



Anisotropic behavior of universe in $f(R, L_m)$ gravity with varying deceleration parameter

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Abstract In this research study, we investigate the anisotropic behavior of the Universe using the framework of $f(R, L_m)$ gravity. This modified gravity theory, represented by $f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta$, incorporates both the Ricci scalar (R) and matter Lagrangian density (L_m). Our exploration centers around understanding the Universe's dynamics through the variable deceleration parameter. Additionally, we employ energy conditions, the jerk parameter, equation-of-state (EoS) parameter and statefinder parameters as analytical tools to gain insights into the evolution of the Universe within this modified gravity scenario. Our findings are then compared to recent observational data and are found to be in agreement with the Λ CDM model. This research contributes valuable insights into the anisotropic nature of the Universe in the context of $f(R, L_m)$ gravity and highlights its deviations from the Λ CDM model, providing a deeper understanding of the fundamental dynamics of our cosmic evolution.

1 Introduction

In recent years, extensive research has been conducted in the fields of astrophysics and cosmology, driven by compelling observational evidence suggesting that our Universe is currently experiencing a phase of accelerated expansion. This intriguing phenomenon has been primarily substantiated through multiple sources of observational data,

including the observations of like type Ia supernovae [1–4], Baryonic Acoustic Oscillations [5,6], Wilkinson Microwave Anisotropy Probe [7], Large Scale Structure [8,9] and the Cosmic Microwave Background Radiation [10,11]. It is also widely accepted that the cosmos underwent a transformation over time, transitioning from an initial deceleration phase to a later period characterized by acceleration [12]. The most encouraging approach for explaining the accelerating expansion of the Universe, without invoking the mysterious dark energy possesses a positive energy density and negative pressure.

In the field of theoretical cosmology, there has been a growing interest in developing cosmological models that explain the accelerated expansion of the universe. Researchers have explored various approaches, including modifying Einstein's field equations and introducing alternative gravitational theories. While classical-level approaches have been proposed to account for cosmological observations, a comprehensive theory of gravity remains elusive. One way to modify Einstein's gravity is by introducing an arbitrary function of the Ricci scalar (R) into the gravitational action [13]. This modification takes into account the interaction between matter and geometry, leading to the development of $f(R)$ gravity models that involve non-minimal coupling [14] and arbitrary coupling [15]. In the realm of gravitational theories, the modified GB gravity has successfully undergone solar system tests comparable to General Relativity and effectively accommodates late-time cosmic acceleration. It demonstrates versatility by accounting for various effective equations of state, including a cosmological constant, quintessence, or phantom. Moreover, it exhibits the capability to describe the transition from deceleration to acceleration, including the crossing of the phantom divide [16]. A comprehensive formulation for modified $f(R)$ gravity, adaptable to any FRW metric, has been proposed by some

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authors [17]. This modification proves to be a viable alternative to General Relativity, particularly in dark epochs, as certain implicit forms of $f(R)$ gravity can depict Λ CDM cosmology without introducing a cosmological constant and without encountering singularities near $R = 0$. A thorough overview of $f(R)$ gravity is presented by another group of authors, aiming to be a valuable resource for both novices and experts. Despite acknowledging review limitations, they classify $f(R)$ theories as “toy theories,” emphasizing significant progress and insights in understanding classical gravity over the past five years [18]. Researchers have explored general properties, different representations, and derived equations for various modified gravity theories, including $f(R)$ theory, modified Gauss–Bonnet gravity, scalar-Gauss–Bonnet gravity, non-minimal models, non-local gravity, and covariant gravity. The study encompasses accelerating cosmological solutions, the de Sitter universe, and attempts to provide a unified description of inflation with dark energy [19]. Extended theories of gravity, aligning with modern gauge theories, offer a promising approach to gravitational challenges across scales. However, their validation awaits confirmation or rejection through future experiments and astrophysical observations [20]. Contemporary advancements in gravitational physics, influenced by the revelation of the cosmos’s enigmatic dark side in the late 1990s, are delved into by several authors [21]. Notably, recent advances in modified gravity, along with a toolbox for exploring inflation, dark energy, and bouncing cosmologies, are provided by certain scholars [22]. The $f(R)$ model aligns with the 2018 Planck constraints on cosmological parameters, resembling the Λ CDM model in the late stages of evolution up to the present day ($z = 0$) and avoiding dark energy oscillations [23]. A numerical analysis of late-time fiber inflationary potential indicates that the $f(R)$ scalar theory can generate a plausible dark energy model consistent with the 2018 Planck data, closely resembling the Λ CDM model [24]. Exploration of string-inspired inflation models, including Gauss–Bonnet couplings and higher-order derivatives of the scalar field, is undertaken by some scholars [25]. This analysis extends to inflation within Chern–Simons theories of gravity, covering various subcases and generalizations of string-corrected modified gravity. Diverse $f(R)$ gravity models with the potential to generate a plausible dark energy era in late times and an R2-like inflationary era in the early universe are examined by researchers. The investigation spans a variety of functional expressions for $f(R)$ gravities, incorporating logarithmic, exponential, and power-law models [26, 27]. Moreover, a more extensive framework called $f(R, L_m)$ gravity [28] has recently emerged. In this context, $f(R, L_m)$ represents a flexible function that depends on both the Ricci scalar (R) and the Lagrangian density (L_m) associated with matter. This approach holds promise for further exploring the relationship between matter and gravity in cosmology.

The field equations of $f(R, L_m)$ gravity are analogous to the field equations of the $f(R)$ model when applied in an empty space-time context. However, they exhibit notable distinctions from both the $f(R)$ model and General Relativity (GR) when confronted with the presence of matter [29]. Certain researchers have discovered that as a result of the coupling between curvature and matter in the cosmos, a substantial quantity of comoving entropy is generated throughout the course of cosmological evolution [30]. Certain authors have made noteworthy observations in the context of the modified theory of gravity known as $f(R, L_m)$ and noted that as the coupling constant within $f(R, L_m)$ gravity increases, the maximum mass of objects can be enhanced. They have noticed that $f(R, L_m)$ gravity may offer an improved framework for understanding the universe and its gravitational dynamics than GR [31]. Some researchers have conducted an analysis of the energy conditions within the framework of $f(R, L_m)$ gravity, adding to our understanding of this model [32]. Additionally, Gonclaves and Moraes have also utilized the $f(R, L_m)$ gravity model in their work [33]. Furthermore, in the field of cosmology, researchers have delved into the combination of non-minimum matter geometries with a specific version of the $f(R, L_m)$ function, opening up new avenues of exploration [34]. Researchers in the field of cosmology are exploring a transit dark energy model in $f(R, L_m)$ gravity. They have established a noteworthy connection between energy density parameters through observational constraints [35]. The $f(R, L_m)$ cosmological model, which includes bulk viscosity, provides a strong explanation for recent observations and successfully describes cosmic expansion [36]. In $f(R, L_m)$ gravity, researchers have studied the anisotropic properties of the Universe within the context of spatially homogeneous and isotropic FRW cosmological models. They have also determined the Universe’s current phase. This approach, based on FRW metric solutions, effectively avoids the Big-Bang singularity and can predict cosmic acceleration, all without the need for a cosmological constant, thanks to the inclusion of geometry-matter coupling terms in the Friedmann-like equations [37, 38]. The $f(R, L_m)$ cosmological model, which includes bulk viscosity, provides a strong explanation for recent observations related to cosmic expansion. It has been used to study wormhole solutions, particularly with a static and spherically symmetric Morris–Thorne wormhole metric, leading to the derivation of field equations for the general $f(R, L_m)$ function [39, 40].

Several investigations have been carried out on the LRS Bianchi type-I cosmological model, which incorporates a linearly changing deceleration parameter. These studies have yielded the conclusion that the Universe’s fate will eventually lead to a scenario known as the Big Rip [41, 42]. Another avenue of research involves the assessment of stability during the transition from the Universe’s early decelerating phase to its current accelerating phase. This analysis is

conducted within the framework of the $f(R, T)$ theory and centers around the perfect fluid cosmological model featuring local rotational symmetry (LRS) in the Bianchi-I geometry [43]. It's worth noting that researchers have observed that the LRS Bianchi type-I model exhibits a point-type singularity at its initial epoch, followed by a transformation from an early deceleration to the current acceleration phase [44]. In addition, scholars have explored spatially homogeneous and anisotropic Bianchi type-I spacetime within the context of $f(R, T)$ theories, investigating various cosmological distance parameters and state-finder parameters as part of their analysis [45].

The organization of the current manuscript is structured as follows: In Sect. 2, we delve into the fundamental action and basic formulation that govern the $f(R, L_m)$ gravity theory. Moving on to Sect. 3, we focus on anisotropic Bianchi type-I cosmological model and motion equations of $f(R, L_m)$ gravity. Section 4 is dedicated to the exploration of a specific $f(R, L_m)$ functional and cosmological solutions for $f(R, L_m)$ gravity. In Sect. 5, we discussed about the physical and dynamical properties of the model. During Sect. 6, we analysis the EoS parameter. Section 7 and Sect. 8, takes us further into the depths of our $f(R, L_m)$ model. Here, we investigate jerk parameter and the trajectory of the $r - s$ and $r - q$ parameters, a key aspect in understanding the dark energy behavior envisioned by our assumed model. In Sect. 9, we apply the stringent energy condition criteria to further validate our results. Finally, in Sect. 10, we bring our journey to a close. Here, we provide a concise and insightful conclusion that encapsulates the key findings and contributions of our study.

2 Basic formalism of $f(R, L_m)$ gravity theory

With the matter Lagrangian density L_m and the Ricci scalar R , the action principle for $f(R, L_m)$ gravity model proposed by Harko and Lobo (2010) [28] is given by

$$S = \int f(R, L_m) \sqrt{-g} d^4x \quad (1)$$

where f is an arbitrary function of R and L_m .

The Ricci-scalar R is defined in terms of metric tensor g_{ij} and Ricci-tensor R_{ij} as below

$$R = g^{ij} R_{ij} \quad (2)$$

where the Ricci-tensor is given by

$$R_{ij} = \partial_\kappa \Gamma_{ij}^\kappa - \partial_j \Gamma_{\kappa i}^\kappa + \Gamma_{ij}^\lambda \Gamma_{\lambda \kappa}^\kappa - \Gamma_{j\lambda}^\kappa \Gamma_{\kappa i}^\lambda \quad (3)$$

Here $\Gamma_{j\kappa}^i$ represents the components of well-known Levi-Civita connection as indicated by

$$\Gamma_{j\kappa}^i = \frac{1}{2} g^{i\lambda} [g_{\kappa\lambda,j} + g_{\lambda j,\kappa} - g_{j\kappa,\lambda}] \quad (4)$$

The field equation for $f(R, L_m)$ gravity, derived through the variation of action principle (1) with respect to the metric tensor g_{ij} , is expressed as follows.

$$\begin{aligned} f_R R_{ij} - \frac{1}{2} (f - f_{L_m} L_m) g_{ij} + (g_{ij} \square - \nabla_i \nabla_j) f_R \\ = \frac{1}{2} f_{L_m} T_{ij} \end{aligned} \quad (5)$$

where, $f_R(R, L_m) = \frac{\delta f(R, L_m)}{\delta R}$, $f_{L_m}(R, L_m) = \frac{\delta f(R, L_m)}{\delta L_m}$, $\square = \nabla_i \nabla^i$ and T_{ij} is the Energy Momentum Tensor (EMT) for perfect fluid, given by

$$T_{ij} = \frac{-2}{\sqrt{-g}} \frac{\delta(\sqrt{-g} L_m)}{\delta g^{ij}} = g_{ij} L_m - 2 \frac{\partial L_m}{\partial g^{ij}} \quad (6)$$

Deriving from the explicit expression of field equation (5), we can ascertain the covariant divergence of the energy-momentum tensor T_{ij} as follows:

$$\nabla^i T_{ij} = 2 \nabla^i \ln(f_{L_m}) \frac{\partial L_m}{\partial g^{ij}} \quad (7)$$

Furthermore, by contracting the field equation Eq. (5), we can establish the connection between the Ricci scalar denoted as R , the Lagrangian density of matter denoted as L_m , and the trace T of the stress-energy-momentum tensor T_{ij} as follows

$$R f_R + 3 \square f_R - (f - f_{L_m} L_m) = \frac{1}{2} f_{L_m} T \quad (8)$$

where, $\square F = \frac{1}{\sqrt{-g}} \partial_i (\sqrt{-g} g^{ij} \partial_j F)$ for any function of F .

3 Metric and motion equations in $f(R, L_m)$ gravity

It's important to note that anisotropic models introduce additional complexity and degrees of freedom into the equations governing the Universe's behavior. This makes them more challenging to work with and test against observational data compared to the simpler isotropic models like FLRW models.

Hence, in order to explored the anisotropic characteristics of the Universe, our focus turns towards investigating the anisotropic and spatially homogeneous LRS Bianchi Type-I metric described by,

$$ds^2 = -dt^2 + A^2(t) dx^2 + B^2(t) (dy^2 + dz^2) \quad (9)$$

Where, A and B are the functions of cosmic time (t) alone. If $A(t) = B(t) = a(t)$, we retain to FLRW cosmological model.

The Ricci scalar corresponding to the LRS Bianchi-I spacetime can be formulated as follows:

$$R = -2 \left[\frac{\ddot{A}}{A} + 2 \frac{\ddot{B}}{B} + 2 \frac{\dot{A} \dot{B}}{AB} + 2 \frac{\dot{B}^2}{B^2} \right] \quad (10)$$

Here we are assuming that the matter distribution of Universe can be characterized by the EMT of an perfect fluid.

Therefore, *EMT* for the perfect-fluid corresponding to the line element (9) is given by

$$T_{ij} = (p + \rho) u_i u_j + p g_{ij} \quad (11)$$

Here, p and ρ are thermodynamic pressure and energy density of the matter and $u^i = (1, 0, 0, 0)$ components of co-moving four velocity vectors in cosmic fluid with $u_i u^j = 0$ and $u_i u^i = -1$.

The modified Friedmann equations (5) which describe the dynamics of Universe in $f(R, L_m)$ gravity are given by

$$\begin{aligned} -\left(\frac{\ddot{A}}{A} + 2\frac{\dot{A}\dot{B}}{AB}\right) f_R - \frac{1}{2} (f - f_{L_m} \cdot L_m) \\ -2\frac{\dot{B}}{B} \dot{f}_R - \ddot{f}_R = \frac{1}{2} f_{L_m} p \end{aligned} \quad (12)$$

$$\begin{aligned} -\left(\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} + \frac{\dot{A}\dot{B}}{AB}\right) f_R - \frac{1}{2} (f - f_{L_m} \cdot L_m) \\ -\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \dot{f}_R - \ddot{f}_R = \frac{1}{2} f_{L_m} p \end{aligned} \quad (13)$$

$$\begin{aligned} -\left(\frac{\ddot{A}}{A} + 2\frac{\ddot{B}}{B}\right) f_R - \frac{1}{2} (f - f_{L_m} \cdot L_m) \\ -\left(\frac{\dot{A}}{A} + \frac{\dot{B}}{B}\right) \dot{f}_R = -\frac{1}{2} f_{L_m} \rho \end{aligned} \quad (14)$$

4 Cosmological solutions for $f(R, L_m)$ gravity

To examine the dynamics of Universe we employ the functional form of $f(R, L_m)$ gravity of the form

$$f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta \quad (15)$$

where α and β are free parameters and one can retain to standard Friedmann equations of GR for $\alpha = 1$.

For this particular functional form of $f(R, L_m)$ gravity, we have considered a dusty Universe scenario where we take $L_m = \rho$ [14, 46–49] and hence the Friedmann equations (12), (13) and (14) yields,

$$2\frac{\ddot{B}}{B} + \frac{\dot{B}^2}{B^2} - \beta - (1 - \alpha)\rho^\alpha = \alpha\rho^{\alpha-1} p \quad (16)$$

$$\frac{\ddot{A}}{A} + \frac{\ddot{B}}{B} + \frac{\dot{A}\dot{B}}{AB} - \beta - (1 - \alpha)\rho^\alpha = \alpha\rho^{\alpha-1} p \quad (17)$$

$$\frac{\dot{B}^2}{B^2} + 2\frac{\dot{A}\dot{B}}{AB} - \beta = (1 - 2\alpha)\rho^\alpha \quad (18)$$

We are confronted with a system of three equations as presented in Eqs. (16)–(18), involving four unknowns: A , B , p , and ρ . Consequently, to achieve a complete solution for the system, an additional plausible condition becomes imperative to ascertain the unique deterministic solution for this system of equations.

In this particular study, the focus a special form of deceleration parameter (q) defined by Singha and Debnath (2009) [50] and used by [51, 52] is linear in time with a negative slope given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{\eta}{1 + a^\eta} \quad (19)$$

where, η is a positive constant and a be the scale factor of the Universe.

The sign of parameter q in the model determines if the Universe is expanding or inflating. When $q > 0$, it signifies a typical decelerating model, while $q < 0$ indicates acceleration. It's worth noting that current observations like SNe-Ia and CMBR tend to support accelerating models with $q < 0$, but both do not altogether.

Solving Eq. (19) we obtained the Hubble parameter (H) as

$$H = \frac{\dot{a}}{a} = \kappa(1 + a^{-\eta}) \quad (20)$$

where, κ is constant of integration.

The scale factor is of paramount importance in cosmology, especially for understanding the late-time dynamics and fate of the Universe. It is a critical component in describing the expansion of the Universe and its interaction with dark energy, and it plays a central role in our models of the cosmos.

Therefore, integrating Eq. (20) we get the average scale factor (a) as

$$a = (e^{\eta t} - 1)^{\frac{1}{\eta}} \quad (21)$$

5 Physical and dynamical properties of the model

The spatial volume (V) is given by

$$V = a^3 = (e^{\eta t} - 1)^{\frac{3}{\eta}} \quad (22)$$

The Hubble parameter in terms of time t and redshift z is,

$$H = \frac{e^{\eta t}}{e^{\eta t} - 1} = \frac{H_0}{2} (1 + (1 + z)^\eta) \quad (23)$$

From Eq. (23), we obtained the Scalar expansion (θ) as

$$\theta = 3H = \frac{3e^{\eta t}}{e^{\eta t} - 1} = \frac{3H_0}{2} (1 + (1 + z)^\eta) \quad (24)$$

From Eqs. (19) and (21), the deceleration parameter is given by

$$q = -\frac{a\ddot{a}}{\dot{a}^2} = -1 + \frac{\eta}{e^{\eta t}} = -1 + \frac{\eta}{1 + \left(\frac{1}{z+1}\right)^\eta} \quad (25)$$

The anisotropic parameter is given by

$$\Delta = \frac{1}{3} \sum_{i=1}^3 \left(\frac{H_i}{H} - 1 \right)$$

$$\Delta = \frac{1}{2} \quad (26)$$

The shear scalar is given by

$$\begin{aligned} \sigma^2 &= \frac{3}{2} \Delta H^2 \\ \sigma^2 &= \frac{3}{4} \left(\frac{e^{\eta t}}{e^{\eta t} - 1} \right)^2 = \frac{3H_0}{8} (1 + (1+z)^\eta)^2 \end{aligned} \quad (27)$$

From definition of spatial volume in terms of average scale factor i.e. $V = a^3 = AB^2$ and using Eq. (21), the corresponding metric potentials A and B are obtained as,

$$A = (e^{\eta t} - 1)^{\frac{2}{\eta}}, \quad B = (e^{\eta t} - 1)^{\frac{1}{2\eta}} \quad (28)$$

Corresponding to above metric potentials, Eq. (9) yields

$$\begin{aligned} ds^2 &= -dt^2 + (e^{\eta t} - 1)^{\frac{4}{\eta}} dx^2 \\ &+ (e^{\eta t} - 1)^{\frac{1}{2\eta}} (dy^1 + dz^2) \end{aligned} \quad (29)$$

Solving Eqs. (16) and (18) and using Eq. (28), we have

The Energy Density (ρ) in terms of t and z is

$$\begin{aligned} \rho &= \left\{ \frac{1}{1-2\alpha} \left[\frac{9}{4} \frac{e^{2\eta t}}{(e^{\eta t} - 1)^2} \right] - \beta \right\}^{\frac{1}{\alpha}} \\ &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \end{aligned} \quad (30)$$

The Pressure (p) in terms of t and z is

$$\begin{aligned} p &= \frac{\frac{-9}{4} \frac{\alpha}{1-2\alpha} \frac{e^{2\eta t}}{(e^{\eta t} - 1)^2} - \frac{4\eta e^{\eta t}}{(e^{\eta t} - 1)^2} + \frac{\alpha\beta}{1-2\alpha}}{\alpha \left\{ \frac{1}{1-2\alpha} \left[\frac{9}{4} \frac{e^{2\eta t}}{(e^{\eta t} - 1)^2} \right] - \beta \right\}^{\frac{\alpha-1}{\alpha}}} \\ &= \frac{\left(\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta} \right)}{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{-\frac{\alpha-1}{\alpha}}} \end{aligned} \quad (31)$$

6 Equation of state (EoS)

An equation of state in cosmology is a mathematical expression that relates the pressure (p) and energy density (ρ) of a particular component of the Universe, such as dark energy, matter, or radiation. It helps to describe the behavior of the Universe's expansion over time. One of the most commonly used equations of state in cosmology is associated with dark energy and is referred to as the "dark energy equation of state." It is usually denoted by the parameter ω , and it relates the pressure and energy density as follows:

$$\omega = \frac{p}{\rho} \quad (32)$$

This equation of state parameter (ω) characterizes the nature of the component:

1. When $\omega = -1$, it signifies a cosmological constant (Λ) or vacuum energy, associated with dark energy. This leads to accelerated expansion, as predicted by the ΛCDM model.
2. $\omega > -1$, but not equal to -1 , it represents dynamic dark energy, like quintessence, where dark energy's properties change with time.
3. When $\omega = 0$, it denotes non-relativistic matter, such as dark matter or baryonic matter.
4. For $\omega = 1/3$, it signifies radiation, like photons or neutrinos, which behave relativistically and impact the early Universe's dynamics.

Using Eqs. (30) and (31) in Eq. (32), we have

$$\begin{aligned} \omega &= \frac{\frac{-9}{4} \frac{\alpha}{1-2\alpha} \frac{e^{2\eta t}}{(e^{\eta t} - 1)^2} - \frac{4\eta e^{\eta t}}{(e^{\eta t} - 1)^2} + \frac{\alpha\beta}{1-2\alpha}}{\frac{\alpha}{1-2\alpha} \left[\frac{9}{4} \frac{e^{2\eta t}}{(e^{\eta t} - 1)^2} - \beta \right]} \\ &= \frac{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{\frac{\alpha-1}{\alpha} + \frac{1}{\alpha}}}{\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta}} \end{aligned} \quad (33)$$

7 Jerk parameter

In the field of cosmology, the term "jerk parameter" denotes the third time derivative of the scale factor of the Universe concerning cosmic time. This parameter, expressed as a dimensionless quantity, serves as a crucial metric for quantifying the pace of alteration in the acceleration of the Universe's expansion. Researchers employ the jerk parameter to delve into the intricate dynamics of the cosmos and to differentiate among various cosmological models. Within the realm of cosmology, the precise value of the jerk parameter takes on significant importance as it plays a pivotal role in unraveling the enigma of dark energy. Dark energy is the mysterious force thought to be responsible for the remarkable phenomenon of the Universe's accelerated expansion.

The jerk parameter (j), can be defined as follows:

$$\begin{aligned} j &= \frac{\ddot{a}}{aH^3} \\ j &= 1 + \frac{\eta^2}{\left(\frac{1}{z+1} \right)^\eta + 1} + \frac{\eta^2}{\left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2} - \frac{3\eta}{\left(\frac{1}{z+1} \right)^\eta + 1} \end{aligned} \quad (34)$$

8 Statefinder parameters

In cosmology, the concept of a “statefinder pair” refers to a pair of dimensionless parameters that can help distinguish between different cosmological models based on the evolution of the cosmic scale factor $a(t)$ and its time derivatives. These parameters were introduced by Sahni et al.(2003) [53].

The statefinder pair $\{r, s\}$ can be calculated from observational data to probe the nature of dark energy and the expansion history of the Universe. Different cosmological models, including those with different forms of dark energy or modified gravity, can produce distinct trajectories in the $\{r, s\}$ plane, allowing researchers to constrain and compare these models with observations.

The statefinder parameters are defined as follows:

$$r = \frac{\ddot{a}}{aH^3}, \quad s = \frac{r-1}{3(q-\frac{1}{2})} \quad (35)$$

with the help of Eqs. (21), (23) and (25), above equations can be written as,

$$r = 1 + \frac{\eta^2}{\left(\frac{1}{z+1}\right)^\eta + 1} + \frac{\eta^2}{\left(\left(\frac{1}{z+1}\right)^\eta + 1\right)^2} - \frac{3\eta}{\left(\frac{1}{z+1}\right)^\eta + 1} \quad (36)$$

$$s = \frac{2\eta^2 \left(\frac{1}{\left(\frac{1}{z+1}\right)^\eta + 1} + 1 \right) - 6\eta}{6\eta - 9 \left(\left(\frac{1}{z+1}\right)^\eta + 1 \right)} \quad (37)$$

9 Energy conditions

The energy conditions represent a vital framework for characterizing the behavior of matter and energy within the Uni-

verse, serving as essential tools in various cosmological investigations. In this study, the primary role of these energy conditions is to ascertain the existence and implications of the accelerated expansion of the Universe. These conditions find their origin in the well-established Raychaudhury equations, which take the forms:

$$\frac{d\theta}{d\tau} = -\frac{1}{3}\theta^2 - \sigma^2 + \omega^2 - R_{ij}u^i u^j \quad (38)$$

$$\frac{d\theta}{d\tau} = -\frac{1}{2}\theta^2 - \sigma^2 + \omega^2 - R_{ij}\zeta^i \zeta^j \quad (39)$$

Here, θ represents the expansion factor, ζ^i is the null vector, and σ and ω denotes the shear and rotation associated with the vector field u^i .

The attractive gravity satisfy the following energy conditions

- Strong Energy Conditions (SEC): $\rho + 3p \geq 0$
- Weak Energy Conditions (WEC): $\rho \geq 0, p + \rho \geq 0$
- Null Energy Conditions (NEC): $p + \rho \geq 0$
- Dominant Energy Conditions (DEC): If $\rho \geq 0, |p| \leq \rho$

These energy conditions provide essential criteria for the behavior of gravitational interactions and matter content in the Universe, offering valuable insights into the viability of various cosmological models and their ability to account for the observed accelerated expansion. They play a pivotal role in the ongoing quest to comprehend the dynamics and evolution of our cosmos.

Using Eqs. (30) and (31) in above conditions leads to

SEC:

$$\begin{aligned} \rho + 3p &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \\ &+ 3 \frac{\left(\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta} \right)}{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{-\frac{\alpha-1}{\alpha}}} \geq 0 \end{aligned} \quad (40)$$

WEC:

$$\begin{aligned} \rho + p &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \\ &+ \frac{\left(\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta} \right)}{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{-\frac{\alpha-1}{\alpha}}} \geq 0 \\ \rho &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \geq 0 \end{aligned} \quad (41)$$

NEC:

$$\begin{aligned} \rho + p &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \\ &+ \frac{\left(\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta} \right)}{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{-\frac{\alpha-1}{\alpha}}} \geq 0 \end{aligned} \quad (42)$$

DEC:

$$\begin{aligned} \rho - p &= \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{1/\alpha} \\ &- \frac{\left(\frac{\alpha\beta}{1-2\alpha} - \frac{9\alpha \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2 \left(\frac{1}{z+1} \right)^{-2\eta}}{4(1-2\alpha)} - 4\eta \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right) \left(\frac{1}{z+1} \right)^{-2\eta} \right)}{\alpha \left(\frac{9 \left(\frac{1}{z+1} \right)^{-2\eta} \left(\left(\frac{1}{z+1} \right)^\eta + 1 \right)^2}{4(1-2\alpha)} - \beta \right)^{-\frac{\alpha-1}{\alpha}}} \geq 0 \end{aligned} \quad (43)$$

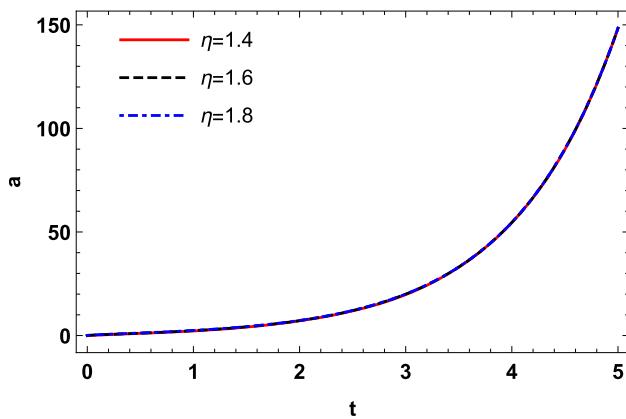


Fig. 1 Variation of average scale factor $a(t)$ against cosmic time t (Gyr)

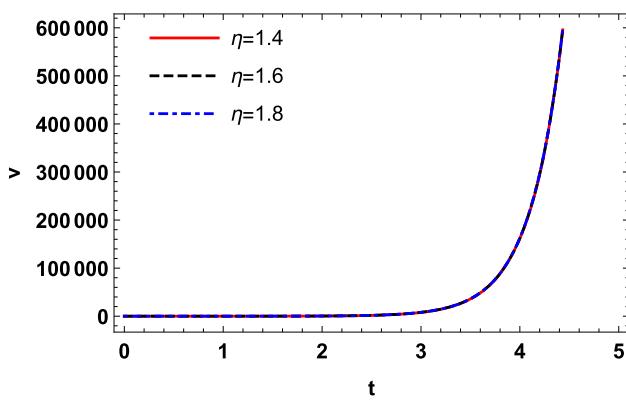


Fig. 2 Variation of spatial volume $V(t)$ against cosmic time t (Gyr)

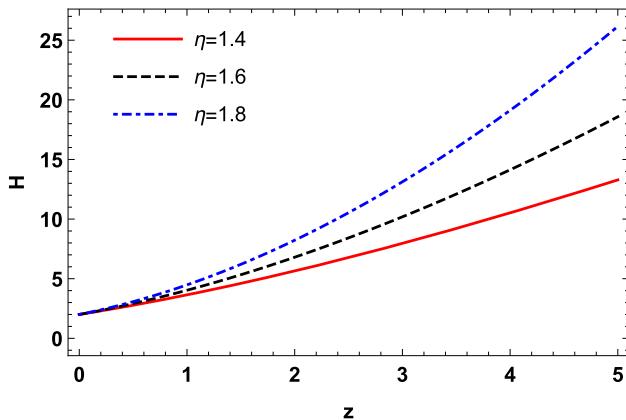


Fig. 3 Variation of hubble parameter (H) against redshift (z)

10 Figures

In the context of depicted figure, several noteworthy observations can be made:

- In Figs. 1 and 2, it becomes evident that the average scale factor and spatial volume exhibit a strikingly sta-

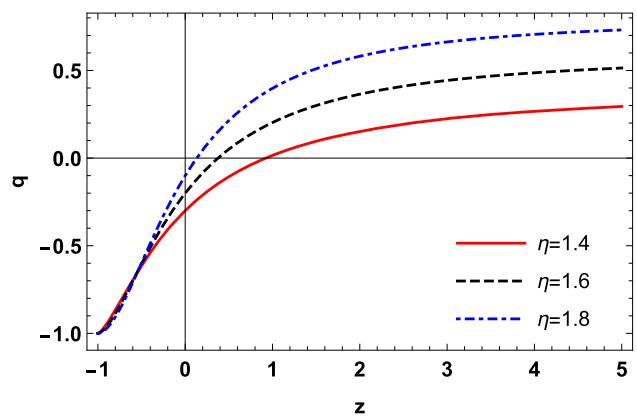


Fig. 4 Variation of Deceleration parameter (q) against redshift (z)

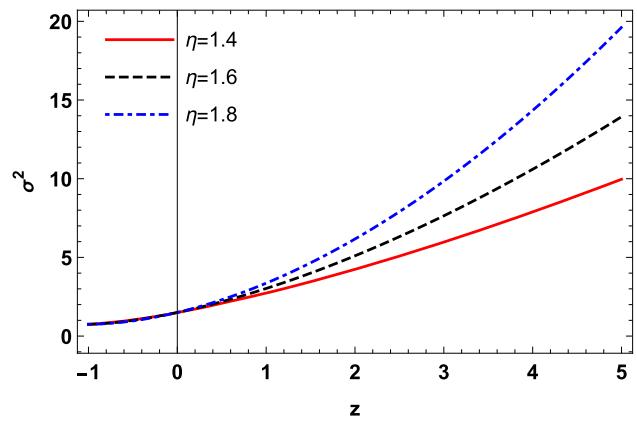


Fig. 5 Variation of shear scalar (σ^2) against redshift (z)

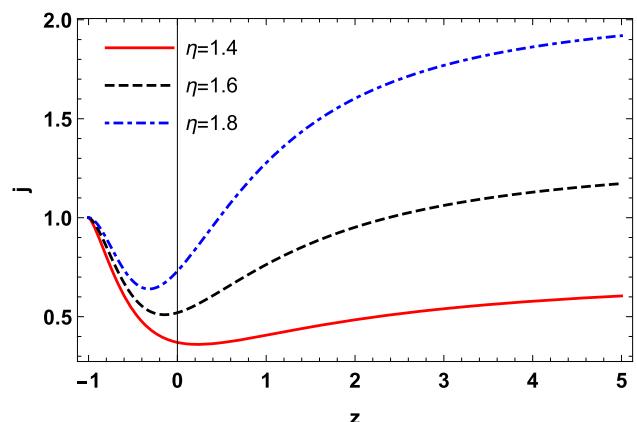
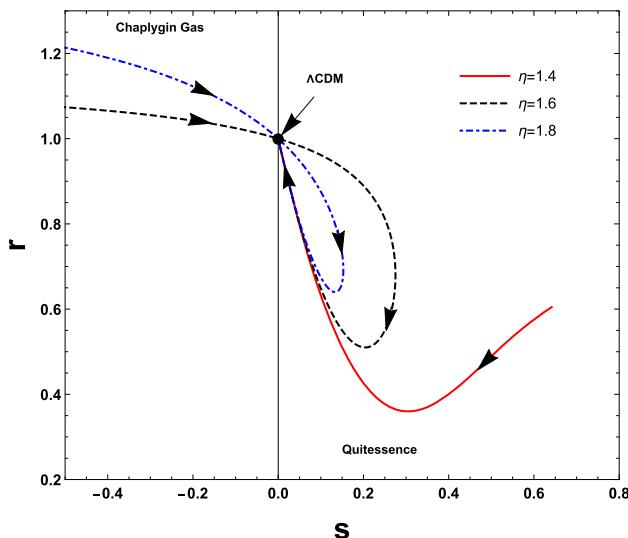
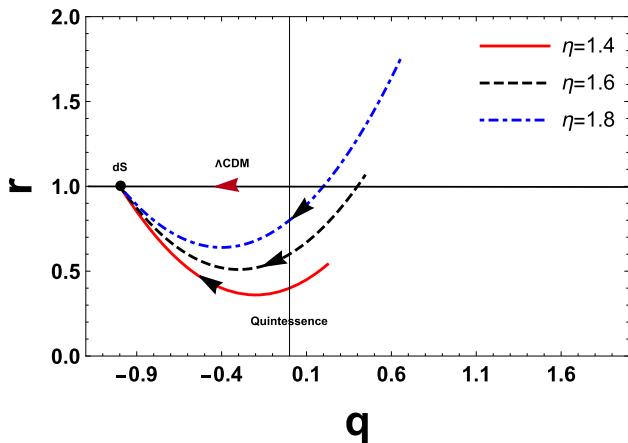
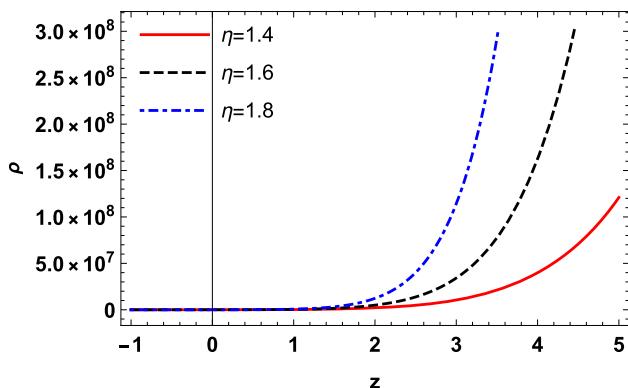
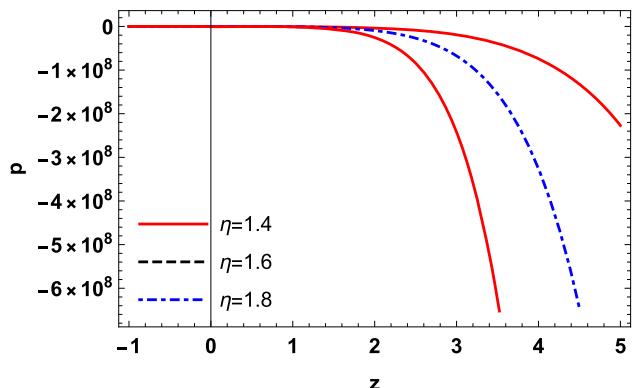
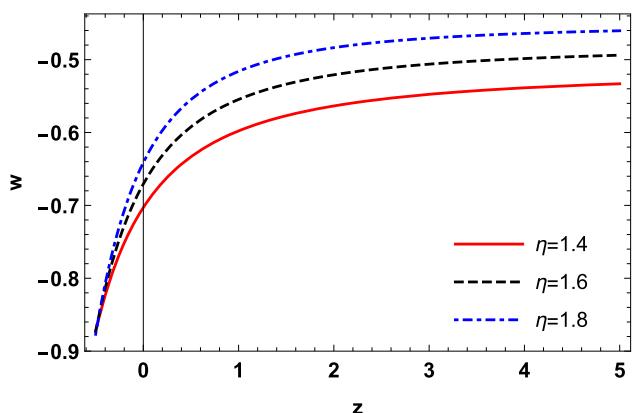
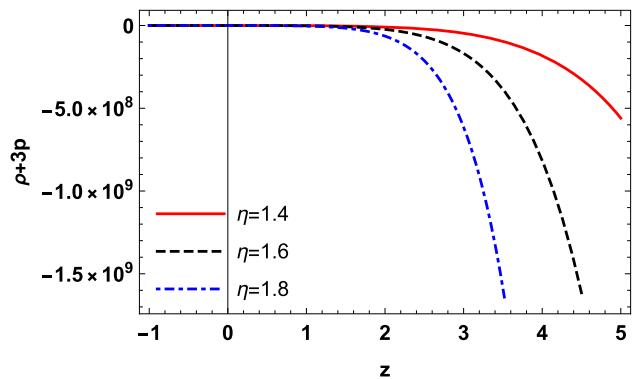


Fig. 6 Variation of jerk parameter (j) against redshift (z)

ble trend at the initial time point ($t = 0$). Nonetheless, as time advances, these two characteristics exhibit an unwavering and gradual increase, eventually stretching towards boundless extents over prolonged periods (t). This remarkable observation underscores an enduring and uninterrupted expansion of the Universe. In Figs. 1 and 2, we have utilized “Gyr” (Gigayear) as the unit of

Fig. 7 Evolution trajectory of $(r - s)$ planeFig. 8 Evolution trajectory of $(r - q)$ planeFig. 9 Variation of energy density (ρ) against redshift (z)

time for representing time scales. By using Gigayears as our time unit, we can effectively convey the long-term evolutionary processes and dynamics depicted in the figures.

Fig. 10 Variation of pressure (p) against redshift (z)Fig. 11 Variation of equation of state parameter (w) against redshift (z)Fig. 12 Variation of $\rho + 3p$ against redshift (z)

- The Hubble parameter, relative to redshift (z), yields crucial insights into the universe's expansion rate across cosmic epochs, shedding light on its dynamic evolution. In Fig. 3, we can see the behaviour of Hubble parameter. At the present time, the value of Hubble parameter is $H_0 = 69 \text{ km s}^{-1} \text{ Mpc}^{-1}$.
- Figure 4 illustrated that the deceleration parameter faces the transition phase. As, $q(z) \geq 0$ at early time, which shows the decelerating phase of the universe. But, in

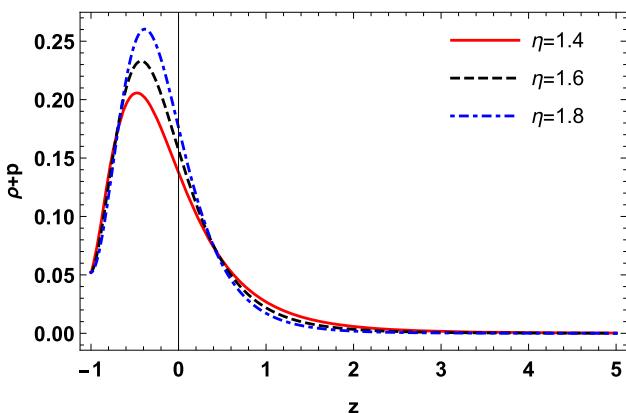


Fig. 13 Variation of $\rho + p$ against redshift (z)

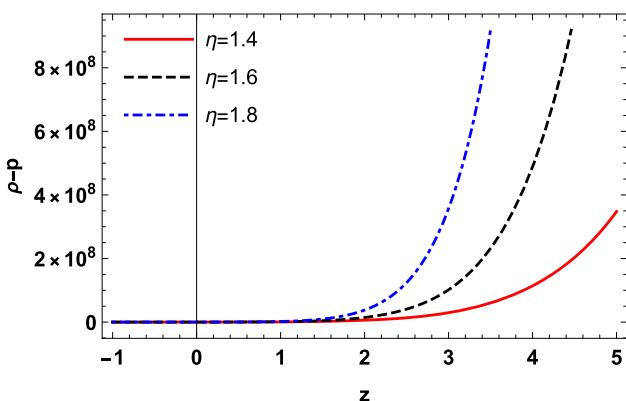


Fig. 14 Variation of $\rho - p$ against redshift (z)

the present and late time it signifying the onset of the Universe's accelerating phase $q(z) < 0$, which aligns with the characteristics of the Λ CDM model. Further, the value of transition redshift z_t are 0.92, 0.4 and 0.15 corresponding the value of $\eta = 1.4$, $\eta = 1.6$ and $\eta = 1.8$, respectively. Also, the present value of the deceleration parameter corresponding to the $\eta = 1.4$, $\eta = 1.6$ and $\eta = 1.8$ is $q_0 = -0.32$, $q_0 = -0.21$ and $q_0 = -0.11$, respectively.

- Figure 5 demonstrates that the shear scalar decreases over time and approaches zero for large time values. This trend signifies the Universe's evolution towards isotropic and homogeneous expansion, consistent with observations of Large Scale Structure [8,9] and the Cosmic Microwave Background Radiation [10,11].
- In Fig. 6, we can clearly discern that the jerk parameter (j) maintains a positive value throughout the entire Universe and aligns with the Λ CDM model for the given values of η . It's noteworthy that the model closely approximates the behavior of the Λ CDM model in this regard.
- In Fig. 7, the s-r plane serves as a graphical representation where each point corresponds to a particular dark energy model or cosmological scenario. Notably, the coordinates

$(0, 1)$ denote the standard cosmological model known as Λ CDM model. Within this framework, dark energy is postulated as a cosmological constant (Λ), while cold dark matter (CDM) predominates the matter content of the universe. The trajectories of dark energy models plotted in the $s - r$ plane offer valuable insights into their fundamental characteristics. For instance, quintessence, characterized by a dynamic form of dark energy governed by a scalar field, occupies the region to the right of the Λ CDM model for $\eta = 1.4$ in the $s - r$ plane. This alignment implies that quintessence entails unique evolutionary properties, potentially resulting in deviations from the conventional cosmological model. Conversely, the Chaplygin gas model, which posits a unified depiction of dark matter and dark energy, is positioned to the left of the Λ CDM model. This spatial arrangement suggests that Chaplygin gas exhibits distinct expansion dynamics compared to Λ CDM.

- Figure 8, illustrates the evolutionary trajectory of the model within the $q - r$ parameter space. Initially, the model is observed to originate from the point $(q, r) = (0.5, 1)$ in the past. This point corresponds to a phase in the universe's history characterized by a matter-dominated Standard Cold Dark Matter ($SCDM$) scenario. As the evolution progresses, the trajectory of the model traverses through the parameter space, eventually culminating at the point $(q, r) = (-1, 1)$, representing the de Sitter (dS) point. At this final stage of evolution, both points tend to resemble the characteristics of a Λ CDM universe or a de Sitter point. In terms of the statefinder parameter space, this convergence is denoted by the condition $(r, s) = (1, 0)$, or equivalently $(q, r) = (-1, 1)$, in the future. This convergence suggests that irrespective of the specific initial conditions or the detailed dynamics of the model, the late-time evolution tends towards a state reminiscent of a cosmological constant-dominated universe or a de Sitter spacetime, characterized by accelerated expansion driven by a constant vacuum energy density.
- As depicted in Fig. 9, the energy density exhibits a captivating pattern, commencing with a notable initial magnitude. Nevertheless, with the passage of time, the energy density steadily wanes and ultimately converges towards zero for present time and future. This remarkable phenomenon strongly indicates the ongoing expansion of the Universe.
- The pressure demonstrates a spectrum of fluctuations, transitioning from notably negative values to marginally negative values, as illustrated in Fig. 10. This characteristic negativity of the pressure signifies the presence of dark energy.
- Figure 11 depicts the evolution of equation of state (EoS) over redshift (z). The evolution trajectory of the param-

eter ω , illustrates the dynamic behavior of the expanding universe. Within the framework of cosmology, the parameter ω assumes one of three possible states: the cosmological constant ($\omega = -1$), quintessence ($-1 < \omega < -\frac{1}{3}$), or the phantom era ($\omega < -1$). These states represent different scenarios for the nature of dark energy and its influence on the universe's expansion

11 Concluding remark

In this article, we explore the Universe's accelerated expansion by employing a non-linear $f(R, L_m)$ gravity model, specifically $f(R, L_m) = \frac{R}{2} + L_m^\alpha + \beta$, where α and β are the free model parameters. Throughout our investigation, we derive the motion equations for the anisotropic LRS Bianchi type-I cosmological model in the presence of a perfect fluid. By utilizing a special form of the deceleration parameter, we obtain cosmological solutions that align with the dark energy Λ CDM model.

The study's findings are highly convincing, leading to the following conclusions:

The average scale factor (a) and spatial volume (V) remain finite near $t = 0$, but expand continuously as time (t) progresses, ultimately approaching infinity as $t \rightarrow \infty$. This suggests that the Universe begins with a finite volume and gradually expands over time, consistent with previous research [54–58]. Our findings reveal that both the Hubble parameter (H) and scalar expansion (θ) factor decrease, ultimately approaching zero in future. This indicates that the Universe is signifying a gradual slowdown in the expansion rate. As a consequence, the Universe contracts, potentially setting the stage for future cosmological scenarios, such as the "Big Crunch" [59–62]. Our findings reveal that the deceleration parameter (q) defined in Eq. (25) undergoes a transition from positive to negative values, ultimately converging towards -1 . This convergence corresponds to the characteristics to dark energy Λ CDM model. Consequently, our model of the Universe undergoes a transition from an early deceleration phase to its current phase of acceleration, a conclusion that aligns well with recent observational [63–67]. The decreasing shear scalar over cosmic time, approaching zero in future, reflects the Universe's transition toward an isotropic and homogeneous expansion. These results align with the recent findings [68–70]. In our findings the jerk parameter approaches unity in the later time, mirroring observational values. This reaffirms the positivity of the cosmic jerk parameter throughout the cosmic timeline and its convergence toward 1 during late epochs, consistent with the Λ CDM model and corroborating findings from [71, 72]. In the current framework, we have undertaken an evaluation of the statefinder parameters, and our analysis has produced outcomes suggesting that $\{r, s\} = \{1, 0\}$ which serves as

a compelling indication of Λ CDM model, closely aligns with recent observations [73–77]. Also, we have observed the Λ CDM model places quintessence to the right of the model, and chaplygin gas to the left. We found that the energy density (ρ) remains positive and undergoes a decrease as cosmic time progresses. Initially, during the early epoch, ρ exhibits a high value, but as we look into the distant future, ρ diminishes significantly over extended time periods. Our research reveals that the Universe's pressure (p) is beginning from a highly negative value and converging towards zero today. This observation is consistent with recent cosmological observations linking accelerated expansion to dark energy [1–3]. It is evident that ω takes on values less than zero, indicating a quintessence dark energy phase. This signifies an accelerating phase in the expansion of the universe. The quintessence phase suggests the presence of a dynamic form of dark energy with an equation of state lying within the range of $(-1 < \omega < -\frac{1}{3})$. It is noteworthy that within our model, the equation of state parameter does not transition across the phantom divide $\omega = -1$. The phantom divide represents a significant threshold in cosmology, beyond which the behavior of dark energy becomes particularly exotic, potentially leading to issues such as the "Big Rip" scenario, where the universe's expansion accelerates to the point of tearing apart all bound structures. Our model's consistency in maintaining ω above this divide suggests a more conventional evolution of dark energy, mitigating the extreme consequences associated with the phantom era. The fulfillment of energy conditions with respect to redshift (z), namely SEC, WEC, NEC, and DEC, collectively validates our current cosmological understanding, as demonstrated in Figs. 9, 12, 13 and 14. These conditions have been rigorously examined and affirmed in diverse scenarios, such as the investigation of dark energy, dark matter, and cosmic expansion, with results consistent with prior studies [78–82].

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Code availability Code/software will be made available on reasonable request. [Authors' comment: All the figures were generated using Mathematica program which is available from the authors upon request].

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