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Vacuum energy of the supersymmetric $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ in the $1/N$ expansion

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By employing the $1/N$ expansion, we compute the vacuum energy $E(\delta\epsilon)$ of the two-dimensional supersymmetric (SUSY) $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in a SUSY-breaking parameter $\delta\epsilon$. This quantity was vigorously studied recently by Fujimori et al. using a semi-classical approximation based on the bion, motivated by a possible semi-classical picture on the infrared renormalon. In our calculation, we find that the parameter $\delta\epsilon$ receives renormalization and, after this renormalization, the vacuum energy becomes ultraviolet finite. To the next-to-leading order of the $1/N$ expansion, we find that the vacuum energy normalized by the radius of the S^1, R, $RE(\delta\epsilon)$ behaves as inverse powers of ΛR for ΛR small, where Λ is the dynamical scale. Since Λ is related to the renormalized 't Hooft coupling λ_R as $\Lambda \sim e^{-2\pi/\lambda_R}$, to the order of the $1/N$ expansion we work out, the vacuum energy is a purely non-perturbative quantity and has no well-defined weak coupling expansion in λ_R.

Subject Index B06, B16, B32, B34, B35

1. Introduction

In this paper, by employing the $1/N$ expansion (for a classical exposition, see Ref. [1]), we compute the vacuum energy $E(\delta\epsilon)$ of the two-dimensional (2D) supersymmetric (SUSY) $\mathbb{C}P^{N-1}$ model [2–4] on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in a SUSY-breaking parameter $\delta\epsilon$. This quantity was vigorously studied recently by Fujimori et al. [5] (see also Refs. [6–8]) using a semi-classical approximation based on the bion [9–14]. One of the motivations for their study was a possible semi-classical picture on the infrared (IR) renormalon [15,16] advocated in Refs. [17–20]. In these works, in the context of the resurgence program (for a review, see Ref. [21] and the references cited therein), it is proposed that the ambiguity caused by the IR renormalon through the Borel resummation (for a review, see Ref. [22]) be cancelled by the ambiguity associated with the integration of quasi-collective coordinates of the bion; this scenario is quite analogous to the Bogomolny–Zinn-Justin mechanism for the instanton–anti-instanton pair [23,24].

In Ref. [5], by using the Lefschetz thimble method [25–27], the integration over quasi-collective coordinates of the bion is explicitly carried out and it was found that the vacuum energy $E(\delta\epsilon)$ possesses the imaginary ambiguity which is of the same order as that caused by the so-called $u = 1$ IR renormalon. On the other hand, for the four-dimensional $SU(N)$ gauge theory with the adjoint fermion (4D QCD(adj.)), for $N = 2$ and 3, it has been found [28] that when the spacetime is compactified as $\mathbb{R}^3 \times S^1$, the logarithmic behavior of the vacuum polarization of the gauge boson associated with the Cartan subalgebra ("photon") disappears. Since the IR renormalon is attributed

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to such a logarithmic behavior, in Ref. [28] it is concluded that the circle compactification generally eliminates the IR renormalon. This appears inconsistent with the renormalon interpretation of the result in Ref. [5].

The original motivation in a series of works [29–31] by a group including the present authors was to investigate the fate of the IR renormalon under the circle compactification to understand the above inconsistency.[1] For this, we employed the $1/N$ expansion (i.e. the large-N limit), in which

$$\Lambda R = \text{const. as } N \to \infty, \tag{1.1}$$

where Λ is a dynamical scale and R is the S^1 radius. We expected that in this way the IR renormalon and the bion can be highlighted, because the beta function of the 't Hooft coupling and the bion action remain non-trivial in the large-N limit, Eq. (1.1), whereas other sources to the Borel singularity such as the instanton–anti-instanton pair are suppressed. This intention was not so successful, because the calculations in Refs. [29–31] show that the behavior of the IR renormalon rather depends on the system; in the 2D SUSY $\mathbb{C}P^{N-1}$ model, the compactification from $\mathbb{R}^2$ to $\mathbb{R} \times S^1$ shifts the location of the Borel singularity associated with the IR renormalon [29,31]. In the 4D QCD(adj.), because of the twisted momentum of the gauge boson associated with the root vectors ("W boson"), $\mathbb{R}^3 \times S^1$ is effectively decompactified in the large-N limit [35–37] and the IR renormalon gives rise to the same Borel singularity as the uncompactified $\mathbb{R}^4$ [30].[2] It appears that a unified picture on the semi-classical understanding of the IR renormalon is still missing.

In the present paper, as announced in Ref. [29], in the $1/N$ expansion with Eq. (1.1), we compute the vacuum energy $E(\delta E)$ of the 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in a SUSY-breaking parameter $\delta\epsilon$; this is the quantity computed in Ref. [5] by the bion calculus. First, we find that the parameter $\delta\epsilon$ receives renormalization and, after this renormalization, the vacuum energy becomes ultraviolet (UV) finite. To the next-to-leading order of the $1/N$ expansion, we find that the vacuum energy is IR finite, as should be the case for a physical quantity. Finally, we find that the vacuum energy normalized by the radius of the S^1, $RE(\delta\epsilon)$ behaves as inverse powers of ΛR for ΛR small, as shown in Eqs. (3.51)–(3.56) and Figs. 2 and 3. Since Λ is related to the renormalized 't Hooft coupling λ_R as $\Lambda \sim e^{-2\pi/\lambda_R}$, to the order of the $1/N$ expansion we work out, the vacuum energy is a purely non-perturbative quantity and has *no well-defined weak coupling expansion in* λ_R. This implies that one cannot even define the perturbative expansion for this quantity computed in the $1/N$ expansion and cannot even discuss the renormalon problem.[3] Therefore, although our $1/N$ calculation is robust, it does not give any clue to the issue. We do not yet fully understand why the semi-classical calculation on the basis of the bion cannot be observed in the $1/N$ expansion. Nevertheless, we believe that it is worthwhile to report our $1/N$ calculation for future consideration because our calculation itself is rather non-trivial.

2. Two-dimensional SUSY $\mathbb{C}P^{N-1}$ model

2.1. *Action and boundary conditions*

Our spacetime is $\mathbb{R} \times S^1$, and $-\infty < x < \infty$ denotes the coordinate of $\mathbb{R}$ and $0 \leq y < 2\pi R$ the coordinate of S^1. The Euclidean action of the 2D SUSY $\mathbb{C}P^{N-1}$ model in terms of the homogeneous

[1] Recent related works are Refs. [32–34].

[2] In this analysis, we relied on the so-called large-β_0 approximation [38–40].

[3] In Appendix A, by taking a particular limit $R \to \infty$, we illustrate that the perturbative part of the vacuum energy contains IR divergences, although when including the non-perturbative part it becomes IR finite.

coordinate variables [2–4] is, in the notation of Eq. (2.24) of Ref. [29],

$$S = \int d^2x \, \frac{N}{\lambda}\big[-f + \bar{\sigma}\sigma + \bar{z}^A(-D_\mu D_\mu + f)z^A + \bar{\chi}^A(\not{D} + \bar{\sigma}P_+ + \sigma P_-)\chi^A + 2\bar{\chi}^A z^A\eta + 2\bar{\eta}\bar{z}^A\chi^A\big] - \int d^2x \, \frac{i\theta}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu. \tag{2.1}$$

Here, and in what follows, it is understood that repeated indices are summed over; the lower Greek indices, μ, ν, ..., take the value x or y and the uppercase Roman indices, A, B, ..., run from 1 to N. λ is the bare 't Hooft coupling and θ is the theta parameter.[4] Also,

$$D_\mu z^A \equiv (\partial_\mu + iA_\mu)z^A, \qquad \not{D}\chi^A \equiv \gamma_\mu(\partial_\mu + iA_\mu)\chi^A,$$
$$P_\pm \equiv \frac{1 \pm \gamma_5}{2}, \qquad \gamma_5 \equiv -i\gamma_x\gamma_y, \qquad \gamma_x \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \qquad \gamma_y \equiv \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{2.2}$$

and $\epsilon_{xy} = -\epsilon_{yx} = +1$.

For the fields with index A (we call them N-fields), we impose the $\mathbb{Z}_N$ twisted boundary conditions along S^1:

$$z^A(x, y + 2\pi R) = e^{2\pi i m_A R} z^A(x, y),$$
$$\chi^A(x, y + 2\pi R) = e^{2\pi i m_A R}\chi^A(x, y), \qquad \bar{\chi}^A(x, y + 2\pi R) = e^{-2\pi i m_A R}\bar{\chi}^A(x, y), \tag{2.3}$$

where the twist angle m_A in these expressions depends on the index A as

$$m_A \equiv \frac{A}{NR} \text{ for } A = 1, \ldots, N-1, \qquad m_N \equiv 0. \tag{2.4}$$

These twisted boundary conditions allow the fractional instanton/anti-instanton, the constituent of the bion.

For the auxiliary fields, f, σ, $\bar{\sigma}$, A_μ, η, and $\bar{\eta}$, on the other hand, we assume periodic boundary conditions along S^1.

For the calculation below, however, it turns out that an alternative form of the action, obtained by

$$f \to f + \bar{\sigma}\sigma \tag{2.5}$$

from Eq. (2.1), that is,

$$S = \int d^2x \, \frac{N}{\lambda}\big[-f + \bar{z}^A(-D_\mu D_\mu + f + \bar{\sigma}\sigma)z^A + \bar{\chi}^A(\not{D} + \bar{\sigma}P_+ + \sigma P_-)\chi^A + 2\bar{\chi}^A z^A\eta + 2\bar{\eta}\bar{z}^A\chi^A\big] - \int d^2x \, \frac{i\theta}{2\pi}\epsilon_{\mu\nu}\partial_\mu A_\nu, \tag{2.6}$$

is more convenient. This is because renormalization with the action in Eq. (2.1) requires an infinite shift of the field f in addition to the multiplicative renormalization of the 't Hooft coupling (Eq. (2.10)

[4] The theta parameter θ may be eliminated by the anomalous chiral rotation $\chi^A \to e^{i\alpha\gamma_5}\chi^A$, $\bar{\chi}^A \to \bar{\chi}^A e^{i\alpha\gamma_5}$, $\eta \to e^{-i\alpha\gamma_5}\eta$, $\bar{\eta} \to \bar{\eta}e^{-i\alpha\gamma_5}$, and $\sigma \to e^{2i\alpha}\sigma$.

below), whereas the action in Eq. (2.6) does not require such a shift. This difference comes from the fact that $\bar{\sigma}\sigma$ in Eq. (2.5) is a composite operator and UV divergent. In fact, the action in Eq. (2.6) can be obtained by the dimensional reduction of a manifestly SUSY-invariant non-linear sigma model in four dimensions [41]; we thus expect a simpler UV-divergent structure. For this reason, we adopt the action in Eq. (2.6) in the present paper.

2.2. *Saddle point and propagators in the leading order of the* $1/N$ *expansion*

Now, since the action of Eq. (2.1) (i.e. Eq. (2.24) of Ref. [29]) and the action of Eq. (2.6) are simply related by the change of variable in Eq. (2.5), we can borrow the results in Ref. [29] in the leading order of the $1/N$ expansion.[5]

First, setting

$$A_\mu = A_{\mu 0} + \delta A_\mu, \qquad f = f_0 + \delta f, \qquad \sigma = \sigma_0 + \delta\sigma, \tag{2.7}$$

where the subscript 0 indicates the value at the saddle point in the $1/N$ expansion and δ denotes the fluctuation, in the leading order of the $1/N$ expansion in Eq. (1.1) we have

$$A_{\mu 0} = A_{y0}\delta_{\mu y}, \qquad f_0 = 0, \qquad \bar{\sigma}_0\sigma_0 = \Lambda^2, \tag{2.8}$$

where Λ is the dynamical scale

$$\Lambda = \mu e^{-2\pi/\lambda_R} \tag{2.9}$$

defined from the renormalized 't Hooft coupling λ_R in the "$\overline{\text{MS}}$ scheme,"

$$\lambda = \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{\varepsilon} \lambda_R \left(1 + \frac{\lambda_R}{4\pi}\frac{1}{\varepsilon}\right)^{-1}. \tag{2.10}$$

Here, we have used dimensional regularization with the complex dimension $D = 2 - 2\varepsilon$; μ is the renormalization scale. In Eq. (2.8), the constant A_{y0} is not determined from the saddle point condition in the present supersymmetric theory and, for $\mathbb{Z}_N$-invariant quantities such as the partition function and the vacuum energy considered below, it should be integrated over with the measure [29]

$$\int_0^1 d(A_{y0}RN). \tag{2.11}$$

Next, we need the propagators among fluctuations of the auxiliary fields. To obtain these, we add the gauge-fixing term

$$S_{\text{gf}} = \frac{N}{4\pi}\int d^2x\, d^2x' \,\frac{1}{2}\partial_\mu \delta A_\mu(x)\partial_\nu \delta A_\nu(x') \int \frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y} e^{-ip(x-x')}\,\mathcal{L}(p) \tag{2.12}$$

and a local counter term

$$S_{\text{local}} \equiv \frac{N}{4\pi}\int d^2x \left(-\frac{1}{2}\right)[\delta\sigma(x) - \delta\bar{\sigma}(x)]^2 \tag{2.13}$$

[5] With the twisted boundary conditions of Eq. (2.3), as we will note in Sect. 3.1, the effective action arising from the Gaussian integration over N-fields is not simply proportional to N but depends nontrivially on N. Such a non-trivial dependence on N in the Gaussian determinant is, however, exponentially suppressed in the large-N limit of Eq. (1.1) [29] and can be neglected in calculations in the $1/N$ expansion.

to the action in Eq. (2.6) [29]. Then, in the leading order of the $1/N$ expansion, we have

$$\begin{aligned}
&\left\langle \delta A_\mu(x)\delta A_\nu(x')\right\rangle \\
&\quad = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}\left\{\delta_{\mu\nu}+4\left[\Lambda^2+\frac{\bar{p}_y^2}{p^2}\frac{\mathcal{K}(p)^2}{\mathcal{L}(p)^2}\right]\frac{p_\mu p_\nu}{(p^2)^2}\right\},\\
&\left\langle \delta A_\mu(x)\delta R(x')\right\rangle = \left\langle \delta R(x)\delta A_\mu(x')\right\rangle = 0,\\
&\left\langle \delta A_\mu(x)\delta I(x')\right\rangle = -\left\langle \delta I(x)\delta A_\mu(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}\frac{2\Lambda^2\bar{p}_\mu}{p^2},\\
&\left\langle \delta A_\mu(x)\delta f(x')\right\rangle = \left\langle \delta f(x)\delta A_\mu(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{K}(p)}{\mathcal{D}(p)}\frac{-2\bar{p}_\mu\bar{p}_y}{p^2},\\
&\left\langle \delta R(x)\delta R(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}\Lambda^2,\\
&\left\langle \delta R(x)\delta I(x')\right\rangle = -\left\langle \delta I(x)\delta R(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{K}(p)}{\mathcal{D}(p)}\frac{-2\Lambda^2\bar{p}_y}{p^2},\\
&\left\langle \delta R(x)\delta f(x')\right\rangle = \left\langle \delta f(x)\delta R(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}(-2\Lambda^2),\\
&\left\langle \delta I(x)\delta I(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}\Lambda^2,\\
&\left\langle \delta I(x)\delta f(x')\right\rangle = -\left\langle \delta f(x)\delta I(x')\right\rangle = 0,\\
&\left\langle \delta f(x)\delta f(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{\mathcal{L}(p)}{\mathcal{D}(p)}(-p^2),\\
&\left\langle \eta(x)\bar{\eta}(x')\right\rangle \\
&\quad = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{(i\not{p}+2\bar{\sigma}_0P_+ +2\sigma_0P_-)\mathcal{L}(p)+2i\not{\bar{p}}\bar{p}_y/p^2\mathcal{K}(p)}{\mathcal{D}(p)}\left(-\frac{1}{2}\right),
\end{aligned}\tag{2.14}$$

where the Kaluza–Klein (KK) momentum along S^1, p_y, takes discrete values $p_y = n/R$ with $n \in \mathbb{Z}$. We have also introduced the notations

$$\bar{p}_\mu \equiv \epsilon_{\nu\mu}p_\nu \tag{2.15}$$

and

$$\delta R(x) \equiv \frac{1}{2}\left[\bar{\sigma}_0\delta\sigma(x)+\sigma_0\delta\bar{\sigma}(x)\right], \qquad \delta I(x) \equiv \frac{1}{2i}\left[\bar{\sigma}_0\delta\sigma(x)-\sigma_0\delta\bar{\sigma}(x)\right]. \tag{2.16}$$

From the above results, we also have

$$\left\langle \delta\sigma(x)\delta\bar{\sigma}(x')\right\rangle = \frac{4\pi}{N}\int\frac{dp_x}{2\pi}\,\frac{1}{2\pi R}\sum_{p_y} e^{ip(x-x')}\,\frac{1}{\mathcal{D}(p)}\left[2\mathcal{L}(p)+4i\frac{\bar{p}_y}{p^2}\mathcal{K}(p)\right]. \tag{2.17}$$

Various functions used in the above expressions are defined by

$$
\begin{aligned}
\mathcal{L}(p) &\equiv \mathcal{L}_\infty(p) + \hat{\mathcal{L}}(p), \\
\mathcal{L}_\infty(p) &\equiv \frac{2}{\sqrt{p^2(p^2+4\Lambda^2)}} \ln\left(\frac{\sqrt{p^2+4\Lambda^2}+\sqrt{p^2}}{\sqrt{p^2+4\Lambda^2}-\sqrt{p^2}}\right), \\
\hat{\mathcal{L}}(p) &\equiv \int_0^1 dx \sum_{m\neq 0} e^{-iA_{y0}2\pi RNm} e^{ixp_y 2\pi RNm} \\
&\qquad \times \frac{2\pi RN|m|}{\sqrt{\Lambda^2+x(1-x)p^2}} K_1(\sqrt{\Lambda^2+x(1-x)p^2}2\pi RN|m|), \\
\mathcal{K}(p) &\equiv i\int_0^1 dx \sum_{m\neq 0} e^{-iA_{y0}2\pi RNm} e^{ixp_y 2\pi RNm} 2\pi RNm K_0(\sqrt{\Lambda^2+x(1-x)p^2}2\pi RN|m|), \\
\mathcal{D}(p) &\equiv (p^2+4\Lambda^2)\mathcal{L}(p)^2 + 4\frac{\bar{p}_y^2}{p^2}\mathcal{K}(p)^2,
\end{aligned}
\tag{2.18}
$$

where $K_\nu(z)$ denotes the modified Bessel function of the second kind. For later calculations, it is important to note the properties

$$
\mathcal{L}(p) = \mathcal{L}(-p), \qquad \mathcal{K}(p) = \mathcal{K}(-p). \tag{2.19}
$$

These can be shown by the change of the Feynman parameter, $x \to 1-x$, noting that $p_y \in \mathbb{Z}/R$.

Going back to the action S in Eq. (2.6), with the saddle point values in Eq. (2.8), the propagators of the N-fields in the leading order of the $1/N$ expansion are given by

$$
\begin{aligned}
\left\langle z^A(x)\bar{z}^B(x')\right\rangle &= \delta^{AB}\frac{\lambda}{N}\int \frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y} e^{ip_x(x-x')} e^{i(p_y+m_A)(y-y')} \\
&\qquad \times \left[p_x^2 + (p_y+A_{y0}+m_A)^2+\Lambda^2\right]^{-1}, \\
\left\langle \chi^A(x)\bar{\chi}^B(x')\right\rangle &= \delta^{AB}\frac{\lambda}{N}\int \frac{dp_x}{2\pi}\frac{1}{2\pi R}\sum_{p_y} e^{ip_x(x-x')} e^{i(p_y+m_A)(y-y')} \\
&\qquad \times \left[i\gamma_x p_x + i\gamma_y(p_y+A_{y0}+m_A) + \bar{\sigma}_0 P_+ + \sigma_0 P_-\right]^{-1}.
\end{aligned}
\tag{2.20}
$$

To obtain these, we noted the twisted boundary conditions of Eq. (2.3).

3. Computation of the vacuum energy

3.1. General strategy

Our objective in this paper is to compute the vacuum energy of the present system as a power series of the coefficient $\delta\epsilon$ of a SUSY-breaking term—the quantity computed in Ref. [5]:

$$
E(\delta\epsilon) = E^{(0)} + E^{(1)}\delta\epsilon + E^{(2)}\delta\epsilon^2 + \cdots. \tag{3.1}
$$

Here, the supersymmetry-breaking term introduced in Ref. [5] is

$$
\delta S \equiv \int d^2x\, \frac{\delta\epsilon}{\pi R}\sum_{A=1}^{N} m_A\left(\bar{z}^A z^A - \frac{1}{N}\right). \tag{3.2}
$$

Note that this depends on the twist angles in Eq. (2.4). A quick way to incorporate the effect of Eq. (3.2) is to regard δS as a mass term of the z^A-field, as

$$S + \delta S = \int d^2x \, \frac{N}{\lambda} \bar{z}^A \left(-\partial_\mu \partial_\mu + \Lambda^2 + \delta_A\right) z^A + \cdots , \tag{3.3}$$

where

$$\delta_A \equiv \frac{\lambda \delta\epsilon}{\pi R N} m_A. \tag{3.4}$$

With this modification, the vacuum energy is given by

$$-\int dx \, E(\delta\epsilon) = \int d^2x \, \frac{1}{\lambda} \sum_A \delta_A - \sum_A \ln \mathrm{Det}(-\partial_\mu \partial_\mu + \Lambda^2 + \delta_A) + (\text{connected vacuum bubble diagrams}). \tag{3.5}$$

Here, the vacuum bubble diagrams, which start from two-loop order, are computed by using the modified z^A-propagator

$$\left\langle z^A(x) \bar{z}^B(x') \right\rangle = \delta^{AB} \frac{\lambda}{N} \int \frac{dp_x}{2\pi} \, \frac{1}{2\pi R} \sum_{p_y} e^{ip_x(x-x')} e^{i(p_y+m_A)(y-y')} \left[p_x^2 + (p_y + A_{y0} + m_A)^2 + \Lambda^2 + \delta_A\right]^{-1} \tag{3.6}$$

instead of the one in Eq. (2.20). Then, by expanding Eq. (3.5) with respect to δ_A, we have the series expansion in Eq. (3.1). In the following calculations, we set $E^{(0)} = 0$ assuming that the bare vacuum energy at $\delta\epsilon = 0$ is chosen so that the system is supersymmetric for $\delta\epsilon = 0$. This amounts to computing the difference $E(\delta\epsilon) - E(\delta\epsilon = 0)$.

If all the N-fields obey the same boundary conditions along S^1, all z^A (or χ^A and $\bar{\chi}^A$) contribute equally and the order of the loop expansion with the use of the auxiliary fields and the order of the $1/N$ expansion would coincide [1]. With the twisted boundary conditions in Eq. (2.3), however, not all N-fields contribute equally. The SUSY-breaking term in Eq. (3.2) also treats each of N-fields differently. For these reasons, in the present system the order of the loop expansion and that of the $1/N$ expansion do not necessarily coincide; we have to distinguish both expansions. For instance, although the one-loop Gaussian determinant in Eq. (3.5) gives rise to the contribution of $O(1/N)$, it also contains terms of subleading orders, $O(1/N^2)$ and $O(1/N^3)$ (see Eq. (3.48), for instance).

3.2. *One-loop Gaussian determinant*

Let us start with the one-loop Gaussian determinant in Eq. (3.5). We first note that

$$\begin{aligned}
&-\sum_A \ln \mathrm{Det}(-\partial_\mu \partial_\mu + \Lambda^2 + \delta_A) \\
&= -\sum_A \int d^2x \int \frac{dp_x}{2\pi} \, \frac{1}{2\pi R} \sum_{p_y} \ln \left[p_x^2 + (p_y + m_A + A_{y0})^2 + \Lambda^2 + \delta_A\right] \\
&= -\int d^2x \sum_A \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} \, e^{i(p_y - m_A - A_{y0})2\pi R n} \ln(p^2 + \Lambda^2 + \delta_A),
\end{aligned} \tag{3.7}$$

where we have used the identity

$$\frac{1}{2\pi R}\sum_{n=-\infty}^{\infty} F(n/R) = \sum_{n=-\infty}^{\infty}\int \frac{dp_y}{2\pi}\, e^{ip_y 2\pi Rn} F(p_y). \tag{3.8}$$

Hence, subtracting the logarithm of the Gaussian determinant at $\delta\epsilon = 0$, we have

$$\begin{aligned}
&-\sum_A \ln \mathrm{Det}\left(\frac{-\partial_\mu\partial_\mu + \Lambda^2 + \delta_A}{-\partial_\mu\partial_\mu + \Lambda^2}\right)\\
&= -\int d^2x \sum_A \sum_{n=-\infty}^{\infty}\int \frac{d^2p}{(2\pi)^2}\, e^{i(p_y - m_A - A_{y0})2\pi Rn} \ln\left(\frac{p^2+\Lambda^2+\delta_A}{p^2+\Lambda^2}\right).
\end{aligned} \tag{3.9}$$

In this expression, since the $n \neq 0$ terms are Fourier transforms, only the $n = 0$ term is UV divergent. Under the dimensional regularization with $D = 2 - 2\varepsilon$, the momentum integration yields

$$\begin{aligned}
&-\sum_A \ln \mathrm{Det}\left(\frac{-\partial_\mu\partial_\mu + \Lambda^2 + \delta_A}{-\partial_\mu\partial_\mu + \Lambda^2}\right)\\
&= -\int d^2x \frac{1}{4\pi}\left[\frac{1}{\varepsilon} - \ln\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)\right]\sum_A \delta_A\\
&\quad - \int d^2x \sum_A \frac{1}{4\pi}\left[\delta_A - (\Lambda^2+\delta_A)\ln\left(1+\frac{\delta_A}{\Lambda^2}\right)\right]\\
&\quad - \int d^2x \sum_A\sum_{n\neq 0} e^{-i(m_A + A_{y0})2\pi Rn}\\
&\qquad \times \frac{1}{4\pi}(-4)\frac{1}{2\pi R|n|}\left[\sqrt{\Lambda^2+\delta_A}K_1(\sqrt{\Lambda^2+\delta_A}2\pi R|n|) - \Lambda K_1(\Lambda 2\pi R|n|)\right].
\end{aligned} \tag{3.10}$$

Since Eqs. (2.9) and (2.10) imply that

$$\frac{1}{4\pi}\left[\frac{1}{\varepsilon} - \ln\left(\frac{e^{\gamma_E}\Lambda^2}{4\pi}\right)\right] = \frac{1}{\lambda}, \tag{3.11}$$

we see that the first term on the right-hand side of Eq. (3.10) is precisely canceled by the first term on the right-hand side of Eq. (3.5).

In this way, from Eq. (3.5) we have

$$\begin{aligned}
&E(\delta\epsilon)|_{\text{1-loop}}\\
&= 2\pi R \sum_A \frac{1}{4\pi}\left[\delta_A - (\Lambda^2+\delta_A)\ln\left(1+\frac{\delta_A}{\Lambda^2}\right)\right]\\
&\quad + 2\pi R \sum_A\sum_{n\neq 0} e^{-i(m_A + A_{y0})2\pi Rn}\\
&\qquad \times \frac{1}{4\pi}(-4)\frac{1}{2\pi R|n|}\left[\sqrt{\Lambda^2+\delta_A}K_1(\sqrt{\Lambda^2+\delta_A}2\pi R|n|) - \Lambda K_1(\Lambda 2\pi R|n|)\right].
\end{aligned} \tag{3.12}$$

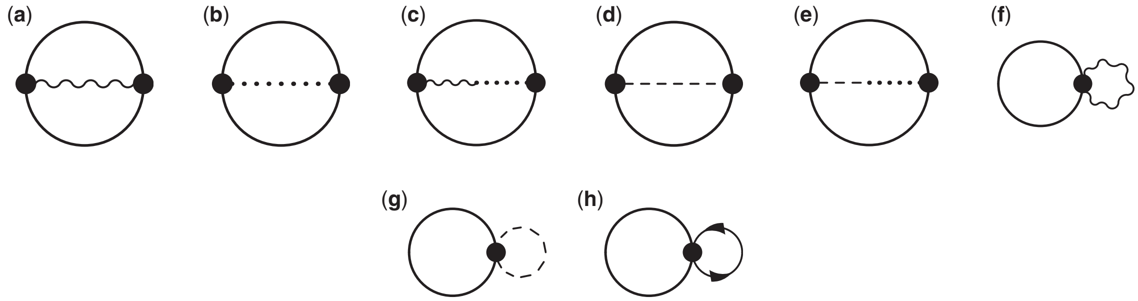

Fig. 1. Two-loop vacuum bubble diagrams that contribute to $E(\delta\epsilon)|_{\text{2-loop}}$ in Eq. (3.13). The solid line denotes the z^A-propagator of Eq. (3.6). The wavy line denotes the δA_μ-propagator, the dotted line the δf-propagator, the broken line the $\delta\sigma$-propagator, and the arrowed solid line the η-propagator in Eqs. (2.17) and (2.14).

3.3. *Two-loop vacuum bubble diagrams*

Next, we work out the vacuum bubble diagrams in the two-loop level; they are depicted in Fig. 1. By using the propagators in Eqs. (2.14), (2.17), (2.20), and (3.6), and interaction vertices in Eq. (2.6), from Eq. (3.5) we have

$$
\begin{aligned}
&E(\delta\epsilon)|_{\text{2-loop}} \\
&= -2\pi R\frac{4\pi}{N}\sum_A\sum_{n=-\infty}^{\infty}\int\frac{d^2p}{(2\pi)^2}\,e^{i(p_y-A_{y0}-m_A)2\pi Rn}\frac{1}{p^2+\Lambda^2+\delta_A} \\
&\quad\times\Bigg[\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\frac{1}{(p-\ell)^2+\Lambda^2+\delta_A} \\
&\qquad\times\Bigg(\frac{1}{2}(2p_\mu-\ell_\mu)(2p_\nu-\ell_\nu)\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\left\{\delta_{\mu\nu}+4\left[\Lambda^2+\frac{\bar{\ell}_y^2}{\ell^2}\frac{\mathcal{K}(\ell)^2}{\mathcal{L}(\ell)^2}\right]\frac{\ell_\mu\ell_\nu}{(\ell^2)^2}\right\} && \text{(Fig. 1a)} \\
&\qquad\quad-\frac{1}{2}\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\ell^2 && \text{(Fig. 1b)} \\
&\qquad\quad-4p_\mu\frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)}\frac{\bar{\ell}_\mu\bar{\ell}_y}{\ell^2} && \text{(Fig. 1c)} \\
&\qquad\quad+2\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\Lambda^2 && \text{(Fig. 1d)} \\
&\qquad\quad-4\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\Lambda^2\Bigg) && \text{(Fig. 1e)} \\
&\qquad+\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y} \\
&\qquad\quad\times\left(-\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\left\{2+4\left[\Lambda^2+\frac{\bar{\ell}_y^2}{\ell^2}\frac{\mathcal{K}(\ell)^2}{\mathcal{L}(\ell)^2}\right]\frac{1}{\ell^2}\right\}\right. && \text{(Fig. 1f)} \\
&\qquad\qquad-2\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} && \text{(Fig. 1g)}
\end{aligned}
$$

$$
+ \frac{1}{(p-\ell)^2+\Lambda^2} 4 \left\{ [-(p-\ell)\cdot\ell + 2\Lambda^2] \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} + 2p_\mu \frac{\bar{\ell}_\mu \bar{\ell}_y}{\ell^2} \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \right\} \Bigg) \Bigg] \qquad \text{(Fig. 1h)}
$$

$$
- (\text{terms with } \delta\epsilon = 0), \tag{3.13}
$$

where the contributions of each diagram in Fig. 1 are separately indicated by the equation numbers. The total sum is

$$
\begin{aligned}
&E(\delta\epsilon)|_{\text{2-loop}} \\
&= -2\pi R \frac{4\pi}{N} \sum_A \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} e^{i(p_y - A_{y0} - m_A)2\pi R n} \frac{1}{p^2+\Lambda^2+\delta_A} \\
&\times \Bigg(\int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \frac{1}{(p-\ell)^2+\Lambda^2+\delta_A} \\
&\quad \times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[2p^2 - 2p\cdot\ell - 8\Lambda^2 \frac{p\cdot\ell}{\ell^2} + 8\Lambda^2 \frac{(p\cdot\ell)^2}{(\ell^2)^2} \right] \right. \\
&\quad\quad \left. + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\bar{\ell}_y^2}{\ell^2} \left[2 - 8\frac{p\cdot\ell}{\ell^2} + 8\frac{(p\cdot\ell)^2}{(\ell^2)^2} \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} (-4) \frac{p\cdot\bar{\ell}\,\bar{\ell}_y}{\ell^2} \right\} \\
&\quad + \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \frac{1}{(p-\ell)^2+\Lambda^2} \\
&\quad\quad \times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[-4p^2 + 4p\cdot\ell + 8\Lambda^2 \frac{p\cdot\ell}{\ell^2} - 4\Lambda^2 \frac{p^2}{\ell^2} - 4\Lambda^4 \frac{1}{\ell^2} \right] \right. \\
&\quad\quad\quad \left. + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\bar{\ell}_y^2}{\ell^2} \left[-4 + 8\frac{p\cdot\ell}{\ell^2} - 4\frac{p^2}{\ell^2} - 4\Lambda^2 \frac{1}{\ell^2} \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} (8) \frac{p\cdot\bar{\ell}\,\bar{\ell}_y}{\ell^2} \right\} \Bigg) \\
&- (\text{term with } \delta\epsilon = 0).
\end{aligned} \tag{3.14}
$$

To examine the renormalizability of this expression, we first note that this can be written as

$$
\begin{aligned}
&E(\delta\epsilon)|_{\text{2-loop}} \\
&= -2\pi R \frac{4\pi}{N} \sum_A \left\{ \left(e^{\delta_A \partial_\xi} e^{\delta_A \partial_\eta} - 1 \right) I(\xi,\eta) + \left(e^{\delta_A \partial_\xi} - 1 \right) [-2I(\xi,0) + J(\xi)] \right\} \Big|_{\xi=\eta=0},
\end{aligned} \tag{3.15}
$$

where

$$
\begin{aligned}
I(\xi,\eta) \equiv& \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \sum_{n=-\infty}^{\infty} \int \frac{d^2p}{(2\pi)^2} e^{i(p_y - A_{y0} - m_A)2\pi R n} \frac{1}{p^2+\Lambda^2+\xi} \frac{1}{(p-\ell)^2+\Lambda^2+\eta} \\
&\times \left\{ \frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \left[2p^2 - 2p\cdot\ell - 8\Lambda^2 \frac{p\cdot\ell}{\ell^2} + 8\Lambda^2 \frac{(p\cdot\ell)^2}{(\ell^2)^2} \right] \right. \\
&\quad \left. + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\bar{\ell}_y^2}{\ell^2} \left[2 - 8\frac{p\cdot\ell}{\ell^2} + 8\frac{(p\cdot\ell)^2}{(\ell^2)^2} \right] + \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} (-4) \frac{p\cdot\bar{\ell}\,\bar{\ell}_y}{\ell^2} \right\}
\end{aligned} \tag{3.16}
$$

and

$$J(\xi) \equiv \int \frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\sum_{n=-\infty}^{\infty}\int \frac{d^2p}{(2\pi)^2}\, e^{i(p_y-A_{y0}-m_A)2\pi Rn}\frac{1}{p^2+\Lambda^2+\xi}\frac{1}{(p-\ell)^2+\Lambda^2}$$
$$\times\left[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\Lambda^2+\frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)}\frac{\ell_y^2}{\ell^2}\right]\left[-8\frac{p\cdot\ell}{\ell^2}-4\frac{p^2}{\ell^2}-4\Lambda^2\frac{1}{\ell^2}+16\frac{(p\cdot\ell)^2}{(\ell^2)^2}\right]. \tag{3.17}$$

From Eq. (2.18), we see that, for $|\ell|\to\infty$, $\hat{\mathcal{L}}(p)$ and $\mathcal{K}(p)$ are exponentially small because of the Bessel functions, and thus

$$\mathcal{L}(\ell)\overset{|\ell|\to\infty}{\to}\frac{2}{\ell^2}\ln(\ell^2/\Lambda^2),\qquad \mathcal{D}(\ell)\overset{|\ell|\to\infty}{\to}\ell^2\mathcal{L}(\ell)^2. \tag{3.18}$$

From these, we see that, in $I(\xi,\eta)$ of Eq. (3.16), the integration over ℓ as well as the integration over p are logarithmically UV divergent. In $J(\xi)$ of Eq. (3.17), the integration over p is logarithmically UV divergent but the integration over ℓ is UV convergent. Assuming (say) the dimensional regularization, the change of integration variables $(p,\ell)\to(p-\ell,-\ell)$ in $I(\xi,\eta)$, Eq. (3.16), shows that

$$I(\xi,\eta)=I(\eta,\xi). \tag{3.19}$$

Now, in Eq. (3.15), using the identity

$$e^{\delta_A\partial_\xi}e^{\delta_A\partial_\eta}-1=\left(e^{\delta_A\partial_\xi}-1\right)\left(e^{\delta_A\partial_\eta}-1\right)+e^{\delta_A\partial_\xi}+e^{\delta_A\partial_\eta}-2 \tag{3.20}$$

and noting the property in Eq. (3.19), we have the following very convenient representation:

$$E(\delta\epsilon)|_{\text{2-loop}}$$
$$=-2\pi R\frac{4\pi}{N}\sum_A\left[\left(e^{\delta_A\partial_\xi}-1\right)\left(e^{\delta_A\partial_\eta}-1\right)I(\xi,\eta)+\left(e^{\delta_A\partial_\xi}-1\right)J(\xi)\right]\Big|_{\xi=\eta=0}. \tag{3.21}$$

This shows that $E(\delta\epsilon)|_{\text{2-loop}}$ is UV finite *provided that the parameter δ_A is UV finite*. That is, the operator $e^{\delta_A\partial_\xi}-1$ acting on $J(\xi)$ increases the power of $p^2+\Lambda^2$ in the denominator in Eq. (3.17) and makes the p integration UV finite. Similarly, the operator $(e^{\delta_A\partial_\xi}-1)(e^{\delta_A\partial_\eta}-1)$ acting on $I(\xi,\eta)$ increases the power of $(p^2+\Lambda^2)[(p-\ell)^2+\Lambda^2]$ in the denominator of Eq. (3.16) and makes the integrations over p and ℓ UV convergent.

3.4. Renormalizability to the two-loop order

So far, we have observed that, from Eq. (3.12),

$$E(\delta\epsilon)|_{\text{1-loop}}$$
$$=2\pi R\sum_A\frac{1}{4\pi}\left[\delta_A-(\Lambda^2+\delta_A)\ln\left(1+\frac{\delta_A}{\Lambda^2}\right)\right]$$
$$+2\pi R\sum_A\sum_{n\neq 0}e^{-i(m_A+A_{y0})2\pi Rn}$$
$$\times\frac{1}{4\pi}(-4)\frac{1}{2\pi R|n|}\left[\sqrt{\Lambda^2+\delta_A}K_1(\sqrt{\Lambda^2+\delta_A}2\pi R|n|)-\Lambda K_1(\Lambda 2\pi R|n|)\right], \tag{3.22}$$

and, from Eq. (3.21),

$$E(\delta\epsilon)|_{\text{2-loop}} = -2\pi R \frac{4\pi}{N} \sum_A \left[\left(e^{\delta_A \partial_\xi} - 1\right)\left(e^{\delta_A \partial_\eta} - 1\right) I(\xi, \eta) + \left(e^{\delta_A \partial_\xi} - 1\right) J(\xi) \right]\Big|_{\xi=\eta=0}. \tag{3.23}$$

These representations show that the vacuum energy to the two-loop order is UV finite, if the parameter δ_A defined in Eq. (3.4) is UV finite. This implies that the parameter $\delta\epsilon$ must receive a non-trivial renormalization, as

$$\delta_A = \frac{\lambda\delta\epsilon}{\pi RN} m_A \text{ is UV finite} \Rightarrow \delta\epsilon = \left(\frac{e^{\gamma_E}\mu^2}{4\pi}\right)^{-\varepsilon}\left(1 + \frac{\lambda_R}{4\pi}\frac{1}{\varepsilon}\right)\delta\epsilon_R, \tag{3.24}$$

so that $\lambda\delta\epsilon = \lambda_R\delta\epsilon_R$ is UV finite; here we have used Eq. (2.10).

In terms of the renormalized parameters, the expansion of Eq. (3.22) with respect to $\delta\epsilon$ yields

$$E^{(1)}\delta\epsilon\Big|_{\text{1-loop}} = N\Lambda \frac{1}{\Lambda R}\frac{\lambda_R\delta\epsilon_R}{\pi N}\frac{R}{N}\sum_A m_A \sum_{n\neq 0} e^{-i(m_A + A_{y0})2\pi Rn} K_0(2\pi\Lambda R|n|),$$

$$E^{(2)}\delta\epsilon^2\Big|_{\text{1-loop}} = N\Lambda \frac{1}{(\Lambda R)^3}\left(\frac{\lambda_R\delta\epsilon_R}{\pi N}\right)^2 \frac{R^2}{N}\sum_A m_A^2 \left(-\frac{1}{4}\right)$$
$$\times \left[1 + \sum_{n\neq 0} e^{-i(m_A + A_{y0})2\pi Rn} 2\pi\Lambda R|n| K_1(2\pi\Lambda R|n|)\right]. \tag{3.25}$$

For Eq. (3.23), we need to carry out momentum integrations in Eqs. (3.16) and (3.17). This is the subject of the next subsection.

3.5. *p-integration in* $E^{(1)}\delta\epsilon|_{\text{2-loop}}$ *and* $E^{(2)}\delta\epsilon^2|_{\text{2-loop}}$

Let us next consider $E^{(1)}\delta\epsilon|_{\text{2-loop}}$, which is given by the $O(\delta_A)$ term of Eq. (3.23). By using the formulas in Appendix B, p-integration in Eq. (3.17) yields

$$E^{(1)}\delta\epsilon\Big|_{\text{2-loop}} = 2\pi R\frac{1}{N}\sum_A \delta_A \int \frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\left[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\Lambda^2 + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)}\frac{\ell_y^2}{\ell^2}\right]$$
$$\times \int_0^1 dx\, \frac{1}{2}\sum_{n\neq 0} e^{-i(m_A + A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn}$$
$$\times \Bigg\{(2\pi Rn)^2 \left[K_0(z) - K_2(z)\right]\frac{2}{\ell^2} + (2\pi Rn)^2 K_0(z)(-8)\frac{\ell_y^2}{(\ell^2)^2}$$
$$+ \frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} K_1(z)\left[\frac{4}{\ell^2} + i2\pi Rn\frac{\ell_y}{\ell^2}(-4)(1-2x)\right]\Bigg\}, \tag{3.26}$$

where

$$z \equiv \sqrt{x(1-x)\ell^2 + \Lambda^2}\, 2\pi R|n|. \tag{3.27}$$

Actually, the form of the integrand in the above expression depends on the choice of the Feynman parameter x. It can be changed by the change of variables $x \to 1 - x$ and $\ell_y \to -\ell_y$, which keeps the integration region and the factor $e^{ix\ell_y 2\pi Rn}$ intact.[6] It is convenient to fix the form of the integrand $\mathcal{I}(x, \ell_y)$ by

$$\int_0^1 dx \sum_{\ell_y} \mathcal{I}(x, \ell_y) \to \int_0^1 dx \sum_{\ell_y} \frac{1}{2} \left[\mathcal{I}(x, \ell_y) + \mathcal{I}(1-x, -\ell_y)\right], \tag{3.28}$$

so that the form of the integrand is invariant under the above change of variables. The particular expression in Eq. (3.26) has been obtained in this way.

Next, in Eq. (3.26) we use the identity

$$K_{\nu-1}(z) - K_{\nu+1}(z) = -\frac{2\nu}{z} K_\nu(z) \tag{3.29}$$

with $\nu = 1$. Then, by further using

$$K_0'(z) = -K_1(z) \tag{3.30}$$

and

$$\frac{\partial z}{\partial x} = \frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2 + \Lambda^2}} (1-2x) \frac{\ell^2}{2}, \tag{3.31}$$

which follows from Eq. (3.27), we have

$$\begin{aligned} E^{(1)}\delta\epsilon\Big|_{\text{2-loop}} &= 2\pi R \frac{1}{N} \sum_A \delta_A \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \left[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \Lambda^2 + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\bar{\ell}_y^2}{\ell^2}\right] \\ &\quad \times \int_0^1 dx \, \frac{1}{2} \sum_{n \neq 0} e^{-i(m_A + A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn} \\ &\quad \times \left[2\pi Rn\ell_y K_0(z) - i\frac{\partial}{\partial x} K_0(z)\right] 2\pi Rn(-8) \frac{\ell_y}{(\ell^2)^2}. \end{aligned} \tag{3.32}$$

Finally, integration by parts with respect to x yields

$$E^{(1)}\delta\epsilon\Big|_{\text{2-loop}} = 0. \tag{3.33}$$

Next, let us consider $E^{(2)}\delta\epsilon^2|_{\text{2-loop}}$, which is given by the $O(\delta_A^2)$ terms in Eq. (3.23). First, the p-integration in the function J in Eq. (3.17) gives

$$\begin{aligned} &E^{(2)}\delta\epsilon^2\Big|_{\text{2-loop}}^{(J)} \\ &= -2\pi R \frac{1}{N} \sum_A \delta_A^2 \int \frac{d\ell_x}{2\pi} \frac{1}{2\pi R} \sum_{\ell_y} \int_0^1 dx \left[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)} \Lambda^2 + \frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)} \frac{\bar{\ell}_y^2}{\ell^2}\right] \\ &\times \left(\frac{1}{\left[x(1-x)\ell^2 + \Lambda^2\right]^3} \left[-x(1-x)(3 - 10x + 10x^2) - (1 - 2x + 2x^2)\frac{\Lambda^2}{\ell^2}\right]\right. \end{aligned}$$

[6] Recall that $\ell_y \in \mathbb{Z}/R$.

$$
\begin{aligned}
&+\frac{1}{4}\sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn}\\
&\quad\times\Bigg\{\left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^3 K_3(z)\\
&\qquad\qquad\times\left[-2x(1-x)(1-3x+3x^2)-(1-2x+2x^2)\frac{\Lambda^2}{\ell^2}\right]\\
&\qquad+\frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2}K_2(z)\\
&\qquad\qquad\times\left[2(1-2x+2x^2)\frac{1}{\ell^2}+i2\pi Rn\frac{\ell_y}{\ell^2}(-2)(1-2x)(1-3x+3x^2)\right]\\
&\qquad+\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)\\
&\qquad\qquad\times\left[(1-2x+2x^2)\frac{1}{\ell^2}-4(1-2x+2x^2)\frac{\ell_y^2}{(\ell^2)^2}\right]\Bigg\}\Bigg).
\end{aligned}
\tag{3.34}
$$

On the other hand, from the function I in Eq. (3.16),

$$
\begin{aligned}
&E^{(2)}\delta\epsilon^2\Big|^{(I)}_{\text{2-loop}}\\
&=-2\pi R\frac{1}{N}\sum_A\delta_A^2\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\int_0^1 dx\\
&\quad\times\Bigg[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\ell^2\Bigg(\frac{1}{\left[x(1-x)\ell^2+\Lambda^2\right]^3}(-2)x(1-x)\left[x(1-x)-(1-6x+6x^2)\frac{\Lambda^2}{\ell^2}-2\frac{\Lambda^4}{(\ell^2)^2}\right]\\
&\qquad+\frac{1}{4}\sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn}\\
&\qquad\quad\times\Bigg\{\left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^3 K_3(z)(-2)x^2(1-x)^2\left(1+4\frac{\Lambda^2}{\ell^2}\right)\\
&\qquad\qquad+\frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2}K_2(z)\\
&\qquad\qquad\quad\times 2x(1-x)\left\{2\frac{1}{\ell^2}+4\frac{\Lambda^2}{(\ell^2)^2}-i2\pi Rn\frac{\ell_y}{\ell^2}(1-2x)\left(1+4\frac{\Lambda^2}{\ell^2}\right)\right\}\\
&\qquad\qquad+\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)(-2)x(1-x)\left[\frac{1}{\ell^2}+4\Lambda^2\frac{\ell_y^2}{(\ell^2)^3}\right]\Bigg\}\Bigg)\\
&\qquad+\frac{\mathcal{K}(\ell)^2}{\mathcal{D}(\ell)\mathcal{L}(\ell)}\frac{\bar{\ell}_y^2}{\ell^2}\Bigg(\frac{1}{\left[x(1-x)\ell^2+\Lambda^2\right]^3}4x(1-x)\left(1-3x+3x^2+\frac{\Lambda^2}{\ell^2}\right)\\
&\qquad+\frac{1}{4}\sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn}
\end{aligned}
$$

$$
\begin{aligned}
&\times \left\{\left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^3 K_3(z)2x(1-x)(1-2x)^2 \right.\\
&\quad + \frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2}K_2(z)8x(1-x)\left[\frac{1}{\ell^2}-i2\pi Rn\frac{\ell_y}{\ell^2}(1-2x)\right]\\
&\quad \left.\left. + \frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)(-8)x(1-x)\frac{\ell_y^2}{(\ell^2)^2}\right\}\right)\\
&+ \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)}\frac{\bar{\ell}_y^2}{\ell^2}\frac{1}{4}\sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn}e^{ix\ell_y 2\pi Rn}\\
&\quad \times \left.\frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2}K_2(z)i2\pi Rn(-4)x(1-x)\right].
\end{aligned}
\tag{3.35}
$$

To obtain the expressions in Eqs. (3.34) and (3.35), we applied the procedure in Eq. (3.28).

To further simplify the above expressions, we first note that all the terms linear in ℓ_y are proportional to $1-2x$, and thus to $\partial z/\partial x$ as in Eq. (3.31). Using this fact and the identity

$$
K_2(z) = -z\left[\frac{1}{z}K_1(z)\right]', \tag{3.36}
$$

we can carry out the integration by parts with respect to x in those terms linear in ℓ_y. We then use the identity in Eq. (3.29) with $\nu = 2$ to express $K_3(z)$ in terms of $K_1(z)$ and $K_2(z)$. The resulting expression contains the term $K_1(z)x(1-x)(1-2x)^2$, for which we use Eq. (3.31). We repeat the integration by parts as long as the factor $1-2x$ remains. In an intermediate step, we use

$$
K_0(z) = -\frac{1}{z}\left[zK_1(z)\right]'. \tag{3.37}
$$

Finally, we can carry out the x-integration in terms that do not contain the Bessel function.[7] In this way, we have the following rather simple expression:

$$
\begin{aligned}
&E^{(2)}\delta\epsilon^2\Big|_{\text{2-loop}}\\
&= -2\pi R\frac{1}{N}\sum_A \delta_A^2\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\\
&\quad \times\left[\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\frac{4-2(\ell^2+2\Lambda^2)\mathcal{L}_\infty(\ell)}{\ell^2(\ell^2+4\Lambda^2)}\right.\\
&\qquad + \int_0^1 dx\sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn}e^{ix\ell_y 2\pi Rn}\\
&\qquad\quad \times\left(\frac{\mathcal{L}(\ell)}{\mathcal{D}(\ell)}\left\{-\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)x(1-x)\right.\right.
\end{aligned}
$$

[7] We note that

$$
\tanh^{-1}\left(\sqrt{\frac{\ell^2}{\ell^2+4\Lambda^2}}\right) = \frac{1}{4}\sqrt{\ell^2(\ell^2+4\Lambda^2)}\mathcal{L}_\infty(\ell).
$$

$$
\begin{aligned}
&\qquad\qquad - \frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2} K_2(z) x(1-x) \\
&\qquad\qquad + \frac{\ell_y^2}{\ell^2}\Bigg[\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}} K_1(z) x(1-x) \\
&\qquad\qquad\qquad + (2\pi Rn)^2 K_0(z) \frac{2}{\ell^2} \Bigg]\Bigg\} \\
&- \frac{\mathcal{K}(\ell)}{\mathcal{D}(\ell)} \frac{\overline{\ell}_y^2}{\ell^2} \frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2} K_2(z) i2\pi Rnx(1-x) \Bigg)\Bigg].
\end{aligned} \tag{3.38}
$$

This completes the p-integration in $E^{(2)}\delta\epsilon^2|_{2\text{-loop}}$.

Let us examine whether the expression in Eq. (3.38) is IR finite or not. From the expressions in Eq. (2.18) and

$$
\mathcal{L}_\infty(\ell) = \frac{1}{\Lambda^2} - \frac{1}{6}\frac{\ell^2}{\Lambda^2} + O((\ell^2)^2), \tag{3.39}
$$

we see that the above ℓ_x-integral for $E^{(2)}\delta\epsilon^2|_{2\text{-loop}}$ is IR finite, as should be the case for any physical quantity.

In what follows, we carry out the summation over the index A in Eqs. (3.25) and (3.38), and integrate the resulting expressions over the "vacuum moduli" A_{y0} as in Eq. (2.11). Then, we organize them according to the powers of $1/N$. Before doing these, however, it is helpful to further simplify Eq. (3.38) by noting that $\hat{\mathcal{L}}(p)$ and $\mathcal{K}(p)$ in Eqs. (2.18) are exponentially suppressed for $N \to \infty$ as $\lesssim e^{-\Lambda RN}$ because of the asymptotic behavior of the Bessel function, $K_\nu(z) \overset{z\to\infty}{\sim} \sqrt{\pi/(2z)}e^{-z}$. Therefore, these functions can be neglected in the power series expansion in $1/N$ and we can set $\mathcal{L}(\ell) \to \mathcal{L}_\infty(\ell)$, $\mathcal{K}(\ell) \to 0$, and $\mathcal{D}(\ell) \to (p^2+4\Lambda^2)\mathcal{L}_\infty(\ell)^2$ in Eq. (3.38) to yield

$$
\begin{aligned}
&E^{(2)}\delta\epsilon^2\Big|_{2\text{-loop}} \\
&\quad = -2\pi R \frac{1}{N}\sum_A \delta_A^2 \int \frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y} \\
&\qquad \times \Bigg[\frac{4-2(\ell^2+2\Lambda^2)\mathcal{L}_\infty(\ell)}{\ell^2(\ell^2+4\Lambda^2)^2\mathcal{L}_\infty(\ell)} \\
&\qquad\quad + \int_0^1 dx \sum_{n\neq 0} e^{-i(m_A+A_{y0})2\pi Rn} e^{ix\ell_y 2\pi Rn} \frac{1}{(\ell^2+4\Lambda^2)\mathcal{L}_\infty(\ell)} \\
&\qquad\qquad \times \Bigg\{ - \frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}} K_1(z) x(1-x) - \frac{(2\pi Rn)^2}{x(1-x)\ell^2+\Lambda^2} K_2(z) x(1-x) \\
&\qquad\qquad\quad + \frac{\ell_y^2}{\ell^2}\Bigg[\frac{(2\pi R|n|)^3}{\sqrt{x(1-x)\ell^2+\Lambda^2}} K_1(z) x(1-x) + (2\pi Rn)^2 K_0(z)\frac{2}{\ell^2}\Bigg]\Bigg\}\Bigg],
\end{aligned} \tag{3.40}
$$

up to exponentially small terms.

3.6. Summation over A and integration over A_{y0}

We thus consider the sum over the index A and the integration over the vacuum moduli A_{y0} in Eq. (2.11). The summation over A can be carried out as

$$\sum_A e^{-im_A 2\pi Rn} = \sum_{j=0}^{N-1} \left(e^{-2\pi ni/N}\right)^j = N \begin{cases} 1, & \text{for } n = 0 \bmod N, \\ 0, & \text{for } n \neq 0 \bmod N, \end{cases} \tag{3.41}$$

$$\sum_A m_A e^{-im_A 2\pi Rn} = \frac{N}{2R} \begin{cases} 1 - \dfrac{1}{N}, & \text{for } n = 0 \bmod N, \\ \dfrac{2}{N} \dfrac{1}{e^{-2\pi ni/N} - 1}, & \text{for } n \neq 0 \bmod N, \end{cases} \tag{3.42}$$

and

$$\sum_A m_A^2 e^{-im_A 2\pi Rn} = \frac{N}{3R^2} \begin{cases} 1 - \dfrac{3}{2N} + \dfrac{1}{2N^2}, & \text{for } n = 0 \bmod N, \\ \dfrac{3}{N} \dfrac{1}{e^{-2\pi ni/N} - 1} \left(1 - \dfrac{2}{N} \dfrac{1}{1 - e^{2\pi ni/N}}\right), & \text{for } n \neq 0 \bmod N. \end{cases} \tag{3.43}$$

On the other hand, the integration over A_{y0} with the measure in Eq. (2.11) results in

$$\int_0^1 d(A_{y0}RN)\, e^{-iA_{y0}2\pi Rn} = \begin{cases} 1, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0,\ n = 0 \bmod N, \\ \dfrac{iN}{2\pi n} \left(e^{-2\pi ni/N} - 1\right), & \text{for } n \neq 0 \bmod N. \end{cases} \tag{3.44}$$

The combination of the above two operations therefore yields

$$\int_0^1 d(A_{y0}RN) \sum_A m_A\, e^{-i(m_A + A_{y0})2\pi Rn} = \frac{N}{2R} \begin{cases} 1 - \dfrac{1}{N}, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0,\ n = 0 \bmod N, \\ \dfrac{i}{\pi n}, & \text{for } n \neq 0 \bmod N, \end{cases} \tag{3.45}$$

and

$$\int_0^1 d(A_{y0}RN) \sum_A m_A^2\, e^{-i(m_A + A_{y0})2\pi Rn}$$
$$= \frac{N}{3R^2} \begin{cases} 1 - \dfrac{3}{2N} + \dfrac{1}{2N^2}, & \text{for } n = 0, \\ 0, & \text{for } n \neq 0,\ n = 0 \bmod N, \\ \dfrac{3i}{2\pi n}\left(1 - \dfrac{1}{N}\right) + \dfrac{3}{2N} \dfrac{1}{\pi n} \dfrac{1}{\tan(\pi n/N)}, & \text{for } n \neq 0 \bmod N. \end{cases} \tag{3.46}$$

Using Eqs. (3.45) and (3.46) for Eq. (3.25), under the integration over A_{y0},

$$E^{(1)}\delta\epsilon\Big|_{\text{1-loop}} = N\Lambda \frac{1}{\Lambda R} \frac{\lambda_R \delta\epsilon_R}{\pi N} \frac{1}{2} \sum_{n \neq 0 \bmod N} \frac{i}{\pi n} K_0(2\pi\Lambda R|n|) = 0, \tag{3.47}$$

and

$$E^{(2)}\delta\epsilon^2\Big|_{\text{1-loop}} = N\Lambda\frac{1}{(\Lambda R)^3}\left(\frac{\lambda_R\delta\epsilon_R}{\pi N}\right)^2\left(-\frac{1}{12}\right)\left[1-\frac{3}{2N}+\frac{1}{2N^2}+\frac{6}{N}\sum_{n>0,n\neq 0 \bmod N}\frac{\Lambda R K_1(2\pi\Lambda Rn)}{\tan(\pi n/N)}\right]. \tag{3.48}$$

For the two-loop corrections, from Eq. (3.33),

$$E^{(1)}\delta\epsilon\Big|_{\text{2-loop}} = 0, \tag{3.49}$$

and for Eq. (3.39) we have

$$\begin{aligned}
E^{(2)}\delta\epsilon^2\Big|_{\text{2-loop}}
&= -\frac{2\pi}{3}\left(\frac{\lambda_R\delta\epsilon_R}{\pi RN}\right)^2\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\\
&\times\Bigg(\frac{1}{R}\left(1-\frac{3}{2N}+\frac{1}{2N^2}\right)\frac{4-2(\ell^2+2\Lambda^2)\mathcal{L}_\infty(\ell)}{\ell^2(\ell^2+4\Lambda^2)^2\mathcal{L}_\infty(\ell)}\\
&\quad+\int_0^1 dx\sum_{n>0,n\neq 0 \bmod N}\\
&\quad\times\left[\frac{6}{N}\frac{\cos(x\ell_y 2\pi Rn)}{\tan(\pi n/N)}-6\left(1-\frac{1}{N}\right)\sin(x\ell_y 2\pi Rn)\right]\frac{1}{(\ell^2+4\Lambda^2)\mathcal{L}_\infty(\ell)}\\
&\quad\times\Bigg\{-\frac{(2\pi Rn)^2}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)x(1-x)-\frac{2\pi Rn}{x(1-x)\ell^2+\Lambda^2}K_2(z)x(1-x)\\
&\quad+\frac{\ell_y^2}{\ell^2}\left[\frac{(2\pi Rn)^2}{\sqrt{x(1-x)\ell^2+\Lambda^2}}K_1(z)x(1-x)+2\pi RnK_0(z)\frac{2}{\ell^2}\right]\Bigg\}\Bigg),
\end{aligned} \tag{3.50}$$

up to exponentially small terms.

3.7. *Final results*

Finally, we arrange the above results in powers of $1/N$. From Eqs. (3.47) and (3.49), we have

$$E^{(1)}\delta\epsilon = 0\cdot N^0 + 0\cdot N^{-1} + O(N^{-2}). \tag{3.51}$$

Thus, $E^{(1)}\delta\epsilon$ vanishes to the order we worked out.

For $E^{(2)}\delta\epsilon^2$, setting

$$E^{(2)}\delta\epsilon^2 = E^{(2)}\delta\epsilon^2\Big|_{O(N^{-1})} + E^{(2)}\delta\epsilon^2\Big|_{O(N^{-2})} + O(N^{-3}), \tag{3.52}$$

from Eq. (3.48),

$$RE^{(2)}\delta\epsilon^2\Big|_{O(N^{-1})} = N^{-1}(\lambda_R\delta\epsilon_R)^2(\Lambda R)^{-2}F(\Lambda R), \tag{3.53}$$

where

$$F(\xi) \equiv -\frac{1}{12\pi^2}[1 + c(\xi)], \qquad c(\xi) \equiv \lim_{N\to\infty} \frac{6}{N} \sum_{n>0, n\neq 0 \bmod N} \frac{\xi K_1(2\pi\xi n)}{\tan(\pi n/N)}. \tag{3.54}$$

From Eqs. (3.48) and (3.50), on the other hand,

$$RE^{(2)}\delta\epsilon^2\Big|_{O(N^{-2})} = N^{-2}(\lambda_R\delta\epsilon_R)^2(\Lambda R)^{-3}G(\Lambda R), \tag{3.55}$$

where

$$\begin{aligned} G(\xi) \equiv & -\frac{1}{12\pi^2}\left\{-\frac{3}{2}\xi + \lim_{N\to\infty}\left[6\sum_{n>0,n\neq 0 \bmod N}\frac{\xi^2 K_1(2\pi\xi n)}{\tan(\pi n/N)} - N\xi c(\xi)\right]\right\} \\ & - \frac{1}{6\pi^3}\xi^3\int_{-\infty}^{\infty} d\tilde{\ell}_x \sum_{\tilde{\ell}_y\in\mathbb{Z}}\left(\frac{4 - 2(\tilde{\ell}^2 + 2\xi^2)\tilde{\mathcal{L}}_\infty(\tilde{\ell},\xi)}{\tilde{\ell}^2(\tilde{\ell}^2 + 4\xi^2)^2\tilde{\mathcal{L}}_\infty(\tilde{\ell},\xi)}\right. \\ & + \lim_{N\to\infty}\int_0^1 dx \sum_{n>0,n\neq 0 \bmod N} \\ & \times\left[\frac{6}{N}\frac{\cos(x\tilde{\ell}_y 2\pi n)}{\tan(\pi n/N)} - 6\sin(x\tilde{\ell}_y 2\pi n)\right]\frac{1}{(\tilde{\ell}^2 + 4\xi^2)\tilde{\mathcal{L}}_\infty(\tilde{\ell},\xi)} \\ & \times\left\{-\frac{(2\pi n)^2}{\sqrt{x(1-x)\tilde{\ell}^2 + \xi^2}}K_1(z)x(1-x)\right. \\ & - \frac{2\pi n}{x(1-x)\tilde{\ell}^2 + \xi^2}K_2(z)x(1-x) \\ & \left.\left. + \frac{\tilde{\ell}_y^2}{\tilde{\ell}^2}\left[\frac{(2\pi n)^2}{\sqrt{x(1-x)\tilde{\ell}^2 + \xi^2}}K_1(z)x(1-x) + 2\pi n K_0(z)\frac{2}{\tilde{\ell}^2}\right]\right\}\right). \end{aligned} \tag{3.56}$$

In this expression, we have defined

$$\tilde{\mathcal{L}}_\infty(\tilde{\ell},\xi) \equiv \frac{2}{\sqrt{\tilde{\ell}^2(\tilde{\ell}^2 + 4\xi^2)}}\ln\left(\frac{\sqrt{\tilde{\ell}^2 + 4\xi^2} + \sqrt{\tilde{\ell}^2}}{\sqrt{\tilde{\ell}^2 + 4\xi^2} - \sqrt{\tilde{\ell}^2}}\right) \tag{3.57}$$

and

$$z \equiv \sqrt{x(1-x)\tilde{\ell}^2 + \xi^2}\,2\pi|n|. \tag{3.58}$$

We plot the function $F(\Lambda R)$ in Eq. (3.53) in Fig. 2 and the function $G(\Lambda R)$ in Eq. (3.55) in Fig. 3. These plots clearly show that, to the order of the $1/N$ expansion we worked out, the vacuum energy is a well-defined finite quantity under the parameter renormalization in Eqs. (2.10) and (3.24). Equations (3.51)–(3.56) and Figs. 2 and 3 are the main results of this paper. Since Figs. 2 and 3 show that the functions $F(\Lambda R)$ and $G(\Lambda R)$ remain finite as $\Lambda R \to 0$, Eqs. (3.53) and (3.55) [and Eq. (3.51)] show that the vacuum energy normalized by the radius of the S^1, $RE(\delta\epsilon)$, behaves as

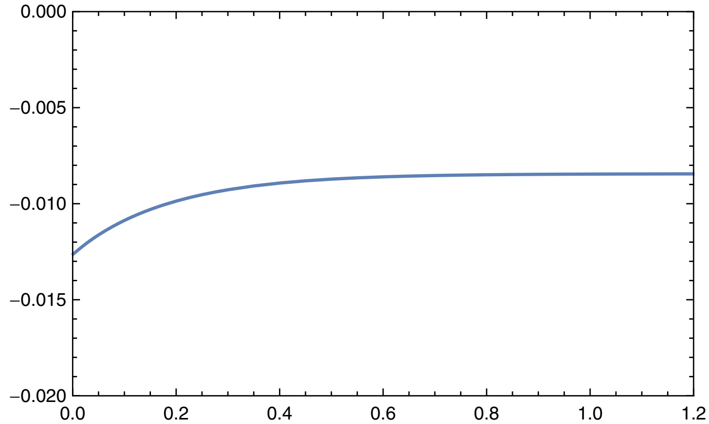

Fig. 2. The function $F(\Lambda R)$ from Eq. (3.53).

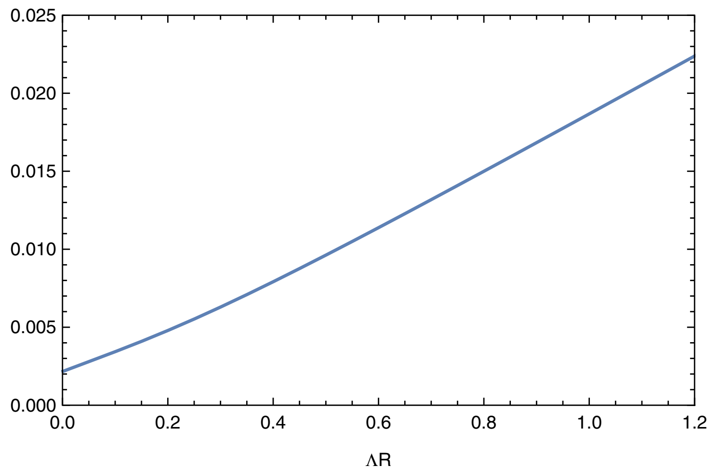

Fig. 3. The function $G(\Lambda R)$ from Eq. (3.55).

inverse powers of ΛR for ΛR small, the $O(N^{-1})$ term behaves as $(\Lambda R)^{-2}$, and the $O(N^{-2})$ term behaves as $(\Lambda R)^{-3}$. Since Λ is given by Eq. (2.9), this result implies that to the order of the $1/N$ expansion we worked out, the vacuum energy is a purely non-perturbative quantity and it has no well-defined weak coupling expansion in λ_R.

4. Conclusion and discussion

By employing the $1/N$ expansion, we have computed the vacuum energy $E(\delta\epsilon)$ of the 2D SUSY $\mathbb{C}P^{N-1}$ model on $\mathbb{R} \times S^1$ with $\mathbb{Z}_N$ twisted boundary conditions to the second order in the SUSY-breaking parameter $\delta\epsilon$ in Eq. (3.2). We found that the vacuum energy is purely non-perturbative and, although it is a perfectly well-defined physical quantity in the $1/N$ expansion, it has no sensible weak coupling expansion in λ_R.

Our original intention was to compare our result in the $1/N$ expansion with the result by the bion calculus in Ref. [5], because it appears that the calculation in Ref. [5] holds even under the limit in Eq. (1.1).

According to Ref. [5], the contribution of a single bion to the vacuum energy in Eq. (3.1) is given by ($E^{(0)}$ is set to be zero)

$$RE^{(1)}\delta\epsilon = -R\sum_{b=1}^{N-1} 2m_b\mathcal{A}_b(\Lambda R)^{2Rm_bN}\delta\epsilon \tag{4.1}$$

and

$$RE^{(2)}\delta\epsilon^2 = -R\sum_{b=1}^{N-1} 2m_b\mathcal{A}_b(\Lambda R)^{2Rm_bN}\left[-2\gamma_E - 2\ln\left(\frac{4\pi Rm_bN}{\lambda_R}\right) \mp \pi i\right]\delta\epsilon^2, \tag{4.2}$$

where the last $\mp\pi i$ term is the imaginary ambiguity caused by the integration over quasi-collective coordinates of the bion. In these expressions, the index b corresponds to the "species" of the bion and the coefficient $\mathcal{A}_b$ is given by using the twist angle m_A in Eq. (2.4) as

$$\begin{aligned}\mathcal{A}_b &= \left[\frac{\Gamma(1-m_bR)}{\Gamma(1+m_bR)}\right]^2 \prod_{a=1,a\neq b}^{N-1} \frac{m_a}{m_a-m_b}\frac{\Gamma(1+(m_a-m_b)R)}{\Gamma(1-(m_a-m_b)R)}\frac{\Gamma(1-m_aR)}{\Gamma(1+m_aR)}\\ &= (-1)^{b+1}\frac{N^{2b}}{(b!)^2}.\end{aligned} \tag{4.3}$$

Using this, the coefficient of the imaginary ambiguity in Eq. (4.2) is given by

$$-R\sum_{b=1}^{N-1} 2m_b\mathcal{A}_b(\Lambda R)^{2Rm_bN} = \frac{2}{N}\sum_{b=1}^{N-1}(-1)^b\frac{b}{(b!)^2}(\Lambda RN)^{2b}. \tag{4.4}$$

When N is fixed, in the weak coupling limit $\Lambda R \ll 1$ for which the semi-classical approximation should be valid, the $b = 1$ term $-2\Lambda^2R^2N$ dominates the sum in Eq. (4.4). $\Lambda^2 = \mu^2 e^{-4\pi/\lambda_R}$ is the exponential of the action of the constituent of the minimum bion (the minimal fractional instanton–anti-instanton pair) and, at the same time, is consistent with the order of the $u = 1$ IR renormalon ambiguity. On the other hand, in the large-N limit in Eq. (1.1), whether Eq. (4.4) possesses a sensible $1/N$ expansion or not is not clear, because each term behaves as $O(N)$, $O(N^3)$, $O(N^5)$, …; we could not estimate the sum as a whole in the large-N limit.

Thus, we cannot compare our result in the $1/N$ expansion with the result in Ref. [5] by the bion calculus. We have no clear idea yet why this comparison is impossible. One phenomenological observation from Eq. (4.4) is that it is a series in the combination ΛRN and thus the result in Ref. [5] seems meaningful for $\Lambda RN \ll 1$ instead of our large-N limit in Eq. (1.1), with which $\Lambda RN \gg 1$.[8] More thought seems to be necessary to clearly understand the relation between bions, the IR renormalon, and the $1/N$ expansion.

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[8] We would like to thank Aleksey Cherman, Yuya Tanizaki, and Mithat Ünsal for providing us with suggestive arguments on this point.

Note added

In this paper we considered the large-N limit specified by Eq. (1.1), with which $N\Lambda R \to \infty$. On the other hand, Ref. [42] discussed that the semi-classical picture such as that in Refs. [17–20] holds only for $N\Lambda R \ll 1$. This is natural because the characteristic mass scale with the twisted boundary condition can be $N\Lambda R$ instead of ΛR, and in the weak coupling limit $\Lambda \to 0$. In this paper, we also observed that the perturbative analyses cannot be available reasonably for $N\Lambda R \gg 1$; our approximation is basically the expansion in $1/(N\Lambda R)$ and it is impossible to read how the vacuum energy behaves as $N\Lambda R \to 0$ from our large-N result. In a recent paper [43], perturbation theory with the twisted boundary condition is carefully studied for $N\Lambda R \to 0$ and a picture consistent with the bion calculus has been obtained.

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Appendix A. The perturbative part of the vacuum energy contains IR divergences

In the limit $R \to \infty$, the expression of the vacuum energy is considerably simplified because $n \neq 0$ terms in Eqs. (3.54) and (3.56) are exponentially suppressed in this limit. We have

$$RE^{(2)}\delta\epsilon^2 \overset{R\to\infty}{\to} -\frac{1}{12\pi^2}(\lambda_R\delta\epsilon_R)^2\left\{N^{-1}(\Lambda R)^{-2} + N^{-2}\left[-\frac{3}{2}(\Lambda R)^{-2} + G_\infty\right] + O(N^{-3})\right\}, \quad \text{(A.1)}$$

where

$$G_\infty \equiv \frac{8\pi}{R^2}\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\frac{4-2(\ell^2+2\Lambda^2)\mathcal{L}_\infty(\ell)}{\ell^2(\ell^2+4\Lambda^2)^2\mathcal{L}_\infty(\ell)}. \quad \text{(A.2)}$$

Equation (A.1) is a non-perturbative expression obtained to the next-to-leading order of the $1/N$ expansion. From Eq. (3.39), we see that the ℓ-integration in G_∞ is IR convergent.

To extract the perturbative part from Eq. (A.1), we expand G_∞ with respect to Λ and neglect all terms with positive powers of $\Lambda = \mu e^{-2\pi/\lambda_R}$. Noting the behavior $\mathcal{L}_\infty \sim (2/\ell^2)\ln(\ell^2/\Lambda^2)$ from Eq. (2.18), we obtain the perturbative part as

$$G_\infty \sim \frac{8\pi}{R^2}\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\frac{2}{(\ell^2)^2}\left[\frac{1}{\ln(\ell^2/\Lambda^2)} - 1\right]. \quad \text{(A.3)}$$

The perturbative expansion with respect to $\lambda_R(\mu)$ is then given by

$$G_\infty \sim \frac{8\pi}{R^2}\int\frac{d\ell_x}{2\pi}\frac{1}{2\pi R}\sum_{\ell_y}\frac{2}{(\ell^2)^2}\left[-1 + \sum_{k=0}^{\infty}[-\ln(\ell^2/\mu^2)]^k\left(\frac{\lambda_R}{4\pi}\right)^{k+1}\right], \quad \text{(A.4)}$$

where we have used

$$\ln(\ell^2/\Lambda^2) = \ln(\ell^2/\mu^2) + \frac{4\pi}{\lambda_R(\mu)}. \quad \text{(A.5)}$$

Equations (A.3) and (A.4) show that the perturbative part of G_∞ suffers from IR divergences in the ℓ-integration, although the full G_∞ itself is IR finite.

Appendix B. Integration formulas

In Sect. 3.5 we have used the following integration formulas (in practice, we are interested in the cases $(\alpha, \beta) = (1, 2)$, $(1, 3)$, and $(2, 2)$):

$$
\begin{aligned}
&\int \frac{d^2p}{(2\pi)^2}\, e^{ip_y 2\pi Rn}\, \frac{1}{\left[(p-\ell)^2+\Lambda^2\right]^{\alpha}}\, \frac{1}{(p^2+\Lambda^2)^{\beta}} \begin{cases} 1 \\ p_\mu \\ p_\mu p_\nu \end{cases} \\
&\overset{n=0}{=} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \\
&\quad \times \frac{1}{4\pi} \begin{cases} \Gamma(\alpha+\beta-1)\left[x(1-x)\ell^2+\Lambda^2\right]^{1-\alpha-\beta}, \\ \Gamma(\alpha+\beta-1)\left[x(1-x)\ell^2+\Lambda^2\right]^{1-\alpha-\beta} x\ell_\mu, \\ \Gamma(\alpha+\beta-1)\left[x(1-x)\ell^2+\Lambda^2\right]^{1-\alpha-\beta} x^2\ell_\mu\ell_\nu \\ \qquad + \frac{1}{2}\Gamma(\alpha+\beta-2)\left[x(1-x)\ell^2+\Lambda^2\right]^{2-\alpha-\beta}\delta_{\mu\nu}, \end{cases} \\
&\overset{n\neq 0}{=} \frac{1}{\Gamma(\alpha)\Gamma(\beta)} \int_0^1 dx\, x^{\alpha-1}(1-x)^{\beta-1} \\
&\quad \times \frac{1}{4\pi} 2^{2-\alpha-\beta} e^{ix\ell_y 2\pi Rn} \begin{cases} \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-1} K_{\alpha+\beta-1}(z), \\ \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-1} K_{\alpha+\beta-1}(z) x\ell_\mu \\ \qquad + \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-2} K_{\alpha+\beta-2}(z) i2\pi Rn\delta_{\mu y}, \\ \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-1} K_{\alpha+\beta-1}(z) x^2\ell_\mu\ell_\nu \\ \qquad + \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-2} K_{\alpha+\beta-2}(z) \\ \qquad\qquad \times (\delta_{\mu\nu} + ix\ell_\mu 2\pi Rn\delta_{\nu y} + i2\pi Rn\delta_{\mu y} x\ell_\nu) \\ \qquad - \left(\frac{2\pi R|n|}{\sqrt{x(1-x)\ell^2+\Lambda^2}}\right)^{\alpha+\beta-3} K_{\alpha+\beta-3}(z) \\ \qquad\qquad \times (2\pi Rn)^2\delta_{\mu y}\delta_{\nu y}, \end{cases}
\end{aligned} \tag{B.1}
$$

where

$$
z \equiv \sqrt{x(1-x)\ell^2+\Lambda^2}\,2\pi R|n|. \tag{B.2}
$$

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