General Relativity and Gravitation (2011)
DOI 10.1007/s10714-010-0991-7

RESEARCH ARTICLE

J. T. Firouzjaee · Reza Mansouri

Asymptotically FRW black holes

Received: 16 September 2009 / Accepted: 16 April 2010

Abstract Special solutions of the LTB family representing collapsing over-dense regions corresponding to asymptotically closed, open, or flat FRW models are found. These solutions may be considered as representing dynamical mass condensations leading to black holes immersed in a FRW universe. We study the dynamics of the collapsing region, and its density profile. The question of the strength of the central singularity and its nakedness, as well as the existence of an apparent horizon and an event horizon is dealt with in detail, shedding light to the notion of cosmological black holes. Differences to the Schwarzschild black hole are addressed.

Keywords Cosmological black hole, Horizons and density evolution

1 Introduction

Let us use the term cosmological black hole for any solution of Einstein equations representing a collapsing overdensity region in a cosmological background leading to an infinite density at its center [1]. There have been different attempts to construct solutions of Einstein equations representing such a collapsing central mass. Gluing of a Schwarzschild manifold to an expanding FRW manifold is one of the first attempts to construct such a cosmological black hole, as done first by Einstein and Straus [2; 3]. Different trajectories in this model shows, however, un-physical behaviors [4].

Models not based on a cut-and-paste technology is much more interesting giving more information on the behavior of the mass condensation within a FRW universe model. The first attempt is due to McVittie [6] introducing a spacetime metric that represents a point mass embedded in a Friedmann–Robertson–Walker

J. T. Firouzjaee R. Mansouri Department of Physics Sharif University of Technology Tehran, Iran firouzjaee@physics.sharif.edu · R. Mansouri School of Astronomy Institute for Research in Fundamental Sciences (IPM) Tehran, Iran mansouri@ipm.ir

(FRW) universe. There have been many other attempts to construct cosmological black holes such as Nolan interior solution [5], and Sultana–Dyer solution [1], each of them contrasting some of the features one expect from theory or observation.

The interest for cosmological black holes in the past has been mainly from the theoretical side to understand concepts like black hole, singularity, horizon, and thermodynamics of black holes [7]. Indeed, the conventional definition of black holes implies an asymptotically flat space-time and a global definition of the event horizon. In practice, however, the universe is not asymptotically flat. The need for local definition of black holes and their horizons has led to concepts such as Hayward's trapping horizon [8], Ashtekar's isolated horizon [9], Ashtekar and Krishnan's dynamical horizon [10], and Booth and Fairhurst's slowly evolving horizon [11].

There are cases where both apparent and event horizon maybe defined. For example, for dynamical black holes one may define the event horizon as the very last ray to reach future null infinity or the light ray that divides those observer who cannot escape the future singularity from those that can [12]. Eardley proposed the conjecture that in such cases trapped surfaces can be deformed to get arbitrarily close to the event horizon [13]. Numerical evidence was provided in [14] and later proved analytically for the Vaidya metric [15].

The precision cosmology has opened a new arena for questions like cosmological black holes and their behavior. New observation of our galactic center allow to resolve phenomena near the Schwarzschild horizon of the central black hole [16]. It is therefore desirable to have black hole models embedded in cosmological environment to see if there may be considerable differences to the familiar Schwarzschild black hole. There have been also increasing interest in the gravitational lensing by a cosmological mass condensation such as a cluster of galaxies in a cosmological background. Simplest cases are Kottler and Einstein–Straus models [17]. The more complex situation is lensing by a mass condensation within a dynamical background.

Now, a widely used metric to describe the gravitational collapse of a spherically symmetric dust cloud is the so-called Tolman–Bondi–Lemaitre (LTB) metric [18; 19; 20]. These models have been extensively studied for the validity of the cosmic censorship conjecture [21; 22; 23; 24; 25; 26; 27]. In particular, we know already that, depending on initial conditions defined in terms of initial density and velocity profiles, the central shell-focusing singularity at $r = 0$ can be either a black hole or a locally or globally naked singularity [28; 29]. We may note however, that in all these papers a compact LTB region is glued to the Schwarzschild metric or the FRW outer universe [30; 31]. Therefore, the results have to be taken cautiously: any principally existent event horizon is cut off by the outer static or homogeneous space-time. The statement may still be correct that in a dynamic spacetime the cosmic censorship hypotheses is valid, as discussed in [32]. It is also possible to glue two different LTB metrics to study the structure formation out of an initial mass condensation or the formation of a galaxy with a central black hole [12; 33; 34]. Given that the structure of the metric outside these mass condensations are fixed by hand to match to a specific galaxy or cluster feature, we are again faced with the shortcomings of the cut and paste models. Faraoni et al. [35] have tried to change McVitte metric so that it resemble a collapsing mass condensation. Their solution, however, represents a singularity within a horizon embedded in a universe filled with a non-perfect fluid changing mass due to the heat flow. This metric gives us no clue whatsoever about the dynamics of a possible collapsed mass condensation. Harada et al. [36] being interested in the behavior of primordial black holes within cosmological models with a varying gravitational constant, use LTB solutions to study the evolution of a background scalar field when a black hole forms from the collapse of dust in a flat Friedmann universe probing the gravitational memory.

Our goal is to look for a model of a cosmological black hole, i.e. a mass condensation leading to a singularity within a FRW universe. In this paper we propose models for closed-, open-, and flat FRW universe studying their density profiles, singularities, and horizon behaviors. There are many nontrivial questions to be answered before understanding in detail the differences of these cosmological black holes to the familiar Schwarzschild ones, which are beyond the scope of this paper and are to be dealt with in future publications.

The question of singularities and the definition of a black hole in such a dynamical environment has been subject of different studies in the last 15 years. We review very shortly different definitions of horizons in section II as reference to the properties of model solutions we are proposing. Some initial attempts to model black holes within a FRW universe is introduced in Section 3. Section 4 is devoted to the LTB metric as the generic solution representing a spherically symmetric ideal fluid. Section 5 is devoted to different models of cosmological black holes, their dynamics, density profiles, apparent and event horizons,and singularities. The question of strength and the nakedness of singularities are dealt with in Sect. 6. We then conclude in Sect. 7. Throughout the paper we assume $8\pi G = c = 1$.

2 Local definitions of black holes

Standard definition of black holes [7] needs some global assumptions such as regular predictability and asymptotic flatness. In the cosmological context concepts of asymptotic flatness and regular predictability have no application. This has already been noticed by Demianski and Lasota [37] stressing the fact that in the cosmological context the standard global definition of black holes using event horizons may not be used any more. There is also an early definition of black hole in non asymptotically flat space times by Tipler [38] which does not resolve the complexities of black hole properties such as thermodynamic laws of black holes. In the last decade the interest in a local definition of black holes has led to four different concepts based primarily on the concept of the apparent horizon.

Let us start by assuming a spacelike two surface S with two normal null vectors ℓ^a and n^a on it.The corresponding expansions are then defined as $\theta_{(\ell)}$ and $\theta_{(n)}$.

Definition 1 [8] A *trapping horizon* H is a hypersurface in a 4-dimensional spacetime that is foliated by 2-surfaces such that $\theta_{(\ell)}\mid_H = 0$, $\theta_{(n)}\mid_H \neq 0$, and $\pounds_n\theta_{(\ell)}\mid_H \neq 0$. A trapping horizon is called *outer* if $\pounds_n\theta_{(\ell)}\mid_H < 0$, *inner* if $\pounds_n\theta_{(\ell)}\mid_H > 0$, *future* if $\theta_{(n)}\mid_H < 0$, and *past* if $\theta_{(n)}\mid_H > 0$. The most relevant case in the context of black holes is the *future outer trapping horizon* (FOTH).

Definition 2 [9] A *weakly isolated horizon* is a three-surface H such that:

1. H is null;
2. The expansion $\theta_{(\ell)}\mid_H = 0$ where ℓ^a, being null and normal to the foliations S of H;
3. $-T^b_a\ell^a$ is future directed and causal;
4. $\pounds_\ell\omega_a = 0$, where $\omega_a = -n_b\nabla_{\underleftarrow{a}}\ell^b$, and the arrow indicates a pull-back to H.

Weakly isolated horizon is a useful term to be used for characterization of black holes not interacting with their surroundings, corresponding to isolated equilibrium states in thermodynamics. These definitions do not apply to cosmological mass condensations because of their dynamical behavior.

Definition 3 [10] A *marginally trapped tube* T (MTT) is a hypersurface in a 4-dimensional spacetime that is foliated by two-surfaces S, called *marginally trapped surfaces*, such that $\theta_{(n)}|_T < 0$ and $\theta_{(\ell)}|_T = 0$. MTTs have no restriction on their signature, which is allowed to vary over the hypersurface. This is a generalization of the familiar concept of the apparent horizon [10]. If a MTT is everywhere spacelike it is referred to as a *dynamical horizon*. If it is everywhere timelike it is called a timelike membrane (TLM). In case it is everywhere null and non-expanding then we have an *isolated horizon*. The apparent horizons evolving in our proposed models will not be everywhere spacelike and, therefore, will have a more complex behavior.

Irrespective of different concepts related to the apparent horizon we may still compromise on a definition of event horizon differing principally from the apparent horizon. We follow the definition of [12] as the very last ray to reach future null infinity or the light ray that divides those observer who cannot escape the future singularity from those that can. We will see in the next sections that cosmological black holes may have distinct apparent and even horizons, in contrast to the Schwarzschild black hole.

3 Existing metrics representing over-densities within a cosmological background and their deficiencies

3.1 McVittie's solutions

In 1933, McVittie [6] found an exact solution of Einstein's equations for a perfect fluid mimicking a black hole embedded in a cosmological background. McVittie's solution can be written in the form

$$ds^2 = -\left(\frac{1-\frac{M}{2N}}{1+\frac{M}{2N}}\right)^2 dt^2 + e^{\beta(t)}\left(1+\frac{M}{2N}\right)^4 (dr^2 + h^2\, d\Omega^2), \qquad (1)$$

where $M = me^{\beta(t)/2}$ and m is a constant. Functions $h(r)$ and $N(r)$ depend on a constant k, and are given, respectively, by

$$h(r) = \begin{cases} \sinh(r) & k = 1 \\ r & k = 0 \\ \sin(r) & k = -1 \end{cases}$$
$$N(r) = \begin{cases} 2\sinh(r/2) & k = 1 \\ r & k = 0 \\ 2\sin(r/2) & k = -1 \end{cases} \tag{2}$$

This metric represents a point mass embedded into an isotropic universe. It possesses a curvature singularity at proper radius $R = 2m$, in contrast to the Schwarzschild metric, where there is a coordinate singularity. It has been shown that this singularity is space-like and weak [39; 40; 41]. The interpretation of the metric in the region $R < 2m$ is also not clear [39; 40; 41]. Therefore, McVittie's metric is not a suitable solution of Einstein equations to represent the collapse of a spherical mass distribution with over-density within a cosmological setting.

3.2 Sultana–Dyer solution

Recently Sultana and Dyer [1] found an exact solution representing a primordial cosmological black hole. It describes an expanding event horizon in the asymptotic background of the Einstein–de Sitter universe. The black hole is primordial in the sense that it forms ab initio with the big bang singularity and therefore does not represent the gravitational collapse of a matter distribution. The Sultana-Dyer metric is given by

$$ds^2 = t^4 \left[\left(1 - \frac{2m}{r}\right) dt^2 - \frac{4m}{r}\, dt\, dr - \left(1 - \frac{2m}{r}\right) dr^2 - r^2\, d\Omega^2 \right]. \tag{3}$$

Though the metric has the same causal characteristics as the Schwarzschild spacetime, there are significant differences for timelike geodesics. In particular an increase in the perihelion precession and the non-existence of circular timelike orbits should be mentioned. The matter content is described by a non-comoving two-fluid source, one of which is a dust and the other is a null fluid. At late times the dust becomes superluminal near horizon violating the energy condition.

4 Introducing realistic models of cosmological mass condensation

There maybe different ways of constructing solutions of Einstein equations representing a collapsing mass concentration in a FRW background, as the preceding sections show. We choose the direct way of a cosmological spherical symmetric isotropic solution and look for an overdense mass distribution within the model universe undergoing a collapse to see if and how a singularity representing a black hole emerges. To begin with, we choose a LTB metric. This is the simplest spherically symmetric solution of Einstein equations representing an inhomogeneous dust distribution [18; 19; 20].

4.1 LTB metric

The LTB metric may be written in synchronous coordinates as

$$ds^2 = dt^2 - \frac{R'^2}{1+f(r)}\,dr^2 - R(t,r)^2\,d\Omega^2, \tag{4}$$

representing a pressure-less perfect fluid satisfying

$$\rho(r,t) = \frac{2M'(r)}{R^2R'}, \quad \dot{R}^2 = f + \frac{2M}{R}. \tag{5}$$

Here dot and prime denote partial derivatives with respect to the parameters t and r, respectively. The angular distance R, depending on the value of f, is given by

$$\begin{aligned} R &= -\frac{M}{f}(1-\cos\eta(r,t)), \\ \eta - \sin\eta &= \frac{(-f)^{3/2}}{M}(t - t_b(r)), \end{aligned} \tag{6}$$

$$\dot{R} = (-f)^{1/2}\frac{\sin(\eta)}{1-\cos\eta}, \tag{7}$$

for $f < 0$, and

$$R = \left(\frac{9}{4}M\right)^{\frac{1}{3}}(t - t_b)^{\frac{2}{3}}, \tag{8}$$

for $f = 0$, and

$$\begin{aligned} R &= \frac{M}{f}(\cosh\eta(r,t) - 1), \\ \sinh\eta - \eta &= \frac{f^{3/2}}{M}(t - t_b(r)), \end{aligned} \tag{9}$$

for $f > 0$.

The metric is covariant under the rescaling $r \longrightarrow \tilde{r}(r)$. Therefore, one can fix one of the three free parameters of the metric, i.e. $t_b(r)$, $f(r)$, or $M(r)$. The function $M(r)$ corresponds to the Misner–Sharp mass in general relativity, as shown in the general case of spherically symmetric solutions of Einstein equations [42]. The r dependence of the bang time $t_b(r)$ corresponds to a non-simultaneous big bang- or big-crunch-singularity. Given our goal of modeling the collapse of an overdense region we prefer to choose a constant bang time re-scaled to $t_b(r) \equiv 0$, to get rid of a non-simultaneous collapse. We are then left with two arbitrary metric functions $f(r)$ and $M(r)$.

There are two generic singularities of this metric, where the Kretschmann- and Ricci scalars become infinite: the shell focusing singularity at $R(t,r) = 0$, and the shell crossing one at $R'(t,r) = 0$. However, there may occur that in the case of $R(t,r) = 0$ the density $\rho = \frac{M'}{R^2R'}$ and the term $\frac{M}{R^3}$ remain finite. In this case the

Kretschmann scalar remains finite and there is no shell focusing singularity. Similarly, if in the case of vanishing R' the term $\frac{M'}{R'}$ is finite, then the density remains finite and there is no shell crossing singularity either.

Now, an expanding universe means generally $\dot{R} > 0$. However, in a region around the center it may happen that $\dot{R} < 0$, corresponding to the collapsing region. It is then easy to show that in this collapsing region $\theta_{(\ell)} \propto (1 - \frac{\sqrt{\frac{2M}{R}+f}}{\sqrt{1+f}})$, $\theta_{(n)} \propto (-1 - \frac{\sqrt{\frac{2M}{R}+f}}{\sqrt{1+f}}) < 0$. Therefore, $R = 2M$, is obviously a *marginally trapped tube*, as defined in Sect. 2, representing an apparent horizon according to the familiar definitions [7; 10]. It will turn out that this apparent horizon is not always spacelike and can have a complicated behavior for different r, as was first seen in [43].

4.2 Asymptotically FRW LTB solutions

For LTB solutions to be asymptotically FRW certain conditions have to be fulfilled. We first note that FRW spaces are special cases of LTB metrics. In cases where $R(r,t)$ is separated as $R(r,t) = ra(t)$ we obviously get the homogeneous FRW solutions. For the vanishing bang time this corresponds to $M(r) = cr^3$, $f(r) = -r^2, 0, r^2$. Therefore, to have an asymptotically LTB solution we obviously have to ask for the following condition to be valid:

Condition 1:

$$M(r) \propto r^3 \quad \text{at} \quad R \gg 1 \tag{10}$$

$$f(r) \propto \pm r^2, 0 \quad \text{at} \quad R \gg 1 \tag{11}$$

Now, the regularity condition at $r = 0$ leads to the vanishing of Misner–Sharp mass M at $r = 0$. We then see from (6) and (9) that $\frac{f^{3/2}}{M}|_{r=0} = const$, [44]. We have therefore to assume $f(r = 0) = 0$. Now assuming

$$M(r) \propto r^n \quad \text{at} \quad r \ll 1 \tag{12}$$

$$f(r) \propto r^m \quad \text{at} \quad r \ll 1, \tag{13}$$

we conclude that

Condition 2:

$f(r = 0) = 0$, $M(r = 0) = 0$ and $3m \geq 2n$.

We still have to look for conditions leading to overdensities near the center $r = 0$ within an expanding universe with $\dot{R} > 0$, at least far from the center. However, overdensities in a region around the center require a late time behavior $\dot{R} < 0$, corresponding to the collapse phase of the overdensity region, which we may assume to start at a time $t_c > 0$. From equations (6), (8), and (9) it is easily seen that for the collapsing region one has to have $f(r) < 0$. In contrast, for the universe outside the collapsing region one may have $f(r) > 0$, $f(r) = 0$, or

$f(r) < 0$ depending on the model. In an asymptotically flat FRW universe, however, one may have $f(r) > 0$ or $f(r) < 0$ as far as $f(r)$ tends to zero for large r. Possible behaviors of the function $f(r)$ are shown in Fig. 1.

The collapse of the overdense region leads to two new conditions on the metric coefficients. First we see from (7) that at any constant time shells corresponding to $0 < \eta(r) < \pi$ are in an expanding phase and those corresponding to $\pi < \eta(r) < 2\pi$ are in the collapsing phase. Using $\eta - \sin\eta = \frac{(-f)^{3/2}}{M}t$ and noting that $\eta - \sin\eta$ is an increasing function of η, we are led to

Condition 3:

$\frac{d(\eta-\sin\eta)}{dr}|_{t=const} = \frac{d\frac{(-f)^{3/2}}{M}}{dr}t|_{t=const} < 0.$

Now, a collapsing region means a singularity to be formed in the center after some time. Therefore, there should be a big crunch singularity at $\eta = 2\pi$ corresponding to a time $t = t_s$, i.e. $2\pi - \sin 2\pi = \frac{(-f(r=0))^{3/2}}{M(r=0)}t_s$. For this to be the case, one must have

Condition 4:

$3m = 2n.$

We will now look for LTB solutions fulfilling these four conditions. Solutions we are proposing may not be generic but should give us hints related to the generalization of the concept of black hole.

5 Construction of models

We have now the necessary prerequisites to construct our models of mass condensation immersed in FRW models leading to singularities and representing cosmological black holes. Our cosmological black hole solutions evolve from mass condensations within closed, open, or flat FRW universes, leading to singularities having different horizons, and providing us examples of collapsed regions behaving differently to known Schwarzschild ones. We are interested in the collapsing phase starting from a time after the onset of the collapse, say $t_c > 0$.

5.1 Example I: $f < 0$: asymptotically closed LTB metric

As mentioned earlier, we are free to choose one of the three parameters of the LTB metric. Assuming a negative $f(r)$, we may choose r such that $f(r) = -M(r)/r$ [21; 22; 23; 24]. Now, let us choose the mass function M such that

$$M(r) = 2^3 a^2 r^3 \frac{\alpha + r^3}{1 + r^3},$$

Fig. 1 Different behaviors of the curvature function $f(r)$

Fig. 2 Evolution of the Cauchy surfaces

Fig. 3 The case of the asymptotically closed universe: in the central region the density increases with time indefinitely while far from the center the density is decreasing with time. The apparent horizon and the trapped region is shown in the lower diagram

where a and α are constants to be defined properly. We then obtain from (6)

$$\begin{aligned} R &= r(1-\cos\eta(r,t)) \\ \eta - \sin(\eta) &= \sqrt{\frac{\alpha + r^3}{1+r^3}}2^{3/2}at. \end{aligned} \tag{14}$$

We are free to fix a and α such that for the time $t > t_c$ the region around the center of the overdensity, $r = 0$, is collapsing while far from the center the universe expands. Note that in contrast to the familiar FRW universe, where the scale factor is an explicit function time, in the LTB case, $R(r,t)$ playing the role of the scale factor, is an implicit function of time and the comoving coordinate r given by (6). We now fix a and α such that $r = 0$ corresponds to $\eta = \frac{3\pi}{2}$, and $r \gg 1$ corresponds to $\eta = \frac{5\pi}{6}$ (Fig. 2). Note that these η values are chosen in accordance with our choice of t_c and also for simplicity. We then find $a \simeq \frac{0.75}{t_c}$ and $\alpha \simeq 7$.

We then see from (7) that the region around $r = 0$, corresponding to $\eta \sim \frac{3\pi}{2}$, is always collapsing for any time t, while the regions far from the center, $r \ll 0$, at the initial time, corresponding to $\eta \sim \frac{5\pi}{6}$, are expanding. Note that this bound LTB model, similar to the closed FRW one, has a maximum comoving radius corresponding to $f(r) = -1$.

The density evolution and the causal structure of the model is shown in Fig. 3. We see clearly how the central overdensity region collapses to a singularity at $r = 0$, while the universe is expanding. Note also how the slope of outgoing null geodesics tend to infinity in the vicinity of the singularity, i.e. $R' \to +\infty$ at $R = 0$.

5.2 Example II: $f < 0, \lim_{r\to\infty} f(r) \to 0^-$; asymptotically flat LTB model 1

Our favorite choice is a solution representing a collapsing overdensity region at the center and a flat FRW far from the overdensity region. Of course the overdensity region may take part in the expansion of the universe at early times but gradually reversing the expansion and start collapsing. To achieve this, we require $f(r) < 0$ and $f(r) \to 0$ when $r \to \infty$.

Let us now make the ansatz $f(r) = -re^{-r}$ leading to

$$M(r) = \frac{1}{a}r^{3/2}(1+r^{3/2}),$$

where a is a constant having the dimension $[a] = [L]^{-2}$. We fix a by $at_c = 3\pi/2$. Similar to our previous model I, this value of a corresponds to the collapsing mass condensation around $r=0$ starting in the expanding phase of the bound LTB model.

Fig. 4 The density profile for the cosmic black hole within a closed but asymptotically flat universe (*bolded red line* shows the density profile at t_c). The causal structure is shown below. Note the behavior of the event horizon for arbitrary large but finite t

Equation (6), (8) then leads to

$$\begin{aligned} R &= \frac{\sqrt{r}(1+r^{3/2})}{ae^{-r}}(1-\cos\eta(r,t)), \\ \eta - \sin(\eta) &= \frac{e^{-\frac{3}{2}r}}{(1+r^{3/2})}at. \end{aligned} \tag{15}$$

We have plotted the density evolution and casual structure of this model in Fig. 4.

As a result of $R' \to +\infty$ near the singularity, the slope of the outgoing null geodesics becomes infinite at the central singularity. Again we see clearly how the collapse of the central region and the evolution of the apparent horizon separates the overdense region from the expanding universe.

The negativity of the curvature function $f(r)$ means that, although the universe is asymptotically flat, waiting enough, every slice $r = constant$ will collapse to the central region. We may , however, define an event horizon according to the definition of section 2 for any large but finite time, as shown in Fig. 4.

5.3 Example III: $f(r) \to 0^+$ when $r \to \infty$; asymptotically flat LTB model 2

What would happen if we choose the curvature function $f(r)$ such that it tends to zero for large r while it is positive? We still have a model which tends to a flat FRW at large distances from the center having a density less than the critical one.

Let us make the ansatz $f(r) = -r(e^{-r} - \frac{1}{r^n+c})$ with $n = 2$ and $c = 20{,}000$, leading to

$$M(r) = \frac{1}{a}r^{3/2}(1+r^{3/2}),$$

where a is a constant having the dimension $[a] = [L]^{-2}$. Again, we fix a by requiring $at_c = 3\pi/2$. Equations (6)–(8) then lead to

$$\begin{aligned} R &= \frac{\sqrt{r}(1+r^{3/2})}{a\left(e^{-r} - \frac{1}{r^2+20{,}000}\right)}(1-\cos\eta(r,t)), \\ \eta - \sin\eta &= \frac{\left(e^{-r} - \frac{1}{r^2+20{,}000}\right)^{1.5}}{(1+r^{3/2})}at, \end{aligned} \tag{16}$$

Fig. 5 Evolution of the cosmic black hole within an open but asymptotically flat universe is similar to the closed case. The causal structure, however, is significantly different, as seen from the lower diagram. Result of the numerical calculation of the locations of the event horizon, apparent horizon and the singularity is also shown

for $f<0$ and

$$\begin{aligned} R &= \frac{\sqrt{r}(1+r^{3/2})}{a\left(\frac{1}{r^2+20{,}000}-e^{-r}\right)}(\cosh\eta(r,t)-1), \\ \eta-\sinh\eta &= \frac{\left(\frac{1}{r^2+20{,}000}-e^{-r}\right)^{1.5}}{(1+r^{3/2})}at, \end{aligned} \tag{17}$$

for $f>0$. The solution is continuous at $r=1$, as can be checked by evaluating $\dot{R}$, R', $\dot{R}'$, $\ddot{R}$, and $\ddot{R}'$ at $r=1$ (see Appendix A). The density evolution and casual structure of the model is plotted in Fig. 5.

The term $\frac{1}{r^n+c}$ is responsible for $f(r)$ being positive and tending to zero for large r given $n\geq 2$ and $c\ll 1$. Let us check if this may cause shell crossing in the region where $f'(r)<0$ while $f>0$. Using (9) we obtain

$$\frac{R'}{R}=\frac{M'}{M}(1-\Phi)+\frac{f'}{f}\left(\frac{3}{2}\Phi-1\right), \tag{18}$$

where $\frac{2}{3}\leq\Phi=\frac{\sinh\eta(\sinh\eta-\eta)}{(\cosh\eta-1)^2}\leq 1$. The condition for no shell crossing singularity is then $\frac{M|f'|}{fM'}<\frac{1-\Phi}{\frac{3}{2}\Phi-1}$. For $\Phi\sim 1$, corresponding to $\eta\gg 1$ or $t\gg 1$ the inequality breaks down leading to a shell crossing singularity. The shell crossing, however, can be shifted to arbitrary large t by choosing $f'\ll 1$ corresponding to $n\gg 1$ and $c\gg 1$ [45; 46]. Therefore, for the model we are proposing the shell crossing will happen out of the range of applicability of it.

As a result of $R'\to+\infty$ near the singularity, the slope of the outgoing null geodesics become infinite at the central singularity. Again we see clearly how the collapse of the central region and the evolution of the apparent horizon separates the overdense region from the expanding universe. There is an event horizon defined by the very last ray to reach future null infinity and separates those observer who can not scape the future singularity from those that can. A fixed $r=r_0$, being the non-trivial root of $f(r)=0$, divides the absolute collapsing region from the absolute expanding region. We may be living in a region inside the event horizon but outside the apparent one without noticing it soon!

This solution represents a collapsing mass within an asymptotically flat FRW universe. The collapsed region is dynamical in the sense that its mass is not constant. In fact the rate of change of the Misner-Sharp energy is given by $\frac{dM(r)}{dt}|_{R=const}=\frac{dM(r)}{dr}\frac{dr}{dt}|_{R=const}>0$ because $\frac{dM(r)}{dr}>0$, $R'dr+\dot{R}dt=0$, $R'>0$, and $\dot{R}<0$ for collapsing region, so $\frac{dr}{dt}|_{R=const}>0$. Therefore, it is clear that concepts such as isolated horizon and slowly evolving horizon do not apply to this case.

Fig. 6 The case of asymptotically open model: the density profile is similar to the previous open but asymptotically flat case, except for the mass concentrated in the central region being less than the previous case. The causal structure is also similar. The locations of the event horizon, apparent horizon and the singularity are calculated numerically. The separation between the singularity and the apparent horizon is not clear due to the scale chosen

5.4 Example IV: $f > 0$ for $r \gg 1$: asymptotically open FRW metric

Now we look for a solution which goes to an open FRW metric at distances far from the center. At the same time one should take care of the conditions $M(0) = 0$ and $\frac{f(0)^{3/2}}{M(0)} \neq \infty$. Let us choose

$$f(r) = -r(1-r),$$

and

$$M(r) = \frac{1}{a} r^{3/2}(1 + r^{3/2}),$$

where a is a constant fixed by the assumption that $r = 0$ corresponds to $\eta = \frac{3\pi}{2}$ at the time t_c. This leads to $at_c = 3\pi/2 + 1$. We then obtain from (9)

$$\begin{aligned} R &= \frac{\sqrt{r}(1+r^{3/2})}{a(1-r)}(1-\cos\eta(r,t)), \\ \eta - \sin(\eta) &= \frac{(1-r)^{3/2}}{(1+r^{3/2})} at, \end{aligned} \tag{19}$$

for $r < 1$, and

$$\begin{aligned} R &= \frac{\sqrt{r}(1+r^{3/2})}{a(r-1)}(\cosh\eta(r,t) - 1) \\ \sinh\eta - \eta &= \frac{(r-1)^{3/2}}{(1+r^{3/2})} at, \end{aligned} \tag{20}$$

for $r > 1$. The solution is again continuous at $r = 1$, as can be checked by evaluating $\dot{R}$, R', $\dot{R}'$, $\ddot{R}$, and $\ddot{R}'$ at $r = 1$ (see Appendix A).

The resulting density profile and the causal structure is plotted in Fig. 6. Obviously a singularity at the origin forms gradually while the universe is expanding. The causal structure is also similar to the open but asymptotically flat case.

This solution represents a collapsing mass within an open FRW universe. The collapsed region is again dynamical in the sense that its mass is not constant, and the rate of change of the Misner-Sharp energy is given by the same amount as the previous model. Therefore, concepts of isolated horizon and slowly evolving horizon do not apply to this case either.

6 Characteristics of singularities of proposed models

We have avoided in the models proposed the shell crossing singularities except example III with a late time shell crossing singularity.

The shell focusing singularities, however, are unavoidable and in fact it is what we are looking for to study characteristics of cosmological black holes. An important aspect of such a singularity is its gravitational strength [47; 48; 49], which is an important differentiating feature of black holes.

6.1 Strength of the shell focusing singularities

Heuristically, a singularity is termed gravitationally strong, or simply strong, if it destroys any object which falls into it by crushing or stretching. The prototype of such a singularity is the Schwarzschild one: a radially infalling object is infinitely stretched in the radial direction and crushed in the tangential directions, with the net result of crushing to zero volume. In contrast, a singularity is termed weak if objects falling into it are not destroyed. To check the strength of singularities of our models we use the criteria defined by Clarke [47; 48; 49].

Let k^{μ} be the tangent vector to the ingoing null geodesic, λ the corresponding affine parameter being zero at the center, and $R_{\mu\nu}$ the Ricci tensor. Now the Clarke condition states that the singularity is said to be strong if

$$\Psi = \lim_{r\to 0} \lambda^2 k^{\mu} k^{\nu} R_{\mu\nu} \neq 0. \tag{21}$$

For a general LTB metric one obtains easily $k^{\mu} k^{\nu} R_{\mu\nu} = 2(k^t)^2 \frac{M'}{R^2 R'}$. In general, dust cosmological black holes can have either strong or week singularities [25; 26]. For the cosmological black hole models proposed in this paper we have done the calculation along the lines of the [25; 26] using appropriate coordinates near singularity. In the cases (15–19) we obtain after some calculation (see Appendix B) $\Psi = 0$ for $r \to 0$. Therefore, shell focusing singularities occurring in the center of the models are week. This is in contrast to the Schwarzschild singularity which is a strong one. We leave it to future studies to see if this weakness is generic of any cosmological black hole.

6.2 Nakedness of singularities

We know already from Oppenheimer–Snyder collapse of a homogeneous dust distribution how the shells become singular at the same time, and thus none of them crosses. In the case of spherically symmetric inhomogeneous matter configurations, however, the proper time of collapse depends on the comoving radius r. Thus the piling up of neighboring matter shells at finite proper radius can occur, thereby producing two-dimensional caustics where the energy density and some curvature components diverge. These singularities can be locally naked, but they are gravitationally weak [21; 22; 23; 24; 50], i.e. curvature invariants and tidal forces remain finite. It has also been shown that analytic continuations of the metric, in a distributional sense, can always be found in the neighborhood of the singularity [51; 52].

Models proposed in this paper are, however, free from shell crossing singularities. The shell crossing singularity of example III at late times does not influence the following argumentation. Conditions for the absence of shell crossing singularities have been studied in detail in [45; 46]. In our case these conditions are equivalent to $M'(r) > 0$ and $R' > 0$, which are satisfied by the models discussed above. We may then conclude that

$$\frac{\frac{dt}{dr}|_{AH}}{\frac{dt}{dr}|_{null}} = \left(1 - \frac{2M'}{R'}\right) < 1. \tag{22}$$

Therefore, the condition for the apparent horizon $R = 2M$ to be spacelike is, i.e. $-1 < \frac{\frac{dt}{dr}|_{AH}}{\frac{dt}{dr}|_{null}} < 1$, leads to the condition $R' - M' > 0$, which is not everywhere satisfied in our model. As a result we notice that apparent horizons of the models proposed here are not spacelike everywhere. Such a behavior has already been discussed in [43].

The case of shell focusing singularity is, however, a different one. Irrespective of the behavior of the apparent and event horizons, it is then a relevant question if the shell focusing singularity could be a naked one. We notice that the slope of the outgoing null geodesics at the singularity are greater than the slope of the singularity itself, i.e. the singularity is **spacelike**. Therefore, no timelike or null geodesic can come out of the singularity, and we conclude that the singularities in our proposed models can not be *naked*. This can also be checked by the test given in [25; 26].

7 Discussion and conclusions

We have constructed models of mass condensation within the FRW universe leading to cosmological black holes without having the usual pathologies we know from other models: the cosmic fluid is dust and ideal producing a singularity at the center in the course of time. The central singularity is spacelike and not naked. In the case of flat or open universe models the singularity is weak and has distinct apparent and event horizons. The apparent horizons are not everywhere spacelike, to be compared with the Schwarzschild one which is null everywhere. This has already been noticed in a general context by [43]. While the apparent horizon is defined by the surfaces $R = 2M$, similar to the Schwarzschild horizon, the even horizon is further away. Models we have proposed show that one has to expect new effects while considering dynamical cosmic black holes. The simple Schwarzschild static model may not reflect all the phenomena one may expect in observational cosmology, and the black hole thermodynamics. Even the simple concept of mass is not a trivial one in such a dynamical environment. The study of these questions is beyond the scope of this paper and will be dealt with in future publications.

Appendix A

The curvature function $f(r)$ has a zero point where it changes sign for models III and IV, corresponding to two different solutions. Therefore, we have to take care of joining continuously two LTB solutions across the hypersurface defined by $f(r)=0$. This is done by looking at the metric functions and their derivatives.

Let us first look at the model IV. There we have to look at the metric function R and its derivatives, R, R', $\dot{R}$, $\ddot{R}$ and $\ddot{R}'$, at the point $r=1$ where f vanishes. From the following relations derived from the Einstein equations (5)

$$\ddot{R} = -\frac{M}{R^2}, \tag{A1}$$

$$\dot{R}' = \frac{M'}{R\dot{R}} - \frac{MR'}{\dot{R}R^2} + \frac{f'}{2\dot{R}}, \tag{A2}$$

and

$$\ddot{R}' = -\frac{M'}{R^2} + \frac{2MR'}{R^3}, \tag{A3}$$

we infer that these second derivatives relevant for the continuity of Einstein equations across the hypersurface $f(r)=0$ are continuous if f, R, R', $\dot{R}$, M', and M are continuous. Now, because of the continuity of f, M', and M, we just have to prove the continuity of R, $\dot{R}$, and R'.

Let us look first at R and its derivative R'. In the case of $r<1$ we have

$$\begin{aligned} R &= \frac{a(r)}{1-r}(1-\cos\eta), \\ \eta - \sin\eta &= \frac{(1-r)^{1.5}}{b(r)}t, \end{aligned} \tag{A4}$$

where $a(r)=\sqrt{r}+r^2$, $b(r)=1+r^{1.5}$, and $a(1)=2$, $a'(1)=2.5$, $b(1)=2$, $b'(1)=1.5$, and

$$\dot{R} = \frac{a\sqrt{1-r}}{b}\frac{\sin\eta}{1-\cos\eta}. \tag{A5}$$

$$\begin{aligned} R' = {} & \frac{a'(1-r)+a}{(1-r)^2}(1-\cos\eta) - \frac{a}{1-r} \\ & \times \frac{\sin\eta}{1-\cos\eta}\frac{1.5(1-r)^{0.5}b + b'(1-r)^{1.5}}{b^2}t. \end{aligned} \tag{A6}$$

Defining $1-r=x$, we have

$$\eta - \sin\eta = \frac{\eta^3}{6} - O(\eta^5) = \frac{x^3}{2}t. \tag{A7}$$

Therefore, to first order in η we have $\eta = \sqrt[3]{3t}\sqrt{x}$. Now taking the limit $x \to 0^-$ we obtain

$$\lim_{x\to 0^-} R(x) = \lim_{x\to 0^-}\frac{2}{x}(1-\cos\eta) = \lim_{x\to 0^-}\left(\frac{\eta^2}{x} - O(\eta^4)/x\right) = (3t)^{2/3}. \tag{A8}$$

which is a well defined quantity.

In the case of $r > 1$ we have

$$
\begin{aligned}
R &= \frac{a(r)}{r-1}(\cosh\eta - 1), \\
\sinh\eta - \eta &= \frac{(r-1)^{1.5}}{b(r)} t,
\end{aligned}
\tag{A9}
$$

$$
\dot{R} = \frac{a\sqrt{r-1}}{b}\frac{\sinh\eta}{\cosh\eta - 1}, \tag{A10}
$$

and

$$
\begin{aligned}
R' = {} & \frac{a'(r-1) - a}{(r-1)^2}(\cosh\eta - 1) + \frac{a}{r-1} \\
& \times \frac{\sinh\eta}{\cosh\eta - 1}\frac{1.5(r-1)^{0.5} b - b'(r-1)^{1.5}}{b^2} t.
\end{aligned}
\tag{A11}
$$

Now, defining $r - 1 = x$, and noting that

$$
\sinh\eta - \eta = \frac{\eta^3}{6} + O(\eta^5) = \frac{x^3}{2} t, \tag{A12}
$$

we obtain to first order in η the relation $\eta = \sqrt[3]{3t}\sqrt{x}$. We then have

$$
\lim_{x\to 0^+} R(x) = \lim_{x\to 0^+} \frac{2}{x}(\cosh\eta - 1) = \lim_{x\to 0^+} \frac{\eta^2}{x} + O(\eta^4)/x = (3t)^{2/3}. \tag{A13}
$$

Therefore, the continuity of R across $r = 1$ is established.

Similar calculation for the first derivatives shows the continuity of R' and $\dot{R}$ having well defined values on both sides of the $r = r_0$ hypersurface:

$$
R'(1) = 2.5(3t)^{2/3} - \frac{3}{4(3t)^{1/3}} t, \tag{A14}
$$

and

$$
\dot{R}(1) = \frac{2}{(3t)^{1/3}}. \tag{A15}
$$

The case of model III is similar except for the hypersurface defined by $g(r) = \frac{f(r)}{r} = e^{-r} - \frac{1}{r^2 + 20{,}000} = 0$ with the root of $e^{-r_0} - \frac{1}{r_0^2 + 20{,}000} = 0$ being at a point $r = r_0$ different from $r = 1$. It is easy to see that $g(r)$ is an analytic function at $r = r_0$, and can be approximated by $g(r) \approx g'(r_0)(r - r_0) + \frac{g''(r_0)}{2}(r - r_0)^2 + \cdots$ Similar calculations verify the continuity of the metric function R and its relevant derivatives across the hypersurface $r = r_0$.

Appendix B

We follow [25; 26] to specify the strength and type of the singularity. Consider the null geodesic k^μ reaching the central singularity at the affine parameter $\lambda = 0$. Define the function P such that the zero component of the null geodesic is given by $k^t = \frac{dt}{d\lambda} = \frac{P}{R}$. Now, using $k^t = \pm\frac{R'k^r}{\sqrt{1+f}}$ and $k^r = \frac{\sqrt{1+f}P}{R'R}$ derived from the null conditions, we may write the geodesic equation

$$\frac{d^2t}{d\lambda^2} + \Gamma^0_{11}(k^r)^2 = 0 \tag{B1}$$

in the following form:

$$\frac{dP}{dk} = \frac{P^2\sqrt{1+f}}{R^2} + \frac{\dot{R}P^2}{R^2} - \frac{P^2\dot{R}'}{RR'}, \tag{B2}$$

where we have used $\Gamma^0_{11} = \frac{R'\dot{R}'}{1+f}$.

Now, to check the Clarke condition we calculate first

$$\Psi = \lim_{r\to 0} \lambda^2 k^\mu k^\nu R_{\mu\nu} = \frac{F'}{R^4 R'}\left(\frac{\lambda}{\frac{1}{P}}\right)^2, \tag{B3}$$

where the limiting process is taken along the the null geodesic to the central singularity $R(t_s, r = 0) = 0$, or $\lambda = 0$. Note that our Ψ corresponds to $\lambda^2\Psi$ as defined in [25; 26]. We have to know the behavior of different metric functions near this singularity to be able to calculate the above expression. Using definitions of section IV.B, we may write $n = r\frac{M'}{M}$. Note that we have used n instead of η as used in [25; 26].

Assume R behaves as $X_0 r^\alpha$ for $r \to 0$ along the ingoing null geodesic k^μ, where X_0 is a constant. To calculate α we first notice that

$$\begin{aligned}\lim_{r\to 0, R\to 0} \frac{R}{r^\alpha} &= \lim_{r\to 0, R\to 0} \frac{dR}{d(r^\alpha)} \\ &= \frac{R'}{\alpha r^{\alpha-1}}\left(1 - \frac{\dot{R}}{\sqrt{1+f}}\right).\end{aligned} \tag{B4}$$

Next, we define a new coordinate $\theta = 2\pi - \eta$ around the big crunch singularity $\eta \to 2\pi$ or $\theta \to 0$. Then from (6) we have

$$\begin{aligned}&\lim_{r\to 0, \theta\to 0} R = -\frac{M}{2f}(\theta)^2, \quad (\theta)^3 = 6\frac{(-f)^{3/2}}{M} t \; r \ll 1, \\ &\lim_{r\to 0, \theta\to 0} \dot{R} = \frac{2\sqrt{f}}{2\theta} \; r \ll 1,\end{aligned} \tag{B5}$$

and

$$\lim_{r\to 0, \theta\to 0} R' = \frac{M'f - f'M}{2f^2}\theta^2 - \frac{2M}{\theta f} 1.5 \frac{1.5\sqrt{f}f'M - M'f^{1.5}}{M^2} \quad r \ll 1. \tag{B6}$$

Substituting these relations in (B4), using $n = 3/2$ and $m = 1$, we find $\alpha = \frac{7}{6}$.

Looking at the equation (58) of [25; 26], we see $\Psi \propto \Lambda$ where $\Lambda = \frac{2M}{r^{\alpha}}$. Now, in our case there is $\Lambda = \frac{2M}{r^{\alpha}} \propto r^{n-\alpha}$ as $r \to 0$. Given that $n - \alpha = 3/2 - 7/6 > 0$, we conclude that $\Lambda \to 0$ as one approaches the singularity. This means that the singularity is a weak one!

References

1. J. Sultana C.C. Dyer (2005) *Gen. Relativ. Gravit.* **37** 1349
2. A. Einstein E.G. Straus (1945) *Rev. Mod. Phys.* **17** 120
3. A. Einstein E.G. Straus (1946) *Rev. Mod. Phys.* **18** 148
4. Baker, G.A. Jr.: astro-ph/0003152
5. B.C. Nolan (1993) *J. Math. Phys.* **34** 1
6. G.C. McVittie (1933) *Mon. Not. R. Astr. Soc.* **93** 325
7. S.W. Hawking G.F.R. Ellis (1973) The Large Scale Structure of Space-Time Cambridge University Press Cambridge
8. S.A. Hayward (1994) *Phys. Rev. D* **49** 6467
9. A. Ashtekar C. Beetle O. Dreyer S. Fairhurst B. Krishnan J. Lewandowski J. Wisniewski (2000) *Phys. Rev. Lett.* **85** 3564 – 3567
10. A. Ashtekar B. Krishnan (2002) *Phys. Rev. Lett.* **89** 261101
11. I. Booth S. Fairhurst (2004) *Phys. Rev. Lett.* **92** 011102
12. A. Krasinski C. Hellaby (2004) *Phys. Rev. D* **69** 023502
13. D. Eardley (1998) *Phys. Rev. D* **57** 2299
14. E. Schnetter B. Krishnan (2006) *Phys. Rev. D* **73** 021502
15. I. Ben-Dov (2007) *Phys. Rev. D* **75** 064007
16. S.S. Doeleman (2008) *Nature* **455** 478 – 480
17. W. Rindler M. Ishak (2007) *Phys. Rev. D* **76** 043006
18. R.C. Tolman (1934) *Proc. Natl. Acad. Sci. USA* **20** 410
19. H. Bondi (1947) *Mon. Not. R. Astron. Soc.* **107** 343
20. G. Lemaitre (1933) *Ann. Soc. Sci. Bruxelles I* **A53** 51
21. P.S. Joshi T.P. Singh (1995) *Phys. Rev. D* **51** 6778
22. R.P.A.C. Newman (1986) *Class. Quantum Grav.* **3** 527
23. D. Christodoulou (1984) *Commun. Math. Phys.* **93** 171
24. D.M. Eardly L. Smarr (1979) *Phys. Rev. D* **19** 2239
25. P.S. Joshi I.H. Dwivedi (1993) *Phys. Rev. D* **47** 5357
26. Jhingan, S., Joshi, P.S.: gr-qc/9701016
27. A. Chamorro S.S. Deshingkar I.H. Dwivedi P.S. Joshi (2001) *Phys. Rev. D* **63** 084018
28. I.H. Dwivedi P.S. Joshi (1997) *Class. Quantum Grav.* **14** 1223
29. S.S. Deshingkar S. Jhingan P.S. Joshi (1998) *Gen. Relativ. Gravit.* **30** 1477
30. S. Khakshournia R. Mansouri (2002) *Phys. Rev. D* **65** 027302
31. V.A. Berezin V.A. Kuzmin I.I. Tkachev (1987) *Phys. Rev. D* **36** 2919
32. Wald, R.M.: gr-qc/9710068
33. A. Krasinski C. Hellaby (2002) *Phys. Rev. D* **65** 023501
34. C. Hellaby A. Krasinski (2004) *Phys. Rev. D* **73** 023518
35. V. Faraoni A. Jacques (2008) *Phys Rev D* **78** 024008
36. T. Harada C. Goymer B.J. Carr (2002) *Phys. Rev. D* **66** 104023

37. M. Demianski J.P. Lasota (1973) *Nature Phys. Sci.* **241** 53
38. F.J. Tipler (1977) *Nature* **270** 500
39. B.C. Nolan (1999) *Class. Quantum Grav.* **16** 1227
40. B.C. Nolan (1998) *Phys. Rev. D* **58** 064006
41. B.C. Nolan (1999) *Class. Quantum Grav.* **16** 3183
42. C.W. Misner D.H. Sharp (1964) *Phys. Rev.* **136** B571
43. I. Booth L. Brits J.A. Gonzalez C. Van Den Broeck (2006) *Class. Quant. Grav.* **23** 413 – 440
44. N. Mustapha C. Hellaby (2001) *Gen. Relativ. Gravit.* **33** 455 – 477
45. C. Hellaby K. Lake (1985) *Astrophys. J.* **290** 381
46. C. Hellaby K. Lake (1985) *Astrophys. J.* **300** 461(E)
47. C.J.S. Clarke (1993) *Class. Quant. Grav.* **10** 1375
48. F.J. Tipler (1977) *Phys. Lett. A* **64** 8
49. C.J.S. Clarke A. Krolak (1985) *J. Geom. Phys.* **2** 127
50. B.C. Nolan (1999) *Phys. Rev. D* **60** 02014
51. P. Yodzis H.J. Seifert H.M. zum Hagen (1973) *Comm. Math. Phys.* **34** 135
52. A. Papapetrou A. Hamoui (1967) *Ann. Inst. Henr Poincare* **VI** 343