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Nuclear Physics B 961 (2020) 115234

NUCLEAR PHYSICS B

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$\mathcal{N}=4$ supersymmetric U(2)-spin hyperbolic Calogero-Sutherland model

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Received 5 August 2020; accepted 29 October 2020
Available online 3 November 2020
Editor: Stephan Stieberger

Abstract

The $\mathcal{N}=4$ supersymmetric U(2)-spin hyperbolic Calogero-Sutherland model with odd matrix fields is examined. Explicit form of the $\mathcal{N}=4$ supersymmetry generators is derived. The Lax representation for the dynamics of the $\mathcal{N}=4$ hyperbolic U(2)-spin Calogero-Sutherland system is found. The reduction to the $\mathcal{N}=4$ supersymmetric spinless hyperbolic Calogero-Sutherland system is established.

1. Introduction

One of the important developments in the study of the famous many-particle Calogero-Sutherland systems [1,2] (see [3,4] for reviews) is their generalization to supersymmetric cases. Most of the researches in these directions have been devoted to supersymmetrization of the rational Calogero systems (see, for example, the papers [5–19] and the review [20]). Supersymmetric generalizations of the hyperbolic and trigonometric Calogero-Sutherland systems have been studied in a very limited number of works (see, for example, the papers [17–19,21–27] and references therein).

In a recent paper [28], $\mathcal{N}=2$ and $\mathcal{N}=4$ supersymmetric generalizations of the hyperbolic Calogero-Sutherland system were proposed using the gauging procedure [11,29] (see also the matrix description of the Calogero models in [4,30,31]). In the paper [32] the $\mathcal{N}=2$ hyperbolic

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https://doi.org/10.1016/j.nuclphysb.2020.115234

Calogero-Sutherland model [28] was considered. In this paper, the Hamiltonian analysis of the $\mathcal{N}=4$ many-particle system obtained in [28] was studied in detail.

At the component level, the $\mathcal{N}=4$ matrix model obtained in [28] is described by the positive definite Hermitian c-number $(n\times n)$–matrix field $X(t) := \|X_a{}^b(t)\|$, $(X_a{}^b)^* = X_b{}^a$, $\det X \neq 0$, and the Hermitian c-number $(n\times n)$–matrix gauge field $A(t) := \|A_a{}^b(t)\|$, $(A_a{}^b)^* = A_b{}^a$ $(a, b = 1, \dots, n)$. In opposite to the $\mathcal{N}=2$ case [32], the $\mathcal{N}=4$ model uses the complex odd $(n\times n)$–matrix field $\Psi^i(t) := \|\Psi^i{}_a{}^b(t)\|$, $\bar\Psi_i(t) := \|\bar\Psi_i{}_a{}^b(t)\|$, $(\Psi^i{}_a{}^b)^* = \bar\Psi_i{}_b{}^a$, and the complex c-number U(n)-spinor field $Z^k(t) := \|Z^k_a(t)\|$, $\bar Z_k(t) := \|\bar Z^a_k(t)\|$, $\bar Z^a_k = (Z^k_a)^*$, which have additional SU(2)-spinor indices $i, k = 1, 2$. This $\mathcal{N}=4$ n-particle system is described by the on-shell component action $S_{\rm matrix} = \int \mathrm{d}t\, L_{\rm matrix}$ with the Lagrangian (system II in [28])

$$
\begin{aligned}
L_{\rm matrix} = {} & \frac{1}{2}\,\mathrm{Tr}\Big(X^{-1}\nabla X\, X^{-1}\nabla X + 2c\, A\Big) \;+\; \frac{i}{2}\Big(\bar Z_k \nabla Z^k - \nabla \bar Z_k Z^k\Big) \\
& + \frac{i}{2}\,\mathrm{Tr}\Big(X^{-1}\bar\Psi_k X^{-1}\nabla\Psi^k - X^{-1}\nabla\bar\Psi_k X^{-1}\Psi^k\Big) \\
& + \frac{1}{8}\,\mathrm{Tr}\Big(\{X^{-1}\Psi^i, X^{-1}\bar\Psi_i\}\{X^{-1}\Psi^k, X^{-1}\bar\Psi_k\}\Big),
\end{aligned}
\tag{1.1}
$$

where the quantity c is a real constant and the covariant derivatives are defined by $\nabla X = \dot X + i\,[A, X]$ and $\nabla\Psi^k = \dot\Psi^k + i\,[A, \Psi^k]$, $\nabla Z^k = \dot Z^k + i\,A Z^k$ and c.c.

Despite the external similarity of the Lagrangian (1.1) with the $\mathcal{N}=2$ supersymmetric Lagrangian [32], the SU(2)-spinor character of the Grassmann matrix quantities Ψ^i and semi-dynamical even variables Z^i lead to the distinctive properties of the $\mathcal{N}=4$ system under consideration. First, using the SU(2)-spinors Z^i leads to the $\mathcal{N}=4$ matrix system that is supersymmetric generalization of the U(2)-spin hyperbolic Calogero-Sutherland system, and not the spinless hyperbolic Calogero-Sutherland system as in the $\mathcal{N}=2$ case [32]. Second, due to the SU(2)-spinor nature of the Grassmann matrix quantities Ψ^i, the $\mathcal{N}=4$ supercharges contain additional terms in odd variables, that were absent in the $\mathcal{N}=2$ case [32]. This paper examines the $\mathcal{N}=4$ case in detail in order to identify these and other distinctive properties of this $\mathcal{N}=4$ system.

The plan of the paper is as follows. In Section 2, the Hamiltonian formulation of the matrix system (1.1) is presented. Partial gauge fixing eliminates purely gauge bosonic off-diagonal matrix fields and yields a classically-equivalent system, whose bosonic limit is exactly the multi-particle U(2)-spin hyperbolic Calogero-Sutherland system. Using the Noether procedure in Section 3 allows one to find the full set of $\mathcal{N}=4$ supersymmetry generators. The Dirac brackets superalgebra of these generators is closed to first class constraints. Section 4 is devoted to the construction of the Lax representation for the equation of motion of the $\mathcal{N}=4$ supersymmetric U(2)-spin hyperbolic Calogero-Sutherland system. Section 5 presents the reduction of the considered U(2)-spin system that yields the $\mathcal{N}=4$ supersymmetric spinless hyperbolic Calogero-Sutherland system. Section 6 contains summary and outlook.

2. Hamiltonian formulation

Here we present the Hamiltonization of the matrix system (1.1) with the U(n) gauge symmetry and its partial gauge-fixing.

2.1. Hamiltonian formulation of the matrix system

The system with the Lagrangian (1.1) is described by pairs of the phase variables $(X_a{}^b, P_c{}^d)$, $(Z^i_a, \mathcal{P}^b_k)$, $(\bar{Z}^a_i, \bar{\mathcal{P}}^k_b)$, $(\Psi^i{}_a{}^b, \Pi_{kc}{}^d)$, $(\bar{\Psi}_{ia}{}^b, \bar{\Pi}^k{}_c{}^d)$ with the nonvanishing canonical Poisson brackets

$$\{X_a{}^b, P_c{}^d\}_{\rm P} = \delta^d_a\delta^b_c\,, \qquad \{Z^i_a, \mathcal{P}^b_k\}_{\rm P} = \delta^b_a\delta^i_k\,, \quad \{\bar{Z}^a_i, \bar{\mathcal{P}}^k_b\}_{\rm P} = \delta^a_b\delta^k_i\,, \tag{2.1}$$

$$\{\Psi^i{}_a{}^b, \Pi_{kc}{}^d\}_{\rm P} = \delta^d_a\delta^b_c\delta^i_k\,, \quad \{\bar{\Psi}_{ia}{}^b, \bar{\Pi}^k{}_c{}^d\}_{\rm P} = \delta^d_a\delta^b_c\delta^k_i\,. \tag{2.2}$$

The phase variables are subject to the primary constraints

$$G^a_k := \mathcal{P}^a_k - \frac{i}{2}\,\bar{Z}^a_k \approx 0\,, \qquad \bar{G}^k_a := \bar{\mathcal{P}}^k_a + \frac{i}{2}\,Z^k_a \approx 0\,, \tag{2.3}$$

$$\Upsilon_{ka}{}^b := \Pi_{ka}{}^b - \frac{i}{2}\,(X^{-1}\bar{\Psi}_k X^{-1})_a{}^b \approx 0\,, \qquad \bar{\Upsilon}^k{}_a{}^b := \bar{\Pi}^k{}_a{}^b - \frac{i}{2}\,(X^{-1}\Psi^k X^{-1})_a{}^b \approx 0\,. \tag{2.4}$$

Besides, the matrix momentum of $X_a{}^b$ has the form

$$P_a{}^b = (X^{-1}\nabla X X^{-1})_a{}^b \tag{2.5}$$

and the momenta of the coordinates $A_a{}^b$ are zero.

The canonical Hamiltonian of the system has the form

$$H_{\rm matrix} = P_b{}^a\dot{X}_a{}^b + \mathcal{P}^a_k\dot{Z}^k_a + \bar{\mathcal{P}}^k_a\dot{\bar{Z}}^a_k + \Pi_{kb}{}^a\dot{\Psi}^k{}_a{}^b + \bar{\Pi}^k{}_b{}^a\dot{\bar{\Psi}}_{ka}{}^b - L_{\rm matrix} = H + {\rm Tr}\big(AF\big)\,, \tag{2.6}$$

where the first term has the following form

$$H = \frac{1}{2}\,{\rm Tr}\Big(XPXP\Big) - \frac{1}{8}\,{\rm Tr}\Big(\{X^{-1}\Psi^i, X^{-1}\bar{\Psi}_i\}\{X^{-1}\Psi^k, X^{-1}\bar{\Psi}_k\}\Big) \tag{2.7}$$

and the second term ${\rm Tr}\big(AF\big)$ uses the quantities

$$F_a{}^b := i[P,X]_a{}^b + Z^k_a\bar{Z}^b_k - \frac{1}{2}\,\{X^{-1}\Psi^k, X^{-1}\bar{\Psi}_k\}_a{}^b - \frac{1}{2}\,\{\Psi^k X^{-1}, \bar{\Psi}_k X^{-1}\}_a{}^b - c\,\delta_a{}^b\,. \tag{2.8}$$

Vanishing momenta of $A_a{}^b$ indicate that quantities (2.8) are the secondary constraints

$$F_a{}^b \approx 0\,. \tag{2.9}$$

The variables $A_a{}^b$ in the Hamiltonian (2.6) play the role of the Lagrange multipliers for these constraints.

The constraints (2.3) and (2.4) possess the following nonzero Poisson brackets:

$$\{G^a_i, \bar{G}^k_b\}_{\rm P} = -i\delta^a_b\delta^k_i\,, \qquad \{\Upsilon_{ia}{}^b, \bar{\Upsilon}^k{}_c{}^d\}_{\rm P} = -iX^{-1d}_{\ \ a}X^{-1b}_{\ \ c}\delta^k_i \tag{2.10}$$

and are the second class constraints. Using the Dirac brackets for the constraints (2.3), (2.4)

$$\{A,B\}_{\rm D} = \{A,B\}_{\rm P} + i\{A,G^a_k\}_{\rm P}\{\bar{G}^k_a,B\}_{\rm P} - i\{A,\bar{G}^k_a\}_{\rm P}\{G^a_k,B\}_{\rm P} \tag{2.11}$$

$$-i\{A,\Upsilon_{ka}{}^b\}_{\rm P}X_b{}^cX_d{}^a\{\bar{\Upsilon}^k{}_c{}^d,B\}_{\rm P} - i\{A,\bar{\Upsilon}^k{}_a{}^b\}_{\rm P}X_b{}^cX_d{}^a\{\Upsilon_{kc}{}^d,B\}_{\rm P}\,,$$

we eliminate the momenta $\mathcal{P}^a_k$, $\bar{\mathcal{P}}^k_a$, $\Pi_{ka}{}^b$, $\bar{\Pi}^k{}_a{}^b$. The nonvanishing Dirac brackets of residual phase variables take the form

$$\{X_a{}^b, P_c{}^d\}_{\rm D} = \delta_a^d\delta_c^b\,, \tag{2.12}$$

$$\begin{aligned}\{P_a{}^b, P_c{}^d\}_{\rm D} &= -\tfrac{i}{4}[X^{-1}(\Psi^k X^{-1}\bar\Psi_k + \bar\Psi_k X^{-1}\Psi^k)X^{-1}]_a{}^d X^{-1}{}_c^b \\ &\quad + \tfrac{i}{4}X^{-1}{}_a^d[X^{-1}(\Psi^k X^{-1}\bar\Psi_k + \bar\Psi_k X^{-1}\Psi^k)X^{-1}]_c{}^b\,,\end{aligned} \tag{2.13}$$

$$\{Z_a^i, \bar Z_k^b\}_{\rm D} = -i\delta_a^b\delta_k^i\,, \qquad \{\Psi^i{}_a{}^b, \bar\Psi_{kc}{}^d\}_{\rm D} = -iX_a{}^d X_c{}^b\delta_k^i\,, \tag{2.14}$$

$$\begin{aligned}\{\Psi^k{}_a{}^b, P_c{}^d\}_{\rm D} &= \tfrac{1}{2}\delta_a^d(X^{-1}\Psi^k)_c{}^b + \tfrac{1}{2}\delta_c^b(\Psi^k X^{-1})_a{}^d\,, \\ \{\bar\Psi_{ka}{}^b, P_c{}^d\}_{\rm D} &= \tfrac{1}{2}\delta_a^d(X^{-1}\bar\Psi_k)_c{}^b + \tfrac{1}{2}\delta_c^b(\bar\Psi_k X^{-1})_a{}^d\,.\end{aligned} \tag{2.15}$$

The residual constraints $F_a{}^b = (F_b{}^a)^*$, defined in (2.9), form the $u(n)$ algebra with respect to the Dirac brackets (2.11):

$$\{F_a{}^b, F_c{}^d\}_{\rm D} = -i\delta_a{}^d F_c{}^b + i\delta_c{}^b F_a{}^d\,. \tag{2.16}$$

So the constraints (2.8), (2.9) are the first class ones and generate local U(n) transformations

$$\delta C = \sum_{a,b}\alpha_b{}^a\{C, F_a{}^b\}_{\rm D} \tag{2.17}$$

of an arbitrary phase variable C where $\alpha_a{}^b(\tau) = (\alpha_b{}^a(\tau))^*$ are the local parameters. These transformations of the primary phase variables have the form

$$\begin{gathered}\delta X_a{}^b = -i[\alpha, X]_a{}^b\,, \quad \delta P_a{}^b = -i[\alpha, P]_a{}^b\,, \quad \delta Z_a^k = -i(\alpha Z^k)_a\,, \quad \delta\bar Z_{ka} = i(\bar Z_k\alpha)^a\,, \\ \delta\Psi^k{}_a{}^b = -i[\alpha, \Psi^k]_a{}^b\,, \quad \delta\bar\Psi_{ka}{}^b = -i[\alpha, \bar\Psi_k]_a{}^b\,.\end{gathered} \tag{2.18}$$

2.2. Hamiltonian formulation of partial gauge-fixing of the matrix system

The gauges $X_a{}^b = 0$ at $a \neq b$ fix the local transformations (2.18) with the parameters $\alpha_a{}^b(\tau)$, $a\neq b$ generated by the off-diagonal constraints $F_a{}^b \approx 0$, $a \neq b$ in the set (2.8), (2.9). This gauge fixing takes the form [11,28,32]

$$x_a{}^b \approx 0 \tag{2.19}$$

if we apply the expansions

$$X_a{}^b = x_a\delta_a{}^b + x_a{}^b\,, \qquad P_a{}^b = \mathrm{p}_a\delta_a{}^b + \mathrm{p}_a{}^b\,, \tag{2.20}$$

where $x_a{}^b$ and $\mathrm{p}_a{}^b$ represent the off-diagonal matrix quantities. In addition, using the constraints $F_a{}^b \approx 0$, $a \neq b$, we express the momenta $\mathrm{p}_a{}^b$ through the remaining phase variables:

$$\mathrm{p}_a{}^b = -\frac{i\,Z_a^k\bar Z_k^b}{x_a - x_b} + \frac{i\,(x_a + x_b)\,\{\Phi^k, \bar\Phi_k\}_a{}^b}{2(x_a - x_b)\sqrt{x_a x_b}}\,, \tag{2.21}$$

where we use the odd matrix variables $\Phi^k{}_a{}^b$, $\bar\Phi_{ka}{}^b = (\Phi^k{}_b{}^a)^*$ defined by

$$\Phi^k{}_a{}^b := \frac{\Psi^k{}_a{}^b}{\sqrt{x_a x_b}}\,, \qquad \bar\Phi_{ka}{}^b := \frac{\bar\Psi_{ka}{}^b}{\sqrt{x_a x_b}}\,. \tag{2.22}$$

Thus, the partial gauge fixing conditions (2.19) and (2.21) remove the variables $x_a{}^b$ and $\mathrm{p}_a{}^b$.

As a result, after the partial gauge fixing, phase space of the considered system is defined by $2n$ even real variables x_a, p_a, $2n$ even complex variables Z^i_a and $2n^2$ odd complex variables $\Phi^i{}_a{}^b$. Their nonvanishing Dirac brackets are

$$\{x_a, \mathrm{p}_b\}'_{\mathrm{D}} = \delta_{ab}\,, \tag{2.23}$$

$$\{Z^i_a, \bar{Z}^b_k\}'_{\mathrm{D}} = -i\,\delta^b_a\delta^i_k\,, \tag{2.24}$$

$$\{\Phi^i{}_a{}^b, \bar{\Phi}_{kc}{}^d\}'_{\mathrm{D}} = -i\,\delta^d_a\delta^b_c\delta^i_k\,. \tag{2.25}$$

In contrast to (2.13) and (2.15) the momenta p_a commute with each other and with the Grassmannian quantities $\Phi^k{}_a{}^b$. Moreover, due to (2.25), the odd variables $\Phi^k{}_a{}^b$ and $\bar{\Phi}_{ib}{}^a$ form canonical pairs (compare with (2.14)).

In the Hamiltonian (2.7) the momenta p_a are presented in the term $\sum_a (x_a\mathrm{p}_a)^2/2$. Let us represent this term in standard form for particle kinetic energy. For this we introduce the phase variables

$$q_a = \log x_a\,, \quad p_a = x_a\mathrm{p}_a\,, \qquad \{q_a, p_b\}'_{\mathrm{D}} = \delta_{ab}\,. \tag{2.26}$$

In these variables and (2.22) and after the gauge-fixing (2.19), (2.21), the Hamiltonian (2.7) takes the form

$$\mathrm{H} = \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{R_a{}^b R_b{}^a}{\sinh^2\left(\dfrac{q_a-q_b}{2}\right)} - \frac{1}{8}\,\mathrm{Tr}\Big(\{\Phi^i,\bar{\Phi}_i\}\{\Phi^k,\bar{\Phi}_k\}\Big)\,, \tag{2.27}$$

where

$$R_a{}^b := Z^k_a\bar{Z}^b_k - \cosh\left(\frac{q_a-q_b}{2}\right)\{\Phi^k,\bar{\Phi}_k\}_a{}^b\,. \tag{2.28}$$

The residual first class constraints in the set (2.8), (2.9) are n diagonal constraints

$$F_a := F_a{}^a = R_a{}^a - c = Z^k_a\bar{Z}^a_k - \{\Phi^k,\bar{\Phi}_k\}_a{}^a - c \approx 0 \qquad (\text{no summation over } a)\,, \tag{2.29}$$

which form an abelian algebra with respect to the Dirac brackets (2.25)

$$\{F_a, F_b\}'_{\mathrm{D}} = 0 \tag{2.30}$$

and generate the $[\mathrm{U}(1)]^n$ gauge transformations of Z^k_a and $\Phi^k{}_a{}^b$ with the local parameters $\gamma_a(t)$:

$$Z^k_a \to \mathrm{e}^{i\gamma_a}Z^k_a\,, \quad \bar{Z}^a_k \to \mathrm{e}^{-i\gamma_a}\bar{Z}^a_k \qquad (\text{no sum over } a)\,, \tag{2.31}$$

$$\Phi^k{}_a{}^b \to \mathrm{e}^{i\gamma_a}\Phi^k{}_a{}^b\mathrm{e}^{-i\gamma_b}\,, \quad \bar{\Phi}_{ka}{}^b \to \mathrm{e}^{i\gamma_a}\bar{\Phi}_{ka}{}^b\mathrm{e}^{-i\gamma_b} \qquad (\text{no sums over } a,b)\,. \tag{2.32}$$

Similarly to (2.20), we can use the expansions of the Grassmannian matrix quantities (2.22) in the diagonal and off-diagonal parts:

$$\Phi^k{}_a{}^b = \varphi^k_a\delta_a{}^b + \phi^k{}_a{}^b\,, \qquad \bar{\Phi}_{ka}{}^b = \bar{\varphi}_{ka}\delta_a{}^b + \bar{\phi}_{ka}{}^b\,, \tag{2.33}$$

where $\phi^k{}_a{}^a = \bar{\phi}_{ka}{}^a = 0$ at the fixed index a. The Dirac brackets (2.25) of the diagonal quantities φ^k_a, $\bar{\varphi}_{ka}$ and the off-diagonal ones $\phi^k{}_a{}^b$, $\bar{\phi}_{ka}{}^b$ have the form

$$\{\varphi^i_a, \bar{\varphi}_{kb}\}'_{\mathrm{D}} = -i\,\delta_{ab}\delta^i_k\,, \qquad \{\phi^i{}_a{}^b, \bar{\phi}_{kc}{}^d\}'_{\mathrm{D}} = -i\,\delta^d_a\delta^b_c\delta^i_k\,. \tag{2.34}$$

The constraints (2.29) involve only the off-diagonal fermions ϕ, $\bar{\phi}$:

$$F_a = Z^k_a \bar{Z}^a_k - \{\phi^k, \bar{\phi}_k\}_a{}^a - c \approx 0 \qquad \text{(no summation over } a\text{)}\,. \tag{2.35}$$

In the variables φ, $\bar{\varphi}$, ϕ, $\bar{\phi}$ the Hamiltonian (2.27) takes the form

$$\begin{aligned}
\mathrm{H} = \frac{1}{2}\sum_a p_a p_a \; + \; & \frac{1}{8}\sum_{a\neq b}\frac{\bar{Z}^a_i Z^k_a \, \bar{Z}^b_k Z^i_b}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \\
& + \frac{1}{4}\sum_{a\neq b}\frac{\coth\left(\frac{q_a-q_b}{2}\right)}{\sinh\left(\frac{q_a-q_b}{2}\right)} Z^i_a \bar{Z}^b_i \Big[(\varphi^k_a - \varphi^k_b)\bar{\phi}_{kb}{}^a + (\bar{\varphi}_{ka} - \bar{\varphi}_{kb})\phi^k{}_b{}^a - \{\phi^k, \bar{\phi}_k\}_b{}^a\Big] \\
& + \frac{1}{8}\sum_{a\neq b}\frac{1}{\sinh^2\left(\frac{q_a-q_b}{2}\right)}\Big[(\varphi^i_a - \varphi^i_b)(\varphi^k_a - \varphi^k_b)\bar{\phi}_{ia}{}^b\bar{\phi}_{kb}{}^a \\
& \qquad + (\bar{\varphi}_{ia} - \bar{\varphi}_{ib})(\bar{\varphi}_{ka} - \bar{\varphi}_{kb})\phi^{ib}_a\phi^{ka}_b \\
& \qquad + 2(\varphi^i_a - \varphi^i_b)(\bar{\varphi}_{ka} - \bar{\varphi}_{kb})\bar{\phi}_{ia}{}^b\phi^k{}_b{}^a \\
& \qquad + \{\phi^i, \bar{\phi}_i\}_a{}^b\{\phi^k, \bar{\phi}_k\}_b{}^a \\
& \qquad + 2(\bar{\varphi}_{ia} - \bar{\varphi}_{ib})\phi^i{}_a{}^b\{\phi^k, \bar{\phi}_k\}_b{}^a \\
& \qquad + 2(\varphi^i_a - \varphi^i_b)\bar{\phi}_{ia}{}^b\{\phi^k, \bar{\phi}_k\}_b{}^a\Big] \\
& - \frac{1}{8}\sum_a \{\phi^i, \bar{\phi}_i\}_a{}^a\{\phi^k, \bar{\phi}_k\}_a{}^a\,.
\end{aligned} \tag{2.36}$$

In the bosonic limit the Hamiltonian (2.36) takes the form

$$\mathrm{H}_{bose} \; = \; \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{S_{ai}{}^k S_{bk}{}^i}{\sinh^2\left(\frac{q_a-q_b}{2}\right)}\,, \tag{2.37}$$

where the quantities

$$S_{ai}{}^k := \bar{Z}^a_i Z^k_a \tag{2.38}$$

at all values a form the $u(2)$ algebras with respect to the Dirac brackets:

$$\{S_{ai}{}^k, S_{bj}{}^l\}'_{\mathrm{D}} = -i\,\delta_{ab}\left(\delta^k_j S_{ai}{}^l - \delta^l_i S_{aj}{}^k\right)\,. \tag{2.39}$$

Thus, the Hamiltonian (2.37) has the form

$$\mathrm{H}_{bose} \; = \; \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{\mathrm{Tr}(S_a S_b)}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \tag{2.40}$$

and is same as the Hamiltonian of the U(2)-spin hyperbolic Calogero-Sutherland A_{n-1}-root system [4,33,34].

Derivation of this many-particle spin system in the $\mathcal{N}=4$ case is the result of using semi-dynamical SU(2)-spinor variables, which are the field components of the **(4,4,0)** multiplets [28]. In contrast to the $\mathcal{N}=4$ case considered here, the use of semi-dynamical scalar variables in the $\mathcal{N}=2$ case produces "a less rich" supersymmetric system, namely the $\mathcal{N}=2$ spinless hyperbolic Calogero-Sutherland system [32].

3. $\mathcal{N}=4$ supersymmetry generators

As discussed in Sect. 1, the system (1.1) considered here was derived from the $\mathcal{N}=4$ superfield model [28]. Therefore, it is invariant under $\mathcal{N}=4$ supersymmetry transformations of the matrix component fields:

$$\begin{aligned}
\delta X &= -\varepsilon_i \Psi^i + \bar{\varepsilon}^i \bar{\Psi}_i\,,\\
\delta\Psi^i &= i\,\bar{\varepsilon}^i\,\nabla X + \bar{\varepsilon}_k X\Big[X^{-1}\Psi^{(i}, X^{-1}\bar{\Psi}^{k)}\Big],\\
\delta\bar{\Psi}_i &= i\,\varepsilon_i\,\nabla X + \varepsilon^k X\Big[X^{-1}\Psi_{(i}, X^{-1}\bar{\Psi}_{k)}\Big],\\
\delta Z^i &= 0\,,\quad \delta\bar{Z}_i = 0\,,\qquad \delta A = 0\,,
\end{aligned} \tag{3.1}$$

where ε_k, $\bar{\varepsilon}^k = (\varepsilon_k)^*$ is two complex Grassmannian parameters. These transformations are generated by the following Noether charges:

$$\begin{aligned}
Q^i &= \mathrm{Tr}\Big(P\Psi^i + \frac{i}{2}X^{-1}\bar{\Psi}^i X^{-1}\Psi_k X^{-1}\Psi^k\Big),\\
\bar{Q}_i &= \mathrm{Tr}\Big(P\bar{\Psi}_i + \frac{i}{2}X^{-1}\Psi_i X^{-1}\bar{\Psi}^k X^{-1}\bar{\Psi}_k\Big),
\end{aligned} \tag{3.2}$$

where the matrix momentum $P_a{}^b$ is presented in (2.5). The supercharges (3.2) and the Hamiltonian H defined in (2.7) form the $\mathcal{N}=4$ $d=1$ superalgebra

$$\{Q^i, \bar{Q}_j\}_{\mathrm{D}} = -2i\,H\,\delta^i_j\,,\qquad \{Q^i, H\}_{\mathrm{D}} = \{\bar{Q}_i, H\}_{\mathrm{D}} = 0 \tag{3.3}$$

with respect to the Dirac brackets (2.12)-(2.15).

Putting the partial gauge fixing conditions (2.19), (2.21) in expressions (3.2) and going to the variables (2.22), (2.26), we obtain the $\mathcal{N}=4$ supersymmetry generators

$$\begin{aligned}
\mathrm{Q}^i &= \sum_a p_a \Phi^i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{R_a{}^b\Phi^i{}_b{}^a}{\sinh\Big(\frac{q_a-q_b}{2}\Big)} + \frac{i}{2}\sum_{a,b}[\Phi^k, \bar{\Phi}_k]_a{}^b\Phi^i{}_b{}^a\,,\\
\bar{\mathrm{Q}}_i &= \sum_a p_a \bar{\Phi}_i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{R_a{}^b\bar{\Phi}_i{}_b{}^a}{\sinh\Big(\frac{q_a-q_b}{2}\Big)} - \frac{i}{2}\sum_{a,b}[\Phi^k, \bar{\Phi}_k]_a{}^b\bar{\Phi}_i{}_b{}^a
\end{aligned} \tag{3.4}$$

for the partial gauge fixing system, which is described by the Hamiltonian (2.27) and the first class constraints (2.29). Using the Grassmannian variables φ^i_a, $\bar{\varphi}_i{}_a$, $\phi^i{}_a{}^b$, $\bar{\phi}_i{}_a{}^b$, defined in (2.33), we cast the generators (3.4) in the form

$$\begin{aligned}
\mathrm{Q}^i = &\sum_a p_a\varphi^i_a - \frac{i}{2}\sum_{a\neq b}\frac{Z^k_a\bar{Z}^b_k\,\phi^i{}_b{}^a}{\sinh\Big(\frac{q_a-q_b}{2}\Big)}\\
&+\frac{i}{2}\sum_{a\neq b}\coth\Big(\frac{q_a-q_b}{2}\Big)\Big[(\bar{\varphi}_{ka}-\bar{\varphi}_{kb})\phi^k{}_a{}^b + (\varphi^k_a-\varphi^k_b)\bar{\phi}_k{}_a{}^b + \{\phi^k,\bar{\phi}_k\}_a{}^b\Big]\phi^i{}_b{}^a\\
&+\frac{i}{2}\Big[\sum_{a\neq b}\Big((\varphi_{ka}+\varphi_{kb})\phi^k{}_b{}^a\bar{\phi}^i{}_a{}^b + \phi_{ka}{}^b\phi^k{}_b{}^a\bar{\varphi}^i_a\Big) + \sum_{a\neq b\neq c\neq a}\phi_{ka}{}^b\phi^k{}_b{}^c\bar{\phi}^i{}_c{}^a\\
&+\sum_a\varphi_{ka}\varphi^k_a\bar{\varphi}^i_a\Big],
\end{aligned} \tag{3.5}$$

$$\bar{\mathrm{Q}}_i = \sum_a p_a \bar{\varphi}_{i\,a} - \frac{i}{2}\sum_{a\neq b}\frac{Z^k_a \bar{Z}^b_k\, \bar{\phi}_{i\,b}{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} \tag{3.6}$$
$$+\frac{i}{2}\sum_{a\neq b}\coth\left(\frac{q_a-q_b}{2}\right)\Big[(\bar{\varphi}_{k\,a}-\bar{\varphi}_{k\,b})\phi^k{}_a{}^b + (\varphi^k_a-\varphi^k_b)\bar{\phi}_{k\,a}{}^b + \{\phi^k,\bar{\phi}_k\}_a{}^b\Big]\bar{\phi}_{i\,b}{}^a$$
$$+\frac{i}{2}\Big[\sum_{a\neq b}\Big(\phi_{i\,a}{}^b\bar{\phi}^k{}_b{}^a(\bar{\varphi}_{k\,a}+\bar{\varphi}_{k\,b}) + \varphi_{i\,a}\bar{\phi}^k{}_a{}^b\bar{\phi}_{k\,b}{}^a\Big) + \sum_{a\neq b\neq c\neq a}\phi_{i\,a}{}^b\bar{\phi}^k{}_b{}^c\bar{\phi}_{k\,c}{}^a$$
$$+\sum_a \varphi_{i\,a}\bar{\varphi}^k_a\bar{\varphi}_{k\,a}\Big].$$

Taking into account the Dirac brackets (2.25), (2.26) and

$$\{R_a{}^b, R_c{}^d\}'_{\mathrm{D}} = -i\Big(\delta^d_a R_c{}^b - \delta^b_c R_a{}^d\Big)$$
$$-i\,\sinh\left(\frac{q_a-q_b}{2}\right)\sinh\left(\frac{q_c-q_d}{2}\right)\Big(\delta^d_a\{\Phi^k,\bar{\Phi}_k\}_c{}^b - \delta^b_c\{\Phi^k,\bar{\Phi}_k\}_a{}^d\Big), \tag{3.7}$$

we find that the supercharges Q^i, $\bar{\mathrm{Q}}_i$ defined in (3.4) form the $\mathcal{N}=4$ superalgebra

$$\{\mathrm{Q}^i,\mathrm{Q}^k\}'_{\mathrm{D}} = -\frac{i}{4}\sum_{a\neq b}\frac{\phi^{(i}{}_a{}^b\phi^{k)}{}_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)}\Big(F_a-F_b\Big), \tag{3.8}$$

$$\{\mathrm{Q}^i,\bar{\mathrm{Q}}_k\}'_{\mathrm{D}} = -2i\,\mathrm{H}\,\delta^i_k - \frac{i}{4}\sum_{a\neq b}\frac{\phi^i{}_a{}^b\bar{\phi}_{k\,b}{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)}\Big(F_a-F_b\Big), \tag{3.9}$$

$$\{\mathrm{Q}^i,\mathrm{H}\}'_{\mathrm{D}} = -\frac{1}{8}\sum_{a\neq b}\frac{R_a{}^b\phi^i{}_b{}^a}{\sinh^3\left(\frac{q_a-q_b}{2}\right)}\Big(F_a-F_b\Big) \tag{3.10}$$

and c.c., where the Hamiltonian H and the constraints $F_a\approx 0$ are given in (2.27) and (2.29). Thus, the quantities H, Q^i, $\bar{\mathrm{Q}}_i$, defined in (2.27), (3.4), form the $\mathcal{N}=4$ superalgebra with respect to the Dirac brackets on the shell of the first class constraints (2.29). Moreover, the generators H, Q^i, $\bar{\mathrm{Q}}_i$ are gauge invariant: they have the vanishing Dirac brackets with the first class constraints (2.29),

$$\{\mathrm{Q}^i,F_a\}'_{\mathrm{D}} = \{\bar{\mathrm{Q}}_i,F_a\}'_{\mathrm{D}} = \{\mathrm{H},F_a\}'_{\mathrm{D}} = 0\,. \tag{3.11}$$

The form of the first two terms in expressions (3.4) is similar to the $\mathcal{N}=2$ supercharges presented in [32]. But the last terms in the $\mathcal{N}=4$ supercharges (3.4) were absent in the $\mathcal{N}=2$ case. Their appearance is the result of the SU(2) spinor nature of Grassmann variables in the $\mathcal{N}=4$ case. Moreover, the first and last terms in the supercharges (3.5), (3.6)

$$\mathbb{Q}^i = \sum_a\Big(p_a\varphi^i_a + \frac{i}{2}\varphi_{k\,a}\varphi^k_a\bar{\varphi}^i_a\Big), \qquad \bar{\mathbb{Q}}_i = \sum_a\Big(p_a\bar{\varphi}_{i\,a} + \frac{i}{2}\varphi_{i\,a}\bar{\varphi}^k_a\bar{\varphi}_{k\,a}\Big) \tag{3.12}$$

contain only diagonal fermions φ^i_a, $\bar{\varphi}_{i\,a}$ and possess the following Dirac brackets:

$$\{\mathbb{Q}^i,\bar{\mathbb{Q}}_k\}_{\mathrm{D}} = -2i\delta^i_k\mathbb{H}\,, \qquad \{\mathbb{Q}^i,\mathbb{H}\}_{\mathrm{D}} = \{\bar{\mathbb{Q}}_i,\mathbb{H}\}_{\mathrm{D}} = 0\,, \tag{3.13}$$

where $\mathbb{H} = \frac{1}{2}\sum_a p_a{}^2$. Although supercharges (3.12) contain terms trilinear in fermions in contrast to the $\mathcal{N}=2$ case [32], these quantities and $\mathbb{H}$ generate the $\mathcal{N}=4$ supersymmetric system describing n non-interacting free particles. This system is described by the $\mathcal{N}=4$ superfield Lagrangian $\mathcal{L} \sim \sum_a \log \mathcal{X}_a$ (see [20,35,36]).

It should also be noted that the terms of the supercharges (3.5), (3.6), without the first and last terms (3.12), describe the interaction of particles and are zero when the off-diagonal matrix fermions $\phi^i{}_a{}^b$, $\bar{\phi}_i{}_a{}^b$ vanish.

Similarly to the $\mathcal{N}=2$ case [32], we can make gauge-fixing for the residual n real first class constraints (2.29) (or (2.35)). However, in the considered $\mathcal{N}=4$ case, we have $2n$ complex spinor variables Z^i_a in opposite to the $\mathcal{N}=2$ case with n complex spinorial degrees of freedom in the last case. Thus, in the $\mathcal{N}=4$ case considered here the $\mathcal{N}=4$ multiparticle model possesses n complex semi-dynamical degrees of freedom in phase space and describes the $\mathcal{N}=4$ supersymmetrization of the many-particle system which differs from the one in the $\mathcal{N}=2$ case. In Section 5, we use the reduction that eliminates these semi-dynamical degrees of freedom in the $\mathcal{N}=4$ invariant way.

4. Lax representation

Classical dynamics of the system with partial gauge-fixing considered here is defined by the total Hamiltonian

$$\mathrm{H_T} \;=\; \mathrm{H} \;+\; \sum_a \lambda_a F_a\,, \tag{4.1}$$

where the Hamiltonian H is defined in (2.27) and $\lambda_a(t)$ are the Lagrange multipliers for the first class constraints F_a, presented in (2.29). A time derivative of an arbitrary phase variable $B(t)$ takes the form

$$\dot{B} \;=\; \{B, \mathrm{H_T}\}'_{\mathrm{D}}\,. \tag{4.2}$$

Let us represent this dynamics in the Lax representation [37].

To do this, we introduce the $n\times n$ matrix

$$L_a{}^b \;=\; p_a\,\delta_a{}^b \;-\; i\left(1-\delta^b_a\right)\frac{R_a{}^b}{2\sinh\left(\dfrac{q_a-q_b}{2}\right)}\,, \tag{4.3}$$

whose evolution

$$\dot{L}_a{}^b \;=\; \{L_a{}^b, \mathrm{H_T}\}'_{\mathrm{D}} \tag{4.4}$$

is represented by the matrix commutator

$$\dot{L}_a{}^b \;=\; -i[M+\Lambda, L]_a{}^b - i\left(1-\delta^b_a\right)\frac{L_a{}^b\,(F_a-F_b)}{4\sinh^2\left(\dfrac{q_a-q_b}{2}\right)}\,, \tag{4.5}$$

where the matrices M and Λ have the following form:

$$M_a{}^b \;=\; \frac{1}{4}\{\Phi^k, \bar{\Phi}_k\}_a{}^a\delta_a{}^b \;+\; \frac{1}{4}\left(1-\delta^b_a\right)\left(\frac{\cosh\left(\dfrac{q_a-q_b}{2}\right)}{\sinh^2\left(\dfrac{q_a-q_b}{2}\right)}R_a{}^b + \{\Phi^k, \bar{\Phi}_k\}_a{}^b\right), \tag{4.6}$$

$$\Lambda_a{}^b \;=\; \lambda_a\,\delta_a{}^b \tag{4.7}$$

and F_a are the constraints defined in (2.35). The equations of motion of the fermionic matrix variables $\Phi_a^{i\,b}$, $\bar\Phi_{i\,a}{}^b$ are also represented as commutators

$$\begin{aligned}\dot\Phi_a^{i\,b} \;&=\; \{\Phi_a^{i\,b}, \mathrm{H_T}\}'_{\mathrm{D}} \;=\; -i[M+\Lambda, \Phi^i]_a{}^b\,,\\ \dot{\bar\Phi}_{i\,a}{}^b \;&=\; \{\bar\Phi_{i\,a}{}^b, \mathrm{H_T}\}'_{\mathrm{D}} \;=\; -i[M+\Lambda, \bar\Phi_i]_a{}^b\end{aligned} \tag{4.8}$$

with the same matrices M and Λ.

On the shell of the first class constraints (2.35) $F_a \approx 0$, equations (4.5), (4.8) are actually the Lax equations and yield the conserved charges in a simple way. So due to equations (4.5), (4.8), the trace

$$\mathcal{J} := \mathrm{Tr}(\mathcal{F}) \tag{4.9}$$

of any polynomial function $\mathcal{F}(L, \Phi, \bar\Phi)$ of the matrix variables $L_a{}^b$, $\Phi_a^{i\,b}$, $\bar\Phi_{i\,a}{}^b$ is a conserved quantity on the shell of constraints (2.35):

$$\dot{\mathcal{J}} \approx 0\,. \tag{4.10}$$

In particular, on the shell of constraints (2.35), the traces

$$I_{\mathrm{k}} := \mathrm{Tr}(L^{\mathrm{k}})\,, \qquad \mathcal{I}^i_{\mathrm{k}} := \mathrm{Tr}(\Phi^i L^{\mathrm{k}-1})\,, \qquad \bar{\mathcal{I}}_{\mathrm{k}i} := \mathrm{Tr}(\bar\Phi L^{\mathrm{k}-1})\,, \qquad \mathrm{k} = 1, \ldots, n \tag{4.11}$$

are conserved:

$$\dot I_{\mathrm{k}} = \frac{i\mathrm{k}}{4}\sum_{a\neq b}\frac{(L^{\mathrm{k}})_a{}^b}{\sinh^2\Big(\dfrac{q_a-q_b}{2}\Big)}\Big(F_a - F_b\Big) \approx 0\,, \qquad \dot{\mathcal{I}}^i_{\mathrm{k}} = 0\,, \qquad \dot{\bar{\mathcal{I}}}_{\mathrm{k}i} = 0\,. \tag{4.12}$$

The Hamiltonian (2.27) and the supercharges (3.4) have the form

$$\mathrm{H} = \frac{1}{2}\,I_2 + J\,, \qquad \mathrm{Q}^i = \mathcal{I}^i_2 + \mathcal{J}^i\,, \qquad \bar{\mathrm{Q}}_i = \bar{\mathcal{I}}_{2i} + \bar{\mathcal{J}}_i\,, \tag{4.13}$$

where

$$\begin{aligned}&J := -\frac{1}{8}\,\mathrm{Tr}\Big(\{\Phi^i, \bar\Phi_i\}\{\Phi^k, \bar\Phi_k\}\Big)\,, \quad \mathcal{J}^i := \frac{i}{2}\,\mathrm{Tr}\Big([\Phi^k, \bar\Phi_k]\Phi^i\Big)\,,\\ &\bar{\mathcal{J}}_i := -\frac{i}{2}\,\mathrm{Tr}\Big(\bar\Phi_i[\Phi^k, \bar\Phi_k]\Big)\,.\end{aligned} \tag{4.14}$$

The equations of motion of the commuting spinning variables Z^i_a, $\bar Z^a_i$ are represented as

$$\begin{aligned}\dot Z^i_a \;&=\; \{\Phi_a^{i\,b}, \mathrm{H}\}'_{\mathrm{D}} \;=\; -i\sum_b\big(A_a{}^b + \Lambda_a{}^b\big)\,Z^i_b\,,\\ \dot{\bar Z}^a_i \;&=\; \{\bar\Phi_{i\,a}{}^b, \mathrm{H}\}'_{\mathrm{D}} \;=\; i\sum_b \bar Z^b_i\,(A_b{}^a + \Lambda_b{}^a)\,,\end{aligned} \tag{4.15}$$

where the matrix A has the form

$$A_a{}^b \;=\; \Big(1-\delta^b_a\Big)\,\frac{R_a{}^b}{4\sinh^2\Big(\dfrac{q_a-q_b}{2}\Big)} \tag{4.16}$$

and the matrix Λ is defined in (4.7). Due to (4.15) we obtain (see (2.38))

$$\dot{S}_k{}^i = 0\,, \qquad \text{where} \qquad S_k{}^i := \sum_a \bar{Z}^a_k Z^i_a\,. \tag{4.17}$$

It should be noted that the structure of the conserved charges in the considered supersymmetric system (4.10) is similar to the form of the charges in the trigonometric (non-matrix) supersymmetric system studied in [25].

Deriving the Lax pair and finding the set of conserved charges (4.9) paves the way for analyzing the integrability of the $\mathcal{N}=4$ supersymmetric system considered here. Analysis of the superalgebra of conserved charges and integrability of the considered many-particle supersymmetric system will be the subject of the next article.

5. Spinless hyperbolic Calogero-Sutherland system as a result of the reduction procedure

Semi-dynamical variables have the following Dirac brackets with the total Hamiltonian (4.1), (2.27)

$$\{H_T\,, Z^j_a\}'_{\rm D} = \frac{i}{4}\sum_{b(\neq a)} \frac{R_a{}^b Z^j_b}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} + i\lambda_a Z^j_a \tag{5.1}$$

and with the supercharges (3.4)

$$\{Q^i\,, Z^j_a\}'_{\rm D} = -\frac{1}{2}\sum_{b(\neq a)} \frac{\Phi^i{}_a{}^b Z^j_b}{\sinh\left(\frac{q_a-q_b}{2}\right)}\,, \qquad \{\bar{Q}_i\,, Z^j_a\}'_{\rm D} = -\frac{1}{2}\sum_{b(\neq a)} \frac{\bar{\Phi}_{i\,a}{}^b Z^j_b}{\sinh\left(\frac{q_a-q_b}{2}\right)}\,. \tag{5.2}$$

Therefore, the conditions

$$Z^{j=2}_a = 0\,, \qquad \bar{Z}^a_{j=2} = 0\,, \qquad \text{at all } a \tag{5.3}$$

are invariant under the $\mathcal{N}=4$ supersymmetry transformations and we can use them as reduction conditions. Similarly to [38], the reduction (5.3) implies the conditions

$$S^{(\pm)}_a := S_{ai}{}^k \sigma^{\pm}{}_k{}^i \qquad \text{at all } a, \tag{5.4}$$

where the quantities $S_{ai}{}^k$ are defined in (2.38), $\sigma^{\pm} = \sigma^1 \pm i\sigma^2$ and $\sigma^{1,2}$ are the Pauli matrices. So the conditions (5.3) lead to zero two generators in all $u(2)$ algebras (2.38), (2.39).

After reduction with the conditions (5.3) the obtained system involves only half of the initial semi-dynamical variables

$$z_a := Z^{j=1}_a\,, \quad \bar{z}^a := \bar{Z}^a_{j=1}\,, \qquad \{z_a\,, \bar{z}^b\}'_{\rm D} = -i\delta^b_a\,. \tag{5.5}$$

Reduction of the Hamiltonian (2.27) takes the form

$$\mathcal{H} \;=\; \frac{1}{2}\sum_a p_a p_a \;+\; \frac{1}{8}\sum_{a\neq b} \frac{T_a{}^b T_b{}^a}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \;-\; \frac{1}{8}\,{\rm Tr}\Big(\{\Phi^i\,, \bar{\Phi}_i\}\{\Phi^k\,, \bar{\Phi}_k\}\Big)\,, \tag{5.6}$$

where

$$T_a{}^b := z_a\bar{z}^b - \cosh\left(\frac{q_a-q_b}{2}\right)\{\Phi^k\,, \bar{\Phi}_k\}_a{}^b\,. \tag{5.7}$$

In this case, the $\mathcal{N}=4$ supersymmetry generators (3.4) take the form

$$\begin{aligned}\mathcal{Q}^i &= \sum_a p_a \Phi^i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{T_a{}^b \Phi^i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} + \frac{i}{2}\sum_{a,b}[\Phi^k,\bar\Phi_k]_a{}^b\Phi^i{}_b{}^a\,,\\ \bar{\mathcal{Q}}_i &= \sum_a p_a \bar\Phi_i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{T_a{}^b \bar\Phi_i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)} - \frac{i}{2}\sum_{a,b}[\Phi^k,\bar\Phi_k]_a{}^b\bar\Phi_i{}_b{}^a\,,\end{aligned} \tag{5.8}$$

while the first class constraints (2.29) become

$$\mathcal{F}_a := T_a{}^a - c = z_a\bar z^a - \{\Phi^k,\bar\Phi_k\}_a{}^a - c \approx 0 \qquad \text{(no summation over } a\text{)}\,. \tag{5.9}$$

Similarly to quantities (2.28) with the Dirac brackets (3.7), quantities (5.7) satisfy

$$\begin{aligned}\{T_a{}^b, T_c{}^d\}'_{\rm D} &= -i\left(\delta_a^d T_c{}^b - \delta_c^b T_a{}^d\right)\\ &\quad -i\,\sinh\left(\frac{q_a-q_b}{2}\right)\sinh\left(\frac{q_c-q_d}{2}\right)\left(\delta_a^d\{\Phi^k,\bar\Phi_k\}_c{}^b - \delta_c^b\{\Phi^k,\bar\Phi_k\}_a{}^d\right).\end{aligned} \tag{5.10}$$

As result, the charges (5.8), (5.6) form the same $\mathcal{N}=4$ superalgebra (3.8)-(3.10), up to the first class constraints (5.9).

However this reduced system contains n first class constraints (5.9) which, together with the gauge fixing conditions, can eliminate all n complex semi-dynamical variables z_a. So similarly to the $\mathcal{N}=2$ case considered in [32], we can make the gauge-fixing

$$\bar z^a = z_a \qquad \text{(for all } a\text{)} \tag{5.11}$$

for the first class constraints (5.9). Then, the components of the spinor z_a become real and are expressed through the remaining variables by the following expressions:

$$z_a = \sqrt{c + \{\Phi^k,\bar\Phi_k\}_a{}^a} \qquad \text{(no summation over } a\text{)}\,. \tag{5.12}$$

In this gauge the supercharges (3.5), (3.6) take the form

$$\begin{aligned}\mathcal{Q}^i &= \sum_a p_a \Phi^i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{\sqrt{c+\{\Phi^k,\bar\Phi_k\}_a{}^a}\sqrt{c+\{\Phi^j,\bar\Phi_j\}_b{}^b}\,\Phi^i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\\ &\quad + \frac{i}{2}\sum_{a\neq b}\coth\left(\frac{q_a-q_b}{2}\right)\{\Phi^k,\bar\Phi_k\}_a{}^b\,\Phi^i{}_b{}^a + \frac{i}{2}\sum_{a,b}[\Phi^k,\bar\Phi_k]_a{}^b\Phi^i{}_b{}^a\,,\end{aligned} \tag{5.13}$$

$$\begin{aligned}\bar{\mathcal{Q}}_i &= \sum_a p_a \bar\Phi_i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{\sqrt{c+\{\Phi^k,\bar\Phi_k\}_a{}^a}\sqrt{c+\{\Phi^j,\bar\Phi_j\}_b{}^b}\,\bar\Phi_i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\\ &\quad + \frac{i}{2}\sum_{a\neq b}\coth\left(\frac{q_a-q_b}{2}\right)\{\Phi^k,\bar\Phi_k\}_a{}^b\,\bar\Phi_i{}_b{}^a - \frac{i}{2}\sum_{a,b}[\Phi^k,\bar\Phi_k]_a{}^b\bar\Phi_i{}_b{}^a\,.\end{aligned} \tag{5.14}$$

Moreover, in this gauge and in a pure bosonic limit, the reduced Hamiltonian (5.6) takes the form

$$\mathcal{H}_{bose} = \frac{1}{2}\sum_a p_a p_a + \frac{1}{8}\sum_{a\neq b}\frac{c^2}{\sinh^2\left(\frac{q_a-q_b}{2}\right)} \tag{5.15}$$

and is the Hamiltonian of the standard spinless hyperbolic Calogero-Sutherland system. Thus, the reduction (5.3) of the considered system yields gauge formulation of the $\mathcal{N}=4$ spinless hyperbolic Calogero-Sutherland system [1–4].

Due to the presence of the square roots in the second terms in the supercharges (5.13), (5.14) they contain higher degrees with respect to the Grassmannian variables. To avoid this, new variables

$$\xi^i{}_a{}^b = \Phi^i{}_a{}^b \sqrt{\frac{c+\{\Phi^j,\bar\Phi_j\}_b{}^b}{c+\{\Phi^k,\bar\Phi_k\}_a{}^a}}\,, \qquad \bar\xi_{ia}{}^b = \bar\Phi_{ia}{}^b \sqrt{\frac{c+\{\Phi^j,\bar\Phi_j\}_b{}^b}{c+\{\Phi^k,\bar\Phi_k\}_a{}^a}} \tag{5.16}$$

were introduced in [16]. In these quantities the supercharges (3.5), (3.6) take the form

$$\begin{aligned}\mathcal{Q}^i &= \sum_a p_a \xi^i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{\left(c+\{\xi^k,\bar\xi_k\}_b{}^b\right)\xi^i{}_b{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\\ &\quad + \frac{i}{2}\sum_{a\neq b}\coth\left(\frac{q_a-q_b}{2}\right)\{\xi^k,\bar\xi_k\}_a{}^b\,\xi^i{}_b{}^a - \frac{i}{2}\,\beta\sum_{a,b}[\xi^k,\bar\xi_k]_a{}^b\,\xi^i{}_b{}^a\,,\end{aligned} \tag{5.17}$$

$$\begin{aligned}\bar{\mathcal{Q}}_i &= \sum_a p_a \bar\xi_i{}_a{}^a - \frac{i}{2}\sum_{a\neq b}\frac{\left(c+\{\xi^k,\bar\xi_k\}_b{}^b\right)\bar\xi_{ib}{}^a}{\sinh\left(\frac{q_a-q_b}{2}\right)}\\ &\quad + \frac{i}{2}\sum_{a\neq b}\coth\left(\frac{q_a-q_b}{2}\right)\{\xi^k,\bar\xi_k\}_a{}^b\,\bar\xi_{ib}{}^a + \frac{i}{2}\,\beta\sum_{a,b}[\xi^k,\bar\xi_k]_a{}^b\,\bar\xi_{ib}{}^a\,,\end{aligned} \tag{5.18}$$

where $\beta=-1$, and coincide exactly with the $\mathcal{N}=4$ supersymmetry generators presented in [19].[1] Point out that in contrast to the properties of the Grassmannian variables (2.22), quantities (5.16) do not form pairs with respect to complex conjugation, that is some obstacle in quantization of the system in such representation.

6. Concluding remarks and outlook

In this paper, the Hamiltonian description of the $\mathcal{N}=4$ supersymmetric multi-particle hyperbolic Calogero-Sutherland system is presented, which was obtained from the matrix superfield model by the gauging procedure [28]. In contrast to the $\mathcal{N}=2$ case, the $\mathcal{N}=4$ supersymmetric generalization of the gauged model has the U(2) spin hyperbolic Calogero-Sutherland system as a bosonic core.

In the presented paper, there are obtained explicit expressions of the $\mathcal{N}=4$ supersymmetry generators for different descriptions of the system under consideration. The supercharges (3.2) and the Hamiltonian (2.7) of the fully matrix system have a simple form, but this system contains a large number of auxiliary degrees of freedom, which can be eliminated by n^2 first class constraints (2.9). After the partial gauge fixing (2.19), eliminating off-diagonal even matrix variables, we obtain the formulation in which the $\mathcal{N}=4$ supersymmetry generators (3.5), (3.6) have the Calogero-like form and are closed on the Hamiltonian (2.27) (or (2.36)) and n first class constraints (2.29) generating the residual $[\mathrm{U}(1)]^n$ gauge symmetry. Without off-diagonal odd

[1] The author thanks Sergey Krivonos for the information that the value $\beta=-1$ is also valid in the hyperbolic case of the model presented in [19].

variables in the classical supercharges (3.4) (or (3.5), (3.6)), the nontrivial interaction terms disappear in them.

It is possible to impose the reduction conditions (5.3) that are $\mathcal{N}=4$ supersymmetry invariant and eliminate half of the spinning variables. As result, we get the $\mathcal{N}=4$ supersymmetric system with n first class constraints (5.9), which allows us gauging of the remaining spinning variables. Such a reduced system is in fact the $\mathcal{N}=4$ generalization of the spinless hyperbolic Calogero-Sutherland system equivalent to the model presented in [19].

In addition, the Lax representation (4.5), (4.8), (4.15) of the equations of motion for the system under consideration is presented. The set of conserved quantities (4.10), (4.11), (4.17) is found. Analysis of the classical integrability of the $\mathcal{N}=4$ system considered here will be the subject of the next paper.

Moreover, a further research will be devoted to quantum integrability of the supersymmetric $\mathcal{N}=2$ and $\mathcal{N}=4$ systems constructed here. Supersymmetry quantum generators are obtained using the Weyl ordering in quantum analogs of quantities such as the $\mathcal{N}=2$ supersymmetric case. However, in contrast to the $\mathcal{N}=2$ case [32], due to the SU(2)-doublet nature of odd variables in the $\mathcal{N}=4$ case, the separation of the invariant sector with only diagonal odd variables does not work in the $\mathcal{N}=4$ quantum case.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

Acknowledgements

I would like to thank Evgeny Ivanov and Sergey Krivonos for useful discussions. This work was supported by the Russian Science Foundation, grant no. 16-12-10306.

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