A LOOK AT THE DEVELOPMENT OF SPECTRAL AND SCATTERING THEORY IN JAPAN

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This is not a complete historical survey of spectral and scattering theory in Japan. Many omissions and negligences exist. Specifically, many important contributions from outside this country will be left out.

Although it is not easy to explain in precise terms what main problems of spectral and scattering theory are, we try this taking as an example a simple scattering system. Given two self-adjoint operators H_o (unperturbed) and H (perturbed), we want to compare the spectral structure of one with that of the other. Suppose H_o is known to be absolutely continuous. (For terminology we shall mostly follow Kato[6].) What conditions on the perturbation $H - H_o$ make the spectral structures of these operators similar? More technically:

(1) Does the limiting absorption method apply to H ? (Or, does the limiting value of the resolvent of H exist in a sense or other when its argument approaches a real point in the (absolutely) continuous spectrum?)

(2) Is the absolutely continuous part of H similar to H_o ?

(3) What is the spectral structure of the singular part of H ?

  i) Is it discrete?

  ii) Are any eigenvalues imbedded in the absolutely continuous spectrum?

(4) Does an eigenfunction expansion theorem hold for H provided it does for H_o ?

(5) Do the wave operators $W_\pm = \text{strong}\lim_{t\to\pm\infty} e^{itH}e^{-itH_o}$ exist?

(6) Are the wave operators complete? Is the scattering operator $S = W_+^* W_-$ unitary?

(7) Does the invariance principle of the wave operators hold?

And many other more particular problems might be enumerated.

Toward the above-mentioned problems there are, roughly speaking, two approaches: one is abstract, and the other concrete. The abstract approach deals with operators in an (or more than one) abstract Hilbert space. Concerning this Professor Kuroda will give a more detailed account. The second approach deals mostly with concrete (partial) differential operators appearing in classical as well as modern physics, such as Maxwell's equations, Schrödinger equations, Dirac equations, etc.. Of course, there are some which may be located in between, like Friedrichs' model.

1. The opening of mathematical theory of scattering in this country was, perhaps, called in 1957 by Kato's work[2, 3] on finite-dimensional (or degenerate) and trace-class perturbations. Then Kuroda's work[1, 2, 3] follows. These pieces of work are in an abstract setting, and indicate fundamental ideas and methods of how to solve scattering problems. Stimulated by them and Povzner's 1953 paper (Mat. Sb. 32(74) (1953), 109-156) Ikebe[1] studied in 1960 the Schrödinger operator $H = -\Delta + V(x)$ in the three-dimensional Euclidean space assuming that $V(x) = O(|x|^{-\alpha})$ with $\alpha > 2$. The before-mentioned problems except (7) (while Problem (5) had been treated by Kuroda[1]) were attacked and solved. It should be noted that the basic problem of self-adjointness of the Schrödinger operator had been answered affirmatively almost 10 years before by Kato[1] and there had been an important contribution also by Kato[4] to Problem (3-ii).

After the appearance of Ikebe's paper[1] many an effort has been devoted, in and outside of Japan, to decrease the exponent α which comes out in the potential $V(x) = O(|x|^{-\alpha})$. An important step is made by S. Agmon in 1970 (Actes Congrès intern. Math. t. 2 (1971), 679-683), and Saitō[2, 3], Mochizuki[7] and Kuroda [9, 10] succeeded with $\alpha > 1$. Soon afterward an attempt came to be made at penetrating the Coulomb barrier ($\alpha = 1$). Although the exact solution of the eigenvalue problem for the purely Coulomb potential had been long known, it was not known until the appearance of a paper of Dollard (J. Math. Phys. 5 (1964), 729-738) on modified wave operators whether or not the usual wave operators existed for the Coulomb potential. Problems (1), (2) and (3) concerning long-range potentials ($\alpha > 0$), under an additional assumption on the asymptotic behavior of $V(x)$, were solved by Ikebe-Saitō[1] and also by Lavine (J. Functional Anal. 12 (1973), 30-54). Very recently, a solution to Problem (4) has been obtained by Ikebe[8, 9] and Saitō[6]. But Problem (6) still remains open, while Problem (5) (and also Problem (7)) has already been settled by a time-dependent method.

2. Problem (3-ii) is a hard problem, having a close bearing upon the unique continuation theorem for elliptic differential equations. The first important contribution was, as mentioned above, by Kato[4] who grasped it as a problem in a neighborhood of the point at infinity, and obtained asymptotic growth estimates for solutions to the eigenvalue problem $-\Delta u + V(x)u = \lambda u$ ($\lambda > 0$). There have been many contributions from abroad to this problem. Rather recently, Ikebe-Uchiyama

[1], Masuda[3] and Uchiyama[5] obtained some significant results. This problem displays a greater difficulty if it is not considered in a *full* neighborhood of infinity. Konno[2] and Tayoshi[1, 2] contributed in this direction.

3. The Schrödinger operator might be considered too simple. But methods and techniques developed for Schrödinger operators are applicable to second- and higher-order elliptic differential operators. Moreover, it can be noticed that no essential difficulty arises in the treatment of exterior problems (with compact obstacles) for elliptic operators. For these problems one can count a rather big bunch of researches in this country. E.g., Shizuta[1], Ikebe[3, 4], Oeda[1], Konno[1], Mochizuki[7], Uesaka[1], Ikebe-Tayoshi[1], Ushijima[3], Kuroda[10, 11] and Kako[2]. The work of Oeda, Konno and Kako is interesting in that an exterior problem is viewed as the limiting case of a sequence of whole-space problems.

4. While the limiting absorption method directly asks for the boundary values of the resolvent, there is a method, called the limiting amplitude method, in which one is involved with the asymptotic behavior of solutions to the time-dependent equation of Schrödinger's type: $i\, du/dt = Hu$. To this sort of problem contributed Mizohata-Mochizuki[1], Kiyama[1], Iwasaki[1], Kubota-Shirota[1] and others. In this connection one cannot forget an important contribution of Masuda[1, 2] on the relation between exponential decay and a certain spectral property of H.

5. The Dirac operator $-i \sum_j \alpha_j \partial/\partial x_j + \beta + V(x)$ describing a relativistic system can also be treated in almost the same way as the Schrödinger operator. Mochizuki[1] and Yamada[1] studied the spectral property of the Dirac operator with a short-range potential ($V(x) = O(|x|^{-\alpha})$, $\alpha > 1$). Very recently, Yamada has succeeded in applying the limiting absorption method to the Dirac operator with a long-range potential.

6. The operator $-iE(x) \sum_j A_j \partial/\partial x_j$, with $E(x)$ and A_j matrices, appears in classical wave propagation problems. While a number of papers were published by Wilcox and Schulenberger, the whole-space, exterior and half-space problems were dealt with by Mochizuki[5, 6], Matsumura[1], Wakabayashi[1, 2, 3], Suzuki[1], Yajima[1, 2] and Ikebe[6, 10]. In the whole-space problem the main assumption on $E(x)$ (which describes the medium of propagation) is that $E(x) - I$ be short-range. No long-range theory seems to exist.

7. Scattering theory of Lax-Phillips' type (Scattering theory, Academic Press, 1967) has not been studied very strenuously in this country. But Iwasaki [2] made a notable contribution to this theory. By his result it has been made possible to treat wave equations in even-dimensional spaces along the line laid by

Lax and Phillips, while their original theory was limited to odd-dimensional spaces.

8. It is usual that operators (differential or of other types) describing physical processes are given *formally*. Thus it is important to set up suitable Hilbert spaces and show the (essential) self-adjointness of the operators under consideration. It was in 1951 that Kato[1] published a basic result on this problem for Schrödinger operators. An important progress since then was made by Ikebe-Kato[1] about a decade later. Today we have more sophisticated results by Kato, Simon, Walter, etc.. Arai[2] discussed this problem for Dirac operators.

9. Sometimes it is hard to distinguish the absolutely continuous spectrum from the singular spectrum. In such a case we have to content ourselves with a rougher classification of the spectrum —— into essential and discrete spectra. Essentially following the technique of Žislin, Uchiyama[1] proved that atomic systems (atoms, ions, molecules) with magnetic external fields have a definite threshold dividing the essential and discrete spectra. On the other hand, Arai[1] obtained a proof that the Stark effect on atomic systems through an electric field whose potential is a linear function of the coordinates entails the essential spectrum extending over the whole real line.

10. There are several pieces of work concerning the discrete eigenvalues below the continuous spectrum. Uchiyama[1, 2, 3, 4] gave criteria for the finiteness of eigenvalues of the many-body problem below the threshold. Konno-Kuroda [1] proposed an abstract criterion for the finiteness of perturbed eigenvalues which is applicable to the one-body Schrödinger operator. Setô[1] extended Bargman's inequality to the n-dimensional space in the spherically symmetric potential case. Tamura[1, 2] considered the asymptotic distribution of the negative eigenvalues of Schrödinger (and elliptic) operators in the neighborhood of 0. Among other results related with somewhat general spectral properties we mention here Ichinose's work[1] on the essential spectrum of the tensor product of linear operators.

11. The Friedrichs model is semi-abstract. To this one can find contributions of Mochizuki[2] and Ushijima[2]. The techniques developed for self-adjoint Schrödinger operators can also be applied without any essential alterations to some physical systems governed by non-self-adjoint operators. Mochizuki[3, 4], Ikebe[7] and Saitō[4, 5] are examples.

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The following list contains, besides the papers quoted above, those which are to be quoted in S. T. Kuroda's article given in these Proceedings.

The writer is much indebted to Professor S. T. Kuroda for his painstaking work of preparing these references.

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