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Thermodynamic product formulae for Finslerian Kiselev black hole

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Abstract The main goal of this paper is to solve the Finslerian Kiselev black hole and investigate its thermodynamic characteristics surrounded by dust, radiation, quintessence, and the cosmological constant. Since these black holes have multiple horizons, thus we calculate thermodynamic quantities, including area, horizon radius, Bekenstein–Hawking entropy, Hawking temperature, surface gravity, and Komar energy products. In addition, we study all universal relations of these black holes and discuss the role and effect of the Finslerian Ricci scalar (η) on Finslerian Kiselev black holes and their stability.

1 Introduction

Recent highly accurate observations have established the existence of a gravitationally repelling interaction (cosmic dark energy), which is the cause of the universe's accelerating expansion [1]. The equation of state $p_q = \omega_q \rho_q$, where p_q, ω_q, and ρ_q are the pressure, energy density, and parameter of the equation of state, respectively, with ω_q, assuming values in the interval $1 < \omega_q < \frac{1}{3}$, is one of the candidates that should be responsible for this phenomenon. Quintessential dark energy, or simply quintessence, is the name given to this type of cosmic dark energy. This type of dark energy should have certain gravitational effects on black holes (BHs) in an astrophysical scenario, such as deflecting light from far-off stars [2], and should be considered in this context. As a result, to understand the function the quintessence played in this situation, it is necessary to solve the Einstein equations for this source, which Kiselev did in 2002 [3].

The fundamental components of nature, from the perspective of string theory, are extended one-dimensional objects rather than point particles. According to this paradigm, it is crucial to comprehend the gravitational effects caused by a group of strings. It can be done by resolving the Einstein equations using a limited source of strings. Letelier [4] presented a gauge invariant model of a cloud of surface-forming strings along with this line of investigation. This model is described by taking a bi-vector connected to the string worksheet. With this approach, he was able to solve the Einstein equations for a cloud of strings that had cylindrical, plane, and spherical symmetry. In the first scenario, Letelier discovered the Einstein equations solution, which describes a BH surrounded by a spherically symmetric cloud of strings and is mathematically similar to the Schwarzschild solution but has a larger horizon.

Since the universe can theoretically be described based on fundamental extended objects like one-dimensional strings rather than particles, it seems natural that this idea would be extended to take into account the strings cloud that surrounds BHs and search for quantifiable gravitational effects of these clouds. On the other hand, the existence of quintessence in the vicinity of BHs should also have some astrophysical repercussions, so it must be considered. This topic is among the most significant investigations related to BHs because the study of BHs' thermodynamic properties has revealed many facets of their physics, especially in a background with quintessence and (or) a cloud of strings as additional sources of the gravitational field.

Recently, physics researchers have studied BHs' dynamical and thermodynamic characteristics in great detail. There may be an inner (Cauchy) horizon in some BHs in addition to the outer (event) horizon. Understanding the microscopic

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characteristics of entropies on both the inner and outer horizons would help one better understand BH properties thermodynamics. Thermodynamic product formulas have been researched recently in this area [5–8]. Several physicists have been intrigued to look into the area and entropy products of BHs, which have multiple horizons, as well as other thermodynamic products [9–12]. The basic objective of quantum gravity [13,14] is to comprehend the microscopic details of the BH's exterior and interior entropies. The horizon area product and its mass dependence in larger curvature gravity models are discussed in Ref. [10]. In Ref. [15], a variety of thermodynamic products are assessed, and phase transition is looked at.

We list various reasons why people are interested in studying the entropy relations of several horizons and physics near other horizons. First, we realized from [16–18] that green functions are sensitive to the geometry near all BH horizons in addition to the outermost horizon. As a result, it is reasonable to assume that each horizon's entropy contributes to the microscopically controlled characteristics of BH. Second, a generic definition of extremity can be derived from the entropy inequalities of multi-horizons in Einstein–Maxwell theory [7,8,19–23]. Moreover, they result in the No-Go theorem [13] for the likelihood of force balance between two rotating BHs. At every horizon, this subject makes physics more appealing. Moreover, the influence of other horizons is required to sustain the mass' independence [24–27]. The equations of Clausius–Clapeyron–Ehrenfest [28,29] provide a framework for comprehending the thermodynamic idea of phase transitions. These equations play a crucial role in categorizing the higher-order types (continuous) or first-order phase transitions that occur in thermodynamic systems. Phase transition phenomenon in BHs has never been thoroughly investigated, despite its numerous applicability in various systems [30,31].

A background study of a Finslerian Kiselev black hole (FK BH) surrounded by the fields and the effect of these additional sources on the thermodynamic properties and quasinormal fluctuations is crucial. In this paper, We want to discuss the thermodynamic products on the Cauchy horizon and the event horizon of FK BHs. On the Cauchy horizon and event horizon, we compute the following quantities: area, horizon radii, Bekenstein–Hawking entropy, Hawking temperature, surface gravity, Komar energy, irreducible mass, and specific heat.

2 Finslerian Kiselev black hole

Let F be a Finsler metric on a differentiable manifold M, has the form $F = F(x, y)$ and that it is a function of (x^i, y^i) in TM. The Finslerian geodesic with respect to F is characterized by

$$\frac{d^2x^\upsilon}{d\tau^2} + 2G^\upsilon(x, y) = 0, \tag{1}$$

where the geodesic spray is

$$G^\upsilon = \frac{1}{4}g^{\upsilon\omega}\left(\frac{\partial^2 F^2}{\partial x^k \partial y^\omega}y^k - \frac{\partial F^2}{\partial x^\omega}\right). \tag{2}$$

The metric coefficients can be written as follows,

$$g_{\upsilon\omega} = \frac{\partial^2\left(\frac{F^2}{2}\right)}{\partial y^\upsilon \partial y^\omega}. \tag{3}$$

Ricci scalar of Finsler structure can be expressed as,

$$Ric = R^\mu_\mu = \frac{1}{F^2}\left(2\frac{\partial G^\mu}{\partial x^\mu} - y^\nu\frac{\partial^2 G^\mu}{\partial x^\nu \partial y^\mu} + 2G^\nu\frac{\partial^2 G^\mu}{\partial y^\nu \partial y^\mu} - \frac{\partial G^\mu}{\partial y^\nu}\frac{\partial G^\nu}{\partial y^\mu}\right), \tag{4}$$

where R^μ_μ exclusively relies on the Finsler structure and is insensitive to connections. The Ricci tensor in Finsler geometry is written in the following form,

$$Ric_{\upsilon\omega} = \frac{\partial^2}{\partial y^\upsilon y^\omega}\left(\frac{1}{2}F^2 Ric\right). \tag{5}$$

In this case, the Finsler metric $F = F(x, y)$ is a function of (x^i, y^i) in a standard coordinate system. The angular coordinate was taken into account in the following ansatz as $\bar{F}^2(\theta, \phi, y^\theta, y^\phi)$.

$$F^2 = e^\nu y^t y^t - e^\mu y^r y^r - r^2\bar{F}^2(\theta, \phi, y^\theta, y^\phi), \tag{6}$$

where $\nu = \nu(r)$ and $\mu = \mu(r)$ are functions of the radial coordinate only and $\bar{F}$ does not depend on y^t and y^r. Therefore, Finsler metric coefficients can be obtained as follows,

$$g_{\upsilon\omega} = diag(e^\nu, -e^\mu, -r^2\bar{g}_{ij}), \tag{7}$$
$$g^{\upsilon\omega} = diag(e^{-\nu}, -e^{-\mu}, -r^{-2}\bar{g}^{ij}), \tag{8}$$

where $\bar{g}_{ij}$ and its inverse are the components of metric that derived from $\bar{F}$ and the index i, j run over angular coordinate θ, ϕ. By inserting the Finsler structure (6) into the formula (2), we have

$$\begin{aligned} G^t &= \frac{1}{2}\nu' y^t y^r, \\ G^r &= \frac{\nu'}{4}e^{\nu-\mu}y^t y^t + \frac{\mu'}{4}y^r y^r - \frac{r}{2}e^{-\mu}\bar{F}^2, \\ G^\upsilon &= \frac{1}{r}y^\upsilon y^r + \bar{G}^\upsilon, \quad (\upsilon = \theta, \phi) \end{aligned} \tag{9}$$

where $\bar{G}^\upsilon$ is the geodesic spray coefficient derived by $\bar{F}$, and the prime represents the derivative with regard to r. Substituting the geodesic coefficients (9) in Eq. (4), we have

$$RicF^2 = \left[\frac{1}{2}\left(\nu'' + (\nu')^2\right)e^{\nu-\mu} - \frac{\nu'}{4}e^{\nu-\mu}\left(\mu' + \nu'\right) + \frac{\nu'}{r}e^{\nu-\mu}\right]y^t y^t + \left[-\frac{1}{2}\left(\nu'' + (\nu')^2\right) + \frac{\nu'}{4}\left(\mu' + \nu'\right) + \frac{\mu'}{r}e^{-\mu}\right]y^r y^r + \left[\bar{R}ic - e^{-\mu} + \frac{r}{2}e^{-\mu}\left(\mu' - \nu'\right)\right]\bar{F}^2, \quad (10)$$

and also, in the Finsler structure, the scalar curvature is as follows,

$$S = g^{\upsilon\omega}Ric_{\upsilon\omega} = \left(\nu'' + (\nu')^2\right)e^{-\mu} - \frac{\nu'}{2}e^{-\mu}\left(\mu' + \nu'\right) + \frac{2}{r}e^{-\mu}\left(\nu' - \mu'\right) - \frac{2}{r^2}\bar{R}ic + \frac{2}{r^2}e^{-\mu}. \quad (11)$$

In the context of the Rastall theory of gravity, we are searching for the general non-vacuum spherically symmetric static uncharged-charged BH solutions in this section. The Rastall field equations can thus be expressed as,

$$G_{\upsilon\omega} + \kappa\lambda g_{\upsilon\omega}S = \kappa T_{\upsilon\omega}, \quad (12)$$

where λ is the Rastall parameter, which is a measure of deviance from the fundamental conservation law of general relativity, κ is the Rastall gravitational coupling constant, and $G_{\upsilon\omega} = Ric_{\upsilon\omega} - \frac{1}{2}g_{\upsilon\omega}S$. In the limit of $\lambda \to 0$ and $\kappa = 8\pi G_N$, where G_N is the Newton gravitational coupling constant, these field equations transform into GR field equations. We derive non-vanishing Rastall tensor components using this metric,

$$H_{\upsilon\omega} = G_{\upsilon\omega} + \kappa\lambda g_{\upsilon\omega}S. \quad (13)$$

Let's start by examining the recent solution for a static, spherically symmetric BH immersed in a field. This field can generally be made up of dust (d), radiation (r), quintessence (q), cosmological constant (c), or even any combination of these. It can also include the electrostatic charge, that non-vanishing Maxwell tensor components as $E^i_j = \frac{Q^2}{\kappa r^4}diag(-1, -1, 1, 1)$. Therefore, we can consider the non-zero energy-momentum tensor component as follows

$$T^t_t = T^r_r = \rho_s + \frac{Q^2}{\kappa r^4}, \quad (14)$$

$$T^\theta_\theta = T^\phi_\phi = -\frac{1}{2}(1 + 3\omega_s)\rho_s - \frac{Q^2}{\kappa r^4}. \quad (15)$$

Here, we utilized the index "s" instead of every field stated before, and ρ_s and ω_s are the density and equation of state parameter, respectively. Considering Eqs. (12), (14), and the symmetry of the problem, we can assume that $\mu = -\nu$ without loss of generality. Make the substitution as follows:

$$\mu = -\ln({}_F f_s(r)).$$

Therefore,

$$G^t_t = G^r_r = -\frac{1}{r^2}\left(r\,{}_F f'_s(r) + {}_F f_s(r) - \bar{R}ic\right), \quad (16)$$

$$G^\theta_\theta = G^\phi_\phi = -\frac{1}{r^2}\left(\frac{1}{2}r^2\,{}_F f''_s(r) + r\,{}_F f'_s(r)\right), \quad (17)$$

and

$$S = \frac{1}{r^2}\left(r^2\,{}_F f''_s(r) + 4r\,{}_F f'_s(r) + 2\,{}_F f_s(r) - 2\bar{R}ic\right). \quad (18)$$

$H^t_t = \kappa T^t_t$, $H^r_r = \kappa T^r_r$, and $H^\theta_\theta = \kappa T^\theta_\theta$ components of Rastall field equations of the following differential equations result in

$$-\frac{1}{r^2}\left(r\,{}_F f'_s(r) + {}_F f_s(r) - \bar{R}ic\right) + \frac{\kappa\lambda}{r^2}\left(r^2\,{}_F f''_s(r) + 4r\,{}_F f'_s(r) + 2\,{}_F f_s(r) - 2\bar{R}ic\right) = \kappa\rho_s + \frac{Q^2}{r^4},$$
$$-\frac{1}{r^2}\left(\frac{1}{2}r^2\,{}_F f''_s(r) + r\,{}_F f'_s(r)\right) + \frac{\kappa\lambda}{r^2}\left(r^2\,{}_F f''_s(r) + 4r\,{}_F f'_s(r) + 2\,{}_F f_s(r) - 2\bar{R}ic\right) = -\frac{1}{2}(1 + 3\omega_s)\kappa\rho_s - \frac{Q^2}{r^4}. \quad (19)$$

Now, one can derive the following general solution for the metric function by solving the set of differential equations (19),

$${}_F f_s(r) = \eta - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_s}{r^{\frac{1+3\omega_s-6\kappa\lambda(1+\omega_s)}{1-3\kappa\lambda(1+\omega_s)}}}, \quad (20)$$

with energy density as

$$\rho_s = \frac{-3\mathcal{W}_s N_s}{\kappa r^{\frac{3(1+\omega_s)-12\kappa\lambda(1+\omega_s)}{1-3\kappa\lambda(1+\omega_s)}}}, \quad (21)$$

where the quantities with $_F$ denote the quantities of FKBH, $\bar{R}ic = \eta =$ constant, the BH's mass (M), and the surrounding field structure parameter (N_s), respectively, are two integration constants and $\mathcal{W}_s$ is a geometric constant that is dependent on the Rastall geometrical constants κ, λ, and ω_s of the BH that surrounds the field, that is:

$$\mathcal{W}_s = -\frac{(1 - 4\kappa\lambda)(\kappa\lambda(1 + \omega_s) - \omega_s)}{\left(1 - 3\kappa\lambda(1 + \omega_s)\right)^2}. \quad (22)$$

This BH looks like a charged Kiselev-like BH with a deficit solid angle. Because $\eta = 1 - a^2$ where a is the deficit solid angle parameter. Since the gravitational influence of a cloud of strings is the same as that caused by a solid deficiency angle. If $\eta \neq 1$, this solution also is similar to a BH surrounded by a cloud of strings by comparing with the results of Refs. [32,33].

Calculations show that η appears only in ${}_F f_s(r)$. $\mathcal{W}_s$ and ρ_s remain unchanged in this case (see more in Table 1), the results obtained for $\lambda \neq 0$ are similar to the Riemannian case in Ref. [34]. Considering this, in this study, we assume that $\lambda = 0$.

2.1 Finslerian Kiselev black hole: $\omega_d = \lambda = 0$

The BH horizons for a FK BH that is surrounded by dust corresponds to ${}_F f_d(r) = 0$,

$$ {}_F r_\pm = \frac{M_{eff} \pm \sqrt{M_{eff}^2 - 4\eta Q^2}}{2\eta}, \tag{23} $$

where $M_{eff} = 2M + N_d$. The symbols ${}_F r_+$ and ${}_F r_-$ stand for the event and Cauchy horizons, respectively. Only in circumstances where $M_{eff}^2 - 4\eta Q^2 \geq 0$ holds will assumptions be made. Using Eq. (23), their product obtains the following form,

$$ {}_F r_+ {}_F r_- = \frac{Q^2}{\eta}, \tag{24} $$

which is independent of mass. The area of this BH is given by,

$$ \begin{aligned} {}_F A_\pm &= \int_0^{2\pi} \int_0^{\pi} \sqrt{g_{\theta\theta} g_{\phi\phi}} d\theta d\phi = 4\pi \, {}_F r_\pm^2 \\ &= \frac{4\pi}{\eta} (M_{eff} \, {}_F r_\pm - Q^2), \end{aligned} \tag{25} $$

and their product will be as

$$ {}_F A_+ {}_F A_- = \frac{16\pi^2}{\eta^2} Q^4, \tag{26} $$

Thus the area product formula in FKd BH is universal. The Bekenstein–Hawking entropy is,

$$ {}_F S_\pm = \frac{{}_F A_\pm}{4} = \frac{\pi}{\eta} (M_{eff} \, {}_F r_\pm - Q^2), \tag{27} $$

thus the entropy product formula is given by,

$$ {}_F S_+ {}_F S_- = \frac{\pi^2}{\eta^2} Q^4, \tag{28} $$

it is also mass-independent. The entropy product formula is also universal in FKd BH. The Hawking temperature is calculated using the method below,

$$ {}_F T_\pm = \frac{{}_F k_\pm}{2\pi} = \frac{1}{4\pi} \frac{df(r)}{dr}\bigg|_{{}_F r_\pm} = \frac{1}{4\pi} \frac{2\eta \, {}_F r_\pm - M_{eff}}{{}_F r_\pm^2}, \tag{29} $$

and their product yields,

$$ {}_F T_+ {}_F T_- = \frac{\eta^2}{16\pi^2} \frac{4\eta Q^2 - M_{eff}^2}{Q^4}. \tag{30} $$

By using Eq. (29), the surface gravity is given by,

$$ {}_F k_\pm = 2\pi \, {}_F T_\pm = \frac{1}{2} \frac{2\eta \, {}_F r_\pm - M_{eff}}{{}_F r_\pm^2}, \tag{31} $$

and their product yields,

$$ {}_F k_+ {}_F k_- = \frac{\eta^2}{4} \frac{4\eta Q^2 - M_{eff}^2}{Q^4}. \tag{32} $$

The surface gravity and surface temperature products are not universal since they depend on mass. The BH's Komar energy is described as,

$$ {}_F E_\pm = 2 \, {}_F S_\pm \, {}_F T_\pm = \frac{2\eta \, {}_F r_\pm - M_{eff}}{2}, \tag{33} $$

and their product is,

$$ {}_F E_+ {}_F E_- = \frac{4\eta Q^2 - M_{eff}^2}{4}. \tag{34} $$

We can observe that energy product is not a universal quantity. The surface area ${}_F A_\pm$ of a non-spinning BH is connected to its irreducible mass M_{irr}, which is given by,

$$ {}_F M_{irr_\pm}^2 = \frac{{}_F S_\pm}{4\pi} = \frac{{}_F A_\pm}{16\pi}, \tag{35} $$

the product of the irreducible mass at the horizons is,

$$ {}_F M_{irr_+} {}_F M_{irr_-} = \frac{Q^2}{4\eta}, \tag{36} $$

the above equation is also a universal quantity. The specific heat for FKd BH is given,

$$ {}_F C_\pm = \frac{\partial M}{\partial \, {}_F T_\pm}. \tag{37} $$

Using Eq. (37), we have the following equation,

$$ {}_F C_\pm = \pi \, {}_F r_\pm^2 \frac{2\eta \, {}_F r_\pm - M_{eff}}{M_{eff} - \eta \, {}_F r_\pm}. \tag{38} $$

Further, we study the following cases: **Case I:** If ${}_F C_\pm > 0$, the BH is thermodynamically stable. If we take into account horizon radii for FKd BH in the range $\frac{M_{eff}}{2\eta} < {}_F r_\pm < \frac{M_{eff}}{\eta}$, then ${}_F C_\pm$ is positive, and FKd BH is therefore thermodynamically stable in the selected range of horizon radii ${}_F r_\pm$.

Case II: If ${}_F C_\pm < 0$, the BH is thermodynamically unstable. If we consider horizon radii for FKd BH in the following range, $0 < {}_F r_\pm < \frac{M_{eff}}{2\eta}$, ${}_F r_\pm > \frac{M_{eff}}{\eta}$, then ${}_F C_\pm$ is negative. Thus, in the selected range of ${}_F r_\pm$, FKd BH is thermodynamically unstable.

Table 1 Metric function and energy density versus surrounding fields

ω_s	${}_F f_s(r)$	$\mathcal{W}_s$	ρ_s
$\omega_d = 0$	${}_F f_d(r) = \eta - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_d}{r^{\frac{1-6\kappa\lambda}{1-3\kappa\lambda}}}$	$\mathcal{W}_d = -\frac{(1-4\kappa\lambda)(\kappa\lambda)}{(1-3\kappa\lambda)^2}$	$\rho_d = \frac{3\lambda(1-4\kappa\lambda)N_d}{(1-3\kappa\lambda)^2} r^{-\frac{3-12\kappa\lambda}{1-3\kappa\lambda}}$
$\omega_r = \frac{1}{3}$	${}_F f_r(r) = \eta - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_r}{r^2}$	$\mathcal{W}_r = \frac{1}{3}$	$\rho_r = \frac{-N_r}{\kappa r^4}$
$\omega_q = -\frac{2}{3}$	${}_F f_q(r) = \eta - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_q}{r^{-\frac{1+2\kappa\lambda}{1-\kappa\lambda}}}$	$\mathcal{W}_q = -\frac{(1-4\kappa\lambda)(2+\kappa\lambda)}{3(1-\kappa\lambda)^2}$	$\rho_q = \frac{(1-4\kappa\lambda)(2+\kappa\lambda)N_q}{\kappa(1-\kappa\lambda)^2} r^{-\frac{1-4\kappa\lambda}{1-\kappa\lambda}}$
$\omega_c = -1$	${}_F f_c(r) = \eta - \frac{2M}{r} + \frac{Q^2}{r^2} - \frac{N_c}{r^{-2}}$	$\mathcal{W}_c = -(1 - 4\kappa\lambda)$	$\rho_c = \frac{3(1-4\kappa\lambda)N_c}{\kappa}$

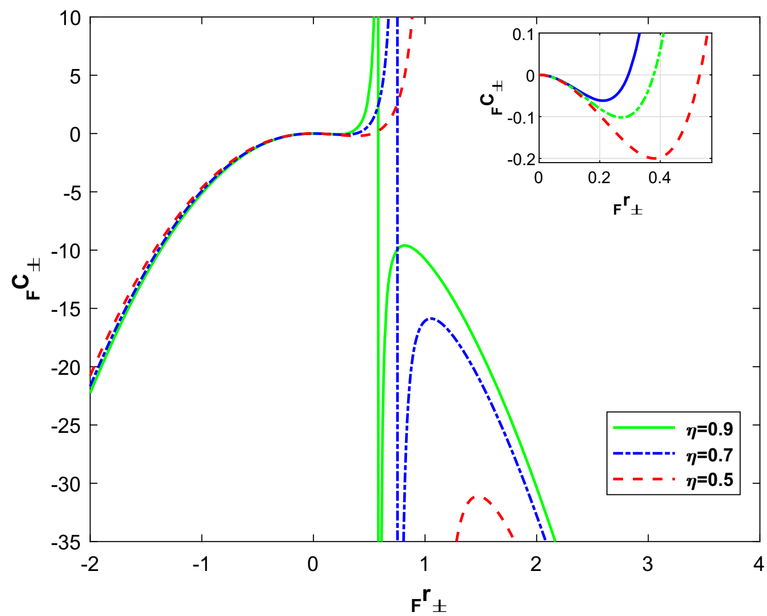


Fig. 1 Specific heat of the FKd BH

In this case, stability and instability distance of BH has been obtained by assuming the Finslerian Ricci scalar is positive ($\eta > 0$).

Case III: If the horizon radius ${}_F r_\pm$ satisfies the following formula, then ${}_F C_\pm$ diverges:

$$M_{eff} - \eta\, {}_F r_\pm = 0.$$

Hence, if ${}_F r_\pm = \frac{M_{eff}}{\eta}$, FKd BH experiences a second-order phase transition. Because whenever the specific heat diverges, BH undergoes a second-order phase transition. Figure 1 shows that as η decreases, the length of the BH stability interval increases. But this relation is directly related to the matter because $\eta = 1 - a^2$ i.e., the length of the interval increases with increasing the matter.

Figure 2a and b show the specific heat behavior in contour and 3-D mode, respectively, and the η range is assumed to be between -1 and 1.

The specific heat of the BH is given by the following form,

$$ {}_F C_\pm = {}_F T_\pm \frac{\partial {}_F S_\pm}{\partial {}_F T_\pm}. \tag{39}$$

Using ${}_F S_\pm = \pi\, {}_F r_\pm^2$, we can obtain the specific heat as follows:

$$ {}_F C_\pm = -{}_F S_\pm \frac{(\eta - 2)\left(\frac{{}_F S_\pm}{\pi}\right) + Q^2}{(\eta - 1)\left(\frac{{}_F S_\pm}{\pi}\right) + Q^2}. \tag{40}$$

Figure 3 shows a first-order phase transition from unstable to stable states. It appears as a point at which the specific heat transitions from negative values to positive ones without discontinuity. When the Finslerian Ricci scalar (η) is increased, the phase transition point is moved to the upper entropies.

2.2 Finslerian Kiselev black hole: $\omega_r = \frac{1}{3}$, $\lambda = 0$

The BH horizons for a FK BH that is surrounded by radiation corresponds to ${}_F f_r(r) = 0$.

$$ {}_F r_\pm = \frac{M \pm \sqrt{M^2 - \eta Q_{eff}}}{\eta}, \tag{41}$$

where $Q_{eff} = Q^2 - N_r$. We only consider the case where $M^2 - \eta Q_{eff} \geq 0$ holds. Using Eq. (41), their product yields,

$$ {}_F r_+ {}_F r_- = \frac{Q_{eff}}{\eta}, \tag{42}$$

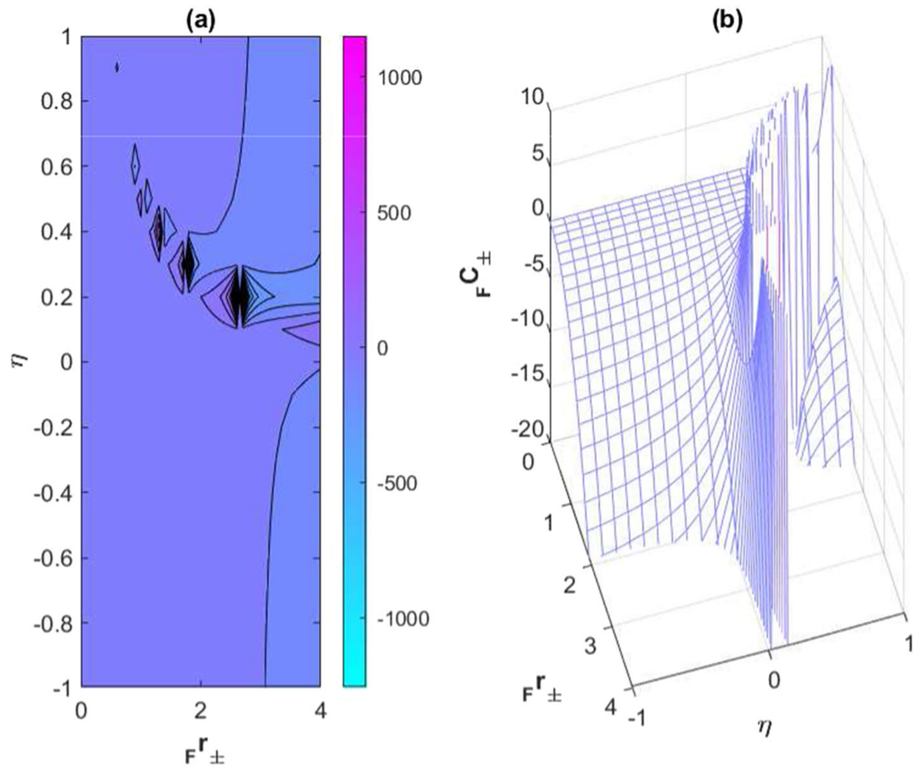


Fig. 2 Specific heat of the FKd BH with respect to η and $_F r_\pm$

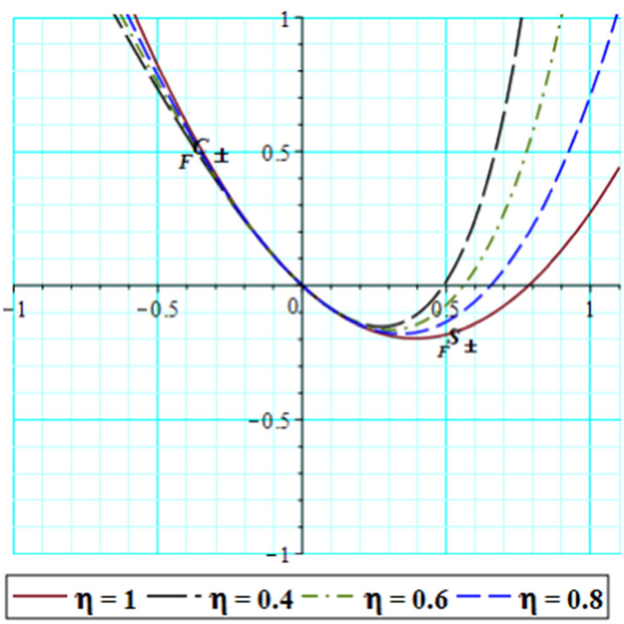


Fig. 3 Specific heat behavior vs entropy for various values of η

Table 2 Area product, Bekenstein–Hawking entropy product, Hawking temperature product, surface gravity product, Komar energy product, irreducible mass product, for FKr BH

$_F A_\pm$	$_F S_\pm$	$_F T_\pm$
$\frac{4\pi}{\eta}(2M\,_F r_\pm - Q_{eff})$	$\frac{\pi}{\eta}(2M\,_F r_\pm - Q_{eff})$	$\frac{\eta\,_F r_\pm - M}{2\pi\,_F r_\pm^2}$
$_F k_\pm$	$_F E_\pm$	$_F M_{irr\pm}$
$\frac{\eta\,_F r_\pm - M}{_F r_\pm^2}$	$\eta\,_F r_\pm - M$	$\frac{_F r_\pm}{2}$
$_F A_{+F} A_-$	$_F S_{+F} S_-$	$_F T_{+F} T_-$
$\frac{16\pi^2}{\eta^2} Q_{eff}^2$	$\frac{\pi^2}{\eta^2} Q_{eff}^2$	$\frac{\eta^2}{4\pi^2} \frac{\eta Q_{eff} - M^2}{Q_{eff}^2}$
$_F k_{+F} k_-$	$_F E_{+F} E_-$	$_F M_{irr+F} M_{irr-}$
$\eta^2 \frac{\eta Q_{eff} - M^2}{Q_{eff}^2}$	$\eta Q_{eff} - M^2$	$\frac{Q_{eff}}{4\eta}$

which is independent of mass. It can be seen in Table 2 that only the products of the area, Bekenstein–Hawking entropy, and irreducible mass are independent of M and are universal quantities. Using Eq. (37), one can obtain specific heat as follows,

$$_F C_\pm = 2\pi\,_F r_\pm^2 \frac{\eta\,_F r_\pm - M}{2M - \eta\,_F r_\pm}. \tag{43}$$

Case I: If we take into account horizon radii for FKr BH in the range $\frac{M}{\eta} < \,_F r_\pm < \frac{2M}{\eta}$, then $_F C_\pm$ is positive, and FKr BH is therefore thermodynamically stable in the selected range of horizon radii $_F r_\pm$.

Case II: If we consider horizon radii for FKr BH in the following range, $0 < \,_F r_\pm < \frac{M}{\eta}$, $_F r_\pm > \frac{2M}{\eta}$, then $_F C_\pm$ is negative. Thus, in the selected range of $_F r_\pm$, FKr BH is thermodynamically unstable.

In this case, stability and instability distance of BH has been obtained by assuming the Finslerian Ricci scalar is positive ($\eta > 0$).

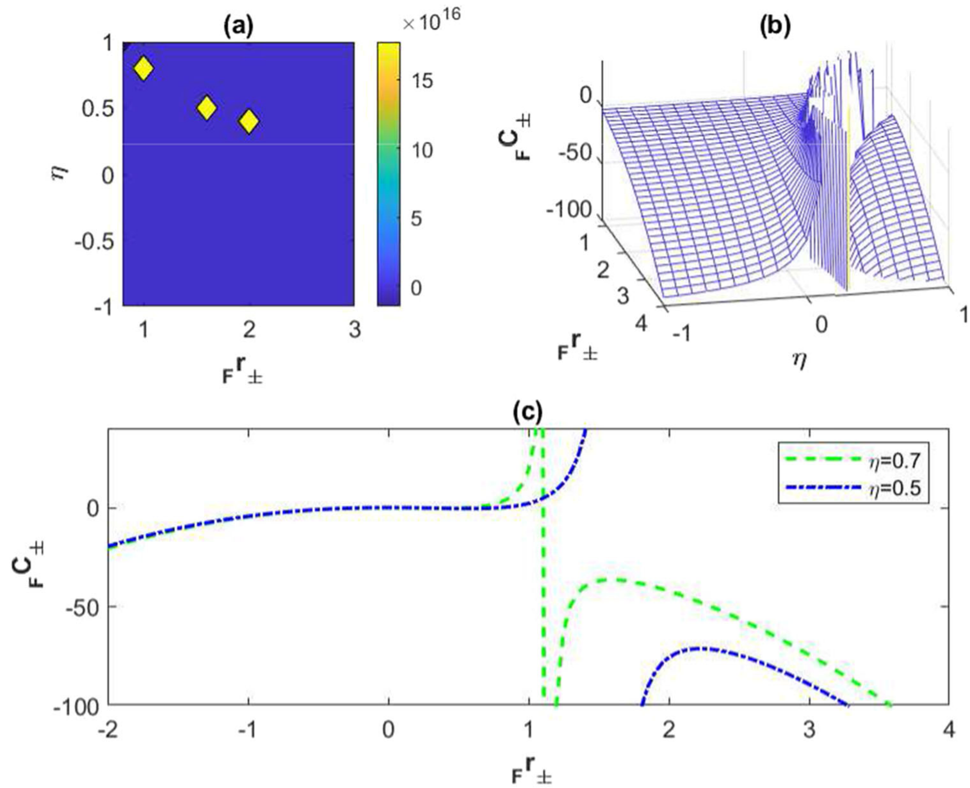


Fig. 4 Specific heat of the FKr BH with respect to η and ${}_F r_\pm$

Case III: If the horizon radius ${}_F r_\pm$ satisfies the following formula, then ${}_F C_\pm$ diverges:

$$2M - \eta {}_F r_\pm = 0.$$

Hence, if ${}_F r_\pm = \frac{2M}{\eta}$, FKr BH experiences a second-order phase transition.

Figure 4c shows that as η decreases, the length of the BH stability interval increases and Fig. 4a and b show the specific heat behavior in contour and 3-D mode, respectively, and the η range is assumed to be between -1 and 1.

Similar to Eq. (40), we can get the specific heat in terms of entropy

$$_F C_\pm = \frac{-2 {}_F S_\pm}{\pi} \frac{M - \eta \left(\frac{{}_F S_\pm}{\pi}\right)^{\frac{1}{2}}}{2M - \eta \left(\frac{{}_F S_\pm}{\pi}\right)^{\frac{1}{2}}}. \tag{44}$$

Figure 5a shows a first-order phase transition from unstable to stable states. It appears as a point at which the specific heat transitions from negative values to positive ones without discontinuity. When the Finslerian Ricci scalar (η) is increased, the phase transition point is moved to the lower entropies. And also, Fig. 5b shows that a second-order transition happens. In this case, also when the Finslerian Ricci scalar (η) increases, the phase transition point is moved to the lower entropies.

2.3 Finslerian Kiselev black hole $\omega_q = \frac{-2}{3}$, $\lambda = 0$

The BH horizons for a FK BH that is surrounded by quintessence corresponds to ${}_F f_q(r) = 0$ in which $Q^2 = 0$ is assumed. We only consider the case where $\eta^2 - 8MN_q \geq 0$,

$$_F r_\pm = \frac{\eta \pm \sqrt{\eta^2 - 8MN_q}}{2N_q}. \tag{45}$$

In this case, although the radius of the FKq BH depends on the Finslerian Ricci scalar (η), their product is independent of η,

$$_F r_+ {}_F r_- = \frac{2M}{N_q}. \tag{46}$$

It can be seen in Table 3 that all the quantities are dependent on M and are not universal. In comparison with the states of dust and radiation, we see that the product of area, entropy, and irreducible mass are independent of the Finslerian Ricci scalar (η). Using Eq. (37), one can obtain specific heat as follows,

$$_F C_\pm = \frac{\pi {}_F r_\pm^3}{M} \left(N_q {}_F r_\pm - \frac{\eta}{2}\right). \tag{47}$$

Case I: If we take into account horizon radii for FKq BH in the range ${}_F r_\pm > \frac{\eta}{2N_q}$, then ${}_F C_\pm$ is positive, and FKq BH is therefore thermodynamically stable in the selected range of horizon radii ${}_F r_\pm$.

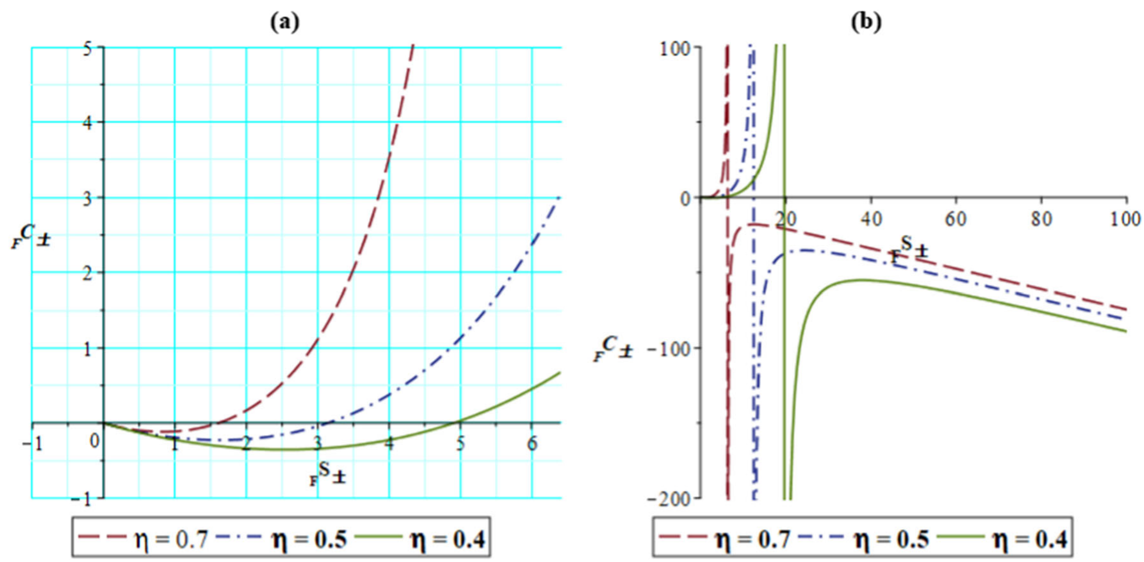


Fig. 5 Specific heat of the FKr BH

Table 3 Area product, Bekenstein–Hawking entropy product, Hawking temperature product, surface gravity product, Komar energy product, irreducible mass product, for FKq BH

${}_F A_\pm$	${}_F S_\pm$	${}_F T_\pm$
$\frac{4\pi}{N_q}(\eta {}_F r_\pm - 2M)$	$\frac{\pi}{N_q}(\eta {}_F r_\pm - 2M)$	$\frac{2M - N_q {}_F r_\pm^2}{4\pi {}_F r_\pm^2}$
${}_F k_\pm$	${}_F E_\pm$	${}_F M_{irr\pm}$
$\frac{2M - N_q {}_F r_\pm^2}{2 {}_F r_\pm^2}$	$\frac{2M - N_q {}_F r_\pm^2}{2}$	$\frac{{}_F r_\pm}{2}$
${}_F A_{+F} A_-$	${}_F S_{+F} S_-$	${}_F T_{+F} T_-$
$\frac{64\pi^2 M^2}{N_q^2}$	$\frac{4\pi^2 M^2}{N_q^2}$	$\frac{16M^2 - \frac{2M\eta^2}{N_q}}{64\pi^2 M^2} N_q^2$
${}_F k_{+F} k_-$	${}_F E_{+F} E_-$	${}_F M_{irr+F} M_{irr-}$
$\frac{16M^2 - \frac{2M\eta^2}{N_q}}{16M^2} N_q^2$	$4M^2 - \frac{M\eta^2}{2N_q}$	$\frac{M}{2N_q}$

Case II: If we consider horizon radii for FKq BH in the following range, $0 < {}_F r_\pm < \frac{\eta}{2N_q}$, then ${}_F C_\pm$ is negative. Thus, in the selected range of ${}_F r_\pm$, FKq BH is thermodynamically unstable.

In this case, stability and instability distance of BH has been obtained by assuming the Finslerian Ricci scalar is positive ($\eta > 0$).

Figure 6 shows that this BH only experiences the first-order phase transition. As η increases, the phase transition transfers to a higher point.

Figure 7a and b show the specific heat behavior in contour and 3-D mode, respectively, and the η range is assumed to be between -1 and 1.

Similar to Eq. (40), one can obtain the specific heat with respect to entropy,

$$ {}_F C_\pm = -{}_F S_\pm \left(1 - \frac{N_q}{2\pi M} {}_F S_\pm\right). \tag{48}$$

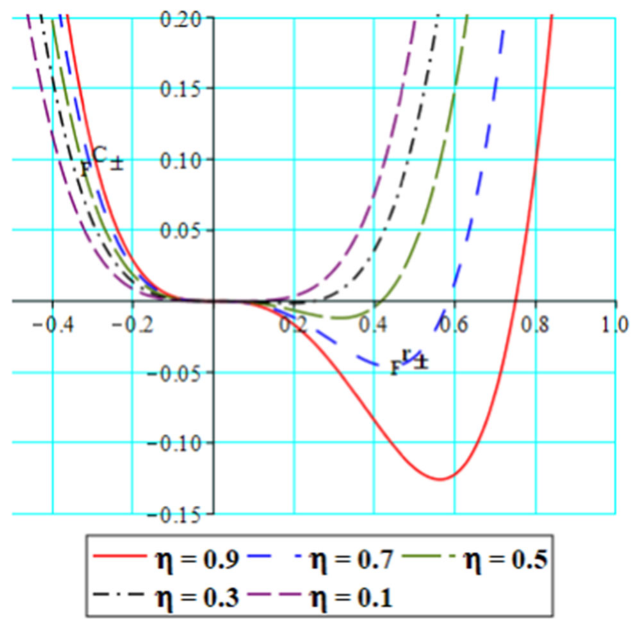


Fig. 6 Specific heat of the FKq BH

Figure 8 shows the changes in specific heat with respect to entropy. It can be seen in Fig. 8, similar to Fig. 6, the BH experiences only the first-order phase change, but it does not depend on the Finslerian Ricci scalar (η).

2.4 Finslerian Kiselev black hole $\omega_c = -1$, $\lambda = 0$

The BH horizon for a FK BH that is surrounded by a cosmological constant corresponds to ${}_F f_c(r) = 0$. In Ref. [5] $\xi(r) = 1 - \frac{2M^*}{r} - \frac{1}{3}\Lambda r^2 + \frac{Q^{*2}}{r^2}$, comparing $\xi(r)$ and ${}_F f_c(r)$, we find that $M^* = \frac{M}{\eta}$, $Q^{*2} = \frac{Q^2}{\eta}$, and $\frac{\Lambda}{3} = \frac{N_c}{\eta}$. Now, we

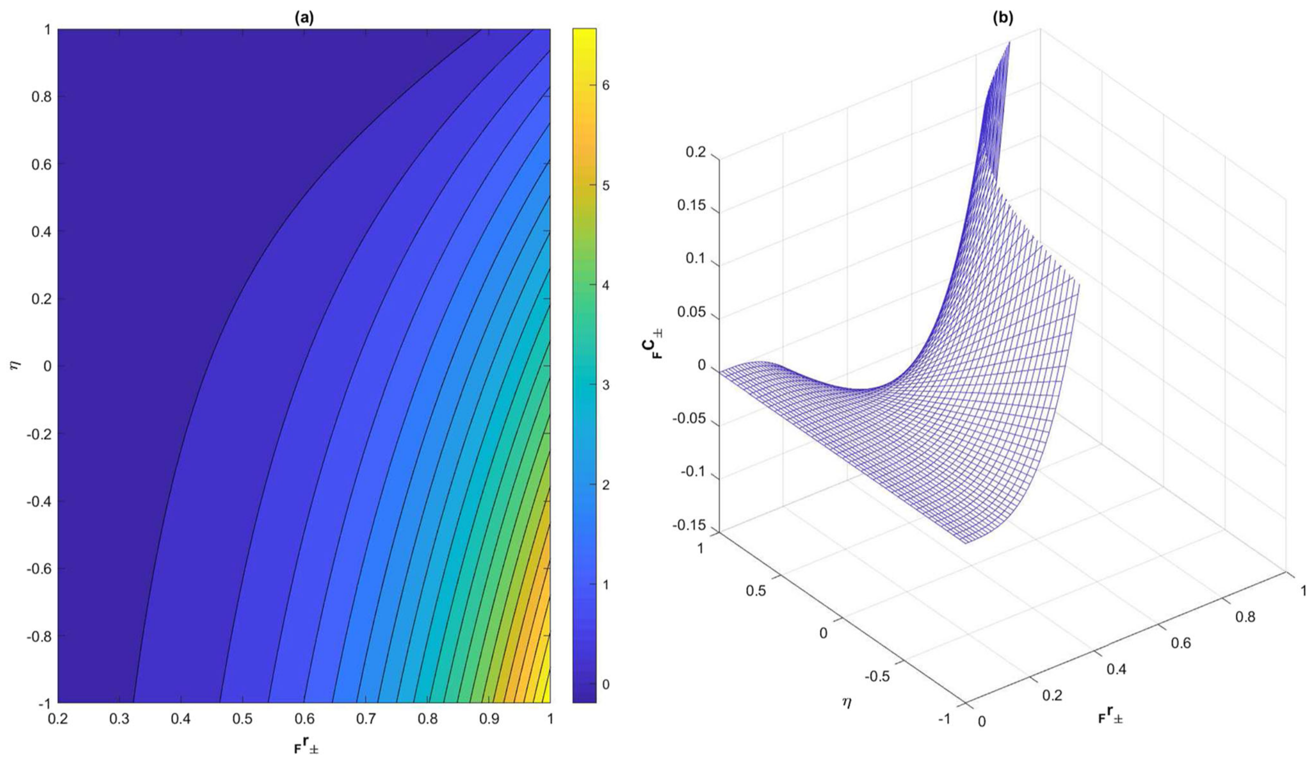


Fig. 7 Specific heat of the FKq BH with respect to η and $_F r_\pm$

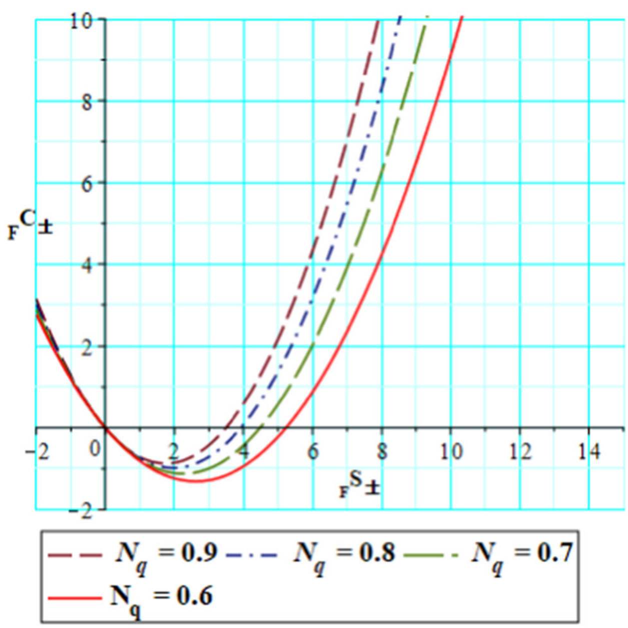


Fig. 8 Specific heat of the FKq BH

can easily obtain the horizons of the BH in the following form, assuming that $\frac{3N_c}{\eta}\frac{M^2}{\eta^2} \ll 1$,

$$_F r_\pm = \frac{M}{\eta} \pm \sqrt{\frac{M^2}{\eta^2} - \frac{Q^2}{\eta} + \frac{N_c}{\eta}\frac{M^4}{\eta^4}}. \tag{49}$$

By using Eq. (23), their product yields,

$$_F r_{+F} r_- = \frac{1}{\eta}\left(Q^2 - \frac{M^4}{\eta^4} N_c\right), \tag{50}$$

which is dependent on both mass and η. It can be seen in Table 4 that all the quantities are dependent on M and are not universal. The difference between this BH and the Reissner–Nordström–de Sitter BH is the presence of the Finslerian Ricci scalar (η), and the effect of η can be observed in all quantities. Using Eq. (37), one can obtain specific heat as follows,

$$_F C_\pm = \frac{2\pi\, _F r_\pm^4 \left(_F r_\pm - \frac{M}{\eta}\right)}{\left(2\frac{M}{\eta}\, _F r_\pm - {_F r_\pm^2} + 6\frac{N_c}{\eta}\frac{M^4}{\eta^4}\right)\left(2\frac{N_c}{\eta}\frac{M^3}{\eta^3} + {_F r_\pm}\right)}. \tag{51}$$

Case I: If we take into account horizon radii for FKc BH in the range $\frac{M}{\eta} < {_F r_\pm} < \frac{M}{\eta}(1 \pm \sqrt{1 + 6\frac{N_c}{\eta}\frac{M^2}{\eta^2}})$, then $_F C_\pm$ is positive, and FKc BH is therefore thermodynamically stable in the selected range of horizon radii $_F r_\pm$.

Case II: If we consider horizon radii for FKc BH in the following range, $0 < {_F r_\pm} < \frac{M}{\eta}$, $_F r_\pm > \frac{M}{\eta}(1 \pm \sqrt{1 + 6\frac{N_c}{\eta}\frac{M^2}{\eta^2}})$, then $_F C_\pm$ is negative. Thus, in the selected range of $_F r_\pm$, FKc BH is thermodynamically unstable.

In this case, stability and instability distance of BH has been obtained by assuming the Finslerian Ricci scalar is positive ($\eta > 0$).

Table 4 Area product, Bekenstein–Hawking entropy product, Hawking temperature product, surface gravity product, Komar energy product, irreducible mass product, for FKc BH

$_FA_\pm$	$_FS_\pm$
$\frac{4\pi}{\eta}(2M\,_Fr_\pm - Q^2 + N_c\frac{M^4}{\eta^4})$	$\frac{\pi}{\eta}(2M\,_Fr_\pm - Q^2 + N_c\frac{M^4}{\eta^4})$
$_FA_{+F}A_-$	$_FS_{+F}S_-$
$\frac{16\pi^2}{\eta^2}(Q^2 - \frac{M^4}{\eta^4}N_c)^2$	$\frac{\pi^2}{\eta^2}(Q^2 - \frac{M^4}{\eta^4}N_c)^2$
$_FT_\pm$	$_Fk_\pm$
$\frac{_Fr_\pm^2-\frac{M}{\eta}\,_Fr_\pm-\frac{2N_c}{\eta}\frac{M^4}{\eta^4}}{2\pi\,_Fr_\pm^3}$	$\frac{_Fr_\pm^2-\frac{M}{\eta}\,_Fr_\pm-2\frac{N_c}{\eta}\frac{M^4}{\eta^4}}{_Fr_\pm^3}$
$_FT_{+F}T_-$	$_Fk_{+F}k_-$
$\frac{(\frac{Q^2}{\eta}-\frac{M^2}{\eta^2})(Q^2+3N_c\frac{M^4}{\eta^4})-N_c(\frac{Q^2}{\eta}-\frac{N_c}{\eta}\frac{M^4}{\eta^4})}{4\pi^2(Q^2-N_c\frac{M^4}{\eta^4})^3}$	$\frac{(\frac{Q^2}{\eta}-\frac{M^2}{\eta^2})(Q^2+3N_c\frac{M^4}{\eta^4})-N_c(\frac{Q^2}{\eta}-\frac{N_c}{\eta}\frac{M^4}{\eta^4})}{(Q^2-N_c\frac{M^4}{\eta^4})^3}$
$_FE_\pm$	$_FM_{irr\pm}$
$\frac{_Fr_\pm^2-\frac{M}{\eta}\,_Fr_\pm-2\frac{N_c}{\eta}\frac{M^4}{\eta^4}}{_Fr_\pm}$	$\frac{_Fr_\pm}{2}$
$_FE_{+F}E_-$	$_FM_{irr+F}M_{irr-}$
$\frac{(\frac{Q^2}{\eta}-\frac{M^2}{\eta^2})(Q^2+3N_c\frac{M^4}{\eta^4})-N_c(\frac{Q^2}{\eta}-\frac{N_c}{\eta}\frac{M^4}{\eta^4})}{(Q^2-N_c\frac{M^4}{\eta^4})}$	$\frac{1}{4\eta}(Q^2 - \frac{M^4}{\eta^4}N_c)$

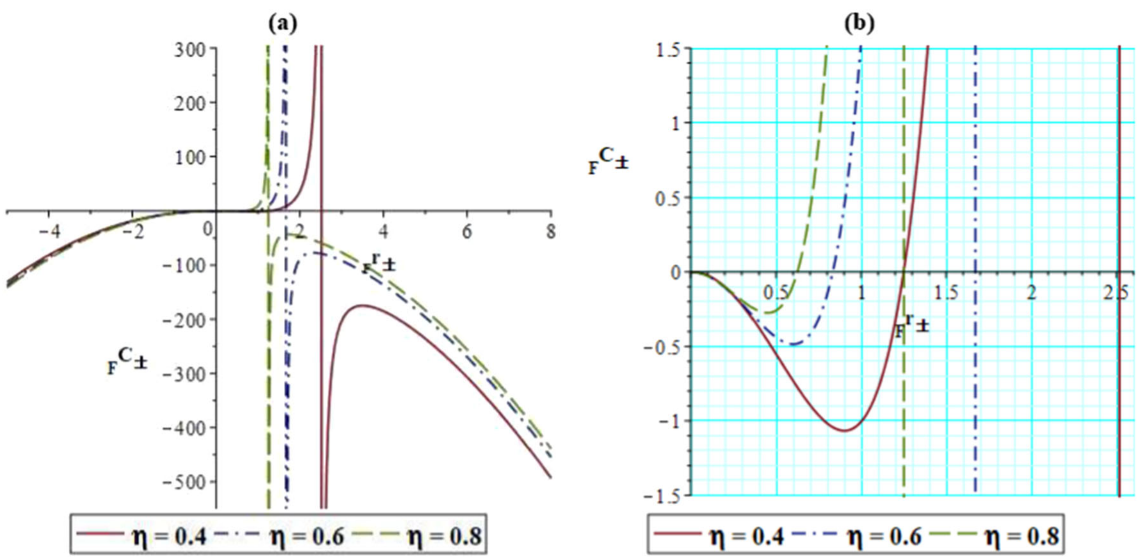


Fig. 9 Specific heat of the FKc BH

Case III: If the horizon radius $_Fr_\pm$ satisfies the following formula, then $_FC_\pm$ diverges:

$$2\frac{M}{\eta}\,_Fr_\pm - \,_Fr_\pm^2 + 6\frac{N_c}{\eta}\frac{M^4}{\eta^4} = 0, \quad 2\frac{N_c}{\eta}\frac{M^3}{\eta^3} + \,_Fr_\pm = 0.$$

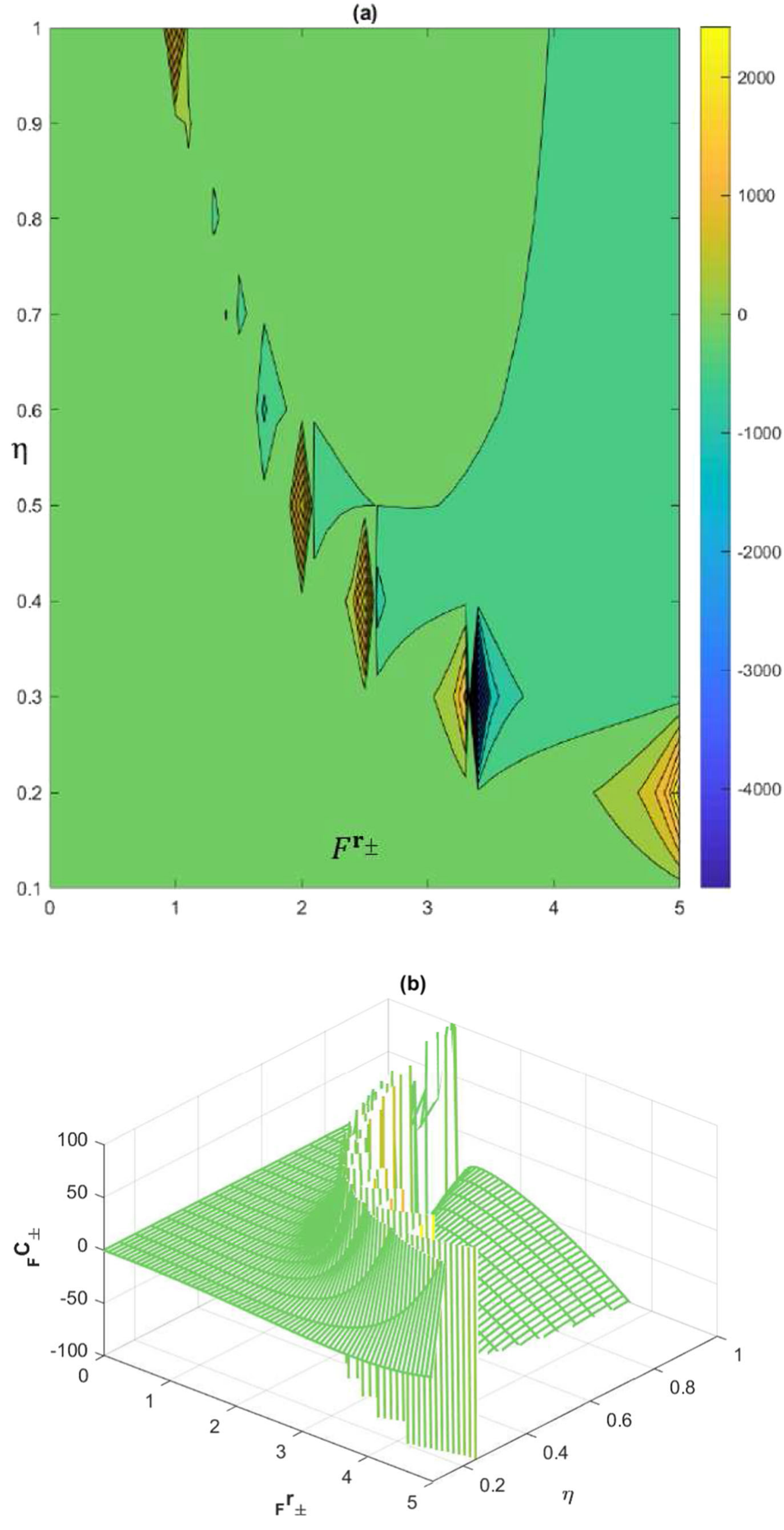
Hence, if $_Fr_\pm = \frac{M}{\eta}(1\pm\sqrt{1+6\frac{N_c}{\eta}\frac{M^2}{\eta^2}})$, FKc BH experiences a second-order phase transition.

Figure 9a shows the first-order and second-order phase transitions, and Fig. 9b clearly shows that when the Finslerian Ricci scalar (η) increases, the first-order phase transition point is shifted to a lower level, and this is also true for the second-order phase transition. Figure 10a and b show the specific heat behavior in contour and 3-D mode, respectively, and the η range is assumed to be between 0 and 1.

3 Conclusion

In the setting of Rastall theory, we find uncharged/charged FK BHs as a novel class of BH solutions encircled by ideal fluid. Then, we focus on the unique situations of charged and uncharged BHs surrounded by common materials like radiation and dust or rare materials like quintessence and cosmological constants. Since the gravitational influence of a cloud of strings is the same as that caused by a solid deficiency

Fig. 10 Specific heat of the FKc BH with respect to η and ${}_F r_\pm$



angle, if $\eta \neq 1$, this solution is similar to a BH surrounded by a cloud of strings. In Eq. (20), we can see that the role of curvature and matter is the same. Instead of adding matter, the same results can be obtained by changing the geometry, and Finsler geometry is a suitable option.

The results are shown that the properties of FK BHs for $\lambda \neq 0$ are similar to Kiselev BH in the non-Finsler state. Because η appears only in the function ${}_F f_s(r)$ and the values of $\mathcal{W}_s$ and ρ_s (see more Eqs. (21, 22)) are independent of η [34]. Therefore, assuming $\lambda = 0$, we have investigated the thermodynamic characteristics of the BH. We have identified several BH thermodynamic products and got some intriguing findings. Quantum field theory and gravity are related through BH thermodynamics. In this regard, evaluating various thermodynamic products can help researchers better understand the microscopic properties of the BH's outer and inner entropies. The findings aid in the microstructure analysis of FK BHs, which in turn identifies a method for comprehending the basic characteristics of BH gravity and establishes quantum gravity.

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Declarations

Conflict of interest The authors declare no competing interests. The authors declare that have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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