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Quantum discord is not extremalized by Gaussian states

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Abstract
Quantum discord is an important measure of quantum correlations that goes beyond the paradigm of quantum entanglement. However, calculating quantum discord involves optimization over measurements, which is computationally challenging and often infeasible. This raises the intriguing question of Gaussian extremality—whether the quantum discord of a reference Gaussian state can provide a meaningful bound to the quantum discord of the original state. In this paper, we investigate this question by comparing the Gaussian discord of a reference Gaussian state with the quantum discord.

Keywords Quantum discord · Gaussian extremality · Continuous variable quantum system

1 Introduction

As Bell has first shown, quantum correlations can exhibit stark differences from classical correlations [1–3]. For pure states, correlations and entanglement are synonymous and are captured by the entropy of entanglement. However, for mixed states, the situation is different: An unentangled state may exhibit finite correlations and furthermore demonstrate nonclassical—i.e., quantum correlations [4, 5]. Quantum discord (QD) is perhaps the most prominent example of correlations of this kind [6, 7]. It is defined as the difference between two classically equivalent expressions for the mutual information and involves a minimization over measurements. While QD is equal to quantum entanglement for pure states, it can be nonzero even for separable mixed states.

The QD is notoriously difficult to calculate due to the required minimization, even when the full density matrix is available. In fact, it is known to be an NP-complete problem [8]. Consequently, closed-form analytical expressions are rare and studies

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on QD primarily focus on a limited class of finite-dimensional systems [9–17]. The challenge is even greater for continuous variable (CV) systems. Numerical optimization is generally impractical for such systems, making it necessary to determine the optimal measurement analytically. Remarkably, this has been accomplished for CV Werner states [18] and a specific class of Gaussian states [19]. In the latter case, the QD is equal to the Gaussian discord (GD), in which the measurement is restricted to Gaussian measurements (and therefore provides an upper bound to the QD) [20, 21]. Computing the QD of states lying outside these classes of CV states remains extremely difficult. One must therefore resort to upper and lower bounds to QD for almost all non-Gaussian CV states. An upper bound can be obtained by restricting the set of measurements, as done in GD, but obtaining a lower bound remains challenging. In this regard, it is interesting to ask whether Gaussian extremality holds for QD.

A measure $\mathcal{M}$ is said to be extremalized by a Gaussian state [22] if either $\mathcal{M}(\rho) \geq \mathcal{M}(\rho_{\mathrm{G}})$ or $\mathcal{M}(\rho) \leq \mathcal{M}(\rho_{\mathrm{G}})$ holds for all states. Here, ρ_{G} is the reference Gaussian state (RGS) whose first- and second-order moments are equal to those of ρ. It was first introduced in Ref. [22], where it was shown that quantum entanglement measures satisfying certain criteria are minimized by Gaussian states when the first- and second-order moments are given. Distillable entanglement and squashed entanglement fall into this class, while logarithmic negativity does not. It was also shown that quantum mutual information, which quantifies the total (both classical and quantum) correlation, does not obey Gaussian extremality [23, 24].

Our primary question is whether the RGS has the minimum QD among the quantum states having the same first- and second-order moments. If it does, we can easily determine whether the QD of a given state is nonzero by calculating the GD of its RGS, as a nonzero GD guarantees a nonzero QD for Gaussian states [25, 26]. We show that this is not the case by providing explicit counterexamples. The rest of this work is organized as follows. In Sect. 2, we first introduce the necessary backgrounds. We then calculate the QD for two types of non-Gaussian states—specifically, a mixture of Bell states in the cat basis and a photon number entangled state—and the GD for their RGSs in Sect. 3. Our results show that no definite ordering exists between them. Pertinent implications of our results are discussed in Sect. 4 before we conclude in Sect. 5.

2 Background

2.1 Quantum discord

The QD is defined as

$$D(\rho_{AB}) = I(\rho_{AB}) - J_B(\rho_{AB}), \tag{1}$$

where $I(\rho_{AB})$ is the quantum mutual information that quantifies the total correlation in ρ_{AB}, while $J_B(\rho_{AB})$ quantifies the classical part of the correlation [7]. The quantum mutual information is defined as

$$I(\rho_{AB}) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}), \tag{2}$$

where $S(\rho) = -\text{Tr}[\rho \ln \rho]$ is the von Neumann entropy and ρ_A (ρ_B) is the reduced density matrix of the subsystem A (B). The classical correlation is defined as

$$J_B(\rho_{AB}) = S(\rho_A) - \min_{\{\Pi_j^B\}} \sum_j f_j S(\rho_{A|j}), \tag{3}$$

which involves a minimization over all sets of measurements $\{\Pi_j^B\}$ on the subsystem B. Given a measurement, f_j and $\rho_{A|j}$ are the probability of the jth measurement outcome and the post-measurement state of the subsystem A, respectively. The two quantities $I(\rho_{AB})$ and $J_B(\rho_{AB})$ are identical in the classical domain; therefore, the difference between them measures the total quantum correlations in ρ_{AB}.

If the measurement is performed on subsystem A instead of B, the definition of QD changes accordingly and is not equivalent to $D(\rho_{AB})$—that is, the QD is not symmetric. However, in this work, we focus exclusively on permutation-invariant quantum states, where $\rho_{AB} = \rho_{BA}$. As a result, the two types of QD are identical for all examples discussed in this study.

2.2 Gaussian discord

An upper bound for the QD can be obtained if the minimization is carried out over a subset of all possible measurements. If this subset is chosen to be Gaussian measurements, the resulting quantity is called the GD [20, 21]. It has been conjectured that for Gaussian states, the GD is equal to the QD [27]. Gaussian states are quantum states whose phase space distributions are Gaussian [28]. They are completely characterized by their first- and second-order moments and are thus mathematically more tractable compared to other quantum states. Given the mean values of the quadrature operators (first-order moments) $\bar{x}_j = \langle \hat{x}_j \rangle$ and $\bar{p}_j = \langle \hat{p}_j \rangle$, the covariance matrix (CM; second-order moments) is defined as

$$(\Sigma_{AB})_{jk} = \text{Tr}[\rho[\hat{R}_j, \hat{R}_k]_+], \tag{4}$$

where $\hat{R} = (\hat{x}_A - \bar{x}_A, \hat{p}_A - \bar{p}_A, \hat{x}_B - \bar{x}_B, \hat{p}_B - \bar{p}_B)$ and $[\hat{R}_j, \hat{R}_k]_+ = (\hat{R}_j \hat{R}_k + \hat{R}_k \hat{R}_j)/2$. The quadrature operators are defined as $\hat{x}_j = \hat{a}_j + \hat{a}_j^\dagger$ and $\hat{p}_j = (\hat{a}_j - \hat{a}_j^\dagger)/i$ with $j \in \{A, B\}$.

The CM of a bipartite quantum state can be represented in a block diagonal form

$$\Sigma_{AB} = \begin{pmatrix} \Sigma_A & \Gamma_{AB} \\ \Gamma_{AB}^T & \Sigma_B \end{pmatrix}, \tag{5}$$

where Σ_A and Σ_B are the local CMs for subsystems A and B, respectively, and Γ_{AB} quantifies the quadrature correlations between the subsystems. Let $\mathcal{A}$, $\mathcal{B}$, $\mathcal{C}$ and $\mathcal{D}$ be the determinants of Σ_A, Σ_B, Γ_{AB} and Σ_{AB}, respectively. The symplectic eigenvalues of the CM are then given by $\nu_\pm = \sqrt{\frac{\Delta \pm \sqrt{\Delta^2 - 4\mathcal{D}}}{2}}$ with $\Delta = \mathcal{A} + \mathcal{B} + 2\mathcal{C}$. Using these, the GD of a Gaussian state ρ_G can be compactly written as [21]

$$D_G(\rho_G) = h(\sqrt{\mathcal{B}}) - h(\nu_+) - h(\nu_-) + \inf_{\sigma_0} h(\sqrt{\det \epsilon}), \tag{6}$$

where $h(x) = (\frac{x+1}{2}) \ln[\frac{x+1}{2}] - (\frac{x-1}{2}) \ln[\frac{x-1}{2}]$ and $\epsilon = \Sigma_A - \Gamma_{AB}(\Sigma_B + \sigma_0)^{-1}\Gamma_{AB}^T$. The last expression, $\inf_{\sigma_0} h(\sqrt{\det \epsilon})$, requires minimization over all pure single-mode Gaussian states σ_0, which can be evaluated as

$$\inf_{\sigma_0} \det \epsilon = \left[\frac{|\mathcal{C}| + \sqrt{\mathcal{C}^2 + (\mathcal{B} - 1)(\mathcal{D} - \mathcal{A})}}{\mathcal{B} - 1}\right]^2, \tag{7}$$

if $(\mathcal{D} - \mathcal{A}\mathcal{B})^2 \leq (1 + \mathcal{B})\mathcal{C}^2(\mathcal{A} + \mathcal{D})$ and

$$\inf_{\sigma_0} \det \epsilon = \frac{\mathcal{A}\mathcal{B} - \mathcal{C}^2 + \mathcal{D} - \sqrt{(\mathcal{A}\mathcal{B} - \mathcal{C}^2 + \mathcal{D})^2 - 4\mathcal{A}\mathcal{B}\mathcal{D}}}{2\mathcal{B}}, \tag{8}$$

otherwise.

2.3 Continuous variable Bell states

Our main objective is to demonstrate that QD is not extremalized by Gaussian states in CV systems. To achieve this, we must first be able to calculate the QD for CV states. Given that this is typically a challenging problem, we mainly focus on a specific example: the Bell diagonal states constructed in the cat basis. These states behave like qubit states, for which the QD can be determined easily.

There are two types of cat states called even and odd cat states:

$$|+\rangle \equiv \frac{1}{\sqrt{2 + 2e^{-2|\gamma|^2}}} \left(|\gamma\rangle + |-\gamma\rangle\right), \tag{9}$$

$$|-\rangle \equiv \frac{1}{\sqrt{2 - 2e^{-2|\gamma|^2}}} \left(|\gamma\rangle - |-\gamma\rangle\right), \tag{10}$$

where $|\gamma\rangle = e^{-\frac{1}{2}|\gamma|^2} \sum_{n=0}^{\infty} \frac{\gamma^n}{\sqrt{n!}} |n\rangle$ is a coherent state with amplitude γ. The two states are orthogonal and can be used to construct the Bell states as follows:

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle|-\rangle \pm |-\rangle|+\rangle), \tag{11}$$

$$|\Phi_\pm\rangle = \frac{1}{\sqrt{2}}(|+\rangle|+\rangle \pm |-\rangle|-\rangle). \tag{12}$$

We will form mixtures of two Bell states with probabilities p and $1 - \mathsf{p}$. The QD of such rank two Bell diagonal states, ρ_{BD}, can be readily worked out [11] and is given by a simple formula

$$D(\rho_{\mathrm{BD}}) = \ln 2 + \mathsf{p} \ln \mathsf{p} + (1 - \mathsf{p}) \ln(1 - \mathsf{p}). \tag{13}$$

Note the state independence—the QD of rank two Bell diagonal states—only depends on the probability p.

3 Results

To prove that Gaussian extremality does not apply to the QD, we compare the QD, $D(\rho)$, with the GD of the RGS, $D_{\mathrm{G}}(\rho_{\mathrm{G}})$. Because GD provides an upper bound to QD, $D(\rho) > D_{\mathrm{G}}(\rho_{\mathrm{G}})$ implies $D(\rho) > D(\rho_{\mathrm{G}})$. Our examples below demonstrate that while $D(\rho) > D_{\mathrm{G}}(\rho_{\mathrm{G}})$ holds for most states, there are some states for which $D(\rho) < D_{\mathrm{G}}(\rho_{\mathrm{G}})$. Note that this condition does not necessarily guarantee $D(\rho) < D(\rho_{\mathrm{G}})$ and cannot be used to conclude that Gaussian extremality does not apply to QD. However, we show that there are cases in which $D(\rho) = 0$ while $D_{\mathrm{G}}(\rho_{\mathrm{G}}) > 0$. This indicates violation of Gaussian extremality for QD because nonzero GD implies nonzero QD for Gaussian states [25, 26].

In the first part of this section, we use the CV Bell states introduced earlier, assuming real γ for simplicity. In the second part, we use a particular superposition of vacuum and two-mode squeezed vacuum (TMSV) states to show that Gaussian extremality does not apply to pure states either.

3.1 Bell states in cat basis

The CMs of the Bell states in the cat basis, with real γ, are given by

$$\Sigma_{AB} = \begin{pmatrix} 1+\alpha_0 & 0 & r\alpha_0 & 0 \\ 0 & 1+\alpha_1 & 0 & s\alpha_1 \\ r\alpha_0 & 0 & 1+\alpha_0 & 0 \\ 0 & s\alpha_1 & 0 & 1+\alpha_1 \end{pmatrix}, \tag{14}$$

where $\alpha_j = 2\gamma^2\{\coth 2\gamma^2 + (-1)^j\}$ and

$$r = \begin{cases} +1 & \text{for } |\Psi_+\rangle \text{ and } |\Phi_+\rangle, \\ -1 & \text{for } |\Psi_-\rangle \text{ and } |\Phi_-\rangle, \end{cases} \tag{15}$$

$$s = \begin{cases} +1 & \text{for } |\Psi_+\rangle \text{ and } |\Phi_-\rangle, \\ -1 & \text{for } |\Psi_-\rangle \text{ and } |\Phi_+\rangle. \end{cases} \tag{16}$$

For pure Bell states $|\Psi_\pm\rangle$ and $|\Phi_\pm\rangle$, $D(\rho) > D_{\mathrm{G}}(\rho_{\mathrm{G}})$ holds in all cases as shown in Fig. 1. Note that the only difference between the CMs of $|\Psi_\pm\rangle$ and $|\Phi_\pm\rangle$ is the signs of the correlations r and s. Consequently, the differences in the GDs of the RGSs arise solely from the determinant of the correlation matrix, Det $[\Gamma_{AB}]$, which means that $D_{\mathrm{G}}(|\Psi_+\rangle\langle\Psi_+|_{\mathrm{G}}) = D_{\mathrm{G}}(|\Psi_-\rangle\langle\Psi_-|_{\mathrm{G}})$ and $D_{\mathrm{G}}(|\Phi_+\rangle\langle\Phi_+|_{\mathrm{G}}) = D_{\mathrm{G}}(|\Phi_-\rangle\langle\Phi_-|_{\mathrm{G}})$. The GDs of the two types of RGSs converge as $\gamma \to \infty$ because Det $[\Gamma_{AB}] \to 0$ in

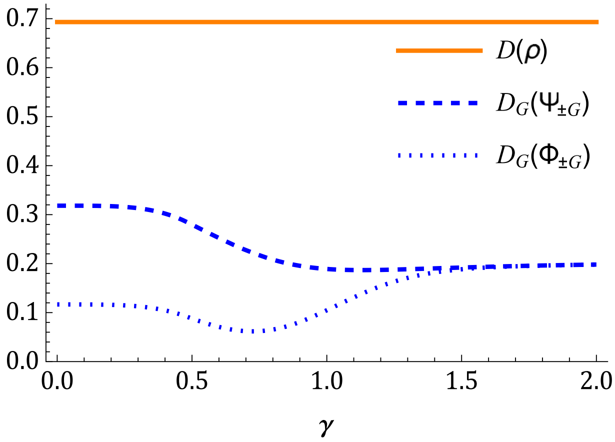

Fig. 1 Discord measures $D(\rho)$ (orange solid curve) and $D_{\mathrm{G}}(\rho_{\mathrm{G}})$ (blue dashed or dotted curve) for $|\Psi_{\pm}\rangle\langle\Psi_{\pm}|$ and $|\Phi_{\pm}\rangle\langle\Phi_{\pm}|$ plotted as functions of γ. In the legend, $\Psi_{\pm\mathrm{G}}$ and $\Phi_{\pm\mathrm{G}}$ denote reference Gaussian states for $|\Psi_{\pm}\rangle$ and $|\Phi_{\pm}\rangle$, respectively (Color figure online)

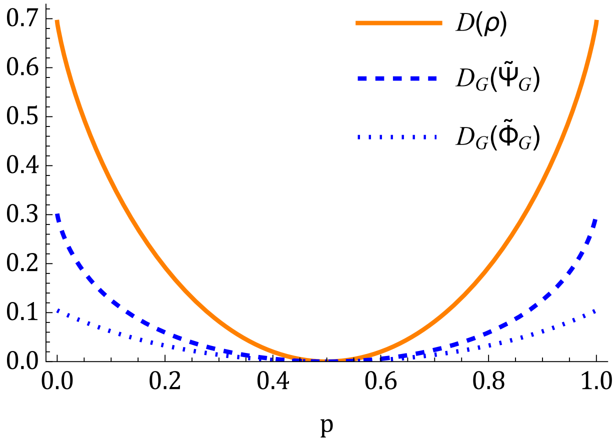

Fig. 2 Discord measures $D(\rho)$ (orange solid curve) and $D_{\mathrm{G}}(\rho_{\mathrm{G}})$ (blue dashed or dotted curve) as functions of probability p for $\tilde{\Psi}$ and $\tilde{\Phi}$ with $\gamma = 0.4$ (Color figure online)

this limit. In the opposite limit of $\gamma \to 0$, $|\Phi_{\pm}\rangle$ become superpositions of $|0\rangle|0\rangle$ and $|1\rangle|1\rangle$ while $|\Psi_{\pm}\rangle$ become superpositions of $|1\rangle|0\rangle$ and $|0\rangle|1\rangle$. The determinants of the correlation matrices for the two types of states are distinct, leading to the difference in their GDs.

Clearly, we need to consider mixtures of these states if we are to observe the violation of Gaussian extremality at all. Considering only rank two mixed states for simplicity, there are six types of mixtures. Of these, two exhibit similar behavior to Fig. 1, i.e., $D(\rho) \geq D_{\mathrm{G}}(\rho_{\mathrm{G}})$, as depicted in Fig. 2. The two types of states are

$$\widetilde{\Psi}(\mathsf{p}) = \mathsf{p}|\Psi_{+}\rangle\langle\Psi_{+}| + (1-\mathsf{p})|\Psi_{-}\rangle\langle\Psi_{-}|, \tag{17}$$

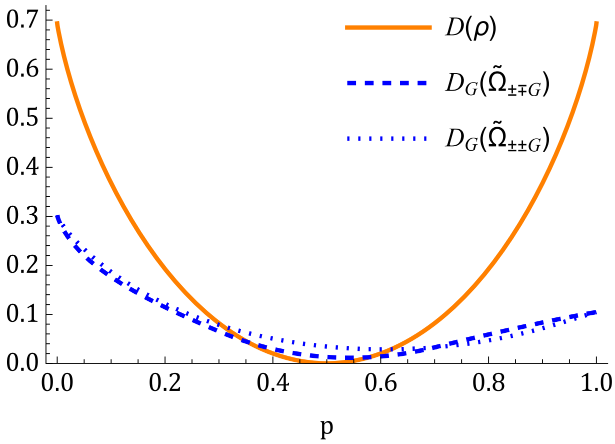

Fig. 3 Discord measures $D(\rho)$ (orange solid curve) and $D_{\rm G}(\rho_{\rm G})$ (blue dashed or dotted curve) as functions of p for $\widetilde{\Omega}_{\pm\pm}$ and $\widetilde{\Omega}_{\pm\mp}$ with $\gamma = 0.4$ (Color figure online)

$$\widetilde{\Phi}(\mathsf{p}) = \mathsf{p}|\Phi_+\rangle\langle\Phi_+| + (1-\mathsf{p})|\Phi_-\rangle\langle\Phi_-|. \tag{18}$$

Note that $D(\rho) = D_{\rm G}(\rho_{\rm G}) = 0$ at $\mathsf{p} = \frac{1}{2}$, as the two types of states are classically correlated and exhibit no quadrature correlations (i.e., Γ_{AB} is a zero matrix) at this point. Explicitly, the states can be expressed as:

$$\widetilde{\Psi}\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(|+\rangle\langle+| \otimes |-\rangle\langle-| + |-\rangle\langle-| \otimes |+\rangle\langle+|\right), \tag{19}$$

$$\widetilde{\Phi}\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(|+\rangle\langle+| \otimes |+\rangle\langle+| + |-\rangle\langle-| \otimes |-\rangle\langle-|\right). \tag{20}$$

The remaining types of rank two mixed states are

$$\widetilde{\Omega}_{\pm\pm}(\mathsf{p}) = \mathsf{p}|\Phi_\pm\rangle\langle\Phi_\pm| + (1-\mathsf{p})|\Psi_\pm\rangle\langle\Psi_\pm|, \tag{21}$$

$$\widetilde{\Omega}_{\pm\mp}(\mathsf{p}) = \mathsf{p}|\Phi_\mp\rangle\langle\Phi_\mp| + (1-\mathsf{p})|\Psi_\pm\rangle\langle\Psi_\pm|, \tag{22}$$

whose discord measures are illustrated in Fig. 3. Note first that $D(\rho) = 0$ at $\mathsf{p} = \frac{1}{2}$, which can be readily checked by rewriting the states as

$$\widetilde{\Omega}_{\pm\pm}\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(|+\rangle_x\langle+| \otimes |\pm\rangle_x\langle\pm| + |-\rangle_x\langle-| \otimes |\mp\rangle_x\langle\mp|\right), \tag{23}$$

$$\widetilde{\Omega}_{\pm\mp}\left(\tfrac{1}{2}\right) = \frac{1}{2}\left(|+\rangle_y\langle+| \otimes |\pm\rangle_y\langle\pm| + |-\rangle_y\langle-| \otimes |\mp\rangle_y\langle\mp|\right), \tag{24}$$

where $|\pm\rangle_j$ are the eigenstates of the Pauli matrix σ_j with $j \in \{x, y\}$. On the contrary, the GDs (of the RGSs) exhibit nonzero values across all ranges of p. Since a nonzero GD implies nonzero QD for Gaussian states [25], these results indicate that, at least at this point, $D(\rho_{\rm G}) > D(\rho)$. This is enough to demonstrate that the Gaussian extremality does not hold for QD.

3.2 Superposition of vacuum and two-mode squeezed vacuum states

By using mixed states constructed from the CV Bell states, we have demonstrated that the QD of the RGS may be either larger or smaller than the QD of the original state, thereby proving that Gaussian extremality does not hold for QD. In this section, we demonstrate that the same conclusion can be reached using pure states only. To this end, we use the following photon number entangled state:

$$|\psi(g,\lambda)\rangle = g|0\rangle|0\rangle + \sqrt{(1-g^2)(1-\lambda^2)}\sum_{n=1}^{\infty}\lambda^{n-1}|n\rangle|n\rangle, \tag{25}$$

with $0 \leq \lambda, g \leq 1$. This state can be rewritten as a superposition of vacuum and TMSV states for $\lambda \neq 0$ as follows:

$$|\psi(g,\lambda)\rangle = \frac{g\lambda - \sqrt{(1-g^2)(1-\lambda^2)}}{\lambda}|0\rangle|0\rangle + \frac{\sqrt{1-g^2}}{\lambda}|\mathrm{TMSV}\rangle, \tag{26}$$

where the TMSV state is given by

$$|\mathrm{TMSV}\rangle = \sqrt{1-\lambda^2}\sum_{n=0}^{\infty}\lambda^n|n\rangle|n\rangle. \tag{27}$$

Depending on the values of g and λ, this state exhibits the following characteristics:

- It represents a TMSV state when $g = \sqrt{1-\lambda^2}$.
- It becomes separable for two cases: (1) $g = 1$, and (2) $g = 0$ and $\lambda = 0$.
- It manifests as an entangled qubit state when $0 < g < 1$ and $\lambda = 0$.

The QD of $|\psi(g,\lambda)\rangle$, which is equal to its entanglement entropy, is given by

$$D(\rho) = -\left\{\frac{\lambda^2}{1-\lambda^2}\ln\lambda^2 + \ln[(1-g^2)(1-\lambda^2)]\right\}(1-g^2) - g^2\ln g^2, \tag{28}$$

and its CM is

$$\Sigma_{AB} = \begin{pmatrix} a & 0 & c & 0 \\ 0 & a & 0 & -c \\ c & 0 & a & 0 \\ 0 & -c & 0 & a \end{pmatrix}, \tag{29}$$

where $a = 1 + 2(1-g^2)/(1-\lambda^2)$ and $c = 2g\sqrt{(1-g^2)(1-\lambda^2)} + 2\lambda(1-g^2)(2-\lambda^2)/(1-\lambda^2)$. Due to the simple structure of the CM, the GD of the RGS can be compactly written as

$$D_{\mathrm{G}}(\rho_{\mathrm{G}}) = h(a) + h\left(a - \frac{c^2}{a+1}\right) - 2h\left(\sqrt{a^2-c^2}\right). \tag{30}$$

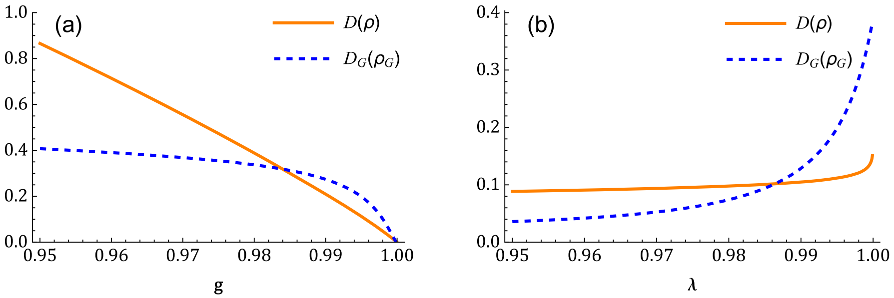

Fig. 4 Discord measures $D(\rho)$ (orange solid curve) and $D_{\text{G}}(\rho_{\text{G}})$ (blue dashed curve) for $|\psi(g, \lambda)\rangle$ **a** as a function of g ($\lambda = 0.995$) and **b** as a function of λ ($g = 0.995$) (Color figure online)

Note that the above RGS is mixed in general, which makes calculating its QD nontrivial in principle. However, it turns out that for Gaussian states with the above CM, i.e., two-mode squeezed thermal states, the GD is equivalent to the QD [19]. Consequently, Fig. 4 clearly demonstrates that Gaussian extremality is violated for pure states as well.

4 Discussion

Our counterexamples have several implications for the properties of QD. In this section, we elaborate on them in detail.

4.1 $D(\rho) = 0 \nRightarrow D(\rho_{\text{G}}) = 0$

For certain measures of correlation, such as several measures of entanglement and quantum mutual information, the vanishing correlation of a quantum state implies vanishing correlation for its RGS. That is, for a measure $\mathcal{M}$, $\mathcal{M}(\rho) = 0 \Rightarrow \mathcal{M}(\rho_{\text{G}}) = 0$. In the case of a quantum entanglement measure E, this holds if the measure obeys Gaussian extremality, i.e., $E(\rho) \geq E(\rho_{\text{G}})$ [22] and nonnegativity, i.e., $E(\rho) \geq 0$. In the case of quantum mutual information I, which fails to satisfy Gaussian extremality, $I(\rho) = 0$ implies $I(\rho_{\text{G}}) = 0$ because (i) $I(\rho) = 0$ only for product states and (ii) the RGS of a product state $\rho = \rho_A \otimes \rho_B$ is a product state $\rho_{\text{G}} = \rho_{A,\text{G}} \otimes \rho_{B,\text{G}}$. On the contrary, our result explicitly demonstrates that $D(\rho) = 0$ is not a sufficient condition for $D(\rho_{\text{G}}) = 0$. Notably, since $D(\rho_{\text{G}})$ becomes zero only when ρ_{G} is a product state [25, 26], one observes this phenomenon for all classically correlated states whose RGS has nonzero quantum mutual information.

4.2 On superactivation of quantum discord

In Ref. [22], it was shown that, for given first- and second-order moments, a correlation measure $\mathcal{M}$ is minimized by Gaussian states, i.e., $\mathcal{M}(\rho) \geq \mathcal{M}(\rho_{\text{G}})$ if it is:

1) continuous, 2) invariant under local unitary operations and 3) strongly superadditive, i.e., $\mathcal{M}(\rho_{A_1B_1A_2B_2}) \geq \mathcal{M}(\rho_{A_1B_1}) + \mathcal{M}(\rho_{A_2B_2})$ [22]. If the third condition is changed to strong subadditivity, i.e., $\mathcal{M}(\rho_{A_1B_1A_2B_2}) \leq \mathcal{M}(\rho_{A_1B_1}) + \mathcal{M}(\rho_{A_2B_2})$, the measure is maximized by Gaussian states, i.e., $\mathcal{M}(\rho) \leq \mathcal{M}(\rho_{\mathrm{G}})$. Since QD satisfies the conditions 1 and 2 [4], our results indicate that the QD is neither strongly superadditive nor strongly subadditive. However, it is known to be subadditive: $D(\rho_{A_1B_1} \otimes \rho_{A_2B_2}) \leq D(\rho_{A_1B_1}) + D(\rho_{A_2B_2})$ [29].

The subadditivity of QD, along with the fact that it is nonnegative, implies that QD cannot be superactivated [30]. In other words, it is impossible to generate QD by preparing multiple copies of a quantum state with zero discord: $D(\rho) = 0 \Rightarrow D(\rho^{\otimes m}) = 0$ for any positive integer m. Now, the Gaussification procedure in [22] can be described as $\widetilde{\rho}_1 = \lim_{m\to\infty} \mathrm{tr}_{2\cdots m}[\hat{U}\rho^{\otimes m}\hat{U}^\dagger] = \rho_{\mathrm{G}}$ where $\hat{U}$ is a product of local unitary operators $\hat{U} = \hat{U}_{A_1A_2\cdots A_m} \otimes \hat{U}_{B_1B_2\cdots B_m}$. Combining the no-go result for superactivation and the local unitary invariance property, we must have $\lim_{m\to\infty} D(\hat{U}\rho^{\otimes m}\hat{U}^\dagger) = 0$ whenever $D(\rho) = 0$ is satisfied. However, we have shown that equal mixtures of certain Bell states have $D(\rho) = 0$ while $D(\rho_{\mathrm{G}}) > 0$. The implication is that while QD cannot be superactivated, it can be *activated* from multiple copies of a classically correlated state by the help of a local unitary operation and partial trace.

4.3 Bounds on the difference between the quantum discords of ρ and ρ_{G}

We have demonstrated that there is no definite ordering between $D(\rho)$ and $D(\rho_{\mathrm{G}})$. Does this mean that their difference can take on any value? Perhaps unsurprisingly, it is bounded both from below and above. To see this, note first that the QD is upper bounded by the local von Neumann entropy, i.e., $D(\rho_{AB}) \leq S(\rho_B)$ [4]. A trivial lower bound can also be obtained as $L(\rho_{AB}) = \max[0, S(\rho_B) - S(\rho_{AB})]$ from the definition of QD. Combining these, we obtain $L(\rho_{AB}) \leq D(\rho_{AB}) \leq S(\rho_B)$ and $L(\rho_{AB,\mathrm{G}}) \leq D(\rho_{AB,\mathrm{G}}) \leq S(\rho_{B,\mathrm{G}})$. Subtracting the second from the first gives

$$L(\rho_{AB}) - S(\rho_{B,\mathrm{G}}) \leq D(\rho_{AB}) - D(\rho_{AB,\mathrm{G}}) \tag{31}$$

for the lower bound and

$$D(\rho_{AB}) - D(\rho_{AB,\mathrm{G}}) \leq S(\rho_B) - L(\rho_{AB,\mathrm{G}}) \tag{32}$$

for the upper bound. The lower bound $L(\rho_{AB}) - S(\rho_{B,\mathrm{G}})$ is always nonpositive due to the nonnegativity and Gaussian extremality of the von Neumann entropy, specifically $S(\rho) \geq 0$ and $S(\rho) \leq S(\rho_{\mathrm{G}})$ [22]. In contrast, the upper bound $S(\rho_B) - L(\rho_{AB,\mathrm{G}})$ is always nonnegative due to the nonnegativity of the von Neumann entropy and the Gaussian extremality of the conditional quantum entropy, i.e., $S(\rho_{AB}) - S(\rho_B) \leq S(\rho_{AB,\mathrm{G}}) - S(\rho_{B,\mathrm{G}})$ [22].

5 Conclusion

In this work, we demonstrated that the QD is not extremalized by Gaussian states. That is, there is no definite ordering between the QD of a bipartite state ρ and that of its RGS ρ_{G}. Our finding suggests that QD-related measures [5] are generally unlikely to be extremalized by Gaussian states. A specific example of this is the measurement-induced disturbance [31], which involves performing measurements on both subsystems in the eigenbasis of their respective reduced density matrices. It is known to be equal to QD for pure states (and to be greater than or equal to QD for mixed states) [32], so our result implies that Gaussian extremality does not hold for measurement-induced disturbance.

Our counterexamples have interesting implications, which are discussed in Sect. 4. We showed that vanishing QD of a bipartite state ρ does not imply vanishing QD of its RGS ρ_{G}. We have also shown that QD cannot be superactivated, but may be activated with the help of local unitary operations and partial trace. Lastly, we derived upper and lower bounds on the difference $D(\rho) - D(\rho_{\mathrm{G}})$. It may be interesting to see whether tighter bounds can be obtained. Another topic worth investigating is Gaussian extremality of stronger quantum correlations beyond quantum entanglement such as quantum steering and nonlocality.

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References

1. Horodecki, R., Horodecki, P., Horodecki, M., Horodecki, K.: Quantum entanglement. Rev. Mod. Phys. **81**, 865 (2009). https://doi.org/10.1103/RevModPhys.81.865
2. Brunner, N., Cavalcanti, D., Pironio, S., Scarani, V., Wehner, S.: Bell nonlocality. Rev. Mod. Phys. **86**, 419 (2014). https://doi.org/10.1103/RevModPhys.86.419
3. Adesso, G., Bromley, T.R., Cianciaruso, M.: Measures and applications of quantum correlations. J. Phys. A: Math. Theor. **49**, 473001 (2016). https://doi.org/10.1088/1751-8113/49/47/473001
4. Modi, K., Brodutch, A., Cable, H., Paterek, T., Vedral, V.: The classical-quantum boundary for correlations: discord and related measures. Rev. Mod. Phys. **84**, 1655 (2012). https://doi.org/10.1103/RevModPhys.84.1655
5. Bera, A., Das, T., Sadhukhan, D., Roy, S.S., SenDe, A., Sen, U.: Quantum discord and its allies: a review of recent progress. Rep. Prog. Phys. **81**, 024001 (2018). https://doi.org/10.1103/RevModPhys.86.419
6. Ollivier, H., Zurek, W.H.: Quantum discord: a measure of the quantumness of correlations. Phys. Rev. Lett. **88**, 017901 (2001). https://doi.org/10.1103/PhysRevLett.88.017901

7. Henderson, L., Vedral, V.: Classical, quantum and total correlations. J. Phys. A: Math. Gen. **34**, 6899 (2001). https://doi.org/10.1088/0305-4470/34/35/315
8. Huang, Y.: Computing quantum discord is NP-complete. New J. Phys. **16**, 033027 (2014). https://doi.org/10.1088/1367-2630/16/3/033027
9. Hamieh, S., Kobes, R., Zaraket, H.: Positive-operator-valued measure optimization of classical correlations. Phys. Rev. A **70**, 052325 (2004). https://doi.org/10.1103/PhysRevA.70.052325
10. Ali, M., Rau, A.R.P., Alber, G.: Quantum discord for two-qubit X states. Phys. Rev. A **81**, 042105 (2010). https://doi.org/10.1103/PhysRevA.81.042105
11. Lang, M.D., Caves, C.M.: Quantum discord and the geometry of bell-diagonal states. Phys. Rev. Lett. **105**, 150501 (2010). https://doi.org/10.1103/PhysRevLett.105.150501
12. Chen, Q., Zhang, C., Yu, S., Yi, X.X., Oh, C.H.: Quantum discord of two-qubit X states. Phys. Rev. A **84**, 042313 (2011). https://doi.org/10.1103/PhysRevA.84.042313
13. Rossignoli, R., Matera, J.M., Canosa, N.: Measurements, quantum discord, and parity in spin-1 systems. Phys. Rev. A **86**, 022104 (2012). https://doi.org/10.1103/PhysRevA.86.022104
14. Huang, Y.: Quantum discord for two-qubit X states: analytical formula with very small worst-case error. Phys. Rev. A **88**, 014302 (2013). https://doi.org/10.1103/PhysRevA.88.014302
15. Yurischev, M.A.: On the quantum discord of general X states. Quantum Inf. Process. **14**, 3399 (2015). https://doi.org/10.1007/s11128-015-1046-5
16. Namkung, M., Chang, J., Shin, J., Kwon, Y.: Revisiting quantum discord for two-qubit X states: the error bound to an analytical formula. Int. J. Theor. Phys. **54**, 3340 (2015). https://doi.org/10.1007/s10773-015-2573-7
17. Chanda, T., Pal, A.K., Biswas, A., SenDe, A., Sen, U.: Freezing of quantum correlations under local decoherence. Phys. Rev. A **91**, 062119 (2015). https://doi.org/10.1103/PhysRevA.91.062119
18. Tatham, R., Mišta, L., Jr., Adesso, G., Korolkova, N.: Nonclassical correlations in continuous-variable non-Gaussian Werner states. Phys. Rev. A **85**, 022326 (2012). https://doi.org/10.1103/PhysRevA.85.022326
19. Pirandola, S., Spedalieri, G., Braunstein, S.L., Cerf, N.J., Lloyd, S.: Optimality of Gaussian discord. Phys. Rev. Lett. **113**, 140405 (2014). https://doi.org/10.1103/PhysRevLett.113.140405
20. Giorda, P., Paris, M.G.A.: Gaussian quantum discord. Phys. Rev. Lett. **105**, 020503 (2010). https://doi.org/10.1103/PhysRevLett.105.020503
21. Adesso, G., Datta, A.: Quantum versus classical correlations in Gaussian states. Phys. Rev. Lett. **105**, 030501 (2010). https://doi.org/10.1103/PhysRevLett.105.030501
22. Wolf, M.M., Giedke, G., Cirac, J.I.: Extremality of Gaussian quantum states. Phys. Rev. Lett. **96**, 080502 (2006). https://doi.org/10.1103/PhysRevLett.96.080502
23. Park, J., Lee, J., Ji, S.-W., Nha, H.: Quantifying non-Gaussianity of quantum-state correlation. Phys. Rev. A **96**, 052324 (2017). https://doi.org/10.1103/PhysRevA.96.052324
24. Park, J.: Estimating quantum mutual information of continuous-variable quantum states by measuring purity and covariance matrix. Entropy **24**, 940 (2022). https://doi.org/10.3390/e24070940
25. Rahimi-Keshari, S., Caves, C.M., Ralph, T.C.: Measurement-based method for verifying quantum discord. Phys. Rev. A **87**, 012119 (2013). https://doi.org/10.1103/PhysRevA.87.012119
26. Mišta, L., Jr., McNulty, D., Adesso, G.: No-activation theorem for Gaussian nonclassical correlations by Gaussian operations. Phys. Rev. A **90**, 022328 (2014). https://doi.org/10.1103/PhysRevA.90.022328
27. Giorda, P., Allegra, M., Paris, M.G.A.: Quantum discord for Gaussian states with non-Gaussian measurements. Phys. Rev. A **86**, 052328 (2012). https://doi.org/10.1103/PhysRevA.86.052328
28. Weedbrook, C., Pirandola, S., García-Patrón, R., Cerf, N.J., Ralph, T.C., Shapiro, J.H., Lloyd, S.: Gaussian quantum information. Rev. Mod. Phys. **84**, 621 (2012). https://doi.org/10.1103/RevModPhys.84.621
29. Madhok, V., Datta, A.: Interpreting quantum discord through quantum state merging. Phys. Rev. A **83**, 032323 (2011). https://doi.org/10.1103/PhysRevA.83.032323
30. Palazuelos, C.: Superactivation of quantum nonlocality. Phys. Rev. Lett. **109**, 190401 (2012). https://doi.org/10.1103/PhysRevLett.109.190401
31. Luo, S.: Using measurement-induced disturbance to characterize correlations as classical or quantum. Phys. Rev. A **77**, 022301 (2008). https://doi.org/10.1103/PhysRevA.77.022301
32. Wu, S., Poulsen, U.V., Mølmer, K.: Correlations in local measurements on a quantum state, and complementarity as an explanation of nonclassicality. Phys. Rev. A **80**, 032319 (2009). https://doi.org/10.1103/PhysRevA.80.032319