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The role of density inhomogeneity and anisotropy in the final outcome of dissipative collapse

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Abstract In this work, we employ the "horizon" function introduced by Ivanov (Int J Mod Phys D 25:1650049, 2016b) to study radiating stellar models with a generalized Vaidya exterior. Since the star is dissipating energy in the form of a radial heat flux, the radial pressure at the boundary is non-vanishing. We study the boundary condition which encodes the temporal behaviour of the model. Utilizing a scheme developed by Ivanov, we provide several solutions to the modified junction condition. We show that the presence of strings, allow for the collapse to a black hole or the complete burning of a star.

1 Introduction

Relativistic astrophysics and cosmology has been at the forefront of numerous studies in recent times. We consider radiating stars and more importantly, the end-states of gravitational collapse. Much work has been covered in earlier times by Oppenheimer and Snyder [1], Vaidya [2], Santos [3], Bonnor et al. [4] and Kolassis et al. [5] on non-adiabatic gravitational collapse and the Cosmic Censorship Conjecture (CCC). The studies undertaken by Herrera [6–12] resulted in a vast extension giving a broader scope of non-adiabatic gravitational collapse and black holes. The study carried out by Penrose [13] led to defining the Cosmic Censorship Conjecture to an extent. The Cosmic Censorship Conjecture states that any reasonable matter distribution that undergoes continued gravitational collapse will always form a black hole. It is important to note that there are various factors that can influence the outcome of such collapse. In the works that followed by Joshi [14,15] and Joshi et al. [16,17], it was suggested that exceptions are in the form of naked singularities. Singularity theorems assume the probability of the formation of singularities which are not covered by an event horizon, though are visible to an external observer. Oppenheimer and Snyder [1] in their work, investigated an idealized spherically symmetric fluid distribution, in which it was concluded that the formation of an event horizon preceding to the formation of a black hole containing the singularity which has been predicted by general relativity. In a study carried out by Joshi [15], it was demonstrated that singularities that do not consist of trapped surfaces are not necessarily a characteristic feature of a naked singularity. Thus a characteristic feature of a naked singularity is the existence of future directed timelike curves whose past arises at the singularity. A key finding in this study was the conditions that were established such that the formation of naked singularities could be avoided. In another investigation, Joshi et al. [18] studied the properties of the respective accretion discs in order to establish black holes from naked singularities. In their analysis, it was determined that accretion disc around a naked singularity is more luminous and contains a series of high frequency power law segments in comparison to that of black holes. In a study by Hussain [19], exact solutions for a collapsing null fluid with pressure P, and density ρ related by a polytropic (EoS) of the form $P = k\rho^a$ were determined. A vital result of this study was that all of the reported solutions were in agreement with the CCC. Maharaj *et al.* [20] considered a model to study radiating stars with generalised Vaidya atmospheres. The exterior geometry consists of the generalised Vaidya spacetime and the interior matter distribution is shear-free and undergoing radial heat flow. In this study, the impact of the generalised Vaidya radiating metric

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on the junction conditions on the boundary of a radiating star is demonstrated. The analysis led to establishing a general atmosphere surrounding the star which resulted in pressureless null dust and a string fluid. They went on to show that the the string density has an influence on the fluid pressure at the surface of the star. A key feature of this model is that the common junction conditions for the Vaidya spacetime are retrieved when the string fluid vanishes. Sharma and Das [21] looked at the collapse of a relativistic self-gravitating star with radial heat flux. In this model the collapse proceeds from an initially static configuration by dissipating energy in the form of radial heat flow. In their analysis they found that the anisotropy has an impact on the collapse rate of gravitationally evolving systems. Govender *et al.* [22] considered the collapse of a spherically symmetric star dissipating energy in the form of a radial heat flux. In this model, the interior is matched to the generalised Vaidya spacetime which consists of a two-fluid atmosphere of null radiation and a string fluid. The generalised junction condition for the matter and gravitational variables on the stellar surface is investigated which resulted in determining an exact solution. In conclusion, it was found that the additional null fluid plays a role on the dynamical evolution of the star. Govender et al. [23] utilized the Karmarkar to demonstrate that the dynamical nature of the collapse affects the initial static configuration and the collapse continues non-adiabatically by releasing energy in the form of a radial heat flux to the exterior Vaidya spacetime. The smooth matching of a spherically symmetric, shear-free spacetime to Vaidya's outgoing metric across a four-dimensional time-like hypersurface was achieved by Govender and Thirukkanesh [24]. Naidu et al. [25] employed a perturbative approach on the thermodynamical variables and gravitational potentials in their model in order to determine the collapse of a self-gravitating sphere. In a more recent study, Maharaj and Brassel [26] considered radiating composite stars with electromagnetic fields. In this model, they determined the junction conditions for a general spherically symmetric radiating star with an electromagnetic field across a comoving surface. The exterior atmosphere is given by the generalised Vaidya spacetime and the interior consists of three components namely, null dust, a null string fluid and a charged field. In their analysis, they found a new physical feature appears which is the surface pressure is dependent on the internal charge distribution for generalised Vaidya spacetimes. Ivanov [27] found all possible solutions for geodesic anisotropic spherical collapse with shear and heat radiation. In this study, a physically important function was presented, called the horizon function where it was utilized in investigating geodesic collapse. In the analysis, a Riccati equation was determined and integrated which resulted in numerous solutions being found. A key finding was that the solutions were in the form of a generating function which allowed for previous solutions to be regained. In another study by Ivanov [28], a different approach was utilized when investigating anisotropic spherical collapse with shear and heat radiation. To gain more insight into the collapse, the horizon function was employed. After utilizing the horizon function, it is clear that the horizon function exists in the expressions for most of the stellar characteristics. In the analysis carried out, the main equation which governs the collapse is much simpler with the horizon function, which consists of simple coefficents. This allowed for various classes of star models in this formulation to be established and the corresponding solutions to be determined. In recents works, the importance of the horizon function was also shown in the study for general spherical fluid collapse. This study was undertaken by Ivanov [29], in which the fluid in question, is anisotropic, consists of shear, bulk and shear viscosities as well as it has charge. The energy is radiated via heat flow. The general junction conditions were presented in which one consisted of the horizon function in the form of a Riccati equation. A linear equation was found with a simple solution. The gravitational collapse and Cosmic Censorship was investigated by Wald [30]. In this study, the status of the weak Cosmic Censorship Conjecture was focused on. It was stated that all singularities of gravitational collapse are hidden within black holes, which has been supported by the findings by Wald [30]. Special cases related to the conjecture were presented and in conclusion, the results obtained enhances that naked singularities cannot arise generically. In the works by East [31], the Cosmic Censorship was held to be true in spheroidal collapse of collisionless matter. In this study, a different approach to that of Shapiro and Teukolsky [32] was utilized. A change of coordinates along with a different range of values were used for the semimajor axis and eccentricity of the initial matter distribution. This led to determining that configurations with a larger semimajor axis can produce strong gravitational radiation. A subsequent study that includes the Cosmic Censorship Conjecture, was carried out by Ong [33]. The focus of this study was on spacetime singularities and the Cosmic Censorship Conjecture. Many theorems have been put forward with regards to singularities being at the forefront of general relativity. It is important to note that singularities occur in gravitational collapses that result in the formation of black holes. After a brief analysis, the results based on various examples and counter examples, have presented that under certain circumstances, naked singularities do form, and in other cases, the predictability is lost inside black holes. Hawking [34] engaged in the study of black holes in general relativity, in which it was stated that the singularities that occur in gravitational collapse are not visible from outside but rather they are hidden behind an event horizon. Firstly, to note, a black hole on a "spacelike" surface is said to be a connected component of the region of the surface bounded by the event horizon. A key aspect of black holes, is that with time, black holes may merge together but can never bifurcate. De Oliveira et al. [35] focused on the physical conditions and thermodynamical relations for the collapse of a radiating star. In their treatment, the surface temperature of the collapsing body and the stage of black hole formation was found from a static distribution of matter which gradually collapses. Ivanov [36] persued an investigation into collapse to a black hole. In this model, the centre of focus was based on collapsing shear-free perfect fluid spheres consisting of heat flow. The simplicity of the main junction condition was established and this allowed the transformation into well known differential equations namely, Riccati, Bernoulli and Abel. A generalisation was determined for separable solutions which consisted of collapse to a naked singularity.

This paper is structured as follows. In Sect. 2 we present the Einstein field equations describing the interior spacetime. In Sect. 3 the exterior spacetime and the junction conditions required for the smooth matching of the interior metric to the generalized Vaidya metric is given. Section 4 features the generalized junction conditions. The geodesic case is studied in Sect. 5. In Sect. 6 we determine a special Riccati equation and establish solutions. In Sect. 7 we consider the linear case and comment on the collapse. In Sect. 8 the physical results are presented and some concluding remarks are made in Sect. 9.

2 Interior spacetime and field equations

The line element for the interior geometry of a radiating star for a general spherically symmetric, spacetime can be written as

$$ds^2 = -A^2dt^2 + B^2dr^2 + Y^2(d\theta^2 + \sin^2\theta d\phi^2)\,, \tag{1}$$

where $A = A(t,r)$, $B = B(t,r)$ and $Y = Y(t,r)$ are potentials. The energy momentum tensor for the interior matter distribution is given by

$$\begin{aligned} T_{\alpha\beta} &= (\mu + P_\perp)V_\alpha V_\beta + P_\perp g_{\alpha\beta} + (P_r - P_\perp)\chi_\alpha\chi_\beta \\ &\quad + q_\alpha V_\beta + q_\beta V_\alpha + \epsilon l_\alpha l_\beta\,, \end{aligned} \tag{2}$$

where μ is the energy density, P_r is the radial pressure, $P_\perp$ is the tangential pressure, q^α is the heat flow vector, ϵ is the energy density of the null fluid and the vector l^α is null. The fluid four-velocity $\mathbf{V}$ is comoving and has the form given by

$$V^\alpha = \frac{1}{A}\delta^\alpha_0\,. \tag{3}$$

The heat flow vector is outgoing and spacelike. It is given in the form

$$q^\alpha = (0, q^1, 0, 0)\,, \tag{4}$$

where $q^\alpha V_\alpha = 0$. In addition we have

$$\chi^\alpha\chi_\alpha = 1, \quad \chi^\alpha V_\alpha = 0, \quad l^\alpha = u^\alpha + \chi^\alpha. \tag{5}$$

The expansion scalar and the fluid four acceleration are defined by

$$\Theta = V^\alpha{}_{;\alpha}, \quad a_\alpha = V_{\alpha;\beta}V^\beta, \tag{6}$$

and the expression

$$\sigma_{\alpha\beta} = V_{(\alpha;\beta)} + a_{(\alpha}V_{\beta)} - \frac{1}{3}\Theta(g_{\alpha\beta} + V_\alpha V_\beta), \tag{7}$$

gives the shear tensor. For the line element (1) these kinematical quantities can be written as

$$a_1 = \frac{A'}{A}, \tag{8}$$

$$\Theta = \frac{1}{A}\left(\frac{\dot{B}}{B} + 2\frac{\dot{Y}}{Y}\right), \tag{9}$$

$$\sigma = -\frac{1}{3A}\left(\frac{\dot{B}}{B} - \frac{\dot{Y}}{Y}\right), \tag{10}$$

in terms of the potentials A, B and Y. In the above dots and primes denote differentiation with respect to t and r respectively.

The Einstein field equations with the coupling constant set to unity is given in the form

$$G_{\alpha\beta} = T_{\alpha\beta}, \tag{11}$$

for the metric (1) and the matter distribution (2), can be written as [37]

$$\begin{aligned} \mu &= -\frac{1}{B^2}\left[2\frac{Y''}{Y} + \left(\frac{Y'}{Y}\right)^2 - 2\frac{B'}{B}\frac{Y'}{Y}\right] \\ &\quad + \frac{1}{A^2}\left(2\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y} + \frac{1}{Y^2}, \end{aligned} \tag{12}$$

$$\begin{aligned} P_r &= -\frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y} - \left(2\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] - \frac{1}{Y^2} \\ &\quad + \frac{1}{B^2}\left(2\frac{A'}{A} + \frac{Y'}{Y}\right)\frac{Y'}{Y}, \end{aligned} \tag{13}$$

$$\begin{aligned} P_\perp &= -\frac{1}{A}\left[\frac{\ddot{B}}{B} + \frac{\ddot{Y}}{Y} - \frac{\dot{A}}{A}\left(\frac{\dot{B}}{B} + \frac{\dot{Y}}{Y}\right) + \frac{\dot{B}}{B}\frac{\dot{Y}}{Y}\right] \\ &\quad + \frac{1}{B^2}\left[\frac{A''}{A} + \frac{Y''}{Y} - \frac{A'}{A}\frac{B'}{B} + \left(\frac{A'}{A} - \frac{B'}{B}\right)\frac{Y'}{Y}\right], \end{aligned} \tag{14}$$

$$q = \frac{2}{AB}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right). \tag{15}$$

We utilize the horizon function "H",

$$H = \frac{Y'}{B} + \frac{\dot{Y}}{A}, \tag{16}$$

to transform the field equations.

We present the form for the mass m, as Cahill and McVittie [38]

$$m(r,t) = \frac{Y}{2}\left[1 + \left(\frac{\dot{Y}}{A}\right)^2 - \left(\frac{\dot{Y}}{B}\right)^2\right]. \tag{17}$$

The compactness of the star is defined as

$$\frac{2m}{Y} = 1 - H^2 + \frac{2\dot{Y}}{A}H. \tag{18}$$

3 Inhomogeneous atmosphere

We assume that the atmosphere is composed of null radiation and anisotropic strings and is described by the metric

$$ds^2 = -\left[1 - \frac{2M(v,r)}{\mathscr{R}}\right]dv^2 - 2dvd\mathscr{R} + \mathscr{R}^2(d\theta^2 + \sin^2\theta d\phi^2), \tag{19}$$

where $M(v,\mathscr{R})$ is the mass of the star, while $\mathscr{R}$ is the radial coordinate in the exterior.

In this section, we present the main junction conditions required for the smooth matching of the interior metric (1) and the exterior generalised Vaidya metric (19). For more insight into the matching of the interior and exterior metric, the reader is directed to the works by Maharaj et al. [20].

The matter source for the generalised Vaidya metric is given by

$$T_{\alpha\beta} = \psi l_\alpha l_\beta + \rho\hat{\beta}_\alpha\hat{\beta}_\beta + p_r\hat{r}_\alpha\hat{r}_\beta + p_\perp(\hat{\theta}_\alpha\hat{\theta}_\beta + \hat{\phi}_\alpha\hat{\phi}_\beta), \tag{20}$$

where l_α and l_β are two null vectors, ρ is the energy density, p_r is the radial pressure and $p_\perp$ is the tangential pressure [39].

The Einstein field equations for the metric (19) and energy momentum tensor (20) gives

$$4\pi\psi = \frac{-\dot{m}}{\mathscr{R}^2}, \tag{21}$$

$$4\pi\rho = -4\pi p_r = \frac{m'}{\mathscr{R}^2}, \tag{22}$$

$$8\pi p_\perp = \frac{-m''}{\mathscr{R}}, \tag{23}$$

keeping in mind that the equation of motion $T^{\alpha\beta}_{;\beta} = 0$ is satisfied.

Considering the case for mass transport, we make the assumption that the strings diffuse which leads us to the following result

$$\partial_u n = \mathcal{D}\nabla^2 n, \tag{24}$$

where

$$\nabla^2 = \mathscr{R}^{-2}\left(\frac{\partial}{\partial\mathscr{R}}\right)\mathscr{R}^2\left(\frac{\partial}{\partial\mathscr{R}}\right), \tag{25}$$

and $\mathcal{D}$ is the coefficient of self-diffusion.

We rewrite Eq. (21) to obtain an expression for $\dot{m}$ and Eq. (22) to obtain the expression for m'. Thus we have the following expressions

$$\dot{m} = -4\pi\mathscr{R}^2\psi, \tag{26}$$

$$m' = 4\pi\mathscr{R}^2\rho. \tag{27}$$

Considering that we now have the expressions for $\dot{m}$ and m', we determine the integrability condition for m which is given by

$$\dot{\rho} + \mathscr{R}^{-2}\partial\mathscr{R}(\mathscr{R}^2\psi) = 0. \tag{28}$$

We then consider the diffusion equation (24) in terms of the string density.

$$\dot{\rho} = \mathcal{D}\mathscr{R}^{-2}\frac{\partial}{\partial\mathscr{R}}\left(\mathscr{R}^2\frac{\partial\rho}{\partial\mathscr{R}}\right). \tag{29}$$

Using $\dot{\rho}$ in Eq. (28), we have the following result

$$\dot{m} = 4\pi\mathcal{D}\mathscr{R}^2\frac{\partial\rho}{\partial\mathscr{R}}, \tag{30}$$

where $\mathcal{D}$ is the coefficient of self-diffusion.

After solving the diffusion equation for ρ, we establish the result for m. Glass and Krisch [39] presented a few analytic solutions of (29) which are given by

$$\rho = \rho_0 + \frac{k_1}{\mathscr{R}}, \tag{31}$$

$$\rho = \rho_0 + k_3 u^{-\frac{3}{2}}\exp\left[\frac{-\mathscr{R}^2}{4\mathcal{D}u}\right], \tag{32}$$

$$\rho = \rho_0 + \frac{k_4}{\mathscr{R}}\frac{\sqrt{\pi}}{2}erf\left(\mathscr{R}(4\mathcal{D}u)^{-\frac{1}{2}}\right). \tag{33}$$

4 Junction conditions

When considering astrophysical models, it is vital to highlight the stellar characteristics, for our model, we present the redshift z_Σ given by

$$z_\Sigma = \frac{1}{H_\Sigma} - 1, \tag{34}$$

we also present the surface luminosity Λ_Σ and the luminosity at infinity Λ_∞. These quantities are given as

$$\Lambda_\Sigma = \left(\frac{1}{2}qBY^2\right)_\Sigma, \tag{35}$$

$$\Lambda_\infty = H^2_\Sigma\Lambda_\Sigma. \tag{36}$$

In our model, we utilize the junction condition

$$(P_r)_\Sigma = q_\Sigma - (\rho_s)_\Sigma, \tag{37}$$

where $(\rho_s)_\Sigma = \rho_s(\mathscr{R}_\Sigma, t)$, which indicates that the radial pressure is not vanishing across the boundary of the star.

We will use results obtained by Herrera et al. [12] for radiating spheres in the presence of shear, thus we have

$$\frac{\dot{m}}{A} = -\frac{1}{2}\left(P_r\frac{\dot{Y}}{A} + qB\frac{Y'}{B}\right)Y^2, \tag{38}$$

$$\frac{m'}{Y'} = \frac{1}{2}\left(\mu + qB\frac{B\dot{Y}}{AY'}\right)Y^2. \tag{39}$$

We focus our attention on the stellar boundary, by utilizing the junction condition (37), Eq. (38) becomes,

$$\left(\dot{m} = -\frac{1}{2}\left(qBAH - \rho_s\dot{Y}\right)Y^2\right)_\Sigma. \tag{40}$$

In the case of vanishing string density ($\rho_s = 0$), and radially directed heat flux to the exterior ($q > 0$), we have $\dot{m} < 0$. In the presence of strings, it is possible to have $\dot{m} > 0$, ie: no possibility of collapse. We can think of the strings stabilizing the matter configuration.

The junction conditions are obtained by matching the metric (1) describing the interior spacetime with the Vaidya exterior metric (19). These are given by

$$A(t, r_\Sigma)dt = \left(1 - \frac{2m(v)}{\mathcal{R}_\Sigma} + 2\frac{d\mathscr{R}_\Sigma}{dv}\right)^{\frac{1}{2}} dv, \tag{41}$$

$$Y(t, r_\Sigma) = \mathscr{R}_\Sigma(v). \tag{42}$$

With the help of (13), (15) and (37), we find that the junction condition becomes

$$\begin{aligned}0 &= \frac{2}{AB}\left(\frac{\dot{Y}'}{Y} - \frac{\dot{B}}{B}\frac{Y'}{Y} - \frac{\dot{Y}}{Y}\frac{A'}{A}\right) - \rho_s \\ &+ \frac{1}{A^2}\left[2\frac{\ddot{Y}}{Y}\right] - \frac{1}{A^2}\left[\left(2\frac{\dot{A}}{A} - \frac{\dot{Y}}{Y}\right)\frac{\dot{Y}}{Y}\right] + \frac{1}{Y^2} \\ &- \frac{1}{B^2}\left(2\frac{A'}{A} + \frac{Y'}{Y}\right)\frac{Y'}{Y},\end{aligned} \tag{43}$$

which governs the temporal evolution of the radiating star across the stellar surface Σ. This equation can be rewritten in the form

$$\begin{aligned}\dot{B} &= \left[\frac{\ddot{Y}}{AY'} - \frac{\dot{A}\dot{Y}}{A^2Y'} + \frac{\dot{Y}^2}{2AYY'} + \frac{A}{2YY'} - \frac{AY\rho_s}{2Y'}\right]B^2 \\ &+ \left(\frac{\dot{Y}'}{Y'} - \frac{\dot{Y}}{Y'}\frac{A'}{A}\right)B - \frac{A}{2}\left(\frac{2A'}{A} + \frac{Y'}{Y}\right).\end{aligned} \tag{44}$$

It is important to note that Eq. (44) is first order in B. A key feature of Eq. (44), once setting $\rho_s = 0$, we achieve the classic Santos junction conditions [3].

Following Ivanov, we define $P \equiv Y'/B$ and substitute into (44) to obtain

$$\begin{aligned}\dot{P} &= \left(\frac{A}{2Y} + \frac{A'}{Y'}\right)P^2 + \frac{A'\dot{Y}}{AY'}P \\ &- \frac{Y}{2A}\left(\frac{2\ddot{Y}}{Y} + \frac{\dot{Y}^2}{Y^2} - \frac{2\dot{A}\dot{Y}}{Y} + \frac{A^2}{Y^2} - \frac{YA^2}{Y}\rho_s\right).\end{aligned} \tag{45}$$

The function P can be written in terms of H as

$$P = H - \frac{\dot{Y}}{A}. \tag{46}$$

We find that using Eq. (46) in Eq. (45) results in a form that consists of the horizon function H on the star's surface given by

$$\begin{aligned}\dot{H} &= \left(\frac{A}{2Y} + \frac{A'}{Y'}\right)H^2 - \left(\frac{A}{Y} + \frac{A'}{Y'}\right)\frac{\dot{Y}}{A}H - \frac{A}{2Y} \\ &+ \frac{1}{2}AY\rho_s.\end{aligned} \tag{47}$$

With inspection, we note that Eq. (47) is a first order equation in H. This equation consists of A which is linked to the four-accceleration, B is simply a metric component and Y is the radius.

We can transform (47) by setting $D \equiv YH$, to obtain

$$\dot{D} = \left(\frac{A'}{YY'} + \frac{A}{2Y}\right)D^2 - \frac{A'\dot{Y}}{AY'}D - \frac{1}{2}A + \frac{AY^2\rho_s}{2}. \tag{48}$$

After inspection of Eq. (48), we notice that switching off the ρ_s term (ie: $\rho_s = 0$), we regain the result from a study carried out by Ivanov [28]. We also note that Eq. (48) can be solved, as it is a consequence of the generating function $D \equiv YH$, which will consist of the geodesic cases that allow for a rich set of solutions to be achieved.

Rearranging Eq. (16), we have a form for B

$$B = \frac{Y'}{H - \dot{Y}/A}. \tag{49}$$

After taking a closer look at Eq. (49), we can state that $\dot{Y} < 0$, during the process of collapse, the radius of the star is decreasing. In this study, we require $Y' > 0$, which is a consequence of B being positive. We perform certain operations on Eq. (17) in order to achieve a different form for the mass. We then rearrange Eq. (47) to yield a differential equation at the stellar boundary

$$\left(\frac{A}{Y} + \frac{A'}{Y'}\right)\frac{2m}{Y} = \frac{A'}{Y,}\left(1 + H^2\right) + AY\rho_s - 2\dot{H}. \tag{50}$$

Making use of Eq. (50), numerous solutions for the collapse are determined.

5 Free-falling matter

If A depends on the function of t only, the four-acceleration vanishes. In this case (47) becomes

$$2Y\dot{H} = H^2 - 2\dot{Y}H - 1 + Y^2\rho_s, \tag{51}$$

and Eq. (48) becomes

$$\dot{D} = \frac{1}{2Y^2}D^2 - \frac{1}{2} + \frac{Y^2}{2}\rho_s. \tag{52}$$

We can treat (52) as quadratic in Y. Solving Y in (52) we obtain

$$Y = \frac{\left[(2\dot{D}+1) \pm \sqrt{(2\dot{D}+1)^2 - 4D^2\rho_s}\right]^{1/2}}{[2\rho_s]^{1/2}}, \tag{53}$$

which can also be recast as

$$H = \frac{D\,[2\rho_s]^{1/2}}{\left[(2\dot{D}+1) \pm \sqrt{(2\dot{D}+1)^2 - 4D^2\rho_s}\right]^{1/2}}. \tag{54}$$

The mass function is obtained from (50),

$$m = -Y^2\dot{H} + \frac{Y^3}{2}\rho_s. \tag{55}$$

It is important to note, by switching off ρ_s (by setting $\rho_s = 0$), in the above equation, we obtain the expression for the mass from the investigation conducted by Ivanov [28].

Equation (55) in the absence of string density ($\rho_s = 0$) indicates that the mass evaporates when $\dot{H} = 0$. Equation (49), with $\rho_s = 0$ shows the relation between rate of mass loss and H. When $H = 0$, a black hole forms. However, in the case of nonvanishing string density, equation (55) shows that a black hole is possible if $\rho_s = \frac{2\dot{H}}{Y}$. Thus the presence of strings may prevent complete burning of the star.

Let us now consider (51). Using the string density profile given in (33) where $r_\Sigma = Y_\Sigma$, (51) reduces to a linear equation in Y when we choose $\rho_s = \frac{K}{Y}$. Integrating, we obtain

$$Y = \frac{e^{\frac{K}{2}\int \frac{1}{H}dt}}{2H}\int (H^2-1)e^{-\frac{K}{2}\int \frac{1}{H}dt}dt, \tag{56}$$

where K is a constant.

In order to complete the integration in (56), we choose

$$H(r,t) = c_1(r) - c_2(r)t, \tag{57}$$

which yields

$$Y(r,t) = -\frac{(c_1(r)-c_2(r)t)^2}{6c_2(r)+K} + \frac{1}{2c_2(r)+K} + F(r), \tag{58}$$

$$\begin{aligned}\dot{m} = \frac{1}{(K+6c_2(r))^2}\Big[&2c_2(r)\,(c_1(r)-c_2(r)t)\\ &\times\Big(K\,(1+KF(r)) - c_1(r)^2\,(K+2c_2(r))\\ &+2c_1(r)c_2(r)t(K+2c_2(r))\\ &\times c_2(r)\,(6+8KF(r))\\ &-c_2(r)^2\left(-12F(r)+Kt^2+2t^2c_2(r)\right)\Big)\Big],\end{aligned} \tag{59}$$

where $F(r)$ is an arbitrary constant of integration. As the mass applies throughout the star, then the collapse depends on $c_1(r)$ and $c_2(r)$. We will discuss the solution in detail by choosing forms for $c_1(r)$ and $c_2(r)$ in Sect. 8.

6 Special Riccati equation

Following Ivanov; we choose

$$\left(\frac{A}{Y} + \frac{A'}{Y'}\right) = 0, \tag{60}$$

in (47) which results in

$$Y = \frac{K(t)}{A}, \tag{61}$$

where $K(t)$ is an integration function.

Equation (47) becomes

$$\dot{H} = \left(\frac{A}{2Y} + \frac{A'}{Y'}\right)H^2 - \frac{A}{2Y} + \frac{1}{2}AY\rho_s. \tag{62}$$

Utilizing Eqs. (61) and (62), we obtain the following

$$\dot{H} + \frac{A^2}{2K(t)}H^2 = -\frac{A^2}{2K(t)} + \frac{K(t)}{2}\rho_s. \tag{63}$$

Choosing $\rho_s = \frac{A^2}{K^2}$ in (63), we obtain

$$\dot{H} + \frac{A^2}{2K(t)}H^2 = 0, \tag{64}$$

which integrates to yield

$$H = \frac{1}{2}\int \frac{A^2}{K}dt + c_1(r), \tag{65}$$

where $c_1(r)$ is an integration function. It is clear from (63) that the presence of strings ($\rho_s \neq 0$) allows us to find exact solutions for the horizon function. This enables us to explore the dynamics of the collapse process.

7 Linear equation

We now consider setting

$$\left(\frac{A}{2Y} + \frac{A'}{Y'}\right) = 0, \tag{66}$$

in Eq. (47), which results in obtaining an expression for Y, ie,

$$Y = \frac{K(t)}{A^2}, \tag{67}$$

where $K(t)$ is an integration function. Using Eq. (67) in Eq. (47), we have the following

$$2Y\dot{H} + \dot{Y}H + A - AY^2\rho_s = 0. \tag{68}$$

We consider the case in which the collapse initially starts at $t = 0$ with q being present. It is vital to note that prior to $t = 0$, we have an initially static configuration. Using $g = 2Y$, $f_0 = A - AY^2\rho_s$ and $f_1 = \dot{Y}$, equation (68) can be recast as

$$g\dot{H} = f_1 H + f_0, \tag{69}$$

which can be integrated to yield

$$H = e^F \left(H_i + \int_0^t e^{-F} \frac{f_0}{g} dt \right), \tag{70}$$

where

$$F = \int_0^t \frac{f_1}{g} dt. \tag{71}$$

Using the following conditions in (70)

$$F = -\frac{1}{2} \ln \frac{Y}{Y_i}, \quad R_i = R(t = 0), \tag{72}$$

we obtain an expression for H

$$H = \sqrt{\frac{Y_i}{Y}} \left(H_i + \frac{1}{2\sqrt{Y_i}} \int_0^t \left(-\frac{A^2}{\sqrt{K}} + \frac{K^{3/2}}{A^2} \rho_s \right) dt \right). \tag{73}$$

It is important to note that once A and K are given, we can determine the other quantities. For the case with shear, we use the following expression for Y

$$Y_\Sigma = Y_i - bt, \tag{74}$$

keeping in mind that Y_i and b are positive. Letting $A = A(r)$, upon integration, Eq. (73) gives

$$H_\Sigma = \frac{A_\Sigma}{b} - \sqrt{\frac{Y_i}{Y_\Sigma}} \left(\frac{A_\Sigma}{b} - H_i \right) + \frac{1}{\sqrt{Y_\Sigma}} \frac{A_\Sigma}{2} \int_0^t Y_\Sigma^{3/2} \rho_s dt, \tag{75}$$

this can be written as

$$H_\Sigma = c - \sqrt{\frac{Y_i}{Y_\Sigma}} \epsilon + \frac{1}{\sqrt{Y_\Sigma}} \frac{A_\Sigma}{2} \int_0^t Y_\Sigma^{3/2} \rho_s dt, \tag{76}$$

where $c = A_\Sigma / b$ and $\epsilon = c - H_i$.

It is clear from (76) that a black hole end state may be possible depending on the choices for Y_i and the string density, ρ_s.

8 Physical results

In this section we will focus on the solution obtained in Sect. 6. In Fig. 1, we have chosen $c_1(r) = -\alpha \cos r$ and $c_2(r) = \beta \sin r$, where α and β are positive constants. It is clear in Fig. 1 that $\dot{m} < 0$ for each interior point of the fluid configuration. In addition, $\dot{m} < 0$ during the collapse epoch. We observe that the rate of energy loss is highest at the core and diminishes as it approaches the boundary of the star. Since the collapse takes place from $-\infty < t \leq 0$, we observe that $\dot{m}$ decreases for late times as expected. It is clear that collapse is possible in the presence of string density whereas in the absence of strings; ie. pure radiation being dissipated, Ivanov showed that collapse to a black hole may not be possible.

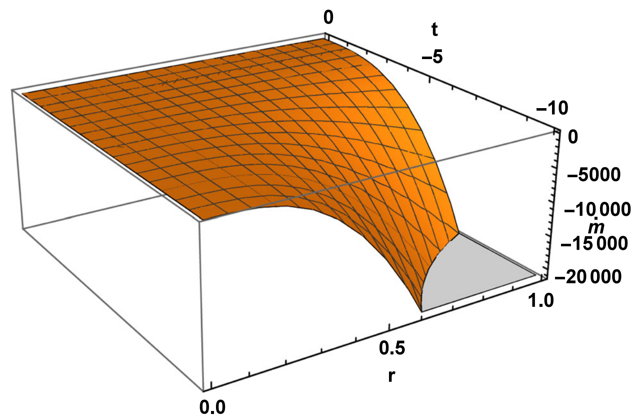

Fig. 1 $\dot{m}$ as a function of the radial (r) and temporal (t) coordinates for $\alpha = 1$ and $\beta = 10$

9 Conclusion

We have utilized the Ivanov algorithm to generate various families of solutions to the junction condition for a radiating star with generalised Vaidya atmosphere. The 'horizon' function introduced by Ivanov simplifies the junction condition which gives a highly non-linear ordinary differential equation encoding the temporal behaviour of the model. The Ivanov algorithm allows us to solve the temporal equation for three special cases: viz., geodesic collapse, collapse leading to a Ricatti equation and lastly, the linear evolution equation. We are able to solve all three cases for specific string density profiles which generalise Ivanov's solutions. We show that the presence of strings drastically alter the end-state of radiative gravitational collapse. In a recent study by Maharaj and Brassels [37], it was shown that a more general interior matter distribution consisting of anisotropic baryoinc matter and a string distribution leads to a more general junction condition in which the interior string density makes its appearance: $p_\Sigma = (q + \mu - \bar{\mu})_\Sigma$ where μ and $\bar{\mu}$ are the interior and exterior string densities respectively. Depending on the choices for μ and $\bar{\mu}$, the outcome of collapse may be significantly different from the scenarios studied here. These cases will be taken up elsewhere.

Data Availability Statement This manuscript has no associated data or the data will not be deposited. [Authors comment: All data was obtained using the formulae explicitly given in the article.]

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