4\, r^2 \sin\theta} \left[\frac{f}{\sqrt{\beta - f^2}} \partial_\theta - \csc\theta\, \partial_\varphi \right]. \tag{7.9}$$

The Lorentz invariant quantity F^2 in this case is given by

$$F_1^2 = -\frac{2\, u_0^2 \left(\alpha^2 \sin^2\theta - \beta r^2\right)}{a(\eta)^4\, r^4 \sin^2\theta}. \tag{7.10}$$

From the above equation, we see that the solution is not well defined when $\sin\theta = 0$. For completeness, we also record the magnitude of the square of the current density vector:

$$j_1^2 = \frac{u_0^2 \left(K\, r^2 - 1\right) f'^2 \beta}{a(\eta)^6\, r^2 \sin^2\theta (f^2 - \beta)}. \tag{7.11}$$

This solution holds some intriguing features that deserve closer inspection. First, we note that the Lorentz transformation that generates the solution is not valid when

$$\chi \equiv \alpha^2 \sin^2\theta - \beta r^2 \le 0\ .$$

Nonetheless, F_1 does not depend on χ. This means that the solution F_1 is defined for all values of χ

(except when $\sin\theta = 0$ for an entirely different reason). As it turns out, examining F_1^2 tells us that our electrically dominated solution smoothly transitions to a null solution when $\chi = 0$ and further a magnetically dominated solution when $\chi < 0$. The tetrad formalism is unsuitable to handle such a transition. So we shift to a different formalism using a foliation-adapted chart which was developed in [71]. About any point in spacetime, there exists a coordinate chart $(x^1, \ldots, x^4)$ that is adapted to field sheets meaning that the tangent space of the submanifolds defined by constant values of x^1 and x^2 contain the kernel of F. Then F takes the form

$$F = u(x^3, x^4)\, dx^3 \wedge dx^4 \ . \tag{7.12}$$

Here, let

$$M^r \equiv g^{r3}\, g^{34} - g^{33}\, g^{r4} \ , \quad \text{and} \quad N^r \equiv g^{r3}\, g^{44} - g^{34}\, g^{r4} \ .$$

Then as it was shown in [71], the equations of FFE in this coordinate system are given by

$$M^4 \, \frac{\partial}{\partial x^4} \ln|u| = -\nabla_r \, M^r \tag{7.13}$$

and

$$N^3 \, \frac{\partial}{\partial x^3} \ln|u| = -\nabla_r \, N^r \ . \tag{7.14}$$

To obtain a foliation-adapted chart for the case of this solution, define vector fields

$$X_1 = \sin\theta \; \partial_\theta - \frac{\sqrt{\beta - f^2}}{f} \, \partial_\varphi \ , \quad X_2 = \partial_\eta - \frac{\alpha}{r^2 f} \, \partial_\varphi \ ,$$

$$X_3 = \partial_\eta + \partial_r + \left[\left(-\beta \ln(\csc\theta + \cot\theta) \frac{f'}{f^2\sqrt{\beta - f^2}} \right) - \alpha\eta \left(\frac{2}{r^3 f} + \frac{f'}{r^2 f^2} \right) \right] \partial_\varphi \ ,$$

and finally

$$X_4 = \partial_\varphi \ .$$

It is easily verified that all of the vector fields $\{X_i\}$ defined above commute with each other. Therefore there exist coordinate functions $\{x^i\}$ such that $X_i = \partial_{x^i}$ for each i. The dual bases $\{dx^i\}$ are such that

$$dx^1 = \csc\theta d\theta\ , \quad dx^2 = d\eta - dr, \quad dx^3 = dr\ ,$$

and

$$dx^4 = -\frac{\alpha}{r^2 f} d\eta + \left[\frac{\alpha}{r^2 f} + \frac{2\alpha\eta}{r^3 f} + \frac{\alpha\eta f'}{r^2 f^2} + \frac{\beta \ln(\csc\theta + \cot\theta) f'}{f^2\sqrt{\beta - f^2}}\right] dr + \frac{\sqrt{\beta - f^2}}{\sin\theta f} d\theta + d\varphi\ .$$

Then

$$dx^i(\partial_{x^j}) = \delta^i_j$$

as required. This will help us compute the determinant of the metric in the adapted basis and also the quantities M^r and N^r as defined above. The relevant quantities in eqs. (7.13) and (7.14) are given by

$$\sqrt{-g} = \frac{a^4(\eta) r^2 \sin^2\theta}{\sqrt{1 - Kr^2}}\ ,$$

$$M^1 = \frac{-\sqrt{\beta - f^2}(1 - Kr^2)}{a^4(\eta) r^2 \sin^2\theta f}\ , \quad M^2 = \frac{-\alpha(1 - Kr^2)}{a^4(\eta) r^2 f}, \quad M^3 = 0\ ,$$

$$M^4 = \frac{(1 - Kr^2)\chi}{a^4(\eta) r^4 \sin^2\theta f^2} = -N^3\ ,$$

$$N^1 = \frac{-(1 - Kr^2)}{a^4(\eta) r^5 \sin^2\theta f^3}\left[\Big(f'\eta r + (r + 2\eta) f\Big)\alpha\sqrt{\beta - f^2} + \beta \ln(\csc\theta + \cot\theta) f' r^3\right]\ ,$$

and finally

$$N^2 = \frac{(Kr^2 - 1)}{a^4(\eta) r^5 f^3 \sin^2\theta}\left[\left(\alpha\eta + \beta r^2 \frac{\ln(\csc\theta + \cot\theta)}{\sqrt{\beta - f^2}}\right)\sin^2\theta r \alpha f' - f\left(\beta r^3 + 2\alpha^2\eta \sin^2\theta\right)\right]\ .$$

Noting that $u = u(x^3, x^4 = \varphi)$, eq.(7.13) becomes

$$M^4 \frac{\partial}{\partial\varphi} \ln|u| = 0\ .$$

I.e., $u_{,\varphi} = 0$ and eq.(7.14) gives

$$\chi \frac{(1-Kr^2)}{a^4(\eta) r^4 \sin^2\theta f^2} \frac{d}{dx^3} \ln|u| = \chi \left[\frac{f'(1-Kr^2)+Krf}{a^4(\eta) r^4 \sin^2\theta f^3} \right] . \tag{7.15}$$

Notice how the factor χ cancels out in the above equation, and hence we can smoothly transition from an electrically dominated solution to a magnetically dominated one. [2]

The above equation is satisfied by setting

$$u = \frac{u_0 f}{\sqrt{1-Kr^2}} .$$

Therefore, in the adapted chart

$$F_1 = \frac{u_0 f}{\sqrt{1-Kr^2}} \, dx^3 \wedge dx^4 .$$

Using the method of tetrads and searching for Lorentz transformations that satisfy eqs.(3.3), (3.8) and (3.4), we have been able to generate several non-trivial solutions in FLRW spacetimes. In the remainder of this section, we simply list the solutions without referring to the generating Lorentz transformation.

Solution II

Using a time-dependent Lorentz transformation we obtain the following electrically dominated solution

$$F_2 = \frac{\alpha}{r^2\sqrt{1-K\,r^2}} d\eta \wedge dr + \frac{\sqrt{\beta - f^2}}{\sin\theta} d\eta \wedge d\theta + f \, d\eta \wedge d\varphi. \tag{7.16}$$

Here $f = f(\eta)$, and, α and β are real constants. The current density is then given by

$$j_2 = \frac{f'}{a^4\, r^2 \sin\theta} \left(\frac{-f}{\sqrt{\beta - f^2}} \partial_\theta + \frac{\partial_\varphi}{\sin\theta} \right) . \tag{7.17}$$

The Lorentz invariant scalars in this case are

$$F_2^2 = -\frac{2\alpha^2 \sin^2\theta + 2\beta\, r^2}{a^4\, r^4 \sin^2\theta}, \tag{7.18}$$

[2]Henceforth such solutions will be referred to as type-changing solutions.

and

$$j_2^2 = \frac{f'^2\,\beta}{a^6\,\sin^2\theta(\beta - f^2)}. \tag{7.19}$$

As in the previous case, this solution is not well defined when $\sin\theta = 0$.

Solution III

For, $K = 1$, and $f = f(\theta)$, and a real constant α, we obtain another type changing solution of the form

$$\begin{aligned} F_3 =& \frac{(\sqrt{K}r + \sqrt{Kr^2-1})^\alpha}{\sin\theta}\Bigg[\Big(f\,\sin(\alpha\sqrt{K}\,\eta) - \sqrt{\beta - f^2}\cos(\alpha\sqrt{K}\,\eta)\Big)\,d\eta\wedge d\theta \\ & - \frac{1}{\sqrt{K\,r^2-1}}\Big(f\,\cos(\alpha\sqrt{K}\,\eta) + \sin(\alpha\sqrt{K}\,\eta)\sqrt{\beta - f^2}\Big)\,dr\wedge d\theta\Bigg] \\ & + C\,\sin\theta\;d\theta\wedge d\varphi. \end{aligned} \tag{7.20}$$

Here, the current density is given by

$$\begin{aligned} j_3 =& \frac{\left(\sqrt{K}\,r + \sqrt{K\,r^2-1}\right)^\alpha f}{a(\eta)^4\sin(\theta)\,r^2\sqrt{\beta - f^2}}\Bigg[-\Big(f\cos\Big(\alpha\sqrt{K}t\Big) - \sin\Big(\alpha\sqrt{K}t\Big)\,\sqrt{\beta - f^2}\Big)\,\partial_\eta \\ & + \Big(-f\sin\Big(\alpha\sqrt{K}t\Big) + \sqrt{\beta - f^2}\,\cos\Big(\alpha\sqrt{K}t\Big)\Big)\,\sqrt{K\,r^2-1}\,\partial_r\Bigg]. \end{aligned} \tag{7.21}$$

In this case

$$F_3^2 = -\frac{2\left(\sqrt{K}\,r + \sqrt{K\,r^2-1}\right)^{2\alpha}\beta\,r^2 - 2C^2\sin^2\theta}{a(\eta)^4\,r^4\sin^2\theta}\,. \tag{7.22}$$

Once again this solution is not well defined when $\sin\theta = 0$. Further, when $K = 0, -1$, the solution above does indeed satisfy the force-free Maxwell's equation. However, the coefficient terms become complex, making the solution physically irrelevant.

Solution IV

We now present the following electrically dominated force-free field

$$F_4 = f_1\,d\eta\wedge d\theta + f_2\,d\eta\wedge d\varphi + f_3\,d\theta\wedge d\varphi, \tag{7.23}$$

where,

$$f_1 = \frac{e^{-\frac{\eta}{\omega_2}}}{\sin\theta}\left(\left(2\beta\alpha\cos^2\theta + (2\alpha k_1 + 2\beta k_2)\cos\theta + k_3\right)e^{\frac{2\eta}{\omega_2}} - (\cos\theta\beta + k_1)^2 e^{\frac{4\eta}{\omega_2}} - (\cos\theta\alpha + k_2)^2\right)^{1/2}, \tag{7.24}$$

$$f_2 = (\cos\theta\alpha + k_2)\,e^{-\frac{\eta}{\omega_2}} + e^{\frac{\eta}{\omega_2}}(\cos\theta\beta + k_1), \tag{7.25}$$

and

$$f_3 = -\omega_2\sin\theta\left(e^{\frac{\eta}{\omega_2}}\beta - \alpha\,e^{-\frac{\eta}{\omega_2}}\right). \tag{7.26}$$

Here k_1, k_2, ω_1 and ω_2 are real constants The current density is given by

$$j_4 = \frac{1}{a(\eta)^4 r^2}\left[-(f_1\cot\theta + \partial_\theta f_1)\,\partial_\eta + (\partial_\eta f_1)\partial_\theta + \frac{\csc\theta^2}{r^2}(\partial_\eta f_2 - \partial_\theta f_3 + f_3\cot\theta)\,\partial_\varphi\right]. \tag{7.27}$$

Here

$$F_4^2 = -\frac{2}{a(\eta)^4 r^4\sin^2\theta}\left[\sin^2\theta\, r^2 f_1^2 + r^2 f_2^2 - f_3^2\right], \tag{7.28}$$

and

$$j_4^2 = \frac{1}{a(\eta)^6 r^6}\left[\csc\theta\,(r^2\partial_\eta f_2 - \partial_\theta f_3 + f_3\cot\theta)^2 + (\partial_\eta f_1)^2 - r^2(f_1\cot\theta + \partial_\theta f_1)^2\right]. \tag{7.29}$$

This solution is not valid during early and late times in addition to the usual pathology when $\sin\theta = 0$.

Solution V and VI

We continue with our presentation by illustrating a few null solutions in FLRW spacetimes. The tetrad formalism for generating null solutions is substantially different from that of non-null solutions. As described in the aforementioned section, we begin with a null pre-geodesic congruence defined by

$$l = \frac{\csc\theta}{r\,a(\eta)^3}\left(\partial_\eta - \sqrt{1 - K r^2}\,\partial_r\right). \tag{7.30}$$

Here, l is a null vector satisfying $\nabla_l l \propto l$. A simple calculation shows that

$$\nabla_l l = \frac{\dot{a}(\eta)\, r + a(\eta)\sqrt{1-k\,r^2}}{r^2 \sin\theta\, a(\eta)^4}\, l. \tag{7.31}$$

The relevant null tetrad $(s, l, \alpha^\sharp, n)$ can be constructed by defining the following vector fields:

$$n = \frac{a(\eta)\, r \sin\theta}{2}\left(\partial_\eta + \sqrt{1-K\,r^2}\,\partial_r\right), \tag{7.32}$$

$$\alpha^\sharp = \frac{1}{r\,a(\eta)}\,\partial_\theta \tag{7.33}$$

$$s = \frac{\csc\theta}{r\,a(\eta)}\,\partial_\varphi \tag{7.34}$$

Here, since

$$g(\nabla_s l, s) = g(\nabla_{\alpha^\sharp} l, \alpha^\sharp) = \frac{\dot{a}(\eta)\, r - a(\eta)\sqrt{a-Kr^2}}{a(\eta)^4\, r^2 \sin\theta}\ , \tag{7.35}$$

we see that the equipartition condition for the null mean curvature is satisfied. Additionally, since l, s forms an involutive distribution, we are guaranteed the existence of a null force-free field solution with a two-parameter prefactor. To isolate the prefactor, we construct a foliation-adapted chart with commuting vector fields defined as follows

$$X_1 = \partial_\eta + \sqrt{1-Kr^2},\quad X_2 = \partial_\varphi,\quad X_3 = \partial_\theta,\quad \text{and}\quad X_4 = l.$$

Our adapted chart is then defined by the following coordinate one-forms

$$dx^1 = d\eta - \frac{dr}{\sqrt{1-Kr^2}},\quad dx^2 = d\varphi,\quad dx^3 = d\theta \quad \text{and} \quad dx^4 = d\eta + \frac{dr}{\sqrt{1-Kr^2}}.$$

The null and force-free field is then given by

$$F_5 = (u\cdot\kappa)\,\alpha\wedge l^\flat = u(x^3, x^4)\left(d\eta\wedge d\theta + \frac{1}{\sqrt{1-Kr^2}}\,dr\wedge d\theta\right)\ . \tag{7.36}$$

Here, $\kappa = -\csc\theta$ and is given by 3.11, and the current density is then given by

$$j_5 = \frac{\partial_\theta f + f \cot\theta}{a(\eta)^4 r^2} \left(-\partial_\eta + \sqrt{1-Kr^2}\,\partial_r\right). \tag{7.37}$$

As required, $F_5^2 = 0$, and since the current density is along l we have that $j_5^2 = 0$. It turns out that this null geodesic congruence also satisfies the uniform equipartition condition described in [73]:

$$g(\nabla_s l, \alpha^\sharp) + g(\nabla_{\alpha^\sharp} l, s) = 0. \tag{7.38}$$

This allows for the possibility of a generalized null solution, which in this case is given by

$$F_6 = \partial_\theta f(\bar{t},\theta,\varphi) \left(d\eta \wedge d\theta + \frac{dr \wedge d\theta}{\sqrt{1-Kr^2}}\right) + \partial_\varphi f(\bar{t},\theta,\varphi) \left(d\eta \wedge d\phi + \frac{dr \wedge d\phi}{\sqrt{1-Kr^2}}\right). \tag{7.39}$$

The t in the above expression is the usual cosmic time given by the relation

$$\bar{t} = \eta + \int \frac{1}{\sqrt{1-K\,r^2}} dr\,. \tag{7.40}$$

The current density in this case is given by

$$j_6 = \frac{\csc\theta \partial_\phi^2 f + \partial_\theta^2 f + \cot\theta \partial_\theta f}{a(\eta)^4 r^2} \left(\partial_\eta - \sqrt{1-K\,r^2}\partial_r\right). \tag{7.41}$$

For clever choices of f, the solution above is valid when $r \neq 0$.

Solution VII

For $f = f(\bar{t},\theta)$, where t is the cosmic time function defined above, we have the following non-null force-free solution with null current

$$F_7 = f \sin\theta \left(d\eta \wedge d\theta + \frac{1}{\sqrt{1-K\,r^2}}\, dr \wedge d\theta\right) + \alpha \sin\theta\, d\theta \wedge d\phi\,. \tag{7.42}$$

Here, as usual, α is a real constant. This solution turns out to be a non-null generalization of our previous null solution F_5. The current density in this case is given by

$$j_7 = -\frac{2\cos\theta\, f + \sin\theta\, \partial_\theta f}{a(\eta)^4 r^2} \left(\partial_\eta + \sqrt{1-Kr^2}\,\partial_r\right). \tag{7.43}$$

We also have that

$$F_7^\sharp = -\frac{f\,\sin\theta}{a(\eta)^4\,r^2}\,\partial_\eta \wedge \partial_\theta + \frac{f\,\sin\theta\sqrt{1-Kr^2}}{a(\eta)^4\,r^2}\,\partial_r \wedge \partial_\theta + \frac{\alpha}{a(\eta)^4\,r^4\sin\theta}\,\partial_\theta \wedge \partial_\phi. \tag{7.44}$$

While it appears that the last term in the right-hand side of the above equation is undefined when $\sin\theta = 0$, the contraction of the Faraday tensor with itself does not suffer from the same pathology i.e.

$$F_7^2 = \frac{2\alpha^2}{a(\eta)^4\,r^4}. \tag{7.45}$$

This is an indication that the solution may just have a coordinate singularity along $\sin\theta = 0$. We demonstrate this fact by explicitly transforming it into a Cartesian coordinate system for the case when $K = 0$. In the usual (η, x, y, z) coordinate system, where we have just transformed the spatial spherical coordinates to Cartesian coordinates,

$$\begin{aligned} F_7 =& \frac{1}{r^3}\left(zxf\,d\eta \wedge dx + zyf\,d\eta \wedge dy - (x^2+y^2)\,f\,d\eta \wedge dz\right. \\ & \left.\alpha z\,dx \wedge dy - (\alpha y + rxf)\,dx \wedge dz + (x\alpha - ryf)\,dy \wedge dz\right). \end{aligned} \tag{7.46}$$

The above expression is well-defined along the z-axis.

Solution VIII

The FLRW metric is further simplified when we set $K = 0$, and in the cartesian coordinates described above, we are able to find two new non-null solutions. First, for constants c_1, c_2, c_3, c_4, we have the following force-free field

$$F_8 = \frac{\sqrt{(-c_1^2+c_2^2)f^2 + c_2 c_4^2}}{c_2}\,d\eta \wedge dx + \frac{c_1\,f}{c_2}\,d\eta \wedge dz + f\,dx \wedge dz, \tag{7.47}$$

where,

$$f = f(c_1\,\eta + c_2\,x + c_3). \tag{7.48}$$

The current density is given by

$$j_8 = \frac{(c_1^2 - c_2^2)(\partial_\eta f)}{a(\eta)^4 \sqrt{(c_2^2 - c_1^2) f^2 + c_4 c_2^2}} \left[\frac{f}{c_1} \partial_\eta - \frac{f}{c_2} \partial_x + \frac{\sqrt{(c_2^2 - c_1^2) f^2 + c_4 c_2^2}}{c_1 c_2} \partial_z \right]. \tag{7.49}$$

The Lorentz scalars of the theory are given by

$$F_8^2 = -\frac{2c_4}{a(\eta)^4}, \tag{7.50}$$

and

$$j_8^2 = -\frac{c_4 (c_1^2 - c_2^2)^2 (\partial_\eta f)^2}{c_1^2 a(\eta)^6 ((c_1^2 - c_2^2) f + c_4 c_2^2)}. \tag{7.51}$$

Solution IX

As in the Cartesian case of the above example, we now provide a secondary non-null solution given by

$$F_9 = \frac{\sqrt{c_4 c_2^2 - (c_2^2 + c_1^2) f^2}}{c_2} dx \wedge dy + \frac{c_1 f}{c_2} dx \wedge dz + f\, dy \wedge dz\,. \tag{7.52}$$

Here $f = f(c_1 x + c_2 y + c_3)$. The current density is given by

$$j_9 = \frac{(c_1^2 + c_2^2)(\partial_x f)}{a(\eta)^4 \sqrt{(c_1^2 + c_2^2) f^2 - c_4 c_2^2}} \left[-\frac{f}{c_1} \partial_x + \frac{f}{c_2} \partial_y - \frac{\sqrt{c_4 c_2^2 - (c_2^2 + c_1^2) f^2}}{c_1 c_2} \partial_z \right]. \tag{7.53}$$

And we have

$$F_9^2 = \frac{2 c_4}{a(\eta)^4}, \tag{7.54}$$

and,

$$j_9^2 = -\frac{(\partial_x f)^2 (c_1^2 + c_2^2)^2 c_4}{c_1^2 a(\eta)^6 ((c_1^2 + c_2^2) f^2 - c_4 c_2^2)}. \tag{7.55}$$

A simple spatial rotation in the x, y plane can simplify the expression for F_9. Consider the transformation given by

$$\begin{bmatrix} x' \\ y' \end{bmatrix} = \frac{1}{\sqrt{c_1^2 + c_2^2}} \begin{bmatrix} c_2 & -c_1 \\ c_1 & c_2 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix}. \tag{7.56}$$

After the transformation $f = f(c_3 + y'\sqrt{c_1^2 + c_2^2})$. The constants can be absorbed by redefining f as $f(c_3' + y')$. The solution F_9 then takes the form

$$F_9' = \frac{\sqrt{c_4 c_2^2 - (c_1^2 + c_2^2) f'^2}}{c_2} dx' \wedge dy' + \frac{f' \sqrt{c_1^2 + c_2^2}}{c_2} dy' \wedge dz. \tag{7.57}$$

By redefining the constants, we can rewrite the solution as

$$F_9' = \sqrt{c^2 - f^2}\, dx' \wedge dy' + f\, dy' \wedge dz\,. \tag{7.58}$$

The co-moving observer with four-velocity $v^\mu = a(\eta)^{-1}\partial_\eta$ does not see an electric field, in F_9', while the magnetic field is given by [3]

$$B^x = -\frac{f}{a(t)^3} \quad \text{and} \quad B^z = -\frac{\sqrt{c^2 - f^2}}{a(t)^3}\,.$$

This solution describes slabs of uniform magnetic field that lie in the xz plane and the field orientation varies in the perpendicular (y) direction. As the magnetic field strength is constant throughout space but the field direction changes, this field configuration describes the magnetohydrodynamic 'tangential discontinuity'. Indeed, the structure is force-free since the field lines have no tension force (there is no field line bending) and there are no magnetic pressure gradients. Now, a posteriori, it seems easy to understand that the tangential discontinuity remains a force-free solution in the FLRW spacetime since uniform expansion doesn't change the field topology but instead simply rescales the field strength.

[3] In the $3+1$ formalism, the electric and magnetic fields are given by the expression $E_\mu = v^\nu F_{\mu\nu}$ and $B_\mu = v^\nu * F_{\mu\nu}$.

Chapter 8

Conclusion

The theoretical study of energy extraction from rotating black holes necessitates solving the equations of force-free electrodynamics in the Kerr spacetime-a notably complex solution to the Einstein field equations.

This pursuit has been made challenging by the intricate nature of coupled nonlinear partial differential equations, requirements of regularity at the horizon and the symmetry axis, and the need for positive energy extraction from black holes. As a result, only two classes of exact solutions were known. By leveraging the foliations of the Kerr spacetime that admit force-free fields we have discovered four new classes of force-free fields in the Kerr spacetime.

Similarly, the considerable literature employing force-free electrodynamics in the study of cosmological magnetic fields necessarily requires the study of force-free fields allowed in an expanding universe described by the Friedmann-Lemaitre-Robertson-Walker (FLRW) spacetime. Apart from finding a wide variety of null, electrically, and magnetically dominated solutions in the FLRW spacetime, our contribution includes presenting, to our knowledge, the first force-free field transitioning from electrically dominated to null and then to a magnetically dominated regime, achieved through a chart adapted to the foliation generating the force-free field-a scenario not accommodated by the tetrad formalism typically used to describe geometric properties of null and non-null solutions.

Our study of foliations liberates us from the burden of grappling with intricate sets of coupled partial differential equations when seeking force-free fields and it allows us to bypass the simplifying assumptions like time independence and axisymmetry which often need to be imposed when solving the equations.

That being said, we still have not found an energy-extracting force-free field that is regular at the event horizon and the symmetry axis and the quest for such a solution continues. There are many more Lorentz transformations to be explored in the search for non-null solutions. Similarly, we have not exhausted the pregeodesic congruences in Kerr geometry that could potentially lead to a null force-free field. Finally, the study of foliations in alternative black hole metrics is also a promising and timely, albeit challenging, pursuit that we relegate to the future.

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Chapter 9

Appendix

9.1 Spacetime as a Manifold

Spacetime is a 4-dimensional topological manifold **M** i.e. a locally Euclidean topological space that is Hausdorff and second-countable. Furthermore, spacetime is endowed with a symmetric bilinear form ”g” called the metric (or a Lorenzian metric) that maps any two vectors from a local tangent space $\mathbf{T_p}(\mathbf{M})$ to a real number.

The existence of the metric g allows us to associate for each vector v in the tangent space $\mathbf{T}(\mathbf{M})$, a corresponding dual or 1-form in the dual space.

There exists a unique torsion-free affine connection that preserves the metric. We call this the covariant derivative.

$$\nabla_\mu g^{\alpha\beta} = 0. \tag{9.1}$$

The action of the covariant derivative, for example, on rank 2 tensors is given by:

$$\nabla_\mathbf{v}\mathbf{T}^{ab} = \partial_\mathbf{v}\mathbf{T}^{ab} + \Gamma^a_{cd}v^c\mathbf{T}^{db} + \Gamma^b_{cd}v^c\mathbf{T}^{ad}$$

$$\nabla_\mathbf{v}\mathbf{T}_{ab} \text{ and } = \partial_\mathbf{v}\mathbf{T}_{ab} + \Gamma^c_{av}\mathbf{T}_{cb} + \Gamma^c_{bv}\mathbf{T}_{ac}$$

$\Gamma^{\mu}_{\alpha\beta}$ are the Christoffel Symbols and are given by:

$$\Gamma^{\mu}_{\alpha\beta} = \frac{1}{2} g^{\mu\sigma} (\partial_{\beta} g_{\sigma\alpha} + \partial_{\alpha} g_{\sigma\beta} - \partial_{\sigma} g_{\alpha\beta}). \tag{9.2}$$

9.2 Exterior Calculus

9.2.1 Forms and Exterior Derivative

Given an n-dimensional manifold, a p-form is a totally antisymmetric covariant (type (0,p)) tensor i.e $\omega_{a_1 \ldots 1_p}$ is a p-form if

$$\omega_{a_1 \ldots a_p} = \omega_{[a_1 .. a_p]}. \tag{9.3}$$

Given a k-form α and a l-form β, the wedge product $\wedge$ is defined as :

$$\alpha \wedge \beta = \frac{(k+l)!}{k!\,l!} (\alpha \otimes \beta - \beta \otimes \alpha). \tag{9.4}$$

The exterior derivative of a m-form ω is given by:

$$d\omega = \frac{\partial \omega}{\partial x^j} \mathrm{d}x^j \wedge \mathrm{d}x^{i_1} \wedge \ldots \wedge \mathrm{d}x^{i_m}. \tag{9.5}$$

Example 1: Exterior Derivative of a 2-form in $\mathbf{R}^3$

$$\beta = x^3 dz \wedge dy + ln(y) dx \wedge dy. \tag{9.6}$$

$$\begin{aligned}
d\beta &= d(x^3 dz \wedge dy) + d(ln(y) dx \wedge dy) \\
&= \frac{\partial x^3}{\partial x} dx \wedge dz \wedge dy + \frac{\partial x^3}{\partial y} dy \wedge dz \wedge dy + \frac{\partial x^3}{\partial z} dz \wedge dz \wedge dy \\
&+ \frac{\partial ln(x)}{\partial x} dx \wedge dx \wedge dy + \frac{\partial ln(x)}{\partial y} dy \wedge dx \wedge dy + \frac{\partial ln(x)}{\partial z} dz \wedge dx \wedge dy \\
&= 3x^2 dx \wedge dz \wedge dy + \frac{1}{x} dx \wedge dx \wedge dy \\
&= 3x^2 dx \wedge dz \wedge dy
\end{aligned}$$

9.2.2 Hodge Dual

The Hodge Dual (or the Hodge Star) of a m form ω is given by:

$$*\omega = \frac{1}{m!\,(n-m)!}\left(\varepsilon_{j_1\dots j_m i_1\dots i_{n-m}}\omega^{j_1\dots j_m}\right)dx_1^i\wedge\dots\wedge dx^{i_{n-m}}. \tag{9.7}$$

In a n-dimensional space, the Hodge Star takes a m form and gives a n-m form.

Example 2: Hodge Star of a 2-form F in Minkowski Space.

$$F = f_1\,dt\wedge dr + f_2\,dt\wedge d\theta.$$

The line element is given by:

$$ds^2 = -dt^2 + dr^2 + r^2 d\theta^2 + r^2\sin\theta^2 d\varphi^2,$$

and, $\sqrt{-|g|} = r^2\sin\theta^2$.

We have,

$$\begin{aligned}
*F &= \frac{1}{2!\,(4-2)!}\sqrt{|g|}\left(\varepsilon_{abcd}F^{ab}\right)dx^c\wedge dx^d\\
&= \frac{r\sin\theta}{4}\left(2\varepsilon_{01cd}F^{01}dx^c\wedge dx^d + 2\varepsilon_{02cd}F^{02}dx^c\wedge dx^d\right)\\
&= \frac{r\sin\theta}{2}\left(\varepsilon_{0123}F^{01}dx^2\wedge dx^3 + \varepsilon_{0213}F^{02}dx^1\wedge dx^3\right)\\
&= \frac{r\sin\theta}{2}\left(F^{01}d\theta\wedge d\varphi - F^{02}dr\wedge d\varphi\right)\\
&= \frac{r\sin\theta}{2}\left(g^{\alpha 0}g^{\beta 1}F_{\alpha\beta}d\theta\wedge d\phi - g^{\gamma 0}g^{\delta 2}F_{\gamma,\delta}dr\wedge d\varphi\right)\\
&= \sin(\theta)\left(-r^2 f_1 d\theta\wedge d\varphi + f_2 dr\wedge d\varphi\right).
\end{aligned}$$