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Schrödinger equation and resonant scattering in the presence of a minimal length

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ABSTRACT

In this work we have studied the consequences of the minimal length, which arises in many theories of quantum gravity, on the Scattering of a point particle by a spherically symmetric potential. The modified Schrödinger equation is factorized to be of second order in position space representation. For the square well potential analytic expressions for the scattering states are obtained. Then the phase shifts are deduced. It is shown that the minimal length has two effects on the resonant scattering. The first one is that the minimal length increases slightly the resonant cross section and the second is the shift of the position of the resonances.

1. Introduction

The scattering theory is of interest to many branches of physics such as nuclear and particle physics, astrophysics, plasma physics, and condensed matter [1]. It has been considered as a powerful technique in experimental physics that provides us with important properties of the microscopic systems. Therefore, it satisfies our deep need to discover the microscopic world. The scattering theory could play also an important role on the experimental justification of various theories. As example, the collision between protons and anti-protons, realized at CERN in 1983, producing gauge bosons W and Z, confirms the Weinberg–Salam–Glashow's unification model [2]. In the scattering processes an important effect may occur when the system under study responds with the interaction. This is known as the resonant scattering [3,4]. In such a case, the cross section which plays the central role on the theory reaches a maximum at certain values of the energy. This effect may be of interest because it gives the chance to observe certain phenomena such us, for example, Z boson production in the electron–positron collision.

In this Letter, we propose to study the resonant scattering in the presence of a nonzero minimal length by considering the simple model of a nonrelativistic particle subjected to a spherically symmetric potential well. In recent years, it has been shown that many theories of quantum gravity implies the existence of a minimal observable length, which is expected to be of the order of the Planck length [5–12]. The study of the influence of the minimal length on a physical effect can be then regarded as a quantum gravitational correction [13]. We expect that the incorporation of this gravitational correction into the scattering theory could be of interest to the experimental observation of the minimal length.

The concept of minimal length can be incorporated on the study of the physical problems by considering the deformed canonical commutation relation

$$[X,P]=i\hbar\left(1+\beta P^2\right), \tag{1}$$

where β is very small positive parameter. This deformation implies a generalized uncertainty principle (GUP)

$$\Delta X\Delta P\geqslant\frac{\hbar}{2}\left(1+\beta(\Delta P)^2\right). \tag{2}$$

Consequently, this GUP leads to a nonzero minimal length given by

$$(\Delta X)_{\min}=\hbar\sqrt{\beta}. \tag{3}$$

Since the elaboration of the fundamental principles of the quantum mechanics with GUP in [14–17], a great effort has been made to study the effect of the minimal length on the quantum systems. Among problems that have been studied in this important version of quantum theory, we cite, for example, the harmonic oscillator [18], the hydrogen atom [19–23], the inverse square potential [24], the scattering problem by Yukawa and Coulomb potentials [25] and some relativistic cases that can be turned to the harmonic oscillator [26–28]. We note that an attempt to solve the Schrödinger

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equation with Woods–Saxon potential in the presence of a minimal length is recently communicated [29]. Elsewhere, the influence of the minimal length on the Casimir effect has been communicated in several works (see for example [30,31]). For recent review on the subject, one can see [32].

In order to study the scattering of nonrelativistic particles by a spherically symmetric potential, one has at his disposal several methods. However, because of the complexity of this novel version of quantum mechanics, we think that the simpler method is the partial waves technique, which permits us to calculate the phase shifts directly. One of the well-known features of this method is that, at low energies, only channels with small angular momentum number contribute to the total cross section. However, the phase shifts method is based on analytic expressions of the scattering states which is not, in general, possible. In order to obtain analytic expressions of the scattering wave functions, we use a position space representation that satisfies the commutation relation (1). This representation leads to a correction with p^4 to the Schrödinger equation. The corresponding differential equation is then of fourth order and its solution is not always possible. For this reason, we factorize, first, the modified Schrödinger equation to write it as a second order equation. Then, we solve the resulting equation for the spherical potential well to obtain analytic expressions for the scattering states and the phase shifts. Finally, we discuss the consequences of the minimal length on the resonance effect.

2. Schrödinger equation with generalized uncertainty principle

As is mentioned above, in order to study the scattering of particles by an external potential in the framework of quantum mechanics, we need the explicit form of the outgoing wave function or at least the asymptotic behavior of this wave. In the presence of a minimal length, this is a very hard task because the wave equation does not admit analytic solutions. Therefore, before approaching the scattering process, let us make some comments about the Schrödinger equation with a generalized uncertainty principle.

2.1. The one-dimensional case

In order to incorporate these gravitational corrections into Schrödinger wave mechanics, we have for the new operators X and P, several representations that satisfy the commutation relation (1). Among these representations, we find in literature the momentum space representation, where

$$P = p, \qquad X = i\hbar\left[\left(1+\beta p^2\right)\frac{\partial}{\partial p}\right], \tag{4}$$

and the position space representation defined by

$$X = x, \qquad P = \left(1+\frac{1}{3}\beta\hat{p}^2\right)\hat{p}, \qquad \hat{p} = -i\hbar\frac{\partial}{\partial x}. \tag{5}$$

In the one-dimensional Schrödinger equation

$$\left[\frac{P^2}{2m} + V(X) - E\right]\psi = 0, \tag{6}$$

the potential $V(X)$ may be developed as

$$V(X) = \sum_n \lambda_n X^n.$$

Therefore, the use of the momentum space representation leads to a differential equation of high order. This makes analytic solutions possible only for some particular cases such as the linear potential and the harmonic oscillator. For $V(X) = \frac{1}{X}$ or $V(X) = \frac{1}{X^2}$, we can multiply the Schrödinger equation by X or X^2 to obtain analytic solutions. However, to our knowledge, except of these 4 potentials, there is no exact solution to the Schrödinger equation in the momentum space representation. This representation is also difficult when we consider simple discontinuous potentials like the square well and the step barrier. With the second representation, we can obtain to the first order on β the following equation

$$\left[\frac{\hat{p}^2}{2m} + \frac{\beta}{3m}\hat{p}^4 + V(x) - E\right]\psi = 0, \tag{7}$$

which is of fourth order. As is communicated in [33] the term with $\hat{p}^4$ is due to quantum-gravitational fluctuations of the background metric. We can see, here, that is also difficult to obtain analytic solutions in this representation because of the $\hat{p}^4$ term.

Recently, an attempt to reproduce a second order Schrödinger equation is proposed [29], where the authors have, simply, replaced the operator $\hat{p}^4$ by $4m^2(E_n^{(0)} - V(x))^2$, where $E_n^{(0)}$ are the eigenvalues of the operator $H_0 = \frac{\hat{p}^2}{2m} + V(x)$. The major drawback of this method is that ψ in (7) is not an eigenvector of H_0 and $\hat{p}^4\psi \neq 4m^2(E_n^{(0)} - V(x))^2\psi$. In addition, when we want to study the scattering process we must take into account that the continuity equation and the conserved current are infected by the minimal length.

To obtain the good approximation of the Schrödinger equation, we introduce an auxiliary wave function φ, so that

$$\psi = \left(1 - \frac{2}{3}\beta p^2\right)\varphi. \tag{8}$$

By substituting (8) into (7) and neglecting terms of higher order on β we get the equation

$$\left[\left(1 + \frac{4m}{3}\beta\left(E - V(x)\right)\right)\frac{p^2}{2m} + \left(V(x) - E\right)\right]\varphi = 0, \tag{9}$$

which is an effective Schrödinger equation with a position dependent effective mass involving quantum gravitational corrections. In the limit $\beta \to 0$, Eq. (9) reduces to the ordinary Schrödinger equation. Since this equation is of second order, we can find analytic solutions for various potentials.

2.2. Generalization to three-dimensional case with a central potential

The generalization of the deformed canonical commutation relation (1) is the string-motivated algebra given by [18]

$$[\hat{X}_i, \hat{P}_j] = i\hbar\left(\delta_{ij} + \beta\hat{P}^2\delta_{ij} + \beta'\hat{P}_i\hat{P}_j\right), \tag{10}$$

where β' is an additional parameter which is of order of β. In this case, the components of the momentum operator commute to one another

$$[\hat{P}_i, \hat{P}_j] = 0. \tag{11}$$

However, the commutator between two position operators is in general different to zero

$$[\hat{X}_i, \hat{X}_j] = i\hbar\frac{(2\beta - \beta') + (2\beta + \beta')\beta\hat{P}^2}{1 + \beta\hat{P}^2}(\hat{P}_i\hat{X}_j - \hat{P}_j\hat{X}_i). \tag{12}$$

It is obvious that the generalized canonical commutation relation (10) leads to the minimal length $(\Delta X_i)_{\min} = \hbar\sqrt{3\beta + \beta'}$.

In this generalization, we have several representations for the canonical operators X_i and P_i. Among these representations, we cite the momentum space representation used by Chang et al. to solve the d-dimensional harmonic oscillator [18]

$$\hat{X}_i = i\hbar\left[\left(1+\beta p^2\right)\frac{\partial}{\partial p_i} + \beta' p_i p_j \frac{\partial}{\partial p_j}\right] + \gamma p_i, \qquad \hat{P}_i = p_i, \quad (13)$$

and the position representation given by [20,21]

$$\hat{X}_i = \hat{x}_i + \frac{2\beta-\beta'}{4}\left(\hat{p}^2\hat{x}_i + \hat{x}_i\hat{p}^2\right), \qquad \hat{P}_i = \hat{p}_i\left(1+\frac{\beta'}{2}\hat{p}^2\right), \quad (14)$$

where $\hat{x}_i$ and $\hat{p}_i$ are the ordinary position and momentum operators $[\hat{x}_i, \hat{p}_j] = i\hbar\delta_{ij}$ and $\hat{p}^2 = \sum_i \hat{p}_i^2$. When $\beta' = 2\beta$, we can see that $[\hat{X}_i, \hat{X}_j] = 0$ to the first order on β and (14) reduces to

$$\hat{X}_i = \hat{x}_i, \qquad P_i = \left(1+\beta\hat{p}^2\right)\hat{p}_i. \quad (15)$$

As in the previous subsection, we consider, in the rest of this Letter, the position space representation (15). Then, for a central potential $V(r)$, the corrected Schrödinger equation to the first order on β can be written as

$$\left[\frac{p^2}{2m} + \frac{\beta}{m}p^4 + V(r) - E\right]\psi(\vec{r}) = 0. \quad (16)$$

If we make the substitution $\psi(\vec{r}) \to \Phi(\vec{r})$, with

$$\psi(\vec{r}) = \left(1-2\beta p^2\right)\Phi(\vec{r}) \quad (17)$$

we see that the function $\Phi(\vec{r})$ satisfies the following equation

$$\left[\left(1+4m\beta\left(E-V(r)\right)\right)\frac{p^2}{2m} + V(r) - E\right]\Phi(\vec{r}) = 0. \quad (18)$$

Besides, if we decompose $\Phi(\vec{r})$ to an angular part and a radial one

$$\Phi(\vec{r}) = R(r)Y_l^m(\theta,\phi) \quad (19)$$

and we take into account that

$$p^2 = p_r^2 + \frac{L^2}{r^2} \quad (20)$$

where L^2 is the angular momentum operator and $p_r = -i\hbar\frac{1}{r}\frac{\partial}{\partial r}r$, we arrive directly at the following modified radial equation

$$\left[\left(1+4m\beta\left(E-V(r)\right)\right)\left(\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} - \frac{l(l+1)}{r^2}\right) - 2m\left(V(r)-E\right)\right]R(r) = 0, \quad (21)$$

which is of second order and expected to have simple and analytic solutions. Here, we remark that the physical solutions that are regular at the origin must obey the boundary condition

$$\lim_{r\to 0}\left[rR(r) + 2\beta\hbar^2\left(r\frac{\partial^2}{\partial r^2}R(r) + 2\frac{\partial}{\partial r}R(r) - l(l+1)\frac{R(r)}{r}\right)\right] = 0 \quad (22)$$

instead of the usual condition $rR(r) \to 0$.

3. Scattering by a spherical well

The study of the scattering process by a square well in one dimension is already studied starting from the modified Schrödinger equation (7) [34]. It is, then, obvious that the attempt to use the present approximation gives the same results to the first order on β. In this section, we consider only the important three-dimensional case which is the more realistic situation. Because of the existence of the nonzero minimal position uncertainty, the definition of a square well with sharp boundary condition does not make sense. So we define the potential well as follows

$$V(r) = \begin{cases} -V_0 f(r) & \text{if } r \leqslant a, \\ 0 & \text{if } r > a, \end{cases} \quad (23)$$

where $f(r)$ is a smooth function fulfilling the conditions $f(0) = 1$ and $f(a) = 0$. The definition (23) divides our space into two parts. The first one is the sphere of radius a, where $V(r) = -V_0 f(r)$ and the second region is outside of the sphere, where the particle behaves like free one. Outside of the sphere, the wave equation can be written as

$$\left[\frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \kappa^2 - \frac{l(l+1)}{r^2}\right]R_{E,l} = 0 \quad (24)$$

where κ is given by

$$\hbar^2\kappa^2 = \frac{2mE}{1+4m\beta E}. \quad (25)$$

Eq. (24) is nothing but the spherical Bessel equation. The general solution of this equation is given in terms of the spherical Bessel function $j_l(\kappa r)$ and the Neumann function $\eta_l(\kappa r)$

$$R_{E,l}^I(r) = A_l j_l(\kappa r) + B_l \eta_l(\kappa r), \quad (26)$$

where A_l and B_l are two constants. When $r \to +\infty$, we can see that $R_{E,l}^I(r)$ has the following asymptotic behavior

$$R_{E,l}^I(r) = \sqrt{A_l^2 + B_l^2}\,\frac{\sin(\kappa r - l\frac{\pi}{2} + \delta_l)}{\kappa r}, \quad (27)$$

where the phase shift δ_l is defined by

$$\tan\delta_l = -\frac{B_l}{A_l}. \quad (28)$$

Since the potential has an arbitrary form for $r < a$, we cannot find closed expression for the solution of the modified Schrödinger equation in this region. Fortunately, this does not prevent from finding the resonance condition and the resonant cross section. For instance, if we note $R_{E,l}^{II}(r)$ the solution for $r < a$ and we define

$$\alpha_l(E) = \frac{\partial}{\partial r}\ln R_{E,l}^{II}(r)\bigg|_{r=a}, \quad (29)$$

we can extract the phase shifts from the continuity conditions of the wave function and its derivative at $r = a$. This gives us the equation

$$\tan\delta_l = \frac{\alpha_l(E)j_l(\kappa a) - \kappa j_l'(\kappa a)}{\alpha_l(E)\eta_l(\kappa a) - \kappa\eta_l'(\kappa a)}, \quad (30)$$

which enables us to study the effect of the minimal length on the resonant scattering. Since the resonant scattering occurs when $\delta_l \to \frac{\pi}{2}$, $\tan\delta_l \to +\infty$ and $\sin\delta_l \to 1$, we find that the resonance condition reads

$$\alpha_l(E) = \kappa\frac{\eta_l'(\kappa a)}{\eta_l(\kappa a)}. \quad (31)$$

Now, by expanding $\alpha_l(E)$ near the resonance energy

$$\alpha_l(E) \approx \alpha_l(E_r) + (E-E_r)\alpha_l'(E_r) + \cdots \quad (32)$$

we can write $\tan\delta_l$ in the form

$$\tan\delta_l = -\frac{\Gamma}{2(E-E_r)}, \quad (33)$$

where Γ is given by

$$\Gamma = \frac{1}{\alpha_l'(E_r)}\frac{\partial}{\partial r}\frac{j_l(\kappa r)}{\eta_l(\kappa r)}\bigg|_{r=a}. \quad (34)$$

The partial cross section will be then given by the Breit–Wigner formula

$$\sigma_l = \frac{4\pi}{\kappa^2}(2l+1)\frac{(\frac{\Gamma}{2})^2}{(E-E_r)^2+(\frac{\Gamma}{2})^2}. \tag{35}$$

Let us note that the resonant cross section is given by

$$\sigma_l^{res} = \frac{4\pi(2l+1)}{\kappa^2} = 4\pi a^2\left[\frac{(2l+1)\hbar^2}{2ma^2E}+2\frac{\beta\hbar^2}{a^2}(2l+1)\right]. \tag{36}$$

In the latter equation the term $2\frac{\beta\hbar^2}{a^2}(2l+1)$ is the correction due to the minimal length. As a result, we conclude that the minimal length increases slightly the resonant cross section. The relative correction can be written as

$$\frac{\Delta\sigma_l^{res}}{\sigma_l^{res}} = 4\beta m E. \tag{37}$$

Since our starting point is the fact that $\beta p^2 \ll 1$, we can see that this effect is not important in the low energy approximation.

4. The square well approximation

Besides the amplification of the resonant cross section the minimal length causes a small shift on the resonance position. Since, in the scattering process, the cross section may change drastically with small variation of the energy, the effect of the GUP may be, then, of interest even if the induced minimal length is very small compared to the potential width. To clarify this point, let us consider the exactly solvable square well defined by

$$V(r) = \begin{cases} -V_0 & \text{if } r \leqslant a, \\ 0 & \text{if } r > a. \end{cases} \tag{38}$$

As is said above, because of the nonzero position uncertainty, this definition makes sense only when $a \gg (\Delta x)_{\min}$.

With this shape of potential, the modified Schrödinger equation in the region $r < a$ takes the same form as (24) with the change $\kappa^2 \to q^2$, where

$$\hbar^2 q^2 = \frac{2m(E+V_0)}{1+4m\beta(E+V_0)}. \tag{39}$$

The physical solution is then given by

$$R_{E,l}^{\mathrm{II}}(r) = C_l j_l(qr), \tag{40}$$

where C_l is a constant. Therefore, $\tan\delta_l$ takes the following expression

$$\tan\delta_l = \frac{\kappa j_l'(\kappa a) j_l(qa) - q j_l'(qa) j_l(\kappa a)}{\kappa \eta_l'(\kappa a) j_l(qa) - q j_l'(qa)\eta_l(\kappa a)}. \tag{41}$$

Let us note that when $\beta \to 0$, κ becomes the same as the ordinary wave vector $k = \sqrt{2mE/\hbar^2}$ and $q = \sqrt{2m(E+V_0)/\hbar^2}$. In such a case, our results (i.e. Eqs. (33), (34), (35), (36) and (41)) reduce to those of the ordinary quantum mechanics.

Let us now consider the s-channel scattering. By the use of the explicit forms of $j_0(x)$ and $\eta_0(x)$ we obtain, for $l = 0$,

$$\tan\delta_0 = \frac{\kappa\tan(qa) - q\tan(\kappa a)}{\kappa\tan(qa)\tan(\kappa a)+q}. \tag{42}$$

The partial cross section σ_0 is then given by

$$\sigma_0 = 4\pi\frac{(\tan(qa) - \frac{q}{\kappa}\tan(\kappa a))^2}{(1+\tan^2 a\kappa)(q^2+\kappa^2\tan^2 qa)}. \tag{43}$$

For low energy case, we have

$$\tan\delta_0 \simeq \kappa a\left(\frac{\tan(qa)}{qa} - 1\right) \tag{44}$$

and

$$\sigma_0 = 4\pi a^2\frac{(\frac{\tan(qa)}{qa}-1)^2}{1+\kappa^2 a^2(\frac{\tan(qa)}{qa}-1)^2}. \tag{45}$$

Here, we can see that σ_0 reach its maximum when $\cos aq = 0$ and consequently, the resonance condition for a given energy is that $aq = (n+\frac{1}{2})\pi$, which gives

$$\frac{2ma^2}{\hbar^2}(V_0+E) = \frac{(n+\frac{1}{2})^2\pi^2}{1-\frac{\hbar^2\beta(2\pi)^2}{a^2}(n+\frac{1}{2})^2}. \tag{46}$$

Because the fact that the scattering energy is always positive (i.e. $E > 0$), the latter condition makes sense only when $\frac{2ma^2}{\hbar^2}V_0 \lesssim (n+\frac{1}{2})^2\pi^2[1-\frac{\hbar^2\beta(2\pi)^2}{a^2}(n+\frac{1}{2})^2]^{-1}$. On the other hand, even if $\frac{\hbar^2\beta}{a^2} \ll 1$, we can find, at least, a value of V_0 for which $(n+\frac{1}{2})^2\pi^2 < \frac{2ma^2}{\hbar^2}V_0 \lesssim (n+\frac{1}{2})^2\pi^2[1-\frac{\hbar^2\beta(2\pi)^2}{a^2}(n+\frac{1}{2})^2]^{-1}$. With such a potential, there is no resonant scattering in ordinary quantum mechanics (i.e. without minimal length) because $\frac{2ma^2}{\hbar^2}V_0 > (n+\frac{1}{2})^2\pi^2$ and the resonance condition, for $\beta = 0$, is $\frac{2ma^2}{\hbar^2}(V_0+E) = (n+\frac{1}{2})^2\pi^2$. In the presence of GUP, however, resonances may be observed because $\frac{2ma^2}{\hbar^2}V_0 \lesssim (n+\frac{1}{2})^2\pi^2[1-\frac{\hbar^2\beta(2\pi)^2}{a^2}(n+\frac{1}{2})^2]^{-1}$. Therefore, the effect of the minimal length on the s-channel scattering becomes important if

$$\left(n+\frac{1}{2}\right)^2\pi^2 < \frac{2ma^2}{\hbar^2}V_0 < \frac{(n+\frac{1}{2})^2\pi^2}{1-\frac{\hbar^2\beta(2\pi)^2}{a^2}(n+\frac{1}{2})^2}. \tag{47}$$

This effect is much important for $l \geqslant 1$, because in this case the resonant scattering becomes very sharp [4]. For example, for $l = 1$, we obtain

$$\sigma_1 = \frac{12\pi}{\tan^2(\kappa a)+1}\frac{((\frac{q^2-\kappa^2}{q\kappa})\tan(\kappa a)+\kappa a\tan(\kappa a)\cot(qa)-qa)^2}{\kappa^2((qa)^2+(\frac{q^2-\kappa^2}{q\kappa}+\kappa a\cot(qa))^2)}. \tag{48}$$

For low energy case, we can see that

$$\tan\delta_1 \simeq \frac{(\kappa a)^3}{(qa)^2}\left(1-(qa)\cot(qa)-\frac{1}{3}(qa)^2\right) \tag{49}$$

and

$$\sigma_1 \simeq 12\pi a^2\left(\frac{\kappa}{q}\right)^4\frac{(1-(qa)\cot(qa)-\frac{1}{3}(qa)^2)^2}{1+\frac{(\kappa a)^6}{(qa)^4}(1-(qa)\cot(qa)-\frac{1}{3}(qa)^2)^2}. \tag{50}$$

Then, the resonance occurs when $\sin(qa) = 0$ or, more explicitly,

$$\frac{2ma^2}{\hbar^2}(V_0+E) = \frac{n^2\pi^2}{1-\frac{\hbar^2\beta(2\pi)^2}{a^2}n^2}. \tag{51}$$

It follows from Eq. (51) that there is no resonant scattering with $\beta = 0$ and $\frac{2ma^2}{\hbar^2}V_0 > n^2\pi^2$. When $\beta > 0$, resonant scattering could take place if $\frac{2ma^2}{\hbar^2}V_0 < n^2\pi^2[1-\frac{\hbar^2\beta(2\pi)^2}{a^2}n^2]^{-1}$. In addition, since we have, at low energies, $\sigma_1 \sim \frac{1}{\kappa^2}$ near the resonance energy and $\sigma_1 \sim \kappa^4$ for other values of κ, the resonance is then very sharp.

As example, let us consider the case where $\frac{2ma^2}{\hbar^2}V_0 = 39.50$. In Fig. 1, we plot $\bar{\sigma} = \frac{\sigma}{4\pi a^2}$ with $\sigma = \sigma_0 + \sigma_1$ as function of

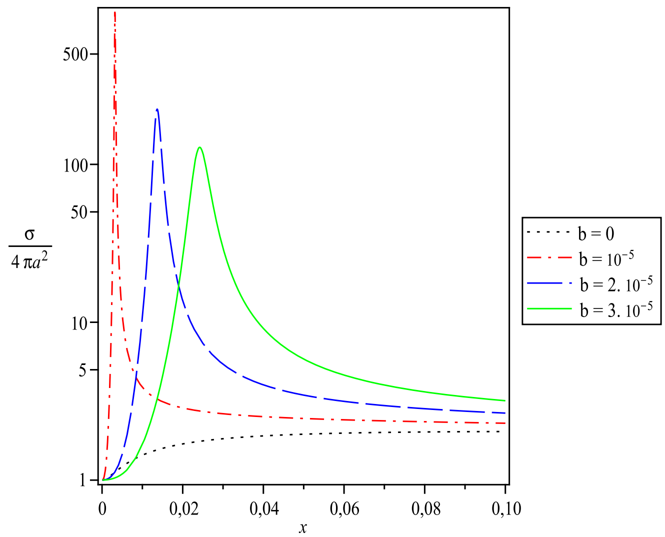

Fig. 1. Plotting $\sigma = \sigma_0 + \sigma_1$, normalized to $4\pi a^2$, as a function of the variable $x = \frac{2ma^2E}{\hbar^2}$. The dimensionless quantity $\frac{2ma^2V_0}{\hbar^2}$ is taken 39.5. The parameter b is defined by $b = \frac{\beta\hbar^2}{a^2}$.

the variable $x = \frac{2ma^2E}{\hbar^2}$ for $\frac{2ma^2V_0}{\hbar^2} = 39.50$ and various values of $\frac{\hbar^2\beta}{a^2} \equiv b$. We remark that the presence of the minimal length may leads to an important resonance in the range $0 \leqslant x \leqslant 0.1$, in contrast to the ordinary case (i.e. without minimal length), where there is no resonance.

Since we have seen that the minimal length causes a small correction to the resonant scattering, the effect shown in Fig. 1 seems to be strange. This effect is due to the fact that the cross sections represented in Fig. 1 are plotted for the same value of V_0 ($\frac{2ma^2V_0}{\hbar^2} = 39.50$), which leads to resonance only for some parameters $\beta > 0$. For $\beta = 0$, resonant scattering could take place if $\frac{2ma^2V_0}{\hbar^2} \lesssim n^2\pi^2$. When $n = 2$, this condition becomes $\frac{2ma^2V_0}{\hbar^2} \lesssim 39.48$. Then, the use of $\frac{2ma^2V_0}{\hbar^2} = 39.50 > 4\pi^2$, shows resonances only in the presence of a nonzero minimal length.

At the end of this work, let us notice that the above formulation permits us to study the so-called Ramsauer–Townsend (RT) effect, where the partial cross section $\sigma_l = 0$. In this case, the l-channel does not contribute to the scattering. In the presence of the minimal length, this effect occurs when $\alpha_l(E)j_l(\kappa a) - \kappa j_l'(\kappa a) = 0$. As for the resonant scattering, we remark, here, that the minimal length causes a shift of the RT position, which may be of interest to experimental physics. Let us note, that the RT effect may be observed in laboratory with electrons of 0.7–1 eV energy scattered by noble gas atoms [35]. This may be a straightforward method to check experimentally the existence of an observable minimal length. However, since the scattering cross section is very small near the RT effect, we expect that the influence of the minimal length on the resonant scattering is more important that its influence on the RT effect.

5. Conclusion

In this work, we have studied the consequences of the minimal length, which arises in many theories of quantum gravity, on the scattering of material particles by a spherically symmetric potential. This minimal length involves a correction with p^4 to the Schrödinger equation in position space representation, what complicates the problem of finding analytic solutions. For this reason, we have factorized the modified equation by introducing an auxiliary wave function in order to eliminate the p^4 term. Since the resulting equation is of second order, we were able to show that the partial cross section is given by the Breit–Wigner formula. For the square well potential, we have obtained analytic expressions for the scattering states and the phase shifts.

We conclude, through this study, that the introduction of the concept of minimal length implies two consequences on the scattering resonances. The first one is that the minimal length increases slightly the resonant cross section. This effect can be explained by the fact that the particle can be considered in this theory as a *ball-point* having a finite size which is of order of the minimal length [15]. Therefore, from classical point of view the minimal length (i.e. the size of the particle) increases the cross section. The second consequence is that the minimal length shifts the position of the resonances. This shift is of interest, because it gives chance to observe resonances even if there is no resonance in the ordinary quantum mechanics (i.e. without minimal length). This corresponds to the values of V_0 that are greater than the critical values leading to resonances in the ordinary quantum mechanics. The importance of this effect in experimental physics depends on the precision with which V_0 is considered. Thus, if in future experiments one finds a similar shift to the resonance energy, it could be a clear signal of this quantum correction.

The effect of the minimal length may be, also, very interesting in quantum field theory; at the electroweak scale, and, especially, the gauge boson production. If we consider, for example, the electron–positron annihilation we can see that a resonant scattering produces a Z boson, where the properties of Z are determined with high precision [36]. In the GUP theory, the Z boson mass is expected to contain small corrections depending on the parameter β. This could play an important role on the examination of the existence of the minimal length experimentally or at least the establishment of an upper bond for it. This problem is actually under consideration.

Let us note that the present methodology gives analytic solutions for a large number of potentials and, consequently, it permits us to study the influence of the generalized uncertainty principle on many quantum effects. It is valid also for bound states problems, when β^2p^6 is negligible compared to βp^4. Furthermore, it may be generalized straightforwardly to the relativistic case.

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Appendix A. The continuity equation, the scattered wave current and scattering amplitude

It is obvious that from Eq. (16) we obtain easily the continuity equation $\frac{\partial}{\partial t}\rho + \operatorname{div}\vec{j} = 0$, with $\rho = \psi^*\psi$ and

$$\vec{j} = \frac{i\hbar}{2m}\big[\big(\psi - 2\hbar^2\beta\Delta\psi\big)\vec{\nabla}\big(\psi^* - 2\hbar^2\beta\Delta\psi^*\big) - \big(\psi^* - 2\hbar^2\beta\big(\Delta\psi^*\big)\big)\vec{\nabla}\big(\psi - 2\hbar^2\beta\big(\Delta\psi\big)\big)\big]. \tag{52}$$

In terms of the auxiliary function $\Phi(\vec{r})$ the quantities ρ and $\vec{j}$ are given by

$$\vec{j} = \frac{i\hbar}{2m}\big[\Phi\vec{\nabla}\Phi^* - \Phi^*\vec{\nabla}\Phi\big] \tag{53}$$

and

$$\rho = \Phi^*\Phi + 2\hbar^2\beta\big(\Phi^*\Delta\Phi + \Phi\Delta\Phi^*\big). \tag{54}$$

Now, if we suppose that the incident particles move along the z direction we can write the general solution of the modified Schrödinger equation as follows

$$\Phi(\vec{r}) \sim e^{i\kappa z} + f(\theta)\frac{e^{i\kappa r}}{r}, \tag{55}$$

where κ is given by Eq. (25) and

$$f(\theta) = \frac{1}{\kappa}\sum_l (2l+1)P_l(\cos\theta)e^{i\delta_l}\sin\delta_l.$$

When $r \gg 1$, the current $\vec{j}$ can be then written in the form $\vec{j}_{inc} + \vec{j}_{scatt}$ where $\vec{j}_{inc} = \frac{\hbar}{m}\kappa\ \vec{u}_z$ and

$$\vec{j}_{scatt} = \frac{|f(\theta)|^2}{r^2}\vec{u}_r + \mathcal{O}\left(\frac{1}{r^3}\right). \tag{56}$$

This leads to the following partial decomposition of the total cross section

$$\sigma = \sum_l \sigma_l,$$

where σ_l is given by

$$\sigma_l = \frac{4\pi}{\kappa^2}(2l+1)\sin^2\delta_l. \tag{57}$$

Let us remark that σ_l given in (57) has the same form as that of the ordinary partial cross section with the change $k \to \kappa$, where k is the ordinary wave vector.

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