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Field Theoretic Approach to Flat Space Holography

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Abstract

Flat space holography proposes a duality between quantum gravity in asymptotically flat spacetimes and field theories defined at null infinity $\mathscr{I}$, offering a non-AdS realisation of the holographic principle. A central feature of this framework is that the dual theory is defined on a null surface, in contrast to the timelike boundary of AdS/CFT. This requires any candidate dual field theory to be compatible with the Carroll limit in which the speed of light is sent to zero. Exploiting the fact that the asymptotic symmetry group of flat space — the BMS group — is the conformal extension of the Carroll group acting on null infinity $\mathscr{I}$, we construct explicit BMS-invariant field theories on the boundary. These models reproduce known results for two- and three-point correlation functions in the literature and provide a first step towards a field-theoretic formulation of flat space holography in the Carrollian framework. Furthermore, we clarify several conceptual and structural aspects of the field-theoretic approach to flat space holography, advancing the development of a more coherent boundary perspective.

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Preface

Looking back at my journey through academia, which has led me to write this thesis, I am deeply grateful to the many individuals who have supported, encouraged, and inspired me along the way.

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1 Introduction

The principle of holography is the idea that a theory of quantum gravity in $(D+1)$-dimensional spacetime can be completely described by a D-dimensional quantum field theory — without gravity — on the boundary of that spacetime. This is similar to how a hologram encodes three-dimensional information on a two-dimensional surface, and it stands as one of the most prominent ideas in the quest for a unified theory of gravity.

The first hint that a theory of gravity might fundamentally be holographic comes from black hole thermodynamics. The Bekenstein-Hawking formula [1, 2]

$$S = \frac{k_B A c^3}{4G\hbar}, \tag{1.1}$$

relates the entropy S of a black hole to the area of its event horizon A, rather than to its volume. This surprising result suggests that all the degrees of freedom inside the black hole seem to be stored at its boundary. Motivated by this observation, 't Hooft and Susskind proposed what is now known as the holographic principle [3, 4]. They conjectured that a complete description of quantum gravity in a given spacetime region could be encoded in a lower-dimensional theory defined on the boundary of that region. A major breakthrough came in 1997, when Maldacena proposed the AdS/CFT correspondence [5], which provides a concrete realisation of the holographic principle. In this duality, type IIB string theory on $\mathrm{AdS}_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ super-Yang-Mills theory in four dimensions — a conformal field theory (CFT) without gravity [6, 7]. This correspondence has since become a central framework for studying quantum gravity in negatively curved (Anti-de Sitter) spacetimes.

A natural question to ask is how far the holographic principle can be generalised. While the AdS/CFT correspondence has provided profound insights into holography in Anti-de Sitter (AdS) spacetime, a similar understanding for asymptotically flat spacetimes (AFS) is lacking (see e.g. [8]). Despite the fact that our universe has a positive cosmological constant ($\Lambda > 0$), AFS still serves as a realistic and valuable model for describing many physical processes observed in astrophysics and cosmology. This motivates the development of flat space holography, which seeks to formulate a precise holographic correspondence for quantum gravity in asymptotically flat spacetimes.

Following the success of the AdS/CFT correspondence, it soon became clear that flat space holography does not directly follow from a simple limit of AdS in which the AdS radius is sent to infinity. A new strategy was therefore required, giving rise to the flat-space holography programme. Fortunately, over the past decades, it has become evident that the symmetries of asymptotically flat spacetime are far richer and more intricate than initially believed [9–15]. In fact, the fundamental symmetry group of asymptotically flat spacetimes is not the finite-dimensional Poincaré group, but rather an infinite-dimensional symmetry group now known as the (extended) BMS group. The constraints imposed by these asymptotic symmetries have inspired two bottom-up approaches to flat space holography: Carrollian and Celestial holography.

The celestial approach to flat space holography proposes that the dual to quantum gravity in asymptotically flat spacetime is a two-dimensional conformal field theory living on the celestial sphere — a codimension-two conformal boundary at null infinity. This approach emerged from Strominger's key insight that the newly recognised BMS symmetries, when combined with suitable matching conditions connecting their action on $\mathscr{I}^+$ and $\mathscr{I}^-$, impose nontrivial constraints on gravitational scattering processes [16, 17]. He further demonstrated that soft theorems for S-matrix elements can be reinterpreted as Ward identities of these asymptotic symmetries [18, 19]. In this framework, the Lorentz group acts as the global conformal group on the celestial sphere, allowing scattering amplitudes to be recast as conformal correlators via a Mellin transform [20–22]. The advantage of this approach is that the dual theory is a 2-dimensional celestial CFT (CCFT) and allows for the use of powerful 2D CFT tools, such as operator product expansions, conformal blocks, and the state-operator correspondence. However, the codimension-two nature of the boundary makes its relation to 'standard holography', such as AdS/CFT, less direct. Nevertheless, it has yielded deep insights into the structure of asymptotic symmetries and infrared behaviour of gauge and gravity theories. The literature on celestial holography is extensive, and we refer the reader to a selection of reviews [14, 23–27]. The connection between celestial and Carrollian holography has also been explored in various works, and the two perspectives have been shown to be complementary [8, 28–31].

The Carrollian approach to flat space holography proposes that the role of the dual theory is played by a conformal Carrollian field theory that lives on the null boundary $\mathscr{I}$ of AFS. While this framework is less developed than its celestial counterpart, its codimension-one boundary makes it more of a direct analogue to AdS/CFT. A key motivation for this approach lies in the observation that the asymptotic symmetry group of flat spacetime —– the BMS group —– is isomorphic to the conformal Carroll group defined on its null boundary [32, 33]. Carrollian physics, originally introduced by Lévy-Leblond in the 1960s [34], arises as a distinct contraction of the Poincaré algebra where the speed of light tends to zero. The term 'Carroll' playfully alludes to the author and mathematician Lewis Carroll, best known for his work Alice in Wonderland, as a nod to the bizarre and seemingly paradoxical nature of this framework. In this thesis, we focus exclusively on Carrollian holography, and for the remainder of the text, this is what we mean when talking about holography.

The main difficulties inherent to the Carrollian approach to flat space holography stem from two key differences compared to the more familiar AdS/CFT framework. First, the conformal boundary $\mathscr{I}$ of AFS is a *null* hypersurface [35], in contrast to the timelike boundary of AdS, which has a clear notion of time evolution. Defining quantum field theories on a null surface requires an understanding of Carrollian physics (see e.g. [33, 36–43]) and the structure of Carrollian CFTs (see e.g. [30, 44–46]), whose dynamics are highly non-intuitive: the light cone collapses, fields exhibit ultralocal behaviour, and the symmetry algebra becomes infinite-dimensional. Therefore, developing a consistent Carrollian CFT on $\mathscr{I}$ poses both technical and conceptual challenges, but offers a promising framework for extending the holographic principles beyond AdS [8, 28–31, 46–52].

A second major difference from the AdS framework is that gravitational charges at null infinity are generally non-conserved, due to the presence of gravitational radiation. In AdS, gravitational radiation does not escape to infinity but is reflected off the timelike boundary, ensuring conservation of global charges and allowing black holes to reach thermal equilibrium. As a result, an asymptotic inertial observer in AFS experiences energy and momentum loss through the flux of gravitational waves across null infinity. This leads to so-called 'leaky boundary conditions', which have been exploited in [28, 30] to compute massless Carrollian scattering amplitudes. These amplitudes have been further analysed in [52–54], and recent developments concerning the flux balance laws of the boundary can be found in [55].

General arguments suggest that the only well-defined observable in a theory of quantum gravity on asymptotically flat spacetime is the S-matrix (see e.g. [56]). This follows from the fact that local observables, which play a central role in ordinary quantum field theory, cannot be defined in a background-independent manner. As a result, any viable holographic dual must be capable of reproducing the (bulk) spacetime S-matrix. Inspired by celestial holography, a proposal has been put forward to relate S-matrix elements to conformal Carroll correlators, via a Mellin-type (Fourier-like) transform [29, 49]. This construction serves as a starting point for developing a holographic dictionary, analogous to that in AdS/CFT, in which boundary correlators encode the full scattering data of the bulk.

The early success of the AdS/CFT correspondence owed much to the fact that it came with a concrete, computable example of a dual CFT. In contrast, the flat space holography program has, thus far, not produced an explicit toy model for the proposed boundary theory. Although significant progress has been made in constructing and analysing Carrollian field theories, a holographic connection linking bulk gravity to a boundary theory is still missing. Instead, much of the existing literature has focused on analysing the structure of scattering amplitudes via Ward identities associated with asymptotic symmetries (see e.g. [28–31, 48, 52]). The underlying hope is that, by uncovering enough properties and constraints of these amplitudes, one can reverse-engineer a viable dual theory. More recently, Carrollian correlation functions have been computed via a flat space limit of AdS/CFT [53, 54, 57], and matched to bulk S-matrix elements through the aforementioned Mellin transform approach. However, none of these developments has produced a concrete boundary theory from which correlation functions can be derived directly.

This thesis aims to construct explicit BMS invariant field theories living on the null boundaries of AFS, intended as duals to field theories in the bulk. Such theories are constructed via Carrollian limits of relativistic theories as well as through symmetry-based arguments. We then examine their suitability for holography by comparing them against criteria from the emerging flat space holographic dictionary [30, 47, 58]. For models that meet these requirements, we compute two- and three-point correlation functions and compare them with known (bulk) S-matrix elements. A crucial step in this analysis is clarifying the relation between conformal Carrollian fields on the boundary and in- and out-states in the bulk. This allows us to reproduce several correlation functions that had previously only been derived either from the large r-limit of bulk fields or as limits of AdS results. In contrast, our approach derives these results directly from explicit boundary Lagrangians, demonstrating that such models can serve as concrete candidates for a dual field theory. Finally, this thesis also clarifies several conceptual and structural aspects of the field-theoretic approach to flat space holography, advancing the development of a more coherent boundary perspective.

The remainder of this thesis is organised as follows. In Chapter 2, we review flat spacetime and introduce flat Bondi coordinates, which are well-suited for establishing a connection between the boundary and the bulk. We then extend this setup to asymptotically flat spacetimes — the relevant setting for flat space holography — and examine their asymptotic symmetries. In Chapter 3, we turn to the boundary perspective. We introduce its geometric structure and demonstrate how this gives rise to Carrollian physics. After a brief overview of Carrollian field theory, we explain how the infinite-dimensional conformal Carroll algebra is isomorphic to the BMS algebra at null infinity. We also derive the transformation law for Carroll primary fields, which plays a central role throughout the thesis. In Chapter 4, we review the holographic dictionary and assess whether known Carrollian field theories are suitable holographic duals. We then construct new BMS-invariant models from symmetry principles and conclude the chapter by analysing their properties. Chapter 5 marks the transition to explicit holographic computations. After reviewing existing results and how they arise from bulk calculations, we relate boundary fields to bulk in- and out-states and compute correlation functions that reproduce these results. This chapter also clarifies several conceptual and structural aspects of the field-theoretic approach to flat space holography. Finally, in Chapter 6, we conclude our analysis with some comments and future directions.

The conventions adopted in this thesis are as follows. In Lorentzian signature, I consistently use the mostly-plus convention, e.g., the Minkowski metric in four dimensions is given by $\eta_{\mu\nu} = \mathrm{diag}(-1, +1, +1, +1)$. I follow the modern convention in which the spacetime dimension is denoted $D = d + 1$, with d the spatial dimensions. Furthermore, Greek indices will be used for spacetime coordinates and Latin indices are reserved for spatial components. Throughout, the symbol Φ will be used to denote an arbitrary (Carroll primary) field. When specific cases are discussed, we will use ϕ, A and σ to refer to scalar, spin-1, and spin-2 Carroll primary fields, respectively.

2 The holographic playing field: The bulk

In this chapter, we lay the groundwork necessary for flat space holography. When constructing a holographic duality for spacetimes with a vanishing cosmological constant, we naturally consider asymptotically flat Minkowski spaces. Such spacetimes can contain highly curved, nontrivial, gravitational fields, e.g. a black hole, but at large distances approach Minkowski space in a well-defined way. Carrollian holography aspires to construct a duality between quantum gravity in asymptotically flat spacetimes (the bulk) and conformal field theories living on the null boundaries $\mathscr{I}^+$ and $\mathscr{I}^-$. A key requirement for such a holographic framework is that the boundary theory must reproduce the asymptotic structure of the bulk. We will therefore begin with a review of Minkowski space $\mathbb{R}^{1,3}$ and analyse its asymptotic behaviour. Subsequently, we will upgrade to asymptotically flat spacetimes (AFS) and examine their asymptotic symmetries and structure.

2.1 Minkowski space

A prerequisite to establishing a duality between the interior (the bulk) of a spacetime and its boundary is understanding the asymptotic behaviour of the bulk. To study the properties of spacetime near infinity, it is useful to introduce Penrose diagrams [59]. Such diagrams manage to depict an infinitely large spacetime in a bounded drawing through the use of a conformal compactification. An example of such a Penrose diagram, in the case of four-dimensional flat Minkowski space, is shown in Figure 1. In this figure, all of Minkowski space is pulled into a finite region, and as a result, distances are not faithfully represented. Instead, conformal transformations preserve angles and thus leave the causal structure of spacetime, characterised by its light cones, unaffected. This is illustrated by the fact that light always moves at a 45-degree angle in such diagrams.

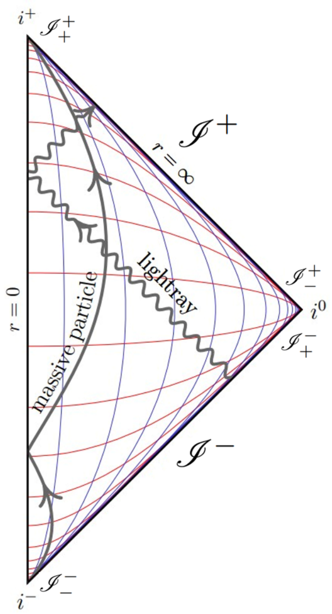

Figure 1. Penrose diagram of Minkowski spacetime [14].

In this Penrose diagram of Minkowski space, the vertical axis is given by a time coordinate T, and the horizontal axis is given by a spatial coordinate often chosen to be a radial coordinate R. Because we are representing a four-dimensional spacetime in only two dimensions, each point in the diagram, other than $R = 0$, should be understood as having an attached 2-sphere S^2 accounting for the angular directions. One of the key features of the Penrose diagram is that it highlights several regions important to the causal structure of spacetime. For instance, massive particles which follow timelike geodesics start at the bottom point i^-, called 'past timelike infinity', and follow their journey through spacetime to the uppermost point in Figure 1, called 'future timelike infinity', denoted by i^+. Similarly, all null geodesics begin at past null infinity $\mathscr{I}^-$ and end at future null infinity $\mathscr{I}^+$. Lastly, there is the point i^0 in the right corner, called 'spatial infinity', and this is where spacelike geodesics begin and end.

In quantum theory, scattering amplitudes are a powerful tool to describe fundamental interactions between particles and can be directly compared to measurable quantities in experiments, such as cross-sections and decay rates. In the context of quantum gravity on asymptotically flat spacetime, their role becomes even more important, as the S-matrix is the only well-defined observable in such a theory. Therefore, in the construction of a holographic duality, we want to study scattering processes in the bulk spacetime and match them to correlation functions of fields living on the boundary. When studying scattering processes, we are interested in knowing how a given initial state of a system transforms into a final state. This is encoded in the scattering amplitude

$$\mathcal{A} = \langle out|in\rangle, \tag{2.1}$$

where the scattering matrix (S-matrix) relates the ingoing and outgoing states, $|out\rangle = \mathcal{S}|in\rangle$. This thesis focuses on the scattering of massless particles — such as gravitational waves or electromagnetic waves — in asymptotically flat spacetimes. To do so, we must specify the ingoing data $|in\rangle$ at $\mathscr{I}^-$, which will then propagate through spacetime, interact in some complicated way, and ultimately emerge at $\mathscr{I}^+$. Since we will only consider massless particles, we do not have to provide initial data at i^- or consider what happens at i^+. From a holographic perspective, these null boundaries $\mathscr{I}^\pm$ will be of essential importance. As we will see in the next chapter, they will serve as the D-dimensional manifolds that support the dual quantum field theory.

To analyse the asymptotic structure of Minkowski space and characterise its null boundaries $\mathscr{I}^\pm$, it is useful to define coordinates. Traditionally, this was done using (round) retarded Bondi coordinates on $\mathscr{I}^+$ and (round) advanced Bondi coordinates on $\mathscr{I}^-$ [9, 14]. Recently, it has become more popular to describe Minkowski space in flat (retarded) Bondi coordinates as these coordinates are capable of describing both $\mathscr{I}^\pm$[1]. In flat Bondi coordinates $(u, r, z, \bar{z}) \in \mathbb{R} \times \mathbb{R} \times \mathbb{C}^2$, u is retarded time, r is a radial coordinate which runs from $-\infty$ to ∞, and $(z, \bar{z})$ are complex coordinates related to the angular coordinates (θ, ϕ).

[1]See [30] Appendix A for a thorough description of all different Bondi coordinate systems.

These flat Bondi coordinates are related to Cartesian coordinates $X^\mu = (t, \vec{x})$ by

$$X^\mu = u\partial_z\partial_{\bar{z}}q^\mu + rq^\mu, \tag{2.2}$$

for

$$q^\mu(z,\bar{z}) = \frac{1}{\sqrt{2}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}). \tag{2.3}$$

The Minkowski metric in flat Bondi coordinates reads

$$d\hat{s}^2 = -2dudr + 2r^2dzd\bar{z}, \tag{2.4}$$

and we can describe both boundaries $\mathscr{I}^+ = \{r \to \infty\}$ and $\mathscr{I}^- = \{r \to -\infty\}$ with these coordinates. Furthermore, lines obtained by keeping $(u, z, \bar{z})$ fixed are null geodesics from $\mathscr{I}^-$ to $\mathscr{I}^+$ [30], and this natural identification between the boundaries will become important in subsequent chapters.

Of course, defining the boundaries by $r \to \pm\infty$ is not very rigorous, as this does not correspond to a point in spacetime. Therefore, to make the definition of $\mathscr{I}^+$ more precise and to obtain a proper spacetime manifold with boundary, we introduce a conformal completion $\Omega = \frac{1}{r}$ [35]. We can now define $\mathscr{I}^+ = \{\Omega = 0\}$ and substituting this in the metric gives

$$d\hat{s}^2 = -\frac{2dud\Omega}{\Omega^2} + \frac{2}{\Omega^2}dzd\bar{z}. \tag{2.5}$$

Then by conformally rescaling the metric $ds^2 = \Omega^2 d\hat{s}^2$ and setting $\Omega = 0$ we get the boundary metric of $\mathscr{I}^+$

$$ds^2 = \lim_{\Omega\to 0}\Omega^2 d\hat{s}^2 = 2dzd\bar{z}. \tag{2.6}$$

This metric is degenerate, as it does not include a time coordinate u. Given that $\mathscr{I}^+$ is a 3-dimensional manifold with coordinates $(u, z, \bar{z})$, this structure naturally leads to Carrollian physics as we will see in Chapter 3. A similar procedure can be followed for past null infinity and will also result in a degenerate conformal metric.

2.2 Asymptotically flat spacetime

We now turn to the study of asymptotically flat spacetimes (AFS), which allow for non-trivial gravitational fields while ensuring asymptotic flatness. In order to ensure flatness at large r (with fixed $(u, z, \bar{z})$), we must impose boundary conditions that lead to fall-off conditions on the metric components. Such boundary conditions are typically chosen to be weak enough to allow for all interesting phenomena, such as gravitational waves, but strong enough to rule out unphysical solutions, e.g. solutions with infinite energy. We will follow the natural choices made by pioneers Bondi, van der Burg, Metzner, and Sachs (BMS) [9–11] (See [60] for a review) but reformulate them in the more state-of-the-art flat

Bondi coordinates.

For the description of AFS by means of suitable coordinates, BMS introduced the so-called Bondi-gauge [9]

$$g_{rr} = g_{rA} = 0 \quad \text{and} \quad \partial_r \det(r^{-2} g_{AB}) = 0. \tag{2.7}$$

It was shown that the four dimensional-asymptotically flat metric in flat Bondi coordinates reads [13, 30]

$$\begin{aligned} ds^2 = &-2dudr + 2r^2 dz d\bar{z} + \frac{2m_B}{r} du^2 + rC_{zz} dz^2 + rC_{\bar{z}\bar{z}} d\bar{z}^2 \\ &+ \left(\frac{1}{2} \partial^z C_{zz} + \frac{2}{3r} \left(N_z + \frac{1}{4} C_{zz} \partial_z C^{zz} \right) \right) dudz + c.c. + \dots \end{aligned} \tag{2.8}$$

The first two terms in the metric (2.8) are simply the flat Minkowski metric, and the remaining terms are leading corrections. Note that all correction terms vanish near the boundary (when we take the limit $\Omega \to 0$) and we thus obtain the same metric (2.6) as before. The AFS metric (2.8) contains three fields m_B, N_z and C_{zz} which depend on $(u, z, \bar{z})$ but not on r. The first two are the Bondi mass aspect m_B and the angular momentum aspect N_z, but since these fields are not relevant for the remainder of this thesis, we will not discuss them in further detail. For us, the most important field is the asymptotic shear C_{AB} $(A = z, \bar{z})$, which is a symmetric trace-free tensor, i.e., $C_{z\bar{z}} = 0$. The fields C_{zz} and $C_{\bar{z}\bar{z}}$, can be seen as the two polarisation modes for gravitational waves, and also encode the helicity modes of the graviton.

Another important field is the 'Bondi news tensor', defined by

$$N_{zz} = \partial_u C_{zz}. \tag{2.9}$$

It plays a central role in the analysis of gravitational radiation at null infinity, as its square, integrated along retarded time u, is proportional to the energy flux across $\mathscr{I}^+$ [14]. To ensure finite outgoing radiation, fall-off conditions must be imposed on N_{zz} at the boundaries of $\mathscr{I}^+$, denoted $\mathscr{I}^+_+ = \{X \in \mathscr{I}^+ | u \to +\infty\}$ and $\mathscr{I}^+_- = \{X \in \mathscr{I}^+ | u \to -\infty\}$[2]. It was demonstrated by Christodoulou and Klainerman [61] that this requires the news to fall off faster than $\frac{1}{|u|}$ at the boundaries of $\mathscr{I}^+$.

Additionally, the Bondi news tensor is also related to the gravitational memory effect, where the passage of gravitational waves produces a permanent shift in the relative positions of a pair of inertial detectors [62]. When you integrate the news over the entire duration of the radiation

$$\Delta C_{zz} = \int_{\mathscr{I}^+} du N_{zz} = C_{zz}|_{\mathscr{I}^+_+} - C_{zz}|_{\mathscr{I}^+_-}, \tag{2.10}$$

[2]In differential geometry, it is a general principle that the boundary of a boundary vanishes, $\partial\partial = 0$. This seems to be contradicted by the presence of $\mathscr{I}^+_+$ and $\mathscr{I}^+_-$ as the boundary of $\mathscr{I}^+$. However, $\mathscr{I}^\pm$ are not genuine boundaries in the traditional sense but arise through conformal compactification and therefore do not violate this principle.

the change ΔC_{zz} is proportional to the displacement of the two inertial detectors [14]. Detection of the gravitational memory effect has been proposed at LIGO and is an exciting experimental prospect for the coming decades.

2.3 Asymptotic symmetries

Over a century ago, in 1905, Poincaré demonstrated that the fundamental symmetry group of four-dimensional Minkowski spacetime is what we now call the Poincaré group

$$\text{Poincaré} = \text{Lorentz} \ltimes \text{Translations}. \tag{2.11}$$

It consists of the Lorentz group (3 boosts and 3 rotations) and 4 spacetime translations. Here, the use of the semi-direct product $\ltimes$ means that the elements of the Poincaré group are pairs consisting of Lorentz transformations and translations, and that the Lorentz transformations act non-trivially on translations.

During the 1960s Bondi, van der Burgh, Metzner, and Sachs (BMS) tried to recover the Poincaré group as the symmetry group of asymptotically flat spacetimes [9–11]. Since they were looking for asymptotic symmetries, at large r where spacetime is almost flat, they expected to recover the same isometries as in Minkowski spacetime. To their surprise, however, they discovered that the fundamental symmetries of four-dimensional asymptotically flat spacetimes are not the Poincaré group. Instead, they found that the AFS metric (2.8) keeps the same form under angle-dependent translations

$$u \mapsto u + \mathcal{T}(z, \bar{z}), \tag{2.12}$$

now known as supertranslations[3]. These supertranslations — parametrised by any smooth real function $\mathcal{T}(z, \bar{z})$ — expand the finite Poincaré group to the infinite-dimensional BMS group

$$\text{BMS} = \text{Lorentz} \ltimes \text{Supertranslations}. \tag{2.13}$$

This new symmetry group consists of globally well-defined, invertible transformations of null infinity [63]. Recently, however, it was suggested by Barnich and Troessaert [12, 13] that the true asymptotic symmetry group of four-dimensional AFS also contains local conformal transformations

$$z \mapsto \mathcal{Y}(z), \tag{2.14}$$

for $\mathcal{Y}(z)$ any meromorphic function. This introduces an extra infinite amount of generators known as 'superrotations', which should be thought of as an infinite-dimensional extension of Lorentz transformations — analogous to how supertranslations extend spacetime translations. This new group is called the 'Extended BMS group'

$$\text{Extended BMS} = \text{Superrotations} \ltimes \text{Supertranslations}, \tag{2.15}$$

[3] Here and in the remainder of this thesis, the terminology of 'super' has nothing to do with supersymmetry. It simply means that certain objects, known in a finite-dimensional context of special relativity, get extended to an infinite-dimensional context in the BMS group.

and is the appropriate symmetry group for asymptotically flat spacetime. Due to its infinite-dimensional structure, this result implies that general relativity does not reduce to special relativity in the regime of weak fields and large distances [14].

The same result can also be obtained starting directly from the AFS metric (2.8) by showing that it is invariant under $\xi = \xi^u \partial_u + \xi^r \partial_r + \xi^z \partial_z + \xi^{\bar{z}} \partial_{\bar{z}}$ with [30]

$$\begin{aligned} \xi^u &= \mathcal{T}(z,\bar{z}) + \frac{u}{2}\left(\partial_z \mathcal{Y}(z) + \partial_{\bar{z}} \bar{\mathcal{Y}}(\bar{z})\right), \\ \xi^z &= \mathcal{Y}(z) + \mathcal{O}(r^{-1}), \qquad \xi^{\bar{z}} = \bar{\mathcal{Y}}(\bar{z}) + \mathcal{O}(r^{-1}), \\ \xi^r &= -\frac{r}{2}\left(\partial_z \mathcal{Y}(z) + \partial_{\bar{z}} \bar{\mathcal{Y}}(\bar{z})\right) + \mathcal{O}(r^0). \end{aligned} \tag{2.16}$$

Again, we recover the extended BMS group as the fundamental symmetry group of AFS. For the remainder of this thesis, we shall refer to it simply as the BMS group.

We can find the infinitesimal transformations of the aforementioned fields in (2.8) by the Lie derivative of the bulk metric (2.8) along the generators (2.16). The most crucial transformation for the purpose of this thesis is the variation of the asymptotic shear, which reads as [16, 17]

$$\begin{aligned} \delta_\xi C_{zz} = &\left[\left(\mathcal{T} + \frac{u}{2}(\partial_z \mathcal{Y} + \partial_{\bar{z}} \bar{\mathcal{Y}}\right)\partial_u + \mathcal{Y}\partial_z + \bar{\mathcal{Y}}\partial_{\bar{z}} + \frac{3}{2}\partial_z \mathcal{Y} - \frac{1}{2}\partial_{\bar{z}} \bar{\mathcal{Y}}\right] C_{zz} \\ &- 2\partial_z^2 \mathcal{T} - u\partial_z^3 \mathcal{Y}, \end{aligned} \tag{2.17}$$

together with the complex conjugate relation for $C_{\bar{z}\bar{z}}$.

3 The holographic playing field: The boundary

In this chapter, we turn our attention to the other side of the proposed duality: the boundary theory. As previously discussed, the boundary of flat space is null, characterised by a degenerate metric with signature $(0, +, +)$. Such a structure precludes the possibility of relativistic dynamics and implies that the boundary theory must instead be of Carrollian nature. We begin by reviewing Carrollian physics and its associated symmetries, and then show how the asymptotic BMS symmetry of flat spacetime emerges as a conformal extension of the Carroll group acting on these null boundaries.

Since we are considering 4-dimensional asymptotically flat spacetime, the boundary[4] $\mathscr{I}$ is a 3-dimensional manifold with the topology $\mathbb{R} \times \mathcal{S}^2$, equipped with a degenerate metric g_{ab} of signature $(0, +, +)$ [35]. Here, $\mathbb{R}$ characterises the time direction and $\mathcal{S}^2$ is a celestial Riemann surface, usually taken to be the celestial sphere S^2. Similarly to the last chapter, we will use flat Bondi coordinates $(u, z, \bar{z}) \in \mathbb{R} \times \mathbb{C}^2$ to parameterise the boundary, as these have the advantage of describing both $\mathscr{I}^+$ and $\mathscr{I}^-$. Due to the degeneracy of the metric, which followed from taking the limit $\Omega \to 0$ in (2.6), we require a vector field n^a in the kernel of the metric, i.e. $g_{ab}n^a = 0$, to characterise the time direction and to fully describe the geometric structure of the boundary [35]. This upgrade of the Riemannian structure at $\mathscr{I}$ is called a Carrollian geometry (g_{ab}, n^a), which in our case given by

$$g_{ab}dx^a dx^b = 0du^2 + 2dzd\bar{z} \qquad \text{and} \qquad n^a\partial_a = \partial_u. \tag{3.1}$$

The term 'Carrollian' refers to theories describing physics in the limit where the speed of light is taken to zero. The above metric (3.1) arises as the $c \to 0$ limit of the 3-dimensional flat Minkowski metric $ds^2 = -c^2du^2 + 2dzd\bar{z}$ written in complex spatial coordinates $z = x + iy$. It follows that a Carrollian geometry is the natural framework for describing null infinity, implying that the dual field theory living on $\mathscr{I}$ must obey Carrollian physics.

3.1 Carrollian physics

Carrollian physics emerges from a limit of spacetime symmetries, in which the speed of light tends to zero $c \to 0$. This curious limit was first explored by Lévy-Leblond in 1965, who identified the Carroll group as a contraction of the Poincaré group [34]. Although it might seem counterintuitive that the limit $c \to 0$ gives something non-trivial, it is well-documented in the literature (see e.g. Ref. [33, 36–43, 45]).

3.1.1 Carroll transformations and symmetry

Consider a Lorentz boosts in the x-direction in $(1 + d)$-dimensional spacetime using Cartesian coordinates

$$ct' = \gamma(ct - \beta x), \quad x' = \gamma(x - \beta ct), \quad y' = y, \quad z' = z, \tag{3.2}$$

[4] The boundary of asymptotically flat spacetime includes both $\mathscr{I}^+$ and $\mathscr{I}^-$. The notation $\mathscr{I}$ may refer to either one or both, depending on the context.

where $\gamma = \frac{1}{\sqrt{1-(\frac{v}{c})^2}}$ and $\beta = \frac{v}{c}$. Replacing the Lorentz boost parameter β with a new Carroll boost parameter b, defined by $\beta = cb$, and taking the limit of $c \to 0$ leads to

$$t' = t - bx, \quad \vec{x}' = x. \tag{3.3}$$

Under these Carroll boosts, time is relative while space is absolute.

Another feature of Carrollian physics is that for $c \to 0$ the light cone closes up, such that particles with timelike worldlines cannot move in the Carroll limit and the theory becomes ultralocal [39]. From the Carroll boosts (3.3), it is possible that $\Delta t > 0$ while $\Delta t' < 0$, i.e., two Carroll observers do not necessarily agree on which event happened first. This seems to violate causality, but it does not, as for causality to be violated, physical information would need to be sent from one event to the other, which is prevented by the light cones closing up [39]. Due to the Carroll light cones being fixed at a point in space, two observers at different points in space will not be in the same light cone, and are therefore never in causal contact with each other.

The Carroll boosts (3.3) can also be recovered starting from the Lorentz boost generator $L_i = \frac{1}{c}x_i\partial_t + ct\partial_i$ and redefining the Carroll boost generator to be $C_i \equiv cL_i$ such that upon taking the limit $c \to 0$ we obtain $C_i = x_i\partial_t$. In addition to Carroll boosts $C_i = x_i\partial_t$, the Carroll algebra consists of the generators $H = \partial_t$ (Hamiltonian), $P_i = \partial_i$ (spatial translations), and $J_{ij} = x_i\partial_j - x_j\partial_i$ (spatial rotations) [8, 39]. The nonzero commutators are given by

$$\begin{aligned}
&[P_i, C_j] = \delta_{ij} H, \\
&[J_{ij}, P_k] = \delta_{jk} P_i - \delta_{ik} P_j, \\
&[J_{ij}, C_k] = \delta_{jk} C_i - \delta_{ik} C_j, \\
&[J_{ij}, J_{kl}] = \delta_{jk} J_{il} - \delta_{ik} J_{jl} + \delta_{il} J_{jk} - \delta_{jl} J_{ik},
\end{aligned} \tag{3.4}$$

where $i, j, k, l \in \{1, \ldots, d\}$. One significant difference between the Carroll group and the Poincaré group is that the Carroll boosts, unlike Lorentz boosts, commute among themselves. Additionally, the Hamiltonian commutes with all generators in the Carroll algebra, making it a central charge.

The presence of symmetries implies conserved Noether currents, and in the case of translation symmetry, this gives the energy-momentum tensor $T^{\mu\nu}$ for which $\partial_\mu T^{\mu\nu} = 0$. In the case of Lorentz symmetry, the Noether current is $T^\mu_\nu L^\nu_i$, where we used $L_i \equiv L^\mu_i \partial_\mu$, which implies that the energy-momentum tensor is symmetric $T^{\mu\nu} = T^{\nu\mu}$. Rotation symmetry still implies that the energy-momentum tensor is symmetric in its spatial indices, but Carroll boosts $C_i \equiv C^\mu_i \partial_\mu$ give Noether currents $T^\mu_\nu C^\nu_i$. Current conservation $\partial_\mu(T^\mu_\nu C^\nu_i) = 0$ then implies $T^i_t = 0$. This result also follows from the Lorentz case by taking $c \to 0$ and is a defining feature for Carrollian theories [39].

3.1.2 Carroll field theories

Another non-intuitive feature of Carroll physics is that upon taking the $c \to 0$ limit of a relativistic field theory, one does not produce a unique Carroll field theory. Instead, there exist two (known) possible ways to take this limit, and Carroll QFTs can roughly be divided into 3 categories [40]. The first, 'electric' theories, are ultralocal in space and have non-trivial time-dependence. The second, 'magnetic' theories, have very simple time dependence but non-trivial space dependence. The third category is a combination of the previous two; however, not much is known about it so far. The nomenclature 'electric'- and 'magnetic' theories originates from considering the Carrollian limits of electromagnetism [38], where in one limit only the electric field survives, and in another only the magnetic field survives. To demonstrate the different $c \to 0$ limits, we will consider the case of a real relativistic massive scalar field, closely following the derivation from [39].

Given a relativistic real scalar field ϕ, it transforms, under Lorentz boosts, as

$$\delta\phi = ct\vec{\beta} \cdot \vec{\partial}\phi + \frac{1}{c}\vec{\beta} \cdot \vec{x}\partial_t\phi. \tag{3.5}$$

The Lagrangian of a relativistic massive scalar field[5]

$$\mathcal{L} = \frac{1}{2c^2}(\partial_t\phi)^2 - \frac{1}{2}(\partial_i\phi)^2 - \frac{m^2c^2}{2\hbar^2}\phi^2, \tag{3.6}$$

then transforms into a total derivative under this Lorentz boost. The electric limit of the relativistic Lagrangian is obtained by substituting $\phi \to c\phi$ and then taking the limit $c \to 0$ while keeping $E_0 := mc^2$ constant. This results in

$$\mathcal{L} = \frac{1}{2}(\partial_t\phi)^2 - \frac{m^2c^4}{2\hbar^2}\phi^2, \tag{3.7}$$

and going back to the common convention of ignoring the constants, gives

$$\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - \frac{1}{2}m^2\phi^2. \tag{3.8}$$

This electric scalar theory contains no spatial derivatives and is, therefore, ultralocal. This is a general feature of electric theories, and it implies that different spatial points are completely independent from one another.

From the previously defined Carroll boost generator $C_i = x_i\partial_t$ with Carroll boost parameter $\vec{b}$, we find that a scalar field transforms infinitesimally as

$$\delta\phi = \vec{b} \cdot \vec{x}\partial_t\phi. \tag{3.9}$$

It is easy to check that our new electric Carroll theory transforms as a total time derivative under this transformation.

[5] Although constants as $\hbar$ and c are often omitted for brevity in theoretical physics, it is very important to keep track of them when working with Carroll limits.

Magnetic theories can be obtained by using the conjugate momentum $\pi_\phi = \frac{1}{c^2}\partial_t\phi$. Starting again from the relativistic Lagrangian (3.6) and defining the auxiliary field $\chi := \pi_\phi$, we obtain

$$\mathcal{L} = -\frac{c^2}{2}\chi^2 + \chi\dot{\phi} - \frac{1}{2}(\partial_i\phi)^2 - \frac{m^2c^2}{2\hbar^2}\phi^2. \tag{3.10}$$

Taking the limit $c \to 0$ while keeping χ, ϕ and the Compton wavelength $\lambda^{-1} = \frac{mc}{\hbar}$ fixed results in

$$\mathcal{L} = \chi\dot{\phi} - \frac{1}{2}(\partial_i\phi)^2 - \frac{m^2c^2}{2\hbar^2}\phi^2, \tag{3.11}$$

where χ acts as a Lagrange multiplier, resulting in the equation of motion $\dot{\phi} = 0$. This simple time dependence is a general feature of magnetic theories.

To show that the magnetic scalar theory is invariant under Carroll boosts, we first need to derive the transformation rules for χ. These can be derived from the transformation rules for a Lorentz boost using the Carroll boost parameter $\beta = cb$

$$\delta\chi = \frac{1}{c^2}\partial_t(\delta\phi) = \vec{b}\cdot\vec{\partial}\phi + c^2 t\vec{\beta}\cdot\vec{\partial}\chi + \vec{b}\cdot\vec{x}\dot{\chi}, \tag{3.12}$$

and taking the limit $c \to 0$

$$\delta\chi = \vec{b}\cdot\vec{x}\dot{\chi} + \vec{b}\cdot\vec{\partial}\phi. \tag{3.13}$$

Using transformations (3.9) and (3.13), it can be checked that the magnetic theory is Carroll boost invariant, as its Lagrangian transforms into a total time derivative.

More generally, magnetic theories can be written as

$$S = \int dt d^d x \, \left(\chi\dot{\phi} + \mathcal{L}(\phi, \partial_i\phi)\right), \tag{3.14}$$

where $\mathcal{L}$ is any Lagrangian depending on fields ϕ and its spatial derivatives [40].

3.2 Equivalence between BMS and conformal Carroll symmetries

The main motivation for studying Carroll field theories is the expectation that they might be dual to quantum gravity in asymptotically flat spacetime. For any holographic duality, it is crucial that the symmetries of the bulk gravitational theory and its dual boundary theory match, as these symmetries reflect the physical content of the theory. In the case of asymptotically flat spacetimes, the extended BMS group governs the asymptotic symmetries at null infinity. In this section, we show that the extended BMS group is isomorphic to the conformal Carroll group, making it a natural candidate for the symmetry algebra of the putative dual theory.

Given the aformentioned Carroll algebra (3.4), where factors of i have been included in the generators to make them Hermitian, we define the time translations $H = -i\partial_t$, spatial translations $P_i = -i\partial_i$, Carroll boosts $C_i = -ix_i\partial_t$ and spatial rotations $J_{ij} = i(x_i\partial_j - x_j\partial_i)$. We can conformally extend this algebra by adding the dilation operator $D = -i(t\partial_t + x^i\partial_i)$, and the Carrollian special conformal generators $K = i\vec{x}^2\partial_t$, and $K_i = -i(2x_ix^j\partial_j + 2x_it\partial_t - \vec{x}^2\partial_i)$ [8, 47]

$$\begin{aligned} &[C_i, K_j] = -i\delta_{ij}K, \quad [D, K] = -iK, \quad [J_{ij}, K_k] = i\delta_{k[j}K_{i]}, \\ &[K, P_i] = -2iC_i, \quad [H, K_i] = -2iC_i, \quad [K_i, P_j] = 2i(D\delta_{ij} - J_{ij}), \\ &[D, P_i] = iP_i, \qquad [D, H] = iH, \qquad [D, K_i] = -iK_i. \end{aligned} \tag{3.15}$$

These conformal generators follow from the $c \to 0$ limits of their relativistic counterparts

$$D = -ix^\mu\partial_\mu \quad \text{and} \quad K_\mu = -i(2x_\mu x^\nu\partial_\nu - x^\nu x_\nu\partial_\mu), \tag{3.16}$$

where $K = K_0$. The conformal Carroll algebra (CCA), (3.4) together with (3.15), is a finite-dimensional algebra, and in $D = 3$ consists of 10 generators. Similar to 2d relativistic CFTs, this group admits an infinite extension [51]

$$\begin{aligned} L_n &= -z^{n+1}\partial_z - (n+1)z^n\frac{t}{2}\partial_t, \\ \bar{L}_n &= -\bar{z}^{n+1}\partial_{\bar{z}} - (n+1)\bar{z}^n\frac{t}{2}\partial_t, \\ M_{p,q} &= z^p\bar{z}^q\partial_t, \end{aligned} \tag{3.17}$$

where for $z = x+iy$, $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ and $r, s, n = -1, 0, 1$, we get back the finite conformal Carroll algebra (CCA)

$$\begin{aligned} M_{00} = iH, \quad M_{10} &= i(C_x + iC_y), \quad M_{01} = i(C_x - iC_y), \quad M_{11} = -iK, \\ L_{-1} = -\frac{i}{2}\left(P_x - iP_y\right), \quad &L_0 = \frac{1}{2i}(D + iJ), \quad L_1 = \frac{1}{2i}\left(K_x + iK_y\right), \\ \bar{L}_{-1} = -\frac{i}{2}\left(P_x + iP_y\right), \quad &\bar{L}_0 = \frac{1}{2i}(D - iJ), \quad \bar{L}_1 = \frac{1}{2i}\left(K_x - iK_y\right). \end{aligned} \tag{3.18}$$

Allowing r, s, n to take arbitrary integer values yields an infinite-dimensional algebra which is closed under commutators

$$\begin{aligned} &[L_m, L_n] = (m-n)L_{m+n}, && [\bar{L}_m, \bar{L}_n] = (m-n)\bar{L}_{m+n}, \\ &[L_m, M_{p,q}] = \left(\frac{m+1}{2} - p\right)M_{m+p,q}, && [\bar{L}_m, M_{p,q}] = \left(\frac{m+1}{2} - q\right)M_{p,m+q}, \\ &[M_{m,n}, M_{p,q}] = 0, && [L_m, \bar{L}_n] = 0. \end{aligned} \tag{3.19}$$

This is the infinite extension of the conformal Carroll algebra, without central extension, and this algebra is often called $\mathfrak{CCarr}$.

Another way to derive this result is to compute the conformal Killing vectors of the metric and thereby determine the symmetry group at the boundary. Given that the boundary $\mathscr{I}$ of (asymptotically) flat spacetime is a 3-dimensional manifold, and in flat Bondi coordinates has Carrollian geometry $ds^2 = 0du^2 + 2dzd\bar{z}$ and $n^a\partial_a = \partial_u$, the conformal Killing equations are[6]

$$\mathscr{L}_\xi g_{ab} = 2\alpha g_{ab}, \qquad \mathscr{L}_\xi n^a = -\alpha n^a, \tag{3.20}$$

where α is a function on $\mathscr{I}$ and $\xi = \xi^u\partial_u + \xi^z\partial_z + \xi^{\bar{z}}\partial_{\bar{z}}$ [35]. Given that the metric is flat, we obtain the Killing equations

$$\mathscr{L}_\xi g_{\mu\nu} = \partial_\mu\xi_\nu + \partial_\nu\xi_\mu = 2\alpha g_{\mu\nu}, \tag{3.21}$$

and

$$\mathscr{L}_\xi n^a = \xi^b\partial_b n^a - n^b\partial_b\xi^a = -\partial_u\xi^a = -\alpha n^a. \tag{3.22}$$

Solving these Killing equations results in

$$\begin{aligned} &\partial_u\xi^u = \alpha, \qquad \partial_u\xi^z = 0, \qquad \partial_u\xi^{\bar{z}} = 0, \\ &\partial_{\bar{z}}\xi^z = 0, \qquad \partial_z\xi^{\bar{z}} = 0, \qquad \partial_z\xi^z + \partial_{\bar{z}}\xi^{\bar{z}} = \alpha, \end{aligned} \tag{3.23}$$

which are solved by

$$\xi^u = \mathcal{T}(z,\bar{z}) + u\alpha, \qquad \xi^z = \mathcal{Y}(z), \qquad \xi^{\bar{z}} = \bar{\mathcal{Y}}(\bar{z}), \tag{3.24}$$

where $\alpha = \frac{1}{2}(\partial\mathcal{Y}(z) + \bar{\partial}\bar{\mathcal{Y}}(\bar{z}))$ and we have adopted the notation $\partial_z \equiv \partial$ and $\bar{\partial} \equiv \partial_{\bar{z}}$. The Killing vector fields (3.24) are precisely the generators of the (extended) BMS algebra (2.16) restricted to the boundary $\mathscr{I}$. Thus, the symmetry group of the boundary corresponds to the (restricted) BMS group.

To make things more apparent, we express these supertranslations and superrotations in terms of Laurent expansions

$$\mathcal{T}(z,\bar{z}) = \sum_{p,q} a_{p,q} z^p \bar{z}^q, \quad \mathcal{Y}(z) = \sum_n b_n z^{n+1}, \quad \bar{\mathcal{Y}}(\bar{z}) = \sum_n \bar{b}_n \bar{z}^{n+1}, \tag{3.25}$$

while rewriting the supertranslations as $M = \mathcal{T}(z,\bar{z})\partial_u = \sum_{p,q} a_{p,q}M_{p,q}$ and superrotations as $L = -\frac{u}{2}\partial\mathcal{Y}(z)\partial_u - \mathcal{Y}(z)\partial = \sum_n b_n L_n$ [46]. This then allows us to directly identify the generators and structure of the BMS algebra[7] [46, 51]

$$\begin{aligned} L_n &= -\left(z^{n+1}\partial_z + (n+1)z^n\frac{u}{2}\partial_u\right), \\ \bar{L}_n &= -\left(\bar{z}^{n+1}\partial_{\bar{z}} + (n+1)\bar{z}^n\frac{u}{2}\partial_u\right), \\ M_{p,q} &= z^p\bar{z}^q\partial_u. \end{aligned} \tag{3.26}$$

[6]The Lie-derivative of a tensor is given by $\mathscr{L}_X T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} = X^\lambda\partial_\lambda T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\nu_q} - \sum_{i=1}^p T^{\mu_1\ldots\lambda\ldots\mu_p}_{\nu_1\ldots\nu_q}\partial_\lambda X^{\mu_i} + \sum_{j=1}^q T^{\mu_1\ldots\mu_p}_{\nu_1\ldots\lambda\ldots\nu_q}\partial_{\nu_j}X^\lambda$, for more details, refer to Appendix B of [64].

[7]Both the use of a different Laurent expansion for $\mathcal{Y}(z)$ and the minus in the generator L_n are a matter of convention and are chosen to be consistent with the literature.

These generators coincide with those of the infinite-dimensional extension of the conformal Carroll algebra (3.17), and obey the same commutators (3.19). The equivalence between the bulk and boundary symmetries was formally established in [32] and can be stated as

$$\mathfrak{bms}_{D+1} \cong \mathfrak{CCarr}_D. \tag{3.27}$$

This is one of the cornerstones in the construction of flat space holography, demonstrating a correspondence between the symmetries in the $(D+1)$-dimensional bulk spacetime and the theories living on the D-dimensional boundary.

We can construct the notion of a Carrollian primary field by labelling Carroll conformal fields Φ living on the boundary $\mathscr{I}$ by their conformal weights under L_0 and $\bar{L}_0$ [29]:

$$[L_0, \Phi(0)] = h\Phi(0), \qquad [\bar{L}_0, \Phi(0)] = \bar{h}\Phi(0). \tag{3.28}$$

Analogous to 2d CFTs, the primary conditions take the form [47, 51]

$$[L_n, \Phi(0)] = 0, \quad [\bar{L}_n, \Phi(0)] = 0, \ \forall n > 0, \quad [M_{r,s}, \Phi(0)] = 0, \ \forall r, s > 0w, \tag{3.29}$$

where the final condition is an additional requirement on these fields and is not present in two-dimensional CFT models. These primary conditions induce the following transformation rules at an arbitrary spacetime point

$$\begin{aligned} \delta_{L_n}\phi(u,z,\bar{z}) &= [L_n, \phi(u,z,\bar{z})] = [L_n, \ e^{-\Lambda}\phi(0,0,0)e^{\Lambda}] \\ &= e^{-\Lambda}[e^{\Lambda}L_n e^{-\Lambda}, \ \phi(\mathbf{0})]e^{\Lambda}, \end{aligned} \tag{3.30}$$

with $\Lambda = -zL_{-1} - \bar{z}\bar{L}_{-1} + uM_{0,0}$. Next we use the Baker-Campbell-Hausdorff identity

$$e^X Y e^{-X} = \sum_{k=0}^{\infty} \frac{1}{k!} \underbrace{[X[X, \ldots [X,Y] \ldots]]}_{k \text{ times}}, \tag{3.31}$$

where the zeroth summand is just Y. Plugging this in and using (3.19) gives

$$e^{\Lambda}L_n e^{-\Lambda} = \sum_{k=0}^{n+1} \frac{z^k}{k!} \frac{(n+1)!}{(n+1-k)!} L_{n-k} + \sum_{k=1}^{n+1} z^{k-1} u \frac{\frac{1}{2}k}{k!} \frac{(n+1)!}{(n+1-k)!} M_{n-(k-1),0}. \tag{3.32}$$

These sums terminate at finite k due to $[L_{-1}, L_{-1}] = 0 = [L_{-1}, M_{0,0}]$ and primary conditions (3.29). The only terms in (3.32) that will contribute to (3.30) are L_0, L_{-1} and $M_{0,0}$

$$[e^{\Lambda}L_n e^{-\Lambda}, \phi(\mathbf{0})] = (n+1)z^n[L_0, \phi(\mathbf{0})] + z^{n+1}[L_{-1}, \phi(\mathbf{0})] + (n+1)z^n \frac{u}{2}[M_{0,0}, \phi(\mathbf{0}]. \tag{3.33}$$

Using $[L_0, \phi(\mathbf{0})] = h\phi(\mathbf{0})$ and the fact that L_{-1} and $M_{0,0}$ are differential operators, we obtain the transformation rule at an arbitrary spacetime point [29]

$$\delta_{L_n}\phi(u,z,\bar{z}) = z^{n+1}\partial_z\phi(u,z,\bar{z}) + (n+1)z^n\left(\frac{u}{2}\partial_u + h\right)\phi(u,z,\bar{z}). \tag{3.34}$$

Similar calculations can be performed for $M_{p,q}$ and $\bar{L}_n$, and after combining all these transformations, we find that a Carroll primary field $\Phi(u, z, \bar{z})$ will have transformation laws of the type

$$\begin{aligned} \delta\Phi &= \left[\mathcal{T}(z,\bar{z})\partial_u + \mathcal{Y}(z)\partial + \bar{\mathcal{Y}}(\bar{z})\bar{\partial} + \partial\mathcal{Y}\left(\frac{u}{2}\partial_u + h\right) + \bar{\partial}\bar{\mathcal{Y}}\left(\frac{u}{2}\partial_u + \bar{h}\right)\right]\Phi \\ &= \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + h(\partial\mathcal{Y}) + \bar{h}(\bar{\partial}\bar{\mathcal{Y}})\right)\Phi, \end{aligned} \tag{3.35}$$

where $f = \mathcal{T}(z, \bar{z}) + \frac{u}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})$, and we again use the notation $\partial_z \equiv \partial$ and $\bar{\partial} \equiv \partial_{\bar{z}}$.

4 BMS invariant field theories

Having established the foundational framework required for flat space holography in the previous chapters, we will now investigate BMS invariant field theories. We begin by reviewing the emerging holographic dictionary that connects bulk fields to Carrollian primary fields on the boundary. Using this dictionary, we will re-evaluate the previously constructed electric and magnetic Carroll theories and determine whether they serve as a promising candidate for holography. Following this, we introduce a method for constructing new BMS-invariant field theories and also check their viability as holographic duals. The chapter concludes with a summary of the theories presented.

4.1 Holographic dictionary

One of the main obstacles in the construction of a flat space holographic framework is relating the bulk fields to conformal Carroll primaries on the boundary. Such a holographic dictionary exists in the successful AdS-CFT correspondence, but this problem is not completely resolved in the asymptotically flat case. A proposal has been made by [30, 47, 58] that starting with relativistic massless spin-s fields in the bulk Φ_{bulk} and pulling them back to the boundary

$$\Phi(u, z, \bar{z}) = \lim_{r \to \pm\infty} r^{1-s} \; \Phi_{\text{bulk}}(u, r, z, \bar{z}), \tag{4.1}$$

results in Carrollian conformal fields Φ on $\mathscr{I}^{\pm}$ of conformal dimension

$$\Delta = \frac{d-1}{2}, \tag{4.2}$$

where d denotes the number of spatial dimensions of the spacetime. The conformal dimension Δ is the eigenvalue of the dilation operator $D = i(L_0 + \bar{L}_0)$, and from the commutation relations

$$[D, \Phi(0)] = i\Delta\Phi(0) \quad \text{and} \quad [J_{ij}, \Phi(0)] = \Sigma_{ij}\Phi(0), \tag{4.3}$$

together with (3.28) we find that for a primary field Φ the conformal dimension is given by the sum of the conformal weights $\Delta = h + \bar{h}$. Similarly, the helicity Σ is identified with the eigenvalue of the rotation operator $J = L_0 - \bar{L}_0$, leading to $\Sigma = h - \bar{h}$. These relations can be combined to express the conformal weights in terms of the conformal dimension and helicity

$$(h, \bar{h}) = \frac{1}{2}(\Delta + \Sigma, \Delta - \Sigma). \tag{4.4}$$

In the case of four-dimensional asymptotically flat spacetimes, this gives an explicit correspondence between relativistic massless fields in the bulk and Carrollian primary fields at the boundary with conformal weights

$$(h, \bar{h}) = \frac{1}{2}\left(1 + \Sigma, 1 - \Sigma\right). \tag{4.5}$$

Following the dictionary, a massless scalar field $\phi_{\text{bulk}}(u,r,z,\bar{z})$ in the bulk, it is dual to a scalar field $\phi(u,z,\bar{z})$, which transforms as a Carroll primary under BMS transformations (3.35) as

$$\delta\phi = \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{1}{2}(\partial\mathcal{Y}) + \frac{1}{2}(\bar{\partial}\bar{\mathcal{Y}})\right)\phi, \tag{4.6}$$

where $f = \mathcal{T}(z,\bar{z}) + \frac{u}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})$.

For higher spin particles, e.g. the spin-1 photon field $A_\mu(u,r,z,\bar{z})$, we can use the equations of motion (the source-free Maxwell equations $\partial^\mu F_{\mu\nu} = 0$) together with the harmonic gauge ($\nabla^\mu A_\mu = 0$) to obtain

$$\Box A_\mu = 0. \tag{4.7}$$

Then using an expansion in r

$$A_\mu(u,r,z,\bar{z}) = \sum_{n=0}^{\infty} \frac{A_\mu^{(n)}(u,z,\bar{z})}{r^n}, \tag{4.8}$$

where the coefficients depend only on the coordinates $(u,z,\bar{z})$ of the boundary $\mathscr{I}$, we find that the asymptotic behaviour of the spin-1 field $A_\mu(u,r,z,\bar{z})$ is given by the leading fall-offs [14, 65]

$$A_u \sim \mathcal{O}(1/r), \quad A_r \sim \mathcal{O}(1/r^2), \quad A_z \sim \mathcal{O}(1). \tag{4.9}$$

Since A_μ scales as r^0 in the limit (4.1), the free data for the spin-1 gauge field on the boundary are given by $A_z^{(0)}(u,z,\bar{z})$ and its complex conjugate $A_{\bar{z}}^{(0)}(u,z,\bar{z})$. Following the holographic dictionary (4.5), these conformal Carroll fields — denoted by $A \equiv A_z^{(0)}$ and $\bar{A} \equiv A_{\bar{z}}^{(0)}$ — transform as Carroll primaries under BMS transformations, with conformal weights (1,0) and (0,1), respectively.

Correspondingly, for spin-2 particles, e.g. the graviton $h_{\mu\nu}$ defined via $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$, we can use the linearised Einstein equations ($G_{\mu\nu} = 0$) together with the harmonic gauge ($\nabla^\mu h_{\mu\nu} - \frac{1}{2}\nabla_\nu h^\mu_\mu = 0$) to find

$$\Box h_{\mu\nu} = 0. \tag{4.10}$$

Applying a similar expansion as before, we find the asymptotic behaviour [14, 65]

$$\begin{aligned} &h_{uu} \sim \mathcal{O}(1/r), \quad h_{uz} \sim \mathcal{O}(1), \quad h_{ur} \sim \mathcal{O}(1/r^2), \quad h_{rr} \sim \mathcal{O}(1/r^3), \\ &h_{rz} \sim \mathcal{O}(1/r), \quad h_{z\bar{z}} \sim \mathcal{O}(1), \quad h_{zz} \sim \mathcal{O}(r), \quad h_{\bar{z}\bar{z}} \sim \mathcal{O}(r). \end{aligned} \tag{4.11}$$

In the case of a spin-2 particle, $h_{\mu\nu}$ scales as r^{-1} in the limit (4.1) and the free data is given by $h_{zz}^{(-1)}(u,z,\bar{z})$ and its complex conjugate $h_{\bar{z}\bar{z}}^{(-1)}(u,z,\bar{z})$. We denote these fields by $\sigma \equiv h_{zz}^{(-1)}$ and $\bar{\sigma} \equiv h_{\bar{z}\bar{z}}^{(-1)}$. According to (4.5), they transform under BMS transformations as conformal Carroll fields with weights $(\frac{3}{2}, -\frac{1}{2})$ and $(-\frac{1}{2}, \frac{3}{2})$, respectively.

Here we need to make an important observation. In Chapter 2, we saw that in the metric of asymptotically flat spacetimes (2.8), the asymptotic shear was defined as

$$C_{zz} = \lim_{r\to\infty} \frac{1}{r} h_{zz}. \tag{4.12}$$

Under BMS transformations (2.16), the asymptotic shear transforms as

$$\delta C_{zz} = \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{2}\partial\mathcal{Y} - \frac{1}{2}\bar{\partial}\bar{\mathcal{Y}} \right) C_{zz} - 2\partial^2\mathcal{T} - u\partial^3\mathcal{Y}, \tag{4.13}$$

and behaves as a quasi-primary field[8]. Therefore, the conformal Carroll field σ, with weights $(\frac{3}{2}, -\frac{1}{2})$, fails to capture the full content of the graviton. We will revisit this issue in Chapter 5. For the time being, we will use σ to describe spin-2 boundary fields.

4.2 Constructing BMS invariant field theories

With the holographic dictionary at our disposal, we must look at the previously mentioned Carroll field theories and see which of them makes a good candidate for holography. This has been done in great detail in [46], and we will review their work first. Additionally, we examine a separate class of first-order theories that are BMS invariant but lack a relativistic origin.

To construct BMS invariant field theories, we must ensure that the action

$$S = \int_{\mathscr{I}+} dudzd\bar{z}\ \mathcal{L}, \tag{4.14}$$

is invariant under BMS transformations of its constituents. Flat spacetime features two null boundaries $\mathscr{I}^-$ and $\mathscr{I}^+$, as can be seen in Figure 1, and the question of how to relate these boundaries will be addressed in the next chapter. For now, we focus on defining our actions solely on the boundary $\mathscr{I}^+$[9].

Because of the additional factor of one-half appearing in the superrotations (3.17), the boundary coordinates $x^\mu = (u, z, \bar{z})$ transform differently under the action of conformal generators L_0 and $\bar{L}_0$ [46]

$$x^\mu \to x'^\mu = e^{L_0} x^\mu = (1 + L_0)x^\mu. \tag{4.15}$$

This results, for variations $\delta \equiv \delta_{L_0}$ and $\bar{\delta} \equiv \delta_{\bar{L}_0}$, in

$$\delta u = L_0 u = -\frac{1}{2}u \quad \text{and} \quad \bar{\delta} u = \bar{L}_0 u = -\frac{1}{2}u, \tag{4.16}$$

[8]'Quasi' refers to the fact that under the global BMS subalgebra — i.e. the Poincaré group — C_{zz} transforms as a primary field. In this restricted case, the supertranslation functions $\mathcal{T}(z, \bar{z})$ are at most linear in z and $\bar{z}$, and the superrotation vector fields $\mathcal{Y}(z)$ and $\bar{\mathcal{Y}}(\bar{z})$ are at most quadratic, causing the non-primary contributions in (4.13) to vanisch (see e.g. [30]).

[9]For now, we will also make the assumption that the fields fall off at the boundaries $\mathscr{I}^+_\pm$, but we will come back to this in Section 4.3.

while

$$\begin{cases} \delta z = -1 \\ \bar{\delta} z = 0, \end{cases} \qquad \text{and} \qquad \begin{cases} \delta \bar{z} = 0 \\ \bar{\delta} \bar{z} = -1. \end{cases} \tag{4.17}$$

Therefore, although each of the boundary coordinates transforms with conformal dimension $\Delta = h + \bar{h} = -1$

$$u : \left(-\frac{1}{2}, -\frac{1}{2}\right), \quad z : (-1, 0), \quad \bar{z} : (0, -1), \tag{4.18}$$

ensuring scale invariance of the action, countering the integrand $dudzd\bar{z}$, requires constructing Lagrangians of conformal weights $(\frac{3}{2}, \frac{3}{2})$.

4.2.1 Electric scalar theory

In the construction of BMS invariant field theories, our starting point will be the Carrollian theories that we have already reviewed. Starting with real electric scalar theory in three dimensions, as considered in [46], which is given by

$$S = \int dudzd\bar{z} \left(\frac{1}{2}\dot{\phi}^2 - V(\phi)\right). \tag{4.19}$$

The kinetic term is invariant under (4.6) for conformal weights

$$h = \bar{h} = \frac{1}{4}. \tag{4.20}$$

With these conformal weights and scaling dimension $\Delta = \frac{1}{2}$, the field ϕ does not meet the holographic requirement (4.2) for a scalar field $(h, \bar{h}) = (\frac{1}{2}, \frac{1}{2})$, suggesting it is not a suitable candidate for holography.

For the potential to be invariant under the BMS transformation (4.6), we find that it must be homogeneous of degree 6,

$$\phi \frac{\partial V}{\partial \phi} = 6V, \tag{4.21}$$

i.e., $V(\phi) = g\phi^6$.

The time-ordered correlation function is computed to be [40]

$$\begin{aligned} \langle 0|T\phi(u, \vec{z})\phi(u', \vec{z}'|0\rangle &= -i \int_{-\infty}^{\infty} \frac{d\omega d^2\vec{p}}{(2\pi)^3} \frac{e^{-i\omega(u-u')+i\vec{p}\cdot(\vec{z}-\vec{z}')}}{-\omega^2 - i\epsilon} \\ &= -i\delta^{(2)}(\vec{z} - \vec{z}') \int \frac{d\omega}{2\pi} \frac{e^{-i\omega(u-u')}}{-\omega^2 - i\epsilon}, \end{aligned} \tag{4.22}$$

where we denoted $\vec{z} = (z, \bar{z})$.

4.2.2 Magnetic scalar theory

The next class of Carroll field theories consists of magnetic theories, which can be written in the general form

$$S = \int dudzd\bar{z} \left(\chi\dot{\Phi} + \mathcal{L}(\Phi)\right), \tag{4.23}$$

where Φ is an arbitrary field and $\mathcal{L}$ is a 'Euclidean' Lagrangian field theory, therefore only depending on Φ and its spatial derivatives $\partial_z\Phi$ and $\partial_{\bar{z}}\Phi$. Starting with the easiest case, $\mathcal{L} = 0$, we find that the Lagrangian is invariant under (3.35) together with the transformation

$$\delta\chi = f\dot{\chi} + \mathcal{Y}\partial\chi + \bar{\mathcal{Y}}\bar{\partial}\chi + (1 - h_\Phi)\partial\mathcal{Y}\chi + (1 - \bar{h}_\Phi)\bar{\partial}\bar{\mathcal{Y}}\chi, \tag{4.24}$$

where again $f = \mathcal{T}(z, \bar{z}) + \frac{u}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})$. In this case, χ transforms a Carroll primary with weights $(1-h_\Phi, 1-\bar{h}_\Phi)$ and its scaling dimension is $\Delta_\chi = 2-\Delta_\Phi$. Following the dictionary (4.2), setting $\Delta_\Phi = h_\Phi + \bar{h}_\Phi = 1$ leads directly to $\Delta_\chi = 1$. This suggests that this model is well-suited for holography, as the correlation functions of the fields Φ, χ meet the holographic requirements. Moreover, when assigning various values of spin to Φ, the resulting weights of χ naturally match those of the complex conjugate field. The main drawback of this model, however, is the absence of spatial derivatives, which makes the relativistic origin unclear and renders the theory ultralocal, thereby exhibiting the typical behaviour of electric Carroll theories.

The time-ordered correlation function is computed to be [40]

$$\begin{aligned} \langle 0|T\chi(u, \vec{z})\Phi(u', \vec{z}')|0\rangle &= -i\int_{-\infty}^{\infty} \frac{d\omega d^d\vec{p}}{(2\pi)^{d+1}} \frac{e^{-i\omega(u-u')+i\vec{p}\cdot(\vec{z}-\vec{z}')}}{-i\omega} \\ &= \frac{-i}{2}\delta^{(2)}(\vec{z} - \vec{z}')\text{sign}(u - u'). \end{aligned} \tag{4.25}$$

In the last step, we used an identity from complex integration. It is important to note that the integral diverges because $\omega = 0$ is a singularity. Therefore, the final result is only meaningful when understood as a distribution, i.e., when integrated against a test function — similar to a delta function. For more context about this and complex integrals, see Appendix A.

When applied to scalar theory, as studied in [46], the above example gives the Lagrangian

$$S = \int dudzd\bar{z}\ \chi\dot{\phi}, \tag{4.26}$$

where the scalar field ϕ has weights $(\frac{1}{2}, \frac{1}{2})$. The theory remains BMS invariant upon adding the interaction potential $V(\phi) = g\phi^3$.

If we want to consider the case of the magnetic theory originating from a relativistic real scalar theory [39], similar to (3.11), we get

$$S = \int dudzd\bar{z} \left(\chi\dot{\phi} - 2(\partial\phi)(\bar{\partial}\phi)\right), \tag{4.27}$$

where we've used complex spatial coordinates $z = x + iy$ and $\partial = \frac{1}{2}(\partial_x - i\partial_y)$. This action is not invariant under the 'standard' Carroll primary transformations due to the spatial derivatives, but can be made invariant under

$$\begin{aligned} \delta\phi &= \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{1}{4}(\partial\mathcal{Y}) + \frac{1}{4}(\bar{\partial}\bar{\mathcal{Y}})\right)\phi, \\ \delta\chi &= \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \frac{3}{4}(\partial\mathcal{Y}) + \frac{3}{4}(\bar{\partial}\bar{\mathcal{Y}})\right)\chi + 2\partial f\bar{\partial}\phi + 2\bar{\partial}f\partial\phi. \end{aligned} \tag{4.28}$$

In this case, the field ϕ is a Carroll primary of weights $(\frac{1}{4}, \frac{1}{4})$, whereas the field χ does not transform as a Carroll primary but may instead realise a different representation of the BMS algebra (3.19). This can be verified by successively applying the symmetry transformations and checking whether their commutators still close. The supertranslations and superrotations of χ are found to satisfy the BMS algebra (3.19)

$$[\delta_{\mathcal{T}_1}, \delta_{\mathcal{T}_2}]\chi = 0, \qquad [\delta_{\mathcal{Y}_1}, \delta_{\mathcal{Y}_2}]\chi = \delta_{\mathcal{Y}_3}, \tag{4.29}$$

with $\mathcal{Y}_3 = \mathcal{Y}_2\partial\mathcal{Y}_1 - \partial\mathcal{Y}_2\mathcal{Y}_1$. We can make the BMS_4 commutation relations more manifest, by writing $\mathcal{Y}_1 = z^{m+1}$ and $\mathcal{Y}_2 = z^{n+1}$ such that $\mathcal{Y}_3 = (m-n)z^{m+n+1}$. However, when combining the supertranslations and superrotations, we obtain

$$[\delta_{\mathcal{T}}, \delta_{\mathcal{Y}}]\chi = \delta_{\tilde{\mathcal{T}}}\chi - \frac{1}{4}\bar{\partial}\mathcal{T}\partial^2\mathcal{Y}\phi, \tag{4.30}$$

with $\tilde{\mathcal{T}} = \mathcal{Y}\partial\mathcal{T} - \frac{1}{2}\mathcal{T}\partial\mathcal{Y}$. The first term once again corresponds to a BMS transformation. Setting $\mathcal{T} = z^p\bar{z}^q$ and $\mathcal{Y} = z^{n+1}$, leads to $\tilde{\mathcal{T}} = \left(p - \frac{n+1}{2}\right)z^{p+n}\bar{z}^q$, which matches the expected structure from the BMS algebra (3.19). The presence of the additional term $-\frac{1}{4}\bar{\partial}\mathcal{T}\partial^2\mathcal{Y}\phi$ suggests that there exists an extra symmetry of the form $\delta_c\chi = c(z,\bar{z})\phi$, which is indeed true in general for magnetic theories. This symmetry can be generalised further to $\delta_c\chi = c(z,\bar{z})\frac{\partial F}{\partial\phi}$ for arbitrary function F. However, incorporating this additional symmetry does not resolve the core issue: the magnetic scalar field with derivatives is not BMS_4 invariant. This result is surprising, given that its three-dimensional analogue was shown to be BMS_3 invariant [66]. Moreover, in [46] the authors demonstrate that the presence of an additional symmetry δ_c generates an extra set of supertranslations which are associated to spin currents. However, for the purpose of this thesis, we will not go in that direction.

We can compute the propagator for this theory by writing the Lagrangian as

$$\mathcal{L} = \Phi^T\mathcal{O}\Phi, \tag{4.31}$$

with

$$\mathcal{O} = \begin{pmatrix} 2\partial\bar{\partial} & -\frac{1}{2}\partial_u \\ \frac{1}{2}\partial_u & 0 \end{pmatrix} \quad \text{and} \quad \Phi = \begin{pmatrix} \phi \\ \chi \end{pmatrix}. \tag{4.32}$$

By computing the inverse operator in momentum space

$$\mathcal{O}^{-1}(\omega,\vec{p}) = \begin{pmatrix} 0 & -\frac{2i}{\omega} \\ \frac{2i}{\omega} & \frac{8|\vec{p}|^2}{\omega^2} \end{pmatrix}, \tag{4.33}$$

where we identify $\vec{p} = (p,\bar{p})$ as the momenta conjugate to $\vec{z} = (z,\bar{z})$, we obtain the corresponding correlation functions

$$\langle \phi(u,\vec{z})\phi(u',\vec{z}')\rangle = -i\mathcal{O}^{-1}_{11} = 0, \tag{4.34}$$

$$\langle \chi(u,\vec{z})\phi(u',\vec{z}')\rangle = -i\mathcal{O}^{-1}_{21} = 2\delta^{(2)}(\vec{z}-\vec{z}')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(u-u')}}{\omega}, \tag{4.35}$$

$$\langle \chi(u,\vec{z})\chi(u',\vec{z}')\rangle = -i\mathcal{O}^{-1}_{22} = -8i\int_{-\infty}^{\infty}\frac{d\omega d^2\vec{p}}{(2\pi)^3}\frac{|\vec{p}|^2 e^{-i\omega(u-u')+i\vec{p}\cdot(\vec{z}-\vec{z}')}}{\omega^2}. \tag{4.36}$$

Although we did not include a kinetic term for χ, the condition $\dot{\phi} = 0$ implies that there is no propagation between two ϕ fields. Instead, propagation occurs between two χ fields and $\langle\chi\phi\rangle$ reproduces (4.25) up to a constant. Moreover, due to the presence of spatial derivatives in the Lagrangian, $\langle\chi\chi\rangle$ does not contain a spatial delta function. This feature will become important later.

Another interesting feature of the Lagrangian (4.27) is that the scalar field ϕ transforms as a Carroll primary with weights $(\frac{1}{4},\frac{1}{4})$, identical to those in the electric case (4.19). Although not much is known about combining electric and magnetic Carroll field theories, this example suggests that they may be directly added. Care must be taken, however, as in the electric theory, the scalar fields have been rescaled $\phi \to c\phi$. To reconcile the units, we can introduce an effective speed of light parameter λ to obtain

$$S = \int dudzd\bar{z}\left(\frac{\lambda^2}{2}\dot{\phi}^2 + \chi\dot{\phi} - 2(\partial\phi)(\bar{\partial}\phi) - V(\phi)\right), \tag{4.37}$$

where $V(\phi) = g\phi^6$. Such a matching between electric and magnetic Carroll theories is also possible in $(1+1)$-dimensions, where both theories contain Carroll primaries ϕ with conformal weight $h = 0$.

4.2.3 First order theories

Following the review of electric and magnetic Carroll field theories, we will now look at another class of BMS invariant field theories. These theories do not follow from the $c \to 0$ limit of some relativistic theory, but are constructed purely from symmetry considerations.

As a starting point in the construction of more general BMS invariant field theories, the most convenient method is to work with Carroll primaries Φ that transform according to (3.35)

$$\delta\Phi = \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + h(\partial\mathcal{Y}) + \bar{h}(\bar{\partial}\bar{\mathcal{Y}})\right)\Phi, \tag{4.38}$$

where $f = \mathcal{T}(z,\bar{z}) + \frac{u}{2}(\partial\mathcal{Y} + \bar{\partial}\bar{\mathcal{Y}})$. It is important to note that, although the notation is implying it, $\bar{h}$ is **not** the complex conjugate of h. Supertranslations are given by a real function $\mathcal{T}(z,\bar{z})$ while super rotations are given by complex holomorphic functions $\mathcal{Y}(z)$ and $\bar{\mathcal{Y}}(\bar{z})$. Therefore, a real Carroll primary must have weights $(\frac{1}{2},\frac{1}{2})$ and can only describe massless scalar fields in the bulk. From a holographic perspective, this makes sense because any massless bulk particle with non-zero spin has two degrees of freedom associated with its helicity modes, and thus cannot be described by a single real scalar field.

Given a complex primary field Φ with weights $(h,\bar{h})$, the complex conjugate of this primary transforms as

$$\delta\bar{\Phi} = \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \bar{h}(\partial\mathcal{Y}) + h(\bar{\partial}\bar{\mathcal{Y}})\right)\bar{\Phi}, \tag{4.39}$$

with weights $(\bar{h},h)$ due to the restriction that $\overline{(\delta\Phi)} = \delta\bar{\Phi}$ which implies that $\overline{(\mathcal{Y})} = \bar{\mathcal{Y}}$. This matches the behaviour of the holographic dictionary (4.5) where fields of opposite helicity have opposite conformal weights.

The product of two Carroll primaries $\Phi : (h_1,\bar{h}_1)$ and $\chi : (h_2,\bar{h}_2)$ is still a Carroll primary

$$\begin{aligned}\delta(\Phi\chi) &= (\delta\Phi)\chi + \Phi(\delta\chi)\\ &= \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + (h_1+h_2)(\partial\mathcal{Y}) + (\bar{h}_1+\bar{h}_2)(\bar{\partial}\bar{\mathcal{Y}})\right)(\Phi\chi),\end{aligned} \tag{4.40}$$

with weights $(h_1+h_2,\bar{h}_1+\bar{h}_2)$. Additionally, the time derivative of a Carroll primary $\partial_u\Phi$ is a Carroll primary

$$\delta(\partial_u\Phi) = \left(f\partial_u + \mathcal{Y}\partial + \bar{\mathcal{Y}}\bar{\partial} + \left(h+\frac{1}{2}\right)(\partial\mathcal{Y}) + \left(\bar{h}+\frac{1}{2}\right)(\bar{\partial}\bar{\mathcal{Y}})\right)\partial_u\Phi, \tag{4.41}$$

with weights $(h+\frac{1}{2},\bar{h}+\frac{1}{2})$. One can even define an inverse time derivative operator [52]

$$\partial_u^{-k}\Phi(u,z,\bar{z}) = \frac{1}{k!}\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}\Phi(u',z,\bar{z}), \tag{4.42}$$

such that $\partial_u\partial_u^{-k}\Phi = \partial_u^{-k}\partial_u\Phi = \partial_u^{-(k-1)}\Phi$, and it transforms as a Carroll primary of weights $(h-\frac{k}{2},\bar{h}-\frac{k}{2})$, provided $\lim_{u\to-\infty}\Phi \sim \mathcal{O}(u^{-k})$[10]. Since both the supertranslations $\mathcal{T}(z,\bar{z})$ and the superrotations $\mathcal{Y}(z),\bar{\mathcal{Y}}(\bar{z})$ are dependent on spatial coordinates, derivatives of Carroll primaries with respect to z or $\bar{z}$ do not remain Carroll primaries. This was also observed in (4.27), where the transformation rules for χ were modified to (4.28) to account for the additional spatial derivative terms coming from ϕ.

From these properties, combined with the knowledge that the integrand of the action (4.14) has conformal weights $(-\frac{3}{2},-\frac{3}{2})$, we can simply construct BMS invariant actions

[10] See Appendix B for more details.

by products of Carroll primaries whose conformal weights add up to $(\frac{3}{2}, \frac{3}{2})$. Lagrangians constructed in this manner transform as a total derivative under BMS transformations

$$\delta\mathcal{L} = \partial_u(f\mathcal{L}) + \partial(b\mathcal{L}) + \bar{\partial}(\bar{b}\mathcal{L}), \tag{4.43}$$

thereby leaving the action invariant. Equipped with this insight and guided by the constraints of the holographic dictionary, we are now able to construct a new class of BMS invariant field theories.

Starting with the construction of a kinetic term, the conformal dimension of two primary fields that satisfy the dictionary (4.2) is $\Delta = 2$. This leaves room for only one Δ-worth of time derivatives, implying that any BMS-invariant kinetic term must be first order. Considering a real Carroll primary field $\phi : \left(\frac{1}{2}, \frac{1}{2}\right)$, we immediately run into trouble as a kinetic term $\mathcal{L} = \phi(\partial_u\phi)$ can be written as a total derivative. It is therefore not possible to construct a non-trivial BMS-invariant Lagrangian using only a single field. Instead, we must consider multiple real scalar fields ϕ^i, in which case we get the Lagrangian

$$\mathcal{L} = \frac{1}{2} a_{ij}\phi^i\dot{\phi}^j, \tag{4.44}$$

where a_{ij} is constant and antisymmetric [46]. This implies that the number of scalar fields must be even and $(h_i, \bar{h}_i) = (\frac{1}{2}, \frac{1}{2})$. The equations of motion read

$$a_{ij}\dot{\phi}^j = 0, \tag{4.45}$$

and if we take a to be invertible, the solution is that all scalar fields are time independent. This u-independence in the equation of motion reflects the presence of soft particles, which correspond to a quanta of zero energy and frequency. This follows from the inverse Fourier transform[11]

$$\tilde{\phi}(\omega, z, \bar{z}) = \int du\ e^{i\omega u}\phi(u, z, \bar{z}), \tag{4.46}$$

where fields $\phi(u, z, \bar{z})$ that are constant or polynomial in u give $\tilde{\phi}(\omega, z, \bar{z}) \propto \delta(w)$ or derivatives thereof. This implies that the frequency behaviour of fields $\phi(u, z, \bar{z})$ corresponds to vanishing ω, indicating particles with zero energy.

Soft particles play an important role in flat space holography, as soft theorems — which govern the emission of low-energy (soft) gravitons and photons — have been shown to be equivalent to the Ward identities of the asymptotic BMS symmetries [18, 19]. This equivalence suggests that soft particles encode information associated with these symmetries and provide insight into the long-range structure of gravitational interactions in the bulk. It is expected that the inclusion of interactions will resolve this issue and enable the description of hard particles.

[11] We adopt the standard physics convention of including a minus sign in the exponential when Fourier transforming with respect to the time coordinate.

This theory leads to the following expression for the propagator

$$\begin{aligned}\langle\phi^i(u,\vec{z})\phi^j(u',\vec{z}')\rangle &= a^{ij}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\int_{-\infty}^{\infty}d^2\vec{k}\;\frac{e^{-i\omega(u-u')}e^{i\vec{k}(\vec{z}-\vec{z}')}}{\omega}\\ &= \frac{-ia^{ij}}{2}\delta^{(2)}(\vec{z}-\vec{z}')\text{sign}(u-u'),\end{aligned} \tag{4.47}$$

where we denoted $\vec{z}=(z,\bar{z})$.

Adding interactions to this theory will result in the Lagrangian

$$\mathcal{L}=\frac{1}{2}a_{ij}\varphi^i\dot{\varphi}^j-\frac{1}{3}b_{ijk}\phi^i\phi^j\phi^k, \tag{4.48}$$

with constants a and b. The resulting equation of motion are given by

$$a_{ij}\dot{\phi}^j=b_{ijk}\phi^j\phi^k, \tag{4.49}$$

which are coupled non-linear equations and therefore extremely hard to solve.

Considering a complex Carroll primary field $\Phi:(h,\bar{h})$ and its complex conjugate $\bar{\Phi}:(\bar{h},h)$, we can construct a first order Lagrangian

$$\mathcal{L}=i(\Phi\partial_u\bar{\Phi}-\bar{\Phi}\partial_u\Phi). \tag{4.50}$$

Due to the restrictiveness of conformal symmetry, this is the only kinetic term that can be constructed (See [67] for a similar result). One can easily check that the conformal weights must satisfy the holographic dictionary relation $h+\bar{h}=1$. Unlike the real case, this exact match between the symmetry constraint and the dictionary relation enables this Lagrangian to describe fields of arbitrary spin. Calculating the equations of motion from this Lagrangian

$$\partial_u\Phi=0,\qquad \partial_u\bar{\Phi}=0, \tag{4.51}$$

we find, similar to the real case, that it describes soft particles.

Computing the propagator for this theory gives

$$\langle\bar{\Phi}(u,\vec{z})\Phi(u',\vec{z}')\rangle=\delta^{(2)}(\vec{z}-\vec{z}')\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{e^{-i\omega(u-u')}}{2\omega}, \tag{4.52}$$

which closely resembles the real scalar first-order theory.

In the case of a complex scalar field $\phi:(\frac{1}{2},\frac{1}{2})$ we can add interactions of the type

$$\mathcal{L}=i(\phi\partial_u\bar{\phi}-\bar{\phi}\partial_u\phi)+g\phi^3+g\bar{\phi}^3+\lambda\phi\bar{\phi}^2+\lambda\bar{\phi}\phi^2. \tag{4.53}$$

It is easy to check that this Lagrangian is invariant under (4.6). Similar to the real scalar field example, this model has coupled nonlinear equations of motion

$$\begin{aligned} \partial_u \phi &= \frac{3g}{2i}\bar{\phi}^2 + \frac{\lambda}{2i}\left(2\phi\bar{\phi} + \bar{\phi}^2\right), \\ \partial_u \bar{\phi} &= -\frac{3g}{2i}\phi^2 - \frac{\lambda}{2i}\left(2\phi\bar{\phi} + \phi^2\right), \end{aligned} \tag{4.54}$$

which are hard to solve in general. From the equation $\partial_u \phi = -\partial_u \bar{\phi}$, it follows that $\phi + \bar{\phi} = C(z, \bar{z})$, and for $\lambda = 0$ we also obtain $\phi^3 + \bar{\phi}^3 = K(z, \bar{z})$.

4.2.4 Higher spin models

So far, we have focused exclusively on scalar fields. However, both the magnetic Lagrangian and, in particular, the complex kinetic term given in (4.50) are capable of describing particles with higher spin. In both cases, the constraint imposed by conformal symmetry aligns precisely with the holographic condition. In this section, we explore possible higher-order spin models, analyse the interactions permitted by conformal symmetry, and, where applicable, consider potential theoretical origins for these theories.

Interacting Spin-1 field theories

For spin-1 Carroll primary fields A and $\bar{A}$, with weights $(1, 0)$ and $(0, 1)$ respectively , we can construct the BMS-invariant Lagrangian

$$\mathcal{L} = i(A\partial_u \bar{A} - \bar{A}\partial_u A). \tag{4.55}$$

Since this Lagrangian is derived purely from symmetry arguments and does not follow from some relativistic theory, it is worth some time to investigate its possible origin. Its structure is very similar to a 3d Chern-Simons model, which is also first order and has a cubic interaction

$$S = \frac{k}{4\pi} \int_M \mathrm{Tr}\left(A \wedge dA + \frac{2}{3} A \wedge A \wedge A\right), \tag{4.56}$$

for a gauge field A. In the abelian case $A \wedge A = 0$ and the action becomes

$$S = \frac{k}{4\pi} \int_M dud^2z \; \epsilon^{\mu\nu\rho} A_\mu \partial_\nu A_\rho. \tag{4.57}$$

In a gauge $A_u = 0$, there are residual time-independent gauge transformations $\delta A_i = \partial_i \varepsilon$. Defining the complex combinations $A \equiv A_z, \bar{A} \equiv A_{\bar{z}}$ from the remaining components, the action in this gauge takes the form

$$S = \frac{k}{4\pi} \int dud^2z \; i\left(A\partial_u \bar{A} - \bar{A}\partial_u A\right). \tag{4.58}$$

This coincides precisely with Lagrangian (4.55), which was derived by requiring BMS invariance. Moreover, the action admits a gauge symmetry

$$\delta A = \partial\varepsilon, \quad \delta\bar{A} = \bar{\partial}\varepsilon, \tag{4.59}$$

with real, time independent, gauge parameter $\varepsilon(z,\bar{z})$.

We can add interactions to Lagrangian (4.55), resulting in

$$\mathcal{L} = i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + i(A\partial_u\bar{A} - \bar{A}\partial_u A) + gA\bar{A}\phi + gA\bar{A}\bar{\phi}, \tag{4.60}$$

where a kinetic term for the fields $\phi, \bar{\phi}$ has also been included to ensure non-trivial dynamics. This gives equations of motion

$$\begin{cases}\partial_u\phi = \frac{g}{2i}A\bar{A}\\ \partial_u\bar{\phi} = -\frac{g}{2i}A\bar{A},\end{cases} \quad \text{and} \quad \begin{cases}\partial_u A = \frac{g}{2i}A(\phi+\bar{\phi})\\ \partial_u\bar{A} = -\frac{g}{2i}\bar{A}(\phi+\bar{\phi}).\end{cases} \tag{4.61}$$

From these equations, it follows that

$$\partial_u^2\phi = \partial_u^2\bar{\phi} = 0, \tag{4.62}$$

and as a result, $\phi = c(z,\bar{z})u + \phi_0(z,\bar{z})$ again takes the form of a polynomial in u, implying that its frequency behaviour remains soft. Furthermore, from equations of motion for the scalar field $\partial_u\phi = -\partial_u\bar{\phi}$ we find $\phi + \bar{\phi} = C(z,\bar{z})$. Using this relation, we can rewrite the equations of motion for A and $\bar{A}$

$$\partial_u A = \frac{gC}{2i}A \quad \text{and} \quad \partial_u\bar{A} = \frac{gC}{2i}\bar{A}, \tag{4.63}$$

which are solved by

$$\begin{aligned} A &= e^{\alpha u}A_0(z,\bar{z}) \approx A_0(z,\bar{z})\left(1 + \alpha u + \mathcal{O}(g^2)\right), \\ \bar{A} &= e^{-\alpha u}\bar{A}_0(z,\bar{z}) \approx \bar{A}_0(z,\bar{z})\left(1 - \alpha u + \mathcal{O}(g^2)\right). \end{aligned} \tag{4.64}$$

for $\alpha = \frac{gC}{2i}$. Unfortunately, this result is also polynomial in u, showing that coupling a scalar field to a spin-1 field fails to produce non-soft dynamics.

Interacting Spin-2 field theories

Primary fields with spin 2 are represented by $\sigma : (\frac{3}{2}, -\frac{1}{2})$ and $\bar{\sigma} : (-\frac{1}{2}, \frac{3}{2})$, which closely resemble the graviton, apart from the inhomogeneous contribution in (2.17). Unlike in the spin-1 case, the origin of the kinetic term (4.50) is less apparent in the spin-2 case. Coupling spin-2 fields to scalar fields, we obtain the interacting Lagrangian

$$\mathcal{L} = i(\sigma\partial_u\bar{\sigma} - \bar{\sigma}\partial_u\sigma) + i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + g\sigma\bar{\sigma}\phi + g\sigma\bar{\sigma}\bar{\phi}, \tag{4.65}$$

where, similar to the spin-1 example, a kinetic term for the fields $\phi, \bar{\phi}$ has been included to ensure non-trivial dynamics. This Lagrangian has the same structure as the spin-1 case coupled to scalar fields, and it therefore reproduces the same dynamics

$$\begin{cases}\phi(u,z,\bar{z}) = c(z,\bar{z})u + \phi_0(z,\bar{z})\\ \bar{\phi}(u,z,\bar{z}) = -c(z,\bar{z})u + \bar{\phi}_0(z,\bar{z}),\end{cases} \quad \text{and} \quad \begin{cases}\sigma(u,z,\bar{z}) = \sigma_0(z,\bar{z})\,(1+\alpha u)\\ \bar{\sigma}(u,z,\bar{z}) = \bar{\sigma}_0(z,\bar{z})\,(1+\alpha u)\,,\end{cases} \tag{4.66}$$

characterised by purely soft behaviour.

Another interacting spin-2 theory can be made by coupling σ and $\bar{\sigma}$ to the spin-1 gauge fields A and $\bar{A}$, resulting in

$$\mathcal{L} = i(\sigma\partial_u\bar{\sigma} - \bar{\sigma}\partial_u\sigma) + i(A\partial_u\bar{A} - \bar{A}\partial_u A) + \sigma\bar{A}\bar{A} + \bar{\sigma}AA. \tag{4.67}$$

This gives coupled nonlinear equations of motion

$$\begin{cases}\partial_u A = -i\bar{A}\sigma \\ \partial_u\bar{A} = iA\bar{\sigma},\end{cases} \quad \text{and} \quad \begin{cases}\partial_u\sigma = -\frac{i}{2}A^2 \\ \partial_u\bar{\sigma} = \frac{i}{2}\bar{A}^2,\end{cases} \tag{4.68}$$

which lead to differential equations

$$\sigma\partial_u^2\bar{\sigma} = \bar{\sigma}\partial_u^2\sigma \quad \text{and} \quad A\partial_u^2\bar{A} = \bar{A}\partial_u^2 A. \tag{4.69}$$

Although these equations are hard to solve in general, a solution is

$$A(u,z,\bar{z}) = A_0(z,\bar{z})e^{i\alpha u} \quad \text{and} \quad \sigma(u,z,\bar{z}) = \sigma_0(z,\bar{z})e^{2i\alpha u}, \tag{4.70}$$

where $\alpha(z,\bar{z}) \in \mathbb{R}$ and with similar solutions for the complex conjugates. Unfortunately, this solution has been encountered previously and provides a description of soft particle dynamics.

4.3 Actions and boundary terms

In this section, following the work of [46], we reconsider our previous assumption that the fields vanish at the boundary of $\mathscr{I}^+$. While this condition is suitable for scalar fields, it proves too restrictive for higher-spin fields such as the photon and graviton, due to the presence of the so-called memory effect. Previously, we have seen that the gravitational memory effect corresponds to variations in the asymptotic shear at the boundaries of $\mathscr{I}^+$

$$\Delta C_{zz} = \int_{\mathscr{I}^+} du N_{zz} = C_{zz}|_{\mathscr{I}^+_+} - C_{zz}|_{\mathscr{I}^+_-}. \tag{4.71}$$

A similar memory effect exists in the electromagnetic case, where the News tensor is given by $N_z = \partial_u A_z$. The presence of the memory effect indicates that the fields A_z and C_{zz} can not vanish at both $\mathscr{I}^+_\pm$.

In the models whe have developed so far to describe high-order spins, the magnetic lagrangian $\mathcal{L}_1 = \chi\partial_u\Phi$ and the first order Lagrangian $\mathcal{L}_2 = i(\Phi\partial_u\bar{\Phi} - \bar{\Phi}\partial_u\Phi)$, both transform under the transformations (3.35) and (4.24) as a total derivative

$$\delta\mathcal{L} = \partial_u(f\mathcal{L}) + \partial(b\mathcal{L}) + \bar{\partial}(\bar{b}\mathcal{L}). \tag{4.72}$$

To ensure that the action vanishes, we must impose boundary conditions such that $\partial_u\Phi$ is equal at the two boundaries $\mathscr{I}^+_\pm$ but Φ itself is not. This can be realised by the fall-off conditions [46]

$$\begin{aligned}\Phi(u \to +\infty, z, \bar{z}) &= u\Phi_1(z,\bar{z}) + \Phi_{0,+}(z,\bar{z}) + \mathcal{O}\left(u^{-1}\right), \\ \Phi(u \to -\infty, z, \bar{z}) &= u\Phi_1(z,\bar{z}) + \Phi_{0,-}(z,\bar{z}) + \mathcal{O}\left(u^{-1}\right),\end{aligned} \tag{4.73}$$

and similar for the complex conjugate. Analogous to (2.9), we define the news tensor

$$N(u, z, \bar{z}) \equiv \partial_u \Phi = N_{\text{vac}}(z, \bar{z}) + \mathcal{O}(u^{-2}), \quad N_{\text{vac}}(z, \bar{z}) = \Phi_1(z, \bar{z}). \tag{4.74}$$

The leading part of the news tensor, $N_{\text{vac}}(z, \bar{z})$, is independent of a time coordinate u, and therefore does not change from $\mathscr{I}^+_+$ to $\mathscr{I}^+_-$. We therefore denote it by the vacuum part of the news. In both the magnetic and the first-order theory, all boundary terms vanish when setting $N_{\text{vac}} = \bar{N}_{\text{vac}} = 0$, ensuring invariance under BMS transformations. This choice still permits fluctuations in Φ_0, allowing the memory effect to be properly described

$$\int_{\mathscr{I}^+} du\ N(u, z, \bar{z}) = \Phi(u, z, \bar{z})|_{\mathscr{I}^+_+} - \Phi(u, z, \bar{z})|_{\mathscr{I}^+_-} = \Phi_{0,+} - \Phi_{0,-}. \tag{4.75}$$

4.4 Summary of candidate holographic duals

In this chapter, we have applied the current holographic dictionary (4.2) to analyse both the electric and magnetic Carroll scalar theories, and to construct new models based on Carroll primary fields. We conclude with a comprehensive overview of the resulting field theories and give some final remarks.

This is an overview of the theories that have been constructed so far.

1. $\mathcal{L} = \frac{1}{2}\dot{\phi}^2 - g\phi^6$
2. $\mathcal{L} = \chi\dot{\phi} - 2(\partial\phi)(\bar{\partial}\phi) - g\phi^6$
3. $\mathcal{L} = \chi\dot{\phi} - g\phi^3$
4. $\mathcal{L} = \frac{1}{2}a_{ij}\phi^i\dot{\phi} - \frac{1}{3}b_{ijk}\phi^i\phi^j\phi^k$
5. $\mathcal{L} = i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + g\phi^3 + g\bar{\phi}^3 + \lambda\phi\bar{\phi}^2 + \lambda\bar{\phi}\phi^2$
6. $\mathcal{L} = i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + i(A\partial_u\bar{A} - \bar{A}\partial_u A) + A\bar{A}\phi + A\bar{A}\bar{\phi}$
7. $\mathcal{L} = i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + i(\sigma\partial_u\bar{\sigma} - \bar{\sigma}\partial_u\sigma) + \sigma\bar{\sigma}\phi + \sigma\bar{\sigma}\bar{\phi}$
8. $\mathcal{L} = i(\sigma\partial_u\bar{\sigma} - \bar{\sigma}\partial_u\sigma) + i(A\partial_u\bar{A} - \bar{A}\partial_u A) + \sigma\bar{A}\bar{A} + \bar{\sigma}AA$

Among the constructed models, the first two (electric and magnetic scalar theories) failed to satisfy the holographic requirement $\Delta = 1$ for the fields ϕ and χ. In contrast, all remaining models met this constraint, which in turn implied that a consistent holographic model cannot be formulated from a single field alone. Theories 4-8 were constructed from pure symmetry considerations and comprise of all combinations permitted by invariance under the BMS transformations (3.35) for the fields $\phi : (\frac{1}{2}, \frac{1}{2})$, $A : (1, 0)$ and $\sigma : (\frac{3}{2}, -\frac{1}{2})$, along with their complex conjugates.

After analysing the dynamical behaviour of these theories, we found that Lagrangians 2, 3, 5-8 exhibit soft particle dynamics, meaning that their fields were constant or polynomial in u. This is a characteristic feature of magnetic theories, where the Lagrange multiplier χ enforces $\dot{\phi} = 0$, and consequently, the resulting equations of motion render χ polynomial in u. For Lagrangians 1, 4, and 5, the dynamics are governed by non-linear differential equations that proved difficult to solve analytically, leaving it unclear whether these models describe soft or hard particles.

A recurring feature across these models is the absence of spatial derivatives, leading to ultralocal dynamics. In Lagrangian 2, the presence of a second independent field χ made it possible to include spatial derivatives. However, this came at the cost of the theory no longer being invariant under the 'standard' BMS transformations (3.35). Nevertheless, since the scalar field ϕ remains a Carroll primary of weights $(\frac{1}{4}, \frac{1}{4})$, the model may still be relevant for holography. The fact that the scalar fields in both the electric and magnetic cases (Lagrangians 1 and 2) have the same weights opens the possibility of combining the theories, effectively coupling the electric scalar theory to the magnetic scalar theory.

We did not examine the boundary conditions of these models in great detail. However, we observed that spin-1 and spin-2 fields exhibit memory effects, which complicate the analysis by requiring the fields to be non-vanishing at the boundaries $\mathscr{I}^+_\pm$. This complication does not arise for scalar fields, which may vanish at the boundary. As a result, the Lagrangians 1-5 are well-behaved. However, extra care is required when handling interactions involving higher-spin fields.

In conventional quantum field theory, it is essential that the potential $V(\phi)$ is bounded from below. Otherwise, the system can become unstable, as it can roll indefinitely towards states of arbitrarily negative energy, making the vacuum ill-defined. Consequently, potentials such as ϕ^3 or $A\bar{A}\phi$ are, in principle, problematic. A way to address this issue is to consider potentials of the form $V(\phi) = |\phi|^3 = (\phi\bar{\phi})^{\frac{3}{2}}$, which are bounded from below. All fields, regardless of spin, satisfy $\phi\bar{\phi} : (1,1)$, $A\bar{A} : (1,1)$, $\sigma\bar{\sigma} : (1,1)$. This allows for the construction of bounded interaction terms of the form $|\Phi|^3$, as well as mixed interaction terms such as $|\Phi_1|^2|\Phi_2|$ or $|\Phi_1||\Phi_2||\Phi_3|$, valid for Carroll fields Φ of arbitrary spin.

It has long been known that in a quantum theory of gravity, one encounters maximally helicity-violating (MHV) diagrams, in which two gravitons have negative helicity and the rest have positive helicity [68]. Since any theory aspiring to describe quantum gravity must reproduce this feature, we attempted to construct an effective 3-point function $\langle\sigma\sigma\bar{\sigma}\rangle$. Unfortunately, the interactions we found are all BMS invariant with conformal weights $(\frac{3}{2}, \frac{3}{2})$, leading to zero helicity. Consequently, no combination of these interactions can yield a helicity-violating diagram. While a term such as $|\sigma|^3$ is a possible 3-point candidate for gravitons, this does not lead to an MHV structure.

5 Holographic correlators and the flat space S-matrix

In this chapter, we compute boundary correlation functions using the toy models constructed in the previous chapter and compare these results to bulk S-matrix elements. This is made possible by a recent proposal that forms part of the emerging holographic dictionary, translating bulk observables to boundary observables. We begin by reviewing the relevant results obtained from bulk calculations. Next, we explore the conceptual and technical challenges involved in formulating a field-theoretic description of flat space holography. Ultimately, we show that our models can reproduce bulk two- and three-point functions, illustrating the potential of this holographic correspondence.

5.1 Flat space S-matrix

To compare our BMS-invariant field theories on the boundary with a theory of quantum gravity in the bulk, it is essential to understand how observables in both frameworks are related. Inspired by celestial holography, a recent proposal relates bulk scattering amplitudes to conformal Carrollian correlation functions by applying a Fourier-like (Mellin) transform to the S-matrix elements themselves [29, 49]

$$\begin{aligned}\left\langle \Phi_1^{\Delta_1,+}\left(x_1^\alpha\right)\dots\Phi_n^{\Delta_n,-}\left(x_n^\alpha\right)\right\rangle \\ \equiv \prod_{k=1}^{n}\int_0^\infty d\omega_k \omega_k^{\Delta_k-1} e^{-i\epsilon_k\omega_k u_k}\left\langle p_1\left(\omega_1,x_1^i\right)\dots\right|\mathcal{S}\left|\dots p_n\left(\omega_n,x_n^i\right)\right\rangle,\end{aligned} \tag{5.1}$$

where $x^\alpha = (u, z, \bar{z})$ and $\epsilon_k = \pm 1$ depending on whether the particle is incoming $(-)$ or outgoing $(+)$. The left-hand side of (5.2) consists of position-space correlation functions of Carrollian conformal primaries with conformal dimension Δ_i. From the dictionary (4.5) we know that the conformal weights of these primary fields are determined by the helicities of the corresponding bulk particles. The right-hand side of (5.2) features bulk S-matrix elements that depend on the momenta p^μ and energies ω of the bulk particles. Following the dictionary (4.2), we will set $\Delta_i = 1$, resulting in

$$\left\langle \Phi_1^{+}\left(x_1^\alpha\right)\dots\Phi_n^{-}\left(x_n^\alpha\right)\right\rangle \equiv \prod_{k=1}^{n}\int_0^\infty d\omega_k e^{-i\epsilon_k\omega_k u_k}\left\langle p_1\left(\omega_1,x_1^i\right)\dots\right|\mathcal{S}\left|\dots p_n\left(\omega_n,x_n^i\right)\right\rangle. \tag{5.2}$$

In the simplest example, free 1-1 scattering, the S-matrix elements are given by

$$\langle p_1|\mathcal{S}|p_2\rangle = \delta^{(4)}(p_1 - p_2) = 2|\vec{p}_1|\delta^{(3)}(\vec{p}_1 - \vec{p}_2). \tag{5.3}$$

When scattering massless particles, it is convenient to parametrise the incoming and outgoing momenta by $p^\mu = \omega q^\mu$, where q^μ is the null vector defined in (2.3). This parametrisation, up to an overall scale, defines a natural map from null directions to points on the celestial sphere (see Figure 2). This implies that the energy measured by an observer at null infinity in the direction $(z, \bar{z})$ is given by ω. Rewriting the S-matrix elements in terms of this parametrisation gives

$$\langle p_1|\mathcal{S}|p_2\rangle = 2\frac{\delta(\omega_1 - \omega_2)}{\omega_1}\delta^{(2)}(\vec{z}_1 - \vec{z}_2). \tag{5.4}$$

Substituting this S-matrix element into (5.2) allows us to evaluate the right-hand side explicitly, yielding a prediction for the two-point correlation function of the putative dual field theory [29, 48, 69]

$$\begin{aligned}\left\langle \Phi^{+}\left(u_1, z_1, \bar{z}_1\right) \Phi^{-}\left(u_2, z_2, \bar{z}_2\right)\right\rangle &= \int_0^\infty d\omega_1 d\omega_2 e^{-i\omega_1 u_1} e^{i\omega_2 u_2} \langle p_1|\mathcal{S}|p_2\rangle \\ &= 2\delta^{(2)}(\vec{z}_1 - \vec{z}_2) \int_0^\infty d\omega \frac{e^{-i\omega(u_1-u_2)}}{\omega}.\end{aligned} \tag{5.5}$$

The spatial delta function appearing in the Carroll 2-point function has the (dual) interpretation that the momentum direction of a free particle in the bulk remains unchanged.

Thus far, this result has been reproduced by evaluating the left-hand side of (5.2) via a large-r expansion of bulk fields. In [30], the two-point function (5.5) was recovered using the operator formalism, where the authors considered complex bulk fields of arbitrary spin and computed boundary correlation functions by sending these bulk fields to future and past null infinity. We will now reproduce their calculation, starting from ordinary quantum field theory in Minkowski spacetime, using coordinates $x^\mu = (t, x, y, z)$. For complex scalar fields, the equations of motion reduce to the massless Klein-Gordon equations $\Box\phi = \Box\bar{\phi} = 0$, whose general solutions are superpositions of plane waves. As previously seen in (4.7) and (4.10), the massless spin-1 photon field A_μ and spin-2 graviton field $h_{\mu\nu}$ also satisfy wave equations $\Box A_\mu = 0$ and $\Box h_{\mu\nu} = 0$, leading to similar plane wave solutions. We first focus on the scalar case and afterwards extend the analysis to higher-spin fields.

For massless scalar fields, the general plane wave solution takes the form[12]

$$\begin{aligned}\phi(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}(\hat{a}_+(\vec{p})e^{ipx} + \hat{a}_-^\dagger(\vec{p})e^{-ipx}), \\ \bar{\phi}(x) &= \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}(\hat{a}_-(\vec{p})e^{ipx} + \hat{a}_+^\dagger(\vec{p})e^{-ipx}).\end{aligned} \tag{5.6}$$

The annihilation operators satisfy $a_\pm|0\rangle = 0$ for all $\vec{p}$, while the creation operators define one-particle states via $|\vec{p}, +\rangle = \hat{a}_+(\vec{p})^\dagger|0\rangle$ and $|\vec{p}, -\rangle = \hat{a}_-(\vec{p})^\dagger|0\rangle$. Here, $\hat{a}_+^\dagger$ creates a particle and $\hat{a}_-^\dagger$ an antiparticle with momentum $\vec{p}$ as indicated by the $\pm$ subscripts. These operators obey canonical commutation relations

$$[a_\alpha(\vec{p}_1), a_\beta^\dagger(\vec{p}_2)] = 2\omega_p(2\pi)^3\delta^{(3)}(\vec{p}_1 - \vec{p}_2)\delta_{\alpha\beta}. \tag{5.7}$$

Since scalar fields decay as $\phi(x) \sim \mathcal{O}(1/r)$, one can push the bulk fields to the null boundaries $\mathscr{I}^\pm$ by taking the asymptotic limit

$$\phi^\pm(u, z, \bar{z}) = \lim_{r\to\pm} r\phi(u, r, z, \bar{z}), \tag{5.8}$$

[12]Here, the subscript $\pm$ does not denote polarisations, as it does in the case of higher-spin. Instead, it distinguishes between particles (+) and antiparticles (-). This convention is adopted to closely resemble the notation used for higher-spin fields.

which results in

$$\phi^{\pm}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\int_0^{+\infty} d\omega \left[a_+(\omega,z,\bar{z})e^{-i\omega u} - a_-^\dagger(\omega,z,\bar{z})e^{i\omega u}\right],$$
$$\bar{\phi}^{\pm}(u,z,\bar{z}) = -\frac{i}{8\pi^2}\int_0^{+\infty} d\omega \left[a_-(\omega,z,\bar{z})e^{-i\omega u} - a_+^\dagger(\omega,z,\bar{z})e^{i\omega u}\right], \tag{5.9}$$

which has the same form on both boundaries $\mathscr{I}^+$ and $\mathscr{I}^-$. A detailed derivation is provided in Appendix C. In this calculation, we parametrised the momentum as $p^\mu = \omega q^\mu$, with q^μ defined in (2.3), which modifies the commutation relations to

$$[a_\alpha(\omega_1,z_1,\bar{z}_1), a_\beta^\dagger(\omega_2,z_2,\bar{z}_2)] = 16\pi^3\frac{\delta(\omega_1-\omega_2)}{\omega_1}\delta^{(2)}(\vec{z}_1-\vec{z}_2)\delta_{\alpha,\beta}. \tag{5.10}$$

This allows us to compute the two-point function directly

$$\langle 0|\phi^{\text{out}}(u_1,z_1,\bar{z}_1)\bar{\phi}^{\text{in}}(u_2,z_2,\bar{z}_2)|0\rangle = \frac{\delta^{(2)}(\vec{z}_1-\vec{z}_2)}{4\pi}\int_0^\infty d\omega \frac{e^{-i\omega(u_1-u_2)}}{\omega}, \tag{5.11}$$

which is the same result as (5.5) for a complex scalar field up to a normalisation factor. Note that the two-point function (5.11) involves ϕ and its complex conjugate, since the two-point function $\langle\phi\phi\rangle$ vanishes.

In the case of higher-spin propagation, pushing the fields A_μ and $h_{\mu\nu}$ to the null boundaries $\mathscr{I}^\pm$ results in[13]

$$A_z^{(0)}(u,z,\bar{z}) = \frac{-ie}{8\pi^2}\int_0^\infty d\omega \left[a_+(\omega,z,\bar{z})e^{-i\omega u} - a_-^\dagger(\omega,z,\bar{z})e^{i\omega u}\right], \tag{5.12}$$

$$C_{zz}(u,z,\bar{z}) = \frac{-i\sqrt{32\pi G}}{8\pi^2}\int_0^\infty d\omega \left[a_+(\omega,z,\bar{z})e^{-i\omega u} - a_-^\dagger(\omega,z,\bar{z})e^{i\omega u}\right]. \tag{5.13}$$

Using (5.10), we can compute their two-point correlation functions and find that they take the same form as in the scalar case (5.11), up to an overall prefactor. This is to be expected, as the S-matrix (5.3) for free 1-1 scattering is independent of spin.

5.2 Boundary correlation functions

Since the central observable in a theory of quantum gravity on asymptotically flat spacetime is the S-matrix, it is crucial that any proposed dual field theory can reproduce bulk S-matrix elements. Up to this point, we have constructed BMS invariant models and explored their properties. In this section, however, we aim to go beyond BMS-invariant field theory and perform actual holographic computations by deriving bulk S-matrix elements directly from our boundary theories. According to the proposal (5.2), this involves computing the left-hand side and reproducing the result (5.5)

$$\langle\Phi^+(u_1,z_1,\bar{z}_1)\Phi^-(u_2,z_2,\bar{z}_2)\rangle = 2\delta^{(2)}(\vec{z}_1-\vec{z}_2)\int_0^\infty d\omega\, \frac{e^{-i(u_1-u_2)}}{\omega}. \tag{5.14}$$

[13] See Appendix C.

However, before we can compute this correlation function of Carroll primaries Φ directly from our boundary theories, we must first address the following two issues.

The S-matrix relates asymptotic in-states at $\mathscr{I}^-$ to asymptotic out-states at $\mathscr{I}^+$. Therefore, any holographic theory trying to reproduce the bulk S-matrix must encode data from both null boundaries. This implies that the putative dual theory should be defined not only on $\mathscr{I}^+$, as we have previously assumed, but also on $\mathscr{I}^-$, making it essential to understand how these two boundaries are connected. Recognising the need for a global formulation of BMS symmetries acting on both past and future null infinity, Strominger proposed an identification between these boundaries by antipodally matching $\mathscr{I}^+_-$ and $\mathscr{I}^-_+$ [16, 17]. The total null boundary can therefore be characterised by $\tilde{\mathscr{I}} = \mathscr{I}^- \bigsqcup \mathscr{I}^+$ (see also Ref. [28, 30]).

So far, we have constructed boundary field theories exclusively on $\mathscr{I}^+$, and matched these fields to bulk fields using the holographic dictionary (4.2). In recent papers [53, 70], it is argued that, for the S-matrix, appropriate boundary conditions require fixing the positive-frequency modes on $\mathscr{I}^-$ and the negative-frequency modes on $\mathscr{I}^+$. As a consequence, at future null infinity $\mathscr{I}^+$, the positive-frequency modes are free to fluctuate and describe outgoing particles. To see this explicitly, consider a Carrollian primary field $\Phi(u, z, \bar{z})$ at $\mathscr{I}^+$ and Fourier transform it with respect to the time coordinate

$$\Phi(u, z, \bar{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Phi_\omega(z, \bar{z}) e^{-i\omega u}. \tag{5.15}$$

We can then decompose the field into its positive and negative frequency parts

$$\Phi(u, z, \bar{z}) = \underbrace{\int_0^{\infty} \frac{d\omega}{2\pi} \Phi_\omega(z, \bar{z}) e^{-i\omega u}}_{\text{free data}} + \underbrace{\int_{-\infty}^{0} \frac{d\omega}{2\pi} \Phi_\omega(z, \bar{z}) e^{-i\omega u}}_{\text{fixed boundary conditions}}. \tag{5.16}$$

This suggests that the free fluctuating data on $\mathscr{I}^-$ and $\mathscr{I}^+$, corresponding to incoming and outgoing particles, is given by

$$\begin{aligned} \Phi^+(u, z, \bar{z}) &= \int_0^{+\infty} \frac{d\omega}{2\pi} \Phi_\omega(z, \bar{z}) e^{-i\omega u} \qquad \text{on } \mathscr{I}^+, \\ \Phi^-(u, z, \bar{z}) &= \int_{-\infty}^{0} \frac{d\omega}{2\pi} \Phi_\omega(z, \bar{z}) e^{-i\omega u} \qquad \text{on } \mathscr{I}^-. \end{aligned} \tag{5.17}$$

Unfortunately, these new 'states' that describe incoming and outgoing particles do not correspond to Carroll primary fields, as we will demonstrate in Section 5.3. As a result, the boundary models constructed so far cannot be used to compute correlation functions between incoming particles from $\mathscr{I}^-$ and outgoing particles from $\mathscr{I}^+$. Fortunately, all these issues can be resolved by using flat Bondi coordinates.

Recently, flat Bondi coordinates have gained popularity due to their natural geometric identification of $\mathscr{I}^-$ to $\mathscr{I}^+$. Following a null geodesic that starts at a point on $\mathscr{I}^-$ with coordinates $(u, z, \bar{z})$ will arrive at a point on $\mathscr{I}^+$ with exactly the same coordinates $(u, z, \bar{z})$. This is illustrated in Figure 2, which is a 3D representation of the Penrose diagram in Figure 1 by reinstating the suppressed angular direction $\phi \in [0, 2\pi)$ and rotating the 2D diagram around its vertical axis.

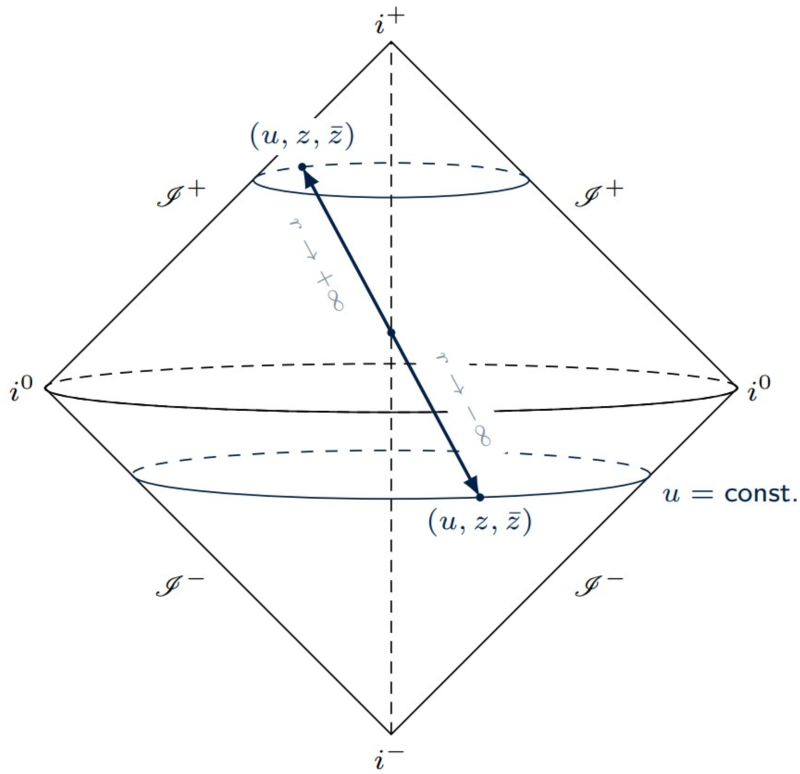

Figure 2. A three-dimensional Penrose diagram of Minkowski space [57].

In flat Bondi coordinates, the retarded time coordinate u parametrises both null boundaries $\mathscr{I}^\pm$. Specifically, u runs from runs from $\mathscr{I}^-_-$ to $\mathscr{I}^-_+$ along past null infinity, and from $\mathscr{I}^+_-$ to $\mathscr{I}^+_+$ along future null infinity. The spatial coordinates $(z, \bar{z})$ label points on the so-called celestial spheres located at $r = \infty$. As depicted in the three-dimensional Penrose diagram of Figure 2, future and past null infinity are foliated by celestial spheres labelled by $(z, \bar{z})$, one for each value of the retarded time u.

The natural identification between $\mathscr{I}^-$ and $\mathscr{I}^+$ enables us to formulate our dual theory on a single boundary, $\mathscr{I}^+$, as long as we correctly distinguish between incoming and outgoing states. This simplification avoids the need to formulate two dual theories on separate boundaries, which are glued together antipodally. The proposal (5.2) already reflects this idea by encoding the in/out nature of particles via $\epsilon = \pm 1$. Using the fact that a null geodesic starting at $\mathscr{I}^-$ and ending on $\mathscr{I}^+$ has the same coordinates $(u, z, \bar{z})$ on both boundaries, we can combine the in- and out-states in (5.17)

$$\Phi(u, z, \bar{z}) = \Phi^+(u, z, \bar{z}) + \Phi^-(u, z, \bar{z}). \tag{5.18}$$

The resulting field $\Phi(u,z,\bar{z})$ is a Carrollian primary field, which contains information about both the incoming and outgoing particles. As long as one remains consistent, the field can be defined on either $\mathscr{I}^-$ or $\mathscr{I}^+$, and we will choose to formulate everything on $\mathscr{I}^+$. Therefore, by combining the previously constructed BMS-invariant field theories with the Carroll primaries defined in (5.18), we obtain an effective framework to describe incoming and outgoing particles in the bulk. This method still requires that the boundary fields have conformal weights that match the corresponding bulk fields through the holographic dictionary (4.2).

When starting directly from a BMS-invariant boundary theory on $\mathscr{I}^+$, one has to decompose the Carroll primary fields as $\Phi = \Phi^+ + \Phi^-$, where Φ^+ encodes outgoing modes and Φ^- incoming ones, distinguished by their frequency. We can formally define these in- and out-states using [46]

$$\Phi^{\pm}(u,z,\bar{z}) \equiv \mp \int_{-\infty}^{+\infty} \frac{\mathrm{d}u'}{2\pi i} \frac{\Phi\left(u',z,\bar{z}\right)}{(u'-u) \pm i\varepsilon}. \tag{5.19}$$

Starting by Fourier transforming the Carroll primary $\Phi(u,z,\bar{z})$ with respect to time

$$\Phi(u,z,\bar{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \Phi_\omega(z,\bar{z}) e^{-i\omega u}, \tag{5.20}$$

and using the integral representation of the Heaviside step function

$$H(x) = \lim_{\epsilon\to 0^+} \mp\frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{1}{\tau \pm i\epsilon} e^{\mp i x\tau} d\tau, \tag{5.21}$$

we then obtain the in- and out states (5.17)

$$\begin{aligned} \Phi^+(u,z,\bar{z}) &= \int_0^{+\infty} \frac{d\omega}{2\pi} \Phi_\omega(z,\bar{z}) e^{-i\omega u}, \\ \Phi^-(u,z,\bar{z}) &= \int_{-\infty}^{0} \frac{d\omega}{2\pi} \Phi_\omega(z,\bar{z}) e^{-i\omega u}. \end{aligned} \tag{5.22}$$

The out-state Φ^+ contains only positive frequencies and, therefore, describes outgoing particles, while the in-state Φ^- contains only negative frequency modes and corresponds to incoming particles.

Upon substituting the in- and out-stats into the first-order BMS-invariant Lagrangian $\mathcal{L} = i(\Phi\partial_u\bar{\Phi} - \bar{\Phi}\partial_u\Phi)$, we find that the in-in and out-out contributions vanish[14], as the resulting delta function lies outside the domain of integration

$$\int_{-\infty}^{\infty} dud^2\vec{z}\;\Phi^+\partial_u\bar{\Phi}^+ = \int_{-\infty}^{\infty} d^2z \int_0^{\infty}\frac{d\omega_1}{2\pi}\int_0^{\infty}\frac{d\omega_2}{2\pi}\Phi_{\omega_1}(z,\bar{z})(-i\omega_2)\bar{\Phi}_{\omega_2}(z,\bar{z})\delta(\omega_1+\omega_2) = 0.$$

[14]We adopt the convention that the complex field Φ is treated as an independent field, with its own Fourier transform. See D for more details.

As a result, the action only includes terms that mediate propagation from in-states to out-states. Using integration by parts to express the Lagrangian in a more compact form, we obtain

$$S = \int_{\mathscr{I}^+} dud^2\vec{z}\; 2i\left(\Phi^+\partial_u\bar{\Phi}^- - \bar{\Phi}^+\partial_u\Phi^-\right). \tag{5.23}$$

Calculating the propagator for this Lagrangian yields[15]

$$\langle\Phi^+(u_1,\vec{z}_1)\bar{\Phi}^-(u_2,\vec{z}_2)\rangle = \frac{i}{2}\delta^{(2)}(\vec{z}_1-\vec{z}_2)\int_0^\infty \frac{d\omega}{2\pi}\frac{e^{-i\omega(u_1-u_2)}}{\omega}, \tag{5.24}$$

which matches the bulk S-matrix result (5.5), providing a realisation of the holographic principle directly from a boundary theory. Similar to the result from the large-r expansion (5.11), the correlation function is between Φ and its complex conjugate $\bar{\Phi}$. It is also possible to calculate a correlation function for two real scalar fields using (4.44), although this would involve two distinct scalar fields. Moreover, the correlation function (5.24) applies to arbitrary Carroll primaries Φ, and thus extends to all spins.

To avoid the divergences that typically arise in the integrals of these Carrollian correlation functions, one can instead consider a correlation function involving at least one descendant of a Carroll primary, such as $\partial_u\Phi$ [48]. Using the regularised integrals

$$I_\beta(x) = \lim_{\epsilon\to 0^+}\int_0^\infty d\omega\;\omega^{\beta-1}e^{-i\omega(x-i\epsilon)} = \lim_{\epsilon\to 0^+}\frac{\Gamma(\beta)(-i)^\beta}{(x-i\epsilon)^\beta}, \tag{5.25}$$

we can compute the two-point function ($\beta = 1$)

$$\begin{aligned}\langle\Phi^+(u_1,z_1,\bar{z}_1)\partial_u\bar{\Phi}^-(u_2,z_2,\bar{z}_2)\rangle &= \frac{1}{2}\delta^{(2)}(\vec{z}_1-\vec{z}_2)\int_0^\infty\frac{d\omega}{2\pi}e^{-i\omega(u_1-u_2)}\\ &= -\frac{i}{2}\frac{\delta^{(2)}(\vec{z}_1-\vec{z}_2)}{(u_1-u_2)}.\end{aligned} \tag{5.26}$$

The use of a descendant field removes the soft pole at $\omega = 0$, ensuring that the integral remains finite.

5.3 In- and out-states

In this section, we elaborate on the methodology used in the previous section to compute boundary correlation functions. Although the in- and out-states are constructed from a Carroll primary $\Phi(u,z,\bar{z})$, these states themselves are not primaries. Starting from the definition (5.19), and using that $\Phi(u,z,\bar{z})$ is a Carroll primary field, we can apply supertranslations $\delta_\mathcal{T}\Phi = \mathcal{T}(z,\bar{z})\partial_u\Phi$ to the out-states. This gives

$$\begin{aligned}\delta_\mathcal{T}\Phi^+(u,z,\bar{z}) &= -\int_{-\infty}^{+\infty}\frac{\mathrm{d}u'}{2\pi i}\frac{\delta_\mathcal{T}\Phi\,(u')}{(u'-u)+i\varepsilon}\\ &= -\mathcal{T}(z,\bar{z})\int_{-\infty}^{+\infty}\frac{\mathrm{d}u'}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{\Phi_\omega(z,\bar{z})(-i\omega)e^{-i\omega u'}}{(u'-u)+i\varepsilon}\\ &= \mathcal{T}(z,\bar{z})\partial_u\Phi^+.\end{aligned} \tag{5.27}$$

[15] A detailed and rigorous path integral calculation can be found in Appendix D.

demonstrating that Φ^+ transforms as a primary under supertranslations. In the case of superrotations, $\delta_{\mathcal{Y}}\Phi = (\mathcal{Y}\partial + h\partial\mathcal{Y} + \frac{u}{2}(\partial\mathcal{Y})\partial_u)\Phi$, we get

$$\begin{aligned}\delta_{\mathcal{Y}}\Phi^+(u,z,\bar{z}) &= \mathcal{Y}\partial\Phi^+ + h\partial\mathcal{Y}\Phi^+ - \frac{\partial\mathcal{Y}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{(u+t)\Phi_\omega(z,\bar{z})(-i\omega)e^{-i\omega u}e^{-i\omega t}}{t+i\varepsilon} \\ &= \left(\mathcal{Y}\partial + h\partial\mathcal{Y} + \frac{u}{2}\partial\mathcal{Y}\partial_u\right)\Phi^+ - \frac{\partial\mathcal{Y}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{t\Phi_\omega(z,\bar{z})(-i\omega)e^{-i\omega u}e^{-i\omega t}}{t+i\varepsilon},\end{aligned} \tag{5.28}$$

where we used a substitution $t = u' - u$ in the first line. From this expression, it follows that Φ^+ transforms as a primary plus additional terms. We will therefore call Φ^+ an 'almost-primary'. The additional term $-\frac{\partial\mathcal{Y}}{2}\Pi$, where

$$\Pi := \int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\Phi_\omega(z,\bar{z})(-i\omega)e^{-i\omega u}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\frac{te^{-i\omega t}}{t+i\varepsilon}, \tag{5.29}$$

is divergent. To see this, observe that we cannot evaluate the t-integral using a complex contour $C_\rho^- = \{z = \rho e^{it} |\ \pi \leq t \leq 2\pi\ \}$, as the contour C_ρ^- does not vanish as a result of Jordan's Lemma[16]. Instead, we can divide the t-integral into

$$I = \int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\frac{(t+i\epsilon-i\epsilon)e^{-i\omega t}}{t+i\varepsilon} = \int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}e^{-i\omega t} - i\epsilon\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\frac{e^{-i\omega t}}{t+i\varepsilon}, \tag{5.30}$$

and compute the first integral directly while using Jordan's lemma to evaluate the second integral

$$\begin{aligned} I &= \lim_{\Lambda\to\infty}\left(\frac{1}{2\pi i}\frac{e^{-i\omega\Lambda} - e^{+i\omega\Lambda}}{-i\omega}\right) - i\epsilon(2\pi i \lim_{u\to i\epsilon} e^{-i\omega u}) \\ &= \lim_{\Lambda\to\infty}\left(\frac{1}{\pi i\omega}\sin(\omega\Lambda)\right) + 2\pi\epsilon e^{-\epsilon\omega}. \end{aligned} \tag{5.31}$$

Clearly, the first term does not converge, making the total integral divergent.

Fortunately, when calculating how Φ^- transforms under BMS transformations, we find

$$\begin{aligned}\delta_{\mathcal{T}}\Phi^- &= \mathcal{T}(z,\bar{z})\partial_u\Phi^-, \\ \delta_{\mathcal{Y}}\Phi^- &= \left(\mathcal{Y}\partial + h\partial\mathcal{Y} + \frac{u}{2}\partial\mathcal{Y}\partial_u\right)\Phi^- + \frac{\partial\mathcal{Y}}{2}\int_{-\infty}^{+\infty}\frac{\mathrm{d}t}{2\pi i}\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}\frac{t\Phi_\omega(z,\bar{z})(-i\omega)e^{-i\omega u}e^{-i\omega t}}{t-i\varepsilon},\end{aligned} \tag{5.32}$$

exactly cancelling the unwanted divergent term in Φ^+. This result shows that BMS-invariant Lagrangians cannot be constructed directly from these in- and out-states. Instead, one must first construct a Lagrangian from pure Carroll primary fields Φ, and only then decompose the fields into outgoing and incoming modes $\Phi = \Phi^+ + \Phi^-$.

The approach of working solely on future null infinity $\mathscr{I}^+$, while keeping track of in- and out-states, can also be applied to the graviton. The asymptotic shear C_{zz}, which characterises the radiative data of the graviton, does not transform as a pure Carroll

[16]See Appendix A for more information on complex integration.

primary, but rather as a quasi-primary. More importantly, its transformation properties differ depending on which boundary the BMS group acts upon [16, 17]. Defining the asymptotic shear on both boundaries

$$\begin{aligned} C_{zz} &= \lim_{r\to\infty} \frac{1}{r} h_{\mu\nu} \quad \text{on } \mathscr{I}^+, \\ D_{zz} &= \lim_{r\to-\infty} \frac{1}{r} h_{\mu\nu} \quad \text{on } \mathscr{I}^-, \end{aligned} \tag{5.33}$$

the BMS transformations of these asymptotic shears are given by the supertranslations

$$\begin{aligned} \delta_{\mathcal{T}} C_{zz}(u,z,\bar{z}) &= \mathcal{T}(z,\bar{z})\partial_u C_{zz}(u,z,\bar{z}) - 2\partial_z^2 \mathcal{T}(z,\bar{z}), \\ \delta_{\mathcal{T}} D_{zz}(u,z,\bar{z}) &= \mathcal{T}(z,\bar{z})\partial_u D_{zz}(u,z,\bar{z}) + 2\partial_z^2 \mathcal{T}(z,\bar{z}), \end{aligned} \tag{5.34}$$

and superrotations

$$\begin{aligned} \delta_{\mathcal{Y}} C_{zz}(u,z,\bar{z}) &= \left[\partial\mathcal{Y}(z)\left(\frac{3}{2} + \frac{u}{2}\partial_u\right) + \mathcal{Y}(z)\partial\right] C_{zz}(u,z,\bar{z}) - u\partial^3\mathcal{Y}(z), \\ \delta_{\mathcal{Y}} D_{zz}(u,z,\bar{z}) &= \left[\partial\mathcal{Y}(z)\left(\frac{3}{2} + \frac{u}{2}\partial_u\right) + \mathcal{Y}(z)\partial\right] D_{zz}(u,z,\bar{z}) + u\partial^3\mathcal{Y}(z). \end{aligned} \tag{5.35}$$

When combining the asymptotic shears from both boundaries, analogous to the in- and out-states, the inhomogeneous terms cancel, resulting in a genuine spin-2 Carroll primary field $\sigma = D_{zz} + C_{zz}$ (a similar construction appears in [52]). This suggests that the models derived from BMS-invariant principles on the boundary $\mathscr{I}^+$ in the previous chapter could provide a holographic description of the graviton. However, as also discussed in the previous chapter, complications arise when trying to define the boundary conditions and also in the construction of MHV diagrams. Therefore, this remains an open problem.

5.4 3-point functions and boundary interactions

Following the successful matching of our propagator (5.24) to the bulk S-matrix, it is natural to ask whether this boundary theory can also compute higher-order n-point functions. Starting with the three-point function, it is well known that the scattering of three on-shell massless particles can only satisfy momentum conservation if they are colinear. The S-matrix element for two outgoing particles and one ingoing particle aligned collinearly is given by [48]

$$\langle p_1, p_2|\mathcal{S}|p_3\rangle^{\text{colinear}} = \frac{\delta(\omega_1+\omega_2-\omega_3)}{\omega_3}\delta^{(2)}(\vec{z}_1-\vec{z}_2)\delta^{(2)}(\vec{z}_1-\vec{z}_3)f(\omega_i,\vec{z}_i). \tag{5.36}$$

The function $f(\omega_i, \vec{z}_i)$ is introduced to preserve Lorentz invariance and must therefore have scaling dimension $1-d$ but is otherwise arbitrary. In [48], it is chosen to be $\frac{1}{\omega_1\omega_2}$, which is the most symmetric option, and we adopt the same choice here. Using (5.36), we can explicitly evaluate the right-hand side of (5.2), yielding a prediction for the two-out one-in three-point function of the proposed dual field theory

$$\begin{aligned} &\langle \Phi^+(u_1,z_1,\bar{z}_1)\Phi^+(u_2,z_2,\bar{z}_2)\Phi^-(u_3,z_3,\bar{z}_3)\rangle \\ &= \delta^{(2)}(z_1-z_2)\delta^{(2)}(z_1-z_3)\int_0^\infty d\omega_1 \frac{e^{-i\omega_1(u_1-u_3)}}{\omega_1}\int_0^\infty d\omega_2 \frac{e^{-i\omega_2(u_2-u_3)}}{\omega_2}\frac{1}{\omega_1+\omega_2}. \end{aligned} \tag{5.37}$$

While the result (5.37) holds for arbitrary primary fields, we focus on the scalar case, as it avoids potential complications with boundary terms at $\mathscr{I}^+_\pm$ and allows for well-defined interactions. We therefore proceed using the scalar Lagrangian

$$\mathcal{L} = i(\phi\partial_u\bar{\phi} - \bar{\phi}\partial_u\phi) + g\phi^3 + g\bar{\phi}^3 + \lambda\phi\bar{\phi}^2 + \lambda\bar{\phi}\phi^2, \tag{5.38}$$

to describe this three-point function. In the case of the three-point function $\langle\phi^+\phi^+\phi^-\rangle$, without complex conjugate fields, the dominant contribution comes from the vertex $g\bar{\phi}^3$. This follows from the fact that the proposal (5.2) relates bulk S-matrix elements to Carrollian correlation functions at the boundary. As such, these are genuine correlation functions which do not require an LSZ procedure, and therefore, the external legs in the diagram remain propagators. In the scalar theory (5.38), the propagators are $\phi^- \to \bar{\phi}^+$ and $\bar{\phi}^- \to \phi^+$, so the relevant first-order interaction vertex is $g\bar{\phi}^3$, as shown in Figure 3.

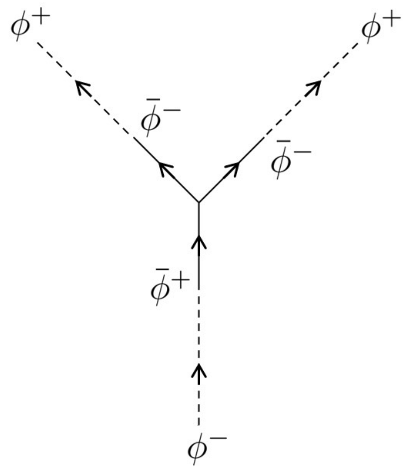

Figure 3. First order 3-point diagram of $\langle\phi^+\phi^+\phi^-\rangle$.

From Appendix D we know that the propagator $\Phi^- \to \bar{\Phi}^+$ has the same form as (5.24)

$$\langle\bar{\Phi}^+(u_1,\vec{z}_1)\Phi^-(u_2,\vec{z}_2)\rangle = -\frac{i}{2}\delta^{(2)}(\vec{z}_1-\vec{z}_2)\int_0^\infty \frac{d\omega}{2\pi}\frac{e^{-i\omega(u_1-u_2)}}{\omega}. \tag{5.39}$$

Upon substituting the in- and out-states $\bar{\phi} = \bar{\phi}^+ + \bar{\phi}^-$ in the interaction term, we arrive at

$$g\bar{\phi}^3 = g\left(\bar{\phi}_+^3 + 3\bar{\phi}_+^2\bar{\phi}_- + 3\bar{\phi}_+\bar{\phi}_-^2 + \bar{\phi}_-^3\right), \tag{5.40}$$

where additional factors of 3 will cancel against the symmetry factors in the final diagram. The three-point function is then obtained by connecting three propagators and integrating over the spacetime point $v^\mu = (v^0, \vec{v})$ where they meet

$$\begin{aligned}\langle\phi^+(u_1,z_2,\bar{z}_1)\phi^+(u_2,z_2,\bar{z}_2)\phi^-(u_3,z_3,\bar{z}_3)\rangle &\propto \int_{-\infty}^{\infty} d^3v\ \delta^{(2)}(\vec{z}_1-\vec{v})\delta^{(2)}(\vec{z}_2-\vec{v})\delta^{(2)}(\vec{z}_3-\vec{v}) \\ \times\, g\int_0^\infty d\omega_1\frac{e^{-i\omega_1(u_1-v)}}{\omega_1}\int_0^\infty d\omega_2\frac{e^{-i\omega_2(u_2-v)}}{\omega_2}&\int_0^\infty d\omega_3\frac{e^{-i\omega_3(v-u_3)}}{\omega_3}.\end{aligned} \tag{5.41}$$

This results in

$$
\begin{aligned}
&\langle \phi^+(u_1, z_2, \bar{z}_1)\phi^+(u_2, z_2, \bar{z}_2)\phi^-(u_3, z_3, \bar{z}_3)\rangle \\
&\propto g\delta^{(2)}(\vec{z}_1 - \vec{z}_3)\delta^{(2)}(\vec{z}_2 - \vec{z}_3) \int_0^\infty \frac{d\omega_1}{\omega_1} e^{-i\omega_1(u_1-u_3)} \int_0^\infty \frac{d\omega_2}{\omega_2} e^{-i\omega_2(u_2-u_3)} \frac{1}{\omega_1+\omega_2},
\end{aligned} \tag{5.42}
$$

which matches the prediction (5.37), up to an overall constant. The boundary theory (5.38) automatically produces a collinear 3-point function. This follows from the presence of spatial delta functions, which enforce that both incoming and outgoing particles lie along the same null geodesic.

The next step would be to compute 4-point functions and other interactions. Due to the holographic requirement that all fields must have conformal dimension $\Delta = 1$ (4.2), all interaction terms in theories consistent with the holographic dictionary involve exactly three fields. From these three-point vertices, one can, in principle, construct any higher-order interaction. Although the inherent collinearity was useful in the case of the three-point function, it presents a significant limitation to n-point functions, as any such vertex constructed from three-point interactions remains collinear. This contrasts with bulk expectations, where, for instance, four-point functions are not constrained to be collinear. This ultralocality arises from the spatial delta functions, which themselves are a consequence of the absence of spatial derivatives in our models. Including spatial derivatives in BMS-invariant models has proven challenging, as spatial derivatives of primary fields are generally not primary. In the magnetic case, we partially succeeded in including spatial derivatives by introducing an auxiliary field χ, but this field does not transform under the BMS algebra. To date, no fully BMS-invariant model with spatial derivatives has been constructed.

6 Conclusion and Outlook

In this thesis, we have explored various aspects of a field-theoretic approach to flat space holography. We began with an overview of asymptotically flat spacetimes in Chapter 2, followed by a review of conformal Carroll field theories in Chapter 3. In Chapter 4, we applied the emerging holographic dictionary (4.2) to known Carrollian models and introduced a novel method for constructing BMS-invariant field theories. The latter were constrained to be first-order, and we analysed their properties, e.g. equations of motion, soft and hard sector decomposition, correlation functions and possible interactions. Finally, in the last chapter, we moved beyond boundary field theory towards a concrete realisation of flat space holography. This was achieved by matching bulk S-matrix elements to boundary correlation functions via the recent proposal (5.2).

The constraints of BMS symmetry, together with the holographic dictionary, forced us to consider only first-order field theories consisting of fields with conformal dimension $\Delta = 1$. Among the Carrollian models, only the magnetic theory satisfied this condition. Extending our analysis to more general BMS-invariant theories, the same constraint limiting us to first-order models also implied that these could not be constructed from a single field. Of these new models, only the scalar theory admits well-defined boundary conditions and potentially has the possibility of describing hard dynamics. Using the geometric identification between boundaries $\mathscr{I}^-$ and $\mathscr{I}^+$ in flat Bondi coordinates, we formulated the dual theory entirely on $\mathscr{I}^+$, provided that we carefully distinguished between incoming and outgoing states. From our boundary theory, we then constructed scalar two- and three-point functions that reproduced known bulk S-matrix elements.

Carrollian holography has so far lacked a concrete toy model suitable for holography. Most research has instead focused on analysing the structural properties of Carrollian field theories, with the hope that once enough constraints have been discovered, we can eventually guess a viable dual field theory. In this thesis, guided by the emerging dictionary (4.2) and the proposal (5.2), we tried to offer a new perspective by explicitly constructing such boundary field theories. While this first attempt at a field-theoretic approach to flat space holography revealed the limitations imposed by BMS symmetry as well as the conceptual challenges of working with a two-sided boundary $\mathscr{I}^- \bigsqcup \mathscr{I}^+$, it provides one of the first concrete examples of flat space holography from an explicit boundary theory.

Despite the progress made in this thesis toward constructing BMS-invariant boundary field theories and connecting them to bulk scattering data, several limitations and unresolved issues remain. A central assumption throughout this thesis was the use of the emerging holographic dictionary with scaling dimension $\Delta = 1$, which constrained us to first-order Lagrangians. Across all models considered, the interactions are limited to three-point vertices, which are strictly collinear. As a result, any higher-order interaction constructed from these three-point vertices remains collinear and therefore cannot account for known, non-collinear scattering processes in the bulk. This limitation arises from the absence of spatial derivatives in our BMS-invariant theories. The only partial success at an attempt to include spatial derivatives came from the magnetic Carroll theory, where we introduced an auxiliary field χ. However, this field failed to transform according to

BMS_4 transformations. Additionally, BMS-invariant models derived from the holographic dictionary fail to capture MHV amplitudes. They also appear to be limited to soft particle dynamics, at least for higher spins and possibly even for scalar fields. Finally, we outlined some ideas on how to construct bounded interaction potentials; however, a clear understanding of how to use them is still lacking.

Beyond the structural limitations of the models, several unresolved conceptual issues stem from the treatment of the boundary itself. In the case of higher-spin particles, which exhibit a memory effect, the boundary conditions become nontrivial: the fields themselves cannot vanish at the boundaries $\mathscr{I}^+_\pm$, but their associated News tensors must cancel. Due to these complications, the structure of interacting higher-spin models remains poorly understood. As a result, we restricted our analysis to the scalar case. To address the issue of the disconnected boundaries $\mathscr{I}^+$ and $\mathscr{I}^-$, we used flat Bondi coordinates and introduced in- and out-states. While this procedure allowed us to reproduce bulk S-matrix elements, it would benefit from a more rigorous treatment in future work. Lastly, we suggested that using flat Bondi coordinates could potentially also resolve the issue of the graviton being a quasi-primary under BMS transformations. While this seems like an interesting idea, it too requires further investigation.

From the current work in this thesis, several directions for future research arise. Most notably, an operator formalism for the boundary theories remains absent. All computations in this thesis were carried out using the path integral approach, as the first-order nature of the Lagrangians prevented us from formulating a consistent operator framework. Although it is known that first-order theories can, in principle, be quantised using the Dirac bracket formalism, we have not implemented this method.

A second challenge encountered in this thesis arises from the fact that all interactions in our models are three-point vertices and strictly collinear. This implies that any higher-order interaction constructed from these three-point vertices will also be collinear. We attempted to resolve this problem by introducing spatial derivatives, which succeeded in the case of magnetic Carroll theory, due to the auxiliary field χ. However, the field χ was found not to transform according to BMS_4 transformations. Since the field ϕ in the same model does transform as a Carroll primary, it remains an open question whether this is a problem — perhaps χ does not need to transform under BMS symmetries. Alternatively, it is possible that one needs to consider further generalising the extended BMS algebra to accommodate such theories (see e.g. [46]).

Thirdly, various papers (e.g. [30]) have suggested that the correct dual theory should be electric, due to the explicit u-dependence observed in propagators. In this thesis, however, we dismissed electric theories early on, based on the conformal weights of the field ϕ : $(\frac{1}{4}, \frac{1}{4})$, which are incompatible with the holographic dictionary. Nevertheless, it may be that — similar to AdS/CFT, where the fundamental observables are often gauge-invariant operators rather than the fields themselves — we must instead consider composite operators with conformal dimension $\Delta = 1$. In the case of electric scalar theory, one such candidate could be the operator $\mathcal{O} = \,:\phi^2:\,$. What appears to be crucial is to take correlation functions of Carroll primaries with conformal dimension one or higher (descendants).

Lastly, little is currently known about how electric and magnetic Carroll field theories might interact. Interestingly, in the magnetic theory with spatial derivatives, the field ϕ is a Carroll primary of weights $(\frac{1}{4}, \frac{1}{4})$, similar to the electric case. This suggests a possible way to connect the two theories is simply adding them. One should be careful, however, as the two theories differ in units. This could be resolved by implementing an effective speed of light parameter, but further research is still required.

This thesis demonstrates that explicit boundary field theories can successfully reproduce aspects of flat space scattering, offering a concrete step toward realising Carrollian holography. In particular, the construction of scalar correlation functions from a boundary theory — matching known bulk S-matrix elements — illustrates how boundary theories provide valuable insight into bulk dynamics. The conceptual and technical challenges encountered in this field-theoretic approach reveal several promising directions for future research. By exploring explicit boundary models, this work contributes to a growing effort to formulate a consistent and predictive holographic framework for asymptotically flat spacetimes.

A Complex integration

In this section, we briefly review complex integration and several techniques relevant to computations throughout this thesis. We will follow the treatment of [71].

Due to the two-dimensional nature of $\mathbb{C} \cong \mathbb{R}^2$, complex integration is performed along one-dimensional curves. Let f be a continuous complex function on a directed smooth curve γ, parameterised by $z = z(t)$ for $a \leq t \leq b$. Then the integral of f along γ is given by

$$\int_\gamma f(z)dz = \int_b^a f(z(t))z'(t)dt. \tag{A.1}$$

In practice, however, integrals are rarely computed directly in this manner. Instead, we make use of the powerful result known as *Cauchy's theorem.*

Theorem 1 *(Cauchy's theorem) Let Γ be a simple closed positively oriented contour. If f is holomorphic (complex differentiable) in some simply connected domain $D \subseteq \mathbb{C}$ containing Γ and z_0 is any point inside this closed contour, then*

$$\int_\Gamma (z - z_0)^n \, dz = \begin{cases} 0 & \text{for } n \neq -1 \\ 2\pi i & \text{for } n = -1, \end{cases} \tag{A.2}$$

such that

$$f(z_0) = \frac{1}{2\pi i} \int_\Gamma \frac{f(z)}{z - z_0} dz. \tag{A.3}$$

This result implies that our main concern is not explicitly computing the integral, but rather ensuring that the contour is chosen correctly and account for the enclosed poles. This process is made significantly easier by the next theorem.

Theorem 2 *(Deformation invariance Theorem) Let f be an analytic function over a domain D containing loops Γ_0 and Γ_1. If these loops can be continuously deformed into one another in D, then*

$$\int_{\Gamma_0} f(z)dz = \int_{\Gamma_1} f(z)dz. \tag{A.4}$$

So far, we have encountered *holomorphic functions* —– functions that are complex differentiable at every point in their domain, i.e., those that satisfy the Cauchy-Riemann equations. We have also seen *analytic functions*, which are characterised by the existence of a convergent Taylor series expansion of the form

$$f(z) = \sum_{n=0}^{\infty} a_n (z - z_0)^n. \tag{A.5}$$

Closely related are poles, which are isolated singularities where a function diverges in a controlled way. Specifically, a function f is said to have a *pole of order* m at z_0 if, in a punctured neighbourhood of z_0, it can be written as

$$f(z) = \frac{g(z)}{(z - z_0)^n}, \tag{A.6}$$

where g is analytic at z_0 and $g(z_0) \neq 0$.

The next class of important functions are the *meromorphic functions*. A function f is meromorphic if it is holomorphic except at isolated points where it has poles of the above form. In such cases, where a function is analytic everywhere except at an isolated singularity z_0, it admits a *Laurent series expansion* around z_0 that converges in a punctured neighbourhood

$$f(z) = \sum_{n=-\infty}^{\infty} a_n (z - z_0)^n. \tag{A.7}$$

The coefficients in a Laurent series carry rich information about the function's behaviour near singularities. In particular, the coefficient a_{-1} in the Laurent expansion of f around a point z_0 is called the *residue* of f at z_0, and plays an important role in evaluating contour integrals

Theorem 3 *Let f be a meromorphic function on some simply connected domain $D \subseteq \mathbb{C}$ containing a positively oriented, simple closed contour Γ. Suppose that f has only finitely many isolated singularities $z_1, z_2, \ldots, z_n$ inside Γ. Then the integral of f around Γ is given by*

$$\oint_{\Gamma} f(z)dz = 2\pi i \sum_{k=1}^{n} Res(f(z), z_k), \tag{A.8}$$

where $Res(f, z_k)$ denotes the residue of f at the point z_k and is given by

$$Res(f, z_0) = \lim_{z \to z_0} \frac{1}{(m-1)} \frac{d^{m-1}}{dz^{m-1}} \left((z - z_0)^m f(z) \right), \tag{A.9}$$

for poles of order m at z_0.

Next, we have some important lemmas that, combined with the Residue theorem, provide powerful tools for evaluating real integrals. The common strategy involves carefully choosing a contour that avoids singularities. These lemmas describe how to handle the additional contours introduced in this process, often involving semicircular arcs. The most commonly used contour is C_ρ^+ parametised by $z = \rho e^{it}$ for $0 \leq t \leq \pi$.

Lemma 1 *If $f(z) = P(z)/Q(z)$ is the quotient of two polynomials such that degree $Q \geq 2 +$ degree P, then*

$$\lim_{\rho \to \infty} \int_{C_\rho^+} f(z)dz = 0. \tag{A.10}$$

Lemma 2 *(Jordan's Lemma) If $m > 0$ and P/Q is the quotient of two polynomials such that degree $Q \leq 1 +$ degree P, then*

$$\lim_{\rho\to\infty} \int_{C_\rho^+} e^{imx} \frac{P(z)}{Q(z)} dz = 0. \tag{A.11}$$

Lemma 3 *If f has a simple pole at $z = c$ and S_r is the circular arc $z = c + re^{i\theta}$, $(\theta_1 \leq \theta \leq \theta_2)$, then*

$$\lim_{r\to 0^+} \int_{S_r} f(z)dz = i(\theta_2 - \theta_1) Res(f, c). \tag{A.12}$$

When evaluating real integrals with singularities on the integration path, most notably on the real axis, the usual Riemann integral is undefined due to divergence at the singularity. In such cases, the Cauchy principal value (p.v.) offers a way to assign a finite, symmetric limit to the integral. For instance, the integral

$$\int_{-1}^{1} \frac{1}{x} dx, \tag{A.13}$$

does not converge in the usual sense because of the pole at $x = 0$. However, the principal value is defined as

$$\text{p.v.} \int_{-1}^{1} \frac{1}{x} dx := \lim_{\epsilon\to 0} \left(\int_{-1}^{-\epsilon} \frac{1}{x} dx + \int_{\epsilon}^{1} \frac{1}{x} dx \right) = 0, \tag{A.14}$$

which yields 0, reflecting the symmetric cancellation of the singularity's contribution. When a pole lies on the real axis, the Cauchy principal value is used to make sense of the resulting divergent integral. Essentially, the principal value treats the pole symmetrically, allowing it to be handled in a well-controlled manner. Consider the following important integral.

$$\text{p.v.} \int_{-\infty}^{\infty} \frac{e^{ix}}{x} dx, \tag{A.15}$$

using the contour

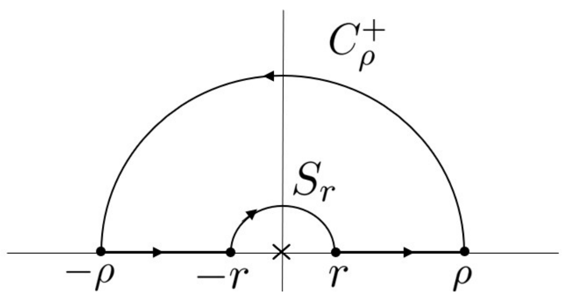

Figure 4.

Since the contour described in Figure 4 is closed and contains no singularities, we have

$$\left(\int_{-\rho}^{-r} + \int_{S_r} + \int_{r}^{\rho} + \int_{C_\rho^+} \right) \frac{e^{iz}}{z} dz = 0. \tag{A.16}$$

By Jordan's Lemma

$$\lim_{\rho\to\infty}\int_{C_\rho^+}\frac{e^{iz}}{z}dz=0, \tag{A.17}$$

and similarly by Lemma 3

$$\lim_{r\to 0^+}\int_{S_r}\frac{e^{iz}}{z}dz=-i\pi\text{Res}(0)=-i\pi. \tag{A.18}$$

Summing up all these results gives

$$\text{p.v.}\int_{-\infty}^{\infty}\frac{e^{ix}}{x}dx=i\pi. \tag{A.19}$$

Using this result, we can rewrite both the sign function sign(x) and the Heaviside step function $H(x)$

$$\text{sign}(x)=\begin{cases}1 & \text{for } x>0\\ 0 & \text{for } x=0\\ -1 & \text{for } x<0,\end{cases}\qquad H(x)=\begin{cases}1 & \text{for } x>1\\ \frac{1}{2} & \text{for } x=0\\ 0 & \text{for } x<0,\end{cases} \tag{A.20}$$

as an integral representation

$$\text{sign}(t)=\pm\frac{1}{i\pi}\int_{-\infty}^{\infty}\frac{e^{\pm itx}}{x}\,dx, \tag{A.21}$$

and

$$H(x)=\lim_{\epsilon\to 0^+}\mp\frac{1}{2\pi i}\int_{-\infty}^{\infty}\frac{1}{\tau\pm i\epsilon}e^{\mp ix\tau}d\tau. \tag{A.22}$$

For the sign function, performing the substitution $x'=tx$ in the integral leads to the expression (A.19). Depending on whether $t>0$ or $t<0$, this yields values of $+1$ or -1, respectively. At $t=0$, the integral reduces to a form similar to that of (A.13) whose principal value evaluates to zero. In the case of the step function, one uses Jordan's Lemma to evaluate the integral. However, when evaluating at $t=0$, the integral again reduces to the form (A.13), whose principal value is zero. This results in a slight deviation from the conventional definition of $H(0)$.

These integrals do not represent standard functions but are interpreted as distributions, which means that they gain meaning only when integrated against smooth test functions. A prime example is the delta function. Although these integrals do not converge in the usual sense, it is important to understand that the Cauchy principal value provides a framework for handling these integrals and takes care of the singularities.

B Carroll primaries with negative derivatives

As we pointed out, following [52], we can define an inverse time derivative operator

$$\partial_u^{-k}\Phi(u,z,\bar{z}) = \frac{1}{k!}\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}\Phi(u',z,\bar{z}), \tag{B.1}$$

such that for a primary Φ of weights $(h,\bar{h})$, the inverse derivative operator $\partial_u^{-k}\Phi$ is also a Carroll primary of weights $(h-\frac{k}{2},\bar{h}-\frac{k}{2})$. First, we check that it acts as an inverse operator

$$\begin{aligned}\partial_u\partial_u^{-k}\Phi &= \frac{1}{k!}\partial_u\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}\Phi(u',z,\bar{z})\\ &= \frac{1}{(k-1)!}\int_{-\infty}^{u} du'(u-u')^{k-1}\partial_{u'}\Phi(u',z,\bar{z}) = \partial_u^{-(k-1)}\Phi,\end{aligned} \tag{B.2}$$

where we used the Leibniz rule

$$\frac{d}{dx}\left(\int_{a(x)}^{b(x)} f(x,t)dt\right) = f(x,b(x))\frac{d}{dx}b(x) - f(x,a(x))\frac{d}{dx}a(x) + \int_{a(x)}^{b(x)}\frac{\partial}{\partial x}f(x,t)dt. \tag{B.3}$$

Similarly,

$$\begin{aligned}\partial_u^{-k}\partial_u\Phi &= \frac{1}{k!}\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}^2\Phi(u',z,\bar{z})\\ &= \frac{1}{(k-1)!}\int_{-\infty}^{u} du'(u-u')^{k-1}\partial_{u'}\Phi(u',z,\bar{z}) = \partial_u^{-(k-1)}\Phi,\end{aligned} \tag{B.4}$$

where we used integration by parts and we assumed $\lim_{u\to-\infty}\Phi \sim \mathcal{O}(u^{-k})$ such that the boundary terms vanish. To show that $\partial_u^{-k}\Phi$ is a Carroll primary we will individually consider supertranslations $\delta_{\mathcal{T}}\Phi = \mathcal{T}(z,\bar{z})\partial_u\Phi$

$$\begin{aligned}\delta_{\mathcal{T}}(\partial_u^{-k}\Phi) &= \frac{1}{k!}\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}\left(\mathcal{T}(z,\bar{z})\partial_{u'}\Phi(u',z,\bar{z})\right)\\ &= \mathcal{T}(z,\bar{z})\partial_u^{-(k-1)}\Phi = \mathcal{T}(z,\bar{z})\partial_u\left(\partial_u^{-k}\Phi\right),\end{aligned} \tag{B.5}$$

and super rotations $\delta_{\mathcal{Y}}\Phi = \left[(\partial\mathcal{Y})\left(h+\frac{u}{2}\partial_u\right)+\mathcal{Y}\partial\right]\Phi$

$$\begin{aligned}\delta_{\mathcal{Y}}(\partial_u^{-k}\Phi) &= \frac{1}{k!}\int_{-\infty}^{u} du'(u-u')^k\partial_{u'}\left(\left[(\partial\mathcal{Y})\left(h+\frac{u'}{2}\partial_{u'}\right)+\mathcal{Y}\partial\right]\Phi(u',z,\bar{z})\right)\\ &= \left(\left(h+\frac{1}{2}\right)(\partial\mathcal{Y})+\mathcal{Y}\partial\right)\partial_u^{-k}\Phi + \frac{\partial\mathcal{Y}}{2k!}\int_{-\infty}^{u} du'u'(u-u')^k\partial_{u'}^2\Phi(u',z,\bar{z})\\ &= \left(\left(h+\frac{1}{2}\right)(\partial\mathcal{Y})+\mathcal{Y}\partial\right)\partial_u^{-k}\Phi + \frac{\partial\mathcal{Y}}{2k!}\int_{-\infty}^{u} du'\left(ku(u-u')^{k-1}-(k+1)(u-u')^k\right)\partial_{u'}\Phi\\ &= \left(\mathcal{Y}\partial+\frac{u}{2}(\partial\mathcal{Y})\partial_u+\left(h-\frac{k}{2}\right)(\partial\mathcal{Y})\right)\partial_u^{-k}\Phi,\end{aligned} \tag{B.6}$$

where we used $(u-u')^{k+1} = u(u-u')^k - u'(u-u')^k$ to go from the second to the third line. Hence, $\partial_u^{-k}\Phi$ is indeed a Carroll primary.

C Asymptotic mode expansions

In this section, we will calculate the asymptotic mode expansions of the scalar field ϕ, the photon field A_μ, and the graviton field $h_{\mu\nu}$ at $\mathscr{I}^+$, using the more traditional round Bondi coordinates, following the method outlined in [14]. For a calculation of the asymptotic mode expansion in flat Bondi coordinates, as used in this thesis, see [30, 47].

Round (retarded) Bondi coordinates $(u, r, z, \bar{z})$ are related to Cartesian coordinates (t, x_1, x_2, x_3) by

$$|\vec{x}|^2 = r^2, \quad t = u + r, \quad x_1 + ix_2 = \frac{2rz}{1+z\bar{z}}, \quad x_3 = \frac{1-z\bar{z}}{1+z\bar{z}}, \tag{C.1}$$

Starting from the free field equation $\Box\phi = 0$, with well-known general solution

$$\phi(x) = \int \frac{d^3\vec{p}}{(2\pi)^3 2p_0}(\hat{a}_+(\vec{p})e^{ipx} + \hat{a}_-^\dagger(\vec{p})e^{-ipx}), \tag{C.2}$$

it is convenient to parameterise the 4-momentum as $p^\mu = \omega q^\mu$ where

$$q^\mu = \frac{1}{1+z\bar{z}}(1+z\bar{z}, z+\bar{z}, -i(z-\bar{z}), 1-z\bar{z}). \tag{C.3}$$

In this parameterisation, q^μ is a null vector ($q^2 = 0$) that specifies a point on the celestial sphere up to an overall scaling, and $\hat{q} = (q^1, q^2, q^3)$ denotes its spatial part and is normalised to unit length. Since massless particles have $p^2 = p_\mu p^\mu = -\omega^2 + |\vec{p}|^2 = 0$, we can write the momentum four-vector as $p^\mu = (\omega, \vec{p}) = \omega(1, \hat{q})$, with $\hat{q}$ the unit direction of photon momentum. Similarly, using $\vec{x} = r\hat{x}$ we can rewrite the exponents as

$$e^{ipx} = e^{-i\omega t + i\vec{p}\vec{x}} = e^{-i\omega u}e^{-i\omega r(1-\hat{q}\cdot\hat{x})} \approx e^{-i\omega u}e^{-\frac{i}{2}\omega r\theta^2}, \tag{C.4}$$

where we used that $\hat{q}\cdot\hat{q} = |\hat{k}||\hat{x}|\cos(\theta) = \cos\theta \approx 1 - \frac{\theta^2}{2}$ with θ small[17]. Furthermore, writing the integration measure in spherical coordinates

$$d^3p = |\vec{p}|^2 dp d\Omega_p = \omega^2 d\omega \sin(\theta) d\theta d\phi \approx \omega^2 d\omega \theta d\theta d\phi. \tag{C.5}$$

Plugging all these changes into our field solution gives

$$\phi(x) = \frac{1}{16\pi^3}\left(\int_0^\infty d\omega\ \omega\hat{a}_+(\omega\hat{q})e^{-i\omega u}\int_0^{2\pi} d\phi \int_0^\pi d\theta\ \theta e^{-\frac{i}{2}\omega r\theta^2} + c.c\right) + \mathcal{O}(r^{-2}), \tag{C.6}$$

where 'c.c.' stands for complex conjugate. Next, using a saddle point approximation for large r [30]:

$$\theta e^{-i\omega r\theta^2} = \frac{-i}{2r\omega}\delta(\theta) + \mathcal{O}(r^{-2}), \tag{C.7}$$

we obtain

$$\phi(x) = \frac{-i}{8\pi^2 r}\int_0^\infty d\omega\left(\hat{a}_+(\omega\hat{q})e^{-i\omega u} + \hat{a}_-^\dagger(\omega\hat{q})e^{i\omega u}\right) + \mathcal{O}(r^{-2}). \tag{C.8}$$

[17] Here we used that for a field propagating to $\mathscr{I}^+$, momentum and the unit direction of the field are colinear and hence θ is small.

Since the scalar field falls off as $\phi(x) \sim \mathcal{O}(1/r)$ we have

$$\phi(u,z,\bar{z}) = \lim_{r\to\infty} r\phi(u,r,z,\bar{z}), \tag{C.9}$$

resulting in

$$\phi(x) = \frac{-i}{8\pi^2}\int_0^\infty d\omega\left(\hat{a}_+(\omega\hat{q})e^{-i\omega u} + \hat{a}_-^\dagger(\omega\hat{q})e^{i\omega u}\right). \tag{C.10}$$

A similar calculation can be performed for the free photon field, which also satisfies a wave equation $\Box A_\mu = 0$. For A_μ, the plane wave solution is given by

$$A_\mu(X) = e\sum_{\alpha\in\pm}\int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}(\hat{a}_\alpha(\vec{p})\epsilon_\mu^{\alpha*}e^{ipx} + \hat{a}_\alpha(\vec{p})^\dagger\epsilon_\mu^\alpha e^{-ipx}). \tag{C.11}$$

Similarly, the graviton field $h_{\mu\nu}$ satisfies $\Box h_{\mu\nu} = 0$ with solution

$$h_{\mu\nu}(x) = \sqrt{32\pi G}\sum_{\alpha\in\pm}\int\frac{d^3\vec{p}}{(2\pi)^3 2p_0}(\epsilon_\mu^{\alpha*}\epsilon_\nu^{\alpha*}\hat{a}_\alpha(\vec{p})e^{ipx} + \epsilon_\mu^\alpha\epsilon_\nu^\alpha\hat{a}_\alpha(\vec{p})^\dagger e^{-ipx}). \tag{C.12}$$

For these fields, the same steps as above apply, with the only difference from the scalar field being the presence of additional polarisation vectors ϵ_μ

$$\begin{aligned}
A_\mu(x) &= \frac{-ie}{8\pi^2 r}\sum_{\alpha\in\pm}\int_0^\infty d\omega\left(\hat{a}_\alpha(\omega\hat{q})\epsilon_\mu^{\alpha*}e^{-i\omega u} + \hat{a}_\alpha^\dagger(\omega\hat{q})\epsilon_\mu^\alpha e^{i\omega u}\right) + \mathcal{O}(r^{-2}), \\
h_{\mu\nu}(x) &= \frac{-i\sqrt{32\pi G}}{8\pi^2 r}\sum_{\alpha\in\pm}\int_0^\infty d\omega\left(\hat{a}_\alpha(\omega\hat{q})\epsilon_\mu^{\alpha*}\epsilon_\nu^{\alpha*}e^{-i\omega u} + \hat{a}_\alpha^\dagger(\omega\hat{q})\epsilon_\mu^\alpha\epsilon_\nu^\alpha e^{i\omega u}\right) + \mathcal{O}(r^{-2}).
\end{aligned} \tag{C.13}$$

Since we started with field operators A_μ and $h_{\mu\nu}$ in the bulk, the index runs over $\mu,\nu \in (t, x_1, x_2, x_3)$. We can change coordinates using a Jacobian $A_\mu = \frac{\partial x^a}{\partial X^\mu}A_a$. Since we are interested in the z-components A_z, C_{zz} we need to compute $A_z = (\partial_z x^\mu)A_\mu$ and $C_{zz} = (\partial_z x^\mu)(\partial_z x^\nu)h_{\mu\nu}$. Starting from the Bondi coordinates

$$t = u + r, \quad x_1 = \frac{r(z+\bar{z})}{1+z\bar{z}}, \quad x_2 = \frac{-ir(z-\bar{z})}{1+z\bar{z}}, \quad x_3 = \frac{r(1-z\bar{z})}{1+z\bar{z}}, \tag{C.14}$$

and taking their z-derivatives we find

$$\partial_z t = 0, \quad \partial_z x = r\frac{1-\bar{z}^2}{(1+z\bar{z})^2}, \quad \partial_z y = -ir\frac{1+\bar{z}^2}{(1+z\bar{z})^2}, \quad \partial_z z = -r\frac{2\bar{z}}{(1+z\bar{z})^2}. \tag{C.15}$$

We can parameterise the polarisation vectors orthogonal to q^μ [14]

$$\epsilon_+^\mu(\vec{q}) = \frac{1}{\sqrt{2}}(\bar{z}, 1, -i, -\bar{z}), \quad \epsilon_-^\mu(\vec{q}) = \frac{1}{\sqrt{2}}(z, 1, i, -z), \tag{C.16}$$

such that $(\epsilon_+^\mu)^* = \epsilon_-^\mu$. Similarly, for a down index, the polarisation is given by

$$\epsilon_\mu^+(\vec{q}) = \frac{1}{\sqrt{2}}(-\bar{z}, 1, -i, -\bar{z}), \quad \epsilon_\mu^-(\vec{q}) = \frac{1}{\sqrt{2}}(-z, 1, i, -z). \tag{C.17}$$

We can now calculate the contributions from the polarisation vectors

$$\begin{aligned}(\partial_z x^\mu)\epsilon^+_\mu &= 0 + \frac{r}{\sqrt{2}}\frac{1-\bar{z}^2}{(1+z\bar{z})^2} - \frac{r}{\sqrt{2}}\frac{1+\bar{z}^2}{(1+z\bar{z})^2} + \sqrt{2}r\frac{\bar{z}^2}{(1+z\bar{z})^2} = 0,\\ (\partial_z x^\mu)\epsilon^-_\mu &= 0 + \frac{r}{\sqrt{2}}\frac{1-\bar{z}^2}{(1+z\bar{z})^2} + \frac{r}{\sqrt{2}}\frac{1+\bar{z}^2}{(1+z\bar{z})^2} + \sqrt{2}r\frac{\bar{z}^2}{(1+z\bar{z})^2} = \frac{\sqrt{2}r}{1+z\bar{z}}.\end{aligned} \tag{C.18}$$

Using that $(\epsilon^\mu_+)^* = \epsilon^\mu_-$ and that the fiels scale as $A_z \sim \mathcal{O}(1)$ and $C_{zz} \sim \mathcal{O}(r)$ we obtain the final result

$$\begin{aligned}A^{(0)}_z(u,z,\bar{z}) &= \lim_{r\to\infty} \partial_z x^\mu A_z(u,r,z,\bar{z})\\ &= \frac{-i}{8\pi^2}\frac{\sqrt{2}e}{1+z\bar{z}}\int_0^\infty d\omega\left(a_+(\omega\hat{x})e^{-i\omega u} - a^\dagger_-(\omega\hat{x})e^{i\omega u}\right),\end{aligned} \tag{C.19}$$

and

$$\begin{aligned}C_{zz}(u,z,\bar{z}) &= \lim_{r\to\infty}(\partial_z x^\mu)(\partial_z x^\nu)\frac{1}{r}h_{\mu\nu}(u,r,z,\bar{z})\\ &= \frac{-i}{4\pi^2}\frac{\sqrt{32\pi G}}{(1+z\bar{z})^2}\int_0^\infty d\omega\left(a_+(\omega\hat{x})e^{-i\omega u} - a^\dagger_-(\omega\hat{x})e^{i\omega u}\right).\end{aligned} \tag{C.20}$$

This result slightly differs from the same result in flat Bondi coordinates [30]

$$\begin{aligned}A^{(0)}_z(u,z,\bar{z}) &= \frac{-ie}{8\pi^2}\int_0^\infty d\omega\left[a_+(\omega,z,\bar{z})e^{-i\omega u} - a^\dagger_-(\omega,z,\bar{z})e^{i\omega u}\right],\\ C_{zz}(u,z,\bar{z}) &= \frac{-i\sqrt{32\pi G}}{8\pi^2}\int_0^\infty d\omega\left[a_+(\omega,z,\bar{z})e^{-i\omega u} - a^\dagger_-(\omega,z,\bar{z})e^{i\omega u}\right].\end{aligned} \tag{C.21}$$

D Computing the boundary propagator

In this section, we compute the propagators (5.24) and (5.39) of the Lagrangian $\mathcal{L} = i(\Phi\partial_u\bar{\Phi} - \bar{\Phi}\partial_u\Phi)$ by using an explicit path integral calculation. Since this calculation requires in- and out-states $\Phi^{\pm}$, which correspond to only half a Fourier transform and therefore lack a true inverse, the calculation becomes significantly more tedious and must be carried out with care.

We start by simplifying the Lagrangian using integration by parts

$$\mathcal{L} = i(\Phi\partial_u\bar{\Phi} - \bar{\Phi}\partial_u\Phi) = 2i\Phi\partial_u\bar{\Phi}. \tag{D.1}$$

We adopt the standard physics convention of including a minus sign in the exponential when Fourier transforming with respect to the time coordinate

$$\Phi(u,z,\bar{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \Phi_\omega(z,\bar{z})e^{-i\omega u}. \tag{D.2}$$

When working with complex fields, one must make a choice in how to define the Fourier transform. For the complex conjugate of $\Phi(u,z,\bar{z})$ we can define both

$$\begin{aligned} \bar{\Phi}(u,z,\bar{z}) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \bar{\Phi}_\omega(z,\bar{z})e^{i\omega u} \qquad \text{with } (\Phi_\omega)^* = \bar{\Phi}_\omega, \\ \bar{\Phi}(u,z,\bar{z}) &= \int_{-\infty}^{\infty} \frac{d\omega}{2\pi}\, \bar{\Phi}_\omega(z,\bar{z})e^{-i\omega u} \qquad \text{with } (\Phi_\omega)^* = \bar{\Phi}_{-\omega}, \end{aligned} \tag{D.3}$$

where the first option corresponds to taking the complex conjugate of the Fourier transform of Φ, and the second option treats $\bar{\Phi}$ as an independent field before Fourier transforming. Introducing the in- and out-states that were defined in (5.19)

$$\begin{aligned} \Phi^+(u,z,\bar{z}) &= \int_0^{+\infty} \frac{d\omega}{2\pi}\, \Phi_\omega(z,\bar{z})e^{-i\omega u}, \\ \Phi^-(u,z,\bar{z}) &= \int_{-\infty}^{0} \frac{d\omega}{2\pi}\, \Phi_\omega(z,\bar{z})e^{-i\omega u}, \end{aligned} \tag{D.4}$$

which satisfy

$$\Phi(u,z,\bar{z}) = \Phi^+(u,z,\bar{z}) + \Phi^-(u,z,\bar{z}), \tag{D.5}$$

yields different results depending on the chosen Fourier transform convention for $\bar{\Phi}$. Choosing the first option in (D.3), together with in- and out-states (D.4), leads to vanishing correlation functions between in-out-states, while the second option in (D.3) results in vanishing correlation functions between in-in and out-out states. For the remainder of this section, we adopt the second convention in (D.3), corresponding to $(\Phi_\omega)^* = \bar{\Phi}_{-\omega}$, where the complex field Φ is treated as an independent field. Substituting the resulting expressions into the action gives

$$\begin{aligned}
S_0 &= \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du\ 2i\Phi\partial_u\bar{\Phi} \\
&= \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{\infty} \frac{d\tilde{\omega}}{2\pi} \Phi_\omega(2i)(-i\tilde{\omega})\bar{\Phi}_{\tilde{\omega}} e^{-i(\omega+\tilde{\omega})u} \\
&= \int_{-\infty}^{\infty} d^2z \left(\int_0^{\infty} \frac{d\omega}{2\pi} \int_{-\infty}^{0} \frac{d\tilde{\omega}}{2\pi} \Phi_\omega(2\tilde{\omega})\bar{\Phi}_{\tilde{\omega}}\delta(\omega+\tilde{\omega}) + \int_{-\infty}^{0} \frac{d\omega}{2\pi} \int_0^{\infty} \frac{d\tilde{\omega}}{2\pi} \Phi_\omega(2\tilde{\omega})\bar{\Phi}_{\tilde{\omega}}\delta(\omega+\tilde{\omega}) \right) \\
&= \int_{-\infty}^{\infty} d^2z \left(\int_0^{\infty} \frac{d\omega}{2\pi} \Phi_\omega(-2\omega)\bar{\Phi}_{-\omega} + \int_{-\infty}^{0} \frac{d\omega}{2\pi} \Phi_\omega(-2\omega)\bar{\Phi}_{-\omega} \right),
\end{aligned}$$

where used that the integrals where the resulting delta function lies outside the domain of integration vanish. The resulting action only contains in-out contributions

$$S_0 = \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du\ 2i(\Phi^+\partial_u\bar{\Phi}^- - \bar{\Phi}^+\partial_u\Phi^-). \tag{D.6}$$

The next step is to introduce source terms

$$S[J, J^\dagger, K, K^\dagger] = \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du(J^\dagger\Phi^+ + J\bar{\Phi}^+ + K^\dagger\Phi^- + K\bar{\Phi}^-), \tag{D.7}$$

which are required to have a complete Fourier transform, ensuring that their inverse transform exists and is well-defined

$$J(u, z, \bar{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} J_\omega(z, \bar{z}) e^{-i\omega u} \quad \text{and} \quad K(u, z, \bar{z}) = \int_{-\infty}^{\infty} \frac{d\omega}{2\pi} K_\omega(z, \bar{z}) e^{-i\omega u}. \tag{D.8}$$

Substituting this into the action $S[\mathbf{J}] = S[J, J^\dagger, K, K^\dagger]$ gives

$$\begin{aligned}
S[\mathbf{J}] &= \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} \frac{d\tilde{\omega}}{2\pi} \left(\int_0^{\infty} \frac{d\omega}{2\pi} (J^\dagger_{\tilde{\omega}}\Phi_\omega + J_{\tilde{\omega}}\bar{\Phi}_\omega)\delta(\omega+\tilde{\omega}) + \int_{-\infty}^{0} \frac{d\omega}{2\pi} (K^\dagger_{\tilde{\omega}}\Phi_\omega + K_{\tilde{\omega}}\bar{\Phi}_\omega)\delta(\omega+\tilde{\omega}) \right) \\
&= \int_{-\infty}^{\infty} d^2z \left(\int_0^{\infty} \frac{d\omega}{2\pi} \left(J^\dagger_{-\omega}\Phi_\omega + K_\omega\bar{\Phi}_{-\omega} \right) + \int_{-\infty}^{0} \frac{d\omega}{2\pi} \left(K^\dagger_{-\omega}\Phi_\omega + J_\omega\bar{\Phi}_{-\omega} \right) \right).
\end{aligned}$$

The full action is obtained by adding the two actions together, $S = S_0 + S[J, J^\dagger, K, K^\dagger]$

$$\begin{aligned}
S = &\int_{-\infty}^{\infty} d^2z \int_0^{\infty} \frac{d\omega}{2\pi} \left(\Phi_\omega(-2\omega)\bar{\Phi}_{-\omega} + J^\dagger_{-\omega}\Phi_\omega + K_\omega\bar{\Phi}_{-\omega} \right) \\
+ &\int_{-\infty}^{\infty} d^2z \int_{-\infty}^{0} \frac{d\omega}{2\pi} \left(\Phi_\omega(-2\omega)\bar{\Phi}_{-\omega} + K^\dagger_{-\omega}\Phi_\omega + J_\omega\bar{\Phi}_{-\omega} \right).
\end{aligned} \tag{D.9}$$

Next, we perform a change of variables, using

$$\begin{aligned}
\Phi_\omega &\to \Phi_\omega + \frac{K_\omega}{2\omega}, \\
\bar{\Phi}_{-\omega} &\to \bar{\Phi}_{-\omega} + \frac{J^\dagger_{-\omega}}{2\omega},
\end{aligned} \tag{D.10}$$

for the positive-frequency integral and

$$\begin{aligned}\Phi_\omega &\to \Phi_\omega + \frac{J_\omega}{2\omega},\\ \bar{\Phi}_{-\omega} &\to \bar{\Phi}_{-\omega} + \frac{K^\dagger_{-\omega}}{2\omega},\end{aligned} \tag{D.11}$$

for the negative-frequency integral. As a result, the action becomes

$$S = \int_{-\infty}^{\infty} d^2z \left(\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \left(\Phi_\omega(-2\omega)\bar{\Phi}_{-\omega}\right) - \int_0^\infty \frac{d\omega}{2\pi} \frac{K_\omega J^\dagger_{-\omega}}{2\omega} - \int_{-\infty}^{0} \frac{d\omega}{2\pi} \frac{J_\omega K^\dagger_{-\omega}}{2\omega} \right). \tag{D.12}$$

We can now construct the generating functional

$$\begin{aligned} Z_0[J, J^\dagger, K, K^\dagger] &= \int \mathcal{D}\Phi e^{iS} \\ &= Z_0[\mathbf{0}] \exp\left\{ -i \int_{-\infty}^{\infty} d^2z \int_0^\infty \frac{d\omega}{2\pi} \frac{K_\omega J^\dagger_{-\omega}}{2\omega} \right\} \exp\left\{ -i \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{0} \frac{d\omega}{2\pi} \frac{J_\omega K^\dagger_{-\omega}}{2\omega} \right\}, \end{aligned} \tag{D.13}$$

where we used the shorthand notation $Z_0[\mathbf{0}] \equiv \int \mathcal{D}\Phi e^{i\int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du\ 2i\left(\Phi^+\partial_u\bar{\Phi}^- - \bar{\Phi}^+\partial_u\Phi^-\right)}$ and $\mathcal{D}\Phi = \mathcal{D}\Phi_+\mathcal{D}\Phi_-\mathcal{D}\bar{\Phi}_+\mathcal{D}\bar{\Phi}_-$. Taking the inverse Fourier transform of the sources

$$J_\omega(z,\bar{z}) = \int_{-\infty}^{\infty} du\ J(u,z,\bar{z})e^{i\omega u}, \tag{D.14}$$

we obtain

$$\begin{aligned} Z_0[J, J^\dagger, K, K^\dagger] &= Z_0[\mathbf{0}] \exp\left\{ -i \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du dv K(u,z,\bar{z}) \Omega^+(u-v) J^\dagger(v,z,\bar{z}) \right\} \\ &\times \exp\left\{ -i \int_{-\infty}^{\infty} d^2z \int_{-\infty}^{\infty} du dv K^\dagger(u,z,\bar{z}) \Omega^-(u-v) J(v,z,\bar{z}) \right\}, \end{aligned} \tag{D.15}$$

with

$$\begin{aligned} \Omega^+(u-v) &= \int_0^\infty \frac{d\omega}{2\pi} \frac{e^{i\omega(u-v)}}{2\omega},\\ \Omega^-(u-v) &= \int_{-\infty}^{0} \frac{d\omega}{2\pi} \frac{e^{-i\omega(u-v)}}{2\omega}. \end{aligned} \tag{D.16}$$

Using functional derivatives, we can obtain correlation functions from the generating functional (D.15)

$$\langle 0|T\Phi_+(u,z,\bar{z})|0\rangle = \int \mathcal{D}\Phi\ \Phi_+\ e^{iS_0} = \left(\frac{1}{i} \frac{\delta}{\delta J^\dagger(u,z,\bar{z})} \int \mathcal{D}\Phi\ e^{i(S_0 + S[J,J^\dagger,K,K^\dagger])} \right) \Big|_{\mathbf{J}=0}.$$

We now have all the necessary tools to compute the propagators, or two-point functions, of the first-order theory $\mathcal{L} = \Phi^+ \partial_u \bar{\Phi}^- - \bar{\Phi}^+ \partial_u \Phi^-$.

$$\begin{aligned}
&\langle 0 | T \Phi^+(u_1, z_1, \bar{z}_1) \bar{\Phi}^-(u_2, z_2, \bar{z}_2) | 0 \rangle = \frac{1}{Z_0[0]} \frac{1}{i} \frac{\delta}{\delta J^\dagger(u_1, \vec{z}_1)} \frac{1}{i} \frac{\delta}{\delta K(u_2, \vec{z}_2)} Z_0[\mathbf{J}] \Big|_{\mathbf{J}=0} \\
&= i \delta^{(2)}(\vec{z}_1 - \vec{z}_2) \int_0^\infty \frac{d\omega}{2\pi} \frac{e^{-i\omega(u_1 - u_2)}}{2\omega},
\end{aligned} \tag{D.17}$$

and similarly

$$\begin{aligned}
&\langle 0 | T \bar{\Phi}^+(u_1, z_1, \bar{z}_1) \Phi^-(u_2, z_2, \bar{z}_2) | 0 \rangle = \frac{1}{Z_0[0]} \frac{1}{i} \frac{\delta}{\delta J(u_1, \vec{z}_1)} \frac{1}{i} \frac{\delta}{\delta K^\dagger(u_2, \vec{z}_2)} Z_0[\mathbf{J}] \Big|_{\mathbf{J}=0} \\
&= -i \delta^{(2)}(\vec{z}_1 - \vec{z}_2) \int_0^\infty \frac{d\omega}{2\pi} \frac{e^{-i\omega(u_1 - u_2)}}{2\omega}.
\end{aligned} \tag{D.18}$$

To compute the two-point function of descendant fields such as $\langle 0 | \Phi^+ \partial_u \bar{\Phi}^- | 0 \rangle$, one should add a source term $K \partial_u \bar{\Phi}^-$ and take the Fourier transform of $\partial_u \bar{\Phi}^-$, which introduces no soft poles during the change of variables and ultimately leads to (5.26).

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