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Chapter 1

Introduction

The Standard Model (SM) of particle physics is the present theory describing extremely successfully the fundamental interactions of Nature (excluding Gravity). It relies on the concept of *gauge field theories* and is based on the gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$, describing respectively Quantum Chromodynamics (QCD) and the Electroweak (EW) interactions. In addition to the gauge bosons associated to these gauge groups, its spectrum contains matter in the form of chiral fermions, splitted into quarks, which feel the colour interaction, and leptons. These fermions are grouped inside three generations, or *families*, which differ only by their masses. The last piece of the SM particle content is the elusive Higgs boson, which is still to be discovered, and accounts for the breaking of the EW sector : $SU(2)_L \times U(1)_Y \to U(1)_{em}$.
Up to now, the SM has proved an extremely robust theory from the experimental side and has been confirmed by many experiments since the 70's. The accidental global symmetries of the model, such as Baryon or Lepton numbers, have been verified with high precision and CP violation in the K or B meson sectors can be entirely explained by the only CP violating phase contained in the unitary CKM matrix [10], which parametrises the mixing between the different generations of quarks.

In spite of all its successes, "anomalies" exist that we cannot explain with the only ingredients of the SM :

- The oldest experimental signature calling for Physics Beyond the SM (BSM) is the value of the asymmetry between the densities of matter and antimatter in the Universe. The recent value of the baryon asymmetry as given by the WMAP experiment :

$$\frac{n_B}{s} = y_B \sim (8.7 \pm 0.3) \times 10^{-11}$$

  is presently too large to be explained by the SM interactions.

- The SuperKamiokande experiment, designed to measure the lifetime of the proton, has established in 1998 an anomaly in atmospheric neutrino generated by collisions of cosmic rays entering the atmosphere. It has established a disappearance of ν_μ's, interpreted as a $\nu_\mu \to \nu_\tau$ conversion [1]. On the other hand, experiments such as SNO [2] or KamLAND [3] have detected an evidence of $\nu_e \to \nu_{\mu,\tau}$ in the flux of ν_e's produced in the Sun. These experiments lead to an interpretation in terms of neutrino oscillations, which implies a non zero mass for at least two neutrinos. This is in direct conflict with the SM, which has been built for accomodating massless neutrinos. Therefore, neutrinos form an experimentally accessible playground to confront BSM theories with experiments, which is why we are going to investigate BSM physics mainly from the neutrino side. Moreover, we will see that the problems of generating neutrino masses consistent with experiments and of providing an explanation to the baryon asymmetry presented above can be elegantly linked in certain models.

The mass matrix m_ν of the neutrinos is parametrised by three mass eigenvalues $m_{1,2,3}$ and a unitary matrix parametrising the neutrino mixings θ_{ij}, which is usually called the PMNS matrix [11], and parametrised in the following way :

$$U_{MNS} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix} \mathrm{diag}(e^{-i\phi_1/2}, e^{-i\phi_2/2}, 1) \; .$$

where $c_{ij} = \cos\theta_{ij}, s_{ij} = \sin\theta_{ij}$. δ is called the Dirac phase and $\phi_{1,2}$ the Majorana phases. The values of the the neutrino sector parameters extracted from atmospheric, solar and reactor experiments [117] are listed in table 1 below. Oscillation experiments only allow to measure mass squared differences $\Delta m^2_{ij} = m_i^2 - m_j^2$, therefore we do not have any direct measure of the neutrino mass scale, although bounds can be inferred from cosmology [52]. We note $|\Delta m^2_{31}| = \Delta m^2_{atm}$ and $\Delta m^2_{sol} = \Delta m^2_{21}$ according to the origin of their measurements. Therefore, the form of the neutrino mass spectrum can be quite different, depending on the lightest eigenvalue m_1 and the sign of Δm^2_{31}. If m_1 is much smaller than $\sqrt{\Delta m^2_{sol}}$ and $\Delta m^2_{31} > 0$, the spectrum is called *hierarchical* and $m_2 \simeq \sqrt{\Delta m^2_{sol}} \simeq 9 \times 10^{-3}$ eV, $m_3 \simeq \sqrt{\Delta m^2_{atm}} \simeq 0.05$ eV. If on the contrary $\Delta m^2_{31} < 0$, the spectrum is called *inverse hierarchical*.

Parameter	Value (90% CL)
$\sin^2(2\theta_{12})$	$0.86(^{+0.03}_{-0.04})$
$\sin^2(2\theta_{23})$	> 0.92
$\sin^2(2\theta_{13})$	< 0.19
Δm^2_{sol}	$(8.0^{+0.4}_{-0.3}) \times 10^{-5} eV^2$
Δm^2_{atm}	1.9 to 3.0×10^{-3} eV^2

Table 1.1: Experimental bounds for the neutrino masses and mixings [117]

- Other cosmological problems are also unexplained at the moment, such as the evidence for the presence of non-baryonic Dark Matter in larger quantities than the measured baryonic matter on galactic scales, or the present acceleration of the Universe known as the *Dark Energy* problem.

There are also well-known theoretical problems to add to the previous list. For example, the infamous hierarchy problem indicates that the Higgs boson mass should receive quantum corrections that scale like the cutoff of any new physics lying above the EW scale. Therefore, if we do note postulate the existence of new degrees of freedom or symmetries to be discovered an order of magnitude above the EW scale or so, we have to tune the bare mass of the Higgs boson at each order of the perturbation expansion. Another question that we would like to elucidate is the origin of the multiplicity of fermion generations, as well as the hierarchy apparent in the masses of the different generations. This calls for a more fundamental theory of flavour, where the Yukawa couplings would be given in terms of a precise symmetry breaking pattern, for example, instead of being free parameters like in the SM.

The purpose of this thesis is to study, in the neutrino sector, the flavour structures at high energy, which is a first step towards a comprehension of the underlying flavour theory. The work is divided into two main parts :

- A well known mechanism to produce small neutrino masses is the seesaw mechanism, which implies the existence of massive particles whose decays violate lepton number. Therefore this mechanism can also be used to generate a net baryon number in the early universe and explain the cosmological observation of the asymmetry between matter and antimatter. However, it is often non-trivial to fulfil the constraints coming at the same time from neutrino oscillations and cosmological experiments, at least in frameworks where the couplings can be somehow constrained, like some Grand Unification models. Therefore we devoted the first part to the study of a certain class of seesaw mechanism which can be found in the context of $SO(10)$ theories for example. We introduce a method to extract the mass matrix of the heavy right-handed neutrinos and explore the phenomenological consequences of this quantity, mainly concerning the production of a sufficient baryon asymmetry.
- When trying to identify the underlying symmetry governing the mixings between the different generations, we see that there is a puzzling difference between the quark and the lepton sectors. However, the quark and lepton parameters have to be compared at the scale of the flavour symmetry breaking, therefore we have to make them run to the appropriate scale. Thus, it is worthwhile investigating models where quantum corrections allow an approximate unification of quark and lepton mixings. This is why the other part of the thesis investigates the running of the effective neutrino mass operator in models with an extra compact dimension, where quantum corrections to the neutrino masses and mixings can be potentially large due to the multiplicity of states.

The manuscript is thus divided into three chapters. The first chapter is an overall introduction to the tools and concepts that we will manipulate in the other parts, such as the Seesaw Mechanism, Supersymmetry or Grand Unification. Then we will turn to the study of the Left-Right symmetric Seesaw formula in $SO(10)$ inspired models and its consequences for baryogenesis through Leptogenesis. Finally, the last chapter will present the necessary notions of extra-dimensional physics before turning to the study of the running of the neutrino mass parameters.

Chapter 2

Seesaw Mechanism and Supersymmetric Grand Unification

The first part of this thesis is an introduction to the necessary ingredients of physics Beyond the Standard Model (BSM) that we are going to use extensively when describing our work [37, 90, 120]. The main part of this work focuses on the study of a certain class of seesaw mechanism. The seesaw mechanism is now more than thirty years old [4] and it is without a doubt the most minimal way to generate small neutrino masses, at least in its simpler version. The seesaw formula for light neutrino masses has been widely studied in the literature, particularly the way low energy and high energy parameters are connected. However, particular features of the seesaw call for an embedding of the mechanism into more motivated theories. This is why we studied seesaw models from Grand Unified Theories (GUTs), in their supersymmetric (SUSY) version.

We will begin this small journey through BSM physics with a basic introduction to the different declinations of the seesaw mechanism.
In order to incorporate them into a coherent GUT picture, we will take some time to explain the basics of supersymmetric theories. At this stage we will also introduce the formalism of supergraphs, quite useful for studying quantum corrections in SUSY theories. This tool will be valuable in the last part of the thesis when we will consider quantum effects in supersymmetric extra-dimensional theories.
Then we will introduce the concept of Grand Unified Theory and more precisely the ones based on the $SO(10)$ gauge group : we will see that it is a very natural setup to realise the seesaw mechanism, and in particular the Left-Right symmetric ones we are interested in.
Since our study of the seesaw aims partly at identifying good candidate GUT theories for a successful generation of the cosmic matter-antimatter asymmetry, we will end this chapter with an overall introduction to the concept of baryogenesis in the early Universe and more specifically to leptogenesis, an appreciable side effect of any generic seesaw mechanism.

2.1 Seesaw basics

The Standard Model (SM) has been built in such a way that no mass term for the neutrinos is possible at the renormalisable level. Although this was perfectly consistent with data in the early days of the SM, it has become clear with the observational evidence for neutrino oscillations that this assumption can no longer hold.

One has then several ways to tackle the problem. Forgetting about renormalisability, one finds that the operator with smaller dimension allowing for neutrino masses is the dimension 5 operator, coupling the lepton doublets l_i to the Higgs doublet H :

$$-\frac{1}{4}\frac{\kappa'_{ij}}{\Lambda}\overline{l^c_i}Hl_jH \quad \longrightarrow \quad -\frac{1}{4}\frac{\kappa'_{ij}}{\Lambda}v^2\overline{\nu^c_i}\nu_j = -\frac{1}{2}m_{ij}\overline{\nu^c_i}\nu_j \tag{2.1}$$

Writing this operator, we made use of the charge conjugation operation : $l^c = C\bar{l}^T$ and C is the charge conjugation operator defined by $C = i\gamma^0\gamma^2$. If we seek to reproduce neutrino masses of the order of an eV or less, as pointed out by experimental data, the scale Λ suppressing the coupling should be of the order 10^{14} GeV approximately, assuming that κ' has $\mathcal{O}(1)$ entries. The mass term induced in eq. (2.1) is called a Majorana mass term. It couples the left-handed neutrino to itself with the use the charge conjugation operator C and the mass eigenstates are their own antiparticle $\nu = \nu^c$.

If we insist on renormalisability, we have to introduce additional particles with which the left-handed (LH) neutrinos will mix. The natural extension is then to add right-handed neutrinos N_R and couple them to left-handed leptons with usual Yukawa couplings, in a similar way to charged fermions. This ingredient is by itself sufficient to generate neutrino masses, but we have to deal with a huge hierarchy in the Yukawa sector : $y_\nu \sim 10^{-12} \sim 10^{-6}y_e$. An SM extended only by a Yukawa coupling for the neutrinos gives rise to Dirac neutrinos.
An important fact to notice here is that the operator $lH = l^\alpha\epsilon_{\alpha\beta}H^\beta$, with $\varepsilon_{\alpha\beta}$ the completely anti-symmetric Levi-Civita tensor, is gauge invariant. For the coupling $\overline{N}_RlH$ to be invariant N_R must be a singlet. Thus, any singlet under the SM gauge group can couple to leptons and mix with the neutrinos when the Higgs boson takes a non-zero vacuum expectation value. This gives to neutrinos the potential to explore physics up to a high energy scale. Indeed if N_R is neutral, nothing prevents us from writing a Majorana mass term for it. The sector giving a mass to neutrinos is then :

$$-(Y_\nu)_{ij}\overline{N}_{Ri}l_jH - \frac{1}{2}M_{ij}\overline{N}_{Ri}N^c_{Rj} + \text{h.c.} \tag{2.2}$$

After Electroweak (EW) symmetry breaking the mass matrix for neutrinos is :

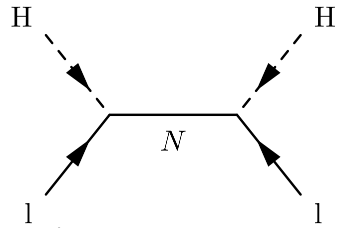
$$-\frac{1}{2}\begin{pmatrix} \nu^T_L & N^{cT}_R \end{pmatrix}\begin{pmatrix} 0 & vY^T_\nu \\ vY_\nu & M \end{pmatrix} C \begin{pmatrix} \nu_L \\ N^c_R \end{pmatrix}$$

To illustrate the consequences of this formula, we can first restrict to one generation. Computation of the eigenvalues in this case is straightforward. If we suppose that $M \gg v$, they can be developed and give approximately :

$$m_{\nu 1} \simeq -Y^2_\nu\frac{v^2}{M} \qquad \text{and} \qquad m_{\nu 2} \simeq M$$

We clearly see that there is a very massive state, since we supposed $M \gg v$, and a state with mass much lighter than v. The light state is mainly composed of ν_L and the heavy one of N_R, and the neutrinos are Majorana particles. When talking about the light and heavy mass eigenstates respectively, we will denote them without left-handed or right-handed subscript : ν and N. Supposing $Y_\nu \sim 1$ we find again that $M \sim 10^{15}$GeV. By introducing a heavy right-handed neutrino we see that we can naturally accommodate a very light neutrino. This extension of the SM, generating a small mass scale with EW and heavy scales only, is called the *Seesaw Mechanism*.

Integrating N_R below M, we actually find the operator of eq. (2.1). The seesaw is thus a natural realisation of the low energy effective theory previously described, which can be seen through the diagram :



(2.3)

Let us now quickly generalise to three families by integrating out the heavy modes. The Lagrangian density and the equations of motion read, for the N_{Ri} :

$$\mathcal{L} \supset i\overline{N}_{Ri}\gamma^\mu\partial_\mu N_{Ri} - (Y_\nu)_{ij}\overline{N}_{Ri}l_j H - \frac{1}{2}M_{ij}\overline{N}_{Ri}N^c_{Rj} + \text{h.c.} \tag{2.4}$$

$$\frac{\delta\mathcal{L}}{\delta\overline{N}_{Ri}} = i\gamma^\mu\partial_\mu N_{Ri} - (Y_\nu)_{ij}l_j H - \frac{1}{2}M_{ij}C\overline{N}^T_{Rj} + \frac{1}{2}M_{ji}C^T\overline{N}^T_{Rj} = 0 \tag{2.5}$$

Supposing then that N_R's do not propagate ($\partial_\mu N_R = 0$), we find :

$$\overline{N}^T_R = M^{-1}Y_\nu C l H \tag{2.6}$$

and injecting in the previous Lagrangian we obtain the non-renormalisable operator :

$$\mathcal{L} \supset \frac{1}{2}l^T H C Y^T_\nu M^{-1} Y_\nu l H \tag{2.7}$$

so that after EWSB we get the seesaw formula for the neutrino mass matrix :

$$m^I_\nu = -v^2 Y^T_\nu M^{-1} Y_\nu \tag{2.8}$$

and the index I is here to keep track of the type I origin of this contribution to the neutrino mass matrix.

There are other ways to generate a seesaw for neutrinos. Clearly, we need to couple either lH to a fermion singlet under $U(1)_Y$ or directly ll to a scalar taking a vev. The second alternative can only be achieved with a triplet of $SU(2)_L$ with hypercharge $y_{\Delta_L} = -2y_l$. As $SU(2)$ does not have complex representations, the triplet corresponds to the adjoint representation :

$$\Delta_L = \begin{pmatrix} \Delta^+ & \sqrt{2}\Delta^{++} \\ \sqrt{2}\Delta^0 & -\Delta^+ \end{pmatrix} \tag{2.9}$$

where we displayed as indices the electromagnetic charges of the three components. Now if $\langle\Delta^0\rangle = v_L \neq 0$, we obtain directly a Majorana mass term for neutrinos :

$$-\frac{1}{2}f_{ij}l^T_i\Delta_L\epsilon C l_j \quad\longrightarrow\quad -\frac{1}{2}(m^{II}_\nu)_{ij}\nu^T_{Li}C\nu_{Lj} \qquad \text{with : } m^{II}_\nu = v_L f \tag{2.10}$$

The problem is here to generate a vev sufficiently small, $v_L \sim 0.1$ eV, and introduce a mass term $M^2_\Delta\text{Tr}(\Delta^\dagger_L\Delta_L)$ large enough to decouple Δ_L from the low energy physics. The most straightforward way to do this is to couple Δ_L to the Higgs boson. The Higgs potential for Δ_L is then :

$$V_{\Delta_L} = -\mu H\Delta_L H - M^2_\Delta\text{Tr}(\Delta^\dagger_L\Delta_L)$$

When H^0 takes a vev v, it generates a tadpole for Δ^0. The equations of motion give the expression for v_L :

$$v_L = -\frac{\mu v^2}{M_\Delta^2} \tag{2.11}$$

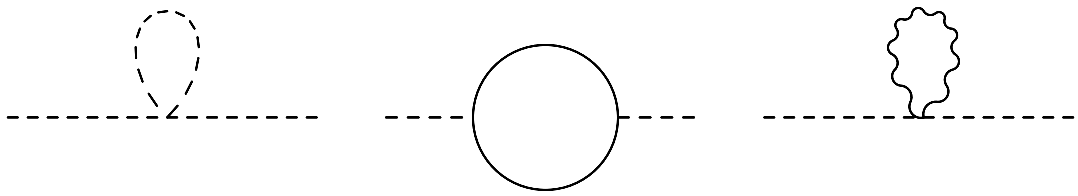
Once again, if the mass of the extra field is large enough, we obtain a small mass scale for neutrinos. The seesaw is here applied to the vev of the scalar field but the idea is similar. Thus we call this mechanism *type II seesaw*, as opposed to the classical seesaw mechanism, called *type I*.
We note that a free dimensionful parameter μ has appeared in the Higgs potential. Making the choice of a small μ could help us lower the mass scale of the triplet. However we will often work in supersymmetric theories where quartic interaction terms derive from trilinear terms in the superpotential. Therefore the cases we will consider will impose the constraint $\mu = M_\Delta$.
We note once again that integrating out the heavy triplet will leave us with the same non-renormalisable operator (2.1).

The last possibility is to couple lH to a fermion triplet of $SU(2)_L$ with a zero hypercharge η. This is called *type III seesaw* and leads to the same low energy effective theory as the types I and II seesaw.

We see that there are several ways to give masses to neutrinos. If one simply adds to the Lagrangian a right-handed part to the neutrino field and couples it to ν_L with the usual Yukawa coupling, one is led to a Dirac neutrino mass eigenstate, with the biggest entry of the Yukawa matrix Y_ν of order 10^{-12}. This means that while the six orders of magnitude separating the masses of the electron and the top quark are regularly populated by many other particles, there would be a sudden gap of equal magnitude between the electron and the next light fermion. While this is technically possible, it would further deepen the mystery of the flavour theory governing the structure of the Yukawa couplings. Therefore, when trying to build a theory with neutrino Yukawa couplings not much smaller than the electron, we have to admit that the new fields we introduce to generate neutrino masses have to be extremely massive. The magnitude of these masses will be naturally explained when embedding the SM into a Grand Unified Theory (GUT) at a high energy scale, the breaking of which will generate masses of the wanted order of magnitude. Some grand unified theories, such as $SO(10)$, will even naturally introduce the appropriate field content to perform an efficient seesaw, such as three right-handed neutrinos or a weak triplet.
Nevertheless, if no additional degree of freedom is introduced with a mass smaller than the seesaw scale we keep a very large difference between the EW and the scale of new physics. Thus we are left with a hierarchy problem for the Higgs boson. This can be circumvented by considering theories with supersymmetry broken at low energy, which is why we devote the following section to supersymmetric theories and their appealing features.

2.2 Supersymmetry

When one tries to compute the quantum corrections to the squared mass of a scalar boson, it is well known that corrections scaling quadratically with the cutoff of the theory, as expected by dimensional analysis. At one loop, for example, such quadratically divergent contributions will come not only from the scalar self interactions but also from any Yukawa coupling with fermions or from gauge interactions with vector bosons, as illustrated by the following diagrams :



(2.12)

which all display behaviours in $\int d^4p/p^2 \propto \Lambda^2$. Therefore, any scalar mass in the theory is naturally pushed to the cutoff scale by quantum effects. Any scalar mass significantly smaller than the cutoff is thus the result of extreme cancellations between the bare mass or the different quantum corrections. Therefore, when integrating over the high energy degrees of freedom up to the Planck scale or to the scale of Grand Unification for example, we need to tune the bare mass of the Higgs boson to an extremely unnatural level to maintain a Higgs mass of the order of the EW scale. Moreover, quadratic divergences occur at each order in perturbation theory, therefore the tuning has to be done at each order of the perturbative development.
The necessary stabilisation of the Higgs mass can be obtained by imposing a symmetry which enforces an exact cancellation between any non-logarithmic divergence of the theory. This symmetry consists in extending the Poincaré group to the super-Poincaré invariance. The theory is then called supersymmetric (SUSY).

2.2.1 The supersymmetry algebra and its representations

Making a theory supersymmetric amounts to extending the Poincare algebra with generators in the spinorial representation of $SO(1,3)$ ($SO(1,d-1)$ if we consider theories with spacetime dimension $d > 4$). As $SO(1,3)$ is chiral, these generators are Weyl spinors with two degrees of freedom. We denote them Q^I_α, where $\alpha = 1,2$ is a spinor component and $I = 1...\mathcal{N}$ when we have $\mathcal{N}$ different charges of SUSY. As they are fermionic operators, the Q's are defined by anti-commutation relations :

$$\{Q^I_\alpha, \bar{Q}_{\dot\beta J}\} = 2\sigma^\mu_{\alpha\dot\beta}\delta^I_J P_\mu \tag{2.13}$$

$$\{Q^I_\alpha, Q^J_\beta\} = \epsilon_{\alpha\beta}(U^{IJ} + iV^{IJ}) \tag{2.14}$$

$$\left[Q^I_\alpha, M_{\mu\nu}\right] = \frac{1}{2}(\sigma_{\mu\nu})^\beta_\alpha Q^I_\beta \tag{2.15}$$

$$\left[Q^I_\alpha, P_\mu\right] = 0 \tag{2.16}$$

Intuitively we see that the Q's can be interpreted as "square roots" of translations. We can then formulate the theory such that the Q's act on the fields as derivatives. SUSY transformation parameters are spinors, so in order to do this, we have to add fermionic coordinates θ and $\overline{\theta}$ to the usual space-time coordinates x^μ. Usual fields are promoted to *superfields* $F(x^\mu, \theta, \overline{\theta})$ [102, 108].
The Q's transform as spinors under the Lorentz group, they carry a spin charge $\frac{1}{2}$. Therefore, when acting on a field with definite transformation properties under the Lorentz group, they will transform it into a field with different spin. More precisely they will relate bosons and fermions.
For 4D model building, only $\mathcal{N} = 1$ SUSY is really interesting, which means that $U = V = 0$ in the algebra. However we will have to deal with $\mathcal{N} = 2$ SUSY when exploring extra-dimensional models, and we will come back to $\mathcal{N} = 2$ representations in the last chapter.

Now, the $(\theta, \overline{\theta})$ coordinates we have introduced are Grassman variables, so for any $n > 1$, $(\theta_\alpha)^n = 0$. This means that the Taylor expansion of the superfields in θ and $\overline{\theta}$ has a finite number of terms, namely :

$$
\begin{aligned}
F(x,\theta,\bar{\theta}) &= f(x) + \theta\phi(x) + \bar{\theta}\bar{\chi}(x) + \theta\theta m(x) + \overline{\theta\theta}n(x) \\
&+\theta\sigma^{\mu}\bar{\theta}v_{\mu}(x) + \theta\theta\bar{\lambda}(x) + \theta\bar{\theta}\bar{\theta}\psi(x) + \theta\theta\bar{\theta}\bar{\theta}d(x) \qquad (2.17)
\end{aligned}
$$

We note by convention $\theta\theta = \theta^2 = \theta^\alpha\theta_\alpha = \theta^\alpha\theta^\beta\epsilon_{\alpha\beta}$ and $\bar{\theta}\bar{\theta} = \bar{\theta}^2 = \bar{\theta}_{\dot{\alpha}}\bar{\theta}^{\dot{\alpha}} = \bar{\theta}_{\dot{\alpha}}\bar{\theta}_{\dot{\beta}}\epsilon^{\dot{\alpha}\dot{\beta}}$. The former expression is the most general expression for a superfield but it is not yet an irreducible representation, we can add constraints at the only condition that they are invariant under SUSY transformations.

Chiral superfields

By defining covariant derivatives, a first possible constraint is : $D_\alpha\Phi = 0$ or $\bar{D}_{\dot{\alpha}}\Phi = 0$, with D and $\bar{D}$ covariant derivatives which commute with SUSY transformations. The derivative expressions for the Q's that were our primary goal for introducing Grassman space-time coordinates, can be chosen as :

$$
Q_\alpha = -i\left(\frac{\partial}{\partial\theta^\alpha} - i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu\right) \qquad (2.18)
$$

$$
\bar{Q}_{\dot{\alpha}} = -i\left(-\frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu\right) \qquad (2.19)
$$

Since D_α and $\bar{D}_{\dot{\alpha}}$ have to anti-commute with Q_β and $\bar{Q}_{\dot{\beta}}$, they can be expressed as :

$$
D_\alpha = \frac{\partial}{\partial\theta^\alpha} + i\sigma^\mu_{\alpha\dot{\alpha}}\bar{\theta}^{\dot{\alpha}}\partial_\mu \qquad (2.20)
$$

$$
\bar{D}_{\dot{\alpha}} = \frac{\partial}{\partial\bar{\theta}^{\dot{\alpha}}} + i\theta^\alpha\sigma^\mu_{\alpha\dot{\alpha}}\partial_\mu \qquad (2.21)
$$

We can now apply the additional constraints to reduce the number of components of a superfield Φ. We will call *chiral superfield* a superfield Φ defined by :

$$
\bar{D}_{\dot{\alpha}}\Phi = 0 \qquad (2.22)
$$

while an *antichiral superfield* $\bar{\Phi}$ obeys the constraint :

$$
D_\alpha\bar{\Phi} = 0 \qquad (2.23)
$$

Defining new variables $y^\mu = x^\mu - i\theta\sigma^\mu\bar{\theta}$ and $\bar{y}^\mu = x^\mu + i\theta\sigma^\mu\bar{\theta}$ and noting that :

$$
D_\alpha\bar{y}^\mu = \bar{D}_{\dot{\alpha}}y^\mu = 0, \qquad D_\alpha\bar{\theta} = \bar{D}_{\dot{\alpha}}\theta = 0 \qquad (2.24)
$$

we can guess that a chiral superfield will depend only on y and θ while an antichiral superfield will depend on $\bar{y}$ and $\bar{\theta}$. The expansion in component field for $\Phi(x,\theta,\bar{\theta})$ is :

$$
\begin{aligned}
\Phi(y,\theta) &= \varphi(y) + \sqrt{2}\theta\psi(y) + \theta\theta F(y) \qquad (2.25)\\
&= \varphi(x) + \sqrt{2}\theta\psi(x) + \theta\theta F(x) - i\theta\sigma^\mu\bar{\theta}\partial_\mu\varphi(x) + \frac{1}{\sqrt{2}}i\theta\theta\partial_\mu\psi(x)\sigma^\mu\bar{\theta} \\
&-\frac{1}{4}\theta\theta\bar{\theta}\bar{\theta}\Box\varphi(x) \qquad (2.26)
\end{aligned}
$$

and similarly for antichiral superfields. A chiral superfield on-shell is therefore composed of two bosonic degrees of freedom (a complex scalar) and two fermionic degrees of freedom (a Weyl fermion). This means that for every fermion of the Standard model we have to add a scalar with the same quantum numbers, that we will call *sfermions*, where the s stands for *scalar*. The natural extension of the SM to a supersymmetric theory, usually called the Minimal Supersymmetric Standard Model (MSSM), therefore contains *squarks* and *sleptons*. The complex auxiliary field F is only present in order to equate the numbers of bosonic and fermionic d.o.f. off-shell, since an off-shell Weyl fermion has four real components.

Vector superfields

The other useful constraint we can apply is the one of hermiticity $V = V^\dagger$: we call V a vector superfield since it will contain a vector boson. This can be seen from its expansion in component fields :

$$\begin{aligned} V(x,\theta,\bar\theta) &= C(x) + i\theta\chi(x) - i\bar\theta\bar\chi(x) + \theta\sigma^\mu\bar\theta A_\mu(x) \\ &+\frac{i}{2}\theta\theta[M(x)+iN(x)] - \frac{i}{2}\bar\theta\bar\theta[M(x)-iN(x)] \\ &+i\theta\theta\left[\bar\lambda(x) + \frac{i}{2}\partial_\mu\chi(x)\sigma^\mu\right]\bar\theta - i\bar\theta\bar\theta\theta\left[\lambda(x) - \frac{i}{2}\sigma^\mu\partial_\mu\bar\chi(x)\right] \\ &+\frac{1}{2}\theta\theta\bar\theta\bar\theta\left[D(x) - \frac{1}{2}\Box C(x)\right] \end{aligned} \tag{2.27}$$

With a supersymmetric gauge transformation $V \to V + i\Lambda - i\Lambda^\dagger$ we can actually eliminate some components and one is left, on-shell, with the vector A_μ and the two-component Majorana fermion λ, while off-shell the real auxiliary field D also remains.This gauge is called the *Wess-Zumino gauge* [9], but as we prefer to work with a general superfield formalism and not in a particular choice of supergauge, we will not explicit the Wess-Zumino gauge any further.

2.2.2 Couplings and non-renormalisation

Now that we have our basic building blocks we can write an action in terms of superfields. This will be more convenient than manipulating the bosonic and fermionic components as it allows to write all possible supersymmetric Lagrangians very quickly and in a compact manner. From now on we consider a gauge theory with chiral superfields in the fundamental representation and a vector superfield in the adjoint of the gauge group. The first problem to solve is to find a superfield, formed with $V = V_a T^a$, which contains the appropriate kinetic terms for its components. The good quantities to consider are :

$$W_\alpha = -\frac{1}{4}\bar D^2 e^{-2gV} D_\alpha e^{2gV} = -\frac{g}{2}\bar D\bar D D_\alpha V + \frac{g^2}{2}\bar D\bar D[V, D_\alpha V] + \mathcal{O}(g^3) \tag{2.28}$$

$$\bar W_{\dot\alpha} = -\frac{1}{4}D^2 e^{-2gV} \bar D_{\dot\alpha} e^{2gV} = -\frac{g}{2}DD\bar D_{\dot\alpha} V + \frac{g^2}{2}DD[V, \bar D_{\dot\alpha} V] + \mathcal{O}(g^3) \tag{2.29}$$

which are respectively a chiral and an antichiral superfield (g is the gauge coupling). If we want to write a Lagrangian, starting from superfields, which is invariant under SUSY transformation we must identify expressions which transform with a total divergence. Performing a general SUSY transformation on chiral and vector superfields, using the explicit expressions of eq. (2.19) for the SUSY generators, we can see that only the $\theta\theta$ component of chiral superfields and the $\theta\theta\bar{\theta}\bar{\theta}$ component of vector superfields possess such a property. Therefore, we have to find a way to select the good superfields' components. This can be done by hand but in order to work with a full superspace formulation, we define a measure of integration over Grassman variables. For spinor Grassman components θ_α :

$$\int \mathrm{d}\theta_\alpha \, \theta_\beta = \delta_{\alpha\beta} \qquad \int \mathrm{d}\theta_\alpha = 0 \tag{2.30}$$

and for an integration over the whole space of Grassman coordinates we define the differential element $\mathrm{d}^2\theta = \frac{1}{4}\varepsilon_{\alpha\beta}\mathrm{d}\theta^\alpha \mathrm{d}\theta^\beta$ which yields :

$$\int \mathrm{d}^2\theta \, \theta^2 = 1 \tag{2.31}$$

and we can thus extract easily :

$$[\Phi]_{\theta\theta} = \int \mathrm{d}^2\theta \, \Phi, \qquad [V]_{\theta\theta\bar{\theta}\bar{\theta}} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, V \tag{2.32}$$

The kinetic term for the components of Φ is constructed by extracting the $\theta\theta\bar{\theta}\bar{\theta}$ component of $\bar{\Phi}\Phi$ which is a real superfield. For the vector superfield kinetic term we extract the $\theta\theta$ component of $W_\alpha W^\alpha$, which is a chiral superfield. Thus the free Lagrangian density reads (with gauge indices discarded) :

$$\mathcal{L}_{kin} = \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \bar{\Phi}\Phi + \int \mathrm{d}^2\theta \, \frac{\mathrm{Tr}}{16g^2 C_2(G)}[W^\alpha W_\alpha] + \int \mathrm{d}^2\bar{\theta} \, \frac{\mathrm{Tr}}{16g^2 C_2(G)} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} \tag{2.33}$$

The Yukawa interactions between chiral superfields are constructed by extracting the $\theta\theta$ component of all possible gauge invariant products of chiral superfields. Dimensional analysis indicates that the mass dimension of θ is $1/2$, so the maximum number of chiral fields we can couple to form a renormalisable term is three. As for gauge interactions, the gauge transformation parameter, $\Lambda(x) = \Lambda_a(x)T^a$, it has to be promoted to a superfield. Therefore, to compensate for the gauge transformation $\Phi \to e^{-ig\Lambda}\Phi$, the kinetic term has to become $\bar{\Phi}\Phi \to \bar{\Phi}e^{2gV}\Phi$ and the transformation of V is defined such that $e^{2gV} \to e^{-ig\bar{\Lambda}} e^{2gV} e^{ig\Lambda}$. The total Lagrangian density is thus :

$$\begin{aligned} \mathcal{L} \;=\; & \int \mathrm{d}^2\theta \mathrm{d}^2\bar{\theta} \, \bar{\Phi}_i e^{2gV} \Phi_i + \int \mathrm{d}^2\theta \left[\frac{\mathrm{Tr}}{16g^2 C_2(G)}[W^\alpha W_\alpha] + W(\Phi_i) \right] \\ & + \int \mathrm{d}^2\bar{\theta} \left[\frac{\mathrm{Tr}}{16g^2 C_2(G)} \bar{W}_{\dot{\alpha}} \bar{W}^{\dot{\alpha}} + \bar{W}(\bar{\Phi}_i) \right] \end{aligned} \tag{2.34}$$

The quantity $W(\Phi_i)$ is a chiral superfield formed from products of the elementary superfields Φ_i of the theory. From the dimensional analysis mentioned above, the renormalisable part of W is at most cubic in the superfields :

$$W(\Phi_i) = m_{ij}\Phi_i\Phi_j + \frac{\lambda_{ijk}}{6}\Phi_i\Phi_j\Phi_k + ... \tag{2.35}$$

The dots stand for any non-renormalisable interaction one might want to add. Of course W can only depend on chiral superfields and not also on antichiral superfields since an arbitrary product of chiral and antichiral superfields is not chiral. The holomorphicity of the superpotential has far reaching consequences for supersymmetric theories. First of all, and most obviously, it constrains the couplings as compared to non SUSY theories. For example, in the SM the Higgs cannot couple holomorphically

to all the fermions, so when we extend the SM to the MSSM it is clear that we have to add another Higgs field with an opposite hypercharge. The theory thus contains a Higgs H_u, coupling to the up quarks and the neutrinos, and H_d, coupling the down quarks and the charged leptons. They both take a vev and the parameter $\tan\beta = v_u/v_d$ parametrises the ratio of these vev's. In the rest of the manuscript, when considering a priori non SUSY setups we will denote the Higgs field H. When considering SUSY theories we will distinguish H_u and H_d.
Another, even more important, consequence is the fact that the superpotential is not renormalised to any order of perturbation theory [107]. This can be conveniently proven using supergraphs [108] or holomorphicity [109]. The coupling constants present in the superpotential are thus renormalised only through wave function renormalisation of the chiral superfields.
In fact, when the spectrum of the theory is completely supersymmetric (all bosonic and fermionic degrees of freedom are paired), the well known quadratic divergences arising when we renormalise the mass of a scalar field cancel each other and the theory is free from quadratic divergences. Therefore, SUSY helps us decorrelate the EW scale from any super heavy scale present in the theory (meaning at least the Planck scale M_P). This point will be very welcome when we will consider Grand Unified Theories in the next section of this chapter.

SUSY breaking and R-parity

When supersymmetrised, our SM must be added new particles, namely sleptons and squarks, the scalar partners of the fermions, and gauginos, the fermionic partners of the gauge bosons. In the rest of the manuscript, we will denote the chiral superfields of the MSSM by the capital letter corresponding to the usual notation of the SM. For example, the lepton doublets l_i become L_i. Moreover the SUSY algebra imposes that the superpartners have the same mass since P^2 commutes with the Q's, and the Q's link component fields with different spins. Therefore, supersymmetry must be broken at a scale higher than the EW scale. When broken, SUSY allows the reappearance of quadratic divergences in the mass of the Higgs boson but they will be proportional to the SUSY breaking scale m_{SUSY}. This is why we expect that the superpartner masses will not be much larger than 1 TeV. However the main problem is now to break SUSY in a consistent and phenomenologically viable way. With sfermions this light, and for a generic set of SUSY breaking parameters, clear signals of flavour and CP violating processes should have been detected, such as very fast proton decay. The first danger comes from renormalisable (thus unsuppressed) operators allowed in SUSY models that did not have a counterpart in the SM, such as LH_u, or $U^cU^cD^c$ for example. These operators can be killed by imposing a $U(1)_R$ symmetry on the superfields and on the fermionic coordinates θ_α, which induces different transformations for the bosonic and fermionic components. As the full $U(1)$ symmetry would prevent the gauginos to acquire a mass, we have in fact to impose only the discrete R-parity subgroup of the continuous $U(1)$ transformation and each component field is assigned a parity $(-1)^{B-L+2s}$. In addition to forbidding dangerous baryon number violation, it makes the lightest superpartner stable, providing a natural candidate for the Dark Matter component of the Universe. Even after the renormalisable operators have been eliminated, a generic set of soft SUSY breaking couplings should produce some detectable signal of SUSY in CP violation measurements or Lepton Flavour Violation, clearly above those of the SM. This issue is resolved in certain models of SUSY breaking where the masses of the sfermions are diagonal and universal at the scale of SUSY breaking and off-diagonal flavour violating effects are only generated radiatively. Therefore the additional mixings between the different flavours of quarks or leptons are small enough to suppress any unwanted contribution.

The last important comment we should make concerns the running of the gauge couplings in the

Standard Model. When taking into account quantum corrections in the SM we can check that gauge couplings almost unify at a high scale but they clearly do not intersect at the same point. However, if we include in the loops the supersymmetric partners of the SM particles around 1 TeV, the gauge couplings all meet inside the experimental error bars at a scale $M \sim 2 \times 10^{16}$ GeV. This is a good hint that Grand Unified Theories might have some relevant connection to the real world.

2.2.3 Supergraphs

We will close this section with a quick review of the supergraph formalism, since it will be of great use in the last chapter. The supergraph formalism consists in working with superfields only, never using component fields at any point in the computations. This allows to compute diagrams involving the partners of a same superfield from a single diagram where only the superfield propagate. Hence, as our chiral and vector superfields are Lorentz scalars, only scalar integrals will appear when evaluating diagrams.

In order to compute everything in the superfield formalism it is more convenient to work with integrals over the full superspace $z = (x, \theta, \bar{\theta})$. We must also extend functional differentiation to superfields, so we define Dirac distributions in Grassman spaces :

$$\int \mathrm{d}\theta_\alpha \delta(\theta_\alpha) = 1 \tag{2.36}$$

so that, by definition, the Dirac distribution corresponds to the Grassman variable itself : $\delta(\theta_\alpha) = \theta_\alpha$. We then extend $\delta^2(\theta) = \theta^2$ and $\delta^4(\theta) = \theta^2\bar{\theta}^2$, thus it comes immediatly that $\delta(0) = 0$. The complete Lagrangian can be rewritten over the full superspace (although this is not the expression usually used to derive the Feynman rules) :

$$\mathcal{L} = \int \mathrm{d}^8 z \left[\Phi^\dagger e^{2gV} \Phi + \frac{\mathrm{Tr}}{16 g^2 C_2(G)} W_\alpha W^\alpha \delta(\bar{\theta}) + W(\Phi_i)\delta(\bar{\theta}) + \mathrm{h.c.} \right] \tag{2.37}$$

Before deriving the Feynman rules, the first task is to determine the free propagators. For the chiral superfield there is a small subtlety in the functional derivative. We can see it by deriving the expression $\int \Phi f$, where f is any function over the superspace :

$$\begin{aligned} \frac{\delta}{\delta\Phi(y,\theta)} \int \mathrm{d}^8 z' \Phi(y', \theta') f(x', \theta', \bar{\theta}') &= \int \mathrm{d}^4 y \mathrm{d}^2\theta' \mathrm{d}^2\bar{\theta} \delta^4(y - y')\delta^2(\theta - \theta') f(y' - i\theta'\sigma\bar{\theta}, \theta', \bar{\theta}) \\ &= \int \mathrm{d}^2\bar{\theta} f(x, \theta, \bar{\theta}) \\ &= -\frac{1}{4}\bar{D}\bar{D} f(x, \theta, \bar{\theta}) \end{aligned} \tag{2.38}$$

In the last line we made use of the property that integration in superspace is equivalent to differentiation, so that $\int \mathrm{d}^2\bar{\theta} f = -\frac{1}{4}\bar{D}\bar{D}f$ and $\int \mathrm{d}^2\theta f = -\frac{1}{4}DDf$. For a chiral superfield Φ it is possible, after some superspace algebra (see [12, 110] for example) to compute the chiral propagators :

$$\langle 0|T[\bar{\Phi}(z_1)\Phi(z_2)]|0\rangle = \frac{-i}{\Box + m^2} \frac{D_1^2 \bar{D}_2^2 \delta^4(\theta_1 - \theta_2)}{16} \tag{2.39}$$

$$\langle 0|T[\Phi(z_1)\Phi(z_2)]|0\rangle = \frac{-im}{\Box + m^2} \frac{\bar{D}_1^2 \delta^4(\theta_1 - \theta_2)}{4} \tag{2.40}$$

Going to momentum space, "super"-Fenynman rules can be written. For this, it is necessary that the integration be performed over the whole super space $\mathrm{d}^8 z$, therefore when interactions from the superpotential are involved we will always convert the $\frac{\bar{D}^2 D^2}{16}$ of one chiral propagator into integrals over $\mathrm{d}^2\theta$ and $\mathrm{d}^2\bar{\theta}$ to complete the superspace integration. Therefore when several internal propagators of chiral fields will be present, all but one of them will display the $\frac{\bar{D}^2 D^2}{16}$ factor.

As far as the gauge part is concerned, a gauge fixing term must be added and we choose :

$$\frac{\mathrm{Tr}}{8\xi C_2(G)} \int \mathrm{d}^8 z D^2 V_a \bar{D}^2 V_a = \frac{1}{\xi} \int \mathrm{d}^8 z V_a (P_1 + P_2) V_a \tag{2.41}$$

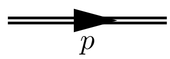
Here we use the three projectors :

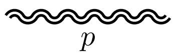
$$P_1 = \frac{D^2 \bar{D}^2}{16} \qquad P_2 = \frac{\bar{D}^2 D^2}{16} \qquad P_T = \frac{D \bar{D}^2 D}{8\Box} = \frac{\bar{D} D^2 \bar{D}}{8\Box} \tag{2.42}$$

which verify $P_i^2 = P_i$ and $P_1 + P_2 + P_T = 1$. Finally the quadratic term for V reads :

$$S_{gauge,kin} = -\int \mathrm{d}^8 z\, V_a \left(P_T - \frac{1}{\xi}(P_1 + P_2) \right) V_a \tag{2.43}$$

In the gauge $\xi = -1$, we end up with propagators for free superfields :

chiral superfield :  $= \dfrac{i}{p^2 - m^2}\delta^4(\theta_1 - \theta_2)$ (2.44)

vector superfield :  $= \dfrac{i}{2p^2}\delta^4(\theta_1 - \theta_2)$ (2.45)

The Feynman rules then proceed as usual, with integration over $\mathrm{d}^4\theta$ at each vertex, and an application of the operators $-D^2/4$ and $-\bar{D}^2/4$ on the chiral propagators, with the rule we mentioned previously. ore details and examples can be found in [12] and particularly [110].

2.3 Grand Unified Theories

If supersymmetry is realised at relatively low energy, we explained that gauge couplings seem to unify at a scale $M_{GUT} \simeq 2 \times 10^{16}$GeV. If not a coincidence, this is a strong hint for an embedding of the SM into a Grand Unified Theory. The principle of a GUT is that the SM gauge group $SU(3)_c \times SU(2)_L \times U(1)_Y$ is the remnant of a simple gauge group (meaning that it is not composed of several factors with different gauge couplings). All fundamental interactions are thus unified at high energy, leaving only one elementary force. Other groups are not simple but semi-simple, such as the Pati-Salam group $SU(4)_c \times SU(2)_L \times SU(2)_R$ [6]. Still they exhibit several interesting properties.
This idea was proposed more than thirty years ago [7], and continues to generate a lot of activity today. As GUTs are based on larger groups than the SM one, they have a number of specific features. To begin with, fermions are grouped inside larger representations which contain at the same time quarks and leptons. This gives a common origin to all the different representations contained inside one family of matter. Another general prediction is the quantification of electric charge. If the group is semi-simple with rank greater than one, i.e. non-Abelian, the eigenvalues of each generators are quantified and cannot vary continuously, as is the case for $U(1)$ groups for example. Therefore, as

$U(1)_Y$ is a subgroup of this larger non-Abelian group, its eigenvalues are also quantised.

The rank of the SM gauge group is four, so every group aiming to contain it has rank four or greater. The smallest viable group is $SU(5)$ [7], which has the minimal rank expected. Thus its breaking to the SM is quite constrained and the minimal version of an $SU(5)$ GUT is quite simple to explore. In order to incorporate type I seesaw we will in the end consider $SO(10)$ models; however, as $SU(5) \subset SO(10)$ it is still interesting to begin with $SU(5)$. After introducing the simple example of $SU(5)$ we will develop $SO(10)$ theories, which have a lot more freedom as concerns the breaking to the SM gauge group. We will see that they are the natural setups to incorporate seesaw neutrino masses. Finally we will present the generic predictions of the simplest GUTs for fermion masses. As the gauge group becomes larger, and fermions are grouped together inside larger representations, the Yukawa couplings become more and more correlated and we may be able to make predictions at the GUT scale, thereby checking the simplest versions of our models.
In the rest of the manuscript we consider mainly supersymmetric GUTs, except when stated otherwise. Much of the analysis we will go through can be done all the same in non-SUSY theories. The main difference, except for the sharp problems of extreme fine-tuning, comes from the fact that we have to deal with more independent couplings since quartic couplings in the Higgs sector are not related to cubic and gauge couplings.

2.3.1 A simple example : $SU(5)$

As explained previously, the smallest simple group embedding of $SU(3)_c \times SU(2)_L \times U(1)_Y$ is $SU(5)$. It is actually quite simple to see the embedding of the Standard Model into $SU(5)$ representations : we simply identify the first three indices as colour indices and the other two as $SU(2)_L$ indices. Let us see this on the adjoint representation first. We can write the gauge bosons of $SU(5)$ in the following way :

$$A = \sum_a A^a \frac{\lambda^a}{2} = \frac{1}{\sqrt{2}} \begin{pmatrix} & & & X_1 & Y_1 \\ & G^\alpha_\beta - \frac{2}{\sqrt{30}} B \delta^\alpha_\beta & & X_2 & Y_2 \\ & & & X_3 & Y_3 \\ X^1 & X^2 & X^3 & \frac{1}{\sqrt{2}} W_3 + \frac{3}{\sqrt{30}} B & W^+ \\ Y^1 & Y^2 & Y^3 & W^- & -\frac{1}{\sqrt{2}} W_3 + \frac{3}{\sqrt{30}} B \end{pmatrix} \tag{2.46}$$

We display the gluons as G and the electroweak bosons W and B as usual. The generators are normalised as $\mathrm{Tr}(\lambda^a \lambda^b) = 2\delta^{ab}$. We see appearing in the "off-diagonal" entries the gauge bosons of $SU(5)/SU(3) \times SU(2) \times U(1)$. Under $SU(3) \times SU(2)$, the 24 dimensional adjoint decomposes as : $\mathbf{24} = (\mathbf{8}, \mathbf{1}) + (\mathbf{1}, \mathbf{3}) + (\mathbf{1}, \mathbf{1}) + (\mathbf{3}, \mathbf{2}) + (\overline{\mathbf{3}}, \mathbf{2})$. The complex bosons X^α and Y^α are thus grouped into doublets of $SU(2)_L$. When $SU(2)_L \times U(1)_Y$ is broken to $U(1)_{em}$ these become vector bosons, triplets under $SU(3)_c$, with electromagnetic charges -4/3 and -1/3.

Matter

The split we used in $SU(5)$ indices allows us to see directly that the fundamental $\mathbf{5}$ of $SU(5)$ will contain a fundamental of $SU(2)$ and a fundamental of $SU(3)$. Thus an anti-fundamental can contain a right-handed quark and a doublet of leptons. The next smallest representation is the anti-symmetric product of two $\mathbf{5}$'s, which is 10-dimensional, and decomposes as : $\mathbf{10} = (\mathbf{5} \times \mathbf{5})_a = (\mathbf{3}, \mathbf{2}) + (\overline{\mathbf{3}}, \mathbf{1}) + (\mathbf{1}, \mathbf{1})$ under

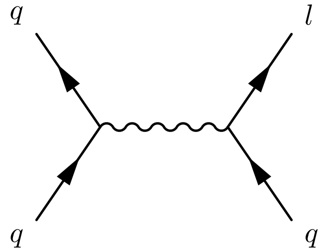


Figure 2.1: proton decay through the tree level exchange of X or Y gauge bosons of $SU(5)$

$SU(3)_c \times SU(2)_L$, which is good enough to incorporate the rest of a matter family. Graphically we have :

$$\overline{\mathbf{5}} = \begin{pmatrix} d^c \\ l \end{pmatrix} \qquad \mathbf{10} = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & u_3^c & -u_2^c & u^1 & d^1 \\ -u_3^c & 0 & u_1^c & u^2 & d^2 \\ u_2^c & -u_1^c & 0 & u^3 & d^3 \\ -u^1 & -u^2 & -u^3 & 0 & e^c \\ -d^1 & -d^2 & -d^3 & -e^c & 0 \end{pmatrix} \tag{2.47}$$

We do not have a complete unification of matter but we traded the five different representations of a single SM family against only two. Here we see that quarks and leptons derive from the same representations and are the same fundamental object. Although satisfying form a theoretical point of view, because we reduce the multiplicity of representations in our theory, this can run pretty quickly into practical problems. Indeed, we cannot anymore define conserved global symmetries such as baryon number or lepton number. A careful analysis shows that only $B-L$ is conserved by $SU(5)$, but L and B alone are not. This is actually a serious problem for the stability of the proton. The gauge bosons X and Y couple quarks and leptons and, when integrated out, will generate dimension 6 operators of the form $qqql$ (see fig.2.1). However, the scale of GUT breaking implied by gauge coupling unification is heavy enough to kill any danger coming from these operators (for the moment).

In order to break $SU(5)$ to $SU(3)_c \times SU(2)_L \times U(1)_Y$ and then to $SU(3)_c \times U(1)_{em}$, we must embed the SM Higgses into complete $SU(5)$ representations. According to what we said earlier the smallest ones are $\mathbf{5}$ and $\overline{\mathbf{5}}$, which contain the necessary weak doublets, along with colour triplets. This is a very delicate problem since these colour triplet fermions, in the SUSY case, create effective interactions proportional to $\frac{1}{M_T} QQQL$ or $\frac{1}{M_T} U^c U^c D^c E^c$ where M_T is the triplet mass. These operators induce at one loop a decay of the proton. As experiments like Super Kamiokande have put severe constraints on the lifetime of the proton, M_T is restricted to a very high energy range. The problem is now to keep the $SU(2)_L$ doublets H_u and H_d light while making the colour triplets heavy enough, which is known as the infamous Doublet-Triplet splitting.

The breaking

Now another representation is needed to break $SU(5)$ at $M_{GUT} \sim 10^{16}$ GeV. As we must not reduce the rank of the gauge group it must be real, so we use a chiral field in the adjoint representation Σ, which is the smallest real one. The Higgs sector consists of $\mathbf{5}_H$, $\overline{\mathbf{5}}_H$ and Σ and its superpotential is :

$$m_5 \mathbf{5}_H \overline{\mathbf{5}}_H + \frac{m_{24}}{2} \mathrm{Tr}(\Sigma^2) + \frac{\lambda_1}{3} \mathrm{Tr}(\Sigma^3) + \lambda_2 \overline{\mathbf{5}}_H \Sigma \mathbf{5}_H \tag{2.48}$$

The minimisation with respect to Σ gives three possible degenerate vacua in the supersymmetric case : unbroken $SU(5)$, $SU(4) \times U(1)$ and $SU(3) \times SU(2) \times U(1)$. This problem can be circumvented by extending SUSY to a local symmetry, called supergravity (see Chapter 6 of [14] and references therein), or by adding more fields to the Higgs sector.
Assuming the appropriate corrections are taken into account, we derive an expression for the vev of Σ :

$$\langle \Sigma \rangle = V \begin{pmatrix} 2 & & & & \\ & 2 & & & \\ & & 2 & & \\ & & & -3 & \\ & & & & -3 \end{pmatrix} \tag{2.49}$$

Combined with the mass term for the **5**'s we obtain masses $m_5 + 2V$ for the colour triplets and $m_5 - 3V$ for the doublets. Fine-tuning the second one we can obtain light doublets while keeping heavy triplets. However, after introducing SUSY to solve our fine-tuning problems with the mass of the Higgs boson, it is not very satisfactory to reintroduce one at this stage[1]. A possible outcome is to add more Higgs representations coupling to **5** and $\overline{\mathbf{5}}$ so that globally there are more doublets than triplets in the spectrum and we cannot couple all the doublets in the $\mathbf{5_H}$ and $\overline{\mathbf{5}}_\mathbf{H}$ to doublets of other representations with large masses. One such configuration consists in adding $\mathbf{50} + \overline{\mathbf{50}} + \mathbf{75}$ to the spectrum. The **75** contains an $SU(5)$ singlet and breaks $SU(5)$ to the SM and we can forget about the **24**. In the mean time, **75** couples to Higgses through $\mathbf{50}.\mathbf{75}.\overline{\mathbf{5}}_\mathbf{H}$ and the **50** contains a colour triplet but no weak doublet.

The main phenomenological problem of this theory comes from the colour triplet part of the $\mathbf{5}_H$ and $\overline{\mathbf{5}}_H$. A good unification of the couplings in fact requires the mass of the colour triplets from the **5** and $\overline{\mathbf{5}}$ Higgses to have a mass smaller than M_{GUT}. Including two-loop renormalisation equations and one-loop threshold effects, the authors of reference [17] find a range :

$$3.5 \times 10^{14} \text{ GeV} \leq M_T \leq 3.6 \times 10^{15} \text{ GeV} \tag{2.50}$$

which is in conflict with the bound on the proton decay channel $\tau(p \to K^+ \overline{\nu})$ from Super-Kamiokande, therefore killing the most minimal SU(5) SUSY-GUT model.

2.3.2 $SO(10)$ and the seesaw

Instead of going to non-minimal $SU(5)$ we can go one step further and consider larger groups. The next simplest possibility to consider is then $S0(10)$. Developing the theory we will realise that it contains naturally many stimulating additional features.
An embedding of $SU(5)$ into $SO(10)$ has been presented in the first reference of [40]. The trick is to consider operators χ_i with $i = 1...N$, defined by anti-commutation relations $\{\chi_i, \chi_j^\dagger\} = \delta_{ij}$. From these operators we can build two sets of generators, the T^i_j, defined by :

$$T^i_j = \chi_i^\dagger \chi_j \tag{2.51}$$

and the Γ_m, $m = 1...2N$:

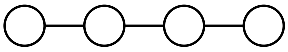
$$\Gamma_{2j-1} = -i(\chi_j - \chi_j^\dagger) \qquad \Gamma_{2j} = \chi_j + \chi_j^\dagger \tag{2.52}$$

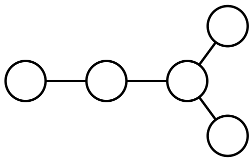
[1]Let us stress, however, that contrary to the non-SUSY case, such a tuning would only need to be made once in the superpotential, and not at each order of the perturbation expansion

Noting that these two sets satisfy the relations :

$$[T^i_j, T^k_l] = \delta^k_j T^i_l - \delta^i_l T^k_j \qquad \text{and :} \qquad \{\Gamma_m, \Gamma_n\} = 2\delta_{mn} \tag{2.53}$$

we see that the T^i_j satisfy an $SU(N)$ algebra while the Γ_m obey a Clifford algebra and therefore generate an $SO(2N)$ algebra with generators $\Sigma_{mn} = \frac{1}{2i}[\Gamma_m, \Gamma_n]$. Thus, we have built an explicit link between $SU(N)$ and $SO(2N)$ and we could even build the spinor representation of $SO(2N)$ by application of the $\chi^\dagger_i$ on a vacuum and see its decomposition under the $SU(N)$ subgroup.
Another way, more visual and direct, is to consider the Dynkin diagrams of these two groups :

$SU(5)$  (2.54)

$SO(10)$  (2.55)

Thus if we severe one of $SO(10)$'s roots we obtain directly $SU(5) \times U(1)$.

The vector multiplet has dimension $\frac{10(10-1)}{2} = 45$ and decomposes as : $\mathbf{45} = \mathbf{24} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{1}$ under $SU(5)$.

Matter representation

We already saw that matter superfields must fall into complex representations of the gauge group. The smallest complex representation of $SO(10)$ is the spinorial one, which contains 16 independent degrees of freedom (since $SO(10)$ is chiral). Decomposing it under the $SU(5)$ subgroup leaves :

$$\mathbf{16} = \mathbf{10} + \overline{\mathbf{5}} + \mathbf{1}$$

We can thus put all the matter of a single family inside one representation of the gauge group ! This is one of the most elegant features of $SO(10)$. First, from a purely aesthetical point of view it reduces considerably the number of representations of the fundamental theory. Second, from a more practical point of view it gives a reason why the SM is free of anomaly at low energy : there is no more miracle cancellation between quarks and leptons' contributions, since $SO(10)$ is free of anomaly.

The other feature of the **16** which can have deep consequences is the presence of a singlet of $SU(5)$ (hence of the SM). This singlet is just the one needed to play the role of the right-handed neutrino we had to introduce for the type I seesaw mechanism. Moreover the fact that we break the Grand Unified gauge group at a high scale, giving large masses to every "non-SM" degree of freedom, gives us the opportunity to justify the large mass scale needed by a successful seesaw. There are mainly two ways to do so. First we write the possible contractions of matter fields :

$$\mathbf{16} \times \mathbf{16} = \mathbf{10_s} + \mathbf{120_a} + \mathbf{126_s}$$

so we must couple fermion bilinears either directly to these representations or to a product of fields containing them. The first option leads directly to the $\overline{\mathbf{126}}$ since it is the only one containing a SM

singlet which can take a large vev. The second one is to resort to non-renormalisable couplings like $\mathbf{16_i 16_j \overline{16}_H \overline{16}_H}$ since of course $\overline{\mathbf{126}} \subset \overline{\mathbf{16}} \times \overline{\mathbf{16}}$. The coupling is then suppressed by a large scale (which is supposed to be smaller than or equal to the Planck mass), but this can be welcome since we generally need right-handed neutrino masses significantly smaller than the GUT scale (at least two orders of magnitude).
A clear disadvantage of the renormalisable solution is the important contribution of the large $\overline{\mathbf{126}}$ to the running of the gauge coupling above the GUT scale (all the more since we must add a **126** in order to prevent SUSY breaking at the scale of the $\overline{\mathbf{126}}$ vev). With representations larger than the **54**, $SO(10)$ becomes strongly coupled before reaching M_P, usually one order of magnitude above M_{GUT}. Thus, non-renormalisable couplings mildly suppressed by the strong coupling scale can be a priori present.

The breaking

Now we must find a way to break $SO(10)$ all the way down to the SM. Although this could be achieved in a quite constrained way for $SU(5)$, we have a lot more freedom for $SO(10)$ because we have multiple possibilities to lower the rank and several possible intermediate gauge groups before obtaining the SM.

The two main subgroups of maximal rank are $SU(5) \times U(1)_X$, as we already mentioned, and $SU(2)_L \times SU(2)_R \times SU(4)_c$, the Pati-Salam group which we also mentioned when introducing Grand Unification. In fact, when embedded into $SO(10)$, the PS group exhibits a parity symmetry between the left and right $SU(2)$ subgroups, often called D-parity. Under $SU(2)_R$, matter is then unified inside doublets, grouping the singlet quarks D^c and U^c, and the right-handed neutrinos N^c with the singlet electrons E^c. This is a more intuitive way to understand why a RH neutrino is introduced for each family. One generation is then incorporated inside the representations $(\mathbf{2}, \mathbf{1}, \mathbf{4}) + (\mathbf{1}, \mathbf{2}, \overline{\mathbf{4}})$ which is the decomposition of the **16** of $SO(10)$ under PS.

Another interesting feature of $SO(10)$ group theory is the incorporation of the $B-L$ symmetry as a subgroup. It is most easily seen when breaking $SU(4)_c$ of PS down to $SU(3)_c$ without lowering the rank. The gauge group is then $SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L}$. This explains the presence of $B-L$ as an accidental global symmetry of the SM and $SU(5)$.
However this is not the most appreciable consequence of gauging $B-L$. Indeed, if we select a breaking mechanism such that only fields with an even $B-L$ charge break the gauge group, we will automatically enforce R-parity in the model, which is quite desirable for SUSY phenomenology.

Let us now turn to the Higgs sector and the different breaking schemes. With the notable exception of the **16**, the usual Higgs representations one can encounter in the different models explored so far are tensorial representations[2], thus they are generated by tensor products of the fundamental **10**. We denote with letters $\{a, b, c, d, ...\}$ tensor indices which run from 1 to 10, and by $\{m, n, ...\}$ spinor indices running from 1 to 16. $[ab]$ stands for an antisymmetrisation of the indices while $\{ab\}$ means a symmetrisation. Starting from the $\mathbf{10_a}$, we can thus form antisymmetric tensor products with two (the adjoint $\mathbf{45_{[ab]}}$), three (the $\mathbf{120_{[abc]}}$), four (the $\mathbf{210_{abcd}}$) and five indices (the $\mathbf{126_{[abcde]}}$). Another frequently used representation is the symmetric contraction of two **10**'s, the $\mathbf{54_{\{ab\}}}$.
From all these representations, only the **16** and the **126** are complex. Thus, when including them in the spectrum one has to include their conjugate representations too, because if a complex representation

[2]Some models exist that use the **144** but they are few.

takes a vev, it must be compensated by the vev of the conjugate field, $\langle\mathbf{16}\rangle = \langle\overline{\mathbf{16}}\rangle$ or $\langle\mathbf{126}\rangle = \langle\overline{\mathbf{126}}\rangle$, in order to preserve D-flatness and prevent SUSY breaking at a high scale. A convenient way to discuss symmetry breaking and couplings is to list the decomposition of these representations under $SU(5)$:

$$\mathbf{10} = \mathbf{5} + \overline{\mathbf{5}} \tag{2.56}$$
$$\mathbf{45} = \mathbf{1} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{24} \tag{2.57}$$
$$\mathbf{54} = \mathbf{15} + \overline{\mathbf{15}} + \mathbf{24} \tag{2.58}$$
$$\mathbf{120} = \mathbf{5} + \overline{\mathbf{5}} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{45} + \overline{\mathbf{45}} \tag{2.59}$$
$$\mathbf{210} = \mathbf{1} + \mathbf{5} + \overline{\mathbf{5}} + \mathbf{10} + \overline{\mathbf{10}} + \mathbf{24} + \mathbf{40} + \overline{\mathbf{40}} + \mathbf{75} \tag{2.60}$$
$$\mathbf{16} = \mathbf{1} + \overline{\mathbf{5}} + \mathbf{10} \tag{2.61}$$
$$\mathbf{126} = \mathbf{1} + \overline{\mathbf{5}} + \mathbf{10} + \overline{\mathbf{15}} + \mathbf{45} + \overline{\mathbf{50}} \tag{2.62}$$

From this listing it appears that the representations able to break $SO(10)$ to $SU(5)\times U(1)$ are the **45** and **210**, while the complex ones **16** and **126** will lower the rank down to $SU(5)$. We can then use the **24** contained in **45**, **54** or **210** to break the remaining $SU(5)$. However, as we need to lower the rank, a complex representation is always needed at some point.
On the other hand, if we wish to break $SO(10)$ to $SU(2)_L \times SU(2)_R \times SU(4)_c$, we will use those representations containing a singlet under PS, the **54** and the **210**.
There are two main chains of $SO(10)$ breaking corresponding to the two maximal subgroups. The first chain is quite simple :

$$SO(10) \longrightarrow \left\{ \begin{array}{c} SU(5)\times U(1) \\ SU(5)'\times U(1)' \end{array} \right. \longrightarrow SU(5) \longrightarrow \text{SM} \tag{2.63}$$

The $SU(5)'$ stands for the flipped $SU(5)$ group.
The other chain, using $PS = SU(2)_L \times SU(2)_R \times SU(4)_c$, is realised by breaking $SO(10)$ to $SO(4)\times SO(6)$, since $SO(4) \sim SU(2)\times SU(2)$ and $SO(6) \sim SU(4)$. The rest of the chain is realised by $SU(2)_R \longrightarrow U(1)_R$ and $SU(4)_c \longrightarrow SU(3)_c \times U(1)_{B-L}$. All the potential steps of the breaking are summarised in the chain below :

$$SO(10) \longrightarrow PS(\times D) \longrightarrow \left\{ \begin{array}{c} SU(3)_c \times SU(2)_L \times SU(2)_R \times U(1)_{B-L} \\ SU(4)_c \times SU(2)_L \times U(1)_R \end{array} \right.$$
$$\longrightarrow SU(3)_c \times SU(2)_L \times U(1)_R \times U(1)_{B-L} \longrightarrow \text{SM} \tag{2.64}$$

The chains are displayed here with a maximum number of breaking steps but several steps can be realised at the same energy. Theoretically, one representation could be enough to break $SO(10)$ to the SM if it contains a SM singlet, but in practice we need several representations to build a realistic GUT model. The models which prefer to keep the gauge coupling perturbative up to the Planck scale use **45**'s, **54**'s and pairs of $\mathbf{16} + \overline{\mathbf{16}}$.

A recurrent problem in $SO(10)$ model building comes from the constraint of neutrino masses. A successful seesaw mechanism requires an intermediate scale, thus implying that some states, in addition to the singlet RH neutrinos, could be lighter than the GUT scale and could contribute to the running of the gauge couplings. If these sates are not full multiplets of the grand unified group, they will contribute to the differential running of the couplings, potentially ruining the unification required in any GUT.

2.3.3 Fermion masses

Realistic fermion masses are always a delicate problem in GUT model building. Indeed, the first thing we are tempted to do is to use the minimal number of representations because it seems at the same time the most aesthetically appealing and the most predictive option.
However, as quarks and leptons are related by the GUT gauge group, their couplings become closely related, so closely in the simplest models that it becomes unrealistic. First, we will quickly develop the $SU(5)$ theory as an example and then switch to the more complicated situation of $SO(10)$.

Yukawa correlations in $SU(5)$

As we saw earlier, the minimal embedding of the Higgs bosons of the MSSM uses a **5** and a $\overline{\mathbf{5}}$ of $SU(5)$. The Yukawa superpotential that we can write is then :

$$(Y_5)_{ij}\mathbf{10_i 10_j 5_H} + (Y_{\overline{5}})_{ij}\mathbf{10_i \overline{5}_j \overline{5}_H} + (Y_\nu)_{ij}\mathbf{\overline{5}_i 1_j 5_H} + \text{h.c.}$$

Y_5 then gives the Yukawa coupling of the up-quarks while $Y_{\overline{5}}$ generates those of the down-quarks and charge leptons. The singlet representations $\mathbf{1}_i$ stand for the singlet RH neutrinos N_i so that Y_ν is the neutrino Yukawa coupling.
As $\overline{\mathbf{5}}$ contains the LH part of charged leptons and RH part of down quarks, in this model we predict the relation : $M_e = M_d^T$. This implies also that charged leptons and down quark masses are equal at the GUT scale, which is now completely experimentally excluded. More precisely, running the masses from M_Z, at which their central values are [87] :

$$\begin{array}{lll} m_d = 3.0 \text{ MeV} & m_s = 54 \text{ MeV} & m_b = 2.87 \text{ GeV} \\ m_e = 0.487 \text{ MeV} & m_\mu = 103 \text{ MeV} & m_\tau = 1.75 \text{ GeV} \end{array} \tag{2.65}$$

to $M_{GUT} \sim 2 \times 10^{16}$ GeV with $M_{SUSY} = 1$ TeV and $\tan\beta = 10$, the value of the masses at M_{GUT} is :

$$\begin{array}{lll} m_d = 0.94 \text{ MeV} & m_s = 17 \text{ MeV} & m_b = 0.98 \text{ GeV} \\ m_e = 0.346 \text{ MeV} & m_\mu = 73.0 \text{ MeV} & m_\tau = 1.25 \text{ GeV} \end{array} \tag{2.66}$$

Therefore, strictly speaking, the minimal SUSY $SU(5)$ model must be extended even without considering proton decay. A possible way out is to introduce a **45** of Higgs [18, 19], which also contains a doublet of $SU(2)_L$ and can couple to the $\overline{\mathbf{5}}$'s and **10**'s of matter. Once decomposed under the SM gauge group, it will contribute differently to the quark and lepton mass matrices, more precisely we obtain :

$$M_d = v_{\overline{5}} Y_{\overline{5}} + v_{45} Y_{45} \qquad M_e^T = v_{\overline{5}} Y_{\overline{5}} - 3 v_{45} Y_{45} \tag{2.67}$$

The factor -3 comes from the following properties of **45** : **45** is a tensor with one fundamental an two anti-fundamental indices obeying $\mathbf{45}^a_{bc} = -\mathbf{45}^a_{cb}$ and $\sum_a \mathbf{45}^a_{ab} = 0$. Its vev will be along $\mathbf{45}^a_{b5}$ and diagonal if we do not want to break $SU(3)_c$, with the constraint over the trace translating into :

$$\langle \mathbf{45} \rangle = v_{45} \begin{pmatrix} 1 & & & & \\ & 1 & & & \\ & & 1 & & \\ & & & -3 & \\ & & & & 0 \end{pmatrix} \tag{2.68}$$

Making some assumptions on the Yukawa couplings (namely putting some entries to zero), leads to the famous Georgi-Jarlskog texture, predicting a factor of 3 between m_μ and m_s at M_{GUT}. However, even this relation is now in conflict with experimental data (cf eq. 2.66).

Yukawa correlations in $SO(10)$

Now we turn to $SO(10)$, trying first to be as minimal as possible. The task is to assign the Higgs doublets to $SO(10)$ representations. The first important remark to be made is the fact that the **10** contains a **5** and a $\overline{\mathbf{5}}$, which is exactly the minimal content compulsory in $SU(5)$ to get the MSSM at low energy. Thus, it appears that $SO(10)$ can even unify the Higgs sector of the MSSM. However, doing so restricts us to the only coupling $Y_{ij}\mathbf{16_i 16_j 10}$, implying immediatly $Y_u = Y_d = Y_e = Y_\nu = Y$ so that the situation is even worse than in minimal $SU(5)$.
Expanding the Higgs sector to several **10**'s will only mildly improve the situation. The generic situation will then be : $Y_u = Y_\nu$ and $Y_e = Y_d$. The first relation is not so bothering (but this is not even the case if we want successful leptogenesis, we will come back to this point later) but the last one leaves us with the same problem as before.

We have to turn to larger representations to do the work. If we want to keep renormalisability, then our choice is limited. From the decomposition of the tensor product $\mathbf{16}\times\mathbf{16}$, the other possibilities we have, apart from the **10**, are the **120** and the $\overline{\mathbf{126}}$. Symmetry properties of $SO(10)$ impose that Y_{10} and $Y_{\overline{126}}$ are symmetric while Y_{120} is anti-symmetric. In order to check what difference these operators will make, let us decompose these representations under the PS group :

$$\mathbf{120} = (\mathbf{3},\mathbf{1},\mathbf{6}) + (\mathbf{1},\mathbf{3},\overline{\mathbf{6}}) + (\mathbf{1},\mathbf{1},\mathbf{20}) + (\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{2},\mathbf{2},\mathbf{15}) \tag{2.69}$$
$$\overline{\mathbf{126}} = (\mathbf{3},\mathbf{1},\mathbf{10}) + (\mathbf{1},\mathbf{3},\overline{\mathbf{10}}) + (\mathbf{1},\mathbf{1},\overline{\mathbf{6}}) + (\mathbf{2},\mathbf{2},\mathbf{15}) \tag{2.70}$$

From this we infer that **120** can get a vev in the $(\mathbf{2},\mathbf{2},\mathbf{1})$ direction, just like the **10**, which means that it doesn't improve much the discrimination between M_e and M_d. The $(\mathbf{2},\mathbf{2},\mathbf{15})$ component is more interesting, since in order not to break $SU(3)_c$ the vev must be in the $B-L$ direction : there again, there is a factor -3 between the contributions to M_e and M_d. This was predictable in the sense that the $\overline{\mathbf{126}}$ contains a $\overline{\mathbf{45}}$ of $SU(5)$ and the **120** a $\mathbf{45}+\overline{\mathbf{45}}$. In the MSGUT model, the $\overline{\mathbf{126}}$ is used in conjunction to a single **10** to generate fermion masses and it appears to fit the data correctly.

For the neutrino sector, we first have to give a large mass to the right-handed neutrinos. The only possibility at the renormalisable level is through the $\overline{\mathbf{126}}$ since it contains a singlet of $SU(5)$. Alternatively we can see it with the PS decomposition for which the vev is carried by a triplet of $SU(2)_R$ with a $B-L$ charge equal to 2. This allows to give the mass scale of the N_i a more physical meaning, namely the breaking scale of the $B-L$ symmetry.

If we want to restrict ourselves to small representations, we have to turn to non-renormalisable interactions. We must then use representations we can contract to obtain **10**, **120** or $\overline{\mathbf{126}}$. One such example is $\mathbf{10}\times\mathbf{45}$, which contains a **120** and a **10**. We will develop later the use of this operator to obtain realistic masses for the model we will consider.
Turning to neutrino masses, we will only obtain a heavy majorana mass if we introduce a $\overline{\mathbf{16}}_\mathbf{H}$ and couple it through :

$$\frac{1}{\Lambda} Y_{NR}\overline{\mathbf{16}}_\mathbf{H}\overline{\mathbf{16}}_\mathbf{H}\mathbf{16_i 16_j} \tag{2.71}$$

The $\overline{\mathbf{16}}_\mathbf{H}$ has to take a vev in the **1** direction of $SU(5)$. The suppression factor $\langle 1_{\overline{\mathbf{16}}} \rangle^2/\Lambda$ immediatly gives intermediate masses for N_i.

2.4 Leptogenesis

Now that we have introduced a natural setup for the seesaw mechanism, we shall turn to one of its most appealing consequences. As we already mentioned, one of the very interesting features of the seesaw mechanism is the generation of a baryon asymmetry through lepton asymmetries in the early universe, thereby explaining the present evidence for an asymmetry in the matter-antimatter distributions [8].
We will begin this section with a review of the baryon asymmetry, its experimental evidence and its generation in the early Universe. Explaining why a correct baryon asymmetry cannot be reproduced by the SM alone will motivate the consideration of the heavy right-handed neutrinos as the source of the cosmic matter-antimatter asymmetry in the present Universe, which is the principle of *leptogenesis*.

2.4.1 Baryon asymmetry

The measured baryon asymmetry of the universe is one of the few concrete evidences for new physics beyond the Standard Model of particle physics, so that predicting a correct asymmetry is a precious test for BSM physics. Before explaining why the SM fails to reproduce the correct baryon asymmetry, let us recall some basic facts.
When trying to deal with matter density in the primordial universe we must work in a non-trivial background, defined by the Friedmann-Robertson-Walker metric :

$$ds^2 = dt^2 - a^2(t) d\vec{x}^2 \tag{2.72}$$

Evolution of the scale factor $a(t)$ is given by the (0,0) component of the Einstein equation :

$$R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} \mathcal{R} = 8\pi G T_{\mu\nu} \tag{2.73}$$

Defining ρ the energy density of the universe and k the curvature constant, equal to +1, -1 or 0 depending if our universe is positively, negatively curved or flat, the equation for a reduces to the Friedmann equation :

$$H^2 + \frac{k}{a^2} = \frac{8\pi G}{3}\rho \qquad \text{with : } H = \frac{\dot{a}}{a} \tag{2.74}$$

H is called the Hubble parameter. As $a(t)$ evolves with time the universe expands, so that the baryon and antibaryon number densities n_b and $n_{\bar{b}}$ do not keep constant during the evolution of the universe. When keeping track of a specific region we see that its volume scales like a^3, so n_B scales like a^{-3} and it will be more convenient to define the comoving number density :

$$y_B = \frac{n_B}{n_s} = \frac{n_b - n_{\bar{b}}}{n_s} \tag{2.75}$$

n_s is the entropy density of the universe and is related to the temperature by the formula :

$$n_s = \frac{2\pi^2}{45} g_{s*} T^3 \tag{2.76}$$

and scales also like a^{-3}. This allows to define a baryon asymmetry y_B which scales as a constant. The factor g_{s*} can be safely replaced, through most of the history of the universe, by g_* which counts the effective number of relativistic degrees of freedom :

$$g_* = \sum_{X=boson} g_X + \frac{7}{8} \sum_{X=fermion} g_X \tag{2.77}$$

g_X being the number of d.o.f. associated with the particle X. Note that the density of photons $n_\gamma = 2\frac{\zeta(3)}{\pi^2}T^3$ is proportional to n_s. This is why we could also work with the quantity :

$$\eta_B = \frac{n_B}{n_\gamma} \simeq 7.04\, y_B \tag{2.78}$$

The value of y_B consistent with the primordial abundance of light elements is [52] :

$$y_B \sim (8.7 \pm 0.3) \times 10^{-11} \tag{2.79}$$

Now, supposing we start with $y_B = 0$[3], baryon and antibaryons are in equilibrium with the plasma until the annihilation rate $\Gamma_{ann} \simeq n_b \langle \sigma_A v \rangle$ becomes smaller than the Hubble rate H, at $T_d \simeq 20$ MeV. At this temperature, the baryon number density freezes out at the value [76] :

$$\frac{n_b}{n_s} \simeq \frac{n_{\bar{b}}}{n_s} \sim \left(\frac{m_p}{T_d}\right)^{3/2} e^{-\frac{m_p}{T_d}} \sim 10^{-18} \tag{2.80}$$

which is is far from what is expected. Supposing that the asymmetry comes from statistical fluctuations in baryon and antibaryon distributions does not help any more.

The necessary conditions for a successful matter-antimatter asymmetry have actually been stated by Sakharov [77] already a long time ago (see also [76] for more detailed considerations), and they are three :

- Baryon number violation :
  Starting with an initial baryon asymmetry $n_B \neq 0$ would seem quite ad hoc. Moreover, any period of inflation would have diluted away the initial condition, leaving only a negligible n_B. Therefore, we prefer to start from a baryon symmetric state ($n_B = 0$), so it is obviously necessary to have baryon number violating interactions. However, one has to be very careful when dealing with brayon violating interactions since they may mediate proton decay, in which case they would be highly constrained by the experimental lower bound on the proton lifetime $\tau_p \geq 10^{33}$ years.

- C and CP violation :
  Since we want to distinguish between matter and antimatter it is also intuitive that charge conjugation C must not be a good symmetry of our theory. Moreover, CP, the product of charge conjugation and parity, must also be broken, otherwise the sum over phase space of any rate is zero. C is maximally violated in the SM and CP is violated through the complex CKM phase in the quark sector.

- Departure from equilibrium :
  If a species is placed at chemical equilibrium, its chemical potential will be zero since it is for this

[3] as suggested by inflationary scenarii

state that the entropy is maximal. The only hope to generate a CP asymmetry is to consider an out of equilibrium phenomenon. We can achieve out of equilibrium either with a first order transition, such as the one which could happen in the EW sector, or if a massive particle decays with CP and B violating interactions and slowly enough for its distribution to depart from the equilibrium value.

Let us develop the latter condition a little further. In the case of a massive particle X interacting with the primordial plasma, it will start to decay when the plasma cools down below its mass $T \leq M_X$[4]. If its coupling to lighter particles in the plasma is not small enough, meaning that its decay width Γ_X is larger than the Hubble rate, $\Gamma_X > H$, X will decay much more rapidly than the typical time of expansion and will adjust its population to the equilibrium value. On the contrary, if $\Gamma_X < H$, X will not decay fast enough and will become overabundant. At the time of their decay, X particles will thus be out of equilibrium. Supposing that baryon-number violating scatterings mediated by X are highly inefficient and we are in the case $\Gamma_X \ll H$, X will decay completely out of equilibrium and its density will not be exponentially suppressed $n_X = n_{\bar{X}} \sim n_\gamma$. Then, if each decay produces, on average, an asymmetry ε, the baryon number density is $n_B \sim \varepsilon n_X \sim \varepsilon n_\gamma$, and since $n_s \sim g_* n_\gamma$:

$$y_B = \frac{n_B}{n_s} \sim \frac{\varepsilon n_\gamma}{g_* n_\gamma} \sim \frac{\varepsilon}{g_*} \tag{2.81}$$

Conversely, with very efficient inverse decays and X-mediated scatterings, the whole asymmetry is washed out as soon as it is created.

A full study of the dynamics and a precise quantitative prediction imposes to solve the Boltzmann Equations of the system. An introduction to Boltzmann Equations is provided in appendix A.

2.4.2 Sphalerons and electroweak phase transition

When looking at the Sakharov conditions, we see that the SM has the necessary qualitative features to generate a B asymmetry without the need for new physics, the third condition being fulfilled only if the Electroweak phase transition is strongly first order.
But let us see first how B-violation appears in the SM. Although perturbatively it may seem that B is accidentally conserved because all interactions in the Lagrangian are B-conserving. However, B is only conserved classically but not quantum mechanically. Transforming the fermions with a general phase factor $\psi'(x) = e^{ia\theta(x)}\psi(x)$, we get extra contributions to the action, the usual one coming from the Lagrangian itself and another one coming from the measure in the generating functional $\mathcal{D}\psi\mathcal{D}\overline{\psi}$. Requiring the action to be invariant then implies the non conservation of the baryonic current $J_B^\mu = \frac{1}{3}\sum_i \bar{q}_i\gamma^\mu q_i$ for $a = 1/3$:

$$\partial_\mu J_B^\mu = i\frac{N_f}{32\pi^2}\left(-g_2^2 F_L^{a\mu\nu}\tilde{F}^a_{L\mu\nu} + g_1^2 F_Y^{\mu\nu}\tilde{F}_{Y\mu\nu}\right) \tag{2.82}$$

with $F_L^{\mu\nu}$ and $F_Y^{\mu\nu}$ the field strength of $SU(2)_L$ and $U(1)_Y$ respectively. The Hodge dual is defined as $\tilde{F}^{\mu\nu} = \frac{1}{2}\epsilon^{\mu\nu\rho\sigma}F_{\rho\sigma}$. When considering lepton number we find similarly :

$$\partial_\mu J_L^\mu = \partial_\mu J_B^\mu \tag{2.83}$$

[4] This discussion, similarly to the one led in appendix A, is inspired by the GUT baryogenesis mechanism, where X stands for the X and Y gauge bosons of $SU(5)$, see section 2.3.1. This mechanism is highly disfavoured in inflationary scenarii since inflation dilutes any asymmetry created before the inflation period.

so that the only conserved global symmetry is $B - L$. The next feature of the SM to be noticed is the non-trivial topology of the non-abelian group $SU(2)_L$. It implies the existence of an infinite set of degenerate pure gauge backgrounds, which cannot be deformed continuously into one another. They are separated by an energy barrier which is roughly 10 TeV in magnitude. Performing a transition from one vacuum to one of its neighbours will involve a variation in B and L :

$$\Delta B = \Delta L = N_F \tag{2.84}$$

which means that one quark per colour and one lepton of each generation will be created, forming an effective 12 particle interaction.
The field configuration which interpolates between two vacua is a static and unstable solution of the equations of motion [79]. It is called a *sphaleron*. At zero temperature it can be shown that the probability to tunnel from one vacuum to the other is proportional to $\exp(-4\pi/\alpha_W) \sim 10^{-150}$ [78]. However, when the temperature reaches the EW scale, sphalerons activate and badly violates B and L. Actually, any asymmetry created in the $B+L$ direction is completely washed-out as $B+L$ is driven to zero. The hope is then to have a strong first order EW transition in order to put the sphaleron processes out of equilibrium. Defining as T_c the critical temperature at which the second vacuum $v \neq 0$ becomes energetically favoured and bubbles of the true vacuum begin to grow, we reach indeed out of equilibrium if :

$$\frac{v(T_c)}{T_c} \geq 1 \tag{2.85}$$

Unfortunately, it requires the Higgs mass to be way too small, $m_H < 80$ GeV from precise computations in the SM [80].
Therefore the experimental result (2.79) undoubtedly calls for new physics. For example, it can be implemented in SUSY theories for which there is more room for a first order EW phase transition [81]. Another nice explanation, which we will develop below, comes out naturally from the seesaw mechanism.

2.4.3 Basics of Leptogenesis

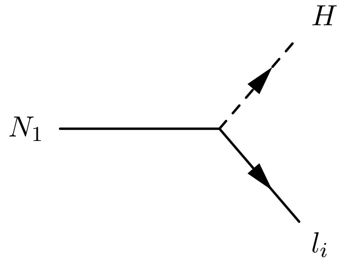
Up to now we have claimed several times that the creation of an efficient matter-antimatter asymmetry is potentially a welcome by-product of the seesaw mechanism. It is now time to explain the principle of leptogenesis and its main features, before digging into the analysis of our benchmark model described in section 3.1.3.

Leptogenesis has been introduced by Fukugita and Yanagida [8] when they realised that when RH neutrinos are heavy enough they can play the role of the particle X mentioned above. As they have a Majorana mass term their mass eigenstates are Majorana particles, meaning that they are their own antiparticle $N_i = N_i^c$. Thus they can decay into leptons and antileptons. Since their Yukawa couplings are a priori complex they violate CP and the decay will differentiate between leptons and antileptons. The initial situation is this one : after inflation, the inflaton decays and reheats the Universe up to some temperature T_{RH}. As far as RH neutrinos are concerned there are two possibilities. Either they are created directly through inflaton decay and their initial value is then highly model dependent. Or T_{RH} is sufficiently large to allow their creation through the interactions of the thermal plasma. The latter is obviously more interesting since the physics is independent from the exact nature of the inflaton and we can study the predictions of leptogenesis in a more model independent way.

In this introduction we will focus on the traditional "one flavour approximation" approach where lepton flavours are assumed to be indistinguishable and the RH neutrino N_1 decays into one flavour l_1.

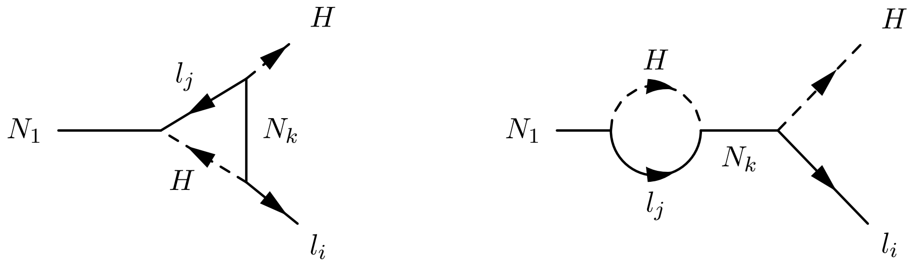
Type I leptogenesis

Let us start with $T_{RH} > M_1$. If $T_{RH} > M_{2,3}$, the decays of $N_{2,3}$ will generate an asymmetry similarly to N_1, and the same could be said of any particle heavier than N_1 with CP and L-violating interactions. However at the time of their decay N_1 is still in equilibrium, so if N_1 couples not too weakly to the plasma, its interactions will washout any lepton asymmetry created at $T > M_1$. Of course, this is only the case if there is some hierarchy between N_1 and the other particles[5]. From now on we will focus on the simple case of hierarchical RH neutrinos. This is anyway the case in most of the spectra that we will consider in the next chapter, so they are the spectra on which we will focus in practice.
The coupling of N_1 to matter was written in eq. (2.2) for the Standard Model. At tree level the amplitude of the decays into l and $\bar{l}$ are complex conjugate so that $\Gamma(N_1 \to lH)$ and $\Gamma(N_1 \to \bar{l}\bar{H})$ are equal and no CP asymmetry is generated. The asymmetry is thus generated by the interference of tree level :



(2.86)

and of the one loop vertex and self-energy corrections :



(2.87)

The CP asymmetry ε_1, is defined by :

$$\varepsilon_1 = \frac{\Gamma(N_1 \to lH) - \Gamma(N_1 \to \bar{l}\bar{H})}{\Gamma(N_1 \to lH) + \Gamma(N_1 \to \bar{l}\bar{H})} \tag{2.88}$$

and at one loop, computation of the previous diagrams gives :

$$\varepsilon_1^I = \frac{1}{8\pi} \frac{\sum_{j\neq 1} \mathrm{Im}[(Y_\nu Y_\nu^\dagger)_{1j}^2]}{(Y_\nu Y_\nu^\dagger)_{11}} f(x_j) \tag{2.89}$$

and we defined the ratios $x_j = M_j^2/M_1^2$. In the limit of hierarchical RH neutrinos : $x_j \gg 1$ and we can use the approximate form for the loop function f_{SM} in the SM :

[5]This not even completely true, however, as has been stated in [58]; we will come back to this point later.

$$f_{SM}(x_j) = \sqrt{x_j}\left[\frac{2-x_j}{1-x_j} + (1+x_j)\text{Ln}\left(1+\frac{1}{x_j}\right)\right] \longrightarrow f_{SM}(x_j \gg 1) \simeq -\frac{3}{2\sqrt{x_j}} \qquad (2.90)$$

and the expression for ε_1 becomes :

$$\varepsilon_1^I \simeq \frac{3}{16\pi}\frac{\text{Im}[(Y_\nu m_\nu^* Y_\nu^T)_{11}]}{(Y_\nu Y_\nu^\dagger)_{11}}\frac{M_1}{v^2} \qquad (2.91)$$

When considering a SUSY theory, the diagrams (2.86) and (2.87) are promoted to superdiagrams. In components, this amounts to consider the decays of the RH neutrinos and their associated RH sneutrinos to leptons plus Higgs and sleptons plus Higgsinos. The form (2.89) of ε_1^I is the same but the loop function is modified, and we give it below, displaying once again its approximate value in the limit $x_j \gg 1$:

$$f_{SUSY}(x_j) = \sqrt{x_j}\left[\frac{2}{1-x_j} - \text{Ln}\left(1+\frac{1}{x_j}\right)\right] \longrightarrow f_{SUSY}(x_j \gg 1) \simeq -\frac{3}{\sqrt{x_j}} \qquad (2.92)$$

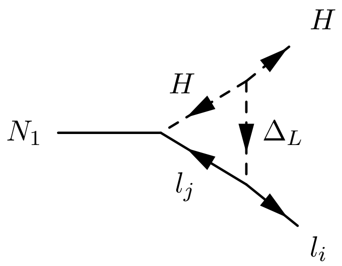
and the approximate form for ε_1 is now :

$$\varepsilon_1^I \simeq \frac{3}{8\pi}\frac{\text{Im}[(Y_\nu m_\nu^* Y_\nu^T)_{11}]}{(Y_\nu Y_\nu^\dagger)_{11}}\frac{M_1}{v_u^2} \qquad (2.93)$$

which is only twice the asymmetry of the Standard Model.
This formula has been widely studied and general statements can be made. For example it can be shown that no successful leptogenesis can be achieved if the R-matrix introduced earlier is real, which means that the low energy CP violating phases cannot be linked to the observation of matter-antimatter asymmetry in the Universe.

Type II leptogenesis

When the $SU(2)_L$ triplet Δ_L of the type II seesaw mechanism is present, and it is much heavier than N_1, the asymmetry is still created by the lightest particle N_1, but there is a new loop contributing to the decays of N_1 [49, 50, 51] :



(2.94)

This diagrams interferes with the tree level decay and adds a contribution to the total CP-asymmetry :

$$\varepsilon_1^{II} = -\frac{3}{8\pi}M_1 v^2 \frac{\sum_{ij}\text{Im}[Y_{1i}Y_{1j}(m_\nu^{II})^*_{ij}]}{(YY^\dagger)_{11}} g(y) \qquad (2.95)$$

With $m_\nu^{II} = v_L f$ and the parameter y stands for the ratio $M_{\Delta_L}^2/M_1^2$. Once again the form of ε_1^{II} is the same with only a change in the loop functions, that are given by :

$$g_{SM}(y) = y\left[-1 + y\,\text{Ln}\left(1+\frac{1}{y}\right)\right] \quad\longrightarrow\quad g_{SM}(y \gg 1) \simeq -\frac{1}{2} \tag{2.96}$$

$$g_{SUSY}(y) = y\,\text{Ln}\left(1+\frac{1}{y}\right) \quad\longrightarrow\quad g_{SUSY}(y \gg 1) \simeq -1 \tag{2.97}$$

The case the most interesting for us is indeed the one where $y \gg 1$, as we will see in the next chapter. In that case, the limit of large y will again lead to an approximate form for ε_1^{II} very similar to the one for ε_1^{I} :

$$\varepsilon_1^{II} = (2\times)\frac{3}{16\pi}\frac{\sum_{ij} Y_{1i}Y_{1j}(m_\nu^{II})^*_{ij}}{(YY^\dagger)_{11}}\frac{M_1}{v_{(u)}^2} \tag{2.98}$$

where the factor of two is applied for the supersymmetric case and ignored in the SM. Summing the two contributions ε_1^{I} and ε_1^{II}, we obtain the total CP asymmetry :

$$\varepsilon_1 = \varepsilon_1^{I} + \varepsilon_1^{II} = (2\times)\frac{3}{16\pi}\frac{\sum_{ij} Y_{1i}Y_{1j}(m_\nu)^*_{ij}}{(YY^\dagger)_{11}}\frac{M_1}{v_{(u)}^2} \tag{2.99}$$

where once again a factor of two is to be applied in the SUSY case.

The baryon asymmetry is finally expressed as :

$$y_B = -1.48 \times 10^{-3}\eta\,\varepsilon_1 \tag{2.100}$$

$\eta \lesssim 1$ being the washout which has to be determined with the Boltzmann equations. As it has been shown [25, 26], this parameter can reach values of about 0.1 or greater on a significant part of the parameter space, so that ε_1 must be strictly greater than 10^{-7} and, more realistically, we can reasonably hope to reach the experimental value (2.79) for y_B, for ε_1 about 10^{-6}.

2.4.4 Boltzmann equations

Usually leptogenesis happens in the range of moderate washout, therefore a numerical value for y_B must be extracted by the resolution of the Boltzmann equations. An introduction to Boltzmann equations is provided in appendix A in a simple case. The processes entering the game for leptogenesis are decays and inverse decays of N_1 : $N_1 \to lH$ and $lH \to N_1$, $\Delta L = 1$ scatterings, for example : $qt^c \to N_1 l$ or $Nq \to l\,t^c$ mediated by Higgs particles, and $\Delta L = 2$ scatterings : $lH \to \bar{l}\bar{H}$ mediated by N_1, or $ll \to \bar{H}\bar{H}$ mediated by Δ_L in the type II mechanism. These processes are drawn in figure 2.2. Usually, if $M_1 \lesssim 10^{14}$ GeV, $\Delta L = 2$ processes are neglected. In the inverse regime of very large M_1, these processes tend to wash out the lepton asymmetry exponentially.

Using, as in the appendix A, $\Delta_{N_1} = Y_{N_1} - Y_{N_1}^{eq}$ and the thermally averaged rates γ_D, $\gamma_{\Delta L=1}$ and $\gamma_{\Delta L=2}$, the equation for the lepton asymmetry and the N_1 number densities are :

$$\Delta'_{N_1} = -\frac{z}{sH(M_1)}(\gamma_D + \gamma_{\Delta L=1})\frac{\Delta_{N_1}}{Y_{N_1}^{eq}} - (Y_{N_1}^{eq})' \tag{2.101}$$

$$Y'_L = \frac{z}{sH(M_1)}\left[\frac{\Delta_{N_1}}{Y_{N_1}^{eq}}\varepsilon_1\gamma_D - \frac{Y_L}{Y_L^{eq}}(\gamma_D + \gamma_{\Delta L=1} + \gamma_{\Delta L=2})\right] \tag{2.102}$$

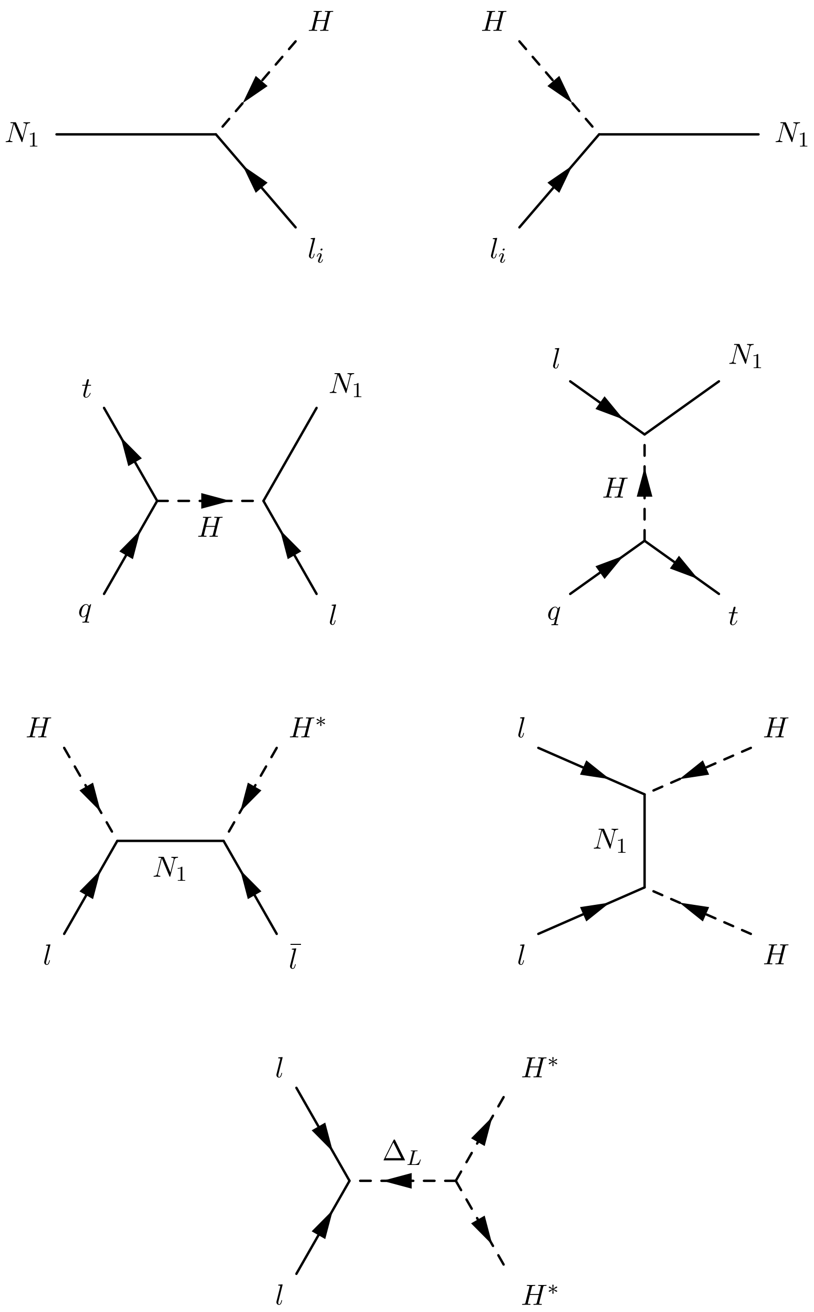


Figure 2.2: The diagrams of the different processes entering the Boltzmann equations of the lepton asymmetry. The four upper diagrams are $\Delta L = 1$ processes whereas the last three diagrams represent $\Delta L = 2$ processes.

An approximate form of these equations can be written which yields solutions in good accordance with numerical resolutions [26, 27, 56]. Using the modified Bessel functions of the second kind K_1 and K_2 we first approximate γ_D and $Y^{eq}_{N_1}$:

$$\gamma_D \simeq s Y^{eq}_{N_1} \frac{K_1(z)}{K_2(z)} \Gamma_{N_1}, \qquad Y^{eq}_{N_1} \simeq \frac{1}{4g_*} z^2 K_2(z) \tag{2.103}$$

and we make use of the parameter κ_1, characterising the out of equilibrium condition for N_1 and defined by :

$$\kappa_1 = \frac{\Gamma_{N_1}}{H(T = M_1)} = \frac{\tilde{m}_1}{m_*} \tag{2.104}$$

and the parameters $\tilde{m}_1$ and m_* are expressed as :

$$\tilde{m}_1 = \frac{(YY^\dagger)_{11} v^2}{M_1}, \qquad m_* = \frac{16\pi^{5/2}\sqrt{g_*}}{3\sqrt{5}} \frac{v^2}{M_P} \simeq 1.08 \times 10^{-3} \text{ eV} \tag{2.105}$$

Using these definitions yields for the asymmetry :

$$\Delta'_{N_1} = -z\kappa_1 \frac{K_1(z)}{K_2(z)} f_1(z) \Delta_{N_1} - (Y^{eq}_{N_1})' \tag{2.106}$$

$$Y'_L = \varepsilon_1 \kappa_1 z \frac{K_1(z)}{K_2(z)} \Delta_{N_1} - \frac{1}{2} z^3 \kappa_1 f_2(z) Y_L \tag{2.107}$$

The functions f_1 and f_2 have the following approximations [27] : $f_{1,2}(z) \simeq 1$ for $z \gg 1$ and $f_{1,2}(z) \propto m_t^2/(v_u^2 z^2)$ for $z \lesssim 1$. From this form of the Boltzmann equations an approximate solution can be found in the strong and weak washout regimes. In the strong washout regime, where $\kappa_1 \gg 1$, the lepton asymmetry will be damped and its final expression is :

$$Y_L \simeq 0.3 \times \frac{\varepsilon_1}{g_*} \left(\frac{0.55 \times 10^{-3}\text{eV}}{\tilde{m}_1} \right)^{1.16} \tag{2.108}$$

In the other case of weak washout, $\kappa_1 \ll 1$, the final asymmetry will depend on whether the N_1 density Y_{N_1} started at its equilibrium value or with a zero abundance. The first case may appear if N_1 has interactions with heavy particles, such as massive gauge bosons from a broken gauge symmetry, which will bring it to equilibrium. It can also happen that the inflaton, whose decay reheats the Universe, couples directly and mainly to N_1, in which case N_1 can even dominate the energy density of the Universe. Obviously this possibility would be quite favourable for leptogenesis since a very large population of N_1 would contribute to the lepton asymmetry. Finally, the case with N_1 starting from a zero abundance is quite realistic when the particles (other than leptons and Higgses) interacting with N_1 have masses bigger than the reheating temperature, but it is less likely to create a sufficient asymmetry than the previous ones. Indeed, in order for Y_{N_1} to reach its equilibrium value before $T < M_1$ when the particle decays, the Yukawa couplings Y_ν of N_1 have to be large enough, otherwise the N_1 population is not numerous enough to generate a reasonable CP asymmetry. However, raising the Yukawa couplings will enhance κ_1 and lead to the regions of strong washout. Therefore, in this case, there is always a tension between large and small values of κ_1.

Since a non-thermal production of N_1 through inflaton decays is quite model dependent we will forget about it in the following, and suppose that Y_{N_1} starts with an equilibrium or zero value. The approximate results can be summarised as follows. In the weak washout regime :

- For an equilibrium initial abundance, there is no washout, so $\eta \simeq 1$, and

$$Y_L \simeq 0.3 \times \frac{\varepsilon_1}{g_*} \tag{2.109}$$

so that any situation with $\tilde{m}_1 \lesssim 10^{-3}$ eV ($\kappa_1 \lesssim 1$) is favourable.

- For a zero initial abundance, Y_L is proportional to κ_1 :

$$Y_L \simeq 0.3 \times \frac{\varepsilon_1}{g_*} \left(\frac{\tilde{m}_1}{3.3 \times 10^{-3} \text{ eV}} \right) \tag{2.110}$$

In this case the optimised situation is the one for $\kappa_1 \sim 1$, where $\eta \sim 0.1$.

$\tilde{m}_1$ is equivalent to κ_1 by definition and we will use either one indifferently when discussing the washout in the analysis of the next chapter.
Finally, the lepton asymmetry thus created will be reprocessed by the sphaleron interactions, which violate L and B. Since they conserve $B-L$, we can express the $B-L$ asymmetry just after the leptogenesis era as $Y_{B-L} = -Y_L$. The asymmetry is then redistributed at low temperature when all Yukawa interactions are in equilibrium. The asymmetry is thus distributed by the Yukawas Y_e, Y_d and Y_u, and the sphalerons, which are effective $qqql$ interactions. Imposing that the plasma be electrically neutral yields a fifth condition and we list the five equations that the effective chemical potentials μ_i of the different species are bound to obey :

$$0 = \mu_e + \mu_l + \mu_H \tag{2.111}$$
$$0 = \mu_d + \mu_q + \mu_H \tag{2.112}$$
$$0 = \mu_u + \mu_q - \mu_H \tag{2.113}$$
$$0 = 3\mu_q + \mu_l \tag{2.114}$$
$$0 = N_g(\mu_q - 2\mu_u + \mu_d - \mu_l + \mu_e) - 2N_H\mu_H \tag{2.115}$$

N_g is the number of matter generations and N_H the number of Higgs doublets. Thus, it is straightforward to replace μ_q and μ_l, for example, by :

$$Y_B = N_g(2\mu_q - \mu_u - \mu_d)\,, \qquad \text{and :} \qquad Y_L = N_g(2\mu_l - \mu_e) \tag{2.116}$$

and express Y_B and Y_L as a function of Y_{B-L} :

$$Y_B = \frac{8N_g + 4N_H}{22N_g + 13n_H} Y_{B-L}\,, \qquad Y_L = -\frac{14N_g + 9N_H}{22N_g + 13N_H} Y_{B-L} \tag{2.117}$$

where n_H is the number of light Higgs doublets.

2.4.5 Gravitino constraints

In supersymmetric models where SUSY is promoted to a local symmetry, a spin 3/2 particle, called the gravitino ψ, is introduced to play the role of the graviton superpartner. When SUSY is broken it takes a mass $m_{3/2}$ which can be as small as a few eV or as large as 100 TeV depending on the models. As every other particle, it is produced after inflation when the Universe is reheated by the inflaton, and it actually puts stringent constraints on the reheating temperature T_{RH} [83, 84, 85].

Let us consider SUSY theories with conserved R-parity. Two cases are then possible :

- The gravitino is not the LSP. One of the superpartners is thus stable and the gravitini will decay into the LSP at some point. Two main constraints then arise. The first one comes from the risk of increasing the density of the LSP above the Dark Matter measure from WMAP [52]. The gravitino is produced by the thermal bath at the T_{RH} as well as from inflaton decay. The latter is again model dependent and is often neglected. ψ is thus produced mainly by gluinos $\tilde{g}$ and a numerical fit [85] gives for its number density (at the 10% level) :

$$Y_{3/2} \simeq 2 \times 10^{-12} \frac{T_{RH}}{10^{10}\ \text{GeV}} \tag{2.118}$$

This translates to the LSP since every ψ will produce one LSP eventually. The non-thermal contribution to add to the relic density of the LSP is :

$$\Delta\Omega \simeq 0.13 \times \frac{m_{LSP}}{100\,\text{GeV}} \frac{T_{RH}}{10^{10}\,\text{GeV}} \tag{2.119}$$

and we can translate the constraint : $\Delta\Omega < \Omega_{DM} \lesssim 0.31$ to T_{RH} :

$$T_{RH} \lesssim 2.4 \times 10^{10}\,\text{GeV} \frac{100\,\text{GeV}}{m_{LSP}} \tag{2.120}$$

Therefore the constraint on T_{RH} is roughly $T_{RH} \lesssim 2 \times 10^{10}$ GeV.
The other constraint that is very dangerous comes the lifetime of the gravitino. As it interacts gravitationally, its width is given by : $\Gamma_{3/2} \sim m_{3/2}^3/M_P^2$. Thus, as $m_{3/2}$ can hardly display a large hierarchy with the other superpartner masses, the lifetime of ψ is :

$$\tau_{3/2} = \Gamma_{3/2}^{-1} \sim \frac{M_P^2}{m_{3/2}^3} \sim 1\,\text{sec} \left(\frac{100\,\text{TeV}}{m_{3/2}} \right)^3 \tag{2.121}$$

and for $m_{3/2} \lesssim 100$ TeV ψ risks to disturb the predictions of BBN. The results are of course dependent of the spectrum of the superpartners, but the numerical study of [85] for several spectra suggests that a mass $m_{3/2} \gtrsim 10$ TeV should be taken if we need a reheating temperature $T_{RH} \sim 10^9$ GeV.

- The gravitino is the LSP. The previous constraints are still to be taken into account in a slightly different form. In this case, it is the density of the gravitino itself that one has to consider, and impose that it does not overclose the Universe [84] :

$$\begin{aligned} \Omega_{3/2} &= \frac{\rho_{3/2}}{\rho_c} = m_{3/2} Y_{3/2}(T) n_{\text{rad}}(T) \rho_c^{-1} \\ &\simeq 0.50 \left(\frac{T_{RH}}{10^{10}\,\text{GeV}} \right) \left(\frac{100\,\text{GeV}}{m_{3/2}} \right) \left(\frac{m_{\tilde{g}}}{1\,\text{TeV}} \right)^2 \end{aligned} \tag{2.122}$$

with ρ_c the critical energy density. Imposing $\Omega_{3/2} < 1$ yields constraints on $m_{3/2}$, for example for $m_{3/2} = 50$ GeV we need $T_{RH} \lesssim 3 \times 10^{10}$ GeV.
As for the BBN constraint, it is now translated to the NLSP (Next to Lightest SUSY Particle), since it couples gravitationally to the LSP ψ and has therefore a quite long lifetime [84] : $\tau = 48\pi m_{3/2}^2 M_P^2 / m_{NLSP}$. The constraints are still model dependent, but it is found generically

that T_{RH} should be less than a few 10^9 GeV.

Successful leptogenesis itself imposes a lower bound on T_{RH} since M_1 must be large enough and N_1's must be sufficiently produced by the thermal plasma. The latest bounds in the framework of flavour leptogenesis are roughly $T_{RH} \gtrsim 2 \times 10^9$ GeV which explains the well known tension existing in supersymmetric leptogenesis scenarii.

On the other hand, absolute bounds have been derived for ε_1 in the type I seesaw model [23], in the limit of infinitely hierarchical RH neutrinos :

$$|\varepsilon_1^I| \leq \varepsilon_1^{max} = \frac{3}{8\pi}\frac{M_1(m_{max} - m_{min})}{v^2} \simeq 2 \times 10^{-7}\left(\frac{M_1}{10^9\text{GeV}}\right)\left(\frac{m_{max} - m_{min}}{0.05 \text{ eV}}\right) \tag{2.123}$$

with m_{max} the maximum eigenvalue of the light neutrino mass matrix m_ν and m_{min} the smallest one. For a hierarchical spectrum, this formula imposes a constraint on M_1, for example $M_1 \geq 2.4 \times 10^9$ GeV if we want to have any chance of solving the problem of the baryon asymmetry. This creates a tension between the gravitino constraint and the need for a successful supersymmetric leptogenesis scenario, as N_1 must be created by the thermal plasma, imposing $T_{RH} \gtrsim M_1$, but there is still some room available in the parameter space. Let us stress again, however, that this is true strictly with infinitely hierarchical RH neutrino masses. If they are only mildly hierarchical, for example $M_2/M_1 \sim M_3/M_2 \sim 100$, these constraints can be somehow relaxed [24].

For completeness we mention that the bound on ε_1 in the type I+II has the form [50, 51] :

$$|\varepsilon_1| \leq \varepsilon_1^{max} = \frac{3}{8\pi}\frac{M_1 m_{max}}{v^2} \simeq 2 \times 10^{-7}\left(\frac{M_1}{10^9\text{GeV}}\right)\left(\frac{m_{max}}{0.05 \text{ eV}}\right) \tag{2.124}$$

which is similar to the bound (2.123), with the change $m_{max} - m_{min} \to m_{max}$. This is an important change since ε_1 does not decrease anymore when the neutrino spectrum becomes quasi-degenerate. Thus, the possibility of successful leptogenesis with quasi-degenerate neutrinos is still viable in this framework.

Chapter 3

Left-Right Symmetric Seesaw and Leptogenesis

Now that we have introduced enough notions of GUTs and leptogenesis, we can go deeper into the study of the seesaw formula. A study of the seesaw formula is of great phenomenological interest since the presence of massive RH neutrinos can lead to different kinds of signatures that are forbidden or suppressed in the SM. In the previous chapter we developed the example of leptogenesis, which is a true signal unexplained in the SM and is therefore extremely interesting for BSM physics, but the fact that RH neutrinos violate lepton flavours leads to flavour violating processes in the lepton sector. Moreover they can participate in Electric Dipole Moments that are very suppressed in the SM. Thus, the knowledge of their mass matrix M_R is really a valuable information.
The formula for type I seesaw as given by equation (2.8), however, does not allow one to extract the RH neutrino mass matrix, even supposing that we know completely the light neutrino mass matrix. A convenient parametrisation was introduced by Casas and Ibarra [20]. Going to the basis where the RH neutrino mass matrix is diagonal $M = \hat{M}$, they show that the matrix $R = \sqrt{\hat{M}^{-1}} Y_\nu U \sqrt{\hat{m}_\nu^{-1}}$ is complex orthogonal and contains all the high-energy parameters, unmeasurable in low energy neutrino oscillations. However if GUT relations allow us to determine Y_ν at M_{GUT} it will be straightforward to extract M.
From the type I formula, it is clear that the structure of Y_ν has a primordial importance on the spectrum of light neutrinos, since it appears "squared". Hence, should Y_ν be as hierarchical as Y_u, as it often comes out in $SO(10)$, the hierarchy in M should be twice that of Y_u in order to keep a light neutrino spectrum with a mild hierarchy [1]. Since the hierarchy in Y_u is roughly $\lambda^8, \lambda^4, 1$, the one in M will be so large that the lightest eigenstate N_1 will have a mass roughly equal to 10^5 GeV. Such a small mass (compared to M_{GUT}) is not in principle a problem since N_1 will not affect the running of the gauge couplings between its mass and the GUT scale. However, as was described in the last chapter, such a low scale of M_1 is quite problematic if we want to rely on the mechanism of leptogenesis for the generation of the baryon asymmetry of the universe. Indeed, due to the huge hierarchy in the RH neutrino sector, the Davidson-Ibarra bound is here very well verified, so that M_1 is three orders of magnitude too small to generate the needed amount of CP-asymmetry. This means that generally, in $SO(10)$ models with type I only, leptogenesis will be quite difficult, except for particular situations of quasi-degeneracy between N_1 and N_2, in which we can rely on resonant leptogenesis [55, 30].

Although not as extensively studied in the literature as the type I, the pure type II seesaw has also

[1] Let us recall that in the case of normal hierarchy, there is a maximum factor of five between m_2 and m_3.

received some attention. It proves quite convenient since, judging from formula (2.10), we can extract immediatly the high energy coupling f_{ij} from the low energy data. However in $SO(10)$, RH neutrinos are always present and in order to consider pure type II we have to neglect the type I contribution. More generally in GUT models we can have both type I and type II at the same time, with no particular reason to neglect one with respect to the other. The situation with both seesaws contributing equally to the neutrino mass matrix had not received much attention until recently [35].

In this part we propose to investigate models with a Left-Right symmetry where type I and II are both present and comparable in magnitude. We will introduce a method to extract the RH neutrino mass matrix in order to study its phenomenological consequences, mainly Lepton Flavour Violation and leptogenesis. The benchmark model we will assume is an $SO(10)$ model and we will first make simplifying assumptions to study the qualitative features of the results before adding the necessary corrections that lead to precise quantitative conclusions.

3.1 Left-Right symmetric seesaw mechanism

3.1.1 Seesaw Duality

Our aim is thus to study the seesaw formula :

$$m_\nu = v_L f_L - \frac{v_u^2}{v_R} Y_\nu^T f_R^{-1} Y_\nu \tag{3.1}$$

v_L and v_R are vevs of $SU(2)_L$ and $SU(2)_R$ triplets Δ_L and Δ_R, and v_u is a vev of an $SU(2)_L \times SU(2)_R$ bidoublet Φ. If f_L and f_R are not related there is no better chance to extract $M = v_R f_R$. However if a Left-Right symmetry between the two $SU(2)$ sectors is imposed, the Lagrangian is invariant under the interchange :

$$l \leftrightarrow \begin{pmatrix} N^c \\ e^c \end{pmatrix} \qquad \Phi \leftrightarrow \Phi^T \qquad \Delta_L \leftrightarrow \Delta_R \tag{3.2}$$

Therefore, the equality $f_L = f_R = f$ will hold. Moreover, the Yukawa couplings are restricted to a symmetric form, $Y_i^T = Y_i$ for $i = u, d, e, \nu$. This restricts the seesaw equation to :

$$m_\nu = v_L f - \frac{v_u^2}{v_R} Y_\nu f^{-1} Y_\nu \tag{3.3}$$

which is somewhat more tractable : indeed, making the necessary assumptions about the unknown parameters of m_ν and an explicit form for Y_ν, only f is left as an unknown matrix variable and the equation now reduces to a matricial second degree equation. This equation has been studied first by Akhmedov and Frigerio in [35] and then in greater detail in [36]. They have found the following interesting property of equ. (3.3). If f is a solution of the equation for given m_ν and Y_ν, then :

$$\hat{f} = \frac{m_\nu}{v_L} - f \tag{3.4}$$

is also a solution. They called the transformation $f \to \hat{f}$ a Seesaw Duality. At this point, solving for f from the seesaw formula amounts to solving the equation :

$$xF(f_{ij} - m_{ij}) = y_i y_j m_{ij} \tag{3.5}$$

where $F = \det f$, $m = m_\nu/v_L$, $x = v_L v_R/v^2$ and we switched to the basis where Y_ν is diagonal with eigenvalues y_i. This is clearly a system of six coupled quartic equations for the f_{ij}. However the authors of ref. [35] found a way to linearise it with respect to the components of f.
They first note that :

$$y_i y_j F_{ij} = -x\hat{F} f_{ij} + y_i y_j (T_{ij} - M_{ij}) \quad \text{with} \quad \begin{cases} F_{ij} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} f_{km} f_{ln} \\ M_{ij} = \frac{1}{2}\epsilon_{ikl}\epsilon_{jmn} m_{km} m_{ln} \\ T_{ij} = \epsilon_{ikl}\epsilon_{jmn} f_{km} m_{ln} \end{cases} \tag{3.6}$$

Moreover taking the determinant of the duality equation (3.4) $x\hat{f} = -Y_\nu f^{-1} Y_\nu$, one obtains :

$$x^3 F\hat{F} = -\det{}^2(Y_\nu) = -y_1^2 y_2^2 y_3^2 \tag{3.7}$$

The trick is finally to perform the rescaling :

$$f = \lambda^{1/3} f' \qquad m = \lambda^{1/3} m' \qquad y = \lambda^{1/3} y' \tag{3.8}$$

and fix the value of λ by $F' = 1$, which yields $\hat{F}' = -(y'_1 y'_2 y'_3)^2/x^3$. The system obtained for f' :

$$(x^3 - (y'_1 y'_2 y'_3)^2) f'_{ij} - x^3 m'_{ij} = x^2 y'_i y'_j (T'_{ij} - M'_{ij}) \tag{3.9}$$

is now linear. All the non-linearity is contained in λ which is the solution of an 8^{th} order equation, yielding generically eight solutions.

The conclusion is that we have now eight potentially different solutions for the heavy neutrino mass matrix $M = v_R f$, where only one is possible for the pure type I or type II seesaw.

3.1.2 Reconstruction procedure

The duality relation (3.4) links together the eight solutions for f into four couples. However we can generalise it and see in a straightforward manner how all the solutions are linked together. This is the basis of the work of ref. [37] and in this section we develop a method to solve for the matrix f, alternative to the one presented above.

We are going to manipulate only equation (3.3), in which all matrices are symmetric. It allows us to decompose Y_ν as $Y_\nu = Y_\nu^{1/2}(Y_\nu^{1/2})^T$. We see clearly that $Y_\nu^{1/2}$ is defined only for a symmetric Y_ν and can be defined as $Y_\nu^{1/2} = U_\nu^T \hat{Y}_\nu$ where U_ν is the unitary matrix that diagonalises Y_ν. Of course we see that $Y_\nu^{1/2}$ is not unambiguously defined, since multiplying $Y_\nu^{1/2}$ by an orthogonal matrix on the right yields the same Y_ν. This is however not physically relevant, as these manipulations will only be a mathematical way to extract the matrix f. Defining then :

$$Z = Y^{-1/2} m_\nu (Y_\nu^{-1/2})^T \qquad X = Y_\nu^{-1/2} f (Y_\nu^{-1/2})^T \tag{3.10}$$

we can write eq. (3.3) in the simple form :

$$Z = \alpha X - \beta X^{-1} \tag{3.11}$$

using the notation $\alpha = v_L$ and $\beta = v^2/v_R$. We assume that we are in the basis where charged leptons are mass eigenstates and that Y_ν is known in this basis, as is possible in $SO(10)$ GUTs. Z is given by the low energy parameters, once we assume a definite value for the yet unknown low energy neutrino

parameters. Solving for f then amounts to solve for X.
Now we can use quite effectively the properties of Z and X. As they are symmetric, it is possible, in the general case, to diagonalise them with a complex orthogonal transformation :

$$Z = O_Z^T \mathrm{Diag}(z_1, z_2, z_3) O_Z \qquad O_Z^T O_Z = \mathrm{Id}_3 \tag{3.12}$$

At this point one has to be careful, since a complex orthogonal transformation is not necessarily well defined. In some cases, at least one eigenvector can be non-normalisable since the orthogonal norm $\vec{u}^2$ is not a norm for complex vectors $\vec{u}$. This means that a non-zero eigenvector of Z can be such that $\vec{u}.\vec{u} = 0$ and we cannot define the matrix O_Z. However all eigenvectors are normalisable when the eigenvalues are note degenerate which is the relevant case for us.

Let us note here that if we use unitary transformations to diagonalise these matrices : $Z = U_Z^T \hat{Z} U_Z$, $X = U_X^T \hat{X} U_X$ and $X^{-1} = U_X^\dagger \hat{X}^{-1} U_X^*$, X and X^{-1} cannot be diagonalised at the same time with real positive eigenvalues. Only using orthogonal matrices can we manage to factorise the diagonalising matrices of X and X^{-1} in the right-hand side.

Moreover, from eq. (3.11), we see that X is diagonal when Z is. Thus X is also diagonalised by O_Z :

$$X = O_Z^T \mathrm{Diag}(x_1, x_2, x_3) O_Z \tag{3.13}$$

Multiplying by O_Z each side of eq. (3.11), the equation transposes to the eigenvalues :

$$z_i = \alpha x_i - \beta x_i^{-1} \tag{3.14}$$

From this equation we see that we have two solutions for each of the x_i's :

$$x_i = \frac{z_i \pm \mathrm{sign}(\mathrm{Re}(z_i))\sqrt{z_i^2 + 4\alpha\beta}}{2\alpha} \tag{3.15}$$

We defined x_i with the factor sign(Re(z_i)) so that we have a good control on the limits $\alpha\beta \gg z_i^2$ and $\alpha\beta \ll z_i^2$. In the case of three generations, we have two possibilities for each eigenvalue which means $2^3 = 8$ solutions for X.
Now that we know X we can extract f :

$$f = \mathrm{Y}_\nu^{1/2} X (\mathrm{Y}_\nu^{1/2})^T \tag{3.16}$$

and the right-handed neutrino masses and mixings can be extracted by physically diagonalising f :

$$f = U_f \hat{f} U_f^T \qquad U_f^\dagger U_f = \mathrm{I}_3 \tag{3.17}$$

The unitary matrix U_f then relates the basis where Y_ν is symmetric and the one where M is diagonal. In the latter basis, where the RH neutrinos are mass eigenstates, the neutrino Yukawa couplings become $Y = U_f^\dagger Y_\nu$.

This procedure of reconstruction is quite adapted to a numerical analysis. Furthermore it is also fit for analytic studies. Let us start by studying the limits of eq. (3.15), which will allow us to identify some known limits.
The different solutions are labeled by $(+,+,+)$ for (x_1^+, x_2^+, x_3^+) and so on. In the limit $4\alpha\beta \ll |z_i|^2$:

$$x_i^+ \simeq \frac{z_i}{\alpha} \qquad x_i^- \simeq -\frac{\beta}{z_i} \tag{3.18}$$

for which we see the use of definition (3.15). In the opposite limit $|z_i|^2 \ll 4\alpha\beta$:

$$x_i^\pm \simeq \pm \text{sign}(\text{Re}(z_i))\sqrt{\frac{\beta}{\alpha}} \tag{3.19}$$

Now, for some solutions, it is possible to interpret the result as a particular dominance of type I or type II seesaw. More precisely, defining the z_i's such that $|z_1| < |z_2| < |z_3|$, in the range of vevs such that $4\alpha\beta \ll |z_1|^2$, the limit (3.18) holds for all z_i's and the $(+,+,+)$ and $(-,-,-)$ solutions give approximately $X^{(+,+,+)} \simeq Z/\alpha$ and $X^{(-,-,-)} \simeq -\beta Z^{-1}$. In this limit, f is then given by :

$$f^{(+,+,+)} \simeq \frac{m_\nu}{v_L} \tag{3.20}$$

$$f^{(-,-,-)} \simeq -\frac{v^2}{v_R} Y_\nu m_\nu^{-1} Y_\nu \tag{3.21}$$

thus in this limit $f^{(+,+,+)}$ corresponds to the pure type I and $f^{(-,-,-)}$ to the pure type II. This will allow us to test our results. For the other solutions the situation is not so clear and both types of seesaw contribute to some or all of the entries of m_ν.
The other interesting limit is the one where $|z_3|^2 \ll 4\alpha\beta$. For the two solutions $\pm(\epsilon_1, \epsilon_2, \epsilon_3)$, where $\epsilon_i = \text{sign}(\text{Re}(z_i))$, we get $X \simeq \pm\sqrt{\frac{\beta}{\alpha}}\,\text{Id}$ and :

$$f^{\pm(\epsilon_1,\epsilon_2,\epsilon_3)} \simeq \sqrt{\frac{\beta}{\alpha}} Y_\nu \tag{3.22}$$

This is expected since when α and/or β becomes much larger than the neutrino mass scale, there must be some kind of cancellation between the two contributions. This is realised only when f aligns on Y_ν. More generally, for any solution and provided the hierarchy in Y_ν is large enough, we still have $f_i \simeq \sqrt{\frac{\beta}{\alpha}}$ in the large $\alpha\beta$ regime.

The duality of ref. [35] is now generalised to the eigenvalues of the matrix X. It takes the form :

$$x_i \to \tilde{x}_i = \frac{z_i}{\alpha} - x_i \tag{3.23}$$

and exchanges x_i^+ and x_i^-. This defines three Z_2 transformations which allow us to link any of the eight solutions, while the seesaw duality $f \to \hat{f} = m_\nu/v_L - f$ exchanges all three eigenvalues at the same time, therefore linking the eight solutions into four pairs.
In ref. [36], the authors make a more extensive study of the solutions for cases different from those here analysed. For example, the Left-Right symmetry in $SO(10)$ implies the equality $f_L = f_R$, but if we consider simply LR theories with a gauge group containing $SU(2)_L \times SU(2)_R$ it is possible to define another kind of LR parity. Denoting again the bidoublet of $SU(2)_L \times SU(2)_R$ containing the Higgs fields as Φ and the $SU(2)_{L/R}$ triplets as $\Delta_{L/R}$, this parity is characterised by an invariance under the transformation :

$$l \leftrightarrow \begin{pmatrix} \bar{N}^c \\ \bar{e}^c \end{pmatrix} \qquad \Phi \leftrightarrow \bar{\Phi} \qquad \Delta_L \leftrightarrow \bar{\Delta}_R \tag{3.24}$$

With this definition, the couplings are related by $f_L = f_R^*$ so that the seesaw formula is now :

$$m_\nu = v_L f - \frac{v^2}{v_R} Y_\nu^T (f^*)^{-1} Y_\nu \tag{3.25}$$

In this case, a "duality" relation can be defined in the same way as in eq. (3.4) in the cases where Y_ν is hermitian or anti-hermitian. The number of solutions is also found to be eight except in the latter case where Y_ν can be such that there is no solution.
The other case studied is the one for which Y_ν is purely anti-symmetric, which can happen in certain Left-Right theories such as $SO(10)$. In this case the three generation case reduces to a two generation case where a simple analysis shows the existence of two solutions.
In the more general case with a version of the Left-Right symmetry not strong enough to constrain the symmetry properties of Y_ν, it is not possible to give any general result on the number of solutions for f. Such a situation could be realised in $SO(10)$ models, for example, where there are both symmetric and anti-symmetric contributions to the Yukawa couplings, coming from the couplings of the **16**'s of matter to a **10** or $\mathbf{1\overline{2}6}$ on the one hand and to a **120** on the other hand.

3.1.3 Spectrum and mixings in a simple $SO(10)$ model

Now that we have introduced a method to deal with the Left-Right symmetric seesaw mechanism, we are going to develop the example of $SO(10)$ theories and investigate the phenomenological consequences of the extracted RH neutrino spectrum. Of course we will restrict to the class of models where a type I+II seesaw is at work, which is expected under some specific conditions on the Higgs sector.
First of all we will put the two Higgs superfields in two ten's $\mathbf{10_u}$ and $\mathbf{10_d}$. Then we have to couple the leptons in the **16**'s to triplets of $SU(2)_L$ and $SU(2)_R$ so we add a $\overline{\mathbf{126}}$[2]. The last condition is to induce a vev $v_L \propto v^2/v_R$ which is accomplished through the addition of a **54** since **54** contains a bitriplet under $SU(2)_L \times SU(2)_R$. The superpotential contains thus :

$$W \supset (\mathbf{Y_u})_{\mathbf{ij}} \mathbf{16_i 16_j 10_u} + (\mathbf{Y_d})_{\mathbf{ij}} \mathbf{16_i 16_j 10_d} + \mathbf{f_{ij} 16_i 16_j} \overline{\mathbf{126}} + \lambda\, \mathbf{54}\, \overline{\mathbf{126}}^{\mathbf{2}} + \kappa\, \mathbf{10_u 10_u 54} \tag{3.26}$$

plus the terms that are needed to break $SO(10)$ and those involving the **126** (it is almost sure that a **45** will be needed in order to break $SO(10)$ correctly for example). In the following we will assume that $SO(10)$ is broken as needed and we will be only interested in the matter sector. Therefore the $B-L$ breaking scale v_R will be chosen as a free parameter, in the range 10^{12} and 10^{17} GeV when possible, since it has been that $SU(2)_R$ can be broken a few orders of magnitude below M_{GUT} without disturbing gauge coupling unification too much [43].
The matrix f will be reconstructed at the seesaw scale, when the RH neutrinos begin to decouple. In the following analysis we will neglect the fact that Y_ν is taken at the GUT scale while f_L and f_R should run appropriately from M_{GUT} to the masses M_{Δ_L} and M_i. We assume everything to be taken at a common "seesaw scale". Using GUT boundary conditions, we use the fact that $M_e = M_d$ to rotate the $\mathbf{16_i}$'s in the basis where the charged leptons and down quarks are diagonal. In this basis we use the other relation $Y_\nu = Y_u$ to write :

$$Y_\nu = U_q^T \mathrm{Diag}(y_u, y_c, y_t) U_q \qquad \text{with : } \quad U_q = P_u V_{CKM} P_d \tag{3.27}$$

$$m_\nu = U_l^* \mathrm{Diag}(m_1, m_2, m_3) U_l^\dagger \qquad \text{with : } \quad U_l = P_e U_{PMNS} P_\nu \tag{3.28}$$

[2]We remind that the conservation of D-flatness imposes to add a **126**.

$m_{1,2,3}$ are light neutrino masses. P_u, P_d and P_e are high energy diagonal matrix phases which can be reabsorbed at low energy but must be taken into account for high energy processes, when quarks and leptons can not be rephased independently. P_ν contains the usual Majorana phases of the neutrino mass matrix. These four matrices contain 3 phases each, from which two global phases can be removed. In total it amounts to 10 independent phases, which can play a role for leptogenesis, for example.
Let us briefly comment on the scale of the vev v_L. Under $SU(2)_L \times SU(2)_R \times U(1)_{B-L}$ the $\overline{\mathbf{126}}$ contains the two triplets $\Delta_L = (\mathbf{3}, \mathbf{1})_{+\mathbf{2}}$ and $\Delta_R = (\mathbf{1}, \mathbf{3})_{-\mathbf{2}}$ while $\mathbf{54}$ contains the bi-triplet $\tilde{\Delta} = (\mathbf{3}, \mathbf{3})_{\mathbf{0}}$ and the $\mathbf{10}$'s contain the bi-doublets $\Phi = (\mathbf{2}, \mathbf{2})_{\mathbf{0}}$. When decomposed under this symmetry the superpotential (3.26) contains for these fields :

$$W \supset \frac{1}{2} f_{ij} l_i l_j \Delta_L + \kappa\, \Phi_u \Phi_u \tilde{\Delta} + \lambda\, \Delta_L \tilde{\Delta} \Delta_R \tag{3.29}$$

so that minimisation under $\tilde{\Delta}$ will transmit the vev of φ_u to Δ_L : $\langle \Delta_L^0 \rangle = v_L \simeq \kappa\lambda v_u^2 v_R / M_{\Delta_L}^2$. As a consequence, if $M_{\Delta_L} \gg v_R$ the type II contribution will be highly suppressed with respect to the type I. For $v_R \ll M_{GUT}$ this a non-trivial constraint since it is usually much easier to give a GUT scale mass to the Higgs multiplets, therefore it may require some tuning in the potential of any specific model. Consequently the ratio β/α can be bigger or smaller than one but unless otherwise specified we will take it to be 1.

The last thing to do is to fix the low energy parameters. In order to be predictive we must assume some specific values for those parameters of m_ν that are still unknown. As far as mass differences and mixing angles are concerned, we rely on the best fit values of [44]. The main characteristic of this fit is the value of θ_{13}, $\sin^2\theta_{13} \simeq 0.009$, not so far from the experimental bound. Unless explicitly stated we consider a normal hierarchy. The renormalisation is taken into account by normalising the neutrinos with a factor 1.2 and the up-quarks with a factor 0.65. The CP phases are left as free parameters.

Now we can use our method and present the spectra as functions of v_R. They are displayed in fig. 3.1. In these plots, we have already taken into account two constraints. First of all we have cut the regions where the biggest eigenvalue of f, f_3, becomes larger than 1. In $SO(10)$ models with representations as large the $\overline{\mathbf{126}}$, the theory becomes strongly coupled before the Planck scale, typically one order of magnitude higher than M_{GUT} : $\Lambda \sim 10 M_{GUT} \sim 2 \times 10^{17}$ GeV. If we want to keep f_3 under perturbative control up to Λ, then $f_3 < 1$ is a wise choice. The immediate consequence of this is a cut of the upper range of v_R for most solutions, with the notable exception of $(-,-,-)$. For the other solutions we are roughly limited to $v_R < 3 \times 10^{14}$ to 3×10^{15} GeV. However this is dependent on the value of $h = \beta/\alpha$.
The second constraint aims at avoiding any tuning in the seesaw formula. It is indeed possible that the type I and type II are much bigger than m_ν, at least in some entries of the mass matrix, while their difference is in the good range. In practice we will measure the fine-tuning in the (33) entry of m_ν and the type I contribution. We want to avoid a 10% fine-tuning, so we want to highlight the region of the parameter space where the (33) entry of the two contributions are larger than ten times the (33) entry of m_ν and cancel each other sufficiently. The fine-tuned region is therefore the one for which :

$$(m_\nu^{II})_{33} = v_L f_{33} > 10 (m_\nu)_{33} \tag{3.30}$$

and the factor of 10 signals a 10% fine-tuning. We will consider that this region is not favoured but we will present later a reason why we still consider it as relevant anyway.

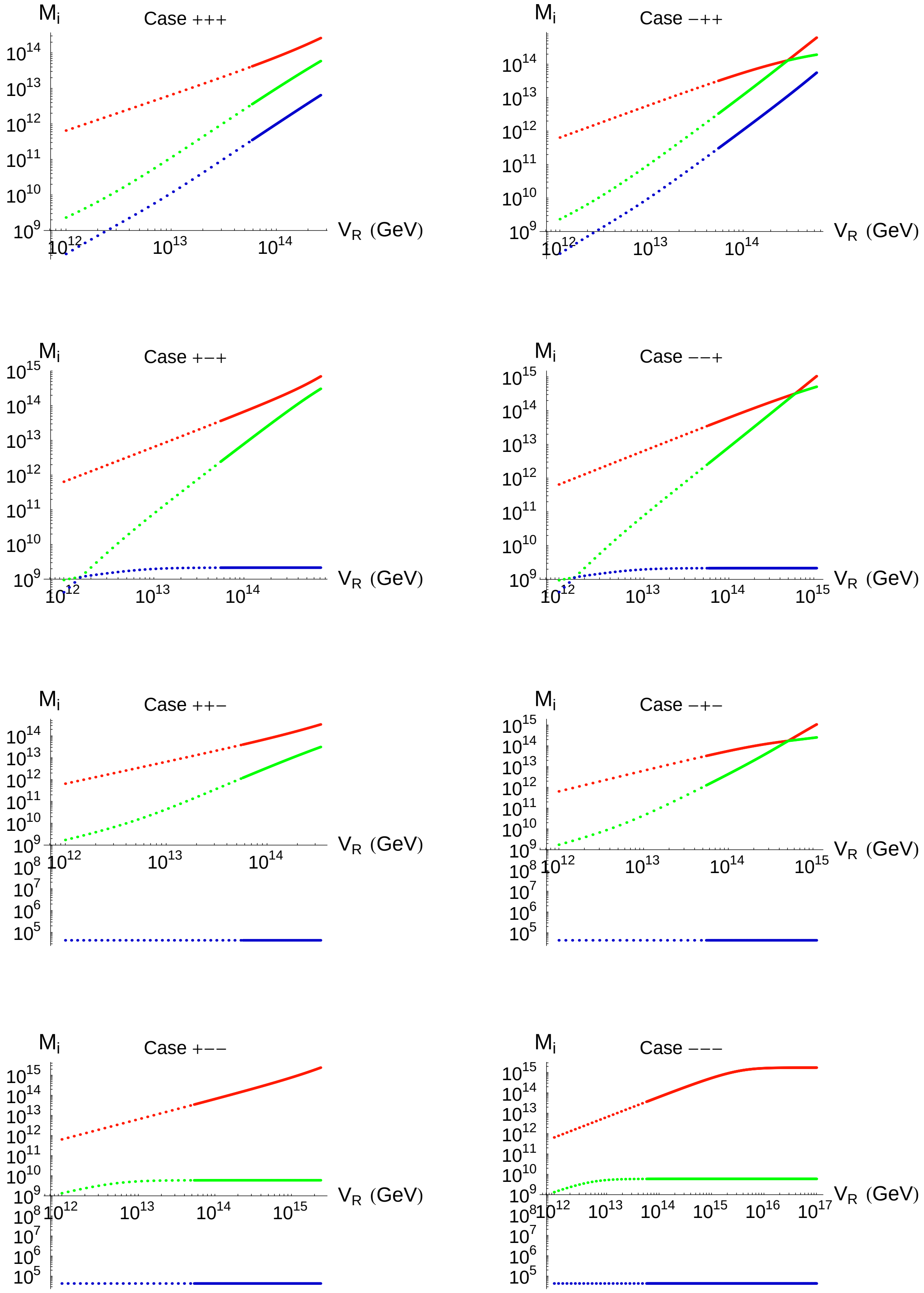


Figure 3.1: Right-handed neutrino masses as a function of v_R for each of the 8 solutions $(+,+,+)$ to $(-,-,-)$ in the reference case of a hierarchical light neutrino mass spectrum with $m_1 = 10^{-3}$ eV, $\beta = \alpha$ and no CP violation beyond the CKM phase ($\delta = \varphi_i^u = \varphi_i^d = \varphi_i^\nu = \varphi_i^e = 0$). The range of variation of v_R is restricted by the requirement that $f_3 \leq 1$. Dotted lines indicate a fine-tuning greater than 10% in the $(3,3)$ entry of the light neutrino mass matrix.

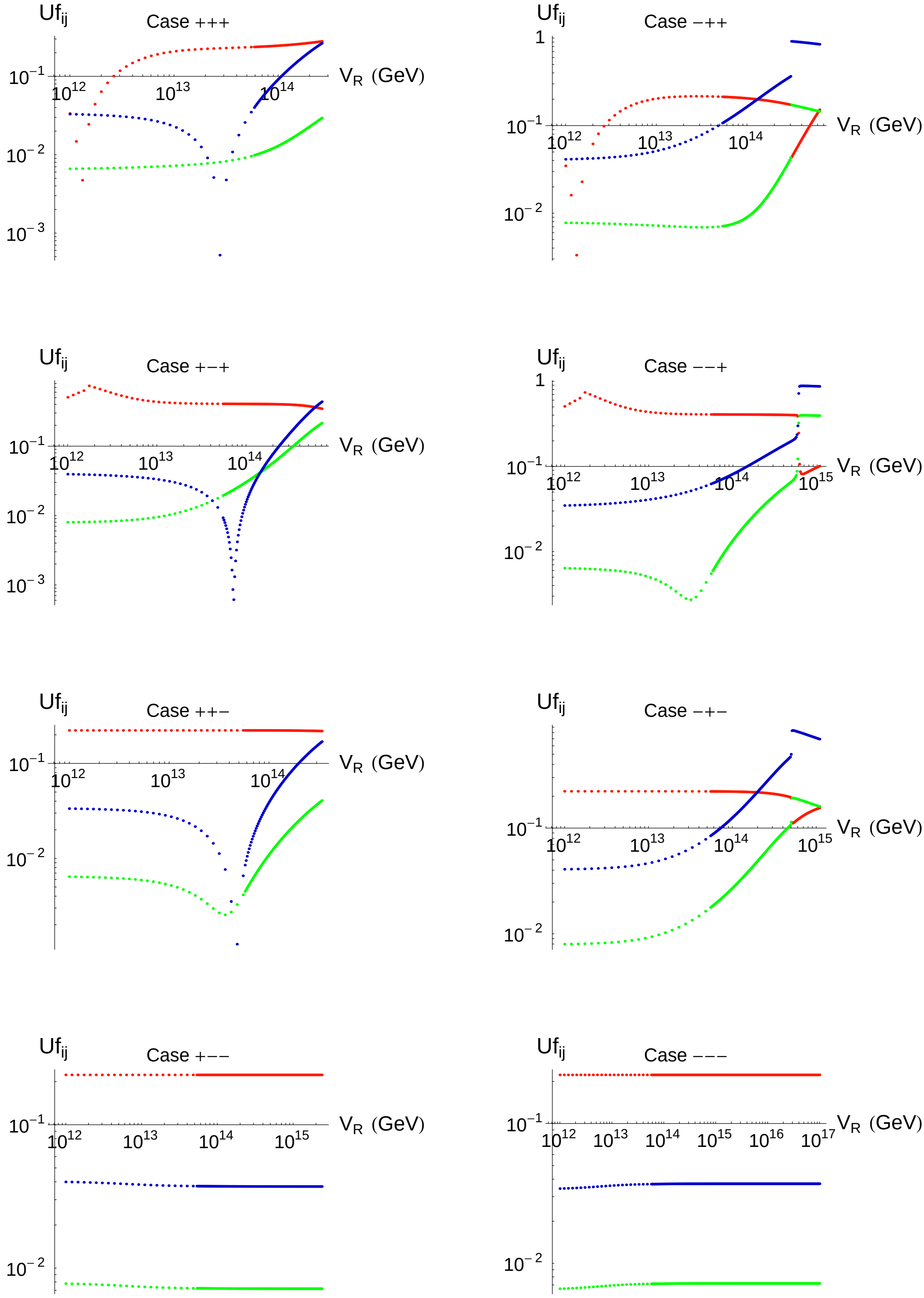


Figure 3.2: Right-handed neutrino mixing angles as a function of v_R for each of the 8 solutions $(+,+,+)$ to $(-,-,-)$ in the reference case of a hierarchical light neutrino mass spectrum with $m_1 = 10^{-3}$ eV, $\beta = \alpha$ and no CP violation beyond the CKM phase ($\delta = \varphi_i^u = \varphi_i^d = \varphi_i^\nu = \varphi_i^e = 0$). The red [dark grey] curve corresponds to $|(U_f)_{12}|$, the green [light grey] curve to $|(U_f)_{13}|$, and the blue [black] curve to $|(U_f)_{23}|$.

Analysing the figures, we immediatly see that a type I-like spectrum appears for $(-,-,-)$ in the large v_R region, with a RH spectrum $M_1 : M_2 : M_3 \propto m_u^2 : m_c^2 : m_t^2$, in order to compensate for the hierarchy in the Yukawa coupling. Examining $(+,+,+)$ we see also that N_i end up with a mild hierarchy $M_1 : M_2 : M_3 \propto m_1 : m_2 : m_3$ for large v_R, compatible with a type II dominance for which $f \propto m_\nu$. This confirms our prediction (3.21).
The appearance of a plateau for the masses associated with an $x_i = x_i^-$ is understood through the approximate form of x_i^- at large v_R, $x_i^- \simeq \beta/z_i$, which implies that $v_R x_i^- \simeq v_u^2/z_i$ is a constant. Whether we reach this plateau for M_1 and its final value separate the spectra in three different classes[3]. Solutions $(+,+,+)$ and $(-,+,+)$ are characterised by a constant rising of M_1, which attains easily values greater than 10^{10} GeV, especially in the region with no fine-tuning. The second class is formed by $(+,-,+)$ and $(-,-,+)$, with an M_1 approximately constant with a value between 10^9 and 10^{10} GeV and an $N_1 - N_2$ degeneracy in the small v_R region. Finally, the remaining four solutions, $(+,+,-)$, $(+,-,-)$, $(-,+,-)$ and $(-,-,-)$ exhibit a constant and very small $M_1 \sim 10^5$GeV.

Turning to the mixing angles of M we can comment on fig. 3.2, where we plot the absolute value of $(U_f)_{12}$, $(U_f)_{13}$ and $(U_f)_{23}$. We recognise immediatly, there again, the dominance of type I in $(+,+,+)$ since the mixings tend to the PMNS values, as well as the type I dominance in $(-,-,-)$ with CKM-like mixings. In the small v_R region an alignment on the CKM matrix is clearly visible for these two solutions, which is expected, but it holds also for the other cases. The discontinuities that can be noticed are due to the level crossings. The angles are generically quite large, at least for v_R large enough, except for $(+,-,-)$ and $(+,+,+)$ where they are quasi-constant and CKM-like.
More can be understood analytically for the behaviour of the masses and mixings and we refer the interested reader to section 3.2 and appendix B of [37].

As for the influence of $h = \beta/\alpha \neq 1$, we found that changing its value will globally shift the curves along the v_R axis and change the value of v_R at which f_3 becomes non-perturbative. Therefore, varying h does not change the form of the results and we can fix it once and for all, for example to $h = 1$.

Up to now all plots are obtained with all CP-violating phases put to zero, except for the CKM one. Varying the phases will not change drastically the form of the curves but only introduces small deformations. This is only interesting for the level crossings since it can lift the degeneracy (thereby smoothing the mixing angles). The different phases are labeled in this way :

$$P_i = \begin{pmatrix} e^{i\varphi_1^i} & 0 & 0 \\ 0 & e^{i\varphi_2^i} & 0 \\ 0 & 0 & e^{i\varphi_3^i} \end{pmatrix} \tag{3.31}$$

with the index $i = e, u, d, \nu$. In fig. 3.3, we display the splitting between M_1 and M_2 for the $(+,-,+)$ with maximal phases φ_2^u or $\varphi_2^d = \frac{\pi}{4}$. The importance of the complex phases will appear later when considering leptogenesis.

Let us come back to low energy parameter that serve as inputs for our reconstruction procedure. In pure type I, it has been shown that the RH neutrino mass matrix can be very sensitive to these parameters [30], for example abrupt level crossings can be obtained. An exhaustive study of their influence could be interesting but is not the aim of our primary work. Still, we show the impact of

[3]This is actually the most relevant criterium for leptogenesis.

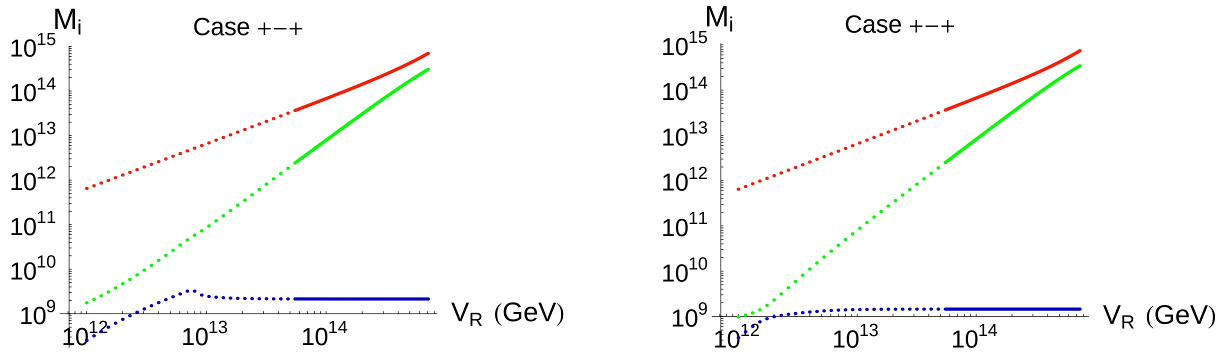


Figure 3.3: Effect of high-energy phases on the right-handed neutrino masses. The input parameters are the same as in Fig. 3.1, except $\varphi_2^u = \pi/4$ (left panel), $\varphi_1^d = \pi/4$ (right panel).

the hierarchy by plotting the solutions $(+,+,+)$ and $(-,+,+)$ in the case of inverted hierarchy 3.4.

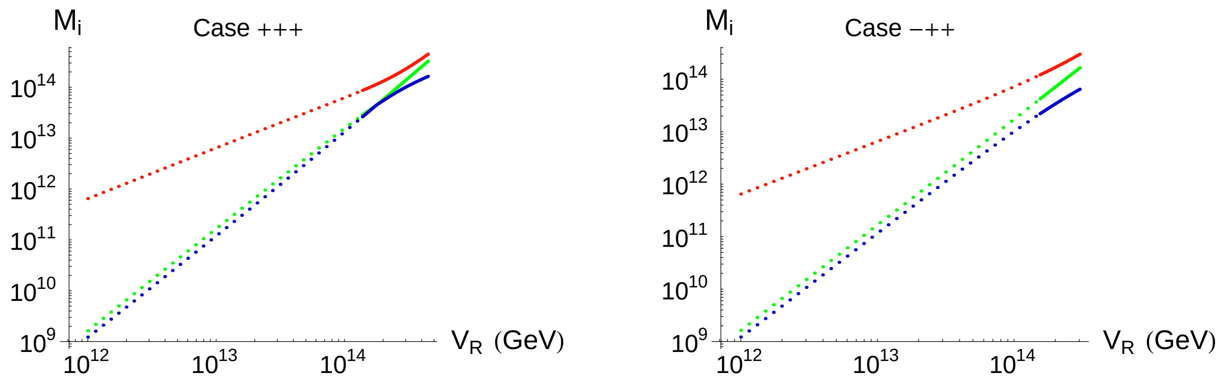


Figure 3.4: Effect of the light neutrino mass hierarchy on the right-handed neutrino masses. The input parameters are the same as in Fig. 3.1, except that the light neutrino mass hierarchy is inverted, with $m_3 = 10^{-3}$ eV and opposite CP parities for m_1 and m_2.

To conclude this section, we mention the paper [38] where a study similar to this one is led. The tuning in m_ν is studied with a more sophisticated criterium. Furthermore the stability of the solutions under small perturbations of f is investigated. All these informations can be encoded into a single parameter Q and it is shown that the four solutions $(\pm,\pm,-)$ are highly unstable, which is due to the large hierarchy required between the eigenvalues. On the contrary, $(+,+,+)$ is found to be the most stable.

It is worth noting that taking into account the analysis of stability as performed in [38], the four $(\pm,\pm,-)$ solutions imply a tuning between the matrix elements of f, since the hierarchy between the matrix elements is smaller than the one present in the spectra, and unfortunately, this is true especially for large values of v_R where typeI-typeII tuning is absent. Even if present at high energies in f for some reason, below v_R, the LR symmetry is broken and f splits into f_L and f_R which run differently. Therefore, it may be that the structure of m_ν will not be conserved when quantum corrections are included in the analysis.

E_6 and a solution to the fine-tuning problem in m_ν

As was explained previously, fine-tuning can appear between the type I and type II contributions. This tuning requires in fact an alignment $f \propto Y_\nu$. Although completely unnatural in $SO(10)$, this alignment can find an explanation if these couplings are related by a symmetry, broken at energies above M_{GUT}. Such a way out can be found by realising that the particle content of our $SO(10)$ model can be incorporated in a simple E_6 model. The smaller representation containing the **16** of $SO(10)$ is the fundamental representation **27**, which decomposes under $SO(10)$ as :

$$\mathbf{27} = \mathbf{16} + \mathbf{10} + \mathbf{1} \tag{3.32}$$

The salient feature of E_6 is the presence of extra matter fields that we introduce to fill the twenty-seven dimensional representations. However, these extra representations are vector-like and can be given large masses when E_6 is broken. For example, coupling bilinears of **27**'s of matter to a $\mathbf{27_H}$ of Higgs and giving a vev to the $\mathbf{1_H}$ will yield a large mass to the **10**'s and **1**'s of matter.
The only other possibility for bilinears of **27** is to couple to **351**, which has anti-symmetric gauge indices, or $\mathbf{351'}$ with symmetric gauge indices. Under $SO(10)$, the $\mathbf{351'}$ decomposes as :

$$\mathbf{351'} = \mathbf{1} + \mathbf{10} + \overline{\mathbf{16}} + \mathbf{54} + \overline{\mathbf{126}} + \mathbf{144} \tag{3.33}$$

Thus, including $\mathbf{10_u}$, **54** and $\overline{126}$ in a $\mathbf{351'}$ of E_6 and the $\mathbf{10_d}$ in a $\mathbf{27_H}$, we will have a common origin for f and Y_ν as follows :

$$f_{ij}\mathbf{27_i 27_j 351'} \rightarrow f_{ij}\mathbf{16_i 16_j 10_u} + f_{ij}\mathbf{16_i 16_j}\overline{\mathbf{126}} \tag{3.34}$$

The same conclusion follows for the couplings κ and λ :

$$\kappa\, \mathbf{351' 351' 351'} \rightarrow \kappa\,(\mathbf{10_u 10_u 54} + \overline{\mathbf{126}}\,\overline{\mathbf{126}}\,\mathbf{54}) \tag{3.35}$$

and the small difference between the couplings is then induced by the running between the breaking scales of E_6 and v_R.

3.2 Phenomenological Consequences

In the previous section we introduced a method of extraction for the RH neutrino mass matrix and used it to study a simple class of $SO(10)$ models. Simple criteria as perturbative couplings and the absence of fine-tuning can already constrain non-trivially the eight solutions.
Now that we have our set of matrices M_R we can investigate their phenomenological consequences. The possibility of a baryon asymmetry coming from the RH neutrino sector as described in section 2.4 is a very appreciable thing. This is why we will focus mainly on the task of obtaining a successful lepton asymmetry in the second subsection, but first we will begin with an analysis of Lepton Flavour Violating processes generated by the couplings f and Y_ν.

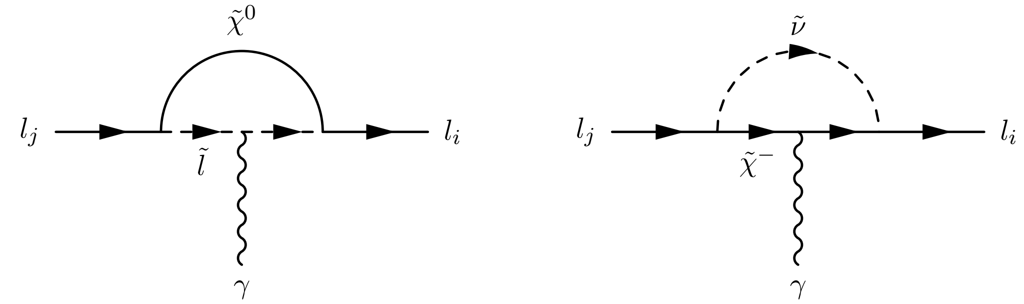


Figure 3.5: Two diagrams potentially creating flavour violating decay processes of the form $l_j \to l_i \gamma$.

3.2.1 Lepton Flavour Violation

An important consequence of lepton number violating interactions in supersymmetric theories is the creation of lepton flavour violating decays from renormalisation group effects. In fact, already at tree level, important contributions to lepton number violation come from the soft SUSY breaking parameters :

$$\mathcal{L} \supset (m_L^2)_{ij} \tilde{l}_{Li}^\dagger \tilde{l}_{Lj} + (m_e^2)_{ij} \tilde{e}_{Ri}^* \tilde{e}_{Rj} + (m_\nu^2)_{ij} \tilde{\nu}_{Ri}^* \tilde{\nu}_{Rj} + A_{ij}^l H_1 \tilde{e}_{Ri}^* \tilde{l}_{Lj} + A_{ij}^\nu H_2 \tilde{\nu}_{Ri}^* \tilde{l}_{Lj} \tag{3.36}$$

If the mass matrices are not diagonal, then the slepton mass eigenstates can couple to different flavours of leptons and the diagrams fig. 3.5 show how to generate flavour violating decays such as $\mu \to e\gamma$ or $\tau \to \mu\gamma$.

This is a generic problem for SUSY theories with scalar masses less than a TeV (which is mandatory for solving the hierarchy problem). However several popular mechanisms of SUSY breaking such as Gauge mediation or the spontaneous breaking of local SUSY in mSUGRA generate universal scalar masses at a high scale and the soft scalar masses are all diagonal. For the applications of interest in this thesis we restrict to minimal Supergravity setups where scalar soft masses are proportional to the identity matrix with a factor m_0^2 and trilinear A_i couplings are proportional to the corresponding Yukawa coupling Y_i with a factor am_0.

It has been noted for a long time [21] that the Yukawa couplings and Majorana mass of RH neutrinos in supersymmetric seesaw models induce non-zero entries in the slepton mass matrix m_L^2 through renormalisation effects between the scale where slepton soft masses are universal, which we call M_U, and the masses of the heavy neutrinos, under which they decouple. The change in the entries of m_L^2 from the running is approximated by (in the basis where the RH neutrino mass matrix M is diagonal) :

$$(\Delta m_L^2)_{ij} \simeq -\frac{1}{16\pi^2}\left(6m_0^2\left[Y_\nu^\dagger \,\mathrm{Ln}\left(\frac{M_U}{M}\right) Y_\nu\right]_{ij} + 2\left[A_\nu^\dagger \,\mathrm{Ln}\left(\frac{M_U}{M}\right) A_\nu\right]_{ij}\right) \tag{3.37}$$

$$\simeq -\frac{(3+2a)m_0^2}{8\pi^2}\left[Y_\nu^\dagger \,\mathrm{Ln}\left(\frac{M_U}{M}\right) Y_\nu\right]_{ij} \tag{3.38}$$

Starting from diagonal matrices at M_U, the off-diagonal entries can thus be expressed through the small parameters $\delta_{ij}^{LL} = (m_L^2)_{ij}/\bar{m}_L^2$ with $\bar{m}_L^2$ the average left slepton mass. In this context, the off-diagonal elements of the slepton mass matrix are known, up to a few SUSY breaking coefficients

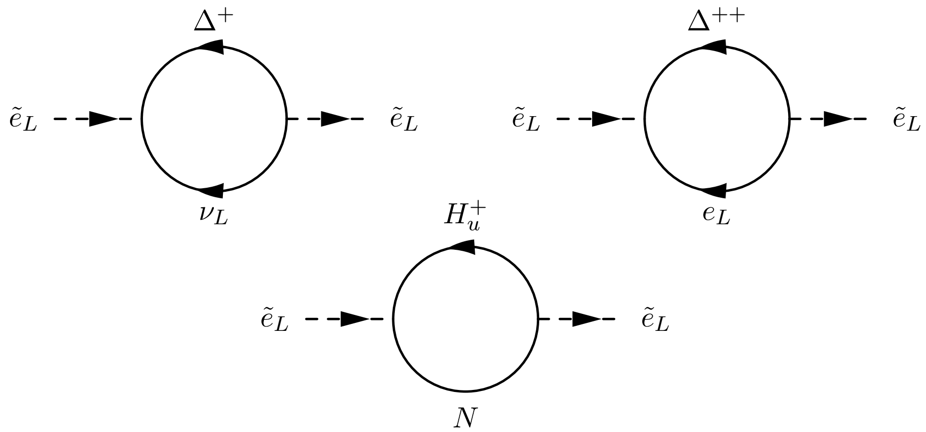


Figure 3.6: Diagrams contributing to the wave function renormalisation of the left-handed sleptons $\tilde{e}_L$. The upper ones correspond to triplet induced corrections while the lower one corresponds to right-handed neutrino corrections.

and several analysis have been performed to study their impact on lepton flavour violating rates [21, 69, 70]. The branching ratios of interest are functions of δ_{ij}^{LL} :

$$\frac{\mathrm{BR}(l_j \to l_i \gamma)}{\mathrm{BR}(l_j \to l_i \bar{\nu}_i \nu_j)} \simeq 10^{-5} \frac{M_W^4}{\bar{m}_L^4} \tan^2\beta |\delta_{ij}^{LL}|^2 F_{susy} \tag{3.39}$$

F_{susy} is a function of SUSY breaking parameters, estimated to be $\mathcal{O}(1)$. We can factorise the dependence in the SUSY breaking parameters and translate the limits on $\mathrm{BR}(l_j \to l_i \gamma)$ to the parameters :

$$C_{ij} = \left[Y_\nu^\dagger \, \mathrm{Ln}\left(\frac{M_U}{M}\right) Y_\nu \right]_{ij} \tag{3.40}$$

If there is also a type II contribution to the neutrino mass $m_\nu^{II} = v_L f$, there are more contributions to the sleptons wave function renormalisation, with weak triplets running in the loops [68]. The diagrams involving Δ_L and the RH neutrinos N_i are recapitulated in fig. 3.6. The coefficients C_{ij} are consequently modified to :

$$C_{ij} = \left[Y_\nu^\dagger \mathrm{Ln}\left(\frac{M_U}{M}\right) Y_\nu \right]_{ij} + 3 \left[f f^\dagger \right]_{ij} \mathrm{Ln}\left(\frac{M_U}{M_{\Delta_L}}\right) \tag{3.41}$$

Upper limits have been experimentally established on LFV ratios [71, 72] : $\mathrm{BR}(\mu \to e\gamma) < 1.2 \times 10^{-11}$ and $\mathrm{BR}(\tau \to \mu\gamma) < 6.8 \times 10^{-8}$, and they have been translated to C_{12} and C_{23} in [73, 74]. Of course, since the branching ratios scale with $\tan^2\beta |C_{ij}|^2$, limits are enhanced for large $\tan\beta$. For the case of interest $\tan\beta = 10$, we get $|C_{12}| \lesssim 0.1$ and $|C_{23}| \lesssim 10$.
The results for the Left-Right symmetric models presented in the previous section are displayed in the fig. 3.7. The curves have two main behaviours, so that we plot only C_{12} and C_{23} for the cases $(+,+,+)$ and $(-,-,-)$. In some cases the signatures are above the experimental limits at large v_R but they are not much more restrictive than the perturbativity constraint. However they are only indicative since they depend on the spectrum of the superpartners. Still they are significantly larger than in the pure type I seesaw model and any experimental improvement will clearly reduce the parameter space.

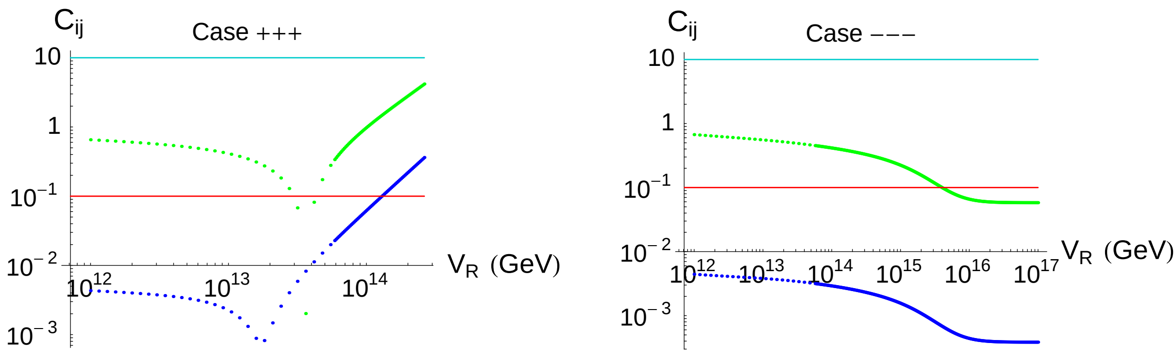


Figure 3.7: Coefficients C_{12} (blue/dark grey) and C_{23} (green/ light grey) for the usual set of parameters ($m_1 = 0.001$ eV and $\beta = \alpha$) with no CP phases except δ_{CKM} and for the cases $(+,+,+)$ (left panel) and $(-,-,-)$ (right panel). The horizontal red and light blue lines represent the experimental limits.

3.2.2 Leptogenesis in the LR symmetric seesaw

Let us turn to the more interesting constraint of leptogenesis, since it is a non-trivial one to fulfil. As already stressed it would be quite appealing to get at the same time small neutrino masses and a realistic baryon asymmetry from the seesaw mechanism and we will use leptogenesis to distinguish among the eight solutions. Referring to the introductory section 2.4 on leptogenesis, it is clear that the bound 2.124 becomes extremely deadly, which is precisely why we splitted the different solutions in the left panel of fig. 3.8 from the value reached by M_1. This led us to classify the spectra into three classes which will have roughly the same behaviour as far as leptogenesis is concerned. Consequently we will display the curves for ε_1 only for the solutions $(+,+,+)$, $(-,-,-)$ and $(+,-,+)$.
Starting with the solutions with $M_1 \sim 10^5$ GeV, we see without surprise in fig.3.8 that we fail to reach the lower bound on ε_1 by several orders of magnitude. Although this figure is plot without any complex phase except for δ_{CKM}, the situation does not improve much better when playing with the free phases and the asymmetry stays in the range $[10^{-14}, 10^{-11}]$. This is reminiscent of the pure type I situation where the RH neutrino spectrum has to be extremely hierarchical to compensate for the up-quark hierarchy (see the large v_R limit of case $(-,-,-)$ in fig. 3.1). Indeed, it is well-known that type I seesaw in $SO(10)$ with a simple link between Y_u and Y_ν cannot yield a good lepton asymmetry through the decays of N_1.

In fig. 3.8 we plot the two contributions ε_1^I (in thin black) and ε_1^{II} (thin red or light grey) of eq. (2.93) and (2.98). As they are the same function of m_ν^I and m_ν^{II} in the hierarchical limit, whenever cancellations occur in m_ν between the two types, it occurs also for leptogenesis. Here, this is graphically seen in a straightforward way since the two thin curves become of the same order of magnitude and even superpose for small v_R while the total asymmetry is one or two orders of magnitude below.

The other extreme class of solutions is the one of $(+,+,+)$, where M_1 grows to quite large values $M_1 \geq 10^{10}$ GeV with a maximum of 10^{13} GeV. It is not a surprise that ε_1 reaches the desired value of 10^{-6} without any problem, even without any phase but δ_{CKM} (see fig. 3.8).
However these cases are not so interesting for several reasons. The first one simply has to do with

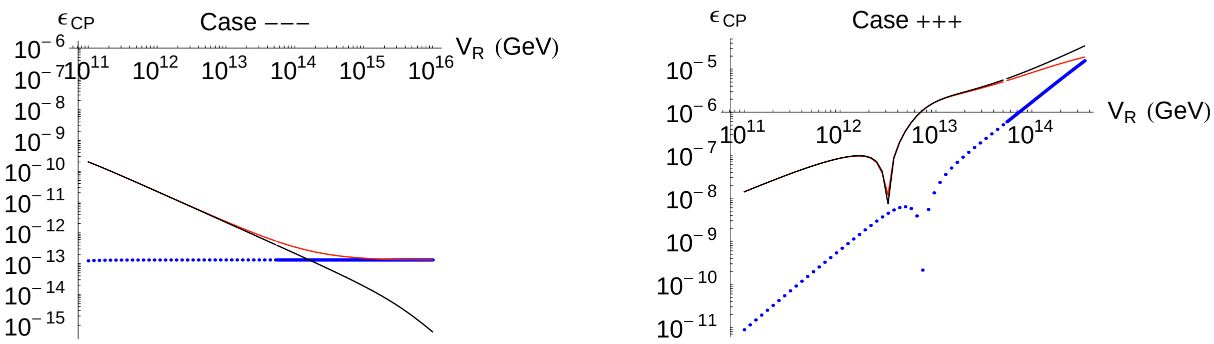


Figure 3.8: CP asymmetry ε_1 as a function of v_R for the solution $(-,-,-)$ (left panel) and $(+,+,+)$ (right panel) in the case of a hierarchical light neutrino mass spectrum with $m_1 = 10^{-3}$ eV, $\beta = \alpha$, and no CP violation beyond the CKM phase. The thin lines correspond to the contribution of right-handed neutrinos (red [grey] curve) and of the heavy triplet (black curve).

the gravitino constraints explained in section (2.4.5). Should the reheating temperature be larger than 10^{10} GeV or even smaller [53], the decays of gravitini would have disturbed BBN processes and this is not affordable. A clear way out of this problem is to invoke an inflaton decaying directly into RH neutrinos but as we stressed earlier, this leads to very model dependent predictions. Moreover, turning on phases (as would be expected for a random point in parameter space) will lift the curve towards upper values and we would risk an asymmetry that is only of the good order of magnitude in the fine-tuned region. Finally, we know that the large v_R region of $(+,+,+)$ is the pure type II limit, therefore this situation is not new to this study.

Last come the cases of highest relevance. In fig. 3.9 are plot the asymmetries for different choices of phases for $(+,-,+)$. As M_1 seems to lie around the lower bound for successful leptogenesis, we expect to find configurations where the bound (2.124) is saturated. Although this is not the case on the whole range of v_R, there is often a peak at $v_R \sim 10^{12-13}$ GeV, due indirectly to the level crossing between M_1 and M_2 (but not to the divergence of the loop function of ε^I : this is apparent from the peak of ε^{II}, since ε^{II} has no pole at $M_1 = M_2$). As these cases are mixed ones which do not appear in type I or II limits, they are very interesting and open new possibilities for successful leptogenesis in $SO(10)$ theories.

The analysis that was made in [37] is only based on these results for the lepton asymmetries. However, at some point we need to compare to the experimental value for y_B, and this is only done with the knowledge of η, known a priori by solving the Boltzmann equations. With help from the approximations of section 2.4 we can however get an idea of the expected outcome by plotting $\tilde{m}_1$ over the appropriate range of v_R. Unfortunately, the link between the up quarks and the neutrino Yukawa sectors lead to neutrino Yukawas close to one, and one can hardly hope to achieve small $\tilde{m}_1$. For example on fig. 3.10, they are displayed for the same choice of phases as in fig. 3.9. These examples illustrate the fact that $\tilde{m}_1$ will generally be confined between 10^{-2} and 10^{-1} eV, in other words in the strong washout regime.

This last remark is valid for every solution and in particular forces to go to large v_R for $(+,+,+)$

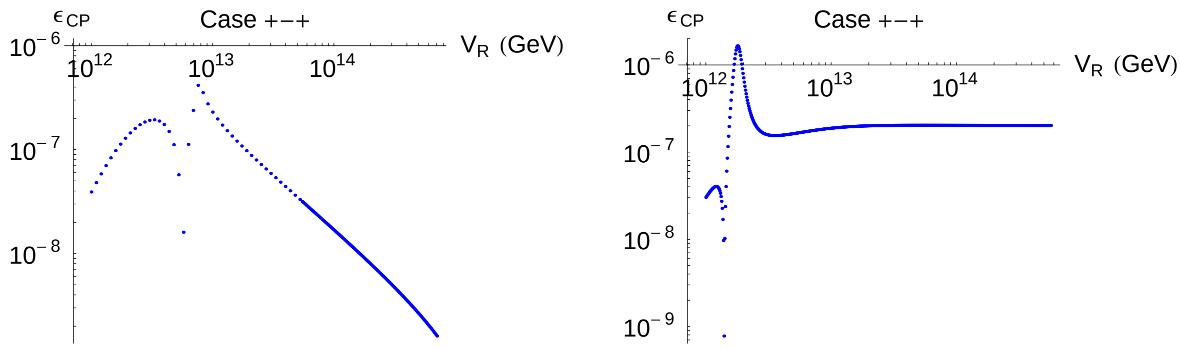


Figure 3.9: Same as fig. 3.8 but for the solution $(+,-,+)$, with $\varphi_2^u = \pi/4$ (left panel) and $\varphi_2^\nu = \pi/4$ (right panel).

where M_1 is particularly large.

One thing to be noticed is the strong dependence of M_1 on m_c, since for a hierarchical spectrum we can see that $M_1 \propto m_c^2(M_{GUT})$ (see appendix B of [37]). Therefore the value of the renormalisation factor applied to m_c plays an important role in the case $(+,-,+)$ where leptogenesis can be marginally allowed.

Although the results for certain solutions seemed appealing, this makes the success of leptogenesis in the considered framework somewhat difficult. However we remind that several approximations have been made so far, and one has to consider the necessary corrections since they can be relevant to either increase the CP asymmetry or decrease the washout factor. In the next two sections we consider the effects of lepton flavours on leptogenesis as well as the necessary corrections to the relation $M_e = M_d$, which will not be negligible as we will see.

3.3 Flavour Effects in Leptogenesis

3.3.1 Equilibrium of charged lepton couplings

Leptogenesis, as every process occurring in the early Universe, demands a lot of attention if one wants to make a proper study, with as less approximations as possible. Indeed, many refinements have been introduced over the years, such as the influence of spectator processes [28, 29] or thermal corrections [26]. However, not until recently have lepton flavours been taken into account [61, 64]. When analysing leptogenesis in [37] we made implicitly the "one flavour approximation", which pretends that charged lepton Yukawa couplings are irrelevant. Thus, leptons are indistinguishable and only the flavour $l_1 = \sum_i (Y_\nu)_{1i} l_i / \sqrt{\sum_j |(Y_\nu)_{1j}|^2}$ with $i = e, \mu, \tau$ interacts with N_1 and will wash away the asymmetry through inverse decays with the Higgs and $\Delta L = 1, 2$ scatterings. It is true that y_e, y_μ and y_τ are irrelevant for the calculation of ε_1 but this is not necessarily the case when considering the dynamical part of the process, in other words the washout. In this case, things will depend on the Yukawa couplings being in equilibrium or not. If charged Yukawas are out of equilibrium, any l_1's

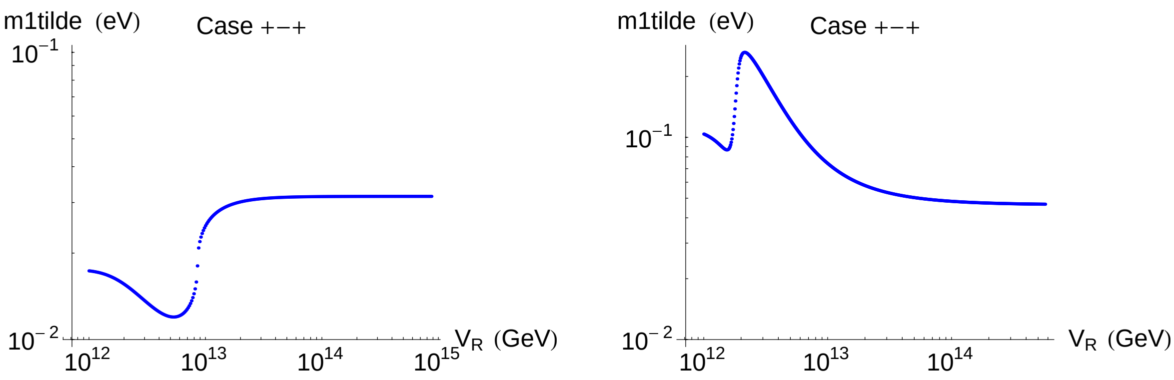


Figure 3.10: Washout parameter $\tilde{m}_1$ with the same parameters and phases as in fig. 3.9. The fine-tuned region is not displayed here.

created by the decays of N_1's will propagate coherently until the next interaction with N_1. However if at least one is in equilibrium, say the τ, then at the moment of leptogenesis it will induce faster processes than the neutrino Yukawa couplings (which are out of equilibrium) and between two interactions with N_1, l_1 will be projected on the τ direction and the orthogonal ones.

To determine the temperatures at which y_e, y_μ and y_τ come into equilibrium we follow ref. [88] where the inverse decay width for $e_{Ri}\bar{e}_{Li} \to H_d$ at finite temperature T is found to be :

$$\Gamma_{ID} = \frac{m_0(T)I(T)}{12\pi\zeta(3)T}\frac{m_0^2(T)h_i^2}{T} \simeq \frac{(\ln 2)^2}{24\pi\zeta(3)}\frac{m_0^2(T)h_i^2}{T} \simeq 5.3\times 10^{-3}\frac{m_0^2(T)h_i^2}{T} \tag{3.42}$$

where I is a function of T defined in [88], and $m_0(T)$ is the Higgs thermal mass [89] :

$$m_0^2(T) = \frac{2m_W^2 + m_Z^2 + 2m_t^2 + m_H^2/2}{4v^2}(T^2 - T_0^2) \tag{3.43}$$

T_0 being the temperature where sphalerons drop out of equilibrium. Hence the criterium used for determining the dynamics of charged Yukawas is :

$$\Gamma_{e,\mu,\tau} \sim 5\times 10^{-3} y_{e,\mu,\tau}^2 T \gg H(T) \tag{3.44}$$

and when translated at the time of leptogenesis $T \sim M_1$ it transforms into :

$$M_1 \lesssim 10^{12}\ \text{GeV} = T_\tau \qquad \text{for } y_\tau \tag{3.45}$$
$$M_1 \lesssim 10^{9}\ \text{GeV} = T_\mu \qquad \text{for } y_\mu \tag{3.46}$$
$$M_1 \lesssim 10^{5}\ \text{GeV} = T_e \qquad \text{for } y_e \tag{3.47}$$

In the MSSM the Yukawa couplings depend on $\tan\beta$ and the previous bounds are rescaled[4] by a factor $(1+\tan^2\beta)$, e.g. for y_τ : $M_1 \lesssim 10^{14}$GeV for moderate $\tan\beta \simeq 10$.

[4] This suggests that leptogenesis, in supersymmetric theories, always takes place in the regime where y_τ is in equilibrium.

When M_1 is greater than T_τ, no charged lepton coupling is in equilibrium and flavour is indistinguishable, justifying the one flavour approximation. When $T_\mu < M_1 < T_\tau$, the τ is distinguished and the decay direction of N_1 is split into $\vec{l_1}.\tau$ and $\vec{l_1} - \vec{l_1}.\tau$. This is equivalent to a two flavour case. Finally, for $M_1 < T_\mu$, all flavours are distinguished since τ and μ define e by orthogonality.

3.3.2 Flavoured Boltzmann Equations

Now that we have some mean to characterise the different flavours, we must "open up" the CP asymmetries :

$$[\varepsilon_1^I]_{\alpha\beta} = \frac{1}{16\pi}\frac{\sum_j \text{Im}\left[Y_{1\alpha}(YY^\dagger)_{1j}Y^*_{j\beta} - Y^*_{1\beta}(Y^*Y^T)_{1j}Y_{j\alpha}\right]}{(YY^\dagger)_{11}} f\left(\frac{M_j^2}{M_1^2}\right) \tag{3.48}$$

and this takes again a simpler form in the hierarchical RH neutrino limit :

$$[\varepsilon_1^I]_{\alpha\beta} = \frac{3}{32\pi}\frac{M_1}{v^2}\frac{\sum_\gamma \text{Im}\left[Y_{1\alpha}Y_{1\gamma}(m^*_\nu)_{\gamma\beta} - Y^*_{1\beta}Y^*_{1\gamma}(m_\nu)_{\gamma\alpha}\right]}{(YY^\dagger)_{11}} \tag{3.49}$$

All the factors become matrices in flavour space [61]. The Boltzmann equations are thus written as matrix equations :

$$\frac{\mathrm{d}}{\mathrm{d}z}[Y_L] = \frac{z}{sH(M_1)}\left([\gamma_D]\frac{\Delta_{N_1}}{Y^{eq}_{N_1}}[\varepsilon_1^I] - \frac{1}{4Y_L^{eq}}\{[\gamma_D],[Y_L]\}\right) - i\big[[\Lambda],[Y_L]\big] \tag{3.50}$$

and γ_D is generalised as :

$$\gamma_D^{\alpha\beta} = \gamma_D\frac{(Y_\nu)_{1\alpha}(Y_\nu)^*_{1\beta}}{(Y_\nu Y_\nu^\dagger)_{11}} \qquad \text{with : } \gamma_D = \sum_\alpha \gamma_D^{\alpha\alpha} \tag{3.51}$$

Λ is a diagonal matrix consisting of the complex thermal masses of the different flavours. More precisely :

$$\Lambda_{\alpha\alpha} = \frac{\omega_{\alpha\alpha} - i\Gamma_{\alpha\alpha}}{H(T)}_{|T=M_1}, \qquad \text{with :} \qquad \omega_{\alpha\alpha} \simeq \frac{y_\alpha^2}{16}T \qquad \Gamma_{\alpha\alpha} \simeq 5\times 10^{-3}y_\alpha^2 T \tag{3.52}$$

From eq. (3.50) we can justify the result of the previous section by noting that the imaginary part of $[\Lambda]$ will damp completely the off-diagonal entries and kill any correlation between the different flavours for $\Gamma_{\alpha\alpha} \gg H(M_1)$. In the opposite limit the last term of the equation becomes negligible and we can forget about flavours. Therefore, any quantum correlation can be forgotten when the charged leptons Yukawa couplings are completely in or out of equilibrium, and the matrices are approximately diagonal (see [67] for a detailed study of quantum correlations in flavour leptogenesis).

In the approximation where flavours are neglected, eq. (3.50) is amputed of its last term. Tracing the equation, and defining $\varepsilon_1^I = \sum_\alpha [\varepsilon_1^I]^{\alpha\alpha}$:

$$\sum_\alpha \frac{\mathrm{d}Y_L^{\alpha\alpha}}{\mathrm{d}z} = \frac{z}{sH(M_1)}\left(\frac{\Delta_{N_1}}{Y^{eq}_{N_1}}\varepsilon_1^I\gamma_D - \sum_\alpha \gamma_D^{\alpha\alpha}\frac{Y_L^{\alpha\alpha}}{Y_L^{eq}}\right) \tag{3.53}$$

So, if the equation is invariant under rotations in flavour space we can place ourselves in a basis where only the last element of the $[Y_L]$ and $[\gamma_D]$ matrices is non-zero, $[\gamma_D] = \text{Diag}(0, 0, \gamma_D)$ and

$[Y_L] = \mathrm{Diag}(0, 0, Y_L)$. Therefore $\mathrm{Tr}\{[Y_L], [\gamma_D]\} = 2\gamma_D Y_L = 2\mathrm{Tr}[Y_L]\mathrm{Tr}[\gamma_D]$.
On the contrary, when flavours are taken into account (and we are in a range of temperature where they are relevant), we cannot rotate indifferently the flavour basis since the equation is diagonal only in the basis where charged lepton are mass eigenstates, and $\mathrm{Tr}([Y_L][\gamma_D]) \neq \mathrm{Tr}([Y_L])\mathrm{Tr}([\gamma_D])$. Thus, as soon as y_τ comes into equilibrium, the situation is clearly different from the one in the one flavour approximation.
Thus, let us write the flavoured BE's for the lepton asymmetry [62], taking into account the decays, inverse decays and $\Delta L = 1$ processes (including for CP violation) :

$$\frac{\mathrm{d}Y_L^{\alpha\alpha}}{\mathrm{d}z} = \frac{z}{sH(M_1)}\left[\left(\frac{Y_{N_1}}{Y_{N_1}^{eq}} - 1\right)[\varepsilon_1^I]^{\alpha\alpha}(\gamma_D + \gamma_{\Delta L=1}) - \frac{Y_L^{\alpha\alpha}}{Y_L^{eq}}(\gamma_D^{\alpha\alpha} + \gamma_{\Delta L=1}^{\alpha\alpha})\right] \tag{3.54}$$

Introducing flavour washout parameters as before, $\kappa_1^{\alpha\alpha} = \kappa_1|(Y_\nu)_{1\alpha}|^2/\sum_\beta|(Y_\nu)_{1\beta}|^2 = \tilde{m}_{1\alpha}/3 \times 10^{-3}$ eV, with $\kappa_1 = \sum_\alpha \kappa_1^{\alpha\alpha}$, we can write the BE's in a simplified, approximate form :

$$(Y_L^{\alpha\alpha})' = (\varepsilon_1^I)^{\alpha\alpha}\kappa_1 z\frac{K_1(z)}{K_2(z)}f_1(z)\Delta_{N_1} - \frac{1}{2}z^3K_1(z)f_2(z)\kappa^{\alpha\alpha}Y_L^{\alpha\alpha} \tag{3.55}$$

$$\Delta'_{N_1} = -z\kappa_1\frac{K_1(z)}{K_2(z)}f_1(z)\Delta_{N_1} - (Y_{N_1}^{eq})' \tag{3.56}$$

This equation shows that the asymmetry created in a specific flavour α is washed-out with a strength that will depend on $\tilde{m}_{1\alpha}$ and not only on $\tilde{m}_1$. In the case where all flavours are weakly or strongly washed out, the final baryon asymmetry may not change much compared to the one flavour case. Approximate solutions can then be found that are similar to previous studies in the one flavour case but they are valid independently for each flavour[5]. The interesting situation is the one where some flavours are strongly washed-out ($\kappa_1^{\alpha\alpha} \gg 1$) while others are weakly washed-out ($\kappa_1^{\beta\beta} \lesssim 1$). In this situation the asymmetries are approximated by :

$$Y_L^{\alpha\alpha} \simeq 0.3\frac{[\varepsilon_1^I]^{\alpha\alpha}}{g_*}\left(\frac{0.55 \times 10^{-3}\mathrm{eV}}{\tilde{m}_1^\alpha}\right)^{1.16} \tag{3.57}$$

$$Y_L^{\beta\beta} \simeq 0.4\frac{[\varepsilon_1^I]^{\beta\beta}}{g_*}\left(\frac{\tilde{m}_1^\beta}{3.3 \times 10^{-3}\mathrm{eV}}\right) \tag{3.58}$$

and $Y_L^{\beta\beta}$ is indeed less suppressed for an optimised $\tilde{m}_1^\beta$ while naively $\kappa_1 \gg 1$ and the total asymmetry ε_1 would be considered as strongly washed-out. This opens a window for new configurations where leptogenesis can be successful when taking into account flavour while it was considered hopeless in the one flavour approximation : it just needs one flavour weakly washed-out, $\tilde{m}_1^\beta \sim 10^{-3}$, while its associated CP asymmetry is reasonably large $\varepsilon_1^{\beta\beta} \geq 10^{-6}$.

Coming back to ε_1 itself, one can infer an interesting bound on the asymmetry. While in the flavourless case the bound (2.123) had been derived for the total asymmetry, a new one has been determined with flavours for the type I seesaw [62] :

$$|\varepsilon_1^{\alpha\alpha}| \leq \frac{3M_1 m_{max}}{8\pi v^2}\sqrt{\frac{\kappa_1^{\alpha\alpha}}{\kappa_1}} \tag{3.59}$$

[5] In [62], one should be careful when comparing the washout factors obtained for democraticly weak washout and the one flavour formula 2.110 we displayed earlier, since we hadn't included CP violation in the $\Delta_L = 1$ scatterings.

We see immediatly that the asymmetry increases this time with the neutrino mass scale and we can hope to find a successful value for y_B with larger neutrino masses compared to the traditional approach (see [63] for a numerical study). Nonetheless, we have now a proportionality to the square root of the branching ratio of N_1 to that flavour, which will limit the value of $\varepsilon_1^{\alpha\alpha}$ for weakly washed-out flavours. Another interesting remark is that even for a real matrix R in the Casas-Ibarra parametrisation, a non-zero asymmetry is possible, contradicting the prediction of the one flavour case [64, 62]. The case with a real matrix R is in fact equivalent to a model for the RH sector is CP-conserving. This can be easily seen in the basis where the charged leptons Yukawa coupling and the RH Majorana mass matrix are diagonal, since in that basis it is clear that the RH unitary rotation that diagonalises $Y_\nu Y_\nu^\dagger$ is real when R is real. Thus, there is still a way for models where CP violation arises only from the LH neutrino sector to achieve successful leptogenesis, i.e. only with the Majorana phases of m_ν.

The analysis carried for the type I case which we have reviewed above has also been extended for the type I+II case in [66]. One opens the asymmetry for ε_1^{II} :

$$(\varepsilon_1^{II})_{\alpha\alpha} = \frac{3}{8\pi}\frac{M_1}{v^2}\frac{\mathrm{Im}[(Y_\nu m_\nu^{II*})_{1\alpha} Y_{\nu 1\alpha}]}{(Y_\nu Y_\nu^\dagger)_{11}}\, g(y) \tag{3.60}$$

and the bound (3.59) is modified to :

$$|\varepsilon_1^{\alpha\alpha}| \leq \frac{3}{8\pi}\frac{M_1}{v^2} m_{max} \tag{3.61}$$

and the dependence on the $K_1^{\alpha\alpha}/K_1$ factor is eliminated. This gives more freedom since it is argued in [66] that this dependence prevents one from optimising the washout by lowering $\tilde{m}_1^{\alpha\alpha}$ since it decreases $\varepsilon_1^{\alpha\alpha}$ automatically.

In the following we will only consider cases when the quantum correlations in the matrices can be ignored and we will avoid repeating the flavour indices : $\varepsilon_1^{\alpha\alpha} \to \varepsilon_1^{\alpha}$.

We can already point out the potential interest of flavour for the cases displayed in the fig. 3.9 by plotting the ε_1^α together with the $\tilde{m}_1^\alpha$ (fig. 3.11). There is clearly a hierarchy between the different flavours but usually $\varepsilon_1^e < \varepsilon_1^\mu < \varepsilon_1^\tau$ while at the same time $\tilde{m}_1^e < \tilde{m}_1^\mu < \tilde{m}_1^\tau$ and obviously this does not go in the right direction. Nevertheless we can potentially play with the inputs to enhance the asymmetry in one flavour α while lowering the $\tilde{m}_1^\alpha$ associated.

3.3.3 Contribution of N_2

Hierarchical N_i

Another assumption that was made earlier, when introducing leptogenesis, was to neglect the contribution of heavier particles decaying before N_1, such as $N_{2,3}$ and Δ_L for type I+II, since their contribution to the asymmetry should be washed-out by N_1. This statement has been argued to fail when N_1 is in the weak washout regime [65], which is actually the case for GUT models where Y_ν is related to Y_u [58]. The reasoning is also based on the difference between the different lepton flavours, and goes as follows. For models with hierarchical Y_ν, the hierarchy in M_R is usually twice as big in order to compensate in the seesaw formula and obtain a mildly hierarchical m_ν. Under these conditions, as we actually stressed in section 3.1.3, M_1 is generally too light to realise a correct value of the lepton asymmetry. Indeed, in the model that we developed, the "type I-like" solutions exhibit an $M_1 \sim 10^5$GeV. Nevertheless, since Y_ν is highly hierarchical, the coupling of N_1 to the thermal plasma

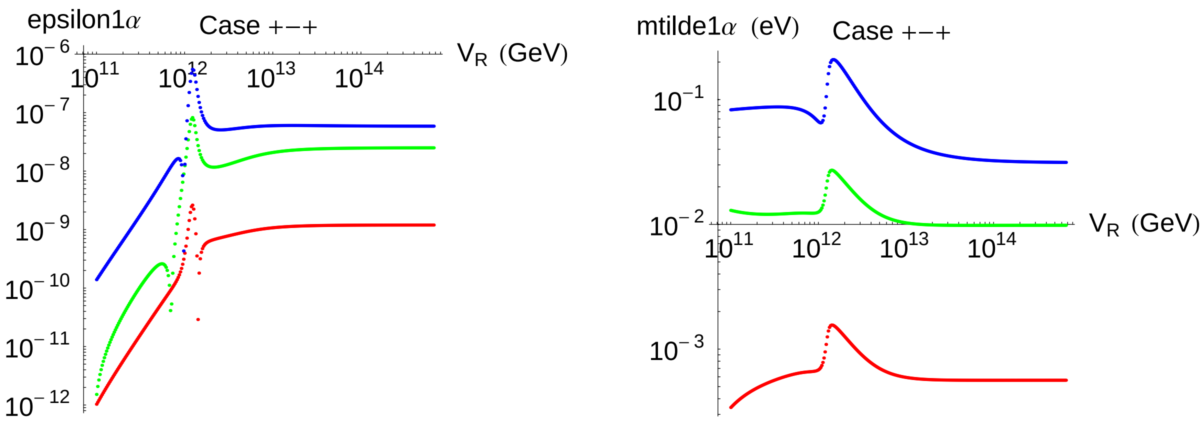


Figure 3.11: The different ε_1^α for the $(+,-,+)$ case with $\varphi_2^\nu = \pi/4$ (left panel) and the corresponding $\tilde{m}_1^\alpha$ (right panel). The red curves (grey) correspond to the e flavour, the green ones (light grey) to the μ and the blue ones (dark grey) to the τ.

is actually quite weak and it is not obvious, starting from a null population of N_1, that any asymmetry in e, μ and τ will be erased completely. Moreover, the CP asymmetry ε_2^α will be enhanced compared to ε_1^α. The expression for ε_2^α can be written just as for ε_1^α, eq. (3.48). When expanded, ε_2^α reads :

$$\varepsilon_2^\alpha = \frac{1}{8\pi(YY^\dagger)_{22}} \left(\frac{3}{2}\frac{M_2}{M_3} \sum_\beta \mathrm{Im}(Y_{2\alpha}Y_{3\alpha}^* Y_{2\beta}Y_{3\beta}^*) + \frac{M_2}{M_1} \sum_\beta \mathrm{Im}(Y_{2\alpha}Y_{1\alpha}^* Y_{2\beta}Y_{1\beta}^*) \right) \tag{3.62}$$

Due to the hierarchy in Y_ν and M_R, and therefore in Y, the first term will dominate, and for the τ flavour, it leads to the bound :

$$\varepsilon_2^\tau \leq 3 \times 10^{-6} \frac{M_2}{10^{10}\ \mathrm{GeV}} \tag{3.63}$$

For $M_2 \gtrsim 10^{10}$ GeV we can hope to achieve a sizeable asymmetry if N_1 does not wash-out too much in the τ direction.

Flavour considerations lead us to consider more general reasons why we can even hope to conserve an asymmetry from the decays of N_2 for a more general Y_ν [59]. The important thing for ε_2^α not to be completely washed-out is that M_1 be larger than the temperature at which the muon comes into equilibrium. If $M_1 > T_\mu$, at most one direction is imposed in flavour space, and we define a flavour basis (l_a, l_b, l_c). Now let us investigate the different possibilities :

- $M_1 > T_\tau$: the τ is not in equilibrium and no direction is preferred. We can choose $l_a = l_1$, the direction in which N_1 decays, while l_b and l_c complete the orthogonal basis, with l_b chosen to realise $\langle l_b | l_2 \rangle = 0$. Therefore the projection of l_2 onto l_c is orthogonal to l_1 and ε_2 will be partly protected from N_1 washout processes and contribute to the CP asymmetry at low energy.

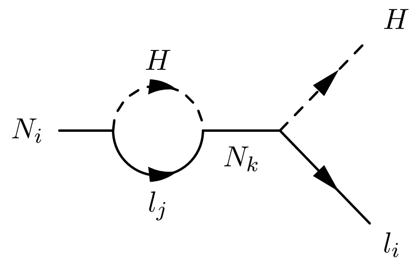
- $T_\tau > M_1 > T_\mu$: here the τ direction is distinguished, $l_a = l_\tau$, and l_c is chosen to be orthogonal to l_1. In a generic situation l_2 will have a non-zero projection on l_c which should survive washout along l_1 and l_τ.

- On the contrary, when $M_1 < T_\mu$ all flavours are distinguished and the charged lepton Yukawa couplings will link the N_1 and N_2 decay directions, erasing generically ε_2^α.

In short, the key argument for a survival of ε_2 is the possibility to have a freedom in flavour space, which implies that l_1 and l_2 cannot be linked sufficiently by the Yukawa interactions to reprocess all the components of $\varepsilon_2^{\alpha\alpha}$. This result is quite interesting but but is more relevant to non-SUSY cases, since in SUSY with $\tan\beta \geq 10$, $T_\mu \gtrsim 10^{11}$ GeV and the reheating temperature can hardly be so large. Thus N_1 cannot be produced if $M_1 > T_\mu$. However, placing ourselves in the non-SUSY and looking back at fig. 3.1, we have several solutions with $M_1 > 10^9$ GeV and these remarks could bear some relevance.

Degenerate N_i [55]

When analysing carefully the loop function of the CP-asymmetry of eq. (2.90) or (2.92), we see that these expressions are singular when $x_i \to 1$, i.e. when the N_i become degenerate. This divergence comes from the self-energy correction, computed from the diagram :



Therefore, when there is a complete degeneracy between some or all of the N_i, the N_k running inside the self-energy diagram will be on-shell and its propagator will diverge. This is simply due to the fact that we considered RH neutrinos as stable particles and did not take into account their decay widths Γ_i. When taking into account the finite decay widths, the resonant contribution to the CP-asymmetry of the decaying neutrino N_i becomes :

$$\varepsilon^I_{N_i} = \frac{\mathrm{Im}(YY^\dagger)^2_{ij}}{(YY^\dagger)_{ii}(YY^\dagger)_{jj}} \frac{(M_i^2 - M_j^2)M_i\Gamma_j}{(M_i^2 - M_j^2)^2 + M_i^2\Gamma_j^2} \tag{3.64}$$

with j the index of the RH neutrino which is degenerate with N_i[6]. The other contributions (vertex corrections and self-energy corrections with non-degenerate RH neutrinos) are the same as before. It is clear that once the finite decay widths are taken into account there is no more divergence in the expression for ε_{CP}. On the contrary, we see that when $N_i \to N_j$ the expression (3.64) goes to zero. The main feature of this new expression is the potential resonant behaviour for $M_i \simeq M_j$, allowing for enhanced CP-asymmetries. This is evident when one compares the second term of eq. (3.64) in the following two cases for the masses :

[6] We will not consider here the case with all three RH neutrino quasi-degenerate

$$M_j \gg M_i \quad \rightarrow \quad \frac{(M_i^2 - M_j^2)M_i\Gamma_j}{(M_i^2 - M_j^2)^2 + M_i^2\Gamma_j^2} \simeq -\frac{(YY^\dagger)_{jj}}{8\pi}\frac{M_i}{M_j} \tag{3.65}$$

$$|M_j - M_i| \simeq \frac{\Gamma_j}{2} \quad \rightarrow \quad \frac{(M_i^2 - M_j^2)M_i\Gamma_j}{(M_i^2 - M_j^2)^2 + M_i^2\Gamma_j^2} \simeq \frac{1}{2} \tag{3.66}$$

where it is apparent that we can gain several orders of magnitude in the CP-asymmetry if N_i and N_j are quasi-degenerate.

However interesting this possibility may be, however, it does not concern the models of interest in this manuscript. Such an enhancement is only realised for degeneracies $10^{-5} - 10^{-6}$ which is quite unnatural except for specific models of flavour, moreover we will not be able to probe the parameter space with a sufficient precision to display points with such a small degeneracy. However the resonant formulae can modify the results of the final baryon asymmetry by an $\mathcal{O}(1)$ factor when there is a level crossing between N_1 and N_2 and we will have to use them for a precise quantitative study. Let us remark that only for N_1 and N_2 will we have to use these formulae since N_3 is usually too big to play any role in the lepton asymmetry in the cases we are interested in. The same remark is also valid for the decay of the $SU(2)_L$ triplet Δ_L. For reasonable values of the reheating temperature, these particles can simply not be produced by the thermal plasma.

3.4 A Model with Realistic Fermion Masses

As it has been pointed out in the last section, taking into account heavier neutrinos and including the effects of lepton flavours in the decays of the RH neutrinos can already modify importantly the results for leptogenesis. These effects are taken into account in [120], where we studied leptogenesis in the same $SO(10)$ framework previously described in section 3.1.3 and paper [37], but in a more quantitative way. In this paper are also taken into account the necessary corrections to the charged leptons and down quark mass matrices in order to depart from the relation (assumed so far) $M_e = M_d$. To achieve this task we must couple the **16**'s of matter to representations with vev's in weak doublet directions, coupling with a different Clebsch-Gordan factor to the leptons and down quarks. Basing ourselves on the discussions of section 2.3.3, we will consider symmetric and anti-symmetric corrections respectively and see that the latter can provide sizeable modifications for the CP asymmetries and washout factors.

3.4.1 Symmetric corrections

Restricting to the strict particle content of the model, it is already possible to modify the fermion mass relations, thanks to the $\overline{\mathbf{126}}$. We remind that $\overline{\mathbf{126}} \supset (\mathbf{2}, \mathbf{2}, \mathbf{15})$, and we suppose that it couples to appropriate representations to get a weak scale vev in this direction. For example, this is achieved through a coupling to a **210** as in the MSGUT model. The vev's of the two weak doublets are denoted v_u^{126} and v_d^{126}, and those of the **10**'s are denoted v_u^{10} and v_d^{10}. The mass matrices now read, after EW symmetry breaking :

$$M_d = v_d^{10} Y_d^{10} + v_d^{126} f \tag{3.67}$$
$$M_e = v_d^{10} Y_d^{10} - 3v_d^{126} f \tag{3.68}$$
$$M_u = v_u^{10} Y_u^{10} + v_u^{126} f \tag{3.69}$$
$$M_D = v_u^{10} Y_u^{10} - 3v_u^{126} f \tag{3.70}$$

Subtracting the equations the relations between the matrices are changed to :

$$M_D = M_u - 4v_u^{126} f \tag{3.71}$$
$$M_e = M_d - 4v_d^{126} f \tag{3.72}$$

This is the most minimal way to modify the masses since we do not introduce any new coupling. The seesaw relation as a function of f and M_u is changed to :

$$m_\nu = v_L f - \frac{1}{v_R} M_D f^{-1} M_D \tag{3.73}$$
$$= v_L - \frac{1}{v_R} \left(M_u - 4v_u^{126} f\right) f^{-1} \left(M_u - 4v_u^{126} f\right) \tag{3.74}$$
$$= \left(v_L - 16\frac{(v_u^{126})^2}{v_R}\right) f + 8\frac{v_u^{126}}{v_R} M_u - M_u \frac{1}{v_R f} M_u \tag{3.75}$$

In order to use our reconstruction procedure we put the equation under the form :

$$m_\nu - 8\frac{v_u^{126}}{v_R} M_u = \left(v_L - 16\frac{(v_u^{126})^2}{v_R}\right) f - M_u \frac{1}{v_R f} M_u \longrightarrow \quad Z = \alpha X - \beta X^{-1} \tag{3.76}$$

where the quantities Z, X, α and β are redefined to be :

$$Z = M_u^{-1/2} m_\nu (M_u^{-1/2})^T - 8\frac{v_u^{126}}{v_R} \tag{3.77}$$
$$X = v_u M_u^{-1/2} f (M_u^{-1/2})^T \tag{3.78}$$
$$\alpha = \frac{v_L}{v_u} + 16\frac{(v_u^{126})^2}{v_u v_R} \tag{3.79}$$
$$\beta = \frac{v_u}{v_R} \tag{3.80}$$

The fact that M_e and M_d are not equal anymore is parametrised by a unitary matrix U_m operating the transitions between the basis in which M_e and M_d are diagonal. We are going to work in the physical basis for neutrinos, which is the charged leptons mass eigenstate basis $M_e = \hat{M}_e$. In this basis the matrix for down quarks decomposes as $M_d = U_m^T \hat{M}_d U_m$, and the matrix U_m is a function of three real angles θ_{ij}^m, for which we note $\cos\theta_{ij}^m = c_{ij}^m$ and $\sin\theta_{ij}^m = s_{ij}^m$, and six imaginary phases δ^m, φ_g^m and φ_i^m, $i = 1...4$:

$$
\begin{aligned}
U_m &= e^{i\varphi_g^m} \begin{pmatrix} e^{i\varphi_1^m} & 0 & 0 \\ 0 & e^{i\varphi_2^m} & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{12}^m & -s_{12}^m & 0 \\ s_{12}^m & c_{12}^m & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} c_{13}^m & 0 & -s_{13}^m e^{i\delta^m} \\ 0 & 1 & 0 \\ s_{13}^m e^{-i\delta^m} & 0 & c_{13}^m \end{pmatrix} \\
&\quad \times \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23}^m & -s_{23}^m \\ 0 & s_{23}^m & c_{23}^m \end{pmatrix} \begin{pmatrix} e^{i\varphi_3^m} & 0 & 0 \\ 0 & e^{i\varphi_4^m} & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad (3.81)
\end{aligned}
$$

U_m enters the definition of the matrices Z and X :

$$Z = \hat{M}_u^{-1/2} U_q^* U_m^* m_\nu U_m^\dagger U_q^\dagger \hat{M}_u^{-1/2} \qquad (3.82)$$

$$X = \hat{M}_u^{-1/2} U_q^* U_m^* f U_m^\dagger U_q^\dagger \hat{M}_u^{-1/2} \qquad (3.83)$$

The corrections being symmetric, they are completely adapted to our reconstruction procedure. However, eqs. (3.76) and (3.72) are matrix coupled equations, which are quite hard to solve, even numerically. Therefore we will not pursue this path any further.

3.4.2 Anti-symmetric corrections

In order to simplify, compared to the situation considered above, we suppose that the $\overline{\mathbf{126}}$ does not take any EW vev and thus f does not contribute to the fermion masses. As we did not include a **120** in the spectrum, the only other way to modify fermion masses is through non-renormalisable couplings. We did not specify completely the Higgs sector of the models under study to keep the analysis general enough. However, at least one adjoint **45** is used in most cases so we will suppose it present for the following analysis. Contracting the **45** with the **10**'s, it is possible to create an effective **120** :

$$\mathbf{10} \times \mathbf{45} = \mathbf{10} + \mathbf{120} + \mathbf{320} \qquad (3.84)$$

It should also be noted that this non-renormalisable operator contains a **10**. If the vev is only directed along the **10**, it will contribute equally to M_e and M_d so we have to give a vev in the **120** direction absolutely. From the decomposition of the product under the Pati-Salam group $SU(2)_L \times SU(2)_R \times SU(4)_c$:

$$\mathbf{10} \times \mathbf{45} = [(\mathbf{2},\mathbf{2},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{6})] \times [(\mathbf{3},\mathbf{1},\mathbf{1}) + (\mathbf{1},\mathbf{3},\mathbf{1}) + (\mathbf{1},\mathbf{1},\mathbf{15}) + (\mathbf{2},\mathbf{2},\mathbf{6})] \qquad (3.85)$$

the **45** can create vev's in the **10** and **120** weak doublets directions if its components take the following vev's : $\langle(\mathbf{1},\mathbf{3},\mathbf{1})\rangle = T_{3R}$ and $\langle(\mathbf{1},\mathbf{1},\mathbf{15})\rangle = B - L$, which couple to the bidoublets in the **10**'s. It is possible to give a vev along the two directions at the same time or introduce two **45**'s, each having a vev in one of the two directions.
We have a model with two **10**'s, $\mathbf{10_u}$ and $\mathbf{10_d}$ each with a vev in one of its doublets. If the **45**'s couple with both **10**'s, not only M_d and M_e will be corrected but also M_D and M_u will receive anti-symmetric contributions, which invalidates the reconstruction procedure using orthogonal diagonalisations. Therefore we will focus on a model where the non-renormalisable operators created by the couplings of **45**'s and $\mathbf{10_u}$ can be supposed negligible. The couplings that are added to the Lagrangian are thus :

$$\frac{Y_{ij}^{1s}}{\Lambda}(\mathbf{10_d}\times\mathbf{45_1})_{|\mathbf{10}}\mathbf{16_i16_j}+\frac{Y_{ij}^{1a}}{\Lambda}(\mathbf{10_d}\times\mathbf{45_1})_{|\mathbf{120}}\mathbf{16_i16_j}+\frac{Y_{ij}^{2s}}{\Lambda}(\mathbf{10_d}\times\mathbf{45_2})_{|\mathbf{10}}\mathbf{16_i16_j}$$
$$+\frac{Y_{ij}^{2a}}{\Lambda}(\mathbf{10_d}\times\mathbf{45_2})_{|\mathbf{120}}\mathbf{16_i16_j} \qquad (3.86)$$

We note the different vev's of $\mathbf{10_d}$ and the $\mathbf{45_i}$'s as follows : $\langle(\mathbf{2},\mathbf{2},\mathbf{1})_{|\mathbf{10_d}}\rangle = v_d$, $\langle(\mathbf{1},\mathbf{3},\mathbf{1})_{|\mathbf{45_1}}\rangle = V_3$ and $\langle(\mathbf{1},\mathbf{1},\mathbf{15})_{|\mathbf{45_2}}\rangle = V_{15}$. V_3 and V_{15} break $SU(2)_R$ and $B-L$ so they have GUT scale values. v_d is the only EW breaking scale in the down quark and charged lepton sector. As for Λ, it is the scale at which the effective non-renormalisable operators are generated. The largest value that Λ can take in 4d GUTs is the Planck scale $M_P \sim 10^{19}$ GeV. However, since $SO(10)$ theories with large representations such as $\mathbf{126}+\overline{\mathbf{126}}$ become strongly coupled at a scale around $10\times M_{GUT}$, Λ can be quite smaller than the Planck scale. Once the gauge symmetry is broken, the masses read :

$$M_d = v_d\left(Y_d^{10}-\frac{V_3}{\Lambda}Y^{1s}-\frac{V_3}{\Lambda}Y^{1a}+\frac{V_{15}}{\Lambda}Y^{2a}\right) \qquad (3.87)$$
$$M_e = v_d\left(Y_d^{10}-\frac{V_3}{\Lambda}Y^{1s}-\frac{V_3}{\Lambda}Y^{1a}-3\frac{V_{15}}{\Lambda}Y^{2a}\right) \qquad (3.88)$$

The coupling Y^{1s} is symmetric and can be absorbed in a redefinition of $Y_d^{10}\to Y_d$ and we forget about it in the following. The anti-symmetric couplings form an effective Y_d^a and Y_e^a. The most practical basis to study the mass matrices is the one where the symmetric contribution is diagonal $Y_d=\hat{Y}_d$, which can be done in an $SO(10)$ symmetric way. As the anti-symmetric parts are changed to $Y_{e,d}^a\to U^T Y_{e,d}^a U$, they stay anti-symmetric so that M_e and M_d differ through their off-diagonal entries :

$$M_d=\begin{pmatrix} y_1 & \varepsilon_1 & \varepsilon_2\\ -\varepsilon_1 & y_2 & \varepsilon_3\\ -\varepsilon_2 & -\varepsilon_3 & y_3\end{pmatrix} \qquad M_e=\begin{pmatrix} y_1 & -x_1\varepsilon_1 & -x_2\varepsilon_2\\ x_1\varepsilon_1 & y_2 & -x_3\varepsilon_3\\ x_2\varepsilon_2 & x_3\varepsilon_3 & y_3\end{pmatrix} \qquad (3.89)$$

y_i are real, while the ε_i and x_i are complex parameters. This parametrisation is the most general one for several $\mathbf{45}$'s, while it simplifies if one couples only one $\mathbf{45}$ containing the two necessary vev's since the anti-symmetric part of M_e and M_d is proportional to the same coupling, $Y^{1a}=Y^{2a}$. Replacing in eq. (3.88), one finds that $x_1=x_2=x_3=x$. We can simplify a bit more by assuming that $V_3=0$ since the real difference between quarks and leptons is introduced with V_{15}, which implies that x is fixed at the value $x=3$.

Let us keep the most general corrections to express the eigenvalues and analyse them. We will do so by exploiting the expressions of $M_x^\dagger M_x$ with $x=e,d$ and their characteristic polynomials :

$$\det\left(M_d^\dagger M_d-\lambda\mathrm{Id}\right)=-\lambda^3+b_x\lambda^2-c_x\lambda+d_x=0 \qquad (3.90)$$

since b, c and d can be expressed with the masses $m_{d,s,b}$ and $m_{e,\mu,\tau}$:

$$
\begin{aligned}
b_d &= m_b^2 + m_s^2 + m_d^2 \simeq m_b^2 && (3.91)\\
b_e &= m_\tau^2 + m_\mu^2 + m_e^2 \simeq m_\tau^2 && (3.92)\\
c_d &= m_b^2 m_s^2 + m_b^2 m_d^2 + m_s^2 m_d^2 \simeq m_b^2 m_s^2 && (3.93)\\
c_e &= m_\tau^2 m_\mu^2 + m_\tau^2 m_e^2 + m_\mu^2 m_e^2 \simeq m_\tau^2 m_\mu^2 && (3.94)\\
d_d &= m_b^2 m_s^2 m_d^2 && (3.95)\\
d_e &= m_\tau^2 m_\mu^2 m_e^2 && (3.96)
\end{aligned}
$$

We will not display the full expressions for b_x, c_x and d_x but we will make the assumption that since $M_{d,e}$ are hierarchical we can take the limit $y_3 \gg y_2 \gg y_1$ and $|\varepsilon_3| \gg |\varepsilon_2| \gg |\varepsilon_1|$. We can then approximate :

$$
\begin{aligned}
m_b^2 &\simeq (v_d)^2 (y_3^2 + 2|\varepsilon_3|^2) && (3.97)\\
m_\tau^2 &\simeq (v_d)^2 (y_3^2 + 2|x_3|^2|\varepsilon_3|^2) && (3.98)\\
m_b m_s &\simeq (v_d)^2 |y_3^2 + \varepsilon_3^2| && (3.99)\\
m_\tau m_\mu &\simeq (v_d)^2 |y_3^2 + x_3^2\varepsilon_3^2| && (3.100)
\end{aligned}
$$

From these expressions we will see quite easily that it is very hard to fit at the same time m_b and m_μ. This is the result of the difference at low $\tan\beta \sim 10$ of m_b and m_τ at the GUT scale. We remind that in this case $m_b(M_{GUT}) \simeq 0.98$ GeV while $m_\tau(M_{GUT}) \simeq 1.25$ GeV if we place all super partner masses at 1 TeV, thus $m_\tau^2 - m_b^2$ is not much smaller that m_τ^2. Roughly, we need $(|x_3|^2 - 1)|\varepsilon_3|^2 \simeq 0.3$ GeV^2 to fit correctly m_b and m_τ, while this would give too large a value for m_μ through the last equation. Even giving up any hierarchy in the parameters will not help much since the expressions are quite symmetric in the matrix elements, so any element large enough to fit $m_{b,\tau}$ would give a large contribution to m_μ.
As charged lepton masses are known with a very good accuracy, we want to fit them quite accurately. A numerical fit will give generally $m_b \gtrsim 1.16$ GeV, again for $\tan\beta = 10$. The case of the Standard Model is even worse since in this case $m_\tau(M_{GUT}) \simeq 1.70$ GeV while $m_b(M_{GUT}) \simeq 1.05$ GeV and the difference between the two masses is significantly larger. No values $m_b < 1.65$ GeV can be numerically obtained.

As far as $m_{d,e}$ are concerned there is no particular problem, so that any mass can be fitted correctly except for m_b. As the reasoning was made in the case of general corrections with several **45**'s, it is clear that the problem remains the same in the simpler cases with only one **45**. Actually it has been checked numerically that the success of the fit does not change between the simple case of a factor -3 in the off-diagonal terms or of several different factors $-x_i$.
Since the error on m_b is not large enough to accommodate the fit, one should invoke something else in order to be fully consistent. In the supersymmetric case these additional contributions can take the form of supersymmetric threshold corrections, as explained below.

3.4.3 SUSY threshold corrections

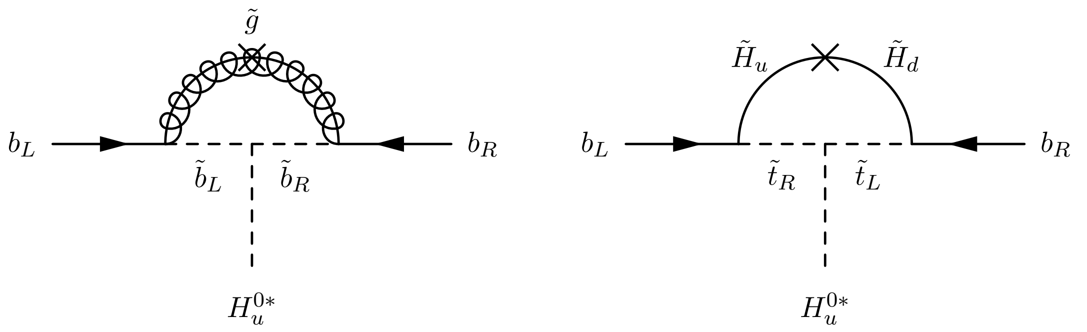
In SUSY, down quarks couple only to one of the two Higgs doublets at tree level, namely H_d. However this is not the case of their scalar partners, the down squarks. Since the superpotential contains :

$$W \supset \mu H_u H_d + Y_d Q D^c H_d \tag{3.101}$$

the scalar potential will contain itself :

$$V \supset \frac{\partial W}{\partial H_d}\frac{\partial W^*}{\partial H_d^*} + \text{h.c.} \supset Y_b \mu \tilde{q}_L \tilde{b}_R^* H_u^* \tag{3.102}$$

This coupling will generate an effective coupling between the down quarks, once SUSY breaking terms are taken into account, through the finite diagrams [82] :



(3.103)

computed in the mass insertion approximation and that we displayed for the b quark, for which the involved corrections are going to be the most important (since it is m_b that raises a problem in our fit). Below the SUSY scale, the Standard Model Lagrangian therefore contains the interaction terms :

$$\mathcal{L} \supset y_b \bar{b}_L b_R H_d^0 + \epsilon_b y_b \bar{b}_L b_R H_u^{0*} + \text{h.c.} \tag{3.104}$$

where ϵ_b is computed from the diagrams drawn in fig. 3.103 :

$$\epsilon_b = \frac{2\alpha_3}{3\pi}\frac{\mu M_3}{m_{\tilde{b}_R}^2} f(M_3^2, m_{\tilde{b}_L}^2, m_{\tilde{b}_R}^2) + \frac{y_t^2}{16\pi^2}\frac{\mu A_t}{m_{\tilde{b}_R}} f(\mu^2, m_{\tilde{t}_L}^2, m_{\tilde{t}_R}^2) \tag{3.105}$$

and the loop function f is expressed as :

$$f(m_1^2, m_2^2, m_3^2) = \left[\frac{m_1^2}{m_3^2 - m_1^2}\text{Ln}\left(\frac{m_1^2}{m_2^2}\right) - \frac{m_2^2}{m_3^2 - m_2^2}\text{Ln}\left(\frac{m_2^2}{m_3^2}\right)\right]\frac{m_3^2}{m_1^2 - m_2^2} \tag{3.106}$$

Since $2\alpha_3/(3\pi)$ and $y_t^2/(16\pi^2)$ are about a percent, ϵ_b is estimated generally around 2%. When the EW symmetry is broken, the b mass is :

$$m_b = y_b v_d + y_b \epsilon_b v_u = y_b(1 + \epsilon_b \tan\beta) v_d \tag{3.107}$$

We see that even if ϵ_b should induce a correction to m_b of the order of a few percent, it is enhanced by $\tan\beta$. Thus, for large $\tan\beta$, m_b receives corrections as large as 50% or more. As we consider theories with $\tan\beta \simeq 10$ the corrections can reach up to 20%. The quantity which runs above the SUSY scale is $y_b v_d$, and we need to enhance it since at the GUT scale $m_b < m_\tau$, so ϵ_b should be negative. Allowing the correction to be 20%, we reach a value $m_b(M_{GUT}) \simeq 1.08$ GeV which is still not enough. Playing with the errors on α_3 and m_t and going to larger $\tan\beta$ we can enhance it a little further.

Finally we can take into account the corrections to m_τ :

$$\epsilon_\tau \tan\beta \simeq \frac{\alpha_2}{8\pi}\frac{M_2\mu}{M_{SUSY}^2}\tan\beta \tag{3.108}$$

which is a few percent. Combining all this it will be possible to reconcile the fit in the preceding section with the data, supposing that the positive contribution coming from the neutrino Yukawa couplings in the running will not enhance $m_\tau(M_{GUT})$ too much.

3.5 Application to the Left-Right symmetric seesaw

Now that we have introduced all the different effects and corrections that have to be taken into account, it is time to display the final results for the different solutions of our Left-Right symmetric seesaw equation.

3.5.1 Inclusion of flavours

Let us first present the computations of y_B for the case where we take into account flavour effects and the decays of N_2, and integrate the BE's numerically but leave untouched the wrong $SO(10)$ mass relation $M_e = M_d$. We concentrate here on the four different cases $(+,+,+)$, $(+,-,+)$, $(+,+,-)$ and $(-,-,-)$, as we estimate that these cases are representative enough of the different behaviours of the spectra with respect to M_1 and M_2. Once again we display the observables as functions of the $B-L$ breaking scale v_R.
We fixed the reheating temperature $T_{RH} = 10^{11}$ GeV but we will comment on the variation of the results when T_{RH} is lowered to 10^{10} GeV. The initial conditions for the N_i are usually chosen to be dynamical since it is the more natural in our setup, given the range of our reheating temperatures. The benchmark case is taken as follows : θ_{12} and θ_{23} are taken as in [37] but θ_{13} is taken to be 0 and we will show the effect of a non-zero θ_{13} later. The spectrum is hierarchical with $m_1 = 10^{-3}$ eV and we will also show the influence of bigger m_1 in the following. Finally, the parameter $h = v_u^2/v_L v_R$ is taken to be 0.1 for practical reasons of numerical integration but a change in h amounts to a translation of the curves as functions of v_R and we refer to the comments in [37].

The curves of y_B for different choices of phases are shown in fig. 3.12. We displayed at the same time the results with flavour effects and the corresponding ones in the one flavour approximation, in order to estimate how valuable flavour effects are indeed in the success of leptogenesis in this context. We see that despite the encouraging values of ε_1 we had found in [37], the case $(+,-,+)$ is clearly not able to reach the WMAP bound on y_B. This is also the conclusion for the $(-,-,-)$ case, for which we could still have some hopes since N_2 was in the good range to generate a sufficient asymmetry. The $(+,+,-)$ case, however, is more encouraging, as we see that for specific choices of phases it is possible to reach the experimental bound. As for the last case $(+,+,+)$, it manages to give the correct value of y_B without any difficulty if v_R is not too big.

The striking difference between the one flavour approximation and the full flavour result in the cases $(-,-,-)$ and $(+,+,-)$ can be understood following the reasoning of [58, 60]. This is due to the fact that in these cases, N_1 is well separated from N_2 and creates very little CP-asymmetry compared to N_2. Therefore in these cases N_1 washes out exponentially any asymmetry created by N_2, down to its own level of CP violation, since we are in the case of a strong total washout ($\kappa_1 \gtrsim 10$). The equation for the one flavour approximation, when the CP-asymmetry created from N_1 is negligible compared to the asymmetry generated previously, reduces to :

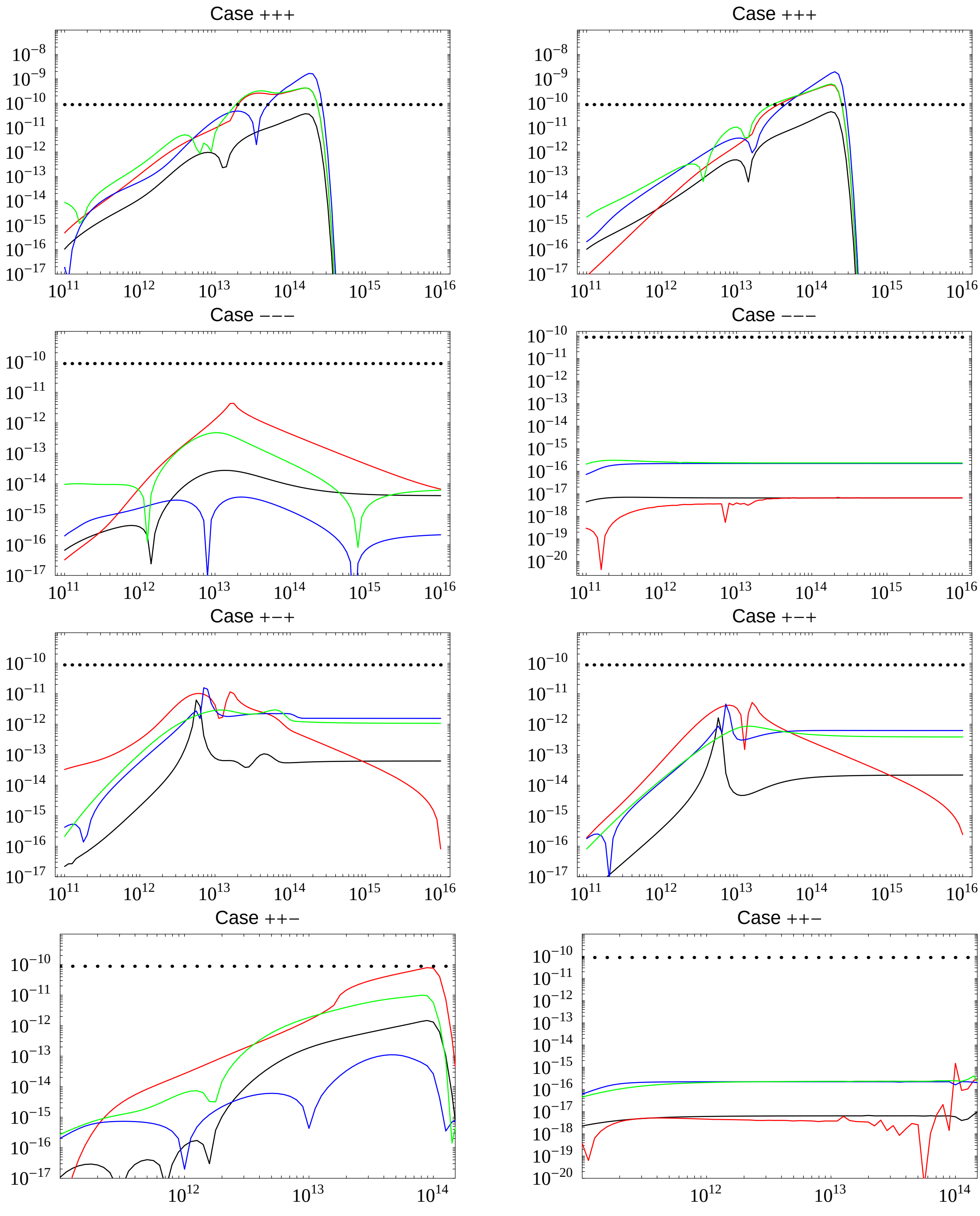


Figure 3.12: Baryon asymmetry y_B as a function of v_R for the solutions $(+,+,+)$, $(-,-,-)$, $(+,-,+)$ and $(+,+,-)$ with a hierarchical light neutrino mass spectrum with $m_1 = 10^{-3}$ eV, $\beta = \alpha$. The four cases are the reference case with no CP violation beyond δ_{CKM} (black), $\varphi_2^u = \pi/4$ (red), $\varphi_2^\nu = \pi/4$ (blue) and $\varphi_2^d = \pi/4$ (green). Left panel : flavour case; right panel : one flavour approximation. The dotted line is the WMAP measure.

$$Y'_{B-L}(z) \simeq -\kappa_1 f(z) Y_{B-L}(z) \qquad \Rightarrow \qquad Y^{fin}_{B-L} \simeq Y^{ini}_{B-L} \exp\left(-3\pi/8 \times \kappa_1\right) \tag{3.109}$$

with f a function of z going to 0 for $z \to 0$ and ∞, which means that if $\kappa_1 \gtrsim 10$, N_1 erases all the existing asymmetry down to its own level of production. However, in the full flavour regime, the equation for the $\Delta_\alpha = B - L_\alpha$ are :

$$Y'_{\Delta\alpha}(z) \simeq -\kappa_{1\alpha} W_1(z) \sum_\beta A_{\alpha\beta} Y_{\Delta\beta}(z) \qquad \Rightarrow \qquad Y^{fin}_{\Delta\alpha} \simeq Y^{ini}_{\Delta\alpha} \times e^{-\frac{3\pi}{8} A_{\alpha\alpha}\kappa_{1\alpha}} \tag{3.110}$$

therefore it is visible that if one of the $\kappa_{1,\alpha} \lesssim 1$, this flavour is not completely washed out by N_1. For the $(-,-,-)$ case, for example, the values of the different washout parameters at $v_R \simeq 10^{14}$ GeV are roughly :

$$\kappa_{1e} \simeq 3, \qquad \kappa_{1\mu} \simeq 10, \qquad \kappa_{1\tau} \simeq 1 \tag{3.111}$$

The evolution of Y_{B-L} in the one flavour approximation and of the $Y_{\Delta\alpha}$ in the flavour regime as functions of z for a fixed value of v_R are displayed in fig. 3.13 for $(-,-,-)$. In this plot we clearly see the first creation of CP-asymmetry due to the decay of N_2, and its further modification due to N_1 at $z \sim 1$. We note that if the matrix A was considered diagonal, the flavour μ for example should be washed away just as in the one flavour case. However, the smaller but non-zero off-diagonal elements of A, as they exhibit an opposite sign compared to their diagonal counterparts, will contribute to enhance the asymmetries. Thus, the term in $A_{\mu\tau}$, for example, will help stabilising the μ asymmetry well above its expected value in the case where we neglect the small entries of the matrix A.
The cases $(-,-,-)$ and $(+,+,-)$ cases are therefore striking illustrations of the importance of flavour effects as described in [58].

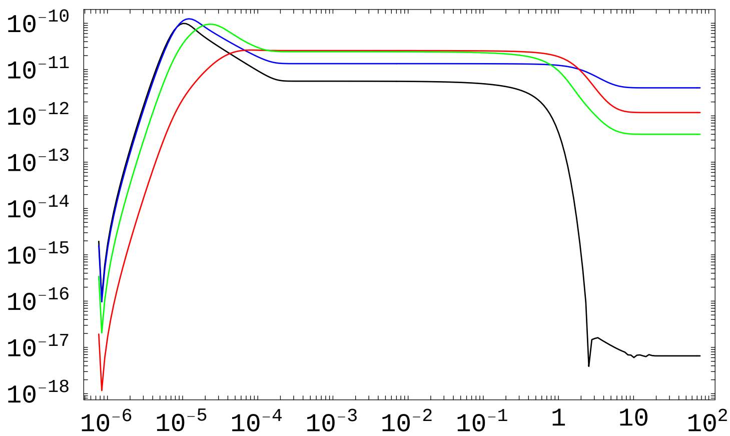


Figure 3.13: Evolution with $z = M_1/T$ of Y_{B-L} in the one flavour approximation (black) compared to the one of the different flavours $Y_{\Delta e}$ (red), $Y_{\Delta\mu}$ (green) and $Y_{\Delta\tau}$ (blue) in the full flavour regime. This result of numerical integration of the Botlzmann Equations was obtained for $v_R = 10^{13}$ GeV in the $(-,-,-)$ case and $\phi_2^u = \pi/4$.

3.5.2 Mass corrections

Now that we checked the role of flavour corrections to the final asymmetry we include the last correction due to the matrix U_m described in section 3.4 and parametrised in eq. (3.81). When fitting the fermion masses, there are basically two possibilities : either matrices M_e and M_d with hierarchical entries, $y_1 \ll y_2 \ll y_3$ and $|\varepsilon_1| \ll |\varepsilon_2| \ll |\varepsilon_3|$, in accordance with the flavour hierarchy in the quark and lepton masses, or solutions with $|\varepsilon_1| \sim |\varepsilon_2|$, implying a large $1-2$ mixing in U_m : $\theta_{12}^m \sim 1$, which might be a little less motivated at first sight but gives interesting results for certain cases. These different patterns of U_m come from the fact that there are more parameters than observables to fit, therefore we have some freedom in the fit to accommodate quark and lepton masses. In particular we have the freedom to choose the parameter ε_1 as we please up to about 0.01 GeV.

In fig. 3.14 we show for our four benchmark solutions the results with two different types of matrix U_m, to be compared with fig. 3.12. What is remarkable is the large enhancement of the U_m's with large 1-2 mixing (in blue) with respect to the more hierarchical structure (red) in some cases. This is partly because a large θ_{12}^m enhances the mass of the RH neutrino relevant to leptogenesis (for example M_1 in the $(+,-,+)$ case). The curves displayed are the most favourable ones we could obtain, although we did not try to scan or optimise the parameters intensively. Therefore we conclude that with our choice of benchmark inputs, $(-,-,-)$ is definitely not viable. However, we will see that when relaxing the constraint of $Y_\nu = Y_u$, even mildly, this failure can be circumvented.
The other three solutions, on the contrary, are found to be viable. This is no real surprise for $(+,+,+)$ which was already working before taking into account any corrections, but this is more than welcome for $(+,-,+)$, all the more since it almost reaches the WMAP constraint at large v_R up to a factor of 2 (and we will see that an order one variation of the Yukawa couplings can enhance y_B sufficiently to fill this gap). The other good surprise comes from $(+,+,-)$ which manages to reach the WMAP bound thanks to its increasing M_2, just before draticscally decreasing due to M_2 becoming larger than the reheating temperature. An appreciable fact is the success of leptogenesis even with small θ_{12}^m for this solution, which are the cases with less tuning in M_d and M_e.

A comment here is in order about the U_m configurations with large 1-2 mixing : checking the effect of U_m on the Lepton Flavour Violating rates $\mu \to e\gamma$ and $\tau \to \mu\gamma$, one can see that the former is greatly enhanced, and can reach the experimental bound for a generic superpartner spectrum, requiring in general heavy superpartners above the TeV. The red curves, nevertheless, lead to reasonable signals of LFV, unfortunately they do not yield a correct y_B for $(+,-,+)$.

3.5.3 Dependence on the different parameters

Dependence on the reheating temperature

Our last task to complete this study of leptogenesis is now to vary some of the parameters kept fixed up to now. The first parameter to relax is the reheating temperature, because of the gravitino problem. Indeed, we have until now supposed that $T_{RH} = 10^{11}$ GeV, which is clearly at odds with gravitino bounds. A more realistic temperature, potentially compatible with leptogenesis, would be $T_{RH} = 10^{10}$ GeV. In fig. 3.15 we plot the same curves as in 3.14 but with this lower reheating scale. The conclusion is now a little less optimistic for $(+,+,-)$ which is incapable of fulfilling the WMAP constraint. Even $(+,+,+)$ starts having difficulties since it needs $M_1 \sim 10^{10}$ GeV for a successful baryon asymmetry. Nevertheless, the interesting $(+,-,+)$ solution survives and once again even for large v_R, a region that, we remind, is better simultaneously for one step gauge coupling unification

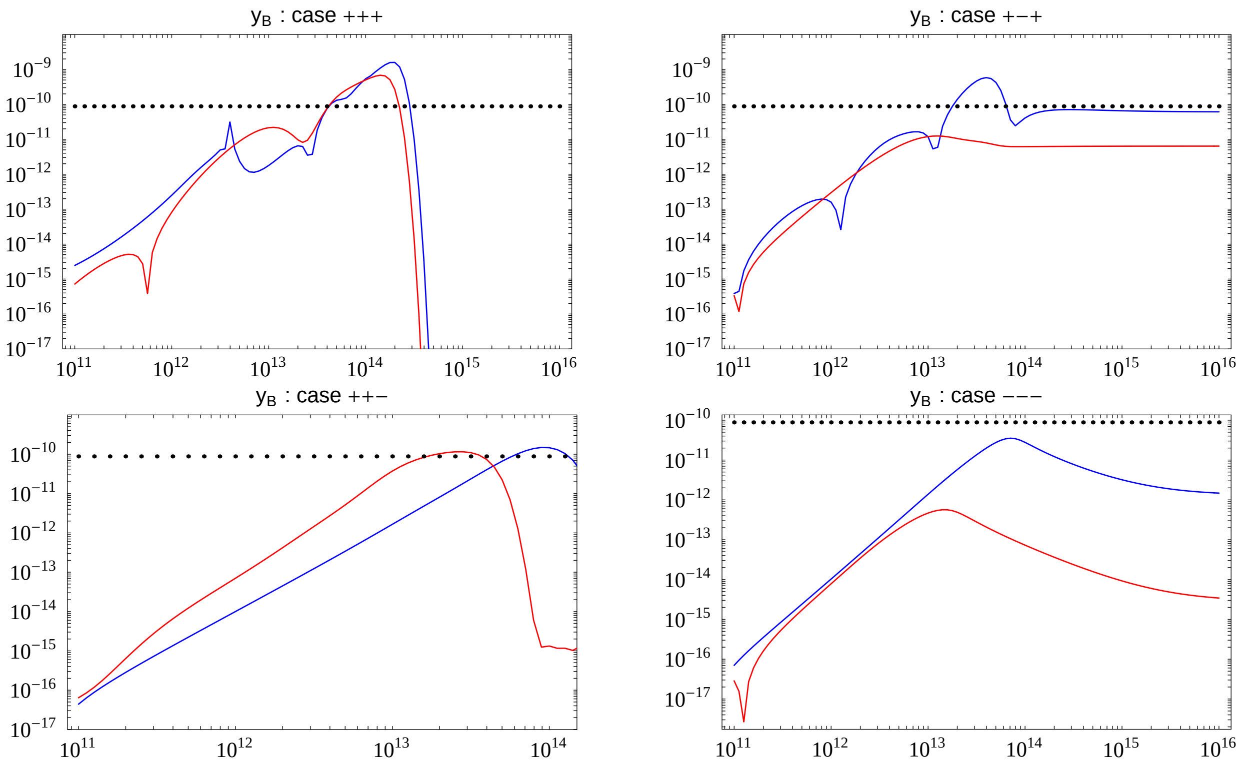


Figure 3.14: Baryon asymmetry y_B as a function of v_R for the solutions $(+,+,+)$, $(+,-,+)$, $(+,+,-)$ and $(-,-,-)$ with a hierarchical light neutrino mass spectrum, $m_1 = 10^{-3}$ eV and $\beta = \alpha$, at $T_{RH} = 10^{11}$ GeV. We display cases with a "natural" U_m (red) with $\theta^m_{12} \sim \theta^m_{23} \sim 0.3$ and $\theta^m_{13} \sim 0.1$, and a more "adjusted" one (blue) with $\theta^m_{12} \sim 1$, $\theta^m_{13} \sim \theta^m_{23} \sim 0.2$ and non-zero phases.

and for avoidance of any tuning between the type I and II seesaw contributions to m_ν.

Dependence on the light neutrino mass parameters

We have also chosen fixed $\theta_{13} = 0$ and $m_1 = 10^{-3}$ eV with a normal hierarchy. The influence of θ_{13}, when varying between 0 and its upper bound, can be rather important on the case $(+,-,+)$ and above all on $(-,-,-)$ which can be decreased by several orders of magnitude (see fig. 3.16). The preferred value to maximise y_B is the extreme value $\theta_{13} = 0$, with the notable exception of $(+,-,+)$ at large v_R, favouring a larger θ_{13}. This is why we chose θ_{13} for our benchmark set of input parameters.
The influence of a change in m_1 is felt mainly for $m_1 \gtrsim 10^{-3}$ eV and is also non-negligible because of the important impact of m_1 on the masses of the RH neutrinos. The generic effect of increasing m_1 is indeed to enhance the masses of a "+ branch" while decreasing those of a "- branch" when it reaches its plateau. Therefore M_1 in $(+,-,+)$ and M_2 in $(-,-,-)$ will decrease when m_1 is taken larger than our benchmark value and the asymmetry will go down, while the M_i will increase for $(+,+,+)$ so that y_B will reach the same maximum but it will do so at smaller v_R (since M_1 crosses the value of T_{RH} for smaller v_R). These characteristic features can be seen on fig. 3.17, while the dramatic fall of y_B at $m_1 \sim 10^{-3}$ in some cases is due to some flavours going from the weak to the strong washout regime (therefore preventing the possibility of N_2 leptogenesis for $(-,-,-)$, for example).

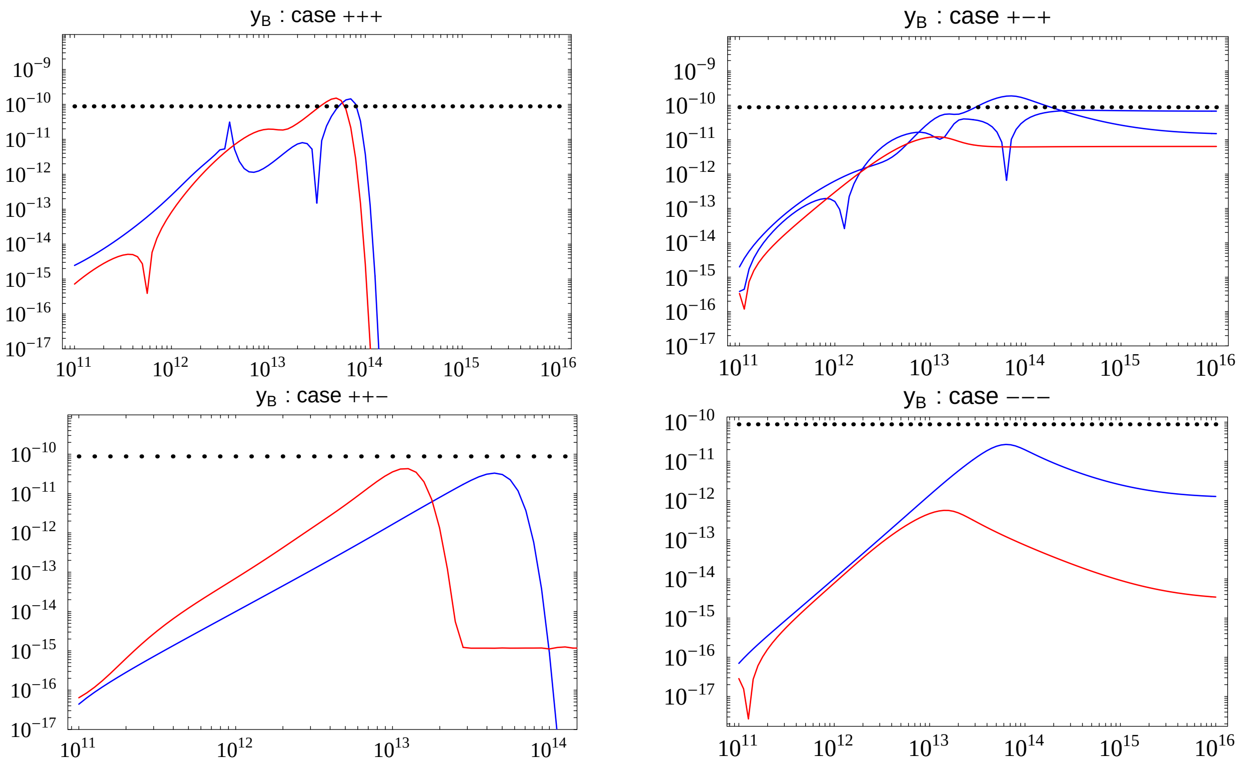


Figure 3.15: Same as fig. 3.14 but with $T_{RH} = 10^{10}$ GeV.

Dependence on the neutrino Yukawa coupling

Finally, a crucial input that we used to restrict the parameter space is the alignement between the up quark Yukawa coupling and the neutrino Yukawa coupling. Moreover, we stuck to the relation $Y_u = Y_\nu$ even when considering corrections to the mass matrices M_e and M_d. However, as was shown in appendix B of [37], some masses of the RH neutrinos in cases with "- branches" can be proportionnal to the square of an eigenvalue of Y_ν when the plateau is reached at large v_R. For example, in the case $(+,-,+)$ we have that $M_1 \propto y_2^2$ while also $M_2 \propto y_2^2$ for $(-,-,-)$. Therefore, a deviation of order one of the relation $y_2 = y_c$ could easily enhance or decrease the mass of the corresponding RH neutrinos by a factor of ten because of the square dependence. This fact is displayed in fig. 3.18 where y_B is displayed for several values of y_2 between $0.1y_c$ and $10y_c$. We perturbed only the eigenvalues and up to a factor of ten only, since in $SO(10)$ theories every contribution to Y_ν contributes also to Y_u with an order one CG coefficient, so that the eigenvalues of M_u and the Dirac mass of the neutrinos M_D should not differ by more than an $\mathcal{O}(1)$ factor[7]. From fig. 3.18 we can see two very interesting things. First, the case $(-,-,-)$ is now able to reach the experimental value thanks to the enhancement provided to M_2. However, let us stress that these curves were obtained for a reheating temperature of 10^{11} GeV, while for $T_{RH} = 10^{10}$ GeV it is not really possible to improve the conclusion we made earlier since $M_2 \sim 10^{10}$ GeV already for $y_2 = y_c$. Therefore, decreasing y_2 with respect to y_c will decrease

[7]To reach this conclusion, we suppose that V_{CKM} mainly comes from M_d, which is legitimate since it is less hierarchical than M_u. This implies that there is no tuning in M_u leading to unnaturally small m_u

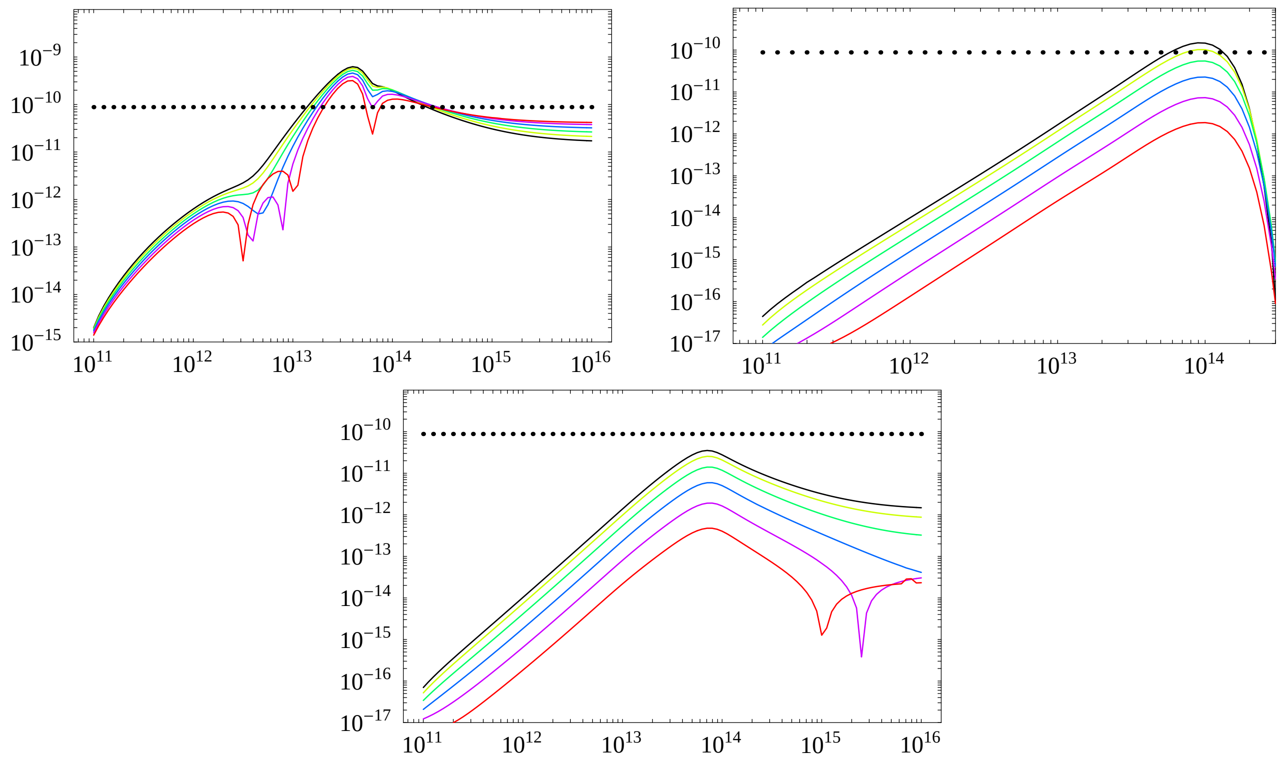


Figure 3.16: Different curves of y_B for various values of θ_{13} between 0 and 13°, for the cases $(+,-,+)$ (upper left), $(+,+,-)$ (upper right) and $(-,-,-)$. The benchmark value of 0° corresponds to the black curve and the red curve corresponds to the maximal value 13°. These curves are plotted with a $U_m \neq I_3$.

the CP asymmetry and increasing it will make $M_2 > T_{RH}$ yielding a Boltzmann suppressed baryon asymmetry. The second thing we should remark is that perturbing the Yukawa matrix can improve y_B in the $(+,-,+)$ case with a U_m matrix containing only small mixing angles ($\theta_{12}^m < 0.3$), which is very appreciable since it creates no tension with $\mu \to e\gamma$ and no tuning in the mass matrices M_e and M_d to obtain small first generation masses. Moreover, even if we only show the plot for a large T_{RH}, we checked that this conclusion holds even with $T_{RH} = 10^{10}$ GeV.

The non-SUSY case

Finally, because of the tension with the gravitino constraint, we should mention that the results of non-SUSY leptogenesis are usually comparable to that of the SUSY case. Nevertheless, the comparison stops when it comes to the mass corrections since the large difference between m_τ and m_b at M_{GUT} prevents one from using the kind of anti-symmetric corrections that we used here. One should then use symmetric corrections, with more free parameters and therefore less predictive in a sense.

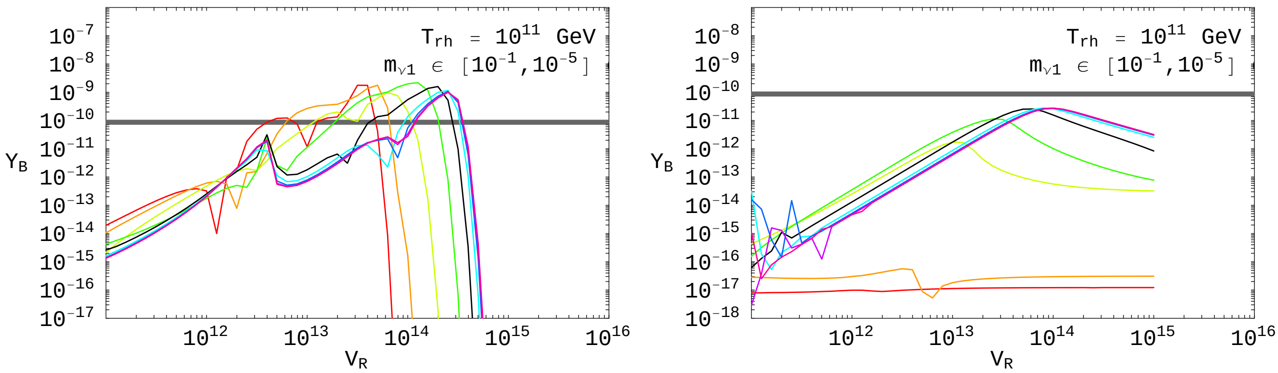


Figure 3.17: Different curves of y_B for various values of m_1 between 0.1 eV and 10^{-5} eV, for the cases $(+,+,+)$ (left) and $(-,-,-)$ (right). The benchmark value of 10^{-3} eV is the black curve in each plot and the red curve corresponds to the quasi-degenerate case 0.1 eV. These curves are plotted with a $U_m \neq I_3$.

3.6 Conclusion

The seesaw mechanism is a most widely used mechanism to generate neutrino masses in BSM models and is often used in its pure type I or type II version, since the type I+II situation seems a priori less predictive. Exploring from another point of view the pioneering work of [35], we proposed a method to investigate efficiently the phenomenology of certain Left-Right symmetric models of seesaw mechanism where couplings of type I and type II are linked. Extracting the mass matrix of the heavy RH neutrinos, one can then investigate the phenomenological consequences, for example concerning Lepton Flavour Violation and leptogenesis. This is what we started to do in [37] in a class of $SO(10)$ models where the neutrino Yukawa coupling is symmetric. The multiplicity of solutions available for the RH spectrum due to the structure of type I+II seesaw provides several interesting "mixed" cases where neither type of seesaw completely dominates over the other. Some of these cases feature a lightest or next to lightest RH neutrino with a mass in the good range to potentially create a sufficient baryon asymmetry complying with the WMAP constraint.

A more detailed study of leptogenesis in the relevant cases has been performed in [120], where we performed a numerical computation of the baryon asymmetry from the Boltzmann Equations, taking into account the full flavour regime. We showed that the flavour regime is extremely relevant to some cases where leptogenesis can be realised thanks to the decays of N_2 (if T_{RH} is favourable enough), whose asymmetry can only be mildly washed out by N_1 because of one flavour being mildly or weakly washed-out. Moreover, we perfected the $SO(10)$ setup presented in [37] by correcting the mass matrices in order to get realistic fermion masses. These corrections introduce further rotations in the flavour sector which lead to considerable enhancements of the final baryon asymmetry. Thanks to these corrections it was shown that an acceptable baryon asymmetry can be obtained for several solutions, even with a reheating temperature of 10^{10} GeV. With such a low reheating temperature, the pure type I seesaw in $SO(10)$, which we find as a limit of one of our solutions, exhibits serious difficulties for leptogenesis, justifying the interest of such type I+II setups.

Even if some solutions were found successful, we remind once again that a reheating temperature of 10^{10} GeV is clearly an upper limit on what is allowed by gravitino constraints in generic models of supersymmetry. However we also mention that some interesting ways out of the graivitino problem

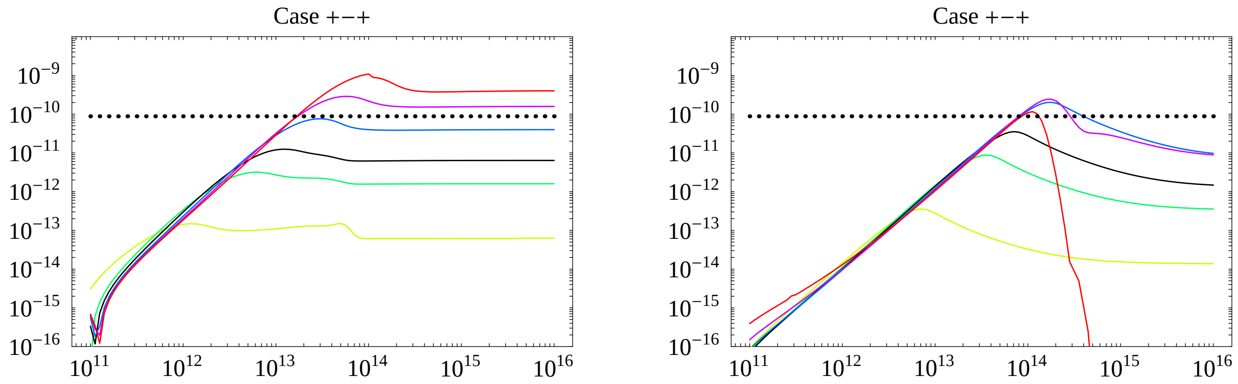


Figure 3.18: Different curves of y_B for various values of y_2 between $0.1y_c$ (yellow) and $10y_c$ (red) for the cases $(+,-,+)$ (left) and $(-,-,-)$ (right). The relation $y_2 = y_c$ used so far is displayed in black curve the red curve correspond to the quasi-degenerate case 0.1 eV. These curves are plotted with a $U_m \neq I_3$ with $\theta_{12}^m \simeq 1$ for $(-,-,-)$ and $\theta_{12}^m \simeq 0.17$ for $(+,-,+)$.

have been proposed, which would allow easily for $T_{RH} \sim 10^{11} GeV$ (see for example the ref. of [86] for mechanisms using either late entropy production due to a long lived particle or a small R-parity breaking). Therefore our choice of reheating temperatures can be embeded in realistic models of SUSY.

Finally, it should be interesting and instructive to construct a more concrete model of $SO(10)$ GUT realising this Left-Right symmetric seesaw mechanism, while at the same time yielding viable fermion masses, GUT breaking and doublet-triplet splitting. Complying with all the constraints a realistic GUT has to fulfill could certainly help testing further the viability of the new solutions exhibited for the RH neutrino mass matrix, as well as clarify the distinctive features of $SO(10)$ models with Left-Right symmetric seesaw.

Chapter 4

Neutrino Masses in 5D Models

4.1 Introduction

The addition of extra spatial dimensions to our usual 4D spacetime is a fascinating subject, not quite recent in fundamental physics. As discussed quite often in introductions to extra-dimensional theories, this idea was first proposed by Kaluza in 1921 and later investigated by Klein [95]. The original idea was to derive the electromagnetic force from the geometry of spacetime and therefore to unify General Relativity with the electromagnetic interaction. This was done by adding a fifth dimension, compact and small so that it would not be noticed in low energy experiments. As this extra-dimension is distinguished from the four usual ones, Lorentz invariance is broken and the metric tensor splits into irreducible representations of the 4D Poincare group. The part $g_{\mu\nu}$ with indices $\mu, \nu = 0...3$ becomes the new four-dimensional metric and its fluctuations stand for the usual spin 2 graviton. The degrees of freedom from the fifth column, however, do not form a spin 2 representation but a Lorentz vector, $A_\mu = g_{\mu 5}$, and a scalar $\phi = g_{55}$ which actually parametrises the size of the fifth dimension through the value of its vev $\langle\phi\rangle \sim R^{-1}$ with R the radius of the compact dimension.
However, at this time, this extra-dimension was thought to be an artificial way to perform calculations, with no real physical content. Indeed, after the works of Kaluza and Klein, this idea has been abandoned for decades before it was revived by high energy theoretical physicists. There is even a setup in which their introduction was required for the theory to be fully consistent, namely superstring theories, which call for the existence of at least six extra dimensions. The presence of these extra dimensions are required for a simple argument which goes as follows. In string theory the fundamental objects are not point particle but small one-dimensional objects, in other words strings, which evolve in spacetime along a two dimensional worldsheet. Their trajectory is therefore described by the coordinates $X^\mu(\sigma, \tau)$ with τ the time parameter of the string. The spectrum of excitations of the string is obtained by quantifying the X^μ's as bosonic fields living in a two dimensional space, and the string interactions in loop expansion is described by two dimensional surfaces. An important symmetry of this two-dimensional field theory is the conformal symmetry, which allows to classify the loop diagrams by their topological properties alone. The main problem arising when the X^μ's are quantified is that each X^μ contributes to the the anomaly breaking the conformal invariance and their number is fixed when one imposes the conservation of conformal symmetry at the quantum level[1]. Among the different theories attempting to quantise the gravitational interaction, string theory is the

[1]This is not completely true, models called "Non-critical string theories" exist where no extra spatial dimension is used to get rid of the conformal anomaly, but due to our lack of knowledge of these models, and to the fact that their phenomenology is not nearly as developed as the one of critical string theory, we will leave them aside in this manuscript.

more advanced and consistent one for the moment, so this is, for example, a strong motivation for exploring models with additional spacetime dimensions.
Therefore the idea of extra dimensions is somehow motivated by UV completions of the SM, and it has begun to develop more than a decade ago, without trying to build and investigate completely realistic string models (which is in itself a highly non-trivial task !) but from a field theoretical point of view. Models with extra dimensions quite diverse in size, number and shape[2] have been explored and provide frameworks with a very rich phenomenology.

In this part we will present a study of the neutrino sector in a 5D SUSY theory, where the extra-dimensional setup provides large quantum corrections to the Yukawa couplings. These quantum corrections can modify consequently the structure of the neutrino mass operator at high energy, and are thus potentially interesting to explore new flavour symmetries realised at a high scale and open more perspectives on flavour physics. To complete this study, we will need to introduce the Kaluza-Klein reduction of higher dimensional theories and the concept of orbifold compactification, which are basic tools needed for any analysis of extra-dimensional theories. We will use them first for a quick survey of interesting ideas that came up from higher-dimensional model building. Next we will present the basic facts one has to deal with when considering higher-dimensional supersymmetry. At last, we will present the way extra-dimensional (and therefore non-renormalisable) theories are treated when we want to compute quantum corrections to the Lagrangian parameters. Once we have surveyed all the necessary ingredients we will discuss the running of the neutrino effective mass operator and the differences between MSSM predictions in 4D and 5D for the neutrino mixing angles.

4.2 Extra dimensions

When one wants to introduce additional spacetime dimensions, the first question one has to address is : how is it that these dimensions were never seen ? If extra dimensions are to be introduced to extend the SM, they must not disturb the latter at low energy. Clearly we cannot switch brutally to 5D Minkowski spacetime, for example. The easier way out is the one of Kaluza : it is sufficient to consider that these dimensions have a finite length and that they are smaller than the precision of our experiments.

4.2.1 Compactification

Let us be a little more precise. We will consider a general situation where we add δ extra dimensions, and the spacetime is $d = 4 + \delta$ dimensional. However, quite often through this chapter we will restrict to $\delta = 1$ for reasons of simplicity, and because our work was limited to a $\delta = 1$ model. When indices run over the d-dimensional spacetime we will denote them with upper case latin indices $(M, N, ...)$, while when we restrict to the usual 4D spacetime we will use greek lower case greek indices $(\mu, \nu, ...)$, and lower case latin indices $(m, n, ...)$ are reserved for indices running on the δ compact dimensions. We split the spacetime coordinates as : $x^M = (x^\mu, y^m)$.
Now, call M the a priori non-compact manifold representing the δ dimensions. We can choose a discrete group G acting freely on M with operators $\zeta_g : M \to M$ for $g \in G$, which means that ζ_g has no fixed point in M except when g is the identity element of G. We can then construct a quotient manifold $C = M/G$ obtained from M by identifying all the points related by a G transformation :

[2] Which is still consistent in string theory since the size and shape of the extra dimensions is not fixed by any known mechanism for the moment.

$$\vec{y} \equiv \zeta_g(\vec{y}) \tag{4.1}$$

A consistent compactification will impose that the physics is the same on every point identified by this equivalence relation (equivalence classes are also called *orbits*), therefore, denoting the fields collectively as ϕ, the following equality must hold for any g :

$$\mathcal{L}[\phi(x,y)] = \mathcal{L}[\phi(x,\zeta_g(y))] \tag{4.2}$$

This condition will be verified if and only if the fields transform as :

$$\phi(x,\zeta_g(y)) = T_g\phi(x,y) \tag{4.3}$$

with T_g a transformation of any global symmetry of the Lagrangian. For a general T_g, this compactification is called *Scherk-Schwarz* compactification. For any $T_g \neq \mathrm{Id}$ some of the fields will be multivalued on C.

Kaluza-Klein reduction

We specialise now to the simple case of a scalar field Φ propagating in δ flat extra dimensions, so $M = \mathbb{R}^\delta$, $G = \mathbb{Z}$ and $C = T^\delta$ (T^δ is the δ-dimensional torus). The identification is made as :

$$y_i \equiv \zeta_{\vec{n}}(y_i) = y_i + 2\pi n_i R_i \tag{4.4}$$

The R_i's are the radii of the extra dimensions. Values of y_i are then constrained to the fundamental domain $]-\pi R_i, \pi R_i]$. Taking the transformation T_n from eq. (4.3) to be the identity for our scalar field : $\Phi(x,\zeta_{\vec{n}}(y)) = \Phi(x,y)$, the problem for the wave function of Φ is similar to the problem of a particle in a box with periodic boundary conditions in Quantum Mechanics so the momentum along y_i will be quantised. Periodicity of Φ along M allows us to decompose its wave function in Fourier series :

$$\Phi(x,y) = \frac{1}{\prod_i \sqrt{2\pi R_i}} \sum_{\vec{n}} \varphi^{(\vec{n})}(x) \exp\left(i \sum_i \frac{n_i y_i}{R_i}\right) \tag{4.5}$$

In order to interpret the theory from a 4D point of view, the strategy will be to integrate the action over the δ extra dimensions. The Lagrangian density is :

$$\mathcal{L}_d = \partial_M \Phi \partial^M \Phi^* - m_0^2 |\Phi|^2 + V(\Phi) \tag{4.6}$$

Introducing the expression (4.5) for Φ and performing the integration yields the 4D Lagrangian density :

$$\mathcal{L}_4 = \int_C \mathcal{L}_d = (\partial_\mu \varphi^{(\vec{n})})^* \partial^\mu \varphi^{(\vec{n})} - \sum_{\vec{n}} \left(m_0^2 + \sum_i \frac{n_i^2}{R_i^2}\right) |\varphi^{(\vec{n})}|^2 + V(\varphi^{(\vec{n})}) \tag{4.7}$$

The 4D effective theory is therefore the theory of an (a priori) infinite tower of states $\varphi^{(\vec{n})}$ with masses $m_{\vec{n}}^2 = m_0^2 + \sum n_i^2/R_i^2$. If we further restrict to the case of a single extra dimension compactified on a circle S^1 with radius R we obtain a tower of states $\varphi^{(n)}$ with square masses separated by a constant factor $1/R^2$. This is a particular feature of flat extra-dimensional theories.

Now if we probe physical phenomena with an energy $\mu > \sqrt{m_0^2 + 1/R^2}$ we will be able to give to our particle enough energy to develop a non-zero momentum along the extra dimension. This will appear in the experiment as an excitation of the well known low energy particle with the same quantum numbers. This reduction of the higher-dimensional theory is called a *Kaluza-Klein reduction* and the infinite tower of resonances as a Kaluza-Klein tower. Since no such tower has been seen in any experiment, it is sufficient to say that the largest radius R is still larger than a few hundred GeV to comply with experimental constraints. The massive states with $n > 0$ will decouple and we can truncate the tower to keep only the zero mode, which leaves the usual 4D scalar theory at low energy.

Gauge fields and fermions

When dealing with gauge or fermionic fields, the procedure will be roughly the same but a clear problem appears at once. Let us start with the gauge fields. As they are promoted to d-dimensional gauge fields they contain additional degrees of freedom $A_5 ... A_d$. Let us stick there again to the case of $\delta = 1$. When expanding in Fourier series and integrating the Lagrangian, the $F_{\mu 5} F^{\mu 5}$ term will create a mass term for the modes $A_\mu^{(n)}$ with $n \neq 0$, and a kinetic mixing between $A_\mu^{(n)}$ and $\partial^\mu A_5^{(n)}$. In fact, the $A_5^{(n)}$ components with $n \neq 0$ will serve as would-be Goldstone bosons for the massive vectors $A_\mu^{(n)}$ and be absorbed as longitudinal components. However the zero mode $A_5^{(0)}$ stays massless, therefore it is present in the spectrum even for energies $\mu < R^{-1}$ where it plays the role of a real scalar field. For several extra dimensions only one combination of the massive $A_m^{(\vec{n})}$ will be absorbed by the massive 4D vectors but the conclusion is the same.
Now switching to fermions we encounter a similar problem : any Lorentz group with $d \geq 5$ has larger spinorial representations than $SO(1,3)$. Let us specialise to $d = 5$ since the discussion for fermions is quite δ-dependant[3]. In 5D, we cannot define a chiral projector such as the four-dimensional γ_5 since it is now part of the 5D Clifford algebra $\Gamma^M = (\gamma^\mu, i\gamma_5)$. This is a well known property of odd-dimensional Special Orthogonal groups. Moreover, there is no possible definition of a Majorana condition that would reduce the number of degrees of freedom. Therefore the 5D spinors are $2^{[5/2]} = 4$ dimensional and when reduced to 4D split into a left-handed and a right-handed field. Both of them having a zero-mode, they can be joined into a Dirac field for each mode of the tower, including the lightest one.
Of course, the SM is constructed with chiral fermions and there is no massless scalar in the adjoint representation of a gauge group, so we have to find a way around this obstacle. The solution takes the form of an *orbifold compactification* or of a *braneworld* model.

4.2.2 Orbifold

Once we have compactified our extra dimensions on a compact manifold C, we can actually impose additional symmetries on the theory. Let us suppose that H is a discrete group with operators ζ_h acting on C, $\zeta_h : G \to G$, with $h \in H$ and they act *non* freely. In much the same way as before, this will identify points in the fundamental region and further reduce it :

$$y \equiv \zeta_h(y) \tag{4.8}$$
$$\phi(x, \zeta_h(y)) = Z_h \phi(x, y) \tag{4.9}$$

[3]In addition to the obvious dependence on the size of Lorentz group, the number of d.o.f. of a fermion depends on whether a chirality projector and/or a Majorana condition can be defined

Since H acts non freely, some points will be fixed points of the transformation ζ_h and become singular points on the manifold. This will break the translational invariance which was still present up to now (and was encoded in the fact that the interactions of the theory were conserving the KK number at each vertex, which is reminiscent of momentum conservation since $p_i = n_i/R_i$). More importantly, it will project the fields on some specific subsets of eigenfunctions of the kinetic operator and will help us solve our chirality problem.

Let us apply this to our 5D model compactified on $C = S^1$. We choose as a discrete group $H = \mathbb{Z}_2$ and obtain the orbifold $S^1/\mathbb{Z}_2$ which identifies $y \equiv -y$. It possesses two fixed points, $y = 0$ and $y = \pi R$ and reduces the fundamental domain to the segment $[0, \pi R[$. Applying twice the orbifold transformation : $\Phi(x, y) = Z^2\Phi(x, y)$ so Z's only eigenvalues are ± 1. Coming back to our scalar field, we have two possibilities for the action of Z :

$$\Phi(x, -y) = \Phi(x, y) \qquad \text{or :} \qquad \Phi(x, -y) = -\Phi(x, y) \tag{4.10}$$

To investigate the consequences of these two choices, it is more convenient to develop Φ on the sin and cos functions since they have a definite parity. This is expressed as :

$$Z_\Phi = +1 \qquad \Longrightarrow \qquad \Phi(x, y) = \frac{1}{\sqrt{\pi R}} \left[\varphi^{(0)}(x) + \sqrt{2} \sum_{n>0} \varphi^{(n)}(x) \cos\left(\frac{n}{R}\right) \right] \tag{4.11}$$

$$Z_\Phi = -1 \qquad \Longrightarrow \qquad \Phi(x, y) = \sqrt{\frac{2}{\pi R}} \sum_{n>0} \varphi^{(n)}(x) \sin\left(\frac{n}{R}\right) \tag{4.12}$$

Now going to gauge fields, the important thing to remark is that once we have chosen a parity for the components A_μ, then A_5 necessarily has the opposite one since it behaves like y. Choosing $Z_{A_\mu} = +1$, from eq. (4.11) we will keep a massless zero-mode for the four-dimensional gauge field while A_5 has no massless component and its massive components are eaten by the massive $A_\mu^{(n)}$. The analysis is similar for the fermions. The kinetic part of the Lagrangian is :

$$\mathcal{L}_{kin} \supset \bar{\Psi} i \Gamma^M \partial_M \Psi - m \bar{\Psi} \Psi \tag{4.13}$$

Since $\partial_y \to -\partial_y$ the term $\bar{\Psi} i \Gamma^5 \Psi = -\bar{\Psi} \gamma_5 \Psi$ must be odd. Therefore fermion fields transform as :

$$\Psi(x, -y) = \pm \gamma_5 \Psi(x, y) \tag{4.14}$$

and the two chiralities will have opposite parities. Therefore when decoupling the massive states at low energy, only one chirality is left and we can recover the SM particle content. The 4D effective Lagrangian for our extra-dimensional fermion reads :

$$\mathcal{L} = \bar{\psi}_L^{(0)} i \gamma^\mu \partial_\mu \psi_L^{(0)} + \sum_{n>0} \left[\bar{\psi}_L^{(n)} i \gamma^\mu \partial_\mu \psi_L^{(n)} + \bar{\psi}_R^{(n)} i \gamma^\mu \partial_\mu \psi_R^{(n)} + \left(\frac{n}{R} \bar{\psi}_L^{(n)} \psi_R^{(n)} + \text{h.c.} \right) \right] \tag{4.15}$$

which is clearly the Lagrangian of a Dirac fermion $\psi^{(n)} = \psi_L^{(n)} + \psi_R^{(n)}$ for the massive excitations and of a Weyl fermion for the zero mode. We note that the 5D mass term has disappeared since it is odd under the orbifold action.

This analysis can be straightforwardly generalised to theories with a larger number of extra dimensions. For a two-torus T^2 for example, there is a larger choice of possibilities for the discrete transformation group H, assuming that the two radii are equal, $R_1 = R_2 = R$. Imposing that the torus lattice on which we compactify be transformed to itself through orbifold operations, we are left with the following list : $\mathbb{Z}_2$, $\mathbb{Z}_3$, $\mathbb{Z}_4$ and $\mathbb{Z}_6$.
As pointed out earlier, the translational invariance along the XD's is now broken, so there is no more conservation of KK number in each interaction term. However, a residual KK parity remains and each KK mode is assigned a parity $(-1)^n$ ($\prod_i(-1)^{n_i}$ for $\delta > 1$). Therefore, it is not sufficient to reach an energy $\mu = R^{-1}$ to produce the first excited states of a KK theory, we would have to go to $\mu = 2/R$ in order to produce either two states with $n = 1$ or a state with $n = 2$.

4.2.3 Branes and field localisation

The other way around a non-chiral low energy effective field theory is to localise the fermions on a three-dimensional subspace of the d-dimensional bulk. In string inspired models this can be done through the introduction of higher dimensional objects called branes. Branes are solitonic objects which fill some subspace of the d-dimensional bulk. Of course, those that are relevant for model building are those branes with three spatial dimension (or more). This will be relevant to us in the following since we will have to localise some fields on a three-dimensional space, therefore we will take some time to describe the possibilities for localising fermions in field and in string theories.

Field theory mechanism

A mechanism known as *Split fermions* has been proposed in [119] (and reviewed in the second reference of [96]) to split the right and left components of 5D fermions. It relies on the particular profile of a boson Φ along the fifth dimension. Introducing this boson with a coupling λ to our fermion Ψ :

$$\mathcal{L} = \bar{\Psi}(i\Gamma^M\partial_M - m + \lambda\Phi(y))\Psi \tag{4.16}$$

Now let us take a closer look at the massless zero-mode $\Psi^{(0)}$ which should obey : $\imath\gamma^\mu\partial_\mu\Psi = 0$. The eigenvalue equation for the two chiralities is therefore :

$$[\pm\partial_y - m + \lambda\Phi(y)]\Psi^{(0)}_{L,R} = 0 \tag{4.17}$$

which we can formally solve :

$$\Psi^{(0)}_{L,R} \propto \exp\left[\mp\int_0^y(\lambda\Phi(y') - m)\mathrm{d}y'\right] \tag{4.18}$$

Now, choosing carefully the shape of Φ along y, there is a way to get rid of $\Psi^{(0)}_R$ making its wave function non-normalisable, and to localise $\Psi^{(0)}_L$ in a particular region. This can be achieved with a linear profile $\Phi(y) \sim 2\mu^2 y$ (at least on a sufficient domain). The wave function for the LH zero mode is then :

$$\Psi^{(0)}_L \propto \exp\left[-\left(\mu y + \frac{m}{2\mu}\right)^2\right] \tag{4.19}$$

and it is peaked around $y = m/(2\mu^2)$. In the case of several fermions, they will be localised around different points according to their mass term and couplings to Φ. The other beneficial feature of this model is to explain naturally the hierarchy in the Yukawa sector, due to the exponentially small intersections of the zero mode wave functions.

D-Branes

A more popular mechanism for field localisations is the use of D-branes. These objects have been found to arise naturally in string theory when studying the strong coupling regime [99]. They are the natural extensions of strings to two or more dimensions. An interesting fact is that open strings, instead of evolving freely in ten dimensions, can have their ends attached to these solitonic objects (the ends of the string obey Dirichlet boundary conditions). The low energy degrees of freedom described by these strings are seen as fields living in a reduced spacetime dimensionality. Thus, a fermion described by an open string attached to a three-dimensional brane is a spinorial representation of $SO(1,3)$ and is naturally chiral. As solitonic objects, branes can also exist in field theory and have been studied in the context of Supergravity. They have been extensively used to build two of the main extra-dimensional setups that we will briefly describe in the following as an illustration to what XD models can help us solve or bring interesting perspectives on.
A last thing to describe concerning the localisation of states in string theory is the concept of a twisted state. We saw that open strings can be constrained to move on higher-dimensional objects, but this possibility is clearly forbidden to any closed string state. However, when a closed string leaves in an orbifold, with singular fixed points, and it surrounds one of the fixed points, it cannot cross the singularity and is compelled to stay localised around the fixed point. Concretely, it cannot have KK excitations, and its low energy degrees of freedom are seen as living on the fixed point, just as open strings would live on a brane. Therefore, every string theory can accommodate localised states, justifying that we localise our fields freely when building an extra-dimensional model (see for example section 7 of [97]).

4.2.4 New possibilities from extra dimensions

In order to conclude this short introduction to extra dimensions, we will outline some of the appealing possibilities that have been proposed to answer some of the main problems of BSM physics. The two classical examples developed usually are devoted to solving the hierarchy problem between the EW scale and the Planck scale.

Large extra dimensions

The model of Arkani-Hamed, Dimopoulos and Dvali (ADD) [124] consists in using the possibility offered by branes to localise the SM fields on a three-dimensional subspace. If the fields with which we probe the physics do not directly feel the extra-dimensions, we can greatly relax the experimental constraint $R^{-1} \gtrsim$ few GeV evoked so far. The only sector which is bound to propagate in the whole bulk of extra dimensions (for obvious reasons) is the gravitational one. If gravitons propagate in the bulk they have KK excitations and additional massive gravitons will mediate gravity interactions. Gravity is thus strengthened between massive bodies, and the Newton law is extended by a tower of Yukawa interactions. Tests of the gravitational laws at microscopic scales is quite arduous due to the weakness of gravity as compared to other forces and Newton's law has been probed down to about 0.1 mm. This opens a possibility for models with dimensions much larger than previously believed. Now, let M_* be the fundamental scale of gravity. Using the fact that for flat extra dimensions, the Ricci scalar verifies $R^{(d)} = R^{(4)}$, dimensional reduction of the action gives :

$$S = -M_*^{d-2} \int \mathrm{d}^d x \sqrt{g^{(d)}} R^{(d)} = -M_*^{d-2} V_\delta \int \mathrm{d}^4 x \sqrt{g^{(4)}} R^{(4)} \tag{4.20}$$

Thus, the 4D Planck scale is given by $M_P^2 = M_*^{2+\delta} V_\delta$. If the volume of the extra dimensions is extremely large compared to the 4D Planck scale, M_* can be significantly reduced compared to M_P. Taking δ and the radii R_i sufficiently large we can go down to $M_* \sim$ a few TeV. The hierarchy between the EW scale and the fundamental scale of quantum gravity does not exist anymore.

There is also a way to explain geometrically the origin of small neutrino masses, using the property stressed earlier that the contraction of lepton doublets with the Higgs field is gauge invariant and can couple to any gauge singlet of the theory. If the SM is restricted to a 3-brane, this might not be the case for gauge singlet fermions ψ. We can then write a coupling $\tilde{y} l H \psi$. The coupling $\tilde{y}$ is dimensionful : if it is generated by gravitational interactions, it can be suppressed by a power of M_*. When dimensionally reducing the action, the effective 4D coupling is suppressed by a certain power of $M_* \times V_\delta^{1/\delta}$ and the smallness of the neutrinos is explained by the suppression of a large volume.

Curved or flat ?

Another extremely popular model is the one of Randall and Sundrum (RS) [125]. It relies on the fact that unlike the three large space dimensions that we know, extra dimensions can have a more complicated geometry. The original RS model is a 5D model based on a slice of AdS_5 bounded by two branes, with the SM field content restricted to one of these branes. The metric along y is not chosen flat but *warped* :

$$\mathrm{d}s^2 = e^{-2k|y|} \mathrm{d}x^\mu \mathrm{d}x^\nu \eta_{\mu\nu} - \mathrm{d}y^2 \tag{4.21}$$

Reducing the 5D action to the effective 4D action will make the exponential factor appear explicitely. Canonically redefining the Higgs field will also redefine its vev :

$$v = e^{-kL} \tilde{v} \tag{4.22}$$

where $\tilde{v}$ is the 5D Higgs vev which is expected to be of the order of the fundamental Planck scale M_*, L is the size of the extra dimension and k^{-1} is the Anti de Sitter radius. A value $kL \sim 30$ is sufficient to explain the 16 orders of magnitude between the EW and the Planck scales.

As the model we are going to investigate considers only a flat XD we will not describe the RS models in any great details. However there is one more interesting possibility worth mentioning. If one relaxes the constraint to put all SM fields on a brane but leave only the Higgs field to extremely localised, it appears that the fermions zero modes have peaked profiles along y, therefore we can explain the Yukawa hierarchies in the same way as for the Split fermion scenario outlined above.

Gauge symmetry breaking and geometrical origin

Eventually, geometrical features of XDs can be used to break gauge symmetries, instead of resorting to tachyonic mass terms. The process is actually simple. Instead of using only the orbifold symmetry to project out the zero mode of the extra components of the gauge fields, we can add another $\mathbb{Z}_2'$ parity that will act as $y \to \pi R - y$ (in the case of the $S^1/\mathbb{Z}_2$ model described earlier). Then, we can choose some of the gauge components A_M^a to be odd under this additional parity. Those that are odd under the two parities (for example the X and Y bosons in the **24** of $SU(5)$, see eq. (2.46)) will not possess any zero mode, including for the 4D vector part A^a_μ. Under the scale R^{-1}, only a part of the gauge bosons will remain in the spectrum and the gauge group is broken [126].

A last point with highly appealing aesthetical features is the geometrical possible origin of gauge symmetries. This was the original motivation of the model of Kaluza and Klein introduced at the beginning of this section and logically reappeared in heterotic string models [127]. Heterotic models start from closed strings with bosonic left movers and supersymmetric right movers, which is technically possible since the left and right propagating excitations on a closed string are independent of each other. However, bosonic strings have to propagate in 26 dimensions while supersymmetric ones only need 10 spacetime dimensions. This problem is resolved by considering the 16 extra spatial coordinates of the bosonic string as "internal" degrees of freedom, compactified on a sixteen-dimensional torus. In the case that the 16 dimensions are orthogonal, and $T^{16} = (S^1)^{16}$, we would obtain, by dimensional reduction of the graviton, sixteen $U(1)$ gauge bosons. However, an internal symmetry of string theory, namely modular invariance, leaves only the possibility to compactify on the root lattice of the Lie groups $E_8 \times E_8$ or $SO(32)$. Analysing the massless spectrum of the string, we can see that the low energy spin 1 fields obtained from dimensional reduction form an adjoint representation of these gauge groups, which are more than large enough to (potentially) recover the SM at low energy. In this setup, the origin of non-abelian gauge groups can be traced back to the isometries of spacetime.

4.3 Supersymmetry in 5 dimensions

4.3.1 $\mathcal{N} = 1$ SUSY in 5D as $\mathcal{N} = 2$ SUSY in 4D

Following [103, 104, 105], we will now give an account of SUSY theories in 5D. Compared to the usual 4D theories, extra-dimensional SUSY theories have very distinctive features, due to the property of spinor representations of the higher dimensional Lorentz group. We have already seen in the previous section that there is an immediate problem, when compactifying from 5D to 4D for example, to be left with chiral fermions, since chirality cannot be defined in 5D. The problem here will of course reappear under the form of extended SUSY. It is seen when one tries to write a SUSY generator, which has to transform as a 5D spinor, meaning that it has four complex components, that is, eight supercharges, while the MSSM consists of four supercharges. It means that when we compactify on a circle, we are left with another SUSY generator and from the 4D point of view, one is left with an extended $\mathcal{N} = 2$ SUSY algebra and the generator are Q^I_α and $\bar{Q}^I_{\dot{\alpha}}$ with $I = \{1, 2\}$.

Now, we can act on any field of a given spin with both types of SUSY generator, so that the SUSY transformations link even more fields than in the $\mathcal{N} = 1$ case that we developed up to now. Therefore, the multiplets will contain more components fields, twice as much as in $\mathcal{N} = 1$ actually, since the fermionic d.o.f. are doubled. We will not write the theory in terms of a pure $\mathcal{N} = 2$ superspace since this requires an additional tool, the Harmonic Superspace. However, following the literature already mentioned, we will decompose the $\mathcal{N} = 2$ multiplets into $\mathcal{N} = 1$, which is convenient since we will have to break $\mathcal{N} = 2$ to $\mathcal{N} = 1$, for $\mathcal{N} = 2$ is a non-chiral theory, with fermion components being Dirac.
A particularity of $\mathcal{N} = 2$ SUSY (with no central charge U or V) is the presence of a symmetry under an $SU(2)$ transformation on the supercharges Q^1 and Q^2. This is the extension of the $U(1)$ R-symmetry of $\mathcal{N} = 1$. It implies that the particle content has to behave as representations of this $SU(2)_R$ symmetry. We will see later that it has important consequences on the possible interactions of the theory.

4.3.2 $\mathcal{N} = 2$ representations

Let us explain a little more the content of the basic $\mathcal{N} = 2$ representations and the embedding of our well-known chiral and vector multiplets.

Vector multiplet

In order to build our gauge theory we start with the extension of vector multiplets. A vector boson A_M in 5D contains three d.o.f. on-shell and four off-shell. Moreover, their fermionic partners λ contain four d.o.f. on-shell and eight off-shell. In order to complete the multiplet we must add a real scalar Σ and three real auxiliary scalars X^a, $a = 1...3$, all of them in the adjoint representation of the gauge group. To display explicitely the behaviour of the components under the $SU(2)_R$ it proves more convenient to write the gaugino λ as two symplectic Majorana spinors λ^i. They form a fundamental doublet of $SU(2)_R$ and are linked by :

$$\lambda^i = \varepsilon^{ij} C \bar{\lambda}_j^T \tag{4.23}$$

with C the usual 4D charge conjugation operator. The vector and real scalar bosons A_M and Σ are singlets under the R-symmetry, while the auxiliary fields form a triplet. The action for the gauge multiplet is written as :

$$\mathcal{L} = \frac{\mathrm{Tr}}{C_2(G)} \left[-\frac{1}{2} F_{MN}^2 - (D_M \Sigma)^2 - \bar{\lambda}_i \Gamma^M D_M \lambda^i + (X^a)^2 + \bar{\lambda}_i [\Sigma, \lambda^i] \right] \tag{4.24}$$

and the SUSY transformation for the components is [104] :

$$\begin{aligned}
\delta_\xi A^M &= i \bar{\xi}_i \Gamma^M \lambda^i && (4.25)\\
\delta_\xi \Sigma &= i \bar{\xi}_i \lambda^i && (4.26)\\
\delta_\xi \lambda^i &= \left(\Gamma^{MN} F_{MN} + \Gamma^M D_M \Sigma \right) \xi^i + i (X^a \sigma^a)^i_j \xi^j && (4.27)\\
\delta_\xi X^a &= \bar{\xi}_i (\sigma^a)^i_j \Gamma^M D_M \lambda^j + i [\Sigma, \bar{\xi}_i (\sigma^a)^i_j \lambda^j] && (4.28)
\end{aligned}$$

These equations allow to identify fields that transform collectively as a 4D vector superfield, namely $(A_\mu, \lambda_L, X^3 - D_5 \Sigma) = V$. The link between the symplectic and the Weyl decomposition of λ is given as :

$$\lambda^1 = \begin{pmatrix} \lambda_L \\ \bar{\lambda}_R \end{pmatrix} \qquad \lambda^2 = \begin{pmatrix} \lambda_R \\ -\bar{\lambda}_L \end{pmatrix} \tag{4.29}$$

The remaining fields transform as a chiral superfield $\chi = (\Sigma + i A_5, -i\sqrt{2}\lambda_R, X^1 + i X^2)$. The scalar field Σ participates at the same time to the auxiliary part of the $\mathcal{N} = 1$ vector multiplet and to the scalar part of the chiral multiplet. Under supergauge transformations, χ will transform as :

$$\chi \to e^{-ig\Lambda} (\partial_y + \chi) e^{ig\Lambda} \tag{4.30}$$

We have thus split the $\mathcal{N} = 2$ vector multiplet into an $\mathcal{N} = 1$ vector multiplet and a chiral multiplet.

Hypermultiplet

Let us now switch to the hypermultiplet, that is, the natural extension of chiral and antichiral multiplets. It is built from a Dirac spinor ψ, and consists, as above, of four d.o.f. on-shell and eight off-shell. Therefore we must add two complex scalars φ^i to form an on-shell multiplet and two other auxiliary complex scalars F^i to close the multiplet off-shell. The Lagrangian is written as :

$$\begin{aligned}\mathcal{L} &= -(D_M\varphi_i)^\dagger(D^M\varphi^i) - i\bar{\psi}\Gamma^M D_M\psi + F_i^\dagger F^i - \bar{\psi}\Sigma\psi + \varphi_i^\dagger(\sigma^a X^a)^i_j\varphi^j \\ &\quad +\varphi_i^\dagger\Sigma^2\varphi^i + (i\sqrt{2}\bar{\psi}\lambda^i\varepsilon_{ij}\varphi^j + \text{h.c.}) \end{aligned} \tag{4.31}$$

The two scalar fields φ^i, as well as the auxiliary fields, transform as doublets of the $SU(2)_R$. As for the spinor itself, it is a singlet. The SUSY transformations from which we will form the $\mathcal{N}=1$ supermultiplets are :

$$\delta_\xi\varphi^i = -\sqrt{2}\varepsilon^{ij}\bar{\xi}_j\psi \tag{4.32}$$
$$\delta_\xi\psi = i\sqrt{2}\Gamma^M D_M\varphi^i\varepsilon_{ij}\xi^j - \sqrt{2}\Sigma\varphi^i\epsilon_{ij}\xi^j + \sqrt{2}F_i\xi^i \tag{4.33}$$
$$\delta_\xi F_i = i\sqrt{2}\bar{\xi}_i\Gamma^M D_M\psi + \sqrt{2}\bar{\xi}_i\Sigma\psi - 2i\bar{\xi}_i\lambda^j\varepsilon jk\varphi^j \tag{4.34}$$

Therefore the grouping is done in two chiral superfields, $\Phi = (\varphi^1, \psi_L, F_1 + D_5\varphi^2 - \Sigma\varphi^2)$ and $\Phi^c = (\varphi_2^\dagger, \psi_R, -F_2^\dagger - D_5\varphi_1^\dagger - \varphi_1^\dagger\Sigma)$. Φ and Φ^c have opposite quantum numbers.

We can write the full Lagrangian in superfield notation and in a 5D invariant way, introducing $\nabla_5 = \partial_y + \chi$:

$$\begin{aligned}\mathcal{L} &= \int \mathrm{d}^8 z \left\{ \frac{\mathrm{Tr}}{2C_2(G)}\left[W^\alpha W_\alpha\,\delta^2(\bar{\theta}) + \text{h.c.}) + (e^{-2gV}\nabla_5 e^{2gV})^2\right] \right. \\ &\qquad \left. +\bar{\Phi}e^{2gV}\Phi + \Phi^c e^{-2gV}\bar{\Phi}^c + (\Phi^c\nabla_5\Phi\,\delta^2(\bar{\theta}) + \text{h.c.}) \right\}\end{aligned} \tag{4.35}$$

When extending the MSSM to 5d, we see that we have to increase the field content. For every vector multiplet that we have in the MSSM we will include an adjoint chiral multiplet. For every chiral multiplet present we have to add another chiral multiplet in the conjugate representation of the gauge group.
The last expression (4.35) is useful to guess the consistency condition of the orbifold projection that we will perform to break the $\mathcal{N}=2$ SUSY to $\mathcal{N}=1$. Obviously, as χ contains A_5, it will transform with an opposite parity compared to V. The same situation arises for the chiral multiplets since they couple with a covariant derivative along the fifth coordinate. Therefore, the orbifold action will project out the zero modes of χ and Φ^c, for example.

4.3.3 R-symmetry and the Yukawa problem

The previous 5D Lagrangians have been written for one hypermultiplet charged under some gauge symmetry. However, when extending the particle content it is most easily seen that Yukawa interactions between several hypermultiplets cannot be written in a fully SUSY invariant way. This is a direct consequence of the $SU(2)_R$ symmetry of the theory. As we displayed earlier, the different components of a supermultiplet carry different $SU(2)_R$ spins. For the chiral multiplets descending from a hypermultiplet, for example, the scalars form a doublet φ^i while the fermions are singlets. Now, the terms that we have to recover at low energy are the Yukawa interactions of the form $\psi\psi\varphi^i$ which is obviously not $SU(2)_R$ invariant. The conclusion is that a naive $\mathcal{N}=2$ theory describes the gauge interaction of a vector multiplet with itself and matter hypermultiplets and is a non-chiral theory.

To circumvent this problem we will have to use the possibilities provided by the compactification tools that were developed earlier. The chirality of the theory at low energy is of course recovered by the use of an orbifold projection. Moreover we will have two possibilities for chiral superfields : either localising all of them on a three-dimensional brane, or letting some of them propagating in the bulk of the extra dimensions while localising their Yukawa interactions on a three-brane, where the SUSY algebra is no longer compelled to display an extended form. It means that we will introduce a term of the form[4] :

$$\int \mathrm{d}^8 z \mathrm{d}y \delta(y) \frac{\tilde{\lambda}_{ijk}}{6} \Phi_i \Phi_j \Phi_k \tag{4.36}$$

This localised term will have important consequences on the calculability of the theory, depending on whether the fields themselves are localised or not.

4.4 Quantum corrections in flat extra-dimensional models

It is now time to come to the very interesting opportunity of flat extra-dimensional models to have an accelerated running compared to renormalisable four-dimensional models. This is why we analyse higher-dimensional theories as Effective Field Theories and recall the usual interpretation of renormalisation in this context [97].

4.4.1 Non-renormalisability of extra-dimensional field theories

The first serious problem that appears when treating higher-dimensional theories quantum mechanically comes from the new dimensionality of their couplings. This stems from the fact that the fields themselves are assigned a different mass dimension. For a theory in a general d-dimensional spacetime, we impose that the action $S = \int \mathrm{d}^d x \partial_M \varphi^\dagger \partial^M \varphi$ or $S = \int \mathrm{d}^d x \bar{\psi} i \Gamma_M \partial^M \psi$ has mass dimension zero, so the bosonic and fermionic fields inherit a mass dimension :

$$[\varphi] = \frac{d-2}{2} \qquad [\psi] = \frac{d-1}{2} \tag{4.37}$$

and the dimension increases with the number of extra dimensions. Therefore, the gauge and Yukawa couplings, dimensionless in 4D, acquire a negative mass dimension to compensate for the fields. Since they both couple two fermions to a boson, their mass dimension is :

$$[\tilde{g}] = [\tilde{Y}_i] = \frac{4-d}{2} = -\frac{\delta}{2} \tag{4.38}$$

From the point of view of the renormalisation procedure, this is theoretically a problem since it renders the theory non-renormalisable, hence at every order in perturbation theory, new counterterms must be added to compensate for the divergences. However, as commonly accepted nowadays, this is

[4]We display the couplings of higher dimensional theories with a tilde since they do not have the same dimensionality than their 4D counterparts, as we will explain right away.

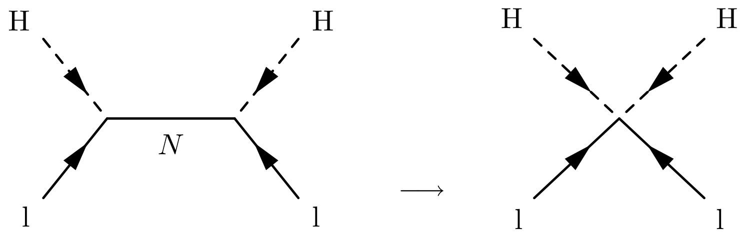


Figure 4.1: The integration of the heavy N field leads to a point-like non-renormalisable interaction

only a problem if we implicitly suppose that our theory is a valid description of the physical world up to arbitrary energies. Most probably, it is valid up to some energy scale where new ingredients come into the game. This is actually what must be done for the Standard Model, since we have little doubt that the gravitational interaction must be quantised somehow. Therefore the effects of gravity should be taken into account in a rigorous treatment, just as those of an elusive Grand Unified Theory or any extension of the SM with new degrees of freedom appearing above the EW scale. In practice, though, the physics ruling at very high energy or very small distance does not affect too much what happens at low energies and we can use safely a simpler, more appropriate theory that mimics the physics of the more fundamental one in a limited range of energy. This is an effective description of the relevant physics, which is why we name this description an effective field theory.

The remnants of the fundamental theory, present at the high scale, say the Planck scale M_P, are captured by "non-renormalisable" operators, that is to say couplings suppressed by powers of the high scale. These couplings arise from the integration performed over the massive or energetic modes of the theory, which amounts to average over regions of space that are too small to be probed accurately with a given precision. One example of this integration was performed in section 2.1 where the massive RH neutrinos are taken as non-propagating for a momentum scale $k \ll M_R$ and their integration generates a non-renormalisable, dimension five, operator. Actually, the small distance over which N propagates is neglected and the Higgs and leptons are assumed to interact at a single point in spacetime, as displayed in fig. 4.1.

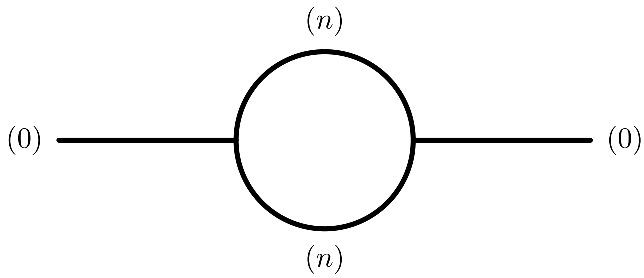
The operators created in this manner form rigorously a whole tower of interactions with higher and higher dimensionality. However, when going to low energies, due to the negative dimensionality of their coupling constants, their importance will decrease and they will become negligible except possibly for the leading order. On the contrary, when increasing the energy, they will become preponderant. This is characteristic of non-renormalisable theories : extrapolating towards very high energy regions, the couplings become uncontrollable unless there is a fixed point in the UV[5]. In fact, the theory stops being valid at the scale where the heavy d.o.f. are integrated out and when these are reintegrated into the spectrum the theory becomes renormalisable again (momentaneously). Under the scale of integration, as we always work with a finite precision, we do not have to rely on an infinite set of observables to fix the free parameters of the theory since we work at a finite order in perturbation theory.
This procedure is actually quite well adapted to renormalisation schemes independent of the particle masses, such as Minimal Subtraction schemes, for which we only extract the divergent part of the diagrams to redefine the Lagrangians parameters. With these procedures, every particle adds its contribution to the beta functions independently of its mass, so in general we have to decouple it by hand for momenta lower than its own mass scale.

[5]For example, hints of the existence of a fixed point for General Relativity have been claimed, see [123].

Integrating out the heavy modes in this way, the physical cutoff is easy to identify, it is the mass scale of the lightest heavy mode under which we define our EFT. Moreover, in 4D renormalisable theories, the results of the running will not depend too much on the way the heavy states are decoupled and the high and low energy theories are matched, since the couplings display a slow logarithmic evolution in the energy. For higher-dimensional theories, though, the problem of decoupling the heavy modes will not prove so straightforward.

4.4.2 Chain of EFTs and power law running

When interpreted from the 4D point of view, extra-dimensional theories are thus composed of towers of fields with masses almost equally spaced. When computing observables and dealing with renormalisation, there is a quite natural way to deal with quantum corrections and the non-renormalisable features of the theory [97]. Usually we use an MS scheme and keep only the divergent part of the loops. Doing so, as already stressed, we decouple by hand particles with masses larger than the energy scale of the process we are interested in. Therefore we can suppose that every KK excitation with a mass greater than the running energy μ is completely decoupled. In this approach, at any given energy, there is only a finite number of modes running in the loops. Roughly, the theory looks just like our old 4D renormalisable theory to which we add a certain finite number of copies of the low energy spectrum.
Thus, it amounts to building a chain of 4D effective theories, for which the particle content is finite. Every time we cross an energy threshold $\mu = n/R$ we add the excitations with a mass $m_n \simeq n/R$. For example, working at an energy μ, let us define N such that $N/R \leq \mu < (N+1)/R$, that is $N = \mathrm{E}[\mu R]$, $\mathrm{E}[x]$ being the integer part of x. As a simplifying assumption, we consider interactions for which the momentum conservation along the fifth dimension is not violated and KK number is conserved. Thus, we have to include loops of the form (thick lines represent any superfield propagator) :



(4.39)

Decoupling the massive states and denoting β_0 the contribution of the zero mode to the beta function, the whole beta function is given by :

$$\beta = \sum_{n<\mu R} \beta_0 = \mathrm{E}[\mu R]\beta_0 \simeq \mu R \times \beta_0 \tag{4.40}$$

The last approximation can be made when $\mu \gg R^{-1}$ and allows to have a smooth beta function. Thus, where the zero mode beta function evolves logarithmically with μ, here it will evolve linearly. Extending straightforwardly to $\delta > 1$ extra dimensions with a common radius R, we consider only particles with masses :

$$m_{\vec{n}} = \sqrt{\sum_i \frac{n_i^2}{R^2}} < \mu \tag{4.41}$$

Thus, in the continuous approximation, the number of particles participating in the total beta function is the volume of the sphere with radius μR in δ dimensions, and :

$$\beta \simeq \beta_0 \frac{2\pi^{\delta/2}}{\Gamma(\delta/2)\delta}(\mu R)^\delta \tag{4.42}$$

This behaviour of the beta functions and, consequently, of the gauge couplings is called a *power law* evolution.

Proper time regularisation

The method of adding naively the individual contributions of the KK excitations, nevertheless, is not the most convenient, particularly when several sums enter the game, with maybe different radii R_i, and because it uses implicitly a hard cutoff along the fifth component of the impulsion while using another type of regularisation along the four non-compactified components p_μ, for example dimensional regularisation. A method that was also introduced in [97] uses a proper time regularisation. This is more convenient, since it uses a global cutoff and does not separate the compactified dimensions from the others. The procedure uses the following identity :

$$\frac{1}{A^2} = \int_0^\infty \mathrm{d}t\, t\, e^{-At} \tag{4.43}$$

for a positive A. The parameter t stands for what we call the proper time. Let us consider a typical contribution from KK excitations to the wave function renormalisation of a zero mode, as displayed in (4.39) :

$$\begin{aligned} \mathcal{I} &= \sum_{n\in\mathbb{Z}} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{i}{k^2 - m_n^2} \frac{i}{(p+k)^2 - m_n^2} \\ &= -i \sum_{n\in\mathbb{Z}} \int \frac{2\pi^2 k^3 \mathrm{d}k}{(2\pi)^4} \int_0^1 \mathrm{d}x \frac{1}{[k^2 + x(1-x)p^2 + m_n^2]^2} \\ &= \frac{-i}{16\pi^2} \sum_{n\in\mathbb{Z}} \int \mathrm{d}x \int \mathrm{d}k\, k^3 \int \mathrm{d}t\, t e^{-t[k^2 + x(1-x)p^2 + m_n^2]} \end{aligned} \tag{4.44}$$

In the first line we switched to Euclidean spacetime in order to use the proper time identity. Next we integrate over k, using :

$$\int_0^\infty \mathrm{d}k\, k^3 e^{-tk^2} = \frac{1}{t^2} \tag{4.45}$$

and introduce the Jacobi function :

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 \tau} \tag{4.46}$$

for $\mathrm{Im}(\tau > 0)$. With the use of (4.45) and (4.46), the expression for $\mathcal{I}$ becomes :

$$\mathcal{I} = \frac{-i}{16\pi^2} \int \frac{\mathrm{d}t}{t} \int \mathrm{d}x\, e^{-x(1-x)p^2 t} \theta_3\left(\frac{it}{\pi R^2}\right) \tag{4.47}$$

Although we have not displayed the boundaries of the integral over the proper time, we have to regularise it since it is divergent. t having a mass dimension -2, its lower bound represents a UV cutoff while its upper bound is an IR cutoff. For the IR cutoff we choose $\xi = rR^2$ while for the UV cutoff we take $\varepsilon = r\Lambda^{-2}$ where Λ is a priori arbitrary since it is a formal cutoff. The parameter r is left free for the moment and will be chosen so as to identify Λ with the physical cutoff very soon. Choosing to regularise at this point amounts in fact to put implicitly a factor $\exp(-m_n^2/\Lambda^2)$ in the propagator of each KK excitation and to decouple their contribution to the total beta function exponentially instead of using a brutal step function :

$$\beta = \sum_n \beta_0 e^{-m_n^2/\Lambda^2} \tag{4.48}$$

The computation of $\mathcal{I}$ is finally achieved with the use of the modular identity for the Jacobi function : $\theta_3(-1/\tau) = \sqrt{-i\tau}\theta_3(\tau)$, and the approximation $\theta_3(\tau) \sim 1$ for $\mathrm{Im}(\tau) \gg 1$ (and we put the external states on-shell $p^2 = 0$) :

$$\mathcal{I} = \frac{-i}{16\pi^2}\int_{r\Lambda^{-2}}^{rR^2} \frac{\mathrm{d}t}{t} R\sqrt{\frac{\pi}{t}} = -i\frac{R}{8\pi^2}\sqrt{\frac{\pi}{r}}\left(\Lambda - \frac{1}{R}\right) \tag{4.49}$$

When using this regularisation, a power law in the cutoff Λ appears. In order to identify Λ with the physical cutoff, namely the running energy μ, as before, the authors of [97] made the choice $r = \pi/4$ for $\delta = 1$ and find the same result as for the case where heavy excitations are decoupled at their mass scale by hand.

The crucial point with power law running is that the exact result depends on the way the massive states are actually decoupled. This issue is discussed in [121] and it is argued that in order to make really precise predictions, it is mandatory to know the UV completion of the theory. Therefore, as we run towards higher energies without supposing any particular fundamental theory, we must admit that our results can only predict a general behaviour but they should not be trusted if one needs precise numerical statements. Still, we can argue that the exponential damping of the massive KK states is well motivated in some stringy setups [122].
Another very important remark is that the use of an explicit cutoff breaks Lorentz invariance. However, proper time computations give the correct power law behaviour of the beta functions, as shown in [118] where a gauge invariant regularisation of 5D abelian theories is performed using a *global cutoff*. Therefore we can safely trust our computations.

Now that we have defined a way to treat extra-dimensional models quantum mechanically, we have to wonder up to which scale we are allowed to run. This is of course not defined if we refuse to consider the embedding of the theory into a particular fundamental setup. However these theories are generally computable on a limited range since some couplings (in particular the coupling of the abelian $U(1)_Y$) become non-perturbative quite quickly. Another, more compelling, scale for the natural cutoff appears in SUSY models, as pointed out in [97] : it is the scale of gauge coupling unification, which is much lowered compared to 4D SUSY models because of the accelerated running. This is the criterium we will adopt to define the cutoff Λ, and we will stop the running at this scale.

4.5 A concrete MSSM model

The power law running of flat extra-dimensional models presents some interesting possibilities for flavour physics. Indeed, when one wants to build a theory a flavour explaining the patterns of the Yukawa sector, one needs the masses and mixings at high energy, where the flavour symmetry should be broken. If the running is accelerated, there might some chance to uncover new possible flavour

structures. Naively, it could even seem trivial that with a power law running the fast evolution will lead to new patterns but we remind that our cutoff is defined by gauge coupling unification, which happens very quickly after the extra dimensions "open up", roughly one order of magnitude above R^{-1} in the case of one extra dimension. We will focus on the running of the neutrino mass operator (2.1) in a 5D SUSY model compactified on $S^1/\mathbb{Z}_2$ [90], while the Yukawa beta functions for non SUSY 5D models have been computed in [100]. After introducing the supersymmetric 4D results, we will discuss the consequences of the localisation of the matter fields and display the most interesting possibilities for the mixing angles of the neutrinos at the cutoff scale, all the time comparing with the 4D MSSM results.
The numerical results obtained in this section have been obtained using the REAP package for Mathematica, developed by the authors of [112] and modified to suit our own needs.

4.5.1 4D results

The renormalisation of the neutrino mass dimension five operator of eq. (2.1) has been studied in [93] and revisited in [94], along with the RG equations in the MSSM which have been extended to two loops in [111]. As a reference we recall the 4D results for the beta functions of the Yukawa sector. Starting from a superpotential :

$$W = \frac{1}{6}\sum_{i,j,k=1}^{N_\Phi} \lambda_{ijk}\Phi_i\Phi_j\Phi_k - \frac{1}{4}\kappa_{fg}L_f H_u L_g H_u \tag{4.50}$$

with N_Φ is the number of chiral fields, and λ_{ijk} regroups all the renormalisable Yukawa couplings. Using non-renormalisation theorems, we only need to compute the wave function counterterms in dimensional regularisation, with $d = 4 - \varepsilon$:

$$\delta Z_{ij} = -\frac{1}{(4\pi)^2}\frac{2}{\varepsilon}\left[\frac{1}{2}\sum_{k,l=1}^{N_\Phi}\lambda^*_{ikl}\lambda_{jkl} - 2\sum_{n=1}^{N_g} g_n^2 C_2(R_n^i)\delta_{ij}\right] \tag{4.51}$$

The coefficient $C_2(R)$ is defined as usual by :

$$C_2(R)\delta_{rs} = \sum_a (T^a T^a)_{rs} \tag{4.52}$$

where T^a are the generator in the representation R. Using this general form. Applying the result to the different superfields yields the one loop beta functions :

$$\begin{aligned}
\beta_{Y_d} &= \frac{1}{(4\pi)^2} Yd\left[3Y_d^\dagger Y_d + Y_u^\dagger Y_u + 3\mathrm{Tr}(Y_d^\dagger Y_d) + \mathrm{Tr}(Y_e^\dagger Y_e) - \frac{7}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2\right] && (4.53)\\
\beta_{Y_u} &= \frac{1}{(4\pi)^2} Y_u\left[Y_d^\dagger Y_d + 3Y_u^\dagger Y_u + 3\mathrm{Tr}(Y_u^\dagger Y_u) - \frac{13}{15}g_1^2 - 3g_2^2 - \frac{16}{3}g_3^2\right] && (4.54)\\
\beta_{Y_e} &= \frac{1}{(4\pi)^2} Y_e\left[3Y_e^\dagger Y_e + 3\mathrm{Tr}(Y_d^\dagger Y_d) + \mathrm{Tr}(Y_e^\dagger Y_e) - \frac{9}{5}g_1^2 - 3g_2^2\right] && (4.55)\\
\beta_\kappa &= X^T\kappa + \kappa X + \alpha\kappa && (4.56)
\end{aligned}$$

We kept a compact form for β_κ since it will have the same form in 5 models with only a change in the coefficients X and α. In the MSSM they are :

$$X = \frac{1}{(4\pi)^2} Y_e^\dagger Y_e \qquad \alpha = \frac{1}{(4\pi)^2}\left[-\frac{6}{5}g_1^2 - 6g_2^2 + 6\mathrm{Tr}(Y_u^\dagger Y_u)\right] \tag{4.57}$$

Exploiting these results, the paper of ref. [94] determines the maximal variations of the neutrino mass parameters as a function of the MSSM free parameters. $\tan\beta$ and m_1 were shown to have an important impact on θ_{12}, for example, which is allowed, in the case of large $\tan\beta$, to approximately vanish at high energy. The two other mixing angles θ_{13} and θ_{23}, on the contrary, are not allowed such freedom, and are bound by $\theta_{23} > 25^\circ$ and $\theta_{13} < 15^\circ$. Analytical approximations have also been derived for the RG equations of the neutrino masses and mixings. The derivations will be the same for 5D models, therefore we display them as a function of the general coefficients C and α :

$$\dot{m}_i = \frac{1}{(4\pi)^2}[\alpha + C y_\tau x_i] m_i \tag{4.58}$$

$$\dot{\theta}_{12} = -\frac{C y_\tau^2}{32\pi^2}\sin(2\theta_{12}) s_{23}^2 \frac{|m_1 e^{i\phi_1} + m_2 e^{i\phi_2}|^2}{\Delta m_{sol}^2} + \mathcal{O}(\theta_{13}) \tag{4.59}$$

$$\dot{\theta}_{13} = \frac{C y_\tau^2}{32\pi^2}\sin(2\theta_{12})\sin(2\theta_{23})\frac{m_3}{\Delta m_{atm}^2(1+\zeta)} \tag{4.60}$$

$$\times[m_1\cos(\phi_1-\delta) - (1+\zeta) m_2 \cos(\phi_2 - \delta) - \zeta m_3 \cos\delta] + \mathcal{O}(\theta_{13}) \tag{4.61}$$

$$\dot{\theta}_{23} = -\frac{C y_\tau^2}{32\pi^2}\sin(2\theta_{23})\frac{1}{\Delta m_{atm}^2}\left[c_{12}^2 |m_2 e^{i\phi_2} + m_3|^2 + s_{12}^2 \frac{|m_1 e^{i\phi_1} + m_3|^2}{1+\zeta}\right] + \mathcal{O}(\theta_{13}) \tag{4.62}$$

The coefficients x_i depend on the mass eigenstate and is an $\mathcal{O}(1)$ quantity that we can write as $x_i = \{2s_{12}^2 s_{23}^2 + \mathcal{O}(\theta_{13}), 2c_{12}^2 s_{23}^2 + \mathcal{O}(\theta_{13}), 2c_{13}^2 c_{23}^2\}$. The parameter ζ stands for $\zeta = \Delta m_{sol}^2/\Delta m_{atm}^2$. These expressions help understand the importance of $\tan\beta$ on the evolution of the mixing angles, as they are proportional to y_τ, as well as the influence of the neutrino mass scale m_1.

4.5.2 Localisation and model building

We wish to extend these results to 5D models where the Yukawa sector exhibits a power law running. This imposes to have some fields of the Yukawa sector to be allowed to propagate in the fifth dimension. We also choose to put the gauge fields in the bulk of the fifth dimension which creates a power law for the gauge couplings and furnishes the physical cutoff where we stop the running in the numerical analysis.

Since we are supersymmetric, every superfield propagating in the bulk is attributed an $\mathcal{N} = 2$ partner superfield, as outlined in section 4.3.2. For every vector field V we add an adjoint chiral superfield χ and for every chiral superfield Φ propagating in the bulk we add another chiral superfield Φ^c. Recovering the MSSM at low energy imposes to eliminate half of the zero modes and we choose a compactification on $S^1/\mathbb{Z}_2$. Under the orbifold action, the superfields V and Φ are assumed to be even while of course their partners χ and Φ^c are odd. The field expansions are thus :

$$
\begin{aligned}
V(x,y) &= \frac{1}{\sqrt{\pi R}}\left[V^{(0)}(x)+\sqrt{2}\sum_{n\geq 1}V^{(n)}(x)\cos\left(\frac{ny}{R}\right)\right] && (4.63)\\
\chi(x,y) &= \sqrt{\frac{2}{\pi R}}\sum_{n\geq 1}\chi^{(n)}(x)\sin\left(\frac{ny}{R}\right) && (4.64)\\
\Phi(x,y) &= \frac{1}{\sqrt{\pi R}}\left[\Phi^{(0)}(x)+\sqrt{2}\sum_{n\geq 1}\Phi^{(n)}(x)\cos\left(\frac{ny}{R}\right)\right] && (4.65)\\
\Phi^{c}(x,y) &= \sqrt{\frac{2}{\pi R}}\sum_{n\geq 1}\Phi^{c(n)}\sin\left(\frac{ny}{R}\right) && (4.66)
\end{aligned}
$$

Once we have chosen the symmetries and the compactification space, we must decide of the localisation of the different fields. The most simple configuration consists in putting every field in the bulk and we will logically investigate this possibility. We will see quickly, however, that this setup is not the most promising since the power law is very strong. Therefore we will turn, in a second time, to the case where all matter fields are confined to a brane while the Higgs fields stay in the bulk.
As the coupling constants in 5D are dimensionful, we denote them with a tilde and reserve the usual notation for the 4D effective couplings. The KK reduction of the different actions and the relations between 5D and 4D couplings can be found in appendix B.
For numerical computations, we suppose that the $\mathcal{N}=1$ SUSY partners of the SM fields have a common mass of 1 TeV.

4.5.3 Model with matter in the bulk

beta functions

The full action for the gauge part S_g is written without the ghost part since we are not interested in computing the gauge beta functions (we refer to [97, 105] for this problem). Moreover, as we restrict our computations to one loop corrections it is sufficient to truncate the expansion of the gauge factors $\exp(2\tilde{g}V)$ at first order in $\tilde{g}$. The part of the action coupling the matter and Higgs part to the gauge fields V and χ is denoted S_{matter} while S_{brane} contains the Yukawa couplings located on the brane at $y=0$.

$$\begin{aligned} S_{gauge} &= \frac{\mathrm{Tr}}{C_2(G)}\int \mathrm{d}^8 z \mathrm{d}y \Big\{ -V\Box(P_T - \frac{1}{\xi}(P_1+P_2))V - V\partial_5^2 V - \frac{\xi}{2}\bar{\chi}\frac{\partial_5^2}{\Box}\chi + \frac{1}{2}\bar{\chi}\chi \\ &\quad + \frac{\tilde{g}}{4}(\bar{D}^2 D^\alpha V)[V, D_\alpha V] + \tilde{g}\left(\partial_5 V[V,\chi+\bar{\chi}] - (\chi+\bar{\chi})[V,\chi+\bar{\chi}]\right) \\ &\quad + \mathcal{O}(g^2) + ghosts \Big\} \end{aligned} \qquad (4.67)$$

$$\begin{aligned} S_{matter} &= \int \mathrm{d}^8 z \mathrm{d}y \left\{ \bar{\Phi}_i\Phi_i + \Phi_i^c\bar{\Phi}_i^c + \Phi_i^c\partial_5\Phi_i\delta(\bar{\theta}) - \bar{\Phi}_i\partial_5\overline{\Phi}_i^c\delta(\theta) \right. \\ &\quad \left. + \tilde{g}(2\bar{\Phi}_i V\Phi_i - 2\Phi_i^c V\bar{\Phi}_i^c + \Phi_i^c\chi\Phi_i\delta(\bar{\theta}) + \bar{\Phi}_i\bar{\chi}\bar{\Phi}_i^c\delta(\theta)) \right\} \end{aligned} \qquad (4.68)$$

$$S_{brane} = \int \mathrm{d}^8 z \mathrm{d}y \delta(y) \left\{ \left(\frac{1}{6}\tilde{\lambda}_{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{4}\tilde{\kappa}_{ij}L_i H_u L_j H_u\right)\delta(\bar{\theta}) + \mathrm{h.c.} \right\} \qquad (4.69)$$

The Feynman rules we are interested in for a one loop computation are also displayed in appendix B, as well as the different one loop diagrams and their computation. The final result for the wave function of chiral superfields is :

$$\delta Z_{ij} = \frac{1}{(4\pi)^2}\left[8\mu R \sum_{n=1}^{N_g} g_n^2 C_2(R_n^{(i)})\delta_{ij} - 2\pi(\mu R)^2 \sum_{k,l=1}^{N_\Phi} \lambda_{ikl}^*\lambda_{jkl} \right] \qquad (4.70)$$

which is obtained by keeping only the divergent contribution in μ of the diagrams. The expressions for the beta functions of the Yukawa couplings are given under a general form by :

$$\beta_{Y_d} = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}(\delta Z_{D^c}^T Y_d + Y_d\delta Z_Q + Y_d\delta Z_{H_d}) \qquad (4.71)$$

$$\beta_{Y_u} = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}(\delta Z_{U^c}^T Y_u + Y_u\delta Z_Q + Y_u\delta Z_{H_u}) \qquad (4.72)$$

$$\beta_{Y_d} = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}(\delta Z_{E^c}^T Y_e + Y_e\delta Z_L + Y_e\delta Z_{H_d}) \qquad (4.73)$$

$$\beta_\kappa = -\frac{1}{2}\mu\frac{\partial}{\partial\mu}(2\kappa\delta Z_{H_u} + \delta Z_L^T\kappa + \kappa\delta Z_L) \qquad (4.74)$$

and when applying the expressions for the δZ, the explicit result for the beta functions is :

$$\begin{aligned}
\beta_{Y_d} &= \frac{1}{(4\pi)^2}\Big[Y_d\left(3\mathrm{Tr}(Y_d^\dagger Y_d)+\mathrm{Tr}(Y_e^\dagger Y_e)+3Y_d^\dagger Y_d+Y_u^\dagger Y_u\right)4\pi(\mu R)^2 \\
&\quad -Y_d\left(\frac{14}{15}g_1^2+6g_2^2+\frac{32}{3}g_3^2\right)\mu R\Big] &(4.75)\\
\beta_{Y_u} &= \frac{1}{(4\pi)^2}\Big[Y_u\left(3\mathrm{Tr}(Y_u^\dagger Y_u)+3Y_u^\dagger Y_u+Y_d^\dagger Y_d\right)4\pi(\mu R)^2 \\
&\quad -Y_u\left(\frac{26}{15}g_1^2+6g_2^2+\frac{32}{3}g_3^2\right)\mu R\Big] &(4.76)\\
\beta_{Y_e} &= \frac{1}{(4\pi)^2}\Big[Y_e(3\mathrm{Tr}(Y_d^\dagger Y_d)+\mathrm{Tr}(Y_e^\dagger Y_e)+3Y_e^\dagger Y_e)4\pi(\mu R)^2 \\
&\quad -Y_e\left(\frac{18}{5}g_1^2+6g_2^2\right)\mu R\Big] &(4.77)
\end{aligned}$$

As for β_κ, it is given by :

$$\begin{aligned}
\beta_\kappa &= \frac{1}{(4\pi)^2}\Big[\Big(\left(-\frac{12}{5}g_1^2-12g_2^2\right)\mu R+24\pi\mathrm{Tr}(Y_u^\dagger.Y_u)(\mu R)^2\Big)\kappa \\
&\quad +4\pi(\mu R)^2\left(\left[Y_e^TY_e^*\right]\kappa+\kappa\left[Y_e^\dagger Y_e\right]\right)\Big] &(4.78)
\end{aligned}$$

Numerical results

The most important feature of these RGEs is the presence of quadratic terms in μR coming from the double KK sums in the loops induced by the localised interactions of the superpotential. This comes indeed from the localised character of these interactions and the non-conservation of the momentum along the fifth dimension. When μ becomes larger than R^{-1} and the fields start to feel the presence of the extra dimensions, the new power law equations superseed the 4D SUSY RGEs and the quadratic part will rapidly become dominant over the linear part. As it is a Yukawa contribution, therefore a positive one, the Yukawa couplings will start to increase very quickly and will become divergent long before any unification of the gauge couplings. Varying the compactification radius R, we observe that y_t diverges at $\mu R \sim 2-3$[6]. The approximations that we made to compute the beta functions with the proper time method, on the other hand, are supposed to be good approximations when $\mu R \gg 1$, which is clearly not the case. This is why we performed new numerical simulations by adding the modes explicitely at every threshold $\mu = nR^{-1}$, instead of using a continuous beta function. This approach leads to a divergence of the top Yukawa coupling at $\mu R \sim 4-5$, which is not so different from the approximate result. The behaviour of y_t in this approach is shown in fig. 4.5.3. While it was noted in [97] that the gauge couplings could become non-perturbative before unification, when every matter family propagates in the bulk and for $R^{-1} \lesssim 10^{10}$ GeV, the situation is in fact even worse when taking into account the Yukawa sector since y_t diverges before gauge coupling unification for $R^{-1} \gtrsim 2.10^{14}$ GeV.

Since calculability is clearly limited in this setup we will not investigate in detail the evolution of the neutrino parameters and switch to the second setup where matter is localised on a brane.

[6]This was already stressed in [106].

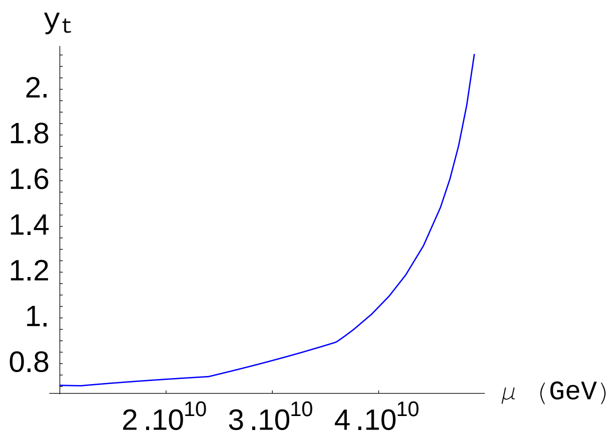


Figure 4.2: Evolution of the top Yukawa coupling with the energy for a choice of $R^{-1} = 10^{10}$ GeV and every matter superfield in the bulk. A clear divergence of y_t occurs after only a few thresholds

4.5.4 Model with matter on the brane

beta functions

When localising the matter fields on the brane the 5D form of the different parts of the action do not change, except for S_{matter} which is affected a $\delta(y)$ for the coupling of the gauge sector to quark and lepton superfields. Once again we refer to appendix B for KK reduction of the action and computation of the wave function renormalisation. The final result for the matter superfields is :

$$\delta Z_{ij} = \frac{1}{(4\pi)^2}\left[16\sum_{n=1}^{N_g} g_n^2 C_2(R_n^{(i)})\delta_{ij} - 4\sum_{k,l=1}^{N_\Phi}\lambda_{ikl}^*\lambda_{jkl}\right]\mu R \tag{4.79}$$

while for the Higgs superfields it is :

$$\delta Z_{H_i} = \frac{1}{(4\pi)^2}\left[\left(8\sum_{n=1}^{N_g} g_n^2 C_2(R_n^{(i)})\delta_{ij}\right)\mu R - \left(\sum_{k,l=1}^{N_\Phi}\lambda_{ikl}^*\lambda_{jkl}\right)\mathrm{Ln}(\mu R)\right] \tag{4.80}$$

The beta functions will not display any quadratic behaviour with the energy since now the only KK excitations that circulate in the chiral loops are those of the Higgses so there is no more double sum arising in the computations. Thus we plug the wave function renormalisation factors into the formulae for the beta function and get :

$$\beta_{Y_d} = \frac{1}{(4\pi)^2}\left[Y_d\left(3\mathrm{Tr}(Y_d^\dagger Y_d) + \mathrm{Tr}(Y_e^\dagger Y_e)\right) + Y_d\left(-\frac{19}{15}g_1^2 - 9g_2^2 - \frac{64}{3}g_3^2 + 12Y_d^\dagger Y_d + 4Y_u^\dagger Y_u\right)\mu R\right] \tag{4.81}$$

$$\beta_{Y_u} = \frac{1}{(4\pi)^2}\left[3Y_u\mathrm{Tr}(Y_u^\dagger Y_u) + Y_u\left(-\frac{43}{15}g_1^2 - 9g_2^2 - \frac{64}{3}g_3^2 + 12Y_u^\dagger Y_u + 4Y_d^\dagger Y_d\right)\mu R\right] \tag{4.82}$$

$$\beta_{Y_e} = \frac{1}{(4\pi)^2}\left[Y_e(3\mathrm{Tr}(Y_d^\dagger Y_d) + \mathrm{Tr}(Y_e^\dagger Y_e)) + Y_e\left(-\frac{33}{5}g_1^2 - 9g_2^2 + 12Y_e^\dagger Y_e\right)\mu R\right] \tag{4.83}$$

while β_κ has the same form as in (4.56) with different coefficients :

$$\alpha = \left(-\frac{18}{5}g_1^2 - 18g_2^2\right)\mu R + 6\mathrm{Tr}(Y_u^\dagger Y_u) \qquad \text{and :} \qquad C = 4\mu R \tag{4.84}$$

Numerical results

From the numerical point of view this model is a lot more comfortable since the Yukawa couplings evolve in the opposite direction compared to the democratical case where every field lives in the bulk. This is due to the fact that the term linear in μR in the Yukawas of charged fermions is dominated by the negative gauge part. Thus, as the order of magnitude of the gauge couplings does not change before unification, and the Yukawa couplings keep on diminishing, they will become more and more perturbative and the divergence of the couplings is not a problem anymore. This is also true for the gauge couplings which will unify perturbatively for any value of R^{-1}.

In practice we will investigate the region of parameter space where the effects are the largest, which will give a (rough) upper bound on the size of the effects. This is why we fix R^{-1} to a relatively small value $R^{-1} = 10$ TeV, for which the cutoff is $\Lambda \simeq 40R^{-1}$, leaving a sizeable energy range with a power law running, and allowing the use of the proper time calculations. We also choose large $\tan\beta = 50$ and $m_1 = 0.1$ eV with a normal hierarchy, except when stated otherwise. These values are acceptable regarding the experimental constraints on the SUSY and neutrino parameters and will lead to interesting results.

The first characteristic of the model is the strong universal diminution of the masses that can be seen in fig. 4.3. The suppression factor of the masses at the cutoff depends on the input value of R^{-1}, but for $R^{-1} = 1$ TeV they decrease by a factor of 5. The universal behaviour is very generic and almost independent of m_1 or $\tan\beta$. It can be easily understood from the RGEs (4.58), replacing the coefficients α and C by the expressions of eq. (4.84), which yields :

$$\dot{m}_i = \frac{1}{(4\pi)^2}\left[-\left(\frac{18}{5}g_1^2 + 18g_2^2\right) + 6\mathrm{Tr}(Y_u^\dagger Y_u) + 4\mu R y_\tau^2 x_i\right] m_i \tag{4.85}$$

As the Yukawa y_τ is driven to zero with a power law behaviour, the Cy_τ^2 and $\mathrm{Tr}(Y_u^\dagger Y_u) \simeq y_t^2$ coefficients becomes quickly negligible compared to the gauge contribution of α. Therefore, the RGE

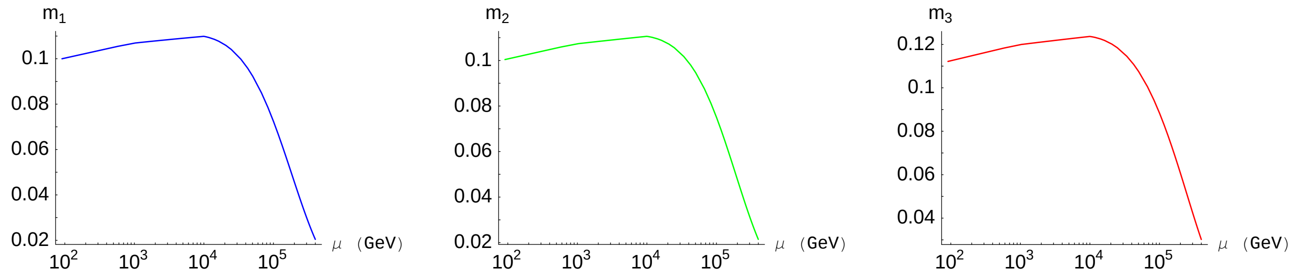


Figure 4.3: Running of the three masses for the values $R^{-1} = 10^4$ GeV, $m_1 = 0.1$ eV, $\tan\beta = 50$, $\theta_{13} = 0$ and all phases vanish at M_Z.

for the mass m_i is governed by the universal gauge part, so that every mass eigenvalue behaves in the same way.

The most interesting results, nevertheless, concern the mixing angles θ_{ij}. One interesting possibility, for example, would be to obtain a CKM-like behaviour for the PMNS matrix, explaining the striking difference between the quark and leptonic mixings at low energy simply by the running of the Yukawa coupling. We display first in fig. 4.4 the influence of $\tan\beta$ on the mixing angle θ_{12} and show that it can be lowered down to 1 degree or smaller. However this is also the case in 4D (see [94]) and is not new to our 5D model.

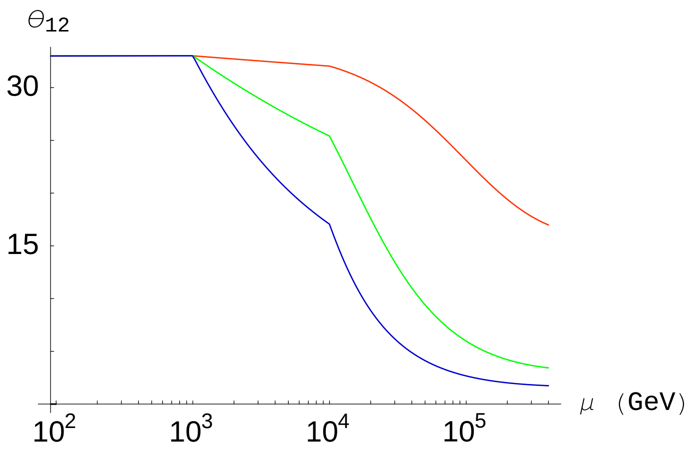


Figure 4.4: Running of θ_{12} for values of $\tan\beta = \{10, 30, 50\}$, from top to bottom. The other parameters are $R^{-1} = 10$ TeV, $m_1 = 0.1$ eV, $\theta_{13} = 0$ and no CP phases at M_Z.

For the two other mixing angles, we performed a scan over random values of m_1, $\tan\beta$, the Majorana and Dirac CP phases, and values of the mixing angles inside the experimental error bars as given by [117]. The scans have been performed for inverted and normal hierarchy and for the 4D an 5D MSSM simultaneously. We plotted the values at the cutoff in the plane $(\theta_{ij}, \tan\beta)$ and noted that there is no significant difference between 4D and 5D for $m_1 \lesssim 0.01$ eV. The results for θ_{23} with a normal hierarchy are displayed in fig. 4.5 while those for θ_{13} with an inverted hierarchy are displayed in fig. 4.6. From these plots it is clear that new possibilities open up for patterns with all three mixing angles either as small as θ_{12}^{CKM} or larger than 30 degrees.

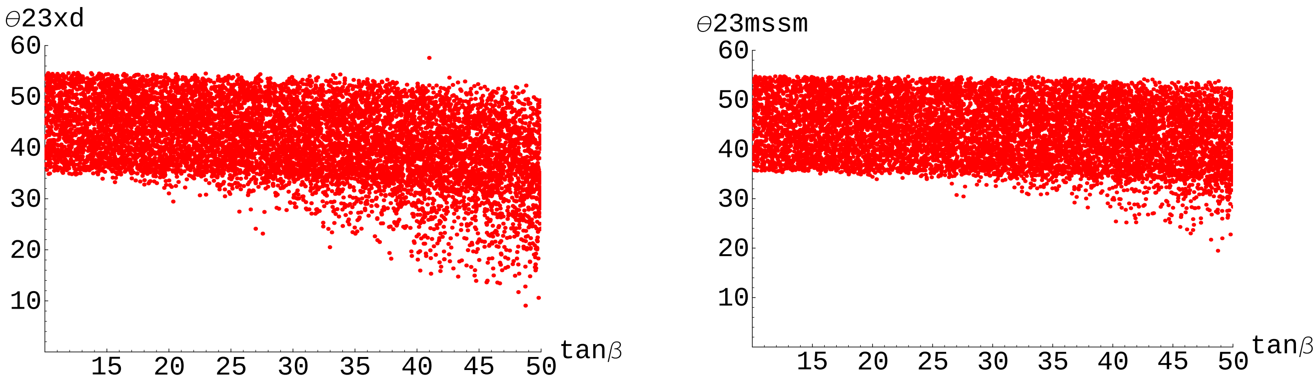


Figure 4.5: Comparison of θ_{23} at the cut-off scale as a function of $\tan\beta$ in our 5D model and in 4D MSSM for random phases and $0.01 < m_1 < 0.1$ eV with normal hierarchy. The cutoff for the 4D MSSM is chosen at the unification scale, $M_{GUT} \simeq 2.10^{16}$ GeV. For smaller values of m_1 the spread is reduced.

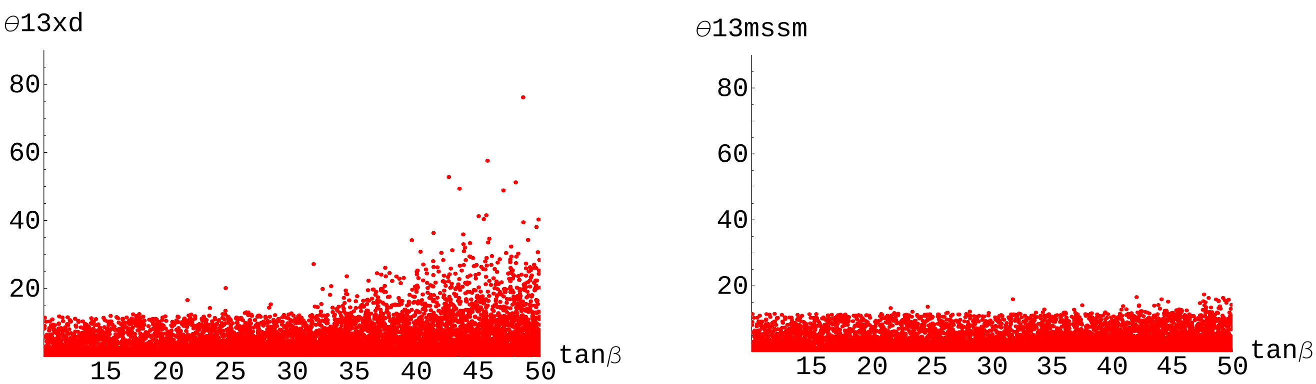


Figure 4.6: Same as Fig. 4.5 for θ_{13} and with an inverse hierarchy.

To illustrate these results in more precise cases, we plot in the left panel of fig. 4.5.4 the three mixing angles in a normal hierarchical case where θ_{23} goes to quite small values at the cutoff scale along with the two other mixing angles, mimicking the situation in the quark sector. On the contrary, the right panel of fig. 4.5.4 shows a case with inverted hierarchy for which θ_{13} starts from a large value at the cutoff scale and runs down to a value smaller than one degree at low energy. We should mention that previous attempts have been made in 4D in [116], to obtain small angles in the lepton sector in order to mimic the CKM mixings. However, we succeed to display such configurations with reasonable values of the neutrino parameters while 4D models such as in [116] have to reach large neutrino mass scales 0.1 eV$< m_1 <$ 0.6 eV, leading to tensions with recent cosmological bounds [52].

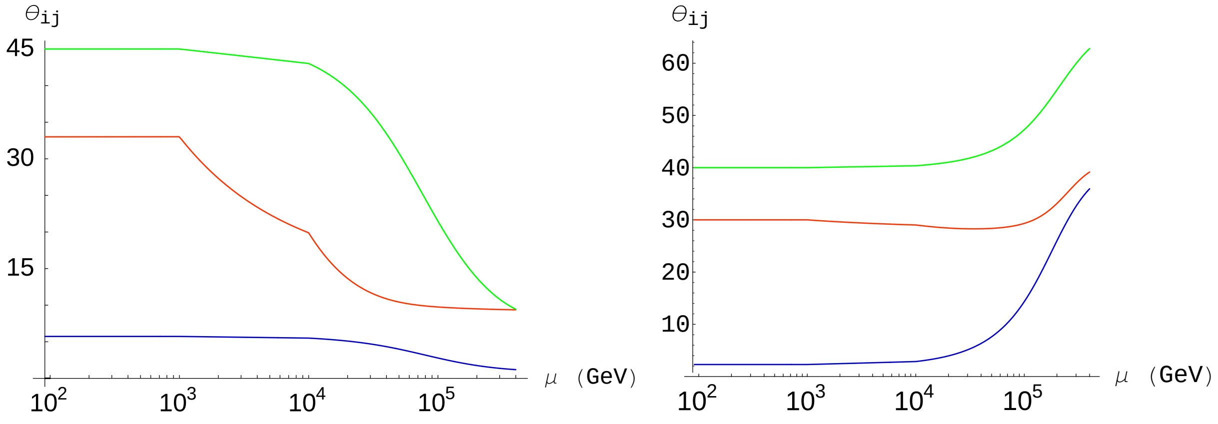


Figure 4.7: Running of the three mixing angles in two cases leading to small θ_{23} (left panel) or large θ_{13} (right panel) at high energy.

4.5.5 Conclusion

We have chosen to study a 5D MSSM model compactified on an orbifold $S^1/\mathbb{Z}_2$, because of its interesting large quantum corrections to the different coupling constants, a phenomenon known as power law running. Despite the smaller cutoff imposed by the theory due to early gauge coupling unification, the quantum corrections to the effective neutrino mass operator are found to be more important than in 4D in some part of the parameter space.

The predictivity of the model was shown to depend greatly on the localisation of the matter fields. When every field is allowed to propagate in the bulk, the quadratic dependence of the beta functions on the energy drives the top Yukawa to the non-perturbative regime too quickly for the model to be calculable. This is a direct consequence of the localisation of the Yukawa couplings, which is itself imposed by the $\mathcal{N} = 2$ supersymmetry of the 4D effective theory.

When accepting to restrict the matter superfields to the brane at $y = 0$ the Yukawa couplings are asymptotically safe and we have extracted predictions for the PMNS angles. Scanning over the parameter space we have displayed regions where it is possible to drive the two large mixing angles to values smaller than 10 degrees, allowing for a CKM-like pattern in the lepton sector, or to drive θ_{13} to values greater than 30 degrees at the cutoff scale, explaining its unobservably small value from pure running effects. These patterns should be interesting for further investigation of new flavour symmetries for example.

Chapter 5

Conclusion

There are several experimental and theoretical problems that lead us to explore physics beyond the Standard Model, and we have presented some of them earlier. A very intriguing one is the search for a flavour theory, explaining in a natural way the patterns of the different Yukawa couplings and the striking difference of the neutrino sector compared to the other ones. Ideas have been proposed to provide answers to some of the flavour puzzles, such as broken flavour symmetries which derive the patterns of Yukawa couplings in terms of an expansion in a small parameter, or string models that predict a net generation number of three as a topological feature of their compactification space.
However, building a theory of flavours requires a precise knowledge of the Yukawa couplings at high energy. This has been the first focus of the work presented here, and which divides itself into the study of two different classes of models. The first part of the work was devoted to the study of the seesaw formula in Left-Right symmetric models. The particular singlet nature of the right-handed part of the neutrinos leads to a neutrino mass matrix quite unlike those of quarks and charged leptons, thereby providing a first hint as to why neutrinos should have such different spectrum and mixings. Moreover, when adding the contribution of a weak triplet to the neutrino mass matrix, known as type II seesaw, the formula complicates further. This is why we provided a method to reconstruct the right-handed neutrino masses at high energy, under the only assumption of a Left-Right symmetry, limiting the number of free couplings. Using this method, it is then seen that several solutions appear for the RH mass spectrum and mixings, quite unlike the cases of pure type I or type II seesaw, and it allows to get additional information on the structure of the Yukawa sector at high energy. Focusing on a certain class of $SO(10)$ models, known to lead to unsuccessful leptogenesis in the case of pure type I seesaw, we studied the consequences of our different solutions for leptogenesis. In addition to several "type I-like" cases, we found that some solutions could lead to interesting predictions. In a second work, we pushed the analysis of leptogenesis to a more detailed and quantitative level, considering the importance of flavours as well as the necessary corrections to fermion masses, and solving the Boltzmann equations numerically. When taking into account all these corrections, some solutions were still found to yield a sufficient baryon asymmetry, even with a reasonable reheating temperature and a large value of the $B - L$ breaking scale. Interestingly enough, these solutions are of the "mixed" type, meaning that they are solutions for which neither types of seesaw dominates every entry of the light neutrino mass matrix, and thus does not correspond to the cases of pure type I or pure type II mechanism.
In the other part of our work, we decided to study the running of neutrino masses in an extra-dimensional model, where quantum corrections are known to lead to a power law running of the couplings. Computing the one loop beta functions for several localisation of the matter fields along the fifth dimension, we displayed the potentially dangerous consequences for Yukawa couplings since they can diverge quite early. When matter fields are restricted to four dimensions, however, an

interesting region in the parameter space has been found where the predictions for the mixing angles are significantly different from the four dimensional MSSM. The possibility exists to lower all the mixing angles to small CKM-like values at the cutoff scale, or even to a large θ_{13} similar to the two others. This could prove an interesting playground for new flavour symmetries, for example.

Appendix A

Boltzmann Equations

The evolution of particle densities in the early Universe is governed by the Boltzmann Equations. Here we will provide a short introduction to these equations by deriving them in simple example. More detailed applications concerning leptogenesis can be found in [26, 27] for example.
Let us consider here a bosonic particle X which will stand for the massive gauge bosons of a GUT for example. We will follow a derivation similar to the one in [76], and we refer to this reference for a more detailed analysis.

Usually the particle X is super-massive and exists in the early Universe with a thermal distribution. When the Universe cools down at a temperature $T < M_X$, the X particles start decaying into leptons and baryons. If the decay rates $X \to ql$ and $\bar{X} \to \bar{q}\bar{l}$ are different, and all the other criteria for successful baryogenesis as described in section2.4.1 are fulfilled, we can hope to generate a net baryon asymmetry. If the X's are completely in equilibrium, $\Gamma_X \gg H(T = M_X)$, the asymmetry is completely washed-out. On the contrary, if $\Gamma_X \ll H(T = M_X)$, then we just have to compute the asymmetry generated in each decay to derive immediately the final baryon asymmetry.
However, if we are not in of these two extreme situations or if we want to be more quantitatively precise, we have to solve for the particle densities during the history of the Universe. Those equations that allow us to do so are the Boltzmann equations. They are formulated primarily for the phase space distribution $f_X(x^\mu, p^\mu)$:

$$\mathbf{L}[f_X] = \mathbf{C}[f_X] \tag{A.1}$$

$\mathbf{L}$ is the time evolution operator and is called the Liouville operator. $\mathbf{C}$ is the collision operator, taking into account the production and annihilation of the species through its interactions with the plasma. For a generic background, $\mathbf{L}$ is expressed as :

$$\mathbf{L} = p^\mu \frac{\partial}{\partial^\mu} - \Gamma^\mu_{\nu\rho} p^\nu p^\rho \frac{\partial}{\partial p^\mu} \tag{A.2}$$

In a Universe obeying the FRW evolution, isotropy implies $f_X = f_X(E, t)$ and the above equation is :

$$\mathbf{L}[f_X] = E\frac{\partial f_X}{\partial t} - \frac{\dot{a}}{a}|\vec{p}|^2 \frac{\partial f_X}{\partial E} \tag{A.3}$$

What we are going to manipulate are the particles' number densities defined by :

$$n_X(t) = \frac{g_X}{(2\pi)^3} \int \mathrm{d}^3 p f_X(E,t) \tag{A.4}$$

g_X measures the number of degrees of freedom for the particle X. Integrating the Boltzmann equation yields the following equation for n_X :

$$\frac{\mathrm{d}n_X}{\mathrm{d}t} + 3Hn_X = \frac{g_X}{(2\pi)^3} \int \mathbf{C}[f] \frac{\mathrm{d}^3 p}{E} \tag{A.5}$$

and the collision term can be expressed through the scattering amplitude for the process (considered to be the unique one for the moment) $X + ... \to i + ...$:

$$\frac{g_X}{(2\pi)^3} \int \mathbf{C}[f] \frac{\mathrm{d}^3 p_X}{E_X} = -\int \mathrm{d}\Pi_X ... \mathrm{d}\Pi_i ... \times (2\pi)^4 \delta^4(p_X + ... - p_i - ...) \tag{A.6}$$

$$\times \left[|\mathcal{M}|^2_{X+...\to i+...} \times f_X ...(1 \pm f_i)... - |\mathcal{M}|^2_{i+...\to X+...} \times f_i ...(1 \pm f_X)... \right] \tag{A.7}$$

the + signs apply to bosons and the − ones to fermions. The symbol $\mathrm{d}\Pi_X$ stands for :

$$\mathrm{d}\Pi_X = \frac{g_X}{(2\pi)^3} \frac{\mathrm{d}^3 p}{2E} \tag{A.8}$$

As already pointed out, it is more interesting to work with a quantity that does not scale with a certain power of a, therefore we introduced the entropy density of the Universe s and built the number density :

$$Y_X = \frac{n_X}{s} \tag{A.9}$$

As $s \propto a^3$, straightforwardly $\dot{n}_X + 3Hn_X = s\dot{Y}_X$. The cosmic time is related to the temperature of the thermal bath by the relation :

$$t = \frac{1}{2H(T)} = 0.301 g_*^{-1/2} \frac{M_P}{T^2} \tag{A.10}$$

Now the better variable to study the dynamics is the quantity $z = m/T$ with m a characteristic mass scale, usually the mass of the particle we are tracking, so that here $z = m_X/T$. In this parametrisation $t = z^2/(2H(m))$ and $\mathrm{d}/\mathrm{d}t = H(m_X)/z \times \mathrm{d}/\mathrm{d}z$. The usual convention is to denote derivatives with respect to t with a dot, while derivatives with respect to z are denoted with a prime.

Armed with these general expressions we will turn to a more specific example. Let us suppose that the interactions of X with baryonic matter happens through the processes $X \to bb$ and $X \to \bar{b}\bar{b}$ and their inverse processes. b is a particle with a baryon charge q_b. For simplicity, still following [76], we suppose that the particles obey Mawxell-Boltzmann statistics, meaning that a species i has a distribution $f_i \simeq \exp[-(E - \mu_i)/T]$. CPT symmetry will enforce the equalities : $|\mathcal{M}(X \to bb)| = |\mathcal{M}(\bar{b}\bar{b} \to X)|$ and $|\mathcal{M}(X \to \bar{b}\bar{b})| = |\mathcal{M}(bb \to X)|$. Since the baryon asymmetry will come from the small difference between $|\mathcal{M}(X \to bb)|$ and $|\mathcal{M}(X \to \bar{b}\bar{b})|$, we introduce the two parameters ε and $\mathcal{M}_0$:

$$\varepsilon = |\mathcal{M}(X \to bb)|^2 - |\mathcal{M}(X \to \bar{b}\bar{b})|^2 \tag{A.11}$$

$$|\mathcal{M}_0|^2 = |\mathcal{M}(X \to bb)|^2 + |\mathcal{M}(X \to \bar{b}\bar{b})|^2 \tag{A.12}$$

What we want is the final baryon density $n_B = q_b n_b - q_b n_{\bar{b}}$ or more precisely $Y_B = n_B/s$. Its evolution will be linked to the evolution of Y_X, so we will derive the system of coupled Boltzmann equations. As we use the Maxwell-Boltzmann distribution and we are interested in processes involving particles with an energy $E \gtrsim m_X$ at a temperature $T \lesssim m_X$ we can approximate $1 \pm f_i \simeq 1$ in eq. (A.7). Applying this equation to the case at hand leads to :

$$\begin{aligned}
\dot{n}_X + 3Hn_X &= \int \mathrm{d}\Pi_X \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \delta(p_x - p_1 - p_2) \\
&\quad \times \left[-f_x(E_X)(|\mathcal{M}(X \to bb)|^2 + |\mathcal{M}(X \to \bar{b}\bar{b})|^2) \right. \\
&\quad \left. + f_{\bar{b}}(p_1) f_{\bar{b}}(p_2) |\mathcal{M}(\bar{b}\bar{b} \to X)|^2 + f_b(p_1) f_b(p_2) |\mathcal{M}(bb \to X)|^2 \right] && \text{(A.13)} \\
s\dot{Y}_X &\simeq \int \mathrm{d}\Pi_X \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \delta(p_x - p_1 - p_2) \times \left[-f_X(p_X) + f_X^{eq}(p_X) \right] |\mathcal{M}_0|^2 && \text{(A.14)} \\
\frac{sH(m_X)}{z} Y_X' &= -\Gamma_X (n_X - n_X^{eq}) = -\Gamma_X s (Y_X - Y_X^{eq}) && \text{(A.15)}
\end{aligned}$$

In the second line we neglected terms in ε so that each amplitude squared reduces to $|\mathcal{M}_0|^2$. Moreover, we neglected terms in $\mu_b/T = -\mu_{\bar{b}}/T$: the distributions for b and $\bar{b}$ become $f_{1,2} = \exp(-E_{1,2}/T)$ and the Dirac function allows to write $f_b(E_1) f_b(E_2) = \exp(-(E_1 + E_2)/T) = \exp(-E_X/T) = f_X^{eq}$, and similarly for $f_{\bar{b}} f_{\bar{b}}$, since complete equilibrium is characterised by $\mu_b = \mu_{\bar{b}} = \mu_X = 0$.
The factor Γ_X in the last line stands for the thermally averaged decay rate of X. In order to study the equations in terms of the more physical quantity :

$$K_X = \frac{\Gamma_X(z=1)}{H(m_X)} \tag{A.16}$$

which parametrises the out-of-equilibrium condition, we also introduce $\gamma_D = \Gamma_X(z)/\Gamma_X(z=1)$ and $\Delta_X = Y_X - Y_X^{eq}$, and the Boltzmann equation becomes :

$$\Delta_X' = -(Y_X^{eq})' - z\gamma_D K \Delta_X \tag{A.17}$$

There remain to write the equation the baryon density $Y_B = n_B/s = q_b(n_b - n_{\bar{b}})/s$. The processes changing the density of b, for example, are of course decays of X and inverse decays $bb \to X$ but also the $2 \leftrightarrow 2$ scatterings $bb \to \bar{b}\bar{b}$ and $\bar{b}\bar{b} \to bb$ mediated by X. Of course those processes the intermediate X state is on-shell have already been counted in decays and inverse decays, therefore we will only take the off-shell part of the amplitude which we denote with a prime $\mathcal{M}'(bb \to \bar{b}\bar{b})$. Thus the Boltzmann equation writes :

$$\begin{aligned}
\dot{n}_b + 3Hn_b &= \int \mathrm{d}\Pi_X \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 (2\pi)^4 \delta^4(p_x - p_1 - p_2) \\
&\quad \times \left[-2|\mathcal{M}(bb \to X)|^2 f_b(E_1) f_b(E_2) + |\mathcal{M}(x \to bb)|^2 f_X(E_X) \right] \\
&\quad + 2 \int \mathrm{d}\Pi_1 \mathrm{d}\Pi_2 \mathrm{d}\Pi_3 \mathrm{d}\Pi_4 (2\pi)^4 \delta^4(p_1 + p_2 - p_3 - p_4) \\
&\quad \times \left[-f_b(p_1) f_b(p_2) |\mathcal{M}'(bb \to \bar{b}\bar{b})|^2 + f_b(p_3) f_b(p_4) |\mathcal{M}'(\bar{b}\bar{b} \to bb)|^2 \right] && \text{(A.18)}
\end{aligned}$$

and a similar equation exists for $n_{\bar{b}}$. By choosing an appropriate normalisation for baryon charges we fix $q_b = 1/2$. Defining the thermally averaged $2 \leftrightarrow 2$ cross section :

$$\langle\sigma|v|\rangle = \int \mathrm{d}\Pi_1\mathrm{d}\Pi_2\mathrm{d}\Pi_3\mathrm{d}\Pi_4(2\pi)^4\delta^4(p_3+p_4-p_1-p_2)\times f_b(p_1)f_b(p_2)|\mathcal{M}'(bb\to\bar{b}\bar{b})|^2\frac{1}{n_\gamma^2} \tag{A.19}$$

we are able to write the evolution equation for Y_B in the compact form :

$$\frac{zs}{H(m_X)}Y_B' = \varepsilon\Gamma_X(n_X - n_X^{eq}) - n_B\Gamma_X\frac{n_X^{eq}}{n_\gamma} - 2n_Bn_b\langle\sigma|v|\rangle \tag{A.20}$$

and it is straightforward to express everything in terms of Δ_X and Y_B :

$$Y_B' = \varepsilon zK_X\gamma_D\Delta_X - zK_X\gamma_BY_B \tag{A.21}$$

and in order to be as compact as possible we made use of the definition :

$$\gamma_B = \frac{g_*Y_X^{eq}\gamma_D + 2n_\gamma\langle\sigma|v|\rangle}{\Gamma_X(z=1)} \tag{A.22}$$

Formal solutions of eqs. (A.17) and (A.21) can expressed by integrating the equations :

$$\Delta_X(z) = \Delta_X^i\exp\left[-\int_{z_i}^z z'K_X\gamma_D(z')\mathrm{d}z'\right] - \int_{z_i}^z (Y_X^{eq})'(z')\exp\left[\int_{z'}^z z\text{"}K_X\gamma_D(z\text{"})\mathrm{d}z\text{"}\right]\mathrm{d}z' \tag{A.23}$$

$$\begin{aligned} Y_B(z) &= Y_B^i\exp\left[-\int_{z_i}^z z'K_X\gamma_B(z')\mathrm{d}z'\right] \\ &\quad +\varepsilon K_X\int_{z_i}^z z'\Delta_X(z')\gamma_D(z')\exp\left[-\int_{z'}^z z\text{"}K_X\gamma_B(z\text{"})\mathrm{d}z\text{"}\right]\mathrm{d}z' \end{aligned} \tag{A.24}$$

with $\Delta_X^i = \Delta_X(z_i)$ and $Y_B^i = Y_B(z_i)$ the initial conditions for the X and baryon asymmetry densities. These equations can then be solved for different regimes of approximation, for example the weak washout regime ($K_X \ll 1$) or the strong washout regime ($K_X \gg 1$).

Applications of these principles are used in section 2.4 to derive the corresponding equations for the lepton asymmetry in the leptogenesis scenario.

Appendix B

One Loop Computations in 5D Models

This appendix contains the computational details needed to derive the one loop results of chapter 4. It will begin with the presentation of the KK reduced action for the two models analysed as well as the relevant propagator and Feynman rules. In a second time we will compute the different KK sums that will appear when considering the different one loop diagrams. Finally, we will present the diagrams participating in the running of the Yukawa couplings and compute them to extract the formula used in section 4.5 for the wave function renormalisation of the chiral superfields.

B.0.6 Actions

Let us write again the actions displayed in eq. (4.67), (4.68) and (4.69) of section 4.5 :

$$
\begin{aligned}
S_{gauge} &= \frac{\mathrm{Tr}}{C_2(G)}\int \mathrm{d}^8 z \mathrm{d}y \Bigg\{ V\Box\left(P_T - \frac{1}{\xi}(P_1+P_2)\right)V + V\partial_y^2 V + \frac{\xi}{2}\bar{\chi}\frac{\partial_y^2}{\Box}\chi + \frac{1}{2}\bar{\chi}\chi \\
&\quad + \frac{\tilde{g}}{4}(\bar{D}^2 D^\alpha V)[V, D_\alpha V] + \tilde{g}\left(\partial_5 V[V,\chi+\bar{\chi}] - (\chi+\bar{\chi})[V,\chi+\bar{\chi}]\right) \\
&\quad + \mathcal{O}(g^2) + ghosts \Bigg\} \\
S_{matter} &= \int \mathrm{d}^8 z \mathrm{d}y \left\{ \bar{\Phi}_i\Phi_i + \Phi_i^c\bar{\Phi}_i^c + \Phi_i^c\partial_y\Phi_i\delta(\bar{\theta}) - \bar{\Phi}_i\partial_y\overline{\Phi}_i^c\delta(\theta) \right. \\
&\quad \left. + \tilde{g}(2\bar{\Phi}_i V\Phi_i - 2\Phi_i^c V\bar{\Phi}_i^c + \Phi_i^c\chi\Phi_i\delta(\bar{\theta}) + \bar{\Phi}_i\bar{\chi}\bar{\Phi}_i^c\delta(\theta)) \right\} \\
S_{brane} &= \int \mathrm{d}^8 z \mathrm{d}y\delta(y)\left\{\left(\frac{1}{6}\tilde{\lambda}_{ijk}\Phi_i\Phi_j\Phi_k + \frac{1}{4}\tilde{\kappa}_{ij}L_iH_uL_jH_u\right)\delta(\bar{\theta}) + \mathrm{h.c.}\right\}
\end{aligned}
$$

We recall that S_{gauge} is the action of the pure gauge, S_{matter} the one which couples the gauge sector the hypermultiplets and S_{brane} the action for the Yukawa couplings. Using the KK expansions (4.63) to (4.64), the 4D effective actions are obtained :

$$\begin{aligned}
S_{gauge} &= \int \mathrm{d}^8 z \left\{ V_a^{(0)} \Box \left(P_T - \frac{1}{\xi}(P_1+P_2) \right) V_a^{(0)} + \sum_{n\geq 1} V_a^{(n)} \Box \left(P_T - \frac{1}{\xi}(P_1+P_2) \right) V_a^{(n)} \right. \\
&\left. - \sum_{n\geq 1} \frac{n^2}{R^2} V_a^{(n)} V_a^{(n)} + \frac{\xi}{2} \sum_{n\geq 1} \frac{n^2}{R^2} \bar{\chi}_a^{(n)} \frac{1}{\Box} \chi_a^{(n)} + \frac{1}{2} \sum_{n\geq 1} \bar{\chi}_a^{(n)} \chi_a^{(n)} \right\} \qquad \text{(B.1)}
\end{aligned}$$

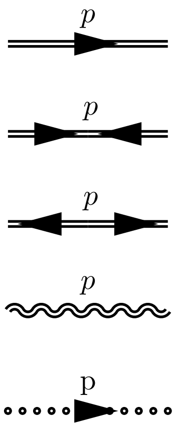
$$\begin{aligned}
S_{matter} &= \int \mathrm{d}^8 z \left\{ \bar{\Phi}^{(0)} \Phi^{(0)} + \sum_{n\geq 1} (\bar{\Phi}^{(n)} \Phi^{(n)} + \Phi^{c(n)} \bar{\Phi}^{c(n)}) - \sum_{n\geq 1} \frac{n}{R} (\Phi^{c(n)} \Phi^{(n)} \delta(\bar{\theta}) + \bar{\Phi}^{(n)} \bar{\Phi}^{c(n)} \delta(\theta)) \right. \\
&+ g \left[2 \bar{\Phi}^{(0)} V^{(0)} \Phi^{(0)} + 2 \sum_{n\geq 1} (\bar{\Phi}^{(0)} V^{(n)} \Phi^{(n)} + \bar{\Phi}^{(n)} V^{(0)} \Phi^{(n)} + \bar{\Phi}^{(n)} V^{(n)} \Phi^{(0)}) \right. \\
&\left. - 2 \sum_{n\geq 1} \Phi^{c(n)} V^{(0)} \bar{\Phi}^{c(n)} + \sum_{n\geq 1} (\Phi^{c(n)} \chi^{(n)} \Phi^{(0)} \delta(\bar{\theta}) + \bar{\Phi}^{(0)} \bar{\chi}^{(n)} \bar{\Phi}^{c(n)} \delta(\theta)) \right] \\
&+ g \left[\sqrt{2} \sum_{m,n\geq 1} \bar{\Phi}^{(m)} V^{(n)} (\Phi^{(m+n)} + \Phi^{(|m-n|)}) + \sqrt{2} \sum_{m,n\geq 1} \Phi^{c(m)} (V^{(m+n)} - V^{(|m-n|)}) \bar{\Phi}^{c(n)} \right. \\
&\left.\left. - \frac{1}{\sqrt{2}} \sum_{m,n\geq 1} \left(\Phi^{c(m)} \chi^{(n)} (\Phi^{(m+n)} - \Phi^{(|m-n|)}) \delta(\bar{\theta}) + \bar{\Phi}^{(m)} \bar{\chi}^{(n)} (\bar{\Phi}^{c(m+n)} - \bar{\Phi}^{c(|m-n|)}) \delta(\theta) \right) \right] \right\} \\
&\qquad \text{(B.2)}
\end{aligned}$$

$$\begin{aligned}
S_{brane} &= \int \mathrm{d}^8 z \left\{ \frac{\lambda_{ijk}}{6} \left[\Phi_i^{(0)} \Phi_j^{(0)} \Phi_k^{(0)} + 3\sqrt{2} \sum_{n\geq 1} \Phi_i^{(n)} \Phi_j^{(0)} \Phi_k^{(0)} + 6 \sum_{m,n\geq 1} \Phi_i^{(0)} \Phi_j^{(m)} \Phi_k^{(n)} \right. \right. \\
&\left. + 2\sqrt{2} \sum_{m,n,p\geq 1} \Phi_i^{(m)} \Phi_j^{(n)} \Phi_k^{(p)} \right] \delta(\bar{\theta}) + \text{h.c.} \\
&+ \frac{\kappa_{ij}}{4} \left[L_i^{(0)} H_u^{(0)} L_j^{(0)} H_u^{(0)} + 2\sqrt{2} \sum_{n\geq 1} (L_i^{(n)} H_u^{(0)} L_j^{(0)} H_u^{(0)} + L_i^{(0)} H_u^{(n)} L_j^{(0)} H_u^{(0)}) \right. \\
&+ 4 \sum_{m,n\geq 1} (L_i^{(m)} H_u^{(n)} L_j^{(0)} H_u^{(0)} + L_i^{(m)} H_u^{(0)} L_j^{(0)} H_u^{(n)}) \\
&+ 4\sqrt{2} \sum_{m,n,p\geq 1} (L_i^{(0)} H_u^{(m)} L_j^{(n)} H_u^{(p)} + L_i^{(m)} H_u^{(0)} L_j^{(n)} H_u^{(p)}) \\
&\left.\left. + 4 \sum_{m,n,p,q\geq 1} L_i^{(m)} H_u^{(n)} L_j^{(p)} H_u^{(q)} \right] \delta(\bar{\theta}) + \text{h.c.} \right\} \qquad \text{(B.3)}
\end{aligned}$$

The gauge parameter is chosen as $\xi = -1$ and we defined the 4D effective couplings from the higher-dimensional ones by :

$$g = \frac{\tilde{g}}{\sqrt{\pi R}}\,, \qquad \lambda = \frac{\tilde{\lambda}}{(\pi R)^{3/2}}\,, \qquad \kappa = \frac{\tilde{\kappa}}{(\pi R)^2} \qquad \text{(B.4)}$$

The propagators are extracted from the quadratic part of the action :



$$\bar{\Phi}_i^{(c)(n)}(-p,\theta)\;\;\Phi_j^{(c)(m)}(p,\theta') \quad = \quad \frac{i}{p^2-\frac{n^2}{R^2}+i\epsilon}\delta_{ij}\delta_{mn}\delta^4(\theta-\theta')$$

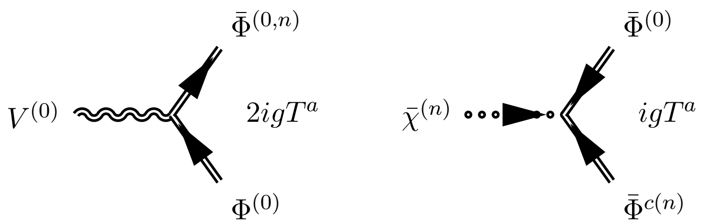
$$\bar{\Phi}_i^{(n)}(-p,\theta)\;\;\bar{\Phi}_j^{c(m)}(p,\theta') \quad = \quad \frac{-i\frac{n}{R}}{p^2(p^2-\frac{n^2}{R^2})+i\epsilon}\delta_{ij}\delta_{mn}\frac{\bar{D}^2(p)}{4}\delta^4(\theta-\theta')$$

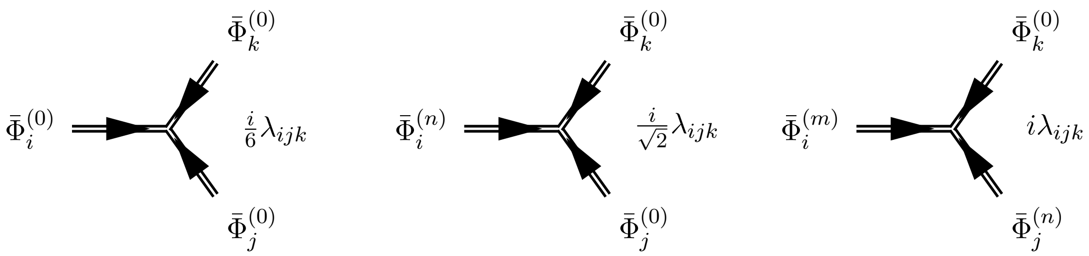
$$\Phi_i^{c(n)}(-p,\theta)\;\;\Phi_j^{(m)}(p,\theta') \quad = \quad \frac{D^2(p)}{4}\frac{-i\frac{n}{R}}{p^2(p^2-\frac{n^2}{R^2})+i\epsilon}\delta_{ij}\delta_{mn}\delta^4(\theta-\theta')$$

$$V_a^{(n)}(-p,\theta)\;\;V_b^{(m)}(p,\theta') \quad = \quad \frac{-i}{2\left(p^2-\frac{n^2}{R^2}+i\epsilon\right)}\delta_{ab}\delta_{mn}\delta^4(\theta-\theta')$$

$$\bar{\chi}_a^{(n)}(-p,\theta)\;\;\chi_b^{(m)}(p,\theta') \quad = \quad \frac{-2i}{p^2-\frac{n^2}{R^2}+i\epsilon}\delta_{ab}\delta_{mn}\delta^4(\theta-\theta')$$

while the Feynman rules relevant for renormalising the wave function of chiral superfields at one loop are :





The same can be done for the model with matter superfields living on the brane. The pure gauge action and the part of S_{matter} concerning the Higgs fields are untouched, while the couplings of the quark and lepton superfields change as follows :

$$
\begin{aligned}
S_{matter} &= \int \mathrm{d}^8 z \mathrm{d}y \delta(y) \left\{ \bar{\Phi}_i \Phi_i + 2\tilde{g} \bar{\Phi}_i V \Phi_i \right\} \\
&= \int \mathrm{d}^8 z \left\{ \bar{\Phi}_i \Phi_i + 2g \bar{\Phi}_i V^{(0)} \Phi_i + 2\sqrt{2} g \sum_{n\geq 1} \bar{\Phi}_i V^{(n)} \Phi_i \right\} \qquad \text{(B.5)}
\end{aligned}
$$

$$
\begin{aligned}
S_{brane} &= \int \mathrm{d}^8 z \mathrm{d}y \delta(y) \left\{ \tilde{Y}_e E^c L H_d + \tilde{Y}_d D^c Q H_d + \tilde{Y}_u U^c Q H_u + \frac{1}{4} \tilde{\kappa} L H_u L H_u + \text{h.c.} \right\} \\
&= \int \mathrm{d}^8 z \left\{ Y_e E^c L H_d^{(0)} + Y_d D^c Q H_d^{(0)} + Y_u U^c Q H_u^{(0)} + \frac{1}{4} \kappa L H_u^{(0)} L H_u^{(0)} \right. \\
&\quad + \sum_{n\geq 1} \sqrt{2} \left(Y_e E^c L H_d^{(n)} + Y_d D^c Q H_d^{(n)} + Y_u U^c Q H_u^{(n)} + \frac{1}{2} \kappa L H_u^{(n)} L H_u^{(n)} \right) \\
&\quad \left. + \sum_{m,n\geq 1} \frac{1}{2} \kappa L H_u^{(m)} L H_u^{(n)} + \text{h.c.} \right\} \qquad \text{(B.6)}
\end{aligned}
$$

The 4D coupling constants are now defined as :

$$
Y_i = \frac{\tilde{Y}_i}{\sqrt{\pi R}} \qquad \kappa = \frac{\tilde{\kappa}}{\pi R} \qquad \text{(B.7)}
$$

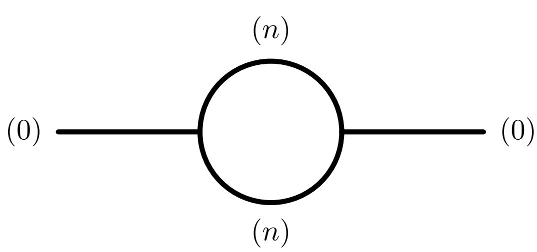
The Feynman rules can be computed in the same way as before but we could not write the Yukawa part in a manner as compact as before so we will not explicitely display the rules.

B.0.7 Useful KK sums

We will suspend the computation of wave function renormalisation factors for the moment and compute the different sums over KK states that will be encountered in the following. We will not keep "finite" terms which do not diverge with the cutoff since we are only interested in the running of the Yukawa couplings. Different types of infinite sums can arise when computing one loop diagrams, depending on whether KK number is conserved or whether every field running in the loop has KK excitation.

KK tower with KK number conservation

The interactions involving two modes which propagate in the bulk conserves the KK number at each vertex[1]. The corresponding diagram is of the form :



(B.8)

[1]This is a remnant of momentum conservation along x^5, even though translational symmetry is broken because of the orbifold compactification

This diagram was already displayed and computed as an example in section 4.4.2 and we will follow here the same path for this appendix to be self-consistent. The total contribution to the self energy is :

$$\mathcal{I}_1 = \sum_{n\geq 1} \int \frac{\mathrm{d}^4 k}{(2\pi)^4} \frac{1}{k^2 - \frac{n^2}{R^2}} \frac{1}{(p+k)^2 - \frac{n^2}{R^2}} \tag{B.9}$$

We recall the definition and approximation of the θ_3 modular function that we use to compute these sums :

$$\theta_3(\tau) = \sum_{n\in\mathbb{Z}} e^{i\pi n^2 \tau} \qquad \theta_3\left(\frac{it}{\pi R^2}\right) \simeq R\sqrt{\frac{\pi}{t}} \tag{B.10}$$

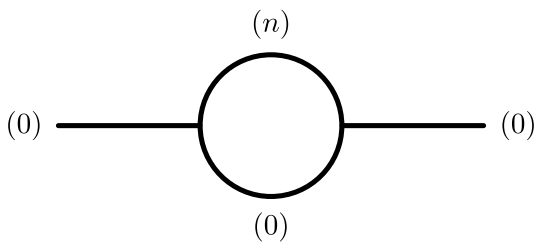
and the approximation holds for $t \gg R^2$ and is derived using the modular approximation $\theta_3(-1/\tau) = \sqrt{-i\tau}\theta_3(\tau)$. Using these properties, the integral $\mathcal{I}_1$ becomes :

$$\begin{aligned} \mathcal{I}_1 &\simeq \frac{i}{(4\pi)^2} \int_\varepsilon^\xi \frac{\mathrm{d}t}{t} \left[\frac{1}{2}\theta_3\left(\frac{it}{\pi R^2}\right) - \frac{1}{2}\right] \\ &\simeq \frac{i}{(4\pi)^2} \int_\varepsilon^\xi \frac{\mathrm{d}t}{t} \left(\frac{1}{2}\sqrt{\frac{\pi R^2}{t}} - \frac{1}{2}\right) \\ &\simeq \frac{i}{(4\pi)^2} \big(2\mu R - \mathrm{Ln}(\mu R)\big) \end{aligned} \tag{B.11}$$

In the last line we have chosen the IR and UV cutoffs as $\xi = \frac{\pi}{4}R^2$ and $\varepsilon = \frac{\pi}{4}\mu^{-2}$ to be consistent with the decoupling method discussed in section 4.4.2.

KK tower with one excited state

The diagrams involved here are the following ones :



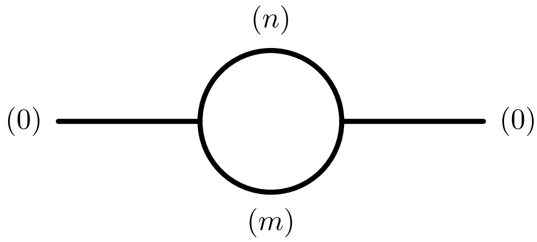
(B.12)

and their expression reads :

$$\begin{aligned}
\mathcal{I}_2 &= \sum_{n\geq 1}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-\frac{n^2}{R^2}}\frac{1}{(k+p)^2} \\
&\simeq \frac{i}{(4\pi)^2}\int \frac{\mathrm{d}t}{t}e^{-\frac{xn^2}{R^2}t} \\
&\simeq \frac{i}{(4\pi)^2}\int \frac{\mathrm{d}t}{t}\frac{1}{2}\left[\theta_3\left(\frac{itx}{\pi R^2}\right)-1\right] \\
&\simeq \frac{i}{(4\pi)^2}\left(4\mu R-\mathrm{Ln}(\mu R)\right) \qquad \text{(B.13)}
\end{aligned}$$

Double KK tower

The last kind of diagram appearing only for localised interactions and involving two excited states running in the loop is :



(B.14)

These diagrams are only relevant for the model with all matter fields living in the bulk, and it is computed as :

$$\begin{aligned}
\mathcal{I}_3 &= \sum_{m,n\geq 1}\int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2-\frac{m^2}{R^2}}\frac{1}{(k+p)^2-\frac{n^2}{R^2}} \\
&\simeq \frac{i}{(4\pi)^2}\int \frac{\mathrm{d}t}{t}e^{-\left(\frac{xn^2}{R^2}+\frac{(1-x)m^2}{R^2}\right)t} \\
&\simeq \frac{i}{(4\pi)^2}\int \frac{\mathrm{d}t}{t}\frac{1}{4}\left[\theta_3\left(\frac{itx}{\pi R^2}\right)-1\right]\left[\theta_3\left(\frac{it(1-x)}{\pi R^2}\right)-1\right] \\
&\simeq \pi\mu^2R^2-4\mu R+\frac{1}{2}\mathrm{Ln}(\mu R) \qquad \text{(B.15)}
\end{aligned}$$

No KK tower in the loop

For completeness, and because we excluded loops with only zero modes in the computations above, we display the usual for a loop where only zero modes run. This will be useful, because the orbifold projection leads to Feynman rules for the zero modes which differ from those for massive KK states by factors of $\sqrt{2}$, due to the different normalisation of the fundamental and excited modes in the KK decomposition.
Thus, with only one zero mode, the integrals are of the form :

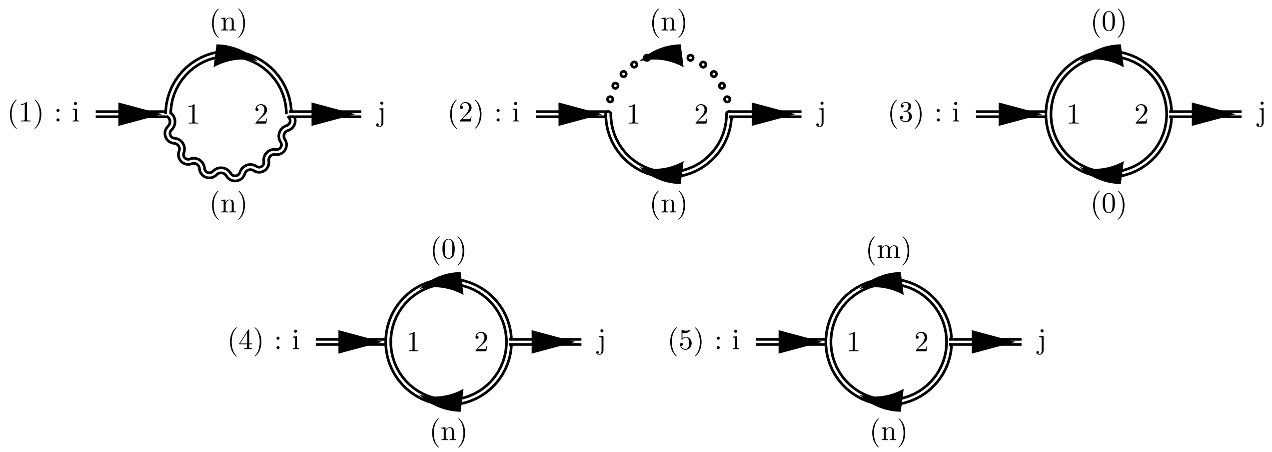
$$
\begin{aligned}
\mathcal{I}_4 &= \int \frac{\mathrm{d}^4k}{(2\pi)^4}\frac{1}{k^2}\frac{1}{(k+p)^2} \\
&\simeq \frac{i}{(4\pi)^2}\int\frac{\mathrm{d}t}{t} \simeq \frac{2i}{(4\pi)^2}\mathrm{Ln}(\mu R)
\end{aligned} \tag{B.16}
$$

B.0.8 Wave function renormalisation

We now possess all the necessary tools, and proceed to compute the wave function counter terms which are necessary to obtain the Yukawa beta functions.

Matter in the bulk

The five types of diagrams involved in the self-energy computation are listed below :



We will compute the first diagram in detail, using the formalism of supergraphs developed in section 2.2.3 and the integral $\mathcal{I}_n$ of the previous section :

$$
\begin{aligned}
-i\delta Z_{ij}^{(1)} &= -4g^2(T^aT^a)_{rs}\delta_{ij}\sum_{n\geq 0}\int\frac{\mathrm{d}^4k}{(2\pi)^4}\mathrm{d}^4\theta_1\mathrm{d}^4\theta_2\frac{-i\delta^4(\theta_1-\theta_2)}{2(k^2-\frac{n^2}{R^2})} \\
&\qquad\times\frac{1}{16}\bar{D}_1^2D_2^2\frac{i\delta^4(\theta_1-\theta_2)}{(k+p)^2-\frac{n^2}{R^2}}\Phi_i^{r(0)}(-p,\theta_1)\bar{\Phi}_j^{s(0)}(p,\theta_2) \\
&= -i2g^2C_2(R)\delta_{ij}^{rs}\left(\mathcal{I}_1+\mathcal{I}_4\right)\int\mathrm{d}^4\theta_1\Phi_i^{r(0)}(-p,\theta_1)\bar{\Phi}_j^{s(0)}(p,\theta_1) \\
&= -i\frac{2g^2C_2(R)\delta_{ij}^{rs}}{16\pi^2}\left(2\mu R+\log(\mu R)\right)\int\mathrm{d}^4\theta_1\phi_i^{r(0)}(-p,\theta_1)\bar{\phi}_j^{s(0)}(p,\theta_1)
\end{aligned}
$$

The indices (r,s) are gauge indices. We displayed the integration factor over the theta coordinate for the first diagram but we will omit it in the other computations. The computation of the the remaining diagrams follows exactly the same lines and we present only the results :

$$
-i\delta Z_{ij}^{(2)} = i\frac{-2g^2C_2(R)\delta_{ij}^{rs}}{16\pi^2}\left(2\mu R-\log(\mu R)\right) \qquad -i\delta Z_{ij}^{(3)} = i\frac{\lambda_{ikl}\lambda_{jkl}^*\delta_{rs}}{16\pi^2}\log(\mu R)
$$

$$
-i\delta Z_{ij}^{(4)} = i\frac{2\lambda_{ikl}\lambda_{jkl}^*\delta_{rs}}{16\pi^2}\left(4\mu R-\log(\mu R)\right) \qquad -i\delta Z_{ij}^{(5)} = i\frac{\lambda_{ikl}\lambda_{jkl}^*\delta_{rs}}{16\pi^2}\left(2\pi(\mu R)^2-8\mu R+\log(\mu R)\right)
$$

The final result is obtained by addition of the five contributions and we note that every subdominant divergence disappears :

$$\delta Z_\Phi^{5D} = \frac{1}{(4\pi)^2}\left[\left(-8\sum_{n=1}^{N_g} g_n^2 C_2(R_n^{(i)})\delta_{ij}\right)\mu R + \left(2\pi\sum_{k,l=1}^{N_\Phi}\lambda_{ikl}^*\lambda_{jkl}\right)\mu^2R^2\right] \tag{B.17}$$

This result is finally applied to the chiral superfields :

$$\delta Z_{H_u} = -\frac{1}{(4\pi)^2}\left[12\pi \mathrm{Tr}(Y_u^\dagger Y_u)\mu^2R^2 - \left(\frac{6}{5}g_1^2 + 6g_2^2\right)\mu R\right] \tag{B.18}$$

$$\delta Z_{H_d} = -\frac{1}{(4\pi)^2}\left[4\pi[3\mathrm{Tr}(Y_d^\dagger Y_d) + \mathrm{Tr}(Y_e^\dagger Y_e)]\mu^2R^2 - \left(\frac{6}{5}g_1^2 + 6g_2^2\right)\mu R\right] \tag{B.19}$$

$$\delta Z_L = -\frac{1}{(4\pi)^2}\left[4\pi(Y_e^\dagger Y_e)\mu^2R^2 - \left(\frac{6}{5}g_1^2 + 6g_2^2\right)\mu R\right] \tag{B.20}$$

$$\delta Z_{E^C} = -\frac{1}{(4\pi)^2}\left[8\pi(Y_e^* Y_e^T)\mu^2R^2 - \left(\frac{24}{5}g_1^2\right)\mu R\right] \tag{B.21}$$

$$\delta Z_{D^c} = -\frac{1}{(4\pi)^2}\left[8\pi(Y_d^* Y_d^T)\mu^2R^2 - \left(\frac{8}{15}g_1^2 + \frac{32}{3}g_3^2\right)\mu R\right] \tag{B.22}$$

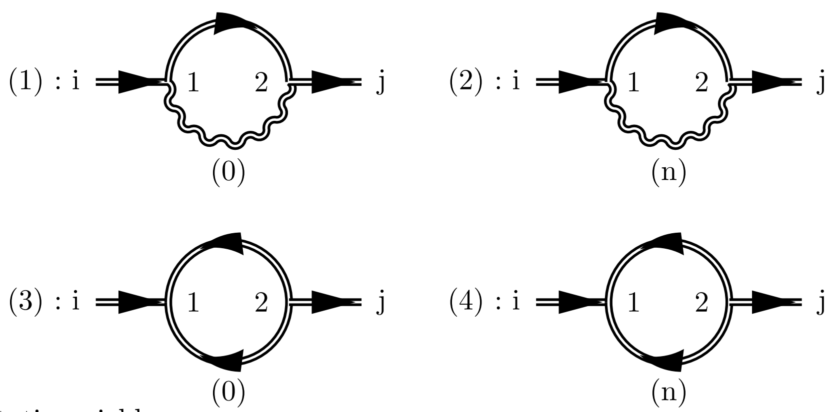
$$\delta Z_Q = -\frac{1}{(4\pi)^2}\left[4\pi(Y_u^\dagger Y_u + Y_d^\dagger Y_d)\mu^2R^2 - \left(\frac{2}{15}g_1^2 + 6g_2^2 + \frac{32}{3}g_3^2\right)\mu R\right] \tag{B.23}$$

$$\delta Z_{U^c} = -\frac{1}{(4\pi)^2}\left[8\pi(Y_u^* Y_u^T)\mu^2R^2 - \left(\frac{32}{15}g_1^2 + \frac{32}{3}g_3^2\right)\mu R\right] \tag{B.24}$$

Using these results we compute the beta functions given in eq. (4.77) and (4.78).

Matter on the brane

In this case the diagrams will be different since for example the lepton and quark superfields do not couple to χ since the latter has a zero wave function at $y = 0$ on the brane. The diagrams are four :



Their computation yields :

$$-i\delta Z_{ij}^{(1)} = i\frac{-4g^2C_2(R)\delta_{ij}^{rs}}{16\pi^2}\log(\mu R) \qquad -i\delta Z_{ij}^{(2)} = i\frac{-8g^2C_2(R)\delta_{ij}^{rs}}{16\pi^2}(4\mu R - \log(\mu R))$$

$$-i\delta Z_{ij}^{(3)} = i\frac{\lambda_{ikl}\lambda_{jkl}^*}{16\pi^2}\log(\mu R) \qquad -i\delta Z_{ij}^{(4)} = i\frac{\lambda_{ikl}\lambda_{jkl}^*}{16\pi^2}(4\mu R - \log(\mu R))$$

the sum of which gives for the matter superfields :

$$\delta Z_{ij}^{\Phi} = -\frac{1}{(4\pi)^2}\left[-16\mu R g^2 C_2(R)\delta_{ij} + 4\mu R \lambda_{ikl}\lambda^*_{jkl}\right] \tag{B.25}$$

The Higgs superfields are renormalised differently. The gauge part is the same as for the previous model, the only change reside in the Yukawa sector :

$$\delta Z^{H} = -\frac{1}{(4\pi)^2}\left[-8\mu R g^2 C_2(R) + 2\log(\mu R)\mathrm{Tr}(Y_i Y_i^\dagger)\right] \tag{B.26}$$

Applying this to the different superfields :

$$\delta Z_{H_u} = -\frac{1}{(4\pi)^2}\left[6\mathrm{Tr}(Y_u^\dagger Y_u)\log\Lambda R - \left(\frac{6}{5}g_1^2 + 6g_2^2\right)\Lambda R\right] \tag{B.27}$$

$$\delta Z_{H_d} = -\frac{1}{(4\pi)^2}\left[[6\mathrm{Tr}(Y_d^\dagger Y_d) + 2\mathrm{Tr}(Y_e^\dagger Y_e)]\log\Lambda R - \left(\frac{6}{5}g_1^2 + 6g_2^2\right)\Lambda R\right] \tag{B.28}$$

$$\delta Z_{L} = -\frac{1}{(4\pi)^2}\left[8(Y_e^\dagger Y_e) - \frac{12}{5}g_1^2 - 12g_2^2\right]\Lambda R \tag{B.29}$$

$$\delta Z_{E^C} = -\frac{1}{(4\pi)^2}\left[16(Y_e^* Y_e^T) - \frac{48}{5}g_1^2\right]\Lambda R \tag{B.30}$$

$$\delta Z_{D^c} = -\frac{1}{(4\pi)^2}\left[16(Y_d^* Y_d^T) - \frac{16}{15}g_1^2 - \frac{64}{3}g_3^2\right]\Lambda R \tag{B.31}$$

$$\delta Z_{Q} = -\frac{1}{(4\pi)^2}\left[8(Y_u^\dagger Y_u + Y_d^\dagger Y_d) - \frac{4}{15}g_1^2 - 12g_2^2 - \frac{64}{3}g_3^2\right]\Lambda R \tag{B.32}$$

$$\delta Z_{U^c} = -\frac{1}{(4\pi)^2}\left[16(Y_u^* Y_u^T) - \frac{64}{15}g_1^2 - \frac{64}{3}g_3^2\right]\Lambda R \tag{B.33}$$

From these, it is straightforward to compute the expressions of eq. (4.83) and (4.84).

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