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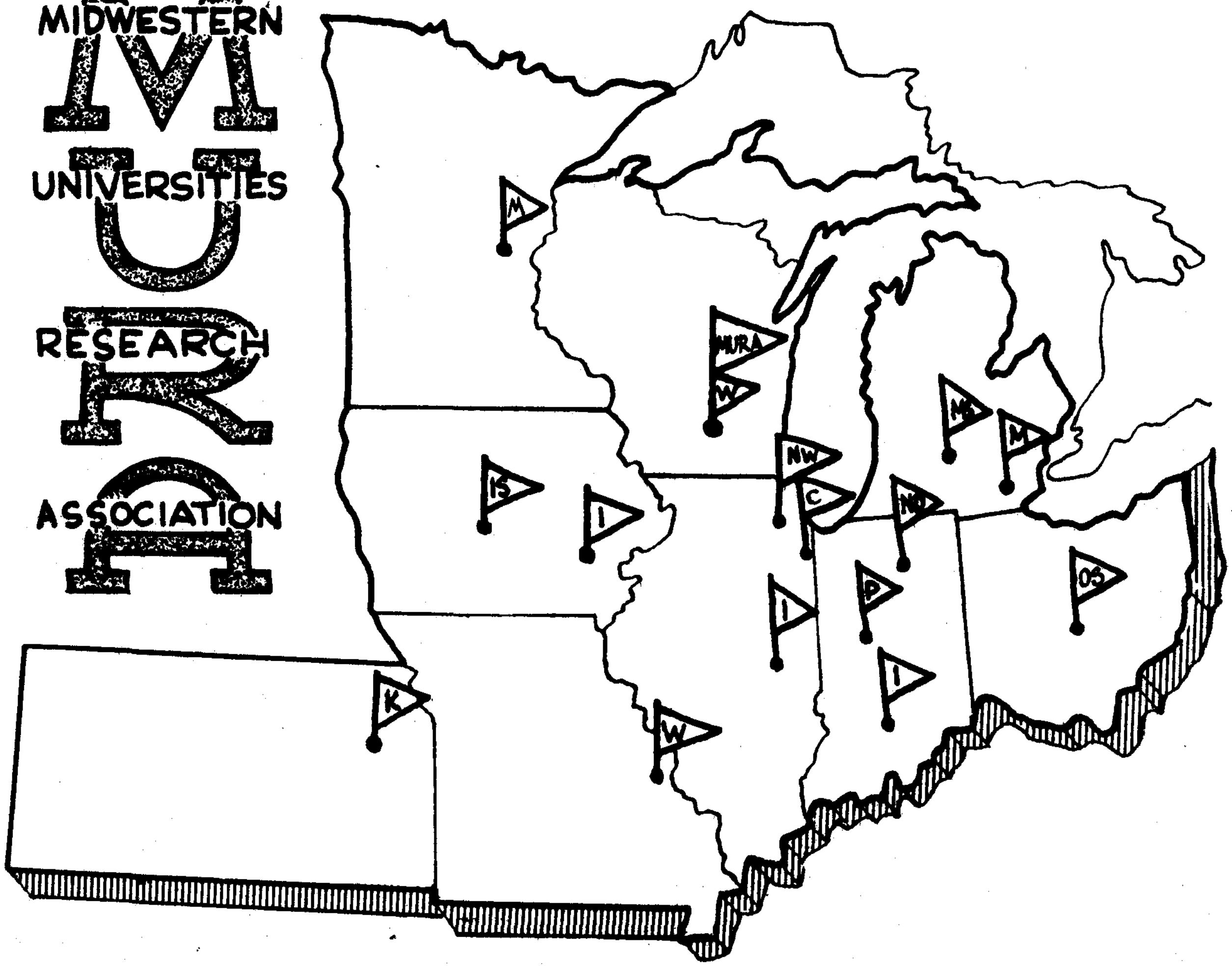

METHODS OF RADIO FREQUENCY ACCELERATION IN FIXED FIELD ACCELERATORS WITH APPLICATIONS TO HIGH CURRENT AND INTERSECTING BEAM ACCELERATORS

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METHODS OF RADIO FREQUENCY ACCELERATION IN FIXED FIELD ACCELERATORS WITH APPLICATIONS TO HIGH CURRENT AND INTERSECTING BEAM ACCELERATORS

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I. A Survey of Ideas for Radio Frequency Acceleration

A Fixed Field Accelerator can accommodate at one time particles circulating at all energies between the injector and output energies. There thus becomes available a whole new class of accelerating mechanisms which appear to promise high intensity beams. Such high intensity, besides being of interest in a single accelerator, is of course essential for the operation of a double accelerator with interacting beams. These accelerating mechanisms are now being studied by analytic means as well as by the digital computer. In general, one is concerned with the energy gain of particles whose frequencies are a function of energy, as these particles are subject to various radio frequency accelerating gaps, whose voltages and frequencies may be secularly changed. Of the many possible arrangements, not all of which have been studied, the following seem to have particular promise. More calculations will have to be done before one can choose which of the following mechanisms or what combination of them is most efficient.

a. Conventional synchrotron acceleration at high repetition rate.

The most straightforward accelerating system is one which uses one or

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several synchronized accelerating gaps supplying a radio frequency voltage whose frequency is modulated as in conventional synchrotrons so as to accelerate a pulse of particles from the injection to the output energy. The only advantage of an FFAG magnet in this case is that the pulse repetition rate is now limited only by the r.f. system, and may, with reasonable requirements on the r.f. system be increased to perhaps several pulses per second as compared with one pulse every few seconds with pulsed magnetic field accelerators.

A typical graph of frequency of revolution versus energy in an FFAG synchrotron is shown in Figure 1. The graph is drawn for a mean field index $k = 82.5$. The energy scale is in rest masses, and the frequency scale is in units of the frequency at transition where the frequency reaches its maximum value. The problem of accelerating through the transition energy will be discussed later. In general it appears that this problem is less difficult in an FFAG synchrotron than in a pulsed field alternating gradient accelerator. For reasons that we shall indicate, conventional synchrotron acceleration is far less efficient in terms of beam current that can be accelerated than is theoretically possible with other schemes.

b. Bucket Lift.

If a radio frequency voltage is applied to an accelerating gap, then in the neighborhood of each energy for which the frequency of revolution of the particles is equal to the radio frequency or to any of its subharmonics, there is a region of particle energies and phases (a bucket) within which particles execute stable phase oscillations around the synchronous energies. If the radio frequency is modulated, buckets move up or down the energy scale. Under suitable conditions particles in any of the buckets can be accelerated by this sytem. Thus it is possible to accelerate a number of buckets of particles at a number of different energies simultaneously, with a single radio frequency accelerating voltage. As the frequency is modulated, each bucket can be filled by the injector when the energy corresponding to that bucket coincides with the injector energy.

c. Phase displacement mechanisms.

These are based on the observation that particles are accelerated if subject to a radio frequency gap which is initially at a frequency corresponding to an energy higher than that of the particles, and then the oscillator frequency is modulated to a frequency corresponding to an energy lower than that of the particles. Note that in this scheme the frequency is modulated in just the reverse direction from that used in conventional synchrotron acceleration, or in the bucket lift. The mechanism may be readily understood, for the oscillator carries virtual particles down in energy, and thus by Liouville's Theorem real particles occupying phase space at a lower energy must be forced upward in energy.

In general, since the current accelerated by phase displacement equals the virtual current which could be carried down by the oscillator, phase displacement and bucket lifts are about equally efficient. The methods vary in the length of time necessary for transit of a given energy interval by any single particle, and as such each method has distinct advantages or disadvantages. It should be clear that the carrying of particles in buckets, and the phase displacement of particles not in buckets are complementary. For any proposed acceleration system involving buckets, one can envision a complementary system involving phase displacement, which is equally efficient if loss of particles to the walls, injector, and gas scattering are neglected. In general, it appears that when particle loss is included, phase displacement acceleration is inferior to acceleration of particles in buckets. There are, however, certain situations in which phase displacement seems to have some advantages. In particular, if one is accelerating particles up to the transition energy with buckets, then there are empty buckets which simultaneously move down from high energy to the transition energy. These empty buckets may be employed to phase displace particles from the transition energy to the output energy of the accelerator, thus increasing particle acceleration efficiency.

It is in any case important to understand the phase displacement process, since it always operates on particles outside of buckets whether one makes use of it or not.

d. Beam stacking.

Particles may be accelerated by a radio frequency cycle as described above, until they reach an energy E_2. On successive cycles buckets full of particles are deposited at the energy E_2. The particles already there are displaced by successive buckets, on the average downward in energy, to make room in phase space for the newly arriving particles according to Liouville's theorem. When a suitable number of buckets of particles has been stacked near the energy E_2, a second radio frequency accelerator may accelerate the particles on to a new energy E_3. If the bucket size for the second cycle is n times the bucket size for the first cycle, then n buckets can be stacked at E_2 during the first cycle. These can then be picked up in a final bucket and carried to E_3 in a single cycle of the second type. The advantage of this system is that the radio frequency schedule can be chosen in the most efficient way to capitalize on the bucket size versus energy relation, which in turn depends upon the frequency of revolution versus energy curve. Thus usually $\frac{d\Omega}{dE}$, where Ω is the frequency of revolution, decreases with energy. (See Figure 1.) This has three consequences:

1. For a given radio frequency voltage, the bucket size increases with energy and hence in the usual acceleration method the buckets are nearly empty when they arrive at the transition energy. By stacking at intermediate energies this can be corrected.

2. For a given radio frequency voltage, the allowable rate of frequency modulation noticeably decreases with increasing energy, and hence the repetition rate is limited. By stacking one can use a higher repetition rate with small buckets at lower energies, and a smaller repetition rate at higher energies, but with larger buckets, so that the output current corresponds to the total injected current at the higher repetition rate.

3. When stacking particles, the particles already at an energy E_2 are displaced in energy by a succeeding bucket by an amount proportional to the phase area occupied by the bucket. Digital computer studies have indicated that the particles are not spread in energy by large amounts, but are kept fairly tightly bunched in energy, with a mean displacement depending only on the total area of the succeeding buckets. Thus if the bucket size is increasing with energy, the energy displacement of particles when one attempts to stack will be very large. Consequently one wants to stack particles using buckets which are full of particles. This can be accomplished by decreasing the cavity voltage or increasing the frequency modulation rate.

e. Multiple oscillators

Schemes have been proposed which involve several independent radio frequency accelerating voltages which act simultaneously on the particles being accelerated. The simplest such scheme, proposed by Darragh Nagle, utilizes a number N of identical oscillators operating simultaneously over the same frequency interval. Each oscillator follows a frequency modulation cycle as in conventional synchrotron acceleration. The N frequency modulation cycles of the different oscillator are staggered so that each oscillator accelerates a pulse of particles, the N pulses following are another in energy from the injection to the output energy. The scheme depends upon the fact that particles are relatively unaffected by radio frequency voltages with which they are not in synchronism. More complicated schemes can be envisioned in which a given particle is accelerated by more than one oscillating voltage, perhaps simultaneously. One may for example use interlaced bucket lift schemes in which several oscillator frequencies are chosen so that their subharmonics are interlaced.

It seems likely that very efficient accelerating schemes using multiple oscillators may be possible. However, too little is known theoretically about the behavior of particles under the action of multiple oscillators to be able to evaluate such schemes at present.

f. Scheduled and stochastic schemes.

Acceleration schemes may be classified as scheduled or stochastic. Scheduled schemes are those in which the radio frequency voltages are programmed in such a way that a particle is accelerated according to a planned schedule. Thus, for example, it is possible to choose the initial and final radio frequencies f_1 and f_2

in a bucket lift so that particles are passed from one bucket to another as they are accelerated. If, for example, for two integers h_1, h_2, we have $f_2/h_1 = f_1/h_2$, then during one frequency modulation cycle, a particle riding in a bucket at harmonic number h_1 is accelerated from an energy corresponding to a frequency of revolution f_1/h_1 to that corresponding to f_2/h_1. On the next FM cycle, the particle rides in a bucket of harmonic number h_2 from the frequency of revolution $f_1/h_2 = f_2/h_1$ to the frequency f_2/h_2. The total energy gain of the particle corresponds to a frequency ratio $\left(\frac{f_2}{f_1}\right)^2$ whereas the oscillator is modulated only over the range f_2/f_1. It is not difficult to find matching systems of harmonic numbers such that particles can be carried in a scheduled way over frequency ranges of many octaves with oscillators modulated over a frequency ratio of a fraction of an octave. Such schemes not only reduce the demands on the rf circuitry with respect to frequency modulation, but they increase the efficiency of the rf system by allowing one rf voltage to accelerate many pulses of particles simultaneously.

In stochastic, or unscheduled schemes, no attempt is made to program the radio frequencies precisely, and the energy of an individual particle varies in an unpredictable or random way. Thus in an unscheduled bucket lift scheme, the initial and final frequencies f_1 and f_2 may bear no particular relationship to each other and may even vary in a random way from cycle to cycle. A particle at the beginning of an FM cycle may or may not find itself in a bucket depending upon whether its frequency of revolution is sufficiently close to f_1/h for some h. If it is in a bucket, it is carried up in energy to a new energy corresponding to f_2/h. If it is not, it will be phase displaced downward in energy by the buckets which pass it during the FM cycle, and it may at the beginning of the next cycle be caught in a bucket. It is convenient to define a mean free path as the average energy increment received by a particle, once caught, before it again has a chance of losing energy. A particle starting at the injection energy has a certain probability of reaching the output energy before being lost. Under certain circumstances, stochastic acceleration schemes yield output currents comparable to those of scheduled schemes, but at the expense of a greater duty factor for the injector. In general, the greater the mean free path in a stochastic scheme, the more efficient is the scheme from the point of view of injector duty factor, and the more rapidly is any given particle carried from the injection to the output energy. If there are no loss mechanisms (orbit instability, gas scattering, etc.) between injector and output, then the time of transit does not affect the theoretical output current; if there are such loss mechanisms, then for long transit times, (short mean free paths) the output current is reduced. Though less efficient in some ways, they have the advantage of simplified rf circuitry. Various partially scheduled schemes are possible in which a particle once caught, may be carried in several successive buckets before being subject again to a chance of being caught or left behind. Recently, E. L. Burshtein, V. I. Veksler, and A. A. Kolomenskii[1] have proposed a stochastic accelerator in which the accelerating voltage is essentially a random noise. Here, of course, the mean free path is simply the mean voltage across the accelerating gap.

g. Intersecting beam experiments.

The proposal to achieve very high energy collisions by directing

1. E. L. Burshtein, V. I. Veksler, A. A. Kolomenskii: U.S.S.R. Academy of Sciences, Moscow, 1955, p. 3-6.

opposing accelerated beams against one another rests on the possibility of stacking successive pulses of particles in FFAG accelerators. Thus, if a circulating beam of particles has a sufficiently long lifetime against orbit instabilities, gas scattering, etc., then very high circulating currents of high energy particles can be built up in this way. Successive pulses of particles may be stacked at the output energy, to build up an intense beam, or they may be stacked at an intermediate energy, and then carried up to the output energy simultaneously in one large bucket. The considerations involved have been discussed under (d) above. The detailed theory will be worked out later.

II. Theory of Radio Frequency Acceleration in Fixed Field Accelerators

a. Frequency versus energy relationship

It is convenient to characterize an equilibrium orbit in a fixed field accelerator by its equivalent radius R defined by

$$L = 2\pi R, \tag{1}$$

where L is the length of the orbit. Each orbit R is traversed by particles of energy E (R). We define the momentum compaction parameter or mean field index k by either of the equivalent forms

$$k = \frac{R}{p}\frac{dp}{dR} - 1 = \frac{R}{\overline{H}}\frac{d\overline{H}}{dR}, \tag{2}$$

where p is the momentum, and $\overline{H}$ is the average magnetic field averaged along the orbit R. If k is constant, we have

$$\frac{p}{p_1} = \left(\frac{R}{R_1}\right)^{k+1} \tag{3}$$

$$\frac{\overline{H}}{\overline{H}_1} = \left(\frac{R}{R_1}\right)^{k} \tag{4}$$

The frequency of revolution of a particle in an orbit R is

$$f = \frac{pc^2}{2\pi RE} , \tag{5}$$

where E is the total energy, including the rest energy $E_o = mc^2$. By squaring Equation (5), differentiating, and rearranging, we obtain a formula for

$$\mathcal{K} = \frac{E}{f} \frac{df}{dE} = \frac{(k + 1) E_o^2 - E^2}{(k + 1) (E^2 - E_o^2)} . \tag{6}$$

The transition energy is given by

$$E_t = \sqrt{(k + 1)} E_o . \tag{7}$$

In a cyclotron $\mathcal{K} = 0$ and Equation (6) then defines k as a function of E. In a synchrotron, k is often constant, and we may then integrate Equation (6) to obtain

$$\frac{f}{f_t} = \left[\sqrt{k + 1} \frac{E_o}{E}\right] \left(\frac{E^2 - E_o^2}{k E_o^2}\right)^{\frac{1}{2} \frac{k}{(k+1)}} \tag{8}$$

where f_t is the frequency of revolution at the transition energy. The quantities $\mathcal{K}$ and f/f_t are plotted in Figure 1 for a typical case (k = 82.5).

b. Canonical form for the acceleration equations.

We neglect coupling between betatron and synchrotron oscillations, and assume that a particle is always on an equilibrium orbit. A particle with energy E travels along an orbit of length $2\pi R(E)$; we will call R(E) the equivalent radius. We define an equivalent angular variable Θ along the orbit by

$$d\Theta = ds/R \tag{9}$$

where ds is the element of arc length. Then if $\mathcal{E}(\Theta, R, t)$ is the electric field component along the orbit, we have

$$\frac{dE}{dt} = e\mathcal{E} R \dot{\text{(H)}} = 2\pi f \, Re\mathcal{E} \tag{10}$$

$$\frac{d\text{(H)}}{dt} = 2\pi f \ , \tag{11}$$

where f(E) is the frequency of revolution for a particle of energy E. If the orbit is not circular, a small oscillatory term in (H) must be added to the right number of Equation (11), but if the origin of (H) is properly chosen, this term has zero mean around the circumference, and we are here ignoring it.

We consider the case when $\mathcal{E}$ has the form

$$\mathcal{E} = \frac{1}{eR} \ F(\text{(H)}, \ t) \ , \tag{12}$$

that is, we assume that the accelerating gaps are radial and have a voltage independent of radius. The case when the voltage varies with radius according to a factor $\mathcal{L}$(R) can easily be treated by a slight modification of the method. We define a new energy variable W(E) as follows:

$$W = \int_{E_o}^{E} \frac{dE}{f(E)} \ , \tag{13}$$

where E_o is arbitrary and may conveniently b e taken as the rest energy if f(E) is extrapolated to that point. We can now rewrite Equations (10) and (11) in the form

$$\frac{dW}{dt} = 2\pi F(\text{(H)}, \ t) \ , \tag{14}$$

$$\frac{d\text{(H)}}{dt} = 2\pi \ f(W) \ , \tag{15}$$

which are derivable from the Hamiltonian function

$$H = -2\pi \int F(\text{(H)}, \ t) \, d\text{(H)} + 2\pi E(W) \ . \tag{16}$$

The variable W, Ⓗ are therefore canonical. It is convenient to think of W, Ⓗ as coordinates in a cylindrical phase space.

The advantage in writing the equations in canonical form is that we can apply certain useful general theorems. We have Liouville's theorem that a closed curve in the W, Ⓗ plane transforms under Equations (14), (15) in such a way that the enclosed area remains constant. If the Hamiltonian function varies sufficiently slowly in time, we may apply the adiabatic theorem which is stated conveniently for our purpose in the form: A set of points which at time t_1 lie along a curve $H(t_1)$ = constant in the W_1, Ⓗ -space, will at a later time t_2 be found to lie along a curve $H(t_2)$ = constant. In order to apply the theorem, it is necessary that H(t) does not change appreciably during the time required for a particle to traverse a typical sample of the curve H(t) = constant.

c. Application to beam stacking.

This result can be applied immediately to the problem of calculating the number of pulses of particles that can be stacked in any given region. Assume that we inject at an energy E_1 where the frequency of revolution is f_1, and that the energy spread from the injection is ΔE_1. A pulse of injected particles, injected for one or more full turns will then occupy an area

$$A_1 = 2\pi \Delta W_1 = \frac{2\pi \Delta E_1}{f_1} \tag{17}$$

in the W, Ⓗ -space. If we wish to stack n pulses at an energy E_2, then these pulses will occupy an area at least equal to

$$A_2 = nA_1 = \frac{2\pi \Delta E_2}{f_2}, \tag{18}$$

so that

$$\frac{\Delta E_2}{\Delta E_1} = n \frac{f_2}{f_1} \quad . \tag{19}$$

If we employ the mean field index defined in Equation (2), then

$$\frac{\Delta R}{R} \quad \frac{1}{(k+1)} \frac{E \Delta E}{(E^2 - E_o^2)} \tag{20}$$

Thus the minimum radial spread of the stacked beam is

$$\Delta R_2 = n \frac{(f_2)(\Delta E_1)}{(f_1)(E_1)} \quad \frac{(E_1 E_2) R_2}{(E_2^2 - E_o^2)(k+1)} \quad , \tag{21}$$

where E_o is the rest energy, and E_2 includes the rest energy.

d. Stationary buckets.

We now assume that we have several oscillators supplying radio frequency voltages at various accelerating gaps, so that

$$F(\Theta, t) = \sum_j F_j(\Theta, t) \cos \left(2\pi \int \nu_j dt\right), \tag{22}$$

where F_j, ν_j are slowly varying functions of t. We expand F_j in a Fourier series:

$$F_j(\Theta, t) = \sum_L A_{jL}(t) \sin (L\Theta - \beta_{jL}), \tag{23}$$

so that

$$H = 2\pi E(W) + \pi \sum_{j,L} \frac{A_{jL}}{L} \left\{ \cos \left[(L\Theta - \beta_{jL} - 2\pi \int \nu_j dt) \right] + \cos (L\Theta - \beta_{jL} + 2\pi \int \nu_j dt) \right\} \tag{24}$$

Let us suppose that $f = \nu_j / h$ for some ν_j and some harmonic number h. Then we introduce a rotating coordinate system on the W, Θ - cylinders:

$$\Theta^* = \Theta - \frac{2\pi}{h}\int \nu_j \, dt - \frac{\beta_{jh}}{h} . \tag{25}$$

For this purpose, we introduce the generating function

$$S = (W)\left(\Theta - \frac{2\pi}{h}\int \nu_j \, dt - \frac{\beta_{jh}}{h} \right), \tag{26}$$

which defines the canonical transformation W, $\Theta \longrightarrow W, \Theta^*$ through the equations

$$\begin{aligned} \Theta^* &= \frac{\partial S}{\partial W} , \\ W &= \frac{\partial S}{\partial \Theta} . \end{aligned} \tag{27}$$

The Hamiltonian becomes

$$\begin{aligned} H^* = H + \frac{\partial S}{\partial t} &= 2\pi E(W) - \left[\frac{2\pi \nu_j + \dot{\beta}_{jh}}{h}\right] W + \frac{\pi A_{jh}}{h} \cos h\, \Theta^* \\ &+ \sum_{j', L} \frac{\pi A_{jL}}{L} \Big\{ \cos\Big[L\Theta^* - \beta_{j'L} + \frac{L}{h}\beta_{jh} - 2\pi \\ &\int (\nu_{j'} - \frac{L}{h}\nu_j)\, dt + \cos\Big[L\Theta^* - \beta_{j'L} + \frac{L}{h}\beta_{jh} \\ &+ 2\pi \int (\nu_{j'} + \frac{L}{h}\nu_j) \, dt \Big] \Big\} , \end{aligned} \tag{28}$$

where the prime in the summation means that the first term in curly brackets is to be omitted in case j' = j, L = h. The terms in the summation are rapidly oscillating and may in many cases be neglected. We then have

$$H^* = 2\pi E(W) - \frac{2\pi\nu}{h} W + \frac{V}{h}\cos h\, \Theta^* , \tag{29}$$

where we have omitted the subscript j, have absorbed the small term $\dot{\beta}_{jh}$ in ν, and have set

$$V = \pi A_{jh} = \int_0^{2\pi} F_j (\Theta, t) \sin(h\Theta - \beta_{jh})\, d\Theta \tag{30}$$

The maximum energy gain per turn is V, as we see from Equations (30), (22) and (12), or from the equations

$$\frac{dW}{dt} = -\frac{\partial H^*}{\partial \Theta^*} = V \sin (h \Theta^*) , \tag{31}$$

$$\frac{d\Theta^*}{dt} = \frac{\partial H^*}{\partial W} = 2\pi(W - \frac{\nu}{h}) , \tag{32}$$

In the case of a single short accelerating gap at $\Theta = \Theta_o$, we have

$$\beta_{jh} = h\Theta_o - \pi/2 , \tag{33}$$

$$F_j(\Theta , t) = V\delta(\Theta - \Theta_o) , \tag{34}$$

$$V = \int e \, \mathcal{E}_{max} R d\Theta , \tag{35}$$

Θ^* in this case is the angular position relative to a particle which is synchronous with the oscillation and arrives at the accelerating gap at a moment when the voltage is zero and decreasing.

If ν and V are constant, then H^* is a constant of the motion. We define the synchronous value W_s by

$$f(W_s) = \frac{\nu}{h} \tag{36}$$

and expand

$$E = E_s + f_s W^* + \frac{1}{2} f'_s W^{*2} + \ldots\ldots , \tag{37}$$

where the prime denotes a derivative with respect to W, and

$$W^* = W - W_s . \tag{38}$$

If we neglect terms of order W^{*3}, we have

$$H^* = \pi f'_s W^{*2} + \frac{V}{h} \cos h \Theta^* . \tag{39}$$

(The term $(2\pi h E_s - 2\pi \nu W_s)$ has been omitted, since it does not contain W^* nor Θ^* and has no influence on the resulting canonical equations.)

A plot of the curves H^* = constant is given in Fig. 2 for the case h = 3. If we set

$$h\Theta^* = \varphi , \qquad (40)$$

$$y = \left[\frac{2\pi h |f_s'|}{V} \right] W^* \qquad (41)$$

$$C = \frac{hH^*}{V} , \qquad (42)$$

then Equation (39) takes on the dimensionless form

$$\pm \frac{1}{2} y^2 + \cos\varphi = C , \qquad (43)$$

where the positive sign applies if $f_s' > 0$, and the negative sign if $f_s' < 0$.

We can write

$$f' = \frac{df}{dW} = f\frac{df}{dE} = \kappa \frac{f^2}{E} , \qquad (44)$$

where κ is given by Equation (6). Curves of constant C according to Equation (43) are plotted in Fig. 3.

We see in Fig. 3 the region of stability or "bucket" within which particles execute stable phase oscillations about the stable phase $\varphi = \pi$ ($\kappa > 0$) or $\varphi = 0$ ($\kappa < 0$). There are h such buckets around the $W\Theta$ cylinder at each F value for which $f = \nu/h$ for some h, and the buckets revolve with frequency f. Outside the buckets, the particles move around the cylinder out of synchronism with the buckets. The bucket boundary, or separatrix is given by Equation (43) with $C = \pm 1$:

$$y = 2 \begin{matrix}\sin \\ \cos\end{matrix} \left(\frac{\varphi}{2}\right) . \qquad (45)$$

The half-height of the bucket is given by $y_m = 2$, and the maximum energy deviation is

$$E_m - E_s = f\, W_m^* = \left[\frac{2V\ E_s}{\pi h\ |K|}\right]^{1/2} . \qquad (46)$$

The area of the buckets, in W^*, Ⓗ space, counting all h buckets at a given harmonic is, (the area of a bucket-shaped figure of half-dimensions a, b is $\frac{8}{\pi}$ ab),

$$A = \frac{8}{f}\left[\frac{2VE}{\pi h K}\right]^{1/2} . \qquad (47)$$

Near the stable point $\phi = 0$ or π, the curves C = constant are ellipses and the frequency of the phase oscillations around these ellipses is

$$\nu_p = f\left[\frac{hKV}{2\pi E}\right]^{1/2} \qquad (48)$$

The above formulas apply only when the contribution from all terms in the summation in Equation (28) may be neglected. In particular, those formulas fail for energies midway between two harmonics, and they certainly fail when the bucket dimensions calculated from Equation (46) are so large that the buckets for adjacent harmonics would overlap. If only one term from the summation is important, say the term j', L, it may be added to the approximate Hamiltonian (29), which now becomes periodic in the time with frequencies $\nu_{j'} - \frac{L}{h}\nu_j$ and $\nu_{j'} + \frac{L}{h}\nu_j$. If only one oscillator is present (j' = j), then all terms in the summation are periodic with frequency ν/h. In such cases, the analytic techniques developed for studying motion under a periodic Hamiltonian are applicable. We find, indeed, that accurate numerical solution of the acceleration equations leads to agreement with these ideas. In particular, when high voltages are used, the buckets no longer agree exactly with the above equations, and we find at the bucket boundaries typical cases of scattering of phase points, appearance of strings of pearls, etc., which arise in studying the non-linear equations for alternating gradient orbits. An example is plotted in Fig. 4a where we show results for a single accelerating gap at a very high voltage. (V = 10 Mev, Estable = 50 Mev, k = 99, h = 2). The phase and energy are plotted at each revolution. For certain starting values, the points lie on invariant curves as drawn. In other cases the points scatter, and a typical set of such points starting from a single central point is shown. Fig. 4b shows a phase plot for a particle subject to two oscillators. ($V_1 = V_2 = 100$ kev, $h_1 = h_2 = 1$, k = 99, oscillators on opposite sides of the accelerator.) The oscillators have frequencies which would lead to the buckets shown, if a particle were subject to each oscillator alone. In Fig. 4c is shown a phase plot in the neighborhood of the 9 and 10 subharmonics of a single oscillator. One notices the various other stable regions occuring between these harmonics. In this case ($V_1 = V_2 = 20$ Mev, k = 99, Estable = 500 Mev. (h = 10), $E_s = 814$ Mev (h = 9).)

e. Adiabatic motion of buckets.

If now the parameters ν_1, V are varied slowly, we may apply the adiabatic theorem to determine the behavior of the particles. A group of particles which at time t_1 lie on a closed curve $H^* (t_1)$ = constant, of area A_1 inside the bucket, will at time t_2 after adiabatic variation of ν, V lie on a closed curve $H^*(t_2)$ = constant, with area A_2; and now by Liouville's theorem $A_1 = A_2$. Thus the adiabatic theorem, together with Liouville's theorem enable us to conclude that a particle inside the bucket remains on a curve H^* = constant, of constant area as the bucket shape or position changes adiabatically.

If the variation of parameters is not adiabatic, Liouville's theorem still applies, so that a group of particles, initially on a closed curve $H^*(t_1)$ =

constant, will remain on a closed curve of constant area, but the curve at a later time t_2 will not be of the type $H^*(t_2)$ = constant. Since particles near the separatrix which bounds the bucket move around a curve with a frequency which approaches zero as the curve approaches the separatrix, the adiabatic theorem cannot be applied to such particles unless the rate of variation of parameters approaches zero. Hence the separatrix does not correctly represent the boundary of the bucket except when the parameters are constant. We will return to this point later.

A particle outside the bucket, but far enough from all other harmonics so that neglect of the summation in Equation (28) is justified, will in the same way remain on a curve H^* = constant, having a constant area beneath it on the W, Ⓗ cylinder. That is, the phase average

$$\overline{W} = \frac{1}{2\pi} \int_0^{2\pi} W \, d\,Ⓗ \qquad (49)$$

remains constant for a particle outside the bucket under adiabatic variation of parameters. Assume now that the frequency is varied so that the bucket approaches the particle from below. The particle then moves along curves which lie closer and closer to the separatrix. The frequency of revolution of the particle relative to the bucket approaches zero and the particle spends most of its time near the unstable fixed point. The rate of frequency modulation must approach zero as the particle approaches the separatrix in order for the adiabatic condition to be satisfied, and the bucket can never pass the particle adiabatically. However, if when the particle is nearly on the upper separatrix, and consequently spends nearly all of its time just above the unstable fixed point, we suddenly change the frequency so as to move the bucket up slightly, the particle will almost certainly find itself just below the fixed point. It now moves just under the lower separatrix. The phase area beneath this curve differs from that beneath the original curve on which the particles lay by the area of the buckets. If we now move the buckets away adiabatically $\overline{W}$ as defined by (49) will remain constant at a value below the initial value by an amount

$$\Delta \overline{W} = \frac{A}{2\pi} , \tag{50}$$

where A is the area of the buckets. This is the process of phase displacement.

In a similar fashion, we can discuss the adiabatic capture and loss of particles near the synchronous energy as the voltage V is increased or reduced. We can show that a group of particles lying in a band of width ΔW around the phase cylinder and centered on the synchronous value W_s, will, if the voltage V is increased adiabatically from zero, be captured into the buckets so that they lie within a closed curve H^* = constant, of area $2\pi\Delta W$. The converse process occurs when the voltage V is turned off adiabatically.

f. Transition energy.

At the transition energy $K = 0$, and we must keep terms up to W^{*3} in Equation (37). By differentiating formula (6), we find at the transition energy

$$\frac{E^2}{f}\left(\frac{d^2 f}{dE^2}\right)_t = -\frac{2}{k} . \tag{51}$$

Hence, if we set

$$E^* = E - E_t , \quad W^* = W - W_t , \tag{52}$$

we have, near the transition energy,

$$f = f_t \left(1 - \frac{E^{*2}}{kE_t^2} \right) , \tag{53}$$

$$W^* = \frac{E^*}{f_t} \left(1 + \frac{1}{3k} \frac{E^{*2}}{E_t^2} \right) \tag{54}$$

$$E^* = f_t W^* - \frac{1}{3k} \frac{f_t^3 W^{*3}}{E_t^2} \tag{55}$$

We introduce the dimensionless variables

$$y = \left(\frac{4\pi}{k}\right)^{1/3} \left(\frac{hE_t}{V}\right)^{1/3} \frac{f_t W^*}{E_t} , \tag{56}$$

$$\varphi = h \, Ⓗ^* \tag{57}$$

and the parameters

$$\eta = (2\pi^2 k)^{1/3} \left(\frac{hE_t}{V}\right)^{2/3} \left(1 - \frac{\nu}{hf_t}\right), \tag{58}$$

$$C = \frac{hH^*}{V} . \tag{59}$$

We may then write the Hamiltonian (29) in the form

$$-\frac{1}{6} y^3 + \eta y + \cos\varphi = C, \tag{60}$$

where we have again omitted terms independent of φ, y.

Graphs of Equation (60) for several values of C are shown in Figs. 5, 6, 7, 8, and 9. We are particularly interested in the separatrices which bound the buckets. The unstable fixed points above and below the transition energy are given respectively by

$$y = \sqrt{2\eta} , \qquad \varphi = \pi , \text{ above and}$$

$$y = \sqrt{2\eta} , \qquad \varphi = 0, \text{ below.} \tag{61}$$

Hence the values of C on the separatrices are

$$C = \pm \left[\frac{1}{3} (2\eta)^{3/2} - 1\right] . \tag{62}$$

For large values of η we have separate buckets above and below the transition energy as shown in Fig. 5. At the critical value

$$\eta_C = \frac{1}{2} (3)^{2/3} = 1.040 \tag{63}$$

the two values of C become equal, and we have the case shown in Fig. 6 where the buckets just touch. For $0 < \eta < 1.040$, the phase plot is as shown in Fig. 7. For $\eta = 0$, when $\nu = hf_t$, the phase plot is as shown in Fig. 8. For $\eta < 0$, there are no separatrices, as shown in Fig. 9. The values y_1, y_2, y_3, y_4 and φ_1 indicated on the figures are plotted in Fig. 10.

The bucket areas, that is, the areas around the stable fixed points and inside the separatrices can be calculated from the formula (W - Ⓗ units)

$$A_1 = \frac{\sqrt[3]{12}\,\pi^2}{2} \left(\frac{kV}{4\pi h E_t}\right)^{1/3} \frac{E_t}{f_t}\, \alpha_1(\eta) \; , \qquad (64)$$

where the function $\alpha_1(\eta)$ is plotted in Fig. 11. Formula (64) gives the area of the h buckets on one side of the transition energy. The area of the region between the outer separatrix on either side and the transition energy is given by

$$A_2 = \frac{\sqrt[3]{12}\,\pi^2}{2} \left(\frac{kV}{4\pi h E_t}\right)^{1/3} \alpha_2(\eta) \; . \qquad (65)$$

The function $\alpha_2(\eta)$ is also plotted in Fig. 10.

By studying the Fig. 5 - 11, we can predict the behavior of the phase points as the frequency is increased adiabatically through the transition frequency. (By adiabatic we mean here that $\dot{\nu}$ is small enough that the adiabatic theorem can be applied except in the immediate neighborhood of the separatrix.) As ν increases η decreases. As $\eta \to \eta_C$, it can be seen from Figs. 5 and 11 that about half of the phase area between the buckets and the transition energy is phase displaced past the buckets and about half is absorbed into the outer region of the growing bucket. Between $\eta = 2.1$ and $\eta = \eta_C$, the bucket area increases about 30%, so that at $\eta = \eta_C$, the outer 30% of the bucket is populated with phase points which were originally between $W = W_t$ and $W = W_t \pm \frac{A_2 - A_1}{2\pi}$ where A_2, and A_1 are evaluated at $\eta = 2.1$. Beyond $\eta = \eta_C$, both A_2 and A_1 decrease, so that when $\nu = \nu_t$, $\eta = 0$, the outer 30% of the bucket area at $\eta = \eta_C$ has been deposited outside the final separatrix on the same side from which the bucket came. Thus phase points initially between the buckets and the transition energy at $\eta = 2.1$ are left in the same region (though not necessarily at the same W) when $\eta = 0$. Phase points which were in the bucket below the transition energy at $\eta = 2.1$ are at $\eta = 0$ in the upper half of the region between the two separatrices (above the dashed curve in Fig. 8), and in such a way that points originally nearer the center of the bucket are left nearer the upper separatrix. If the frequency ν is now modulated beyond ν_t, the curves along which the phase points move straighten out, so that points in the lower bucket at $\eta = 2.1$ are finally deposited in a band above the transition energy extending from $W = W_t$ to $W = W_d$ where

$$W_d - W_t = \frac{A_{2t}}{2\pi} = \frac{\sqrt[3]{12}\,\pi}{4}\left(\frac{kV}{4\pi hE_t}\right)^{1/3} . \tag{66}$$

Furthermore, points originally near the center of the bucket are deposited near W_d, so that if at a point $\eta_1 > 2.1$, the area of the bucket below the transition energy was A_1, particles in the bucket at this time are left finally in a band $W_d \geq W \geq W_d - (A_1/2\pi)$. If the frequency were modulated adiabatically downward from above ν_t, the above process would take place in reverse. In order to accelerate the particles in the band $W_d \geq W \geq W_d - A_1/2\pi$ higher in energy, one may turn off the oscillator voltage, bring the frequency to a point synchronous with particles in this band, increase the voltage adiabatically to capture the band into a bucket in the usual way, and modulate the frequency downward to carry them higher in energy. It is also possible to cross the transition in a non-adiabatic way by modulating the frequency more rapidly up to a value slightly greater than ν_t and then downward again. If the frequency overshoot is properly adjusted relative to the rate of frequency modulation, one can see from the figures that a fraction of the particles can be transferred from the lower to the higher energy buckets.

It is of interest to calculate the frequency at the points W_c and W_d. At $\eta = \eta_c$, according to Equation (53) and Fig. 10 the frequency of revolution is

$$f_c = f_t \left[1 - 2.08\left(\frac{V}{4\pi h E_t}\right)^{2/3} k^{-1/3}\right] , \tag{67}$$

and at W_d it is

$$f_d = f_t \left[1 - 3.23\left(\frac{V}{4\pi h E_t}\right)^{2/3} k^{-1/3}\right] . \tag{68}$$

It should be emphasized that the formulas in this section are correct only when higher order terms in E^*/E_t are neglected. The ratio of the next (cubic) to the quadratic term in formula (53) is

$$\frac{\dfrac{E^{*3}}{6}\,\dfrac{d^3 f}{dE^3}}{\dfrac{E^{*2}}{2}\,\dfrac{d^2 f}{dE^2}} = -\frac{3k}{5k+4}\,\frac{E^*}{E_t} . \tag{69}$$

not very good for present model except very near transition.

g. Synchronous coordinate system.

When the oscillator frequency changes rapidly, the adiabatic theorem as applied in the preceding section breaks down, particularly near the separatrices. Let us make a canonical transformation $W, \text{Ⓗ}^* \longrightarrow W^*, \text{Ⓗ}^*$ via the generating function

$$S = -\text{Ⓗ}^* (W - W_s) , \tag{70}$$

where $W_s(t)$ is defined by Equation (36) when the oscillation frequency is a specified function (t). The new canonical momentum

$$W^{*} = W - W_s(t) \tag{71}$$

is then measured with respect to the value W_s for a synchronous particle. The Hamiltonian (29) becomes under the transformation (70),

$$H^{**} = 2\pi E(W^{*}) - \frac{2\pi \dot{\nu}}{h}\left[W^{*} + W_s(t)\right] + \frac{V}{h}\cos h\,Ⓗ^{*} + \dot{W}_s H^{*}, \tag{72}$$

where, by Equation (36),

$$\dot{W}_s = \frac{\dot{\nu}}{h f'_s} = \frac{\dot{\nu} E_s}{h K_s f_s^2}. \tag{73}$$

If we expand $E(W^{*})$ as in Equation (37), and omit terms independent of W^{*}, $Ⓗ^{*}$, we obtain

$$H^{**} = \dot{W}_s Ⓗ^{*} + \frac{V}{h}\cos h\,Ⓗ^{*} + \pi f'_s W^{*2} + \frac{\pi}{3} f''_s W^{*3} + \ldots \tag{74}$$

The quantities f'_s, f''_s, - - - will be slowly varying, and if $\dot{W}_s$ and V are constant, or slowly varying, we can apply the adiabatic theorem to Equation (74), even when $\dot{\nu}$ is large.

Let us neglect terms of order W^{*3} and make the substitutions (40), (41), (42), so that Equation (74) takes on the dimensionless form

$$\pm 1/2\, y^2 + \cos\varphi + \Gamma\varphi - C, \tag{75}$$

with

$$\Gamma = \frac{\dot{W}_s}{V} = \frac{\dot{\nu}}{h K_s f_s^2}\;\frac{E_s}{V}. \tag{76}$$

The sign of y^2 in Equation (75) is the same as the sign of K. Curves of constant C are plotted in Fig. 12, for $K > 0$, $\Gamma = .5$. Particles within the closed separatrix execute phase oscillations in a clockwise sense about the synchronous point W_s, ϕ_s, where

$$\sin \varphi_s = \Gamma \quad , \tag{77}$$

at a small amplitude frequency

$$\nu_p = \left[\frac{h |K| V \sqrt{1-\Gamma^2}}{2\pi E_s} \right]^{1/2} f_s \tag{78}$$

Particles outside the separatrix move along curves which circle downward around the y, ϕ cylinder, reversing direction of circling as they pass the bucket.

The separatrix in Fig. 12 is given by (for $K > 0$)

$$y^2 = 4 \sin^2 \frac{\varphi}{2} - 2\Gamma\varphi + 2\left[\sqrt{1-\Gamma^2} + \Gamma(\pi - \varphi_s) - 1\right] . \tag{79}$$

The values of y_1, φ_s, φ_1, φ_2 as indicated in Fig. 12 are plotted in Fig. 13 as functions of Γ . The area of the bucket is

$$A = \frac{8}{f_s} \left[\frac{2VE_s}{\pi h |K_s|} \right]^{1/2} \alpha_3(\Gamma) , \tag{80}$$

where the factor $\alpha_3(\Gamma)$ is plotted in Fig. 14. In analyzing bucket lift schemes, it is useful to see how A varies with energy for a given oscillation voltage and rate of frequency modulation. It is then convenient to rewrite Equation (80) in the form

$$A = 8\sqrt{\frac{2}{\pi}} \; V\dot{\nu}^{-1/2} \alpha_3 |\Gamma|^{1/2} . \tag{81}$$

The area A for a given oscillator varies as $\alpha_3 |\Gamma|^{1/2}$ as the parameters E_s, f_s, K_s, h change with energy. The quantity $\alpha_3 |\Gamma|^{1/2}$ is also plotted in Fig. 14. It will be noted that the bucket area changes relatively little over a fairly wide range of Γ .

It is of interest to calculate the energy change suffered by a particle outside of a moving bucket as the bucket goes by. We know the average change in W must agree with that calculated on the basis of Liouville's theorem in Section IIe. However it is clear from Fig. 12 that a particle outside the bucket and near the separatrix will spend a long time in the neighborhood of the unstable fixed point, and hence will be carried along for a considerable distance by the bucket. There will therefore by fluctuations in the energy change about the average value.

By equation (31), we have for the change in W

$$\Delta W = \int_{\varphi}^{\varphi_0} \frac{V \sin \varphi}{\dot{\varphi}} \, d\varphi \quad . \tag{82}$$

We expand the right member of Equation (32), keeping only first order terms in W^*, using $\varphi = h \Theta^*$, to obtain

$$\dot{\varphi} = f \left[\frac{2\pi h K V}{E} \right]^{1/2} y \tag{83}$$

If we take ϕ_0 to be the phase at which the curve (75) passes the bucket (y = 0), we obtain

$$\Delta W = \frac{1}{f} \left[\frac{EV}{4\pi h K} \right]^{1/2} \int_{\varphi}^{\varphi_0} \frac{\sin \varphi \, d\varphi}{\left[\cos \varphi_0 - \cos \varphi + \gamma (\varphi_0 - \varphi)\right]^{1/2}} . \tag{84}$$

The total change in W is twice the limiting value for $\varphi \to -\infty$:

$$\Delta W = \frac{1}{f} \left[\frac{EV}{\pi h K} \right]^{1/2} \int_{-\infty}^{\varphi_0} \frac{\sin \varphi \, d\varphi}{\left[\cos \varphi_0 - \cos \varphi + \Gamma (\varphi_0 - \varphi)\right]^{1/2}} . \tag{85}$$

This integral must be evalutated numerically.

h. Phase Flux.

A useful concept in the analysis of accelerating systems is the phase flux Φ (W) defined as the phase area per unit time which is accelerated past a given value of W. By Liouville's theorem, if there is some value of W at which the phase flux is zero, then the flux of area <u>decelerated</u> per unit time past any value of W must also be equal to Φ (W), that is, the net phase flux past any value of W is zero.

If the phase area being accelerated is filled with an average density, J (particles per unit area), then the current of particles per unit time accelerated past the point W is

$$I(W) = J \; \Phi(W) . \tag{86}$$

Since J can never exceed its value at injection (Equation (17)) , the maximum output current which can be delivered by an accelerating scheme is J Φ min, where Φ min is the minimum phase flux between injection and output. Since any non-adiabatic mishandling of the particles, e. g. jitter in the frequency modulation or in the accelerating voltage V, will reduce the phase density J, in a well designed accelerating system, Φ should increase with W in proportion to the decrease in J. In FFAG synchrotrons, for a given maximum voltage V, Φ can be made much larger at high energies than at low energies because of the decrease in K at higher energies. The theoretical output currents from high energy FFAG synchrotrons calculated according to these principles are very large -- comparable with synchro-cyclotron currents.

If n buckets per unit time of area A pass a given point per second, the phase flux is

$$\Phi = n A \tag{87}$$

If, for example, an accelerating scheme, utilizing a single harmonic number h, accelerates particles from energy E_1 to E_2 at constant [illegible] , the repetition rate is

$$n = \frac{\dot{W}_s}{\Delta W} = \frac{\Gamma V f}{E_2 - E_1}$$

(88)

where $\bar{f}$ is a suitable average value. The phase flux is then, by formulas (80), (87), and (88)

$$\Phi = 8\sqrt{\frac{2}{\pi}}\ \frac{\bar{f}}{f}\ \frac{V^{3/2}E^{-1/2}}{(h|K|)^{1/2}(E_2-E_1)}\ \alpha_3\Gamma . \tag{89}$$

The quantity $\alpha_3\Gamma$ is plotted in Fig. 14. For a given V, the maximum phase flux at any given W is achieved by choosing ν so that $\Gamma = 0.4$. The minimum phase flux Φ then occurs where $|f^2 K/E|$ is a maximum. This quantity is plotted in Fig. 1 for k = 82.5. The repetition rate, and hence the phase flux can be increased somewhat by choosing ν(t) so that the bucket area A remains constant during the acceleration. The maximum phase flux is then more difficult to calculate, but is usually not different in order of magnitude. If instead of Γ, $\dot{\nu}$ is held constant, the phase flux is

$$\Phi = 8\sqrt{\frac{2}{\pi}}\ \frac{|\dot{\nu}|^{1/2}}{\nu_2-\nu_1}\ V\ \alpha_3\,|\Gamma|^{1/2}, \tag{90}$$

As a second example, in a bucket lift scheme which uses all harmonics, the number of harmonics per unit time which pass a given frequency f is, if h is large,

$$n = \left|\frac{dh}{dt}\right| = \frac{h\dot{\nu}}{\nu} \tag{91}$$

and the phase flux is

$$\Phi = 8\sqrt{\frac{2}{\pi}}\quad \frac{V^{3/2}\,(hK)^{1/2}}{E^{1/2}}\cdot\ \alpha_3\Gamma \tag{92}$$

Again the maximum phase flux at any value of W is achieved by choosing $\dot{\nu}$ so that [redacted] = 0.4. We may also write Equation (92) in the form

$$\Phi = 8\sqrt{\frac{2}{\pi}}\,\frac{|\dot{\nu}|^{1/2}}{f_s}\,V\alpha_3\,|\Gamma|^{1/2} \tag{93}$$

which shows that for a given oscillator, Φ is proportional to $\alpha\,|\Gamma|^{1/2}/f_s$ at different energies.

As an illustration of the concept of phase flux, we consider the following case. An accelerating system brings a phase flux Φ_1 filled with particles at a mean density J_1 past a point W_1 beyond which the particles are spilled out in any manner. A second accelerating system carries a phase flux Φ_2 out of the region just beyond W_1. If $\Phi_2 = \Phi_1$, then it is in principle possible to synchronize the two systems in such a way that all of the current accelerated by the first system is picked up by the second. Suppose, however, that the systems are not synchronized, and that an equilibrium exists in which the region just beyond W_1, including the region from which the second system buckets draw their area, is filled with uniform density J_2. Then since the first system decelerates an equal phase flux, we have, by balancing currents,

$$J_1\Phi_1 = J_2\Phi_1 + J_2\Phi_2 . \tag{94}$$

The phase density in the second bucket is therefore

$$J_2 = \frac{\Phi_1}{\Phi_1 + \Phi_2}\,J_1 , \tag{95}$$

and the current accelerated by the second system is

$$J_2\Phi_2 = \frac{\Phi_2}{\Phi_1 + \Phi_2}\,J_1\Phi_1 . \tag{96}$$

Such a random transfer of particles between systems can therefore be effected with an efficiency of about 1/2 if the phase fluxes are equal. Either the current efficiency or the phase density efficiency (but not both) can be made to approach unity by making the ratio Φ_2 / Φ_1 sufficiently large or sufficiently small.

MIDWESTERN UNIVERSITIES RESEARCH ASSOCIATION

BUCKET AREA PARAMETERS

$dW = dE/f, \quad d\Theta = ds/R$

$$dA = dW\,d\Theta, \quad A = \frac{8}{f}\left[\frac{2VE}{\pi h |\mathcal{K}|}\right]^{1/2} \alpha_3(\Gamma)$$

E = particle energy, f = frequency of revolution (cycles per second)

ds = element of arc length along orbit, $2\pi R$ = length of orbit

A = total area of buckets at harmonic number h, V = oscillator voltage

$$\mathcal{K} = \frac{(k+1)\,E_o^2 - E^2}{(k+1)(E^2 - E_o^2)}$$

$$\Gamma = \sin\phi_s = \frac{\dot{W}}{V} = \frac{\dot{\nu}}{h\mathcal{K} f^2}\,\frac{E}{V}$$

ϕ_s = stable phase angle, ν = oscillation frequency.

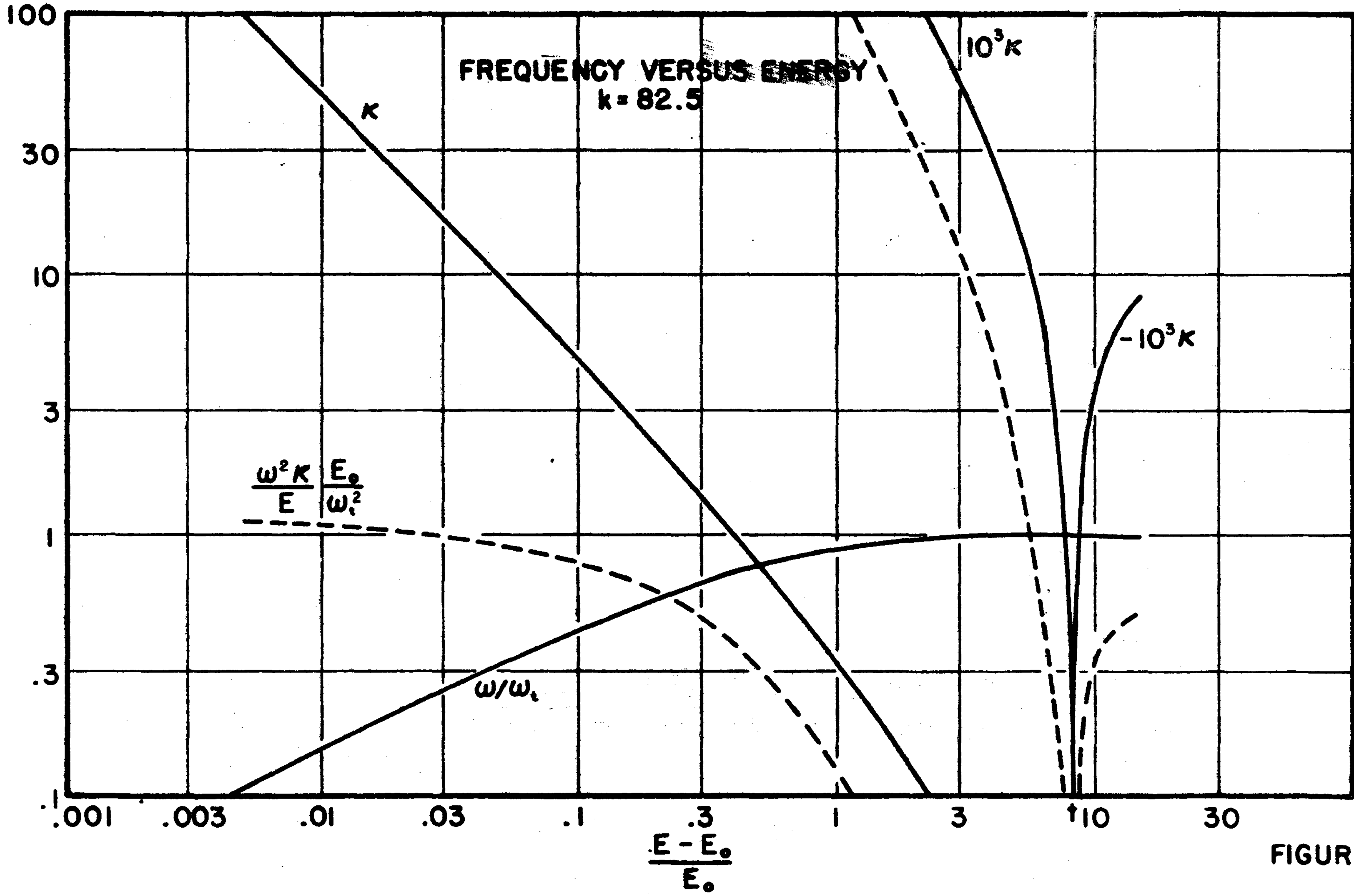

FREQUENCY VERSUS ENERGY
k = 82.5
100
30
10
3
1
.3
.1
.001
.003
.01
.03
.1
.3
1
3
10
30
κ
$10^3 κ$
$-10^3 κ$
$\frac{\omega^2 κ}{E} \frac{E_0}{\omega_t^2}$
ω/ω_t
$\frac{E - E_0}{E_0}$

FIGURE 1

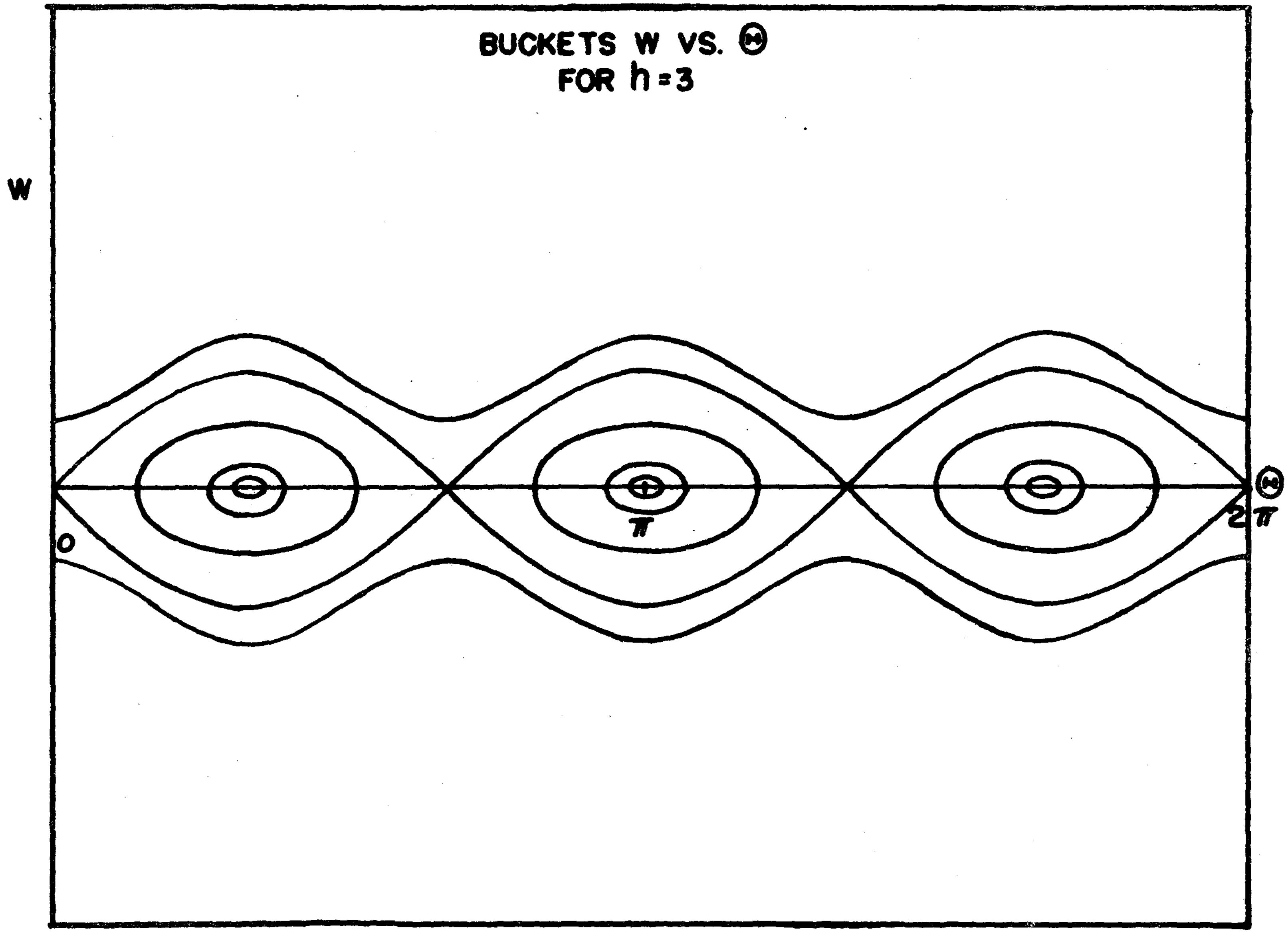


FIG. 2

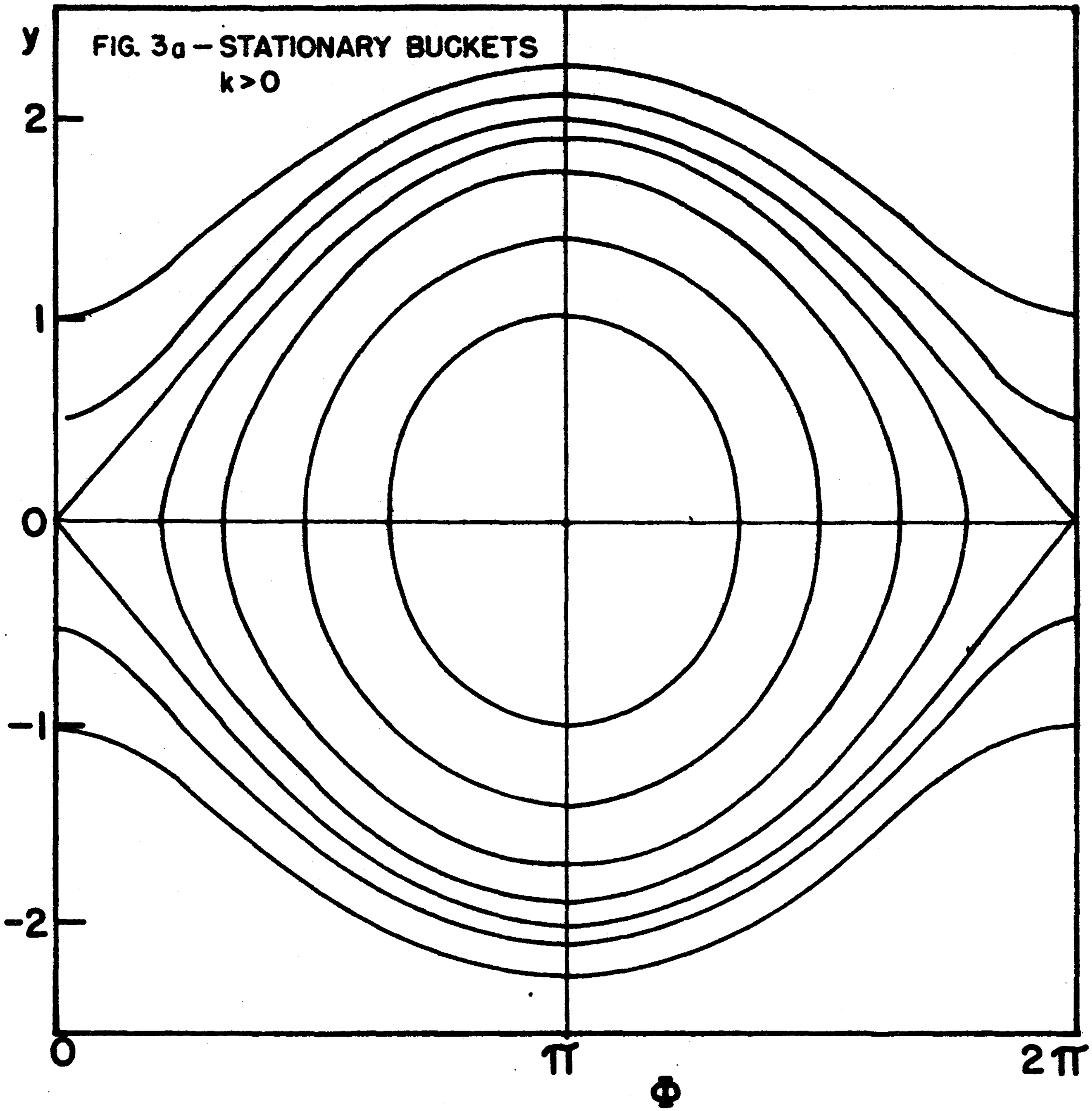


FIG. 3a – STATIONARY BUCKETS
k > 0
y
2
1
0
-1
-2
0
π
2π
Φ

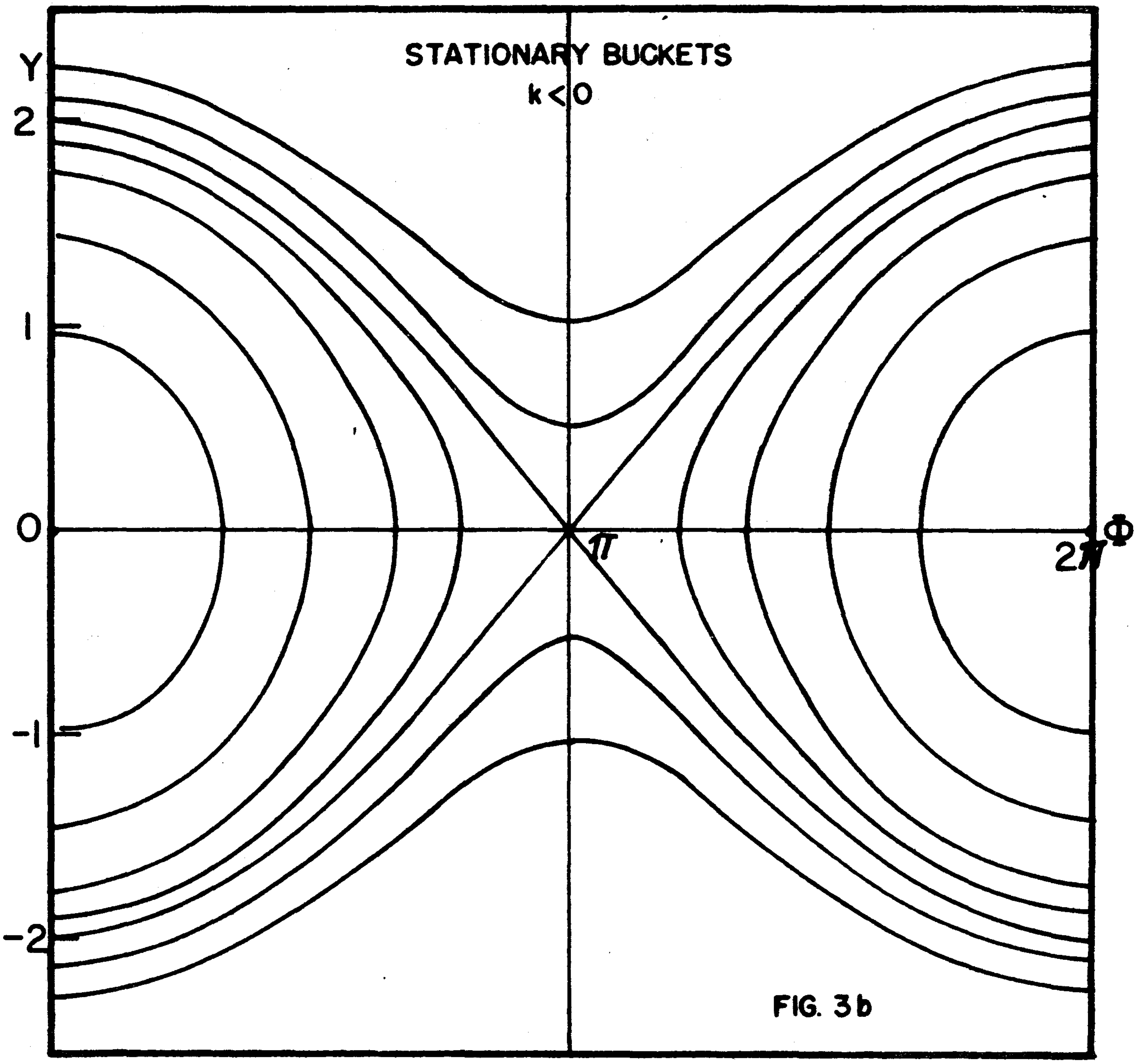
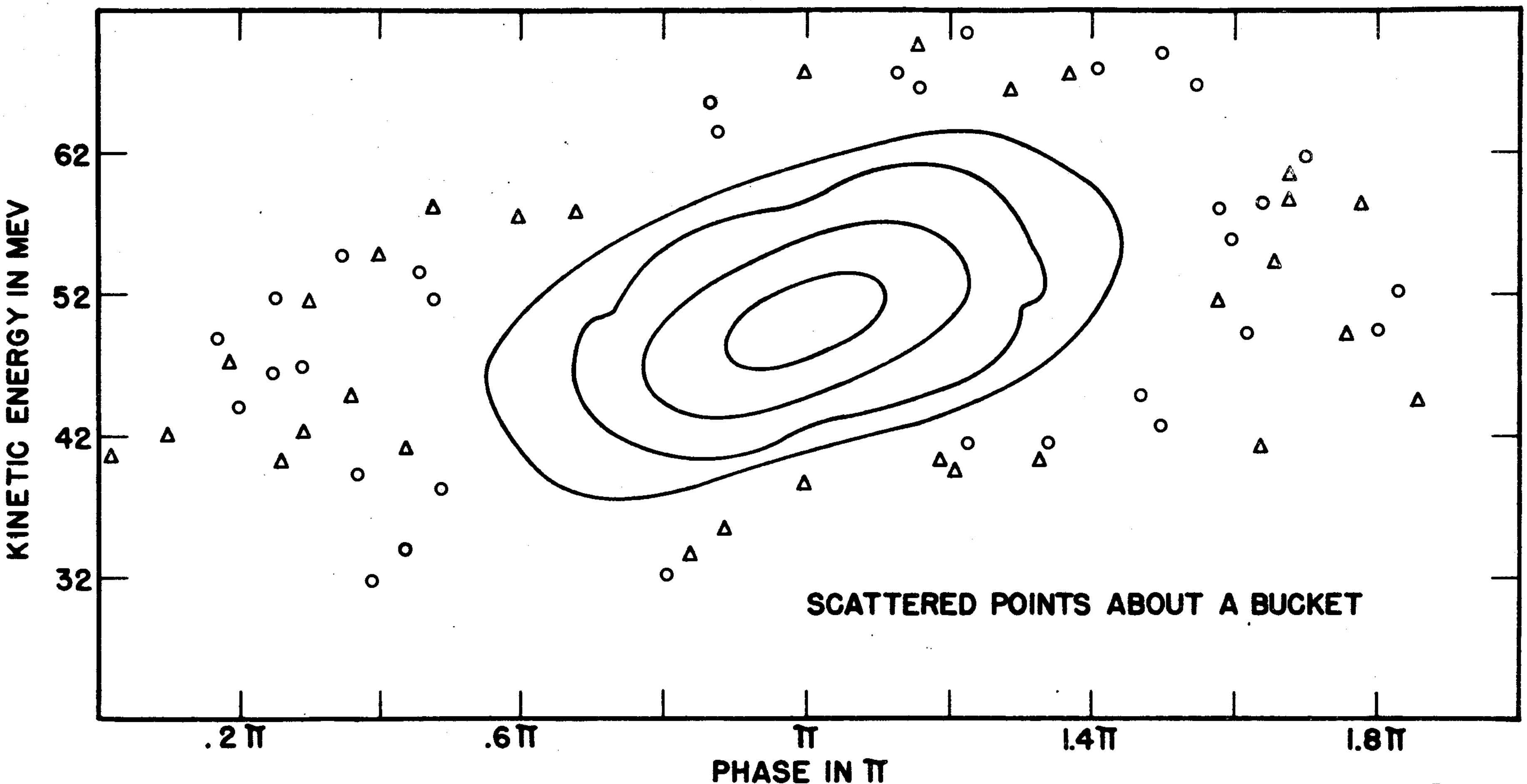


FIG. 3b



KINETIC ENERGY IN MEV
62
52
42
32
SCATTERED POINTS ABOUT A BUCKET
.2π
.6π
π
1.4π
1.8π
PHASE IN π

FIGURE 4A

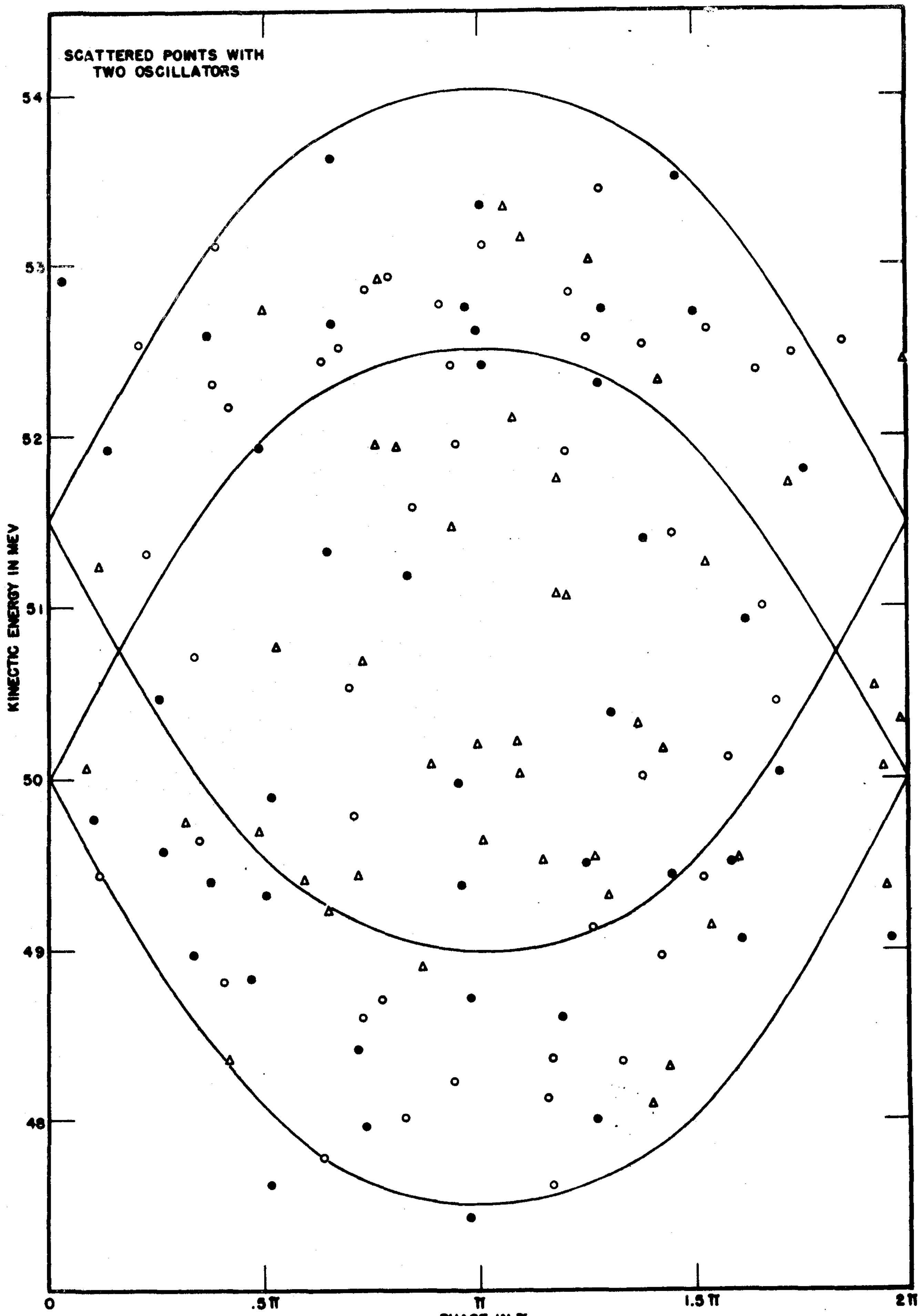

SCATTERED POINTS WITH
TWO OSCILLATORS
54
53
52
51
50
49
48
KINECTIC ENERGY IN MEV
0
.5π
π
1.5π
2π
PHASE IN π

FIGURE 4B

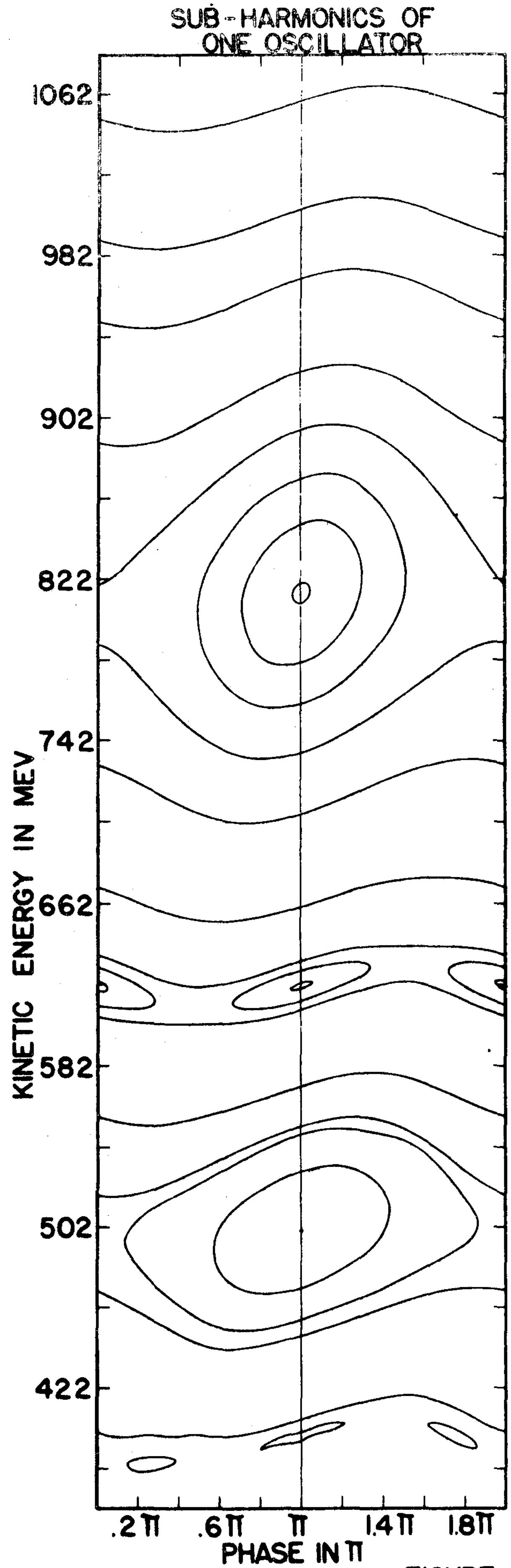
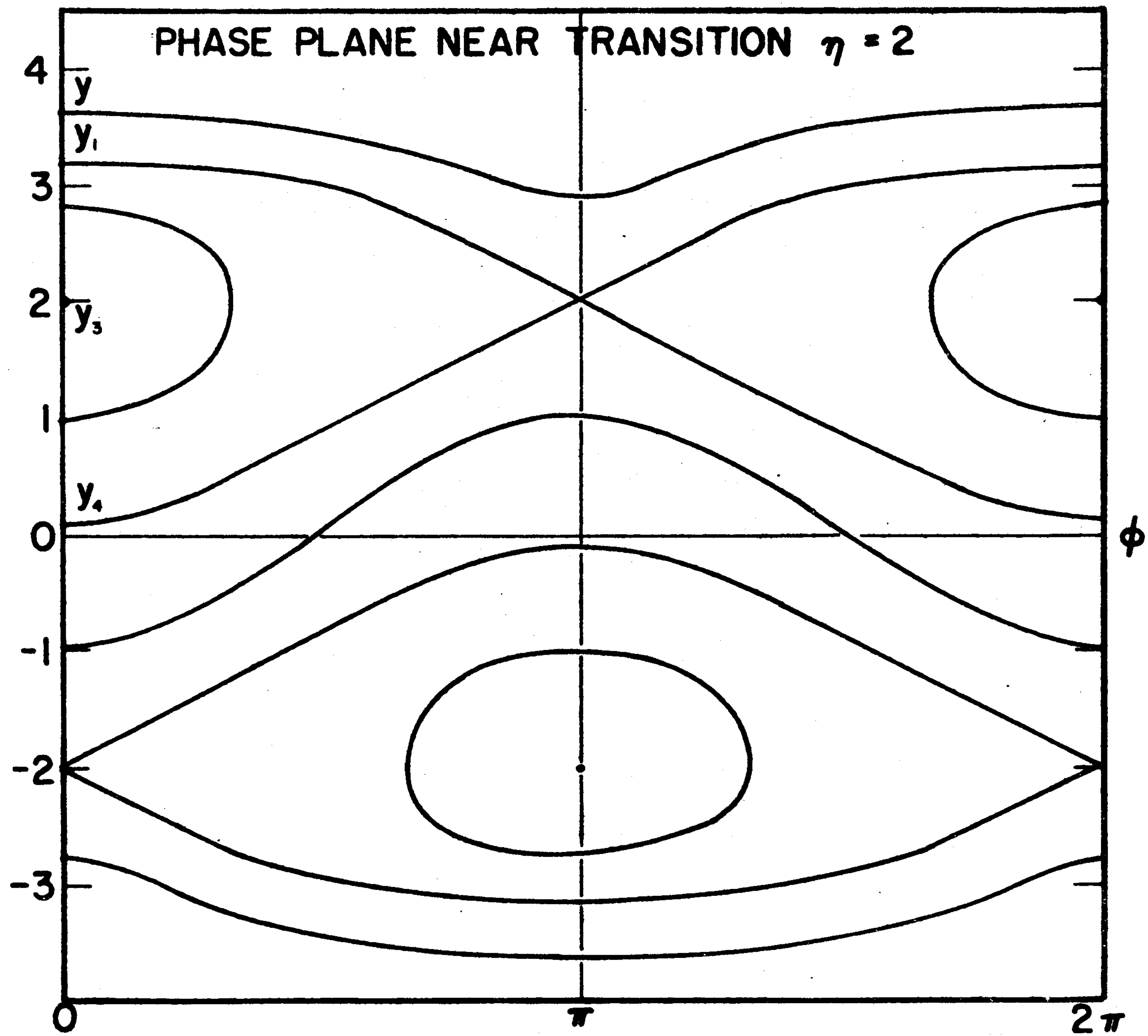

SUB-HARMONICS OF
ONE OSCILLATOR
1062
982
902
822
742
662
582
502
422
KINETIC ENERGY IN MEV
.2 π
.6 π
π
1.4 π
1.8 π
PHASE IN π

FIGURE 4 C



PHASE PLANE NEAR TRANSITION $\eta = 2$
4
3
2
1
0
-1
-2
-3
$\bar{y}$
y_1
y_3
y_4
ϕ
0
π
2π

FIGURE 5

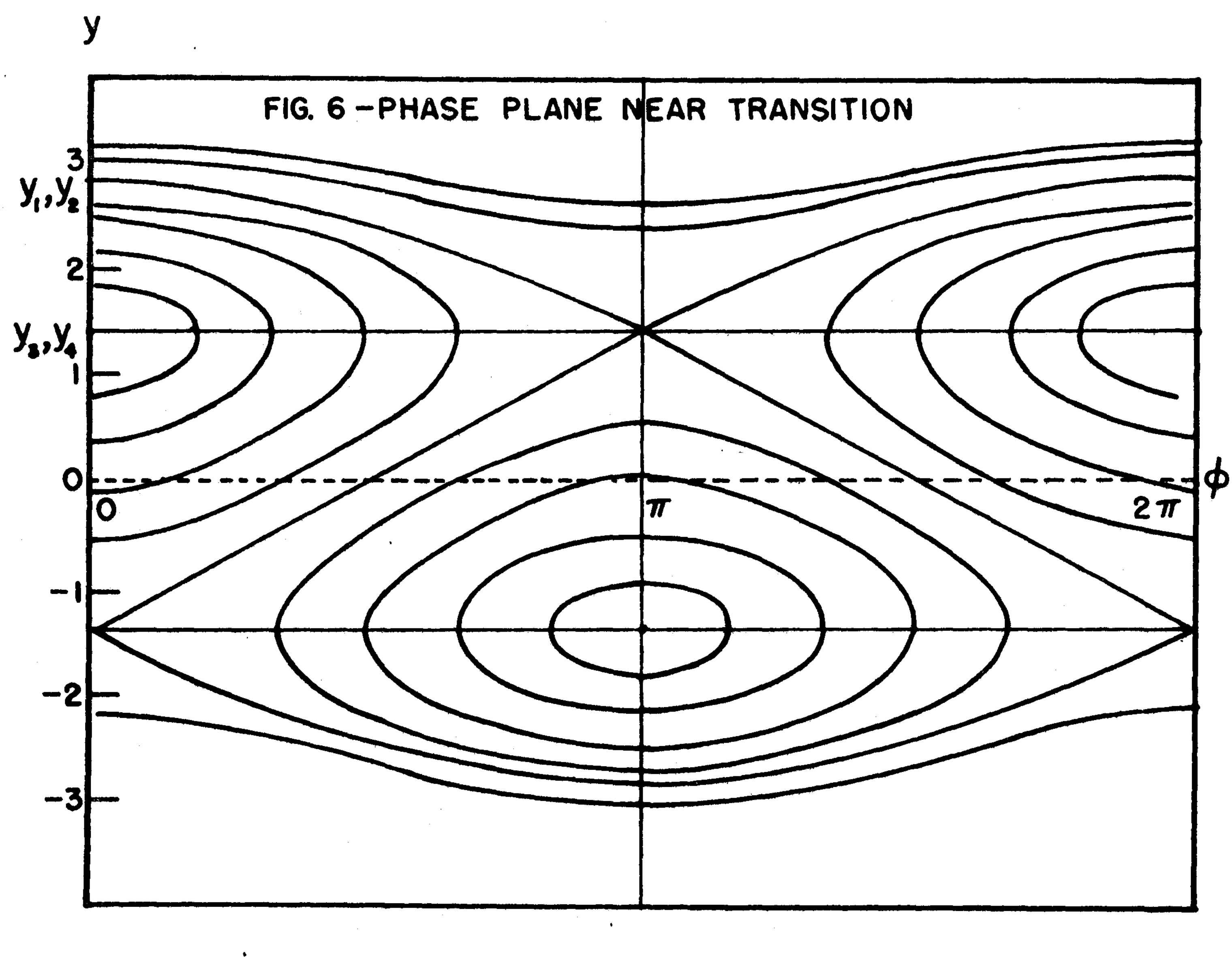


FIG. 6 - PHASE PLANE NEAR TRANSITION
y
3
y_1, y_2
2
y_3, y_4
1
0
-1
-2
-3
ϕ
0
π
2π

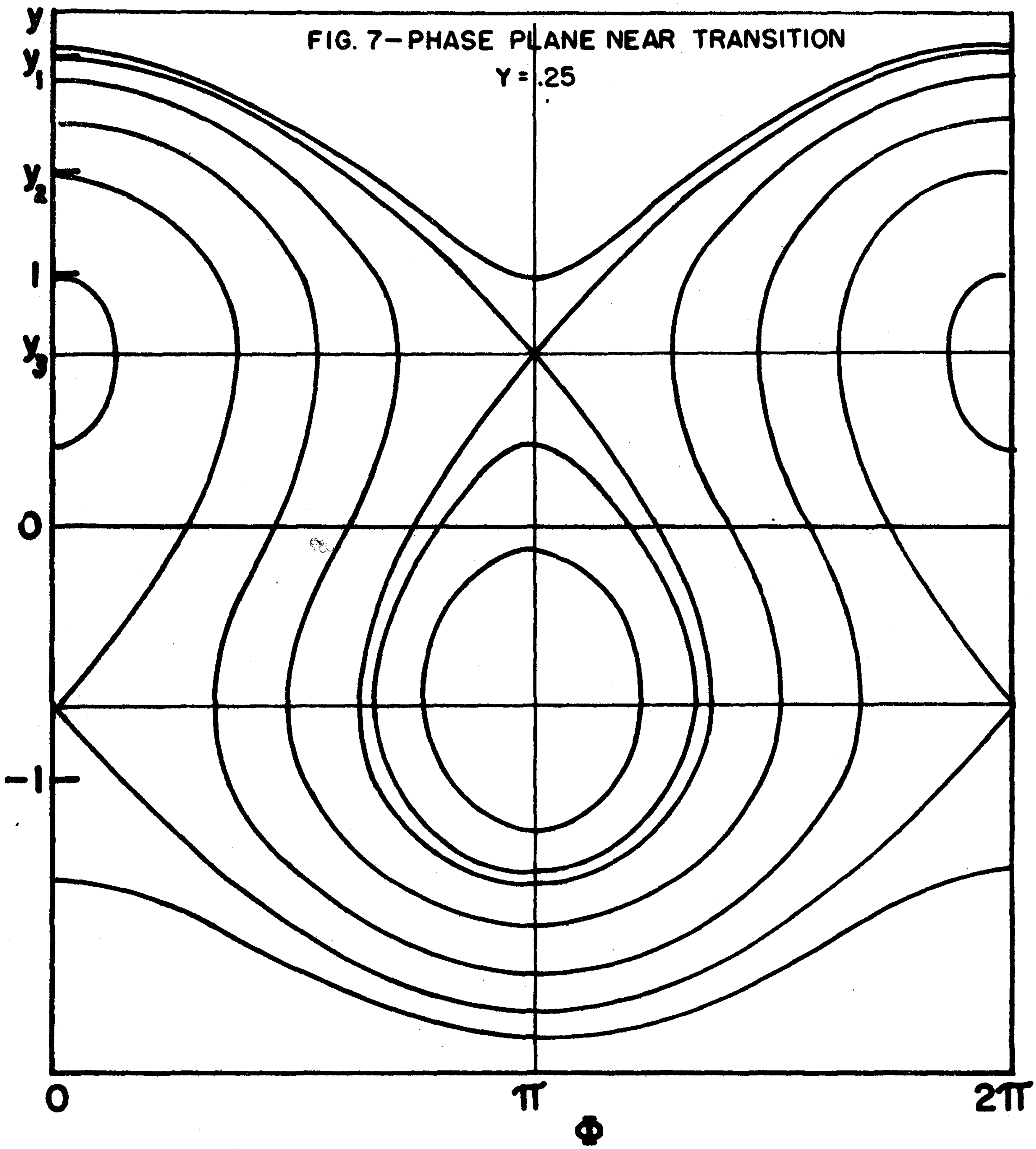


FIG. 7—PHASE PLANE NEAR TRANSITION
Y = .25
y
y1
y2
1
y3
0
-1
0
π
2π
Φ

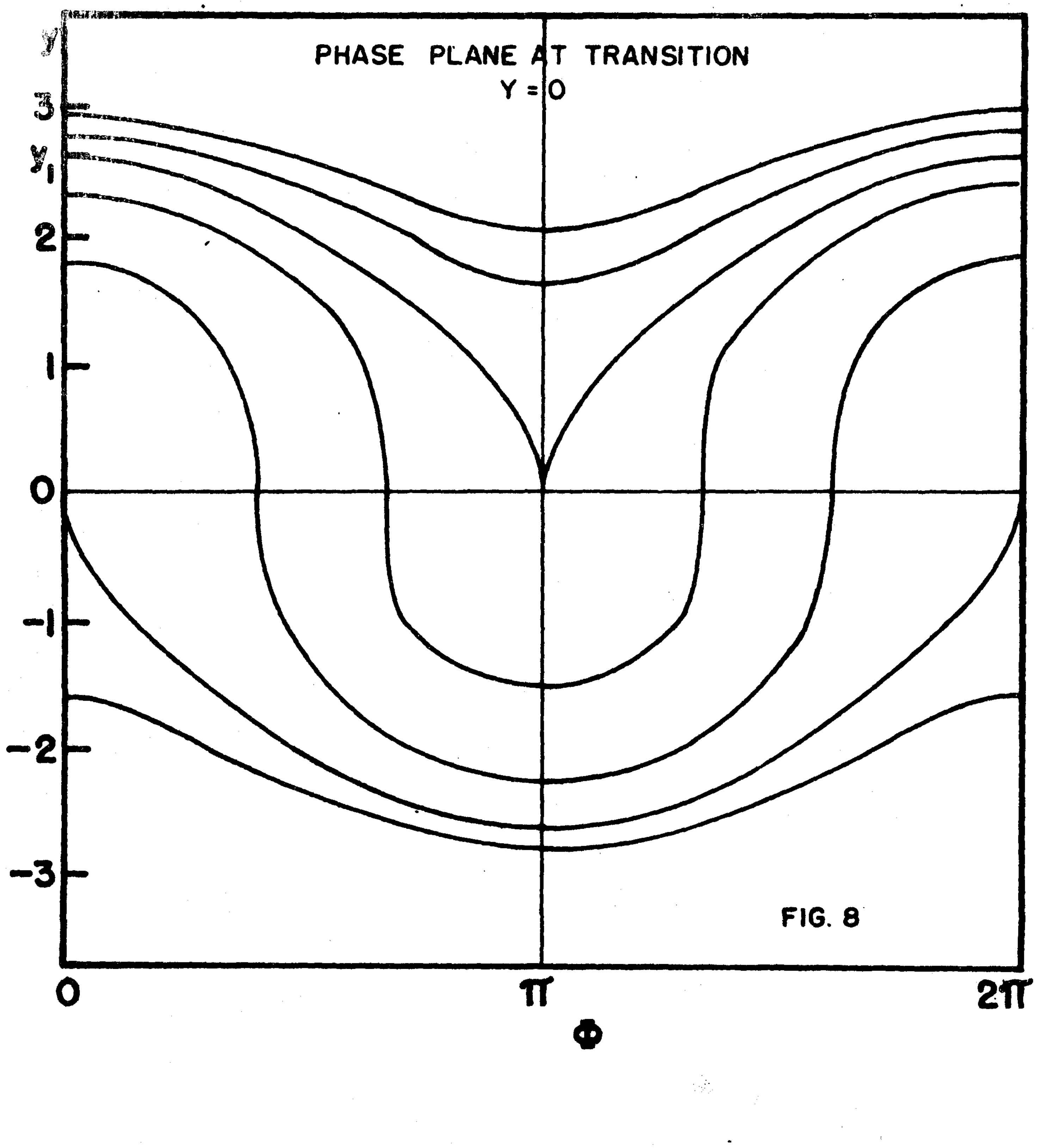
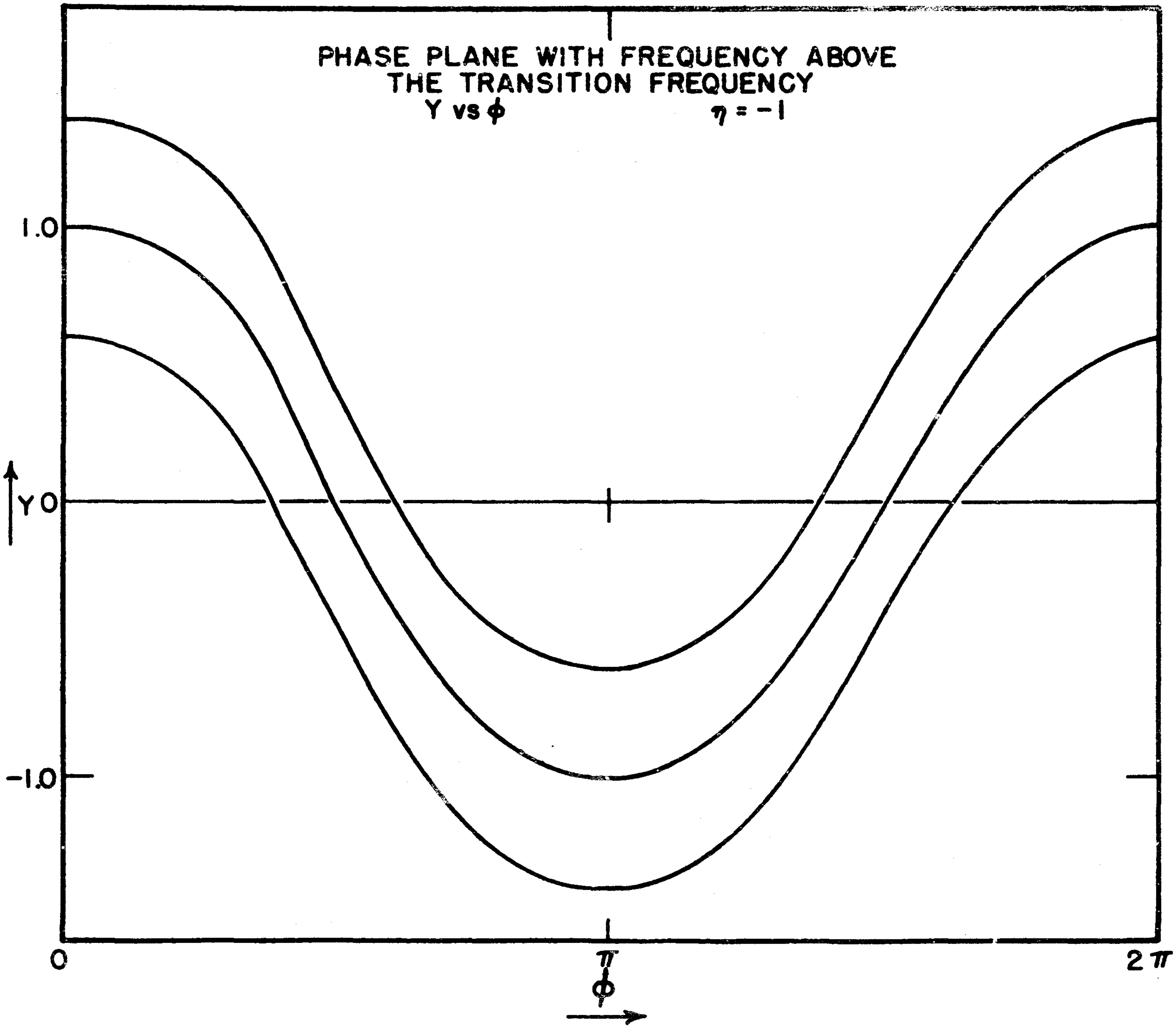


FIG. 8



PHASE PLANE WITH FREQUENCY ABOVE
THE TRANSITION FREQUENCY
Y vs φ
η = -1
1.0
Y 0
-1.0
0
π
2π
φ

FIGURE 9

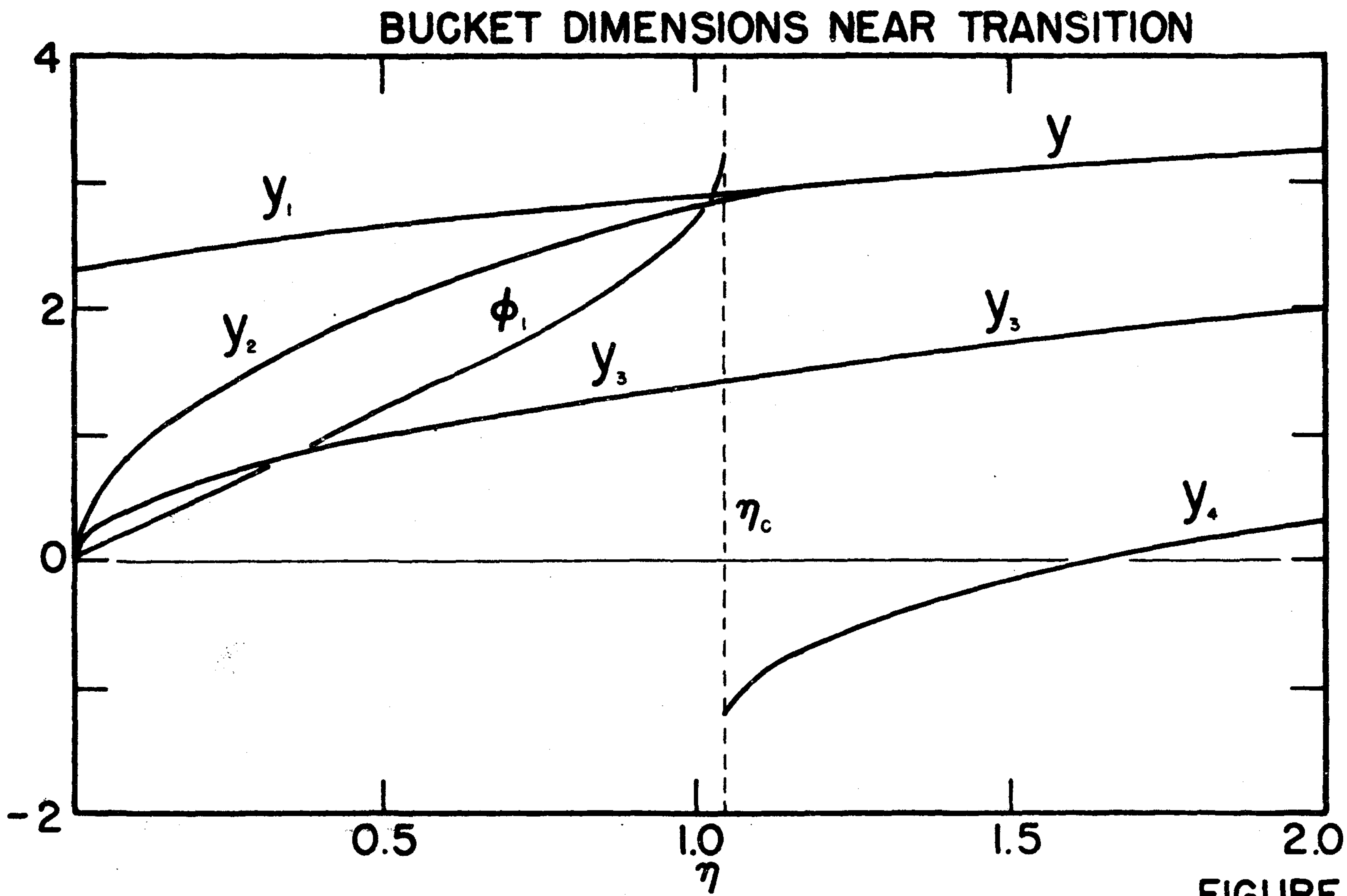
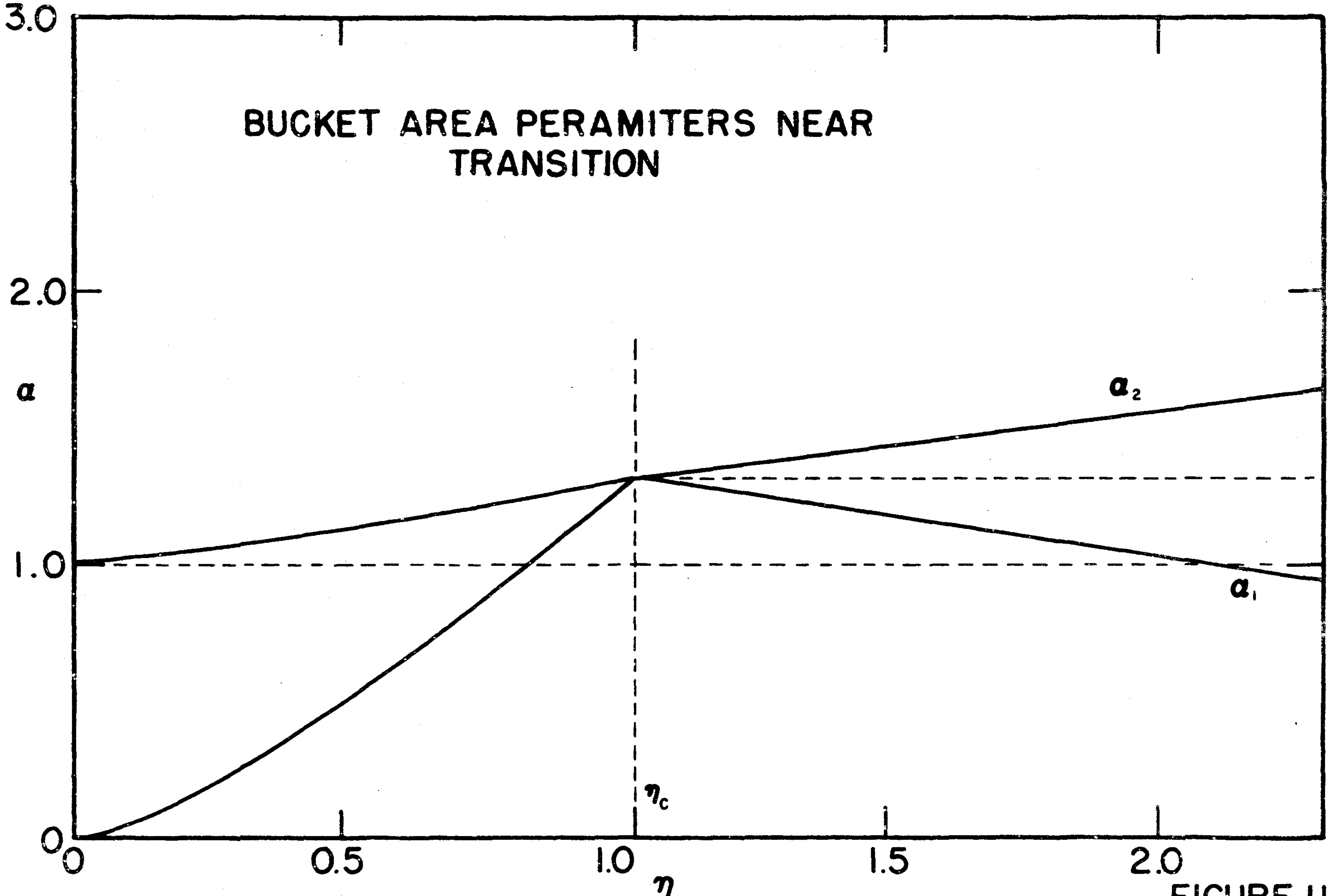

BUCKET DIMENSIONS NEAR TRANSITION
4
2
0
-2
0.5
1.0
1.5
2.0
η
y
y₁
y₂
φ₁
y₃
y₃
ηc
y₄

FIGURE 10



BUCKET AREA PERAMITERS NEAR TRANSITION
3.0
2.0
1.0
0
α
α₂
α₁
η_c
0
0.5
1.0
1.5
2.0
η

FIGURE 11

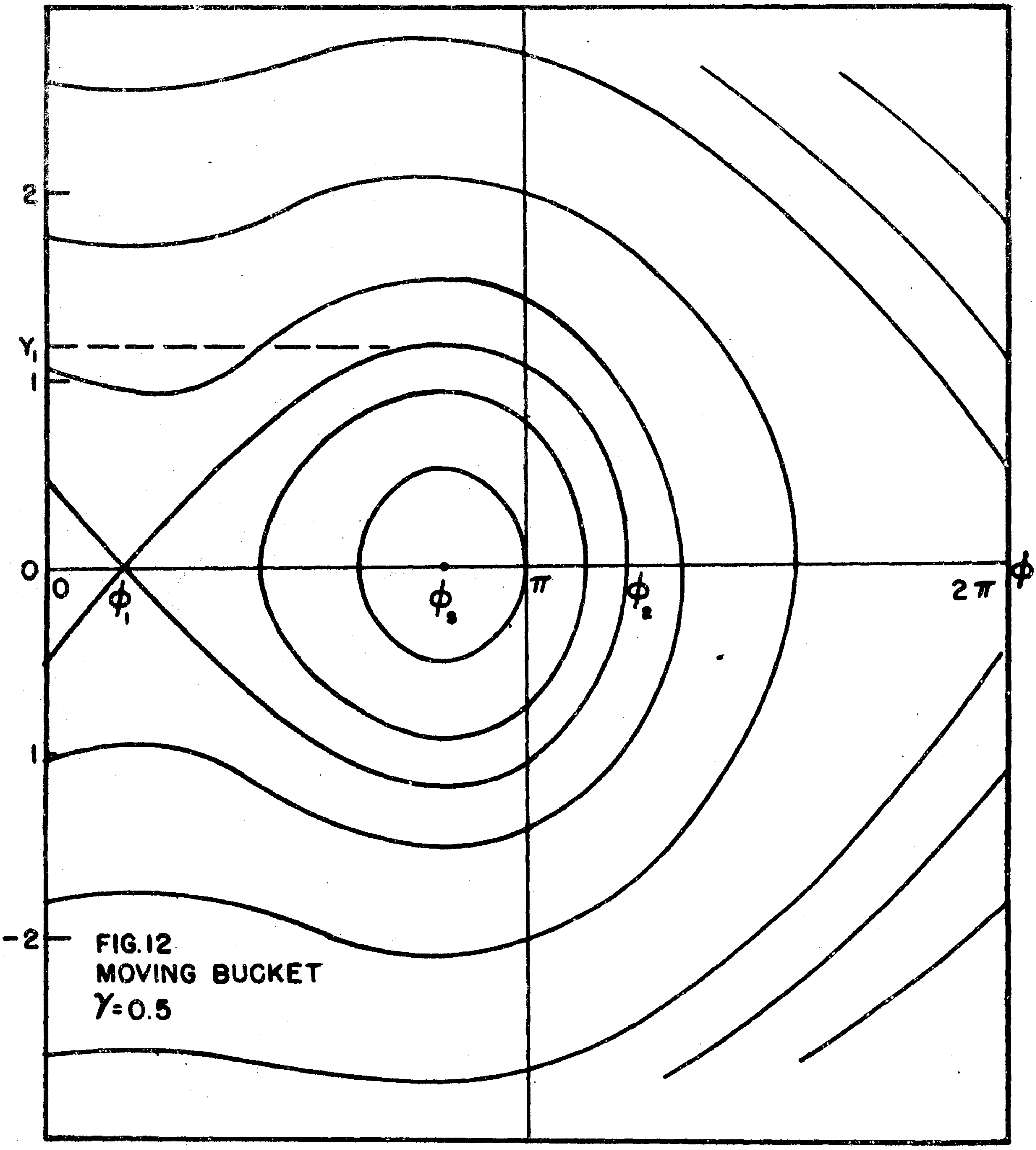


FIG. 12
MOVING BUCKET
$\gamma = 0.5$

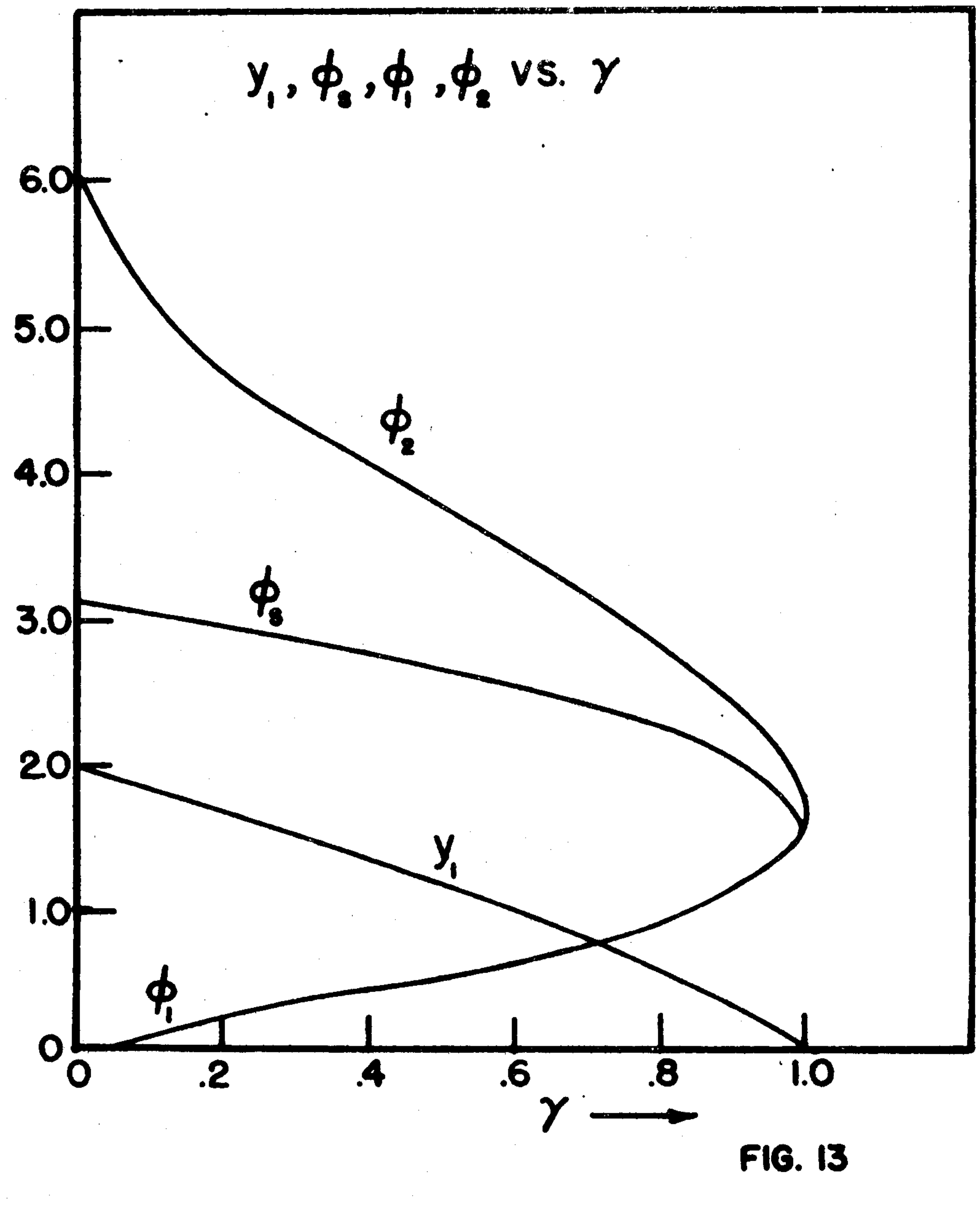
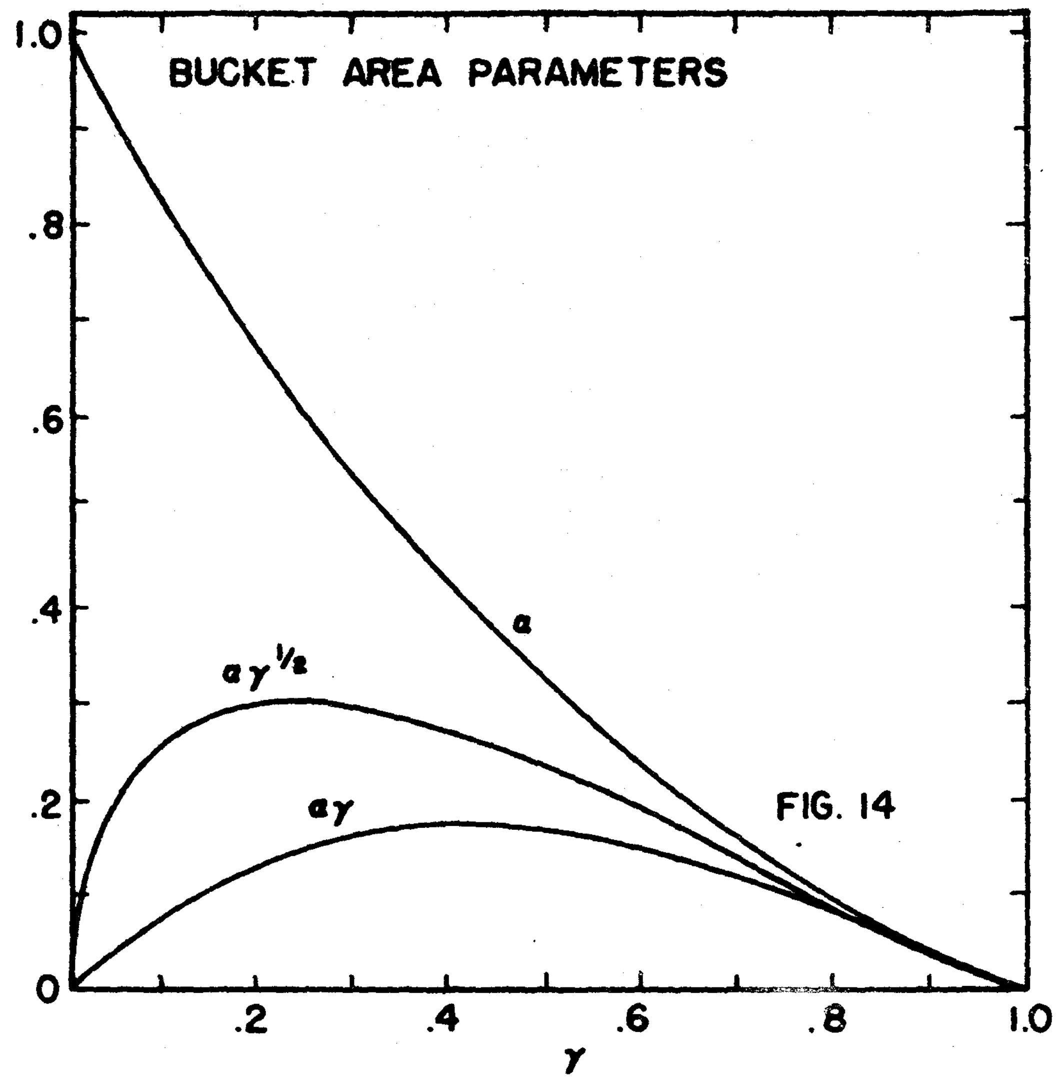
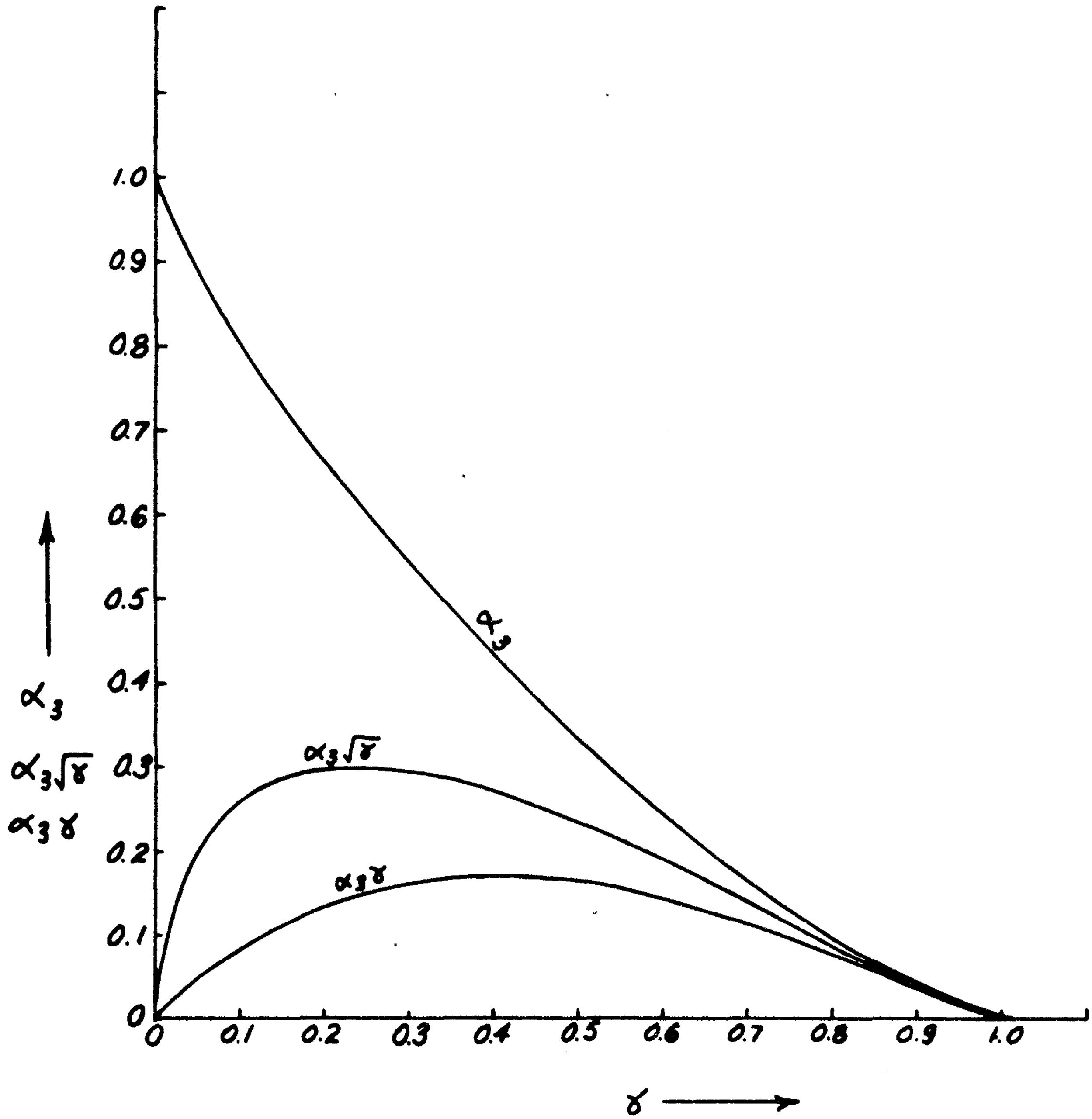


FIG. 13

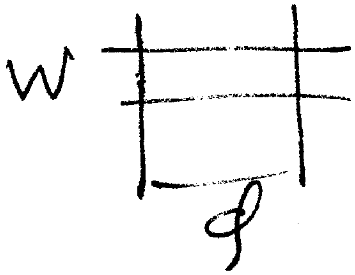

BUCKET AREA PARAMETERS
1.0
.8
.6
.4
.2
0
.2
.4
.6
.8
1.0
γ
a
aγ^1/2
aγ
FIG. 14

γ	$\mathcal{L}_3$	$\gamma \mathcal{L}_3$	$\sqrt{\gamma}$	$\sqrt{\gamma}\,\mathcal{L}_3$
0.00	1.000 00	0.000 00	0.000 00	0.000 00
0.02	0.948 43	0.018 97	0.141 42	0.134 13
0.04	0.907 54	0.036 30	0.200 00	0.181 51
0.06	0.870 68	0.052 24	0.244 95	0.213 28
0.08	0.836 54	0.066 92	0.282 84	0.236 61
0.10	0.803 32	0.080 39	0.316 23	0.254 22
0.12	0.773 19	0.092 78	0.346 41	0.267 84
0.14	0.743 60	0.104 10	0.374 17	0.278 23
0.16	0.715 18	0.114 43	0.400 00	0.286 07
0.18	0.687 88	0.123 82	0.424 26	0.291 84
0.20	0.661 22	0.132 24	0.447 21	0.295 70
0.22	0.635 22	0.139 75	0.469 04	0.297 94
0.24	0.610 34	0.146 48	0.489 90	0.299 01
0.26	0.585 85	0.152 32	0.509 90	0.298 72
0.28	0.562 13	0.157 40	0.529 15	0.297 45
0.30	0.538 85	0.161 66	0.547 72	0.295 14
0.32	0.516 25	0.165 20	0.565 69	0.292 03
0.34	0.494 04	0.167 97	0.583 10	0.288 07
0.36	0.472 46	0.170 09	0.600 00	0.283 48
0.38	0.451 19	0.171 45	0.616 44	0.278 13
0.40	0.430 56	0.172 22	0.632 46	0.272 31
0.42	0.410 17	0.172 27	0.648 07	0.265 82
0.44	0.490 45	0.171 80	0.663 33	0.259 00
0.46	0.370 86	0.170 60	0.678 23	0.251 53
0.48	0.351 99	0.168 96	0.692 82	0.243 87
0.50	0.333 20	0.166 60	0.707 11	0.235 61
0.52	0.315 19	0.163 87	0.721 11	0.227 25
0.54	0.297 55	0.160 46	0.734 85	0.218 36
0.56	0.279 86	0.156 72	0.748 33	0.209 43
0.58	0.262 67	0.152 35	0.761 58	0.200 04
0.60	0.246 11	0.147 67	0.774 60	0.190 64
0.62	0.229 77	0.142 42	0.787 40	0.180 87
0.64	0.213 76	0.136 81	0.800 00	0.171 01
0.66	0.198 07	0.130 73	0.812 40	0.160 91
0.68	0.182 81	0.124 31	0.824 62	0.150 75
0.70	0.167 75	0.117 43	0.836 66	0.140 35
0.72	0.153 44	0.110 48	0.848 53	0.130 20
0.74	0.139 26	0.103 05	0.860 23	0.119 80
0.76	0.125 46	0.095 35	0.871 78	0.109 37
0.78	0.112 08	0.087 42	0.883 18	0.098 99
0.80	0.099 12	0.079 30	0.894 43	0.088 66

γ	α_3	$\gamma\alpha_3$	$\sqrt{\gamma}$	$\sqrt{\gamma}\alpha_3$
0.82	0.086 45	0.070 89	0.905 54	0.078 28
0.84	0.074 38	0.062 48	0.916 52	0.068 17
0.86	0.062 73	0.053 95	0.927 36	0.058 17
0.88	0.051 50	0.045 32	0.938 08	0.048 31
0.90	0.040 84	0.036 76	0.948 68	0.038 74
0.92	0.030 83	0.028 36	0.959 17	0.029 57
0.94	0.021 42	0.020 13	0.969 54	0.020 77
0.96	0.012 83	0.012 32	0.979 80	0.012 57
0.98	0.005 38	0.005 27	0.989 95	0.005 33
1.00	0.000 00	0.000 00	1.000 00	0.000 00



1.0
0.9
0.8
0.7
0.6
0.5
0.4
0.3
0.2
0.1
0
α_3
$\alpha_3\sqrt{\gamma}$
$\alpha_3\gamma$
0
0.1
0.2
0.3
0.4
0.5
0.6
0.7
0.8
0.9
1.0
γ



$$W = \int \frac{dE}{f}$$

MURA $\begin{cases} 254 \\ 255 \\ 260 \end{cases}$