Mixed-State Correlation Functions of Twist Fields in Two-Dimensional Integrable Models of Quantum Field Theory

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Mixed-State Correlation Functions of Twist Fields in Two-Dimensional Integrable Models of Quantum Field Theory

by

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degree of Doctor of Philosophy

in the
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Department Of Mathematics

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Declaration of Authorship

I, Yixiong Chen, declare that this thesis titled, 'Mixed-State Correlation Functions of Twist Fields in Two-Dimensional Integrable Models of Quantum Field Theory' and the work presented in it are my own. I confirm that:

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- Where any part of this thesis has previously been submitted for a degree or any other qualification at this University or any other institution, this has been clearly stated.
- Where I have consulted the published work of others, this is always clearly attributed.
- Where I have quoted from the work of others, the source is always given. With the exception of such quotations, this thesis is entirely my own work.
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Abstract

Theoretical Physics Group
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Doctor of Philosophy

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The aim of this thesis is to evaluate correlation functions of twist fields in mixed states in two-dimensional integrable models of quantum field theory (QFT). We construct the "Liouville space" for general models of QFT in general mixed states associated to diagonal density matrices, and define mixed-state form factors in Liouville space. We then specialize to two concrete models: the Ising model and $U(1)$ Dirac model. Using a novel method based on deriving and solving a system of nonlinear functional differential equations, we obtain exact mixed-state form factors of twist fields, in both models. These form factors are in agreement with finite-temperature form factors which correspond to the thermal Gibbs state. We then write down mixed-state correlation functions for these fields in terms of the full form factor expansions with respect to the vacuum in Liouville space. Under weak analytic conditions on the eigenvalues of the density matrix, they are exact large-distance expansions. We apply the results in the Ising model to analyze large-distance behaviours of two-point functions of order and disorder fields in generalized Gibbs ensembles and non-equilibrium steady states. In particular, we find non-equilibrium form factors have branch cuts in rapidity space and the leading large-distance behaviour of two-point functions admit oscillations in the log of the distance between fields. Using the results in the Dirac model and the relation between the Ising and Dirac models, we deduce the Réyi entropy for even integer n. Finally, as an extra work, we deduce the high- and low-temperature limit of the exact current at non-equilibrium steady states in general integrable models of quantum field theory with diagonal scatterings.

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Chapter 1

Introduction

Quantum field theory (QFT), since its birth, has witnessed a rapid evolution over the years and has occupied a central position in the description of modern theoretical physics. This is partly due to the principle of universality which is a primary property of all local interactions and which naturally arises from the analysis of the renormalization group. Thanks to the principle of universality, it is possible for a given model of quantum field theory to describe the large-distance physics of various systems with very different microscopic details. Moreover, a vast variety of successful applications are also responsible for the popularity of quantum field theory. Originally developed to describe elementary particles and their interactions, quantum field theory has been shown to be a very powerful and efficient method for describing condensed matter physics, especially for the case of low dimensional and strongly correlated systems where traditional methods fail. Quantum field theory also found its applications in quantum electrodynamics (QED), in the description of weak and strong interactions, and in statistic mechanics.

Among the objects in a model of quantum field theory, correlation functions are of crucial importance since they encode in principle all physical information including the dynamics of the physical systems and the response of the model to external perturbations. Correlation functions are the key allowing us to leave the frame of pure theoretical formalism and get into the real world of experiments. For instance, correlation functions, interpreted from statistical mechanics, correspond to linear responses to applied local external fields, which can be directly measured in lab, and the measurement of which is one of the main approach to study a material. Moreover, the computation of all quantities of interest in a model can be in principle achieved from the knowledge of correlation functions. In this sense, we could say that the main problem in the study of a model of quantum field theory can be seen as the reconstruction of its correlation functions.

Unfortunately, it is usually non-trivial to obtain numerically accurate results for correlation functions of a model of quantum field theory. In addition, most models of relativistic quantum field theory, from the mathematical point of view, is not well-defined. However, there exist two types of models in which these two obstacles can be overcome. These two exceptions are quantum field theory for free relativistic particles and two-dimensional quantum field theory with conformal invariance. In the context of these exceptions, correlation functions of more complicated models can be studied with the help of perturbative methods and physical arguments.

Besides models in conformal field theory and free field theory, a special family of two-dimensional massive integrable models of quantum field theory [1–3, 17, 18] is also a very populous research field in modern physics. These models admit the presence of an infinite number of conserved charges and it is these conserved charges that open the possibility for exact evaluations of many quantities, such as scattering matrix, form factors (matrix elements of local fields), and correlation functions, giving rise to a rich mathematical structure to build upon. Successes for these models have been made over the last two decades on the determination of vacuum correlation functions via form factor bootstrap approach. For instance, large distance expansions of two-point correlation functions, which are hardly accessible by perturbation theory, can be obtained by form factor expansion (Källen-Lehmann expansions) [1–3] under the factorized scattering theory and both their large-distance and short-distance asymptotic behaviors agree with general QFT expectations (see for instance, [4, 5]).

In recent years, correlation functions in general mixed states have attracted growing concerns and triggered an enormous amount of work, because of their wide scope of applications both of theoretical and experimental interest. For instance, correlation functions in thermal Gibbs state, which can be related to correlation functions on an infinite cylindrical geometry [6], have been the subject of intense study in general QFT [7], and have been studied more precisely in massive integrable QFT [8–15]. In particular, the Ising model at finite temperature has been widely investigated by employing several approaches including form factor expansions [19–23], integrable differential equations [24, 25], semiclassic methods [26], and the finite volume regularization method [15, 16, 27–30]. On the other hand, more mixed states have been explored, including generalized Gibbs ensembles (GGEs) which have been predicted to occur after quantum quench in integrable models [31–35], non-equilibrium steady states [36–40] and others.

From these works, the structure of finite-temperature two-point correlation functions is relatively well known, although much work still needs to be done in integrable QFT in order to get as powerful large-distance or large-time expansions as for vacuum two-point functions. The study of expectation values and correlation functions in generalized Gibbs ensembles is however much more recent. Analytic leading-asymptotic results for two-point functions are available in the Ising model [34, 41] and series expansions for expectation values in GGEs of general integrable models [42]. Concerning correlation functions in non-equilibrium energy-carrying steady states, there are even fewer results, including a leading-decay result at large distances in the XY spin chain [43]. Hence, obtaining exact QFT expansions for general diagonal mixed states would shed much light onto the structure of correlation functions.

This thesis aims to obtain the exact result for correlation functions of twist fields in general mixed states with diagonal density matrices. The concept of twist fields was originally introduced in [44] as the $\mathbb{Z}_2$ monodromy field of the Majorana fermion, corresponding to the spin operator of the Ising model. It has been shown late in [45] and [46] that the twist fields in the n-copy Ising model can be used to study the quantum entanglement entropy. In fact, twist fields exist in any model possessing a global internal symmetry and each twist field is associated to one element of a symmetry group. Twist fields are "interacting fields" since they are not local with respect to the fundamental fields in quantum field theory. Therefore, correlation functions of twist fields usually exhibit non-trivial behaviors and should contain parts of the complicated structure of correlation functions in integrable models of interacting particles. It has been demonstrated in various ways that these non-trivial correlation functions can be obtained as solutions to non-linear differential equations [47–50]. Correlation functions of twist fields associated to the $U(1)$ symmetry can also be obtained from re-summing the form factor expansion in terms of Fredholm determinants [47, 48, 51, 52]. At zero temperature, from those existing results, large-distance behaviors of correlation functions of twist fields can be analyzed efficiently. At finite temperature or on the cylinder, correlation functions of twist fields can be described by partial differential equation in the coordinates on the cylinder [24, 53, 54], but these equations do not provide a very useful tool for calculating correlation functions and analyzing their large-distance behavior. In general mixed states, results are even less known. This thesis will constitute one step towards this direction.

In this thesis, we consider three integrable models of quantum field theory. The first model is the Ising model, which describes the scaling limit of the Ising quantum chain near the critical value h_c of the external transverse magnetic field h. In this model, the twist fields are associated to the $\mathbb{Z}_2$ symmetry and they represent the scaling limit of the order parameter in the ordered ($h > h_c$) and disordered ($h < h_c$) regimes. The second model is the free Dirac fermion theory, which is equivalent with "doubled" Ising model. In this model, the twist fields are associated to the $U(1)$ symmetry. Finally, we consider a model which is composed of n non-interacting copies of the Ising model. In this model, the twist fields are called branch-point twist fields and they correspond to the Z_n symmetry of the model.

The correlation functions we will obtain in this thesis admit the expressions which can be adapted to a large family of mixed states and offer an efficient way to analyze their large-distance behavior. In the Ising model, for the case of generalized Gibbs ensembles, we will find agreement with leading large-distance results derived in [41], and we will calculate new subleading terms; for the case of non-equilibrium steady state, although our expansion needs further regularization, the leading large-distance behavior indeed agrees with the well-known results presented in [43], and we will conjecture the existence of logarithmic oscillating subleading factors.

Our results are derived by the "Liouville-space method". This method was initially established in [21, 23] in order to derive finite-temperature spin-spin correlation functions, and then further developed in [55] to obtain general diagonal mixed-state spin-spin correlation functions, both in the Ising model of QFT. The Liouville space construction [56–61] is based on the GNS construction [62, 63] of C^*-algebras (see the book [64]) and it has applications in thermal and non-equilibrium physics. In the present thesis, we will apply this method to integrable models of quantum field theory. We will define and evaluate the mixed-state form factors of twist fields, and then formulate mixed-state two-point functions of these fields using form factor expansion with respect to the vacuum in Liouville space.

This thesis is organized as follows:

- In chapter 2, based on [65–68], we will start by providing a detailed review of the most important and most relevant properties of the two-dimensional integrable quantum field theory. After this general part, we will specialize to two particular integrable models: the Ising model, where a fairly rigorous analysis of the connection between the Ising quantum chain and the free massive Majorana field theory will be carried out, and the $U(1)$ Dirac model.
- In chapter 3, based on [66], we will give the explanation of the twist fields in general through the path integral formulation of quantum field theory. Two types of twist fields and their form factors will be reviewed. In addition, we will introduce the branch-point twist fields in the n -copy Ising model and their relation with the $U(1)$ twist fields in the n-copy Dirac model will be reviewed.
- In chapter 4, we will introduce the Liouville space in general, including its construction and properties, and illustrate the concept of mixed-state form factors. Within

this framework, we will then construct the Liouville space in the Ising model and in the Dirac model, and define the associated mixed-state form factors.

- In chapter 5, we will give a review of a method employed in [21], which is to determine finite-temperature form factors of twist fields in the Ising model by setting up and solving a Riemann-Hilbert problem. Following similar lines, we will then derive a similar Riemann-Hilbert problem for twist fields in the Dirac theory. With the help of low temperature expansions, we will calculate the finite-temperature form factors of $U(1)$ twist fields.

- In chapter 6, we will present the exact results for mixed-state form factors of twist fields in the Ising model and in the Dirac theory. Thereafter, we will derive a system of non-linear first-order functional differential equations for mixed-state form factors of twist fields and verify that the exact form factors we obtain indeed satisfy these equations. We close this chapter by presenting a general solution as integral-kernel to this system of equations, which serves as an alternative expression for the mixed-state form factors of twist fields.

- In chapter 7, we will apply our exact results for mixed-state form factors to the evaluation of the corresponding mixed-state correlation functions of twist fields in the Ising model and the Dirac theory, which is the main goal of these thesis. Then, the first application of the general mixed-state correlation functions will be to the Gibbs thermal state in the Ising model, where we will show that our thermal correlation functions indeed reproduce the large-distance correlation functions on the circle. After this, we will enter, in the Ising model, the analysis of the analytic properties of form factors of twist fields and large-distance behaviors of two-point functions, in non-equilibrium steady state. We will also study the large-distance behaviors of two-point functions in the state described by the generalized Gibbs ensemble after a quantum quench. Finally, we will obtain the Rényi entropy for integer n in the Ising model, in virtue of our mixed-state correlation functions of $U(1)$ twist fields. This result can be directly used to compute the bipartite entanglement entropy for the Ising model.

- In chapter 8, we will summarize the main works carried out in this thesis and point out some open problems for future investigations.

- In the appendix A, we will present an extra work concerning the high- and low-temperature limit of the energy current obtained in [69] for non-equilibrium steady states in general integrable models of relativistic quantum field theory with diagonal scattering.

Chapter 2

Two-dimensional massive integrable quantum field theory

2.1 Exact S-matrices

2.1.1 Asymptotic states and scattering matrix

Two-dimensional massive integrable quantum field theory, which we will focus on in this thesis, is built on a basis of the Hilbert space, namely asymptotic states (see, for instance, [70]). With the assumption that interactions in our theory are short-ranged, an asymptotic state represents a set of well-defined particles that are infinitely separated from each other in the infinite past or infinite future, behaving like a collection of free propagating particles. This is an eigenstate of the Hamiltonian as it has a well-defined energy. The n-particle asymptotic states can be written as

$$|A_{a_1}(\theta_1) \cdots A_{a_n}(\theta_n)\rangle$$

where we denote by $A_{a_i}(\theta_i)$ the particle of type a_i traveling with rapidity θ_i. Note that it is convention, in two-dimensional QFT, to characterise the momentum of on-shell particles by invariable θ called rapidity. For a particle of mass m_i, we have

$$p_i^0 = m_i \cosh\theta_i, \qquad p_i^1 = m_i \sinh\theta_i \tag{2.1}$$

with $p^\mu = (p^0, p^1)$ its momentum. Asymptotic states consist of two types of states: *in*- and *out*-states. *In*-states are given by particles at $t \to -\infty$ and *out*-states by particles at $t \to \infty$. In two-dimensional QFT, particles move on a line and this means that, for interactions to happen at finite time, particles must be ordered in space from left to right, with decreasing values of rapidities, in the infinite past $t \to -\infty$, while they must be in opposite order, in the infinite future $t \to \infty$. According to this physical situation, we consider $A_{a_i}(\theta_i)$ as non-commuting symbols, whose orders are associated to the space orderings of the particles they represent. In this way, *in*- and *out*-states can be written respectively as

$$|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in}\,, \quad \text{with} \quad \theta_1 > \cdots > \theta_n\,, \tag{2.2}$$

and

$$|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{out}\,, \quad \text{with} \quad \theta_1 < \cdots < \theta_n\,. \tag{2.3}$$

The asymptotic states are constructed, as a fundamental assumption, by a set of *in*-and *out*-operators of creation and annihilation type, which we denote by $A_a^\dagger(\theta)^{(in,out)}$ and $A_a(\theta)^{(in,out)}$, representing a particle whose quantum numbers are labeled by a and which has rapidity θ. These operators satisfy the canonical (anti-)commutation relations

$$\begin{aligned} \left[A_a(\theta)^{(in,out)}, A_{a'}^\dagger(\theta')^{(in,out)}\right] &= 4\pi\delta_{aa'}\delta(\theta-\theta') \\ \left[A_a(\theta)^{(in,out)}, A_{a'}(\theta')^{(in,out)}\right] &= \left[A_a^\dagger(\theta)^{(in,out)}, A_{a'}^\dagger(\theta')^{(in,out)}\right] = 0\,. \end{aligned} \tag{2.4}$$

The operators $A_a^\dagger(\theta)^{(in,out)}$ and $A_a(\theta)^{(in,out)}$ define the space of physical states. The vacuum state is defined as the one annihilated by any of these operators

$$A_a(\theta)^{(in,out)}|\text{vac}\rangle = 0 = \langle\text{vac}|A_a^\dagger(\theta)^{(in,out)}\,. \tag{2.5}$$

The n-particle states are generated by the action of n *in*- or *out*-operators on the vacuum state. These states are not linearly independent due to the algebra (2.4) and a certain prescription is needed in order to select out a basis of independent states. In fact, the asymptotic state introduced above is a natural choice and it is generated by acting *in*-operators with a decreasing ordering of rapidities in the infinite past or *out*-operators with an increasing ordering of rapidities, on the vacuum state. Since *in*-and *out*-operators are eigenoperators of the Hamiltonian and the momentum

$$\left[H, A_{a_i}^\dagger(\theta_i)^{(in,out)}\right] = m_{a_i}\cosh\theta_i A_{a_i}^\dagger(\theta_i)^{(in,out)}\,, \tag{2.6}$$

$$\left[P, A_{a_i}^\dagger(\theta_i)^{(in,out)}\right] = m_{a_i}\sinh\theta_i A_{a_i}^\dagger(\theta_i)^{(in,out)}, \tag{2.7}$$

the asymptotic states created by these *in*- and *out*-operators are eigenstates of the Hamiltonian and the momentum

$$H|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \sum_{k=1}^{n} m_{a_k}\cosh\theta_k|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out}\,, \quad (2.8)$$

$$P|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \sum_{k=1}^{n} m_{a_k}\sinh\theta_k|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out}\,. \quad (2.9)$$

In this sense, the Hilbert space is the Fock space over the algebra of all such *in*-operators, which is isomorphic to the Fock space over the algebra of all *out*-operators.

Note that if we find a state with a definite number of particles at definite rapidities in the infinite past, then we can find in the infinite future a superposition of such states. This can be implemented by the scattering matrix or the S-matrix which provides the mapping between the *in*-state basis and *out*-state basis. Take a two-particle *in*-state for example,

$$|A_{a_1}(\theta_1)A_{a_2}(\theta_2)\rangle_{in} = \sum_{n=2}^{\infty}\sum_{b_1,\ldots,b_n}\sum_{\theta_1'<\cdots<\theta_n'} S_{a_1a_2}^{b_1\cdots b_n}(\theta_1,\theta_2;\theta_1',\ldots,\theta_n')|A_{b_1}(\theta_1')\cdots A_{b_n}(\theta_n')\rangle_{out} \quad (2.10)$$

where $\theta_1 > \theta_2$ and the sum over θ_i' generally involves a number of integrals, with the rapidities constrained by the overall conservation of left- and right-lightcone momenta:

$$m_{a_1}e^{\pm\theta_1} + m_{a_2}e^{\pm\theta_2} = m_{b_1}e^{\pm\theta_1'} + \cdots + m_{b_n}e^{\pm\theta_n'} \quad (2.11)$$

So we can see that the scattering matrix is the overlap between the associated *in*-state and *out*-state, representing the scattering amplitude from the in-state to the out-state

$$S_{a_1\cdots a_n}^{b_1\cdots b_m}(\theta_1\cdots\theta_n;\theta_1'\cdots\theta_m') = {}_{out}\langle A_{b_1}(\theta_1')\cdots A_{b_m}(\theta_m')|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in}\,. \quad (2.12)$$

2.1.2 Higher spin conserved charges, elasticity and factorisability

The very common conserved quantities in QFT are the energy H and momentum P, which transform under the Lorentz group as scalars and vectors. They are both local and can be expressed as

$$H = \int dx\; h(x), \quad P = \int dx\; p(x) \quad (2.13)$$

where $h(x)$ and $p(x)$ are energy density and momentum density, respectively. They are diagonalised by the basis of asymptotic multi-particle states:

$$H|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \left(\sum_{k=1}^{n} m_{a_k}\cosh\theta_k\right)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} \quad (2.14)$$

$$P|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \left(\sum_{k=1}^{n} m_{a_k}\sinh\theta_k\right)|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} . \quad (2.15)$$

Beyond these, in integrable models there exist also higher spin conserved charges which transform as tensors of higher rank under the Lorentz group

$$Q_s \to \Lambda^s Q_s \quad (2.16)$$

where Λ is a Lorentz boost characterised by a velocity v or a rapidity θ

$$\Lambda = \sqrt{\frac{1+v}{1-v}} = e^{\theta}$$

and integer s is called the (Lorentz) spin of Q_s. These conserved charges Q_s are regarded as tensors of rank s, and, in particular, $Q_{\pm 1}$ coincide with the light-cone components $p^{\pm} = p^0 \pm p^1$ of the energy-momentum operator $p^{\mu} = (p^0, p^1)$. In this thesis we will consider only the local conserved charges Q which are integrals of local charge densities $q(x)$

$$Q = \int dx\; q(x) \quad (2.17)$$

In fact, some integrable models also possess non-local conserved charges which are often associated to operators with fractional spin [71–79]. Taking into account relativistic covariance, the local conserved charges Q_s act as follows on asymptotic *in*- and *out*-states:

$$Q_s|A_a(\theta)\rangle_{in,out} = \xi_a^{(s)}\left(m_a e^{\theta}\right)^s|A_a(\theta)\rangle_{in,out} \quad (2.18)$$

where $\xi_a^{(s)}$ is a non-vanishing Lorentz scalar that depends on quantum number a of the particle and the spin s, and $\xi_a^{(s)}(m_a)^s$ is the one-particle eigenvalue of the charge Q_s on the particle a. The set of spin s and the one-particle eigenvalue $\xi_{a_i}^{(s)}(m_{a_i})^s$ are model-dependent and hence are good fingerprints for an integrable model. Due to the locality, the conserved charges Q_s act additively on multi-particle asymptotic states:

$$Q_s|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} = \left[\sum_{i=1}^{n}\xi_{a_i}^{(s)}\left(m_{a_i}e^{\theta_i}\right)^s\right]|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in,out} . \quad (2.19)$$

It can be seen from (2.19) that the conserved charges Q_s are simultaneously diagonalised by the basis of asymptotic multi-particle states. As a result, they are also in involution

$$[Q_s, Q_n] = 0 \tag{2.20}$$

for all s, n.

The existence of these higher spin local conserved charges has profound consequences on the scattering processes in QFT. In 1967 S. Coleman and J. Mandula demonstrated in their paper [80] that the existence of even just one conserved charge of spin higher than one leads to trivial S-matrix, namely $S = \pm 1$, in a model of QFT in more that one space dimension. This statement does not hold for two-dimensional QFT, but, in this case, an infinite number of conserved charges do impose a series of stringent conditions on the scattering processes [81–85], which are listed below:

- there is no particle production in any scattering process, namely the number of particles in the *in*- and *out*-states is the same; ;
- the set of the initial momenta of the particles is the same as that of the finial momenta, namely the scattering processes are purely elastic;
- any n-particle S-matrix can be factorized into sums of products of two-particle S-matrices.

These conditions enable us to find the full S-matrices of two-dimensional models. Following the Refs mentioned above, I shall give a couple of arguments to explain why these constraints follow from the existence of infinitely many conserved charges. In fact, it has been shown in [86] that the existence of only two conserved charges of spin higher than one is sufficient to lead to these constraints on the S-matrices in two-dimensional models of QFT. But we will employ only the arguments based on the presence of an infinite number of conserved charges.

To explain the first two properties, let us now consider a scattering process with n incoming particles and m outgoing particles, and the scattering amplitudes are

$$S_{a_1,\ldots,a_n}^{b_1,\ldots,b_m}(\theta_1,\ldots,\theta_n;\theta_1',\ldots,\theta_m') = {}_{out}\langle A_{b_1}(\theta_1')\cdots A_{b_m}(\theta_m')|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in}\,.$$

The conserved charges act on the associated *in*- and *out*-states as

$$Q_s|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in} = \left[\sum_{i=1}^{n}\xi_{a_i}^{(s)}\left(m_{a_i}e^{\theta_i}\right)^s\right]|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle_{in}$$

and

$$Q_s|A_{b_1}(\theta'_1)\cdots A_{b_m}(\theta'_m)\rangle_{out} = \left[\sum_{i=1}^{n} \xi^{(s)}_{b_i} \left(m_{b_i} e^{\theta'_i}\right)^s\right] |A_{b_1}(\theta'_1)\cdots A_{b_m}(\theta'_m)\rangle_{out}$$

respectively. In accordance with the fact that Q_s are conserved quantities

$$\frac{dQ_s}{dt} = 0\,, \tag{2.21}$$

an initial eigenstate of Q_s with a given eigenvalue must evolve into a superposition of states all with the same eigenvalue, and this imply a sequence of equations that involves the sum of the higher powers of the momenta of the initial and final particles

$$\sum_{i=1}^{n} \xi^{(s)}_{a_i} \left(m_{a_i} e^{\theta_i}\right)^s = \sum_{i=1}^{m} \xi^{(s)}_{b_i} \left(m_{b_i} e^{\theta'_i}\right)^s\,. \tag{2.22}$$

The number of these equations is infinite since we assume the existence of infinitely many conserved charges. Thus the only solution to these infinite numbers of equations with different values of s is that

$$n = m\,, \qquad \theta_i = \theta'_i\,, \qquad \xi^{(s)}_{a_i}\left(m_{a_i}\right)^s = \xi^{(s)}_{b_i}\left(m_{b_i}\right)^s\,, \tag{2.23}$$

for $i = 1, \ldots, n$. Therefore, there is no particle production and the set of momenta of the particles in *in*- and *out*-states coincide. But this does not suggest that the outgoing set of quantum numbers $\{b_1, \ldots, b_n\}$ must be equal to the ingoing set $\{a_1, \ldots, a_n\}$. The possible exchange of quantum numbers between ingoing particles and outgoing particles is allowed in the case of degenerate spectrum in which more than one particle share the same mass. It should be mentioned that, in some models, there exist also some solutions with $n \neq m$ to those equations (2.22). However, these solutions can be only found for some special sets of initial momenta in the presence of bound states [98].

In addition to being elastic, the scattering processes in two-dimensional integrable QFT are also factorised [81–86]. The proof of the S-matrix factorisability requires the assumption that any asymptotic one-particle state $|A_{a_i}(\theta_i)\rangle$ can be represented by a localised wave packet $\Psi_{a_i}\left(x^0, x^1\right)$

$$\begin{aligned} \Psi_{a_i}\left(x^0, x^1\right) &= N \int dp^1 e^{f(p^1)} \\ f\left(p^1\right) &= -a(p^1 - p^1_{a_i})^2 + i\left[p^1\left(x^1 - x^1_{a_i}\right) - p^0_{a_i}\left(x^0 - x^0_{a_i}\right)\right] \end{aligned} \tag{2.24}$$

where $p^\mu_{a_i} = (p^0_{a_i}, p^1_{a_i})$ is the energy-momentum of the particle a_i and $\left(x^0_{a_i}, x^1_{a_i}\right)$ the coordinates of the central position of the wave packet, namely the approximated time-space position of the particle a_i. N is a normalisation constant and a is a constant expressing the spreading on the velocity of the wave packet. Even though the wave function formalism is not valid in the context of relativistic QFT due to particle production and annihilation, we can exploit it in the framework of asymptotic states in which particles are propagating freely without interactions. In this way, we can associate to each particle in the multi-particle *in*-states or *out*-states a localised wave packet like (2.24). According to the wave packet description, the physical scattering matrix also depends on the impact parameters, which are the central positions of the particles with respect to each other in the asymptotic states. In our construction of the asymptotic states, for convenience, we choose the impact parameters of the scattering so that all particles collide at one point under an extrapolation of their free trajectories, namely the central positions of the wave packets of all particles are same. We could also define the asymptotic states with the central position of each wave packet shifted slightly, which means that some particles could collide first in an extrapolation of the free trajectories. Hence the scattering matrix, which is formed by the overlap between the associated *in*- and *out*-states, is different for different choices of impact parameters. Now, let us consider the action of an operator $e^{i\alpha Q_s}$ for free parameter α on the wave function (2.24) and this amounts to shifting the coordinates of the center of the wave packet as follows

$$x^0_{a_i} \longrightarrow x^0_{a_i} + \alpha s \xi^s_{a_i} \left(m_{a_i} e^{\theta_i} \right)^{s-1}, \tag{2.25}$$

$$x^1_{a_i} \longrightarrow x^1_{a_i} + \alpha s \xi^s_{a_i} \left(m_{a_i} e^{\theta_i} \right)^{s-1}. \tag{2.26}$$

It is worth noting that the shift mentioned above is in general rapidity-dependent, excluding the case $s = 1$ for which all wave packets are shifted by the same amount α. Consequently, it is feasible, via such an operator with $s > 1$, to make an asymptotic multi-particle state in which each wave packet has its central position independently shifted. In this way, the higher spin conserved charges lead to alterations in the impact parameters. According to this formalism, we can demonstrate the feature of the factorization of the S-matrix by analyzing 3 particle $\longrightarrow$ 3 particle scattering processes depicted in Fig. 2.1 The central positions of wave packets can be shifted enough by the action of the operator $e^{i\alpha Q_s}$ so that as time evolves a well-defined first sub-process takes place, in which two wave packets collide while the other one propagate freely. The set of momenta is preserved due to the absence of particle production and hence there is still a well-defined momenta configuration after the collision. Since particles propagate on a line in one space dimension, this two-particle scattering procedure will be repeated until all wave packets have their order inverted. The full scattering process will consists of three two-particle scattering sub-processes separated by free propagation. These two-particle scattering processes will occur in different orders for different ways of shifting central positions of wave packets.

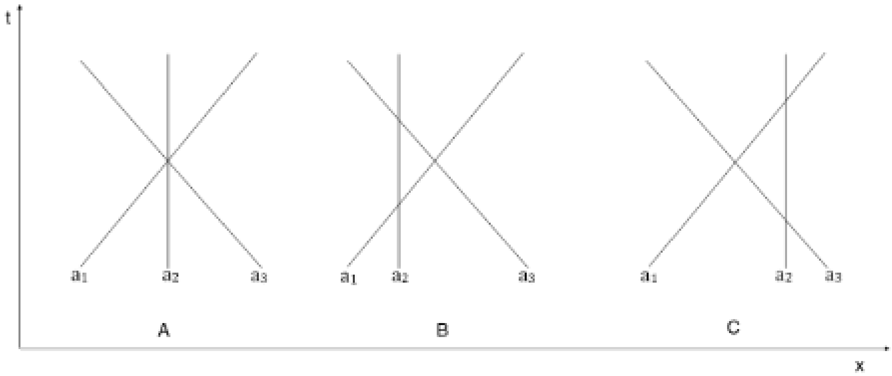

FIGURE 2.1: 3 particle $\longrightarrow$ 3 particle scattering processes

In the case of 3 $\longrightarrow$ 3 scattering, there are two different ways to shift central positions of wave packets, which are depicted as B and C in Fig. 2.1.

In a general QFT, the three processes in Fig. 2.1, including the simultaneous collision of the three particles, are not related to each other and their associated scattering amplitudes are not same. However, the situation is different for a QFT with the presence of higher spin conserved charges. Since the conserved charges Q_s commute with the hamiltonian of the system, their action must result in equivalent physical situation. This means that the scattering amplitude must be invariant under any transformation generated by conserved charges

$$S_{a_1,\ldots,a_n}^{b_1,\ldots,b_n} = e^{-\alpha Q_s} S_{a_1,\ldots,a_n}^{b_1,\ldots,b_n} e^{\alpha Q_s}. \tag{2.27}$$

In this spirit, the three scattering processes in Fig. 2.1 should share the same scattering amplitude. The equality of the amplitude B and C leads to a constraint on the two-particle S-matrices, known as the so-called Yang-Baxter equation [87]

$$S_{a_1a_2}^{kp}(\theta_{12})S_{ka_3}^{b_1r}(\theta_{13})S_{pr}^{b_1b_2}(\theta_{23}) = S_{a_1k}^{rb_3}(\theta_{13})S_{a_2a_3}^{pk}(\theta_{23})S_{rp}^{b_1b_2}(\theta_{12}) \tag{2.28}$$

where a repeated index implies a summation. In particular, if the particle spectrum is non-degenerate, the quantum numbers of the particles are also preserved in the scattering processes. In this case, the S-matrix is diagonal and the Yang-Baxter equation becomes trivial. It is also obvious that the same argument can be applied to any $n \longrightarrow n$ scattering process in which the scattering matrix will factorise into a product of $n(n-1)/2$ two-particle scattering matrices. Finally, from the above arguments, we can see that in any two-dimensional IQFT, with the presence of an finite numbers of local conserved charges, the scattering matrix does not depend on impact parameters.

The factorization properties of the S-matrix can be implemented by assuming the asymptotic operators of creation and annihilation type to obey the highly non-trivial algebra

$$A_{a_i}(\theta_i)A_{a_j}(\theta_j) = \sum_{b_i,b_j} S_{a_i a_j}^{b_i b_j}(\theta_i-\theta_j)A_{b_j}(\theta_j)A_{b_i}(\theta_i)\,, \tag{2.29}$$

$$A^\dagger_{a_i}(\theta_i)A^\dagger_{a_j}(\theta_j) = \sum_{b_i,b_j} S_{a_i a_j}^{b_i b_j}(\theta_i-\theta_j)A^\dagger_{b_j}(\theta_j)A^\dagger_{b_i}(\theta_i), \tag{2.30}$$

$$A_{a_i}(\theta_i)A^\dagger_{a_j}(\theta_j) = \sum_{b_i,b_j} S_{a_i a_j}^{b_i b_j}(\theta_j-\theta_i)A^\dagger_{b_j}(\theta_j)A_{b_i}(\theta_i) + 2\pi\delta_{a_i a_j}(\theta_i-\theta_j)\,. \tag{2.31}$$

where we dropped out the subindices *in* or *out* in these operators since definitions (2.2) and (2.3) allow to distinguish the set of incoming and outgoing particles by the ordering of the rapidities. This algebra is known as Zamolodchikov's algebra [82] and named also as Faddeev-Zamolodchikov algebra [88]. This algebra provides the generalization of the usual bosonic and fermionic algebraic relations. These asymptotic operators define the space of physical states in integrable QFT: the *in*-basis is formed by products of these operators with rapidities in decreasing order from left to right; the *out*-basis is formed by similar products of these operators with rapidities in increasing order. Since the S-matrix is involved in this algebra, any commutation of these operators can be interpreted as a scattering process. It is worth noting that the two-particle S-matrix only depends on the rapidity difference of scattering particles because of its Lorentz invariance. The explicit form of this algebra depends on the S-matrix involved which varies with different theories.

2.2 Analytic structures of the two-particle S-matrix

In light of the properties of elasticity and factorisability, the S-matrix theory of a two-dimensional massive integrable model is drastically simplified and the problem of finding the exact full S-matrices has been reduced to the determination of all two-particle S-matrices which are associated with the different $2 \longrightarrow 2$ scattering processes in the model. The explicit expression of two-particle S-matrices can be obtained by solving, in addition to the highly non-trivial Yang-Baxter equation following from quantum integrability, a set of restrictive equations [82, 89] which arise from general physical principles of QFT:

- Hermitian analyticity: $\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{b_2,b_1}^{a_2,a_1}(-\theta^*)$
- Unitarity: $S_{a_1,a_2}^{b_1,b_2}(\theta)S_{b_1,b_2}^{c_1,c_2}(-\theta) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2}$
- Crossing symmetry: $S_{a_1,a_2}^{b_1,b_2}(i\pi-\theta) = S_{a_1,\bar{b}_2}^{b_1,\bar{a}_2}(\theta).$

Let us begin with a detailed discussion of the physical requirements listed. In a general model of QFT, the scattering matrix is usually expressed in terms of the so-called Mandelstam variables s, t and u, which are defined as

$$s = (p_1 + p_2)^2, \quad t = (p_1 - p_3)^2, \quad u = (p_1 - p_4)^2 \tag{2.32}$$

with $s + t + u = \sum_{i=1}^{4} m_i^2$ for a scattering process of the type $1 + 2 \longrightarrow 3 + 4$. In two-dimensional integrable models, the variable u vanishes, namely $u = 0$, due to the elasticity of the scattering process. Therefore, only one of these three variables is independent and it is convention to focus on s. Consider now a two-particle elastic scattering process $a_1 + a_2 \longrightarrow b_1 + b_2$ with the scattering amplitude $S_{a_1,a_2}^{b_1,b_2}(s) = S_{a_1,a_2}^{b_1,b_2}(\theta)$. In terms of rapidity difference $\theta = \theta_1 - \theta_2$, we have

$$s = m_{a_1}^2 + m_{a_2}^2 + 2m_{a_1}m_{a_2}\cosh\theta \; . \tag{2.33}$$

For a physical process, both θ and s are real. We can analytically continue s to the complex plane, and the two-particle S-matrix, as a function of s, is a multi-valued function. The branch points of $S_{a_1,a_2}^{b_1,b_2}(s)$ start at the values $s = (m_{a_1} + m_{a_2})^2$, and the corresponding branch cuts are on the portions of the real axis $s \leq (m_{a_1} - m_{a_2})^2$ and $s \geq (m_{a_1} + m_{a_2})^2$. Therefore, $S_{a_1,a_2}^{b_1,b_2}(s)$ is not a meromorphic function. The first sheet of the full Riemann surface for $S_{a_1,a_2}^{b_1,b_2}(s)$ is called physical sheet. The physical values of $S_{a_1,a_2}^{b_1,b_2}(s)$ are just above the cut $[(m_{a_1} + m_{a_2})^2, \infty]$ and this "just above the cut" prescription is in the spirit of Feynman's $i\epsilon$ prescription in perturbation theory. The simple poles are in the interval $[(m_{a_1} - m_{a_2})^2, (m_{a_1} + m_{a_2})^2]$ of the real line and they correspond to possible bound states in the scattering process.

Let us turn our attention from the Mandelstam variable s to the rapidity difference θ, via transformation

$$\begin{aligned} \theta &= \cosh^{-1}\left(\frac{s - m_{a_1}^2 - m_{a_2}^2}{2m_{a_1}m_{a_2}}\right) \\ &= \log\left[\frac{1}{2m_{a_1}m_{a_2}}\left(s - m_{a_1}^2 - m_{a_2}^2 + \sqrt{(s - (m_{a_1} + m_{a_2})^2)\,(s - (m_{a_1} - m_{a_2})^2)}\right)\right] . \end{aligned} \tag{2.34}$$

Since $s(\theta) = s(-\theta)$, the mentioned branch cuts do not occur in the θ-plane, and therefore the two-particle S-matrix is a meromorphic function of θ. The physical sheet corresponds to the region $\mathrm{Im}\,\theta \in [0, \pi]$ and this region is called physical strip. The poles corresponding to bound states are located on the imaginary axis $\mathrm{Re}(\theta) = 0$ in the physical strip.

Hermitian analyticity [90–92] is one of general principles of QFT and it postulates that the complex conjugate of the S-matrix on the physical sheet is equivalent with the S-matrix on the complex conjugate argument still on the physical sheet:

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{b_2,b_1}^{a_2,a_1}(-\theta^*)\,. \tag{2.35}$$

Since we are looking at models of relativistic QFT, the assumption of charge-parity-time-reversal (CPT) invariance should be considered

$$\begin{aligned} S_{a_1,a_2}^{b_1,b_2}(\theta) &= S_{\bar{a}_1,\bar{a}_2}^{\bar{b}_1,\bar{b}_2}(\theta), \qquad C && (2.36)\\ S_{a_1,a_2}^{b_1,b_2}(\theta) &= S_{a_2,a_1}^{b_2,b_1}(\theta), \qquad P && (2.37)\\ S_{a_1,a_2}^{b_1,b_2}(\theta) &= S_{b_2,b_1}^{a_2,a_1}(\theta), \qquad T. && (2.38)\end{aligned}$$

Hermitian analyticity, together with time-reversal invariance (2.38), leads to real analyticity

$$\left(S_{a_1,a_2}^{b_1,b_2}(\theta)\right)^* = S_{a_1,a_2}^{b_1,b_2}(-\theta^*)\,. \tag{2.39}$$

In general QFTs, the total probability of producing an arbitrary *out*-state from any *in*-state is postulated to be one. This probability conservation condition requires the S-matrix to be unitary

$$SS^\dagger = 1\,. \tag{2.40}$$

This should be interpreted as a matrix equation, with a sum over a complete set of intermediate states hiding between S and $S^\dagger$. In principle, with sufficiently high energy, any n-particle intermediate state can be allowed in the sum. However, in integrable models, integrability restricts intermediate states to two-particle states, meaning that the amplitudes of producing intermediate states with more than two particles are all zero. In this case, for real θ, unitarity reads

$$S_{a_1,a_2}^{b_1,b_2}(\theta)\left(S_{c_1,c_2}^{b_1,b_2}(\theta)\right)^* = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2}\,. \tag{2.41}$$

In virtue of PT invariance (2.37), (2.38), and real analyticity, this can be rewritten as

$$S_{a_1,a_2}^{b_1,b_2}(\theta)\left(S_{b_1,b_2}^{c_1,c_2}(-\theta)\right) = \delta_{a_1}^{c_1}\delta_{a_2}^{c_2} \tag{2.42}$$

which is usually called unitarity for two-particle S-matrix in integrable models. By analytic continuation to the complex plane, the relation (2.42) is assumed to hold for any complex value of θ.

Crossing symmetry is a fundamentally relativistic property of QFT. In the process of two-particle scattering $a_1 + a_2 \longrightarrow b_1 + b_2$, any particle involved can be replaced by its antiparticle, which is interpreted as the particle traveling back in time with opposite momentum. A new two-particle scattering $a_1 + \bar{b}_2 \longrightarrow b_1 + \bar{a}_2$ can be anticipated from this fact, where the overbar indicates the antiparticle, and crossing symmetry states that the scattering amplitude for this process can be obtained as the analytic continuation of the previous one to negative energies. In terms of equations, crossing symmetry can be expressed as

$$S_{a_1,a_2}^{b_1,b_2}(i\pi - \theta) = S_{a_1,\bar{b}_2}^{b_1,\bar{a}_2}(\theta) \tag{2.43}$$

and this relation can be understood via the following consideration. For the two-particle scattering $S_{a_1,a_2}^{b_1,b_2}(\theta_1 - \theta_2)$, we rotate the time and space arrow by $\pi/2$, which, since the slope of the world line of the particle j is $\coth\theta_j$, amounts to $\theta_j \mapsto i\pi/2 - \theta_j$. After this rotation, particle a_2 and b_2 seem like propagating back in time. Then we transform them, with the help of time-reversal symmetry, into their antiparticles which propagate correctly, and this gives crossing symmetry (2.43).

Finally, in the presence of bound states, a set of additional conditions, called "bootstrap equations" [81, 93, 94], will arise in the integrable theory to further constrain the S-matrix. A bound state is manifested by a virtual particle, which is created in a two-particle scattering process and identified as an asymptotic particle already in the spectrum of the theory, and it generates in the S-matrix a pole lying on the imaginary axis in the physical strip $0 < \mathrm{Im}(\theta) < \pi$. For the scattering process $a_1 + a_2 \longrightarrow c$, the corresponding pole in the S-matrix is given by

$$\theta = i \arccos\left(\frac{m_c^2 - m_{a_1}^2 - m_{a_2}^2}{2m_{a_1}m_{a_2}}\right) . \tag{2.44}$$

In fact, the bootstrap equations not only restrain the S-matrix but also affect the masses of the particles, the set of spin s of the conserved charges and the eigenvalues of the conserved charges [93, 95–97] (see the review [98]).

In light of internal symmetries and bootstrap equations, the work of reconstructing the S-matrix and the spectrum of particles has been performed in several models [99, 100]. In general, it is not trivial to obtain the S-matrix as the solution to those conditions mentioned above. However, the spectrum and S-matrix can sometimes be exactly evaluated by the method of Bethe ansatz [101–104] or the quantum inverse scattering [105] (see also the book [106]).

2.3 Form factors and correlation functions

The scattering matrix is not directly useful for most calculations related to experimental situations. The objects that are of direct importance are the correlation functions of local fields which encode all physical information. In particular, the two-point function

$$\langle \text{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\text{vac}\rangle$$

is a quantity often required, as it is related to the response function of the system at one point once it is disturbed at another point. In principle, all quantities of interest in a model can be obtained from the knowledge of correlation functions. Therefore, the main problem in the study of a model of quantum field theory can then be seen as the reconstruction of its correlation functions. In general, the computation of correlation functions is a difficult task, usually achieved with partial success through perturbative methods. Probably the most fruitful idea is to start from a representation of the two-point function coming from inserting a complete set of energy eigenstates between the operators:

$$\begin{aligned} &\langle \text{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\text{vac}\rangle \\ = &\sum_{n=0}^{\infty}\sum_{a_1,\ldots,a_n}\int_{\theta_1>\cdots>\theta_n}\frac{d\theta_1\cdots d\theta_n}{(2\pi)^n}e^{-iE_nt+ip_nx}\Big[\langle \text{vac}|\mathcal{O}_1(0,0)|\theta_1,\ldots,\theta_n\rangle_{a_1,\ldots,a_n} \\ &{}_{a_1,\ldots,a_n}\langle\theta_1,\ldots,\theta_n|\mathcal{O}_2(0,0)|\text{vac}\rangle\Big] \end{aligned} \tag{2.45}$$

for $E_n = \sum_{j=1}^n m_{a_j}\cosh\theta_j$ and $p_n = \sum_{j=1}^n m_{a_j}\sinh\theta_j$. This is the basis for the usual Källen-Lehmann spectral decomposition. The *in*-states are, as we define, wave packets at minus infinite time with particles ordered from left to right by decreasing rapidity. Of course, we could have used as well the *out*-basis. Using the fact that these bases are not independent, we can rewrite (2.45) in terms of all bases:

$$\begin{aligned} &\langle \text{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\text{vac}\rangle \\ = &\sum_{n=0}^{\infty}\sum_{a_1,\ldots,a_n}\int\frac{d\theta_1\cdots d\theta_n}{(2\pi)^n n!}e^{-iE_nt+ip_nx} \\ &\times\langle \text{vac}|\mathcal{O}_1(0,0)|\theta_1,\ldots\theta_n\rangle_{a_1,\ldots,a_n}\ {}_{a_1,\ldots,a_n}\langle\theta_1,\ldots,\theta_n|\mathcal{O}_2(0,0)|\text{vac}\rangle \end{aligned} \tag{2.46}$$

where the factors $1/n!$ come from overcounting repeated intermediate states. Note that we will from now on employ the notation $|\theta_1,\ldots,\theta_n\rangle_{a_1,\ldots,a_n}$ to represent an n-particle state instead of the notation $|A_{a_1}(\theta_1)\cdots A_{a_n}(\theta_n)\rangle$. The matrix elements between the vacuum state and an n-particle state

$$\langle \mathrm{vac}|\mathcal{O}_1(0,0)|\theta_1,\ldots\theta_n\rangle_{a_1,\ldots,a_n}$$

are called form factors, which are the central quantities in integrable theories. It is convention to denote by

$$f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$$

the form factors of a certain operator $\mathcal{O}$. Thanks to crossing symmetry and the S-matrix, it is possible to deduce all matrix elements of $\mathcal{O}$ between asymptotic states in terms of the form factors. Hence, we can completely describe the local field once its form factors are known. The form factors $f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$, as tensor-value functions, can be analytically continued to complex values of θ_i and it has been shown in [3] that the analytically continued form factors of a local field must satisfy a set of axioms, which arise as the direct consequences of factorized scattering and general principles of QFT. Therefore, similarly to the construction procedure of S-matrices for integrable massive 1+1-dimensional QFTs described before, the form factors associated to a certain operator can be obtained as the solutions to a set of consistency equations. These consistency equations form what is called Riemann-Hilbert problem and are formulated as follows:

1. Meromorphicity: the form factor $f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$, as function of the variable $\theta_i-\theta_j$, for any $i,j\in\{1,\ldots,n\}$, is analytic inside the strip $0<\mathrm{Im}(\theta)<2\pi$, apart from some simple poles corresponding to the bound states.

2. Relativistic invariance:

$$f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1+\beta,\ldots,\theta_n+\beta)=e^{s\beta}f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)\,,$$

   where s is the spin of $\mathcal{O}$.

3. Generalized Watson's theorem:

$$f^{\mathcal{O}}_{a_1,\ldots,a_j,a_{j+1},\ldots,a_n}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_n)= \\ S^{b_j,b_{j+1}}_{a_j,a_{j+1}}(\theta_j-\theta_{j+1})f^{\mathcal{O}}_{a_1,\ldots,b_{j+1},b_j,\ldots,a_n}(\theta_1,\ldots,\theta_{j+1},\theta_j,\ldots,\theta_n)\,.$$

4. Monodromy:

$$f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1+2\pi i,\ldots,\theta_n)=(-1)^{f_{\mathcal{O}}f_{\Psi}}e^{2\pi i\omega(\mathcal{O},\Psi)}f^{\mathcal{O}}_{a_2,\ldots,a_n,a_1}(\theta_2,\ldots,\theta_n,\theta_1)$$

   where $f_{\mathcal{O}}=1$ if $\mathcal{O}$ is fermionic, $f_{\mathcal{O}}=0$ if $\mathcal{O}$ is bosonic, Ψ is the fundamental field that creates the particle a_n, and $\omega(\mathcal{O},\Psi)$ is the semi-locality index of $\mathcal{O}$ with respect to Ψ, which is the phase factor arising in correlation functions when one of the fields is turned once around another field counterclockwise and was originally introduced in [107].

5. Kinematic pole: the form factor $f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$, as function of the variable θ_n, has some simple poles, called kinematic poles, at the points $\theta_n = \theta_j + i\pi$ for $j \in 1,\ldots,n-1$, with residues

$$
if^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \sim C_{a_n,b_j} \frac{f^{\mathcal{O}}_{b_1,\ldots,\hat{b}_j,\ldots,b_{n-1}}(\theta_1,\ldots,\hat{\theta}_j,\ldots,\theta_{n-1})}{\theta_n - \theta_j - i\pi} \times
\Big[\delta^{b_1}_{a_1}\cdots\delta^{b_{j-1}}_{a_{j-1}} S^{b_{j+1},c_j}_{a_{j+1},a_j}(\theta_{j+1}-\theta_j) S^{b_{j+2},c_{j+1}}_{a_{j+2},c_j}(\theta_{j+2}-\theta_j)\cdots S^{b_{n-1},b_j}_{a_{n-1},c_{n-3}}(\theta_{n-1}-\theta_j) -
(-1)^{f_{\mathcal{O}} f_\Psi} e^{2\pi i\omega(\mathcal{O},\Psi)} \delta^{b_{n-1}}_{a_{n-1}}\cdots\delta^{b_{j+1}}_{a_{j+1}} S^{c_j,b_{j-1}}_{a_j,a_{j-1}}(\theta_j-\theta_{j-1}) S^{c_{j-1},b_{j-2}}_{c_j,a_{j-2}}(\theta_j-\theta_{j-2})\cdots S^{b_j,b_1}_{c_3,a_1}(\theta_j-\theta_1)\Big].
$$

6. Bound state: the form factor $f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n)$, as function of the variable $\theta_i - \theta_j$, in the presence of bound states of mass m, has some simple poles at the points $\theta_i - \theta_j = i \arccos\left(\frac{m^2 - m^2_{a_i} - m^2_{a_j}}{2m_{a_i}m_{a_j}}\right) \in i[0,\pi]$.

Axiom 1 is required by the analytic properties of the general QFT. Axiom 2 is a direct consequence of relativistic invariance. Axiom 3 is obtained by the braiding relation (2.30), which means that a commutation of two asymptotic operators is equivalent to a scattering process. Axiom 4 results from the analysis based on LSZ reduction formulae [108] and it states the discontinuity of the form factors at the cuts $\theta_{1i} = 2\pi i$. Concerning axiom 5 and axiom 6, they describe the pole structure of the form factors. Two kinds of simple poles are involved in the form factors. The first are kinematical poles and they come from the one-particle state realized by the three-particle clusters, which corresponds to the crossing channel of the S-matrix. Kinematical poles do not depend on the existence of bound states. The second are poles coming from the two-particle cluster and they are related to bound states of the S-matrix. It is worth noticing that both types of poles of the form factors are determined by the underlying scattering theory and they are operator-independent. Such pole structure plays a very crucial role in finding exactly the solutions to the form factor consistency equations, since form factors with different number of particles can be related by the residues of these poles. This point can be made more clear in the case of diagonal scattering where two sets of recursion relations, relating the $(n+2)$- and the n-particle form factors and the $(n+1)$- and the n-particle form factors respectively, arise from the poles mentioned in axiom 5 and 6. Notice that if we assume the S-matrix to be diagonal, in virtue of Axiom 3, Axiom 4 can be rewritten as

$$
f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1 + 2\pi i,\ldots,\theta_n) = (-1)^{f_{\mathcal{O}} f_\Psi} e^{2\pi i\omega(\mathcal{O},\Psi)} \prod_{i=2}^{n} S_{a_i,a_1}(\theta_{i1}) f^{\mathcal{O}}_{a_1,\ldots,a_n}(\theta_1,\ldots,\theta_n) \quad (2.47)
$$

which turn out to be a set of recursion relations relating the n- and n-particle form factors.

Form factors associated to a certain operator can be obtained as the solution to the set of consistency equations summarised above. Once the solution has been found, the identification of the operator it corresponds to is required, due to the fact that some of the consistency equations, namely axiom 1, 4, 5 and possibly 6 in the presence of bound states, do not refer to any particular nature of the operator involved. It is an assumption that each solution to the form factor consistency equations corresponds to a particular local operator [1, 109]. Based on this assumption, the work of identifying and constraining the specific content of the operator has been carried out in numerous papers [1, 3, 4, 108–114], by exploiting various methods including investigating asymptotic behaviours, performing perturbation theory, taking symmetries into account and formulating quantum equations of motion.

The locality of the operator is a fundamental requirement which needs to be proved in order to guarantee that we are looking at a well-defined QFT. Even though the locality of the operator has been encoded in the form factor consistency equations, since there exists no well established proof for it and several consistency equations do not require any information of the operator content, the verification of it can be of great interest. Inspired by the definition of the locality of bosonic or fermionic operators

$$[\mathcal{O}_1(x_1,t_1),\mathcal{O}_2(x_2,t_2)]=0 \qquad \text{or} \qquad \{\mathcal{O}_1(x_1,t_1),\mathcal{O}_2(x_2,t_2)\}=0 \tag{2.48}$$

with (x_1,t_1), (x_2,t_2) causally disconnected points in Minkowski's space and $\mathcal{O}_1$, $\mathcal{O}_2$ local operators of the QFT, we can verify the locality property by evaluating correlation functions of the form

$$\langle\text{vac}|\,[\mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)]\,|\text{vac}\rangle \qquad \text{or} \qquad \langle\text{vac}|\{\mathcal{O}_1(x_1,t_1)\mathcal{O}_2(x_2,t_2)\}|\text{vac}\rangle \tag{2.49}$$

via the spectral representation series involving form factors associated to the operators $\mathcal{O}_1$, $\mathcal{O}_2$. If such correlation functions vanish for all causally disconnected points (x_1,t_1), (x_2,t_2), the locality of the operators $\mathcal{O}_1$, $\mathcal{O}_2$ will be confirmed.

The work of reconstructing form factors from the form factor consistency equations was initiated, in the context of two-dimensional QFT's, by P. Weisz and M. Karowski [1, 109]. Then, the development of the form factor approach has been performed to a large extent by several authors including F. A. Smirnov, J. L. Cardy and G. Musssardo [3, 110, 115–121]. It turns out that this approach is very powerful within two-dimensional integrable QFT's due to its wide range of applications like determining form factors associated to certain operators, evaluating correlation functions, characterizing the operator content of the perturbed conformal field theory.

As mentioned before, once the matrix elements of the operators, namely the form factors, are known, their correlation functions can be recovered in terms of spectral representation series (2.46). It is worth mentioning that these series present remarkable convergence properties. The form factors, in contrast with Feynman formalism, employ from the beginning all the physical parameters of the theory and hence divergences of the perturbative series can be avoided. Also, it has been proved that these form factor expansions in integrable models allow for evaluating correlation functions in a wide rage of energies with a high numerical precision, and together with conformal perturbation theory they give correlation functions at all energy scales. By using space-time translation and relativistic invariances, form factor expansions (2.46) at space-like separations can be rewritten as

$$\begin{aligned}&\langle \mathrm{vac}|\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)|\mathrm{vac}\rangle \\ =&\sum_{n=0}^{\infty}\sum_{a_1,\ldots,a_n}\int \frac{d\theta_1\cdots d\theta_n}{(2\pi)^n n!}e^{-r\sum_j m_{a_j}\cosh\theta_j} \\ &\quad\times\langle \mathrm{vac}|\mathcal{O}_1(0,0)|\theta_1,\ldots,\theta_n\rangle_{a_1,\ldots,a_n}\ {}_{a_1,\ldots,a_n}\langle\theta_1,\ldots,\theta_n|\mathcal{O}_2(0,0)|\mathrm{vac}\rangle\end{aligned} \tag{2.50}$$

where r is the Minkoswki distance $r=\sqrt{x^2-t^2}$ with $x^2>t^2$. This representation is more useful since it is a large-distance expansion, which converges well at large r and describes the large-distance behaviour of the two-point functions, and which is hardly accessible by perturbation theory.

2.4 Ising model

2.4.1 Ising quantum chain

Let us commence with the generic XY model defined by the infinite-length quantum chain of spin-$\frac{1}{2}$ degree of freedom and the Hamiltonian of the total system is expressed as

$$H_{XY}=-\frac{J}{2}\sum_n\left[\frac{1+\kappa}{2}\sigma_n^x\sigma_{n+1}^x+\frac{1-\kappa}{2}\sigma_n^y\sigma_{n+1}^y+h\sigma_n^z\right]. \tag{2.51}$$

The σ_n^i with $i=x,y,z$ are pauli matrices at site n with the representation

$$\sigma^x=\begin{pmatrix}0&1\\1&0\end{pmatrix},\qquad \sigma^y=\begin{pmatrix}0&-i\\i&0\end{pmatrix},\qquad \sigma^z=\begin{pmatrix}1&0\\0&-1\end{pmatrix}. \tag{2.52}$$

There exists a two-dimensional Hilbert space of spin-$\frac{1}{2}$ degree of freedom at each site of the chain and the full quantum space is the tensor product of all such two-dimensional spaces, for all sites of the chain. In this spirit, σ_n^i can be considered as the operator acting on the n-th site of the chain

$$\sigma_n^i = \mathbf{1} \otimes \mathbf{1} \otimes \cdots \otimes \sigma^i \otimes \cdots \otimes \mathbf{1} \tag{2.53}$$

with $\mathbf{1}$ the identity operator. At the n-th site, the basis of the associated two-dimensional quantum space is given by

$$|\uparrow\rangle_n = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \qquad |\downarrow\rangle_n = \begin{pmatrix} 0 \\ 1 \end{pmatrix} \tag{2.54}$$

which are the two eigenstates of the operator σ_n^z with eigenvalues 1 and -1, respectively. The operator σ_n^x are off-diagonal in the basis of these states and their eigenstates are given by a linear superposition of these basis states

$$\begin{aligned} |\rightarrow\rangle_n &= \left(|\uparrow\rangle_n + |\downarrow\rangle_n\right)/\sqrt{2} \\ |\leftarrow\rangle_n &= \left(|\uparrow\rangle_n - |\downarrow\rangle_n\right)/\sqrt{2} \end{aligned} \tag{2.55}$$

with eigenvalues ± 1. The operators σ^i at different sites will commute with each other, namely

$$\left[\sigma_n^i, \sigma_m^j\right] = 0\,, \qquad n \neq m \tag{2.56}$$

with $i = x, y, z$ and $j = x, y, z$, because they act on different spin states. $\kappa \in [-1, 1]$ is the anisotropy parameter and h is the dimensionless parameter representing the external transverse magnetic field. Due to the invariance of the Hamiltonian (2.51) under the operation $(h, \kappa) \mapsto (-h, -\kappa)$, we can focus on without loss of generality the region $(h \geq 0, \kappa \geq 0)$. J is an energy coupling constant describing the relative interaction between the nearest two sites. The absolute value of J determines the energy scale and the sign of it sets the type of interactions between the spins. In this thesis, we restrict ourselves to the regime $J > 0$, namely the ferromagnetic regime, in which the spins have lower energy when the neighbors are polarized parallel.

At the special value $\kappa = 1$, the XY model is reduced to the quantum Ising model in a transverse field with the Hamiltonian

$$H = -\frac{J}{2} \sum_n \left[\sigma_n^x \sigma_{n+1}^x + h\sigma_n^z\right]. \tag{2.57}$$

One of the most interesting feature of the quantum Ising model is that it exhibits a second order phase transition at a special point $h = h_c$, which is called the critical point. Generally speaking, second order phase transitions are those at which the characteristic energy scale of fluctuations above the ground state goes to zero as h approaches h_c. Let us denote by Δ the energy gap between the lowest excitation and the ground state. In most cases, it can be shown that as h approaches h_c, Δ vanishes as

$$\Delta \sim J|h - h_c|^{zv} \tag{2.58}$$

where zv is a critical exponent. This feature of quantum phase transition is called as a level-crossing between the energy level of the ground state and of the lowest excited state. Moreover, the ground state energy, which is a smooth function of h, is not analytic at the critical point $h = h_c$. It is worth mentioning that the above level-crossing does not generally occur for the case of a finite lattice. Instead, there is an avoided level-crossing between the energy levels of the lowest excited state and the ground state. However, as the lattice size increases, the avoided level-crossing gets sharper and give rise to a point of nonanalyticity at $h = h_c$ in the infinite lattice limit. In this sense, the above level-crossing can be seen as the limiting case of an avoided level-crossing.

In addition to the vanishing energy gap between the ground state and the lowest excited state, and the nonanalyticity in the ground state energy, a second order quantum phase transition is usually accompanied by a diverging correlation length ξ. Correlation length is the length scale which determines the exponential decay of equal-time correlation functions in the ground state and describes the system around the criticality. For instance, in the region $h > h_c$, it has been demonstrated by sophisticated calculations that correlation functions for two separated sites are short-ranged decreasing exponentially at large separations:

$$\langle 0|\sigma_n^x \sigma_m^x|0\rangle \sim e^{-|n-m|/\xi}, \qquad |n-m| \to \infty \tag{2.59}$$

where we denote by $|0\rangle$ the ground state of the system. The correlation length is a function of h and as h approaches h_c it diverges as

$$\xi \sim \frac{1}{|h - h_c|^v} \tag{2.60}$$

Considering (2.58) and (2.60), we can relate the energy gap Δ with the correlation length ξ:

$$\Delta \sim \xi^{-z} \tag{2.61}$$

from which z is called dynamic critical exponent. At the critical point $h = h_c$, the correlation length become infinite and a system exhibiting this type of behaviour is said to undergo a critical phase transition. Correlation functions at the critical point, due to the divergent correlation length, display an algebraic decay with the power law:

$$\langle 0|\sigma_n^x \sigma_m^x|0\rangle \sim |n-m|^{-2d}\,, \qquad |n-m| \to \infty \tag{2.62}$$

where d is a critical exponent with positive value and it is usually referred to as scaling dimension of the observable. It is worth mentioning that the exponents in (2.58), (2.60) and (2.62) are universal, which means that they do not depend on the microscopic details of the Hamiltonian. This is universality and it arises in the scaling limit which will be introduced later. The assertion of universality is that the thermodynamic properties of a system in the scaling limit show the insensitivity to the microscopic details of the system and depend only on a small number of features, such as dimensionality and symmetry. In light of universality, it is plausible to describe different and physically unrelated phenomena with the same theory. In the Ising model, the critical exponents zv, v and d are in the Ising universality class and their effective values are called Ising values.

Then let us turn our attention to the local magnetization M^x along the x-axis. This quantity is defined as the expectation value of σ_n^x in the ground state:

$$M^x = \langle 0|\sigma_n^x|0\rangle$$

and this definition does not depend on the position because of translation invariance. The local magnetization M^x determines the physical properties and characterizes two different phases in which the system displays very different responses to the external disruption, namely the transverse maganetic field, and which are divided by the critical point h_c. In the region $h < h_c$, the expectation value M^x is non-zero. This is called ordered phase. On the other hand, in the region $h > h_c$, there is no magnetization along the x-axis, namely the expectation value M^x is zero. This is called disordered phase. Since these two phases are characterized by the value of M^x, this quantity is termed order parameter. The above arguments (concerning the qualitative change in the ground state) can made more clear by considering two limiting cases $h \to 0$ and $h \to \infty$ since the ground state of the Hamiltonian (2.57) depends only on the value of h. In the limit $h \to \infty$, the second term in (2.57) dominates and the ground state is simply

$$|0\rangle = \prod_n |\uparrow\rangle_n\,. \tag{2.63}$$

Since the σ_n^x are off-diagonal in the basis of states, the order parameter $M^x = 0$, which means that the magnetization of the system along x-axis is zero and the system is magnetically disordered. As a result, the correlation functions $\langle 0|\sigma_n^x \sigma_m^x|0\rangle$ in this ground state are short-ranged and their large-distance behaviours are described by the exponential decay

(2.59). The ground state in this phase possesses a symmetry under an exact global Z_2 symmetry transformation:

$$\sigma_n^x \to -\sigma_n^x\,, \qquad \sigma_n^z \to \sigma_n^z \tag{2.64}$$

which is generated by the unitary operator $\Pi_n \sigma_n^z$ and which leaves the Hamiltonian (2.57) invariant. Because of the symmetric ground state, the disordered phase is also called symmetric phase. On the other hand, in the limit $h \to 0$, the external transverse magnetic field vanishes and the first term in Hamiltonian (2.57) dominates. The system has two possible ground states

$$|0\rangle = \prod_n |\rightarrow\rangle_n \quad \text{or} \quad |0\rangle = \prod_n |\leftarrow\rangle_n \tag{2.65}$$

in which the spins are either up or down along the x-axis. Since these two ground states are both eigenstates of σ_n^x, the order parameter $M^x = \pm 1$, which implies that the system is magnetically ordered and the correlations of σ_n^x are long-ranged:

$$\langle 0|\sigma_n^x \sigma_m^x|0\rangle = 1\,, \qquad |n-m| \to \infty\,. \tag{2.66}$$

Again, the Hamiltonian (2.57) is invariant under the transformation (2.64). But, the symmetry of the ground state has been broken and the two possible ground state are mapped into each other by the transformation (2.64). The system can choose any of these two states as the ground state since they are physically equivalent. This is usually referred to as spontaneously breaking of the Z_2 symmetry. The degeneracy of the ground state can survive even when a small density of external magnetic field h is applied to the system thanks to the Z_2 symmetry.

Now we have observed that there is no possibility that the ground states obeying (2.66) and (2.59) can be transformed into each other analytically as a functions of h. There must be a critical point $h = h_c$ at which the large-distance behaviour of the two-point correlation function changes from (2.66) to (2.59) and which marks the boundary between the two qualitatively different phases discussed above. At this point, the two-point function obey (2.62). This is the position where a second order quantum phase transition occurs. For the Ising model, the value of critical point is specialized as $h_c = 1$. We focus for now on the zero temperature case, so that any phase transition will be driven only by quantum effects, and the system will be in a pure state rather than a thermal ensemble. We take the system to be in its ground state.

2.4.2 Ising model in the scaling limit

As we introduced before, the correlation length can be varied by varying the external magnetic field h. When the system is close sufficiently to the critical point, namely $h \to h_c$, the correlation length tends towards infinity, leading to the long-ranged correlations of the system. Along with the divergent correlation length, the energy gap between the lowest excitation and the ground state becomes almost null in the limit $h \to h_c$, leading to a gapless energy spectrum at $h = h_c$. This implies that near the criticality excitations at arbitrarily low energy are allowed and the system is dominated by the low-energy physics. Compared to the infinite correlation length, the lattice space we denote by a, setting the microscopic length-scale of the system, become infinitesimal, $a \ll \xi$. We can take the lattice space as zero, $a \to 0$, which amounts to looking at large distances, and this is a continuum limit, in which the configurations of the system can be considered as sufficiently smooth on the lattice spacings. It is then natural to assume that the system can be described in the language of the relativistic quantum field theory which are continuous and which looks at sufficiently low energy. In another word, a second order quantum phase transition defines a quantum field theory in the continuum. The process of getting closer to the critical point where a second quantum phase transition happens while looking at large distance is the scaling limit. A lattice system, in the scaling limit, will exhibit a number of features which are independent of the microscopic details of the underlying lattice system. This is universality. In light of universality, it is plausible that various models close to a second order quantum phase transition will share the same set of universal properties, and these models fall into universality classes which are characterized by the critical exponents of models. It turns out that the quantum field theory, as the result of the scaling limit of lattice systems, provides the most natural formulation for the quantitative study of those universal properties in the vicinity of the quantum critical point and for the description of universality classes. Let us make the above argument more explicit. In the scaling limit, the lattice spacing is reduced to zero while the correlation length goes to infinity. There is a charateristic length of the system and it can be defined as

$$\hat{\xi} = a\,\xi(h)\,. \tag{2.67}$$

The scaling limit must be taken in such a way that the characteristic length is kept constant

$$\hat{\xi} = \lim_{\substack{a\to 0 \\ h\to h_c}} a\,\xi(h) = const \tag{2.68}$$

in order to preserve the physics of the system. In fact, we can define the characteristic length in arbitrary way, but no matter in which definition the two limits $h \to h_c$ or $\xi \to \infty$

and $a \to 0$ must be taken keeping length $\hat{\xi}$ fixed. Besides the correlation length, all the lengths must be rescaled. For instance, the position of the site n can be rescaled as $x = an$ and to keep x intact the limit $n \to \infty$ must be performed. Consider the correlation function of local operators σ_n^x, σ_m^x at positions n, m:

$$\langle 0|\sigma_n^x \sigma_m^x|0\rangle \,.$$

We take the scaling limit

$$h \to h_c, \quad a \to 0, \quad n, m \to \infty \tag{2.69}$$

and the correlation function, in this limit, vanishes as $a^{2d} \to 0$. But, the correlation function multiplied by $(m\xi)^{2d}$, where m is defined by $m = 1/\hat{\xi}$, in the same limit, will be non-zero and it can be written in terms of a two-point correlation function in a quantum field theory with the mass scale m:

$$\lim_{h\to h_c} \lim_{a\to 0} \left[(m\xi)^{2d} \langle 0|\sigma_n^x \sigma_m^x|0\rangle\right] = \langle \mathrm{vac}|\mathcal{O}(x)\mathcal{O}(y)|\mathrm{vac}\rangle \tag{2.70}$$

where $\mathcal{O}(x)$ is a local field of QFT and it corresponds to the operator σ_n^x in the lattice model with the identification $x = an$. The quantity m can be interpreted as the mass of the lightest particle in QFT due to its correspondence with the dimensionful mass gap in a relativistic theory. The positive number d making the limit finite is called the dimension of the local field $\mathcal{O}$. The correlation function on the righ-hand side of (2.70) is a scaling function and the resulting QFT is a scaling theory. The quantum field theory, associated with a Hamiltonian defined in the continuum, is a simplification of the lattice theory, as it has no intrinsic short distance or high-energy cutoff and focuses only on the low-energy physics. Notice that the quantum chain, before the scaling limit is performed, can be treated as a regularisation of the quantum field theory, and taking the scaling limit leads to the renormalisation process.

Now we consider the quantum Ising model in a transverse field in the scaling limit. In this model, the Ising quantum chain of spin-$\frac{1}{2}$ particles can be mapped, through the Jordan-Wigner transformation [122–124], to a system of free spinless fermions with the Hamiltonian

$$H = -\frac{J}{2} \sum_n \left[\left(c_n^\dagger - c_n\right) \left(c_{n+1}^\dagger + c_{n+1}\right) - h \left(c_n^\dagger - c_n\right) \left(c_n^\dagger + c_n\right) \right]. \tag{2.71}$$

$c_n^\dagger$ and c_n are fermionic operator satisfying the canonical anti-commutation relation

$$\{c_n, c_{n'}^\dagger\} = \delta_{nn'}, \qquad \{c_n, c_{n'}\} = \{c_n^\dagger, c_{n'}^\dagger\} = 0 \tag{2.72}$$

To relate (2.71) with a free Majorana Hamiltonian, we can rewrite (2.71) as

$$H = -iJ \sum_n \left[\bar{\Psi}(n) \left(\Psi(n+1) - \Psi(n) \right) - (h-1) \bar{\Psi}(n) \Psi(n) \right] \tag{2.73}$$

where we define the two components of a Majorana spinor

$$\Psi(n) = \frac{c_n^\dagger + c_n}{\sqrt{2}}, \qquad \text{and} \qquad \bar{\Psi}(n) = \frac{c_n^\dagger - c_n}{\sqrt{2}i}. \tag{2.74}$$

It can be seen from (2.72) that these two components also satisfy the canonical anti-commutation relation

$$\{\Psi(n), \bar{\Psi}(n')\} = \delta_{nn'}, \qquad \text{and} \qquad \{\Psi(n), \Psi(n')\} = \{\bar{\Psi}(n), \bar{\Psi}(n')\} = 0. \tag{2.75}$$

The energy spectrum of this free-fermion model is described by fermionic excitation modes of energies

$$\epsilon_\phi = J\sqrt{1 + h^2 - 2h\cos\phi} \tag{2.76}$$

where $\phi \in [-\pi, \pi]$ is the wave number. The lowest excitation, equivalent with the energy gap, occurs at $\phi = 0$:

$$\epsilon_{min} = J|h-1|\,. \tag{2.77}$$

We see that the model is gapless at the critical point $h = 1$, implying that it is the critical point $h = 1$ that marks the phase boundary between order and disorder phase. In this regime, fermions with low momenta are permitted to carry arbitrarily low energies. The critical point is in the Ising universality class and it is described by the Ising model of conformal field theory with central charge $c = 1/2$ [159], which is the massless free Majorana field theory.

Further, it is also possible to describe the Ising model close to the criticality by a free massive quantum field theory which is expressed in terms of fermion fields . To achieve this, we take the scaling limit, namely the limits $a \to 0$ and $h \to 1$. We define the velocity of the excitations $v = Ja$ and their mass $m = J(h-1)$. In order to obtain a gapped scaling theory, another limit $J \to \infty$ needs to be taken in such a way that the mass m is constant. In the continuum limit $a \to 0$, we replace the fermionic operators $\Psi(n)$, $\bar{\Psi}(n)$ by the continuum Fermi fields $\Psi(x)$, $\bar{\Psi}(x)$ which obey the continuum anti-commutation relations

$$\{\Psi(x), \Psi(x')\} = \delta(x-x')\,, \quad \{\bar{\Psi}(x), \bar{\Psi}(x')\} = \delta(x-x')\,, \tag{2.78}$$

identifying the coordinates $x = na$. We perform all the aforementioned limits while holding m, v, $\Psi(x)$ and $\bar{\Psi}(x)$ fixed. In these limits, with the replacement $\sum_n \to \int dx$, we expand in (2.73) the continuum fields $\Psi(x)$, $\bar{\Psi}(x)$ to their first order in the spacial gradients, and we then obtain a continuum Hamiltonian

$$H = -i \int dx \left(v\bar{\Psi}(x)\partial_x \Psi(x) - m\bar{\Psi}(x)\Psi(x) \right) \tag{2.79}$$

leading to the vanishing of the irrelevant short wavelength degrees of freedom. We then perform the replacement

$$\psi(x) = \frac{\Psi(x) + \bar{\Psi}(x)}{\sqrt{2}}, \quad \psi(x) = \frac{\Psi(x) - \bar{\Psi}(x)}{\sqrt{2}} \tag{2.80}$$

where $\psi(x)$ and $\bar{\psi}(x)$ are a new set of Majorana components satisfying anti-commutation relations

$$\{\psi(x), \psi(x')\} = \delta(x - x'), \quad \{\bar{\psi}(x), \bar{\psi}(x')\} = \delta(x - x'), \tag{2.81}$$

and this gives rise to the free massive Majorana field theory with the Hamiltonian

$$H = -i \int dx \left[\frac{v}{2} \left(\bar{\psi}(x)\partial_x \bar{\psi}(x) - \psi(x)\partial_x \psi(x) \right) + m\psi(x)\bar{\psi}(x) \right]. \tag{2.82}$$

where in the following we set $v = 1$.

2.4.3 The free massive Majorana theory

In the free Majorana theory with mass m, fermion operators evolving in real time t can be expressed in terms of mode operators $a(\theta)$ and $a^\dagger(\theta)$:

$$\begin{aligned} \psi(x,t) &= \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(a(\theta)\, e^{ip_\theta x - iE_\theta t} + a^\dagger(\theta)\, e^{-ip_\theta x + iE_\theta t} \right) \\ \bar{\psi}(x,t) &= -\frac{i}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{-\theta/2} \left(a(\theta)\, e^{ip_\theta x - iE_\theta t} - a^\dagger(\theta)\, e^{-ip_\theta x + iE_\theta t} \right) \end{aligned} \tag{2.83}$$

where p_θ and E_θ are the relativistic momentum and energy associated to the rapidity θ and with mass m

$$p_\theta = m \sinh\theta, \quad E_\theta = m\cosh\theta \tag{2.84}$$

and where the mode operators $a(\theta)$ and their Hermitian conjugate $a^\dagger(\theta)$ obey the canonical anti-commutation relations

$$\{a(\theta), a(\theta')\} = \{a^\dagger(\theta), a^\dagger(\theta')\} = 0, \quad \{a(\theta), a^\dagger(\theta')\} = \delta(\theta - \theta'). \tag{2.85}$$

The fermion operators are the solution of the equation of motion

$$\bar{\partial}\psi(x,t) = \frac{m}{2}\bar{\psi}(x,t), \quad \partial\bar{\psi}(x,t) = \frac{m}{2}\psi(x,t) \tag{2.86}$$

with

$$\partial \equiv \partial_x - \partial_t \,, \quad \bar{\partial} \equiv \partial_x + \partial_t$$

and they satisfy the equal-time anti-commutation relations

$$\{\psi(x), \psi(x')\} = \delta(x - x') \,, \quad \{\bar{\psi}(x), \bar{\psi}(x')\} = \delta(x - x') \tag{2.87}$$

with other anticommutators vanishing. The Hamiltonian (2.82) can also be given in terms of mode operators:

$$H = \int d\theta \, m \cosh\theta \, a^\dagger(\theta) a(\theta) \tag{2.88}$$

and it is bounded from below on $\mathcal{H}$. Similarly the momentum operator is

$$P = \int d\theta \, m \sinh\theta \, a^\dagger(\theta) a(\theta) \,. \tag{2.89}$$

The spectrum of the free massive Majorana theory contains only one particle type. The space of in-states is a Fock space $\mathcal{H}$ over the canonical anti-commutation relations (2.85) which is in agreement with Zamolodchikov's algebra for the Ising model with two-particle scattering matrix -1. With the vacuum state $|\text{vac}\rangle$ defined by $a(\theta)|\text{vac}\rangle = 0$, the n-particle asymptotic in-states are identified as

$$|\theta_1, \ldots, \theta_n\rangle = a^\dagger(\theta_1) \cdots a^\dagger(\theta_n)|\text{vac}\rangle \,, \qquad \theta_1 > \cdots > \theta_n$$

and the same state with different rapidity orderings can be obtained via the canonical anti-commutation relations (2.85). The inner products are normalized as follow

$$\langle \theta'_1, \ldots, \theta'_n | \theta_1, \ldots \theta_n \rangle = \prod_{i=1}^{n} \delta(\theta_i - \theta'_i), \qquad \theta'_1 > \cdots > \theta'_n \text{ and } \theta_1 > \cdots > \theta_n, \tag{2.90}$$

where we denote $\langle \theta_1, \ldots, \theta_n | \equiv |\theta_1, \ldots, \theta_n\rangle^\dagger$. As we discussed before, the vacuum correlation functions can be obtained via the spectral decomposition which follows the statement that the basis of asymptotic states can be used to resolve the identity. In the Ising model, the resolution of the identity is given by

$$\mathbf{1} = \sum_{n=0}^{\infty} \frac{1}{n!} \int_{-\infty}^{\infty} d\theta_1 \cdots \int_{-\infty}^{\infty} d\theta_n \, |\theta_1, \ldots, \theta_n\rangle\langle\theta_1, \ldots, \theta_n| \tag{2.91}$$

where $n!$ in the denominator comes from overcounting the same state with different orderings of rapidities.

2.5 $U(1)$ Dirac model

In the free massive $U(1)$ Dirac theory, fermion operators with evolution in real time t are given in terms of mode operators $D_\pm(\theta)$ and $D^\dagger_\pm(\theta)$:

$$\begin{aligned}\Psi_R(x,t) &= \sqrt{m}\int d\theta e^{\theta/2}\left(D^\dagger_+(\theta)e^{itE_\theta-ixp_\theta}-iD_-(\theta)e^{-itE_\theta+ixp_\theta}\right)\\ \Psi_L(x,t) &= \sqrt{m}\int d\theta e^{-\theta/2}\left(iD^\dagger_+(\theta)e^{itE_\theta-ixp_\theta}-D_-(\theta)e^{-itE_\theta+ixp_\theta}\right)\end{aligned} \tag{2.92}$$

where

$$\begin{aligned}E_\theta &= m\cosh\theta\ ,\\ p_\theta &= m\sinh\theta\ .\end{aligned}$$

The Hermitian Conjugation of fermion operators can be obtained by directly taking Hermitian Conjugation of (2.92):

$$\begin{aligned}\Psi^\dagger_R(x,t) &= \sqrt{m}\int d\theta e^{\theta/2}\left(iD^\dagger_-(\theta)e^{itE_\theta-ixp_\theta}+D_+(\theta)e^{-itE_\theta+ixp_\theta}\right)\\ \Psi^\dagger_L(x,t) &= \sqrt{m}\int d\theta e^{-\theta/2}\left(-D^\dagger_-(\theta)e^{itE_\theta-ixp_\theta}-iD_+(\theta)e^{-itE_\theta+ixp_\theta}\right)\ .\end{aligned} \tag{2.93}$$

The creation and annihilation operators satisfy canonical anti-commutation relations:

$$\begin{aligned}\{D^\dagger_+(\theta_1),D_+(\theta_2)\} &= \delta(\theta_1-\theta_2)\\ \{D^\dagger_-(\theta_1),D_-(\theta_2)\} &= \delta(\theta_1-\theta_2)\end{aligned} \tag{2.94}$$

with other anti-commutators vanishing. The fermion operators satisfy the equations of motion

$$\begin{aligned}\bar\partial\Psi_R &= m\Psi_L,\quad \partial\Psi_L = m\Psi_R\\ \bar\partial\Psi^\dagger_R &= m\Psi^\dagger_L,\quad \partial\Psi^\dagger_L = m\Psi^\dagger_R\end{aligned} \tag{2.95}$$

and their anti-commutation relations are

$$\begin{aligned}\{\Psi_R(x_1),\Psi^\dagger_R(x_2)\} &= 4\pi\delta(x_1-x_2)\\ \{\Psi_L(x_1),\Psi^\dagger_L(x_2)\} &= 4\pi\delta(x_1-x_2)\end{aligned} \tag{2.96}$$

with other anti-commutators vanishing. The Hamiltonian and momentum operators are:

$$\begin{aligned} H &= \int d\theta\, m \cosh\theta \left(D_+^\dagger(\theta) D_+(\theta) + D_-^\dagger(\theta) D_-(\theta) \right) , \\ P &= \int d\theta\, m \sinh\theta \left(D_+^\dagger(\theta) D_+(\theta) + D_-^\dagger(\theta) D_-(\theta) \right) . \end{aligned} \tag{2.97}$$

The spectrum of the free massive $U(1)$ Dirac theory contains two types of particle with charges $\pm$. The space of in-states is simply the Fock space $\mathcal{H}$ over algebra (2.94) with vacuum state defined by $D_\pm|\text{vac}\rangle = 0$ and with multi-particle states denoted by

$$|\theta_1, \ldots, \theta_N\rangle_{\nu_1,\ldots,\nu_N} := D_{\nu_1}^\dagger \cdots D_{\nu_N}^\dagger |\text{vac}\rangle, \qquad \theta_1 > \cdots > \theta_N \tag{2.98}$$

where ν_i are signs ($\pm$), corresponding with particle type. Again, the asymptotic states with different rapidity orderings can be obtained from the associated in-state by the anti-commutation relations (2.94). The inner products are normalized as follows:

$$_{\nu'_1,\ldots,\nu'_N}\langle\theta'_1, \ldots, \theta'_N|\theta_1, \ldots, \theta_N\rangle_{\nu_1,\ldots,\nu_N} = \prod_{i=1}^{N} \delta_{\nu_i \nu'_i} \delta(\theta_i - \theta'_i) \tag{2.99}$$

with the notation $_{\nu_1,\ldots,\nu_N}\langle\theta_1, \ldots, \theta_N| \equiv |\theta_1, \ldots, \theta_N\rangle^\dagger_{\nu_1,\ldots,\nu_N}$. The resolution of the identity are then written as:

$$\mathbf{1} = \sum_{N=0}^{\infty} \frac{1}{N!} \sum_{\nu_1,\ldots,\nu_N} \int_{-\infty}^{\infty} d\theta_1 \cdots \int_{-\infty}^{\infty} d\theta_N \, |\theta_1, \ldots, \theta_N\rangle_{\nu_1,\ldots,\nu_N} \; {}_{\nu_1,\ldots,\nu_N}\langle\theta_1, \ldots, \theta_N| \tag{2.100}$$

where $N!$ in the denominator comes from overcounting the same state with different orderings of rapidities.

Chapter 3

Twist fields in free fermion models

3.1 General definition

In QFT, if a field $\mathcal{O}(x)$ commutes with the Hamiltonian density $h(x')$, related to the Hamiltonian by

$$H = \int dx\, h(x),$$

at space-like distances:

$$[\mathcal{O}(x), h(x')] = 0\,, \quad x \neq x'\,, \tag{3.1}$$

this field is called a local field. Locality means quantum mechanical independence with respect to the energy field at space-like distances. Let us then introduce the concept of mutual locality: if two fields $\mathcal{O}_1(x)$, $\mathcal{O}_2(x')$ commute (for bosonic fields) or anticommute (for fermionic fields) at space-like distances

$$[\mathcal{O}_1(x), \mathcal{O}_2(x')] = 0 \quad \text{or} \quad \{\mathcal{O}_1(x), \mathcal{O}_2(x')\} = 0\,, \quad x \neq x'\,, \tag{3.2}$$

they are called local with respect to each other. On the other hand, if two fields $\mathcal{O}_1(x)$, $\mathcal{O}_2(x')$ neither commute nor anticommute at space-like distances and they satisfy

$$\mathcal{O}_1(x)\mathcal{O}_2(x') = (-1)^{f_{\mathcal{O}_1} f_{\mathcal{O}_2}} e^{-2\pi i \omega(\mathcal{O}_1,\mathcal{O}_2)\Theta(x-x')} \mathcal{O}_2(x')\mathcal{O}_1(x)\,, \quad x \neq x' \tag{3.3}$$

or

$$\mathcal{O}_1(x)\mathcal{O}_2(x') = (-1)^{f_{\mathcal{O}_1} f_{\mathcal{O}_2}} e^{2\pi i \omega(\mathcal{O}_1,\mathcal{O}_2)\Theta(x'-x)} \mathcal{O}_2(x')\mathcal{O}_1(x)\,, \quad x \neq x' \tag{3.4}$$

where $\omega(\mathcal{O}_1, \mathcal{O}_2)$ is the semi-locality index we introduced before, $\Theta(x - x')$ is Heaviside's step function ($\Theta(x - x')$ is 1 for $x > x'$ and 0 for $x < x'$) , $f_{\mathcal{O}} = 1$ for $\mathcal{O}$ fermionic, $f_{\mathcal{O}} = 0$ for $\mathcal{O}$ bosonic, they are called semi-local with respect to each other. Generalizing the relation (3.3) and (3.4) for semi-locality, we introduce the twist field: it is the field which is associated with a global internal symmetry of a QFT model and exhibit the property of semi-locality generated by this symmetry transformation. To be more explicit, we consider a twist field T_g associated to a global internal symmetry g. If $g\mathcal{O} \neq \mathcal{O}$, where $\mathcal{O}$ is a local field, then the twist field T_g is said to be semi-local with respect to $\mathcal{O}$. The property of this semi-locality is described by the equal-time exchange relation

$$\mathcal{O}(x)T_g(x') = (-1)^{f_{T_g} f_{\mathcal{O}}} g^{\Theta(x-x')} T_g(x')\mathcal{O}(x)\,, \quad x \neq x' \tag{3.5}$$

or

$$\mathcal{O}(x)T_g(x') = (-1)^{f_{T_g} f_{\mathcal{O}}} g^{-\Theta(x'-x)} T_g(x')\mathcal{O}(x)\,, \quad x \neq x'. \tag{3.6}$$

Hence, the twist field T_g is an interacting field. Now we can see that the semi-locality defined through (3.3) or (3.4) is just a special case in which g is the $U(1)$ symmetry.

In order to gain more intuition into the semi-locality of twist fields, we consider it in Feynmann's path integral formulation of QFT, in which a product of operators inside a correlation function $\langle \text{vac}|\mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n)|\text{vac}\rangle$ is required to be time-ordered so that its correlation function can be defined through the functional integral:

$$\int [d\Psi] e^{iS[\Psi]} \mathcal{O}_1(x_1,t_1)\cdots\mathcal{O}_n(x_n,t_n)\,. \tag{3.7}$$

In this way, the semi-locality of the twist field can be reformulated by the following

$$\begin{aligned} & \lim_{\varepsilon\to 0^+}\left[\int [d\Psi] e^{iS[\Psi]} \left(\cdots\mathcal{O}(x,\varepsilon)T_g(x',0)\cdots\right)\right] \\ = \; & \lim_{\varepsilon\to 0^+}\left[\int [d\Psi] e^{iS[\Psi]} \left(\cdots g^{\pm\Theta(\pm(x-x'))}\mathcal{O}(x,-\varepsilon)T_g(x',0)\cdots\right)\right]\,, \quad x \neq x' \end{aligned} \tag{3.8}$$

where $\mathcal{O}$ is a local field and $\cdots$ represents insertions of other local fields at different space-time points. The sign $\pm$ is synchronized and only one choice of sign can be chosen, which is valid for both bosonic and fermionic fields. To further interpret this semi-locality, we exploit the imaginary time formalism, since with an imaginary time two operators are always separated by a space-like distance. The equation (3.8) is then rewritten by replacing ε with $-i\varepsilon$:

$$\begin{aligned} & \lim_{\varepsilon\to 0^+}\left[\int [d\Psi] e^{iS[\Psi]} \left(\cdots\mathcal{O}(x,-i\varepsilon)T_g(x',0)\cdots\right)\right] \\ = \; & \lim_{\varepsilon\to 0^+}\left[\int [d\Psi] e^{iS[\Psi]} \left(\cdots g^{\pm\Theta(\pm(x-x'))}\mathcal{O}(x,i\varepsilon)T_g(x',0)\cdots\right)\right]\,, \quad x \neq x' \end{aligned} \tag{3.9}$$

which is allowed in Feynmann's path integral formalism. To be more precise, we consider the functional integral

$$F(x,t) = \int [d\Psi] e^{iS[\Psi]} \left(\cdots \mathcal{O}(x,t) T_g(0,0) \cdots \right) \tag{3.10}$$

as a function of x and t, and analytically continue it in the variable t from a space-like region $|x| > |t|$ to the purely imaginary region of time $t = -i\tau$, with τ real. Employing the complex coordinates $z = x + i\tau$ and $\bar{z} = x - i\tau$, the semi-locality of the twist field can be interpreted by the discontinuity of the function F when going once around the point $x = 0, \tau = 0$ anti-clockwise:

$$F\left(e^{-i\phi} z, e^{i\phi} \bar{z}\right)\Big|_{\phi: 0 \mapsto 2\pi} = F'(z, \bar{z}) \tag{3.11}$$

where

$$F'(z,\bar{z}) = \int [d\Psi] e^{iS[\Psi]} \left(\cdots g\mathcal{O}(z,\bar{z}) T_g(0,0) \cdots \right).$$

This implies that the function F is defined on a multi-sheeted covering of $\mathbb{R}^2$ with a branch point at $(0,0)$ and it should have a branch cut on $z \in \mathbb{R}^+$ or on $z \in \mathbb{R}^-$ according to the synchronized sign $\pm$ in (3.9). From this, the twist field T_g can be interpreted as a branch point of a cut through which any other local field transforms inside the correlation function as

$$\mathcal{O} \mapsto g\mathcal{O}. \tag{3.12}$$

Let us denote by $\mathcal{O}(x,\tau)$ the fields $\mathcal{O}(x,-i\tau)$ in Euclidean theory which are analytically continued from the space-like region as discussed above and we then can formally define a twist field through the path integral

$$\int [d\Psi] e^{iS[\Psi]} T_g(0,0) \mathcal{O}_1(x_1,\tau_1) \cdots \mathcal{O}_n(x_n,\tau_n) = \int_{\mathcal{C}(0,0)} [d\Psi] e^{iS[\Psi]} \mathcal{O}_1(x_1,\tau_1) \cdots \mathcal{O}_n(x_n,\tau_n) \tag{3.13}$$

where $\mathcal{C}(0,0)$ is a quasi-periodicity condition imposed on the fundamental fields Ψ in the functional integral:

$$\mathcal{C}(0,0): \begin{cases} \Psi(x,0^+) = g\Psi(x,0^-) & (x>0) \\ \Psi(x,0^+) = \Psi(x,0^-) & (x<0). \end{cases} \tag{3.14}$$

As we see, in order for the functional integral (3.13) to represent the insertion a twist field, we have to take a cut in the plane over which the integration is taken. This cut starts at the point (0,0) where the twist field T_g is inserted in the functional integral, and it ends at the infinity point. The fundamental fields or any other fields $\mathcal{O}$ such that $g\mathcal{O} \neq \mathcal{O}$ are discontinuous across this cut. The twist field T_g is local with respect to the Hamiltonian density, namely it is a local field, due to the invariance of the Hamiltonian density under the symmetry transformation g. As a consequence of this, the functional integral (3.13) is independent of the shape of the cut $x > 0$ and the shape changing only produces g transformations of the fields that the cut may cross while moving. To make this more explicit, let us consider two cuts l_1 and l_2 both of which start from $(0, 0)$ and end at the infinity, with $l_3 = l_1 \cup l_2$ a closed path, and the quasi-periodicity condition is satisfied on the each side of the two cuts. We suppose that the cut l_1 is above the cut l_2. When the cut in the path integral is moving from l_1 to l_2 clockwise, all the fields inside l_3 transform according to the symmetry g, adding a contribution to the path integral:

$$\int_{l_1} [d\Psi] e^{iS[\Psi]} \mathcal{O}_1(x_1, \tau_1) \cdots \mathcal{O}_n(x_n, \tau_n) = \int_{l_2} [d\Psi] e^{iS[\Psi]} e^{-\int_{\mathcal{C}_3} ds^\mu j^\nu \epsilon_{\mu\nu}} \mathcal{O}_1(x_1, \tau_1) \cdots \mathcal{O}_n(x_n, \tau_n) \tag{3.15}$$

where ds^μ is the line element along path l_3, j^ν is the Noether current associated to the symmetry g, and the action in the isolated region is preserved under the transformation g. The extra contribution has the effect of performing the transformation g of the fields inside the path l_3. Now we get a similar path integral with the new cut l_2, across which the quasi-periodicity condition still holds, and the only difference stems from the transformation g of all the fields present inside the path l_3.

Now let us turn our attention on the uniqueness of the twist field. It is possible that a set of fields have the same twist-field effect, for instance, the quasi-periodicity condition (3.14), but they have different scaling dimensions. In the language of conformal field theory, among the twist fields associated to the same global symmetry, the one with the lowest scaling dimension is called the primary twist field and those with higher dimensions are called its descendants. Primary twist fields can be uniquely defined by the semi-locality condition (3.5) or (3.6), and the condition that it has the lowest scaling dimension and be invariant under all symmetries of the model that commutes with g. In this thesis, we focus on the primary twist fields.

In the following sections, we will introduce twist fields in the Ising model, in the $U(1)$ Dirac model, and in the n-copy Ising model, which are associated to different global symmetries.

3.2 Twist fields in the Ising model

As we introduced before, a correlation function of local operators in the Ising lattice model can reproduce in the scaling limit a correlation function of local fields in the QFT, with a correspondence between the local fields and the local operators, for instance, (2.70). For the Hamiltonian (2.57), the observable σ^x is related, in the scaling limit, to two fields in the Majorana theory: twist fields σ and μ associated with Z_2 symmetry:

$$(\psi, \bar{\psi}) \mapsto (-\psi, -\bar{\psi}) \tag{3.16}$$

which preserves the Euclidean action

$$S_E[\psi, \bar{\psi}] = -i \int d^2x \, [\psi \bar{\partial}_E \, \psi - \bar{\psi} \partial_E \, \bar{\psi} + 2m\psi\bar{\psi}] \tag{3.17}$$

with notations $\partial_E \equiv (1/2)(\partial/\partial x - i\partial/\partial\tau)$ and $\bar{\partial}_E \equiv (1/2)(\partial/\partial x + i\partial/\partial\tau)$. The twist field σ is obtained in the order phase $h \to 1^+$ and hence called order field. The order field σ only generates even numbers of particles and hence is of bosonic statistics. The order field σ has nonzero vacuum expectation value since it couples states with even numbers of particles. By contrast, the twist field μ is called disorder field since it is obtained in the disorder phase $h \to 1^-$. The disorder field μ only generates odd numbers of particles and hence is of fermionic statistics. The disorder field μ has vanishing vacuum expectation value since it couples states with odd numbers of particles. However, order field σ and disorder field μ share the same scaling dimension 1/8. These field represent the endpoint of branch cuts through which other fields are affected by the Z_2 symmetry transformation (3.16). These branch cuts, in principle, can be taken arbitrarily. Here, for convention, we denote by σ^+, μ^+ twist fields with branch cuts running towards the right direction, and by σ^-, μ^- those with branch cuts running towards the left direction. These two types of twist fields are related to each other by an unitary operator

$$Z := \exp\left[i\pi \int d\theta \, a^\dagger(\theta) a(\theta)\right] \tag{3.18}$$

which implements the Z_2 symmetry transformation and we then have

$$\sigma^-(x,t) = \sigma^+(x,t)Z\,, \quad \mu^-(x,t) = \mu^+(x,t)Z\,. \tag{3.19}$$

where the multiplication by Z does not change the scaling dimension. Twist fields σ^η and μ^η, with $\eta = \pm$, are semi-local with respect to the free Majorana fermion fields and this semi-locality is charaterized by their twist conditions:

$$\psi(x)\sigma^\eta(0) = \begin{cases} (-\delta_{\eta-} + \delta_{\eta+})\, \sigma^\eta(0)\psi(x) & (x < 0) \\ \\ (-\delta_{\eta+} + \delta_{\eta-})\, \sigma^\eta(0)\psi(x) & (x > 0) \end{cases} \tag{3.20}$$

and

$$\psi(x)\mu^{\eta}(0) = \begin{cases} (+\delta_{\eta-} - \delta_{\eta+})\, \mu^{\eta}(0)\psi(x) & (x < 0) \\ \\ (+\delta_{\eta+} - \delta_{\eta-})\, \mu^{\eta}(0)\psi(x) & (x > 0) \end{cases} \tag{3.21}$$

with similar twist conditions holding with the replacement $\psi \mapsto \bar{\psi}$. These twist conditions are in agreement with the relation (3.19). Along with the requirement that twist fields $\sigma^{\pm}$ and $\mu^{\pm}$ be of the lowest scaling dimension, these fields can be uniquely defined, up to the normalization, by their twist conditions (3.20) and (3.21).

Even though the Ising model is free-fermion model with a trivial scattering matrix, the twist field σ^{η} and μ^{η} are interacting fields rather than free fields. As a result, the form factors of these twist fields are nontrivial. Thanks to the twist conditions (3.20) and (3.21), the form factors of order fields σ^{η} and disorder fields μ^{η} [5, 21, 126] are given by

$$f^{\sigma^{\eta}}(\theta_1, \ldots, \theta_N) \overset{N \text{ even}}{=} \left(\frac{i}{\sqrt{2\pi}}\right)^N \langle\sigma\rangle \prod_{1\leq i<j\leq N} \tanh\left(\frac{\theta_j - \theta_i}{2}\right) \tag{3.22}$$

$$f^{\mu^{\eta}}(\theta_1, \ldots, \theta_N) \overset{N \text{ odd}}{=} \eta \frac{1}{\sqrt{i}} \left(\frac{i}{\sqrt{2\pi}}\right)^N \langle\sigma\rangle \prod_{1\leq i<j\leq N} \tanh\left(\frac{\theta_j - \theta_i}{2}\right) \tag{3.23}$$

where $\langle\sigma\rangle := \langle \text{vac}|\sigma^{\eta}|\text{vac}\rangle$ is the vacuum expectation value and it has been computed as $m^{\frac{1}{8}} 2^{\frac{1}{12}} e^{-\frac{1}{8}} A^{\frac{3}{2}}$ with A a Glaisher's constant [125]. The order fields σ^{η} have nonzero form factors for even particle numbers only and the disorder fields μ^{η} have nonzero form factors for odd particle numbers only. All other matrix elements can be evaluated by using crossing relations [1, 3, 5, 126], and that [21]

$$(\sigma^{\eta})^{\dagger} = \sigma^{\eta}, \quad (\mu^{\eta})^{\dagger} = \eta\mu^{\eta}\,. \tag{3.24}$$

The explicit matrix elements for the fields σ^{η} and μ^{η}, provided as above, can be seen as another way of uniquely defining these fields. The form factors (3.22), (3.23) as well as their normalizations are in agreement with (3.19).

Lastly, it is worth noting that higher-particle form factors of the twist fields σ^{η} can be obtained using Wick's theorem on the particles, with a contraction given by the two-particle form factor of σ^{η}. Similarly, the twist fields μ^{η} with higher number of particles can also be factorized into a product of one-particle form factors of μ^{η} and two-particle form factors of σ^{η} with the help of Wick's theorem. The factorization of form factors indicates that the twist fields σ^{η} and μ^{η} are normalized exponentials of bilinear expressions in fermion operators (the overall normalization is made finite by normal ordering). For instance, the order fields σ^η can be expressed as

$$\sigma^\eta = \langle\sigma\rangle : \exp\left[\sum_{\epsilon_1,\epsilon_2}\int d\theta_1 d\theta_2\, F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)a^{\epsilon_1}(\theta_1)a^{\epsilon_2}(\theta_2)\right] : \tag{3.25}$$

where

$$F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) = -\frac{1}{2}\frac{i}{2\pi}e^{-\epsilon_1\frac{i\pi}{4}}e^{-\epsilon_2\frac{i\pi}{4}}\tanh\left(\frac{\theta_2-\theta_1+\eta i(-\epsilon_2+\epsilon_1)}{2}\right)^{\epsilon_1\epsilon_2} \tag{3.26}$$

are the matrix elements of σ^η on the Hilbert space, up to the factor $-\frac{1}{2\langle\sigma\rangle}$. This expression along with the twist conditions (3.20) provide a third way to uniquely define, up to normalization, the order fields σ^η. Similar argument holds for the disorder fields μ^η.

3.3 Twist fields in the $U(1)$ Dirac model

3.3.1 Bosonic primary twist fields

The Dirac theory possesses a $U(1)$ internal symmetry $\Psi_{R,L} \mapsto e^{2\pi i\alpha}\Psi_{R,L}$ where for now we consider $0 \le \alpha < 1$ and there exists a family of primary twist fields $\sigma_\alpha(x,t)$ associated with this symmetry, which are local, Lorentz spinless, and $U(1)$ neutral, with dimension α^2 [48]. These fields generate even number of fermions and hence are of bosonic statistics. The bosonic primary twist fields with negative index can be defined by Hermitian conjugation:

$$\sigma_\alpha^\dagger = \sigma_{-\alpha}\,, \qquad 0 \le \alpha < 1\,. \tag{3.27}$$

Twist fields σ_α are associated with branch cuts through which other fields are affected by $U(1)$ symmetry transformation. These branch cuts, in principle, can be taken arbitrarily. Here, for convention, we denote by σ_α^+ the twist fields with branch cuts running towards the right direction, while by σ_α^- the ones with branch cuts running towards the left direction. These two types of twist fields are related to each other by an unitary operator

$$Z := \exp\left[\sum_\nu 2\pi i\nu\alpha\int d\theta D_\nu^\dagger(\theta)D_\nu(\theta)\right] \tag{3.28}$$

which implements the $U(1)$ symmetry transformation and we have

$$\sigma_\alpha^-(x,t) = \sigma_\alpha^+(x,t)Z\,. \tag{3.29}$$

Twist fields σ_α^η with $\eta=\pm$ are semi-local with respect to the Dirac fermion fields, and are characterized by equal-time exchange relations

$$\Psi_{R,L}(x)\sigma_\alpha^\eta(0)=\begin{cases}\left(\delta_{\eta,-}e^{2\pi i\eta\alpha}+\delta_{\eta,+}\right)\sigma_\alpha^\eta(0)\Psi_{R,L}(x) & (x<0)\\ \left(\delta_{\eta,+}e^{2\pi i\eta\alpha}+\delta_{\eta,-}\right)\sigma_\alpha^\eta(0)\Psi_{R,L}(x) & (x>0)\end{cases} \tag{3.30}$$

and

$$\Psi^\dagger_{R,L}(x)\sigma_\alpha^\eta(0)=\begin{cases}\left(\delta_{\eta,-}e^{-2\pi i\eta\alpha}+\delta_{\eta,+}\right)\sigma_\alpha^\eta(0)\Psi^\dagger_{R,L}(x) & (x<0)\\ \left(\delta_{\eta,+}e^{-2\pi i\eta\alpha}+\delta_{\eta,-}\right)\sigma_\alpha^\eta(0)\Psi^\dagger_{R,L}(x) & (x>0)\ .\end{cases} \tag{3.31}$$

Thanks to these twist conditions, two-particle form factors of twist fields σ_α^η for $-1<\alpha<1$ can be fixed, up to normalization, [44, 132–134] (see also appendix A of [47]):

$$\langle \mathrm{vac}|\sigma_\alpha^\eta(0)|\theta_1,\theta_2\rangle_{\nu_1,\nu_2}=\delta_{\nu_1,-\nu_2}\ \nu_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\alpha(\theta_1-\theta_2)}}{\cosh\frac{\theta_1-\theta_2}{2}}\langle\sigma_\alpha\rangle \tag{3.32}$$

where $\langle\sigma_\alpha\rangle:=\langle\mathrm{vac}|\sigma_\alpha^\eta|\mathrm{vac}\rangle=c_\alpha m^{\alpha^2}$ is the vacuum expectation value. The dimensionless constants c_α are computed in [135, 136]. All other higher-particle form factors can be obtained by Wick's theorem due to the fact that twist fields σ_α^η can be expressed as normal-ordered exponentials of bilinear expressions in Dirac fermion operators:

$$\sigma_\alpha^\eta=\langle\sigma_\alpha\rangle\left(:\exp\left[\sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}\int d\theta_1 d\theta_2 F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)D^{\epsilon_1}_{\nu_1}(\theta_1)D^{\epsilon_2}_{\nu_2}(\theta_2)\right]:\right) \tag{3.33}$$

where

$$\begin{aligned}F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) &= -\frac{1}{2}\left[\delta_{\epsilon_1,\epsilon_2}\delta_{\nu_1,-\nu_2}\,\nu_1\epsilon_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}}{\cosh(\frac{\theta_1-\theta_2}{2})}+\right.\\ &\left.\delta_{\epsilon_1,-\epsilon_2}\delta_{\nu_1,\nu_2}\,i\nu_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}e^{-i\pi\eta\nu_1\alpha}}{\sinh\left(\frac{\theta_1-\theta_2+\eta i(\epsilon_1-\epsilon_2)0^+}{2}\right)}\right]\ .\end{aligned} \tag{3.34}$$

are the matrix elements of σ_α^η on the Hilbert space, up to the factor $-\frac{1}{2\langle\sigma_\alpha\rangle}$. Note that twist fields σ_α^η have non-zero form factors only for even particle numbers since they are $U(1)$ neutral. Other matrix elements can be evaluated using crossing symmetry. Finally, it is natural for us to define twist fields σ_α for all $\alpha\in\mathbb{R}\setminus\mathbb{Z}^*$ by noticing that their form factors for fixed rapidities are analytic functions of α on $\alpha\in\mathbb{C}\setminus\mathbb{Z}^*$, with general poles on $\mathbb{Z}^*:=\mathbb{Z}\setminus\{0\}$.

3.3.2 Fermionic primary twist fields

In the $U(1)$ Dirac theory, there also exist two families of primary twist fields of fermionic statistics, which can be obtained as the coefficients occurring in the operator product expansions (OPEs) of primary twist fields σ_α with the Dirac fields Ψ_R and $\Psi_R^\dagger$, for all $\alpha \in \mathbb{R} \setminus \mathbb{Z}^*$:

$$\sigma_{\alpha+1,\alpha}(x,t) = \lim_{z\to w}(z-w)^{\alpha}\Psi_R^\dagger(x',t')\sigma_\alpha(x,t) \tag{3.35}$$

$$\sigma_{\alpha-1,\alpha}(x,t) = \lim_{z\to w}(z-w)^{-\alpha}\Psi_R(x',t')\sigma_\alpha(x,t) \tag{3.36}$$

where $z = -\frac{1}{2}(x'-t')$ and $w = -\frac{1}{2}(x-t)$ with time ordering $t' > t$. The factors $(z-w)^\alpha$ and $(z-w)^{-\alpha}$ are taken on the principal branch. Twist fields $\sigma_{\alpha\pm1,\alpha}$ have charges $\mp$, spins $\pm\alpha + 1/2$, and dimensions $\alpha^2 \pm \alpha + 1/2$. Their Hermitian conjugations are given by

$$\sigma^\dagger_{\alpha\pm1,\alpha} = \sigma_{-\alpha\mp1,-\alpha}\,. \tag{3.37}$$

Again, we define two types of fermionic primary twist fields: $\sigma^+_{\alpha\pm1,\alpha}$ with branch cuts on the right and $\sigma^-_{\alpha\pm1,\alpha}$ with branch cuts on the left, which are related to each other by the unitary operator Z

$$\sigma^-_{\alpha\pm1,\alpha} = \sigma^+_{\alpha\pm1,\alpha} Z\,. \tag{3.38}$$

From the definitions (3.35) (3.36) and twist properties of σ^η_α (3.30) (3.31), these twist fields should also be parameterized by non-trivial equal-time exchange relations with respect to the Dirac fermion fields

$$\Psi_{R,L}(x)\sigma^\eta_{\alpha\pm1,\alpha}(0) = \begin{cases} -\left(\delta_{\eta,-}e^{2\pi i\eta\alpha} + \delta_{\eta,+}\right)\sigma^\eta_{\alpha\pm1,\alpha}(0)\Psi_{R,L}(x) & (x<0) \\ -\left(\delta_{\eta,+}e^{2\pi i\eta\alpha} + \delta_{\eta,-}\right)\sigma^\eta_{\alpha\pm1,\alpha}(0)\Psi_{R,L}(x) & (x>0) \end{cases} \tag{3.39}$$

and

$$\Psi^\dagger_{R,L}(x)\sigma^\eta_{\alpha\pm1,\alpha}(0) = \begin{cases} -\left(\delta_{\eta,-}e^{-2\pi i\eta\alpha} + \delta_{\eta,+}\right)\sigma^\eta_{\alpha\pm1,\alpha}(0)\Psi^\dagger_{R,L}(x) & (x<0) \\ -\left(\delta_{\eta,+}e^{-2\pi i\eta\alpha} + \delta_{\eta,-}\right)\sigma^\eta_{\alpha\pm1,\alpha}(0)\Psi^\dagger_{R,L}(x) & (x>0)\,. \end{cases} \tag{3.40}$$

One-particle form factors of these twist fields can be deduced in the OPEs [49]:

$$\langle\text{vac}|\sigma^\eta_{\alpha+1,\alpha}(0)|\theta\rangle_\nu = \delta_{\nu,+}\frac{e^{-i\pi\alpha/2}e^{2\pi i\nu\alpha\delta_{\eta,-}}}{\Gamma(1+\alpha)}m^{\alpha+1/2}e^{(\alpha+1/2)\theta}\langle\sigma_\alpha\rangle \tag{3.41}$$

$$\langle\text{vac}|\sigma^\eta_{\alpha-1,\alpha}(0)|\theta\rangle_\nu = -i\delta_{\nu,-}\frac{e^{i\pi\alpha/2}e^{2\pi i\nu\alpha\delta_{\eta,-}}}{\Gamma(1-\alpha)}m^{-\alpha+1/2}e^{(-\alpha+1/2)\theta}\langle\sigma_\alpha\rangle\,. \tag{3.42}$$

Any higher-particle form factors can be factorised into a product of the associated one-particle form factor and two-particle form factors due to Wick's theorem. Other matrix elements can be obtained by crossing symmetry.

It is worth noting that we can obtain the same families of fermionic primary twist fields $\sigma_{\alpha,\alpha-1}$ and $\sigma_{\alpha,\alpha+1}$ [49, 50] by shifting $\alpha \mapsto \alpha - 1$ in $\sigma_{\alpha+1,\alpha}$ and shifting $\alpha \mapsto \alpha + 1$ in $\sigma_{\alpha-1,\alpha}$ respectively. These fields are just a relabeling of the same fermionic primary twist fields.

3.4 Twist fields in the n-copy Ising model

The branch-point twist fields originally arose in CFT, in the context of evaluating partition functions on Riemann surfaces [127]. But their most important application is probably in the evaluation of entanglement entropy [45, 46, 128–131]. In this section, we will present a brief review of the branch-point twist fields, which is mainly based on the literature [46].

Consider a model which is formed by n independent copies of the Ising model. Particles on different copies do not interact. The lagrangian density of this n-copy Ising model is the sum of the lagrangian density of every copy and it can be written as

$$\mathcal{L}^{(n)}[\psi_1, \dots, \psi_n](x) = \mathcal{L}[\psi_1](x) + \cdots + \mathcal{L}[\psi_n](x) \tag{3.43}$$

where ψ_i is the free Majorana field on the i^{th} copy of the model. It is obvious that this model possesses a $\mathbb{Z}_n$ symmetry under cyclic exchange of the copies:

$$\mathcal{L}^{(n)}[g\psi_1, \dots, g\psi_n](x) = \mathcal{L}^{(n)}[\psi_1, \dots, \psi_n](x) \tag{3.44}$$

where g is the transformation that permutes the copy numbers cyclically:

$$g\psi_i = \psi_{i+1} \tag{3.45}$$

with $i = 1, \cdots, n, n+1 \equiv 1$. The branch-point twist field $\mathcal{T}$ is the twist field associated to the symmetry g. Denoting by $\mathcal{T}^+$ and $\mathcal{T}^-$ the twist fields with branch cuts on the right and on the left, respectively, their equal-time exchange relations with respect to the free Majorana fields are:

$$\psi_i(x)\mathcal{T}^\eta(0) = \begin{cases} \mathcal{T}^\eta(0)\,(\delta_{\eta+}\psi_i(x) + \delta_{\eta-}\psi_{i+1}(x)) & (x < 0) \\ \mathcal{T}^\eta(0)\,(\delta_{\eta-}\psi_i(x) + \delta_{\eta+}\psi_{i+1}(x)) & (x > 0) \end{cases} \tag{3.46}$$

with $\eta = \pm$. In addition, we can define another branch-point twist field $\tilde{\mathcal{T}}$ which is associated to the symmetry g^{-1} under the opposite cyclic exchange $g^{-1}\psi_i = \psi_{i-1}$ with $i - n = i$. Similarly, we denote by $\tilde{\mathcal{T}}^+$ and $\tilde{\mathcal{T}}^-$ the twist fields with branch cuts on the right and on the left, respectively. They also have the equal-time exchange relations with respect to the free Majorana fields:

$$\psi_i(x)\tilde{\mathcal{T}}^\eta(0) = \begin{cases} \tilde{\mathcal{T}}^\eta(0)\,(\delta_{\eta+}\psi_i(x) + \delta_{\eta-}\psi_{i-1}(x)) & (x < 0) \\ \tilde{\mathcal{T}}^\eta(0)\,(\delta_{\eta-}\psi_i(x) + \delta_{\eta+}\psi_{i-1}(x)) & (x > 0) \end{cases} \tag{3.47}$$

with the identification $i - n = i$. It is implied from (3.46) and (3.47) that the branch-point twist field $\tilde{\mathcal{T}}^\eta$ is the Hermitian conjugate of the branch-point twist field $\mathcal{T}^\eta$:

$$\tilde{\mathcal{T}}^\eta = (\mathcal{T}^\eta)^\dagger \,. \tag{3.48}$$

These twist fields are spinless and they are primary fields with the lowest possible scaling dimension (they have the same scaling dimension) [127]:

$$d_n = \frac{1}{24}(n - \frac{1}{n})\,. \tag{3.49}$$

The branch-point twist fields $\mathcal{T}^\eta$ and $\tilde{\mathcal{T}}^\eta$, defined by (3.46) and (3.47) respectively, can be uniquely fixed by the requirement that they have the lowest scaling dimension given by (3.49) and they are invariant under all symmetries of the n-copy Ising model which commute with g. For our limited purpose, we will not review in this section the form factors of branch-point twist fields. We refer the reader to the papers [46, 131].

It has been shown in [46] that there exists a relation between the branch-point twist fields in the n-copy Ising model and the $U(1)$ twist fields in the n-copy Dirac theory. To see this relation, we construct an n-copy free Dirac fermion model by doubling the n-copy Ising model. We denote the fundamental real Majorana fermion fields for each n-copy Ising model by $\psi_{a,j}, \bar{\psi}_{a,j}$ and $\psi_{b,j}, \bar{\psi}_{b,j}$ for $j = 1, \cdots, n$, respectively, and the fundamental Dirac spinor fermion field by

$$\Psi_j = \begin{pmatrix} \Psi_{R,j} \\ \Psi_{L,j} \end{pmatrix} .$$

Then we have the identification:

$$\Psi_{R,j} = \frac{1}{\sqrt{2}}(\psi_{a,j} + i\psi_{b,j}), \qquad \Psi_{L,j} = \frac{1}{\sqrt{2}}(\bar{\psi}_{a,j} - i\bar{\psi}_{b,j})\,. \tag{3.50}$$

In order for different copies of the Dirac fermions to anti-commute with each other, we define a new basis

$$\begin{pmatrix} \Psi_1^{\rm ac} \\ \vdots \\ \Psi_n^{\rm ac} \end{pmatrix} \tag{3.51}$$

with scattering matrix -1 among different copies. Accordingly, the branch-point twist field in the n-copy Dirac theory, which we denote by $\mathcal{T}^{\eta}_{\rm Dirac}$, has modified exchange relations with respect to the Dirac fermions:

$$\Psi_j^{\rm ac}(x)\mathcal{T}^{+}_{\rm Dirac}(0) = \begin{cases} \mathcal{T}^{+}_{\rm Dirac}(0)\Psi_j^{\rm ac}(x) & \text{for } j=1,\cdots,n \quad (x<0) \\ \mathcal{T}^{+}_{\rm Dirac}(0)\Psi_{j+1}^{\rm ac}(x) & \text{for } j=1,\cdots,n-1 \quad (x>0) \\ -\mathcal{T}^{+}_{\rm Dirac}(0)\Psi_1^{\rm ac}(x) & \text{for } j=n \quad (x>0) \end{cases} \tag{3.52}$$

and

$$\Psi_j^{\rm ac}(x)\mathcal{T}^{-}_{\rm Dirac}(0) = \begin{cases} \mathcal{T}^{-}_{\rm Dirac}(0)\Psi_j^{\rm ac}(x) & \text{for } j=1,\cdots,n \quad (x>0) \\ \mathcal{T}^{-}_{\rm Dirac}(0)\Psi_{j+1}^{\rm ac}(x) & \text{for } j=1,\cdots,n-1 \quad (x<0) \\ -\mathcal{T}^{-}_{\rm Dirac}(0)\Psi_1^{\rm ac}(x) & \text{for } j=n \quad (x<0)\,. \end{cases} \tag{3.53}$$

Then, we diagonalise the branch-point twist fields in the n-copy Dirac theory by performing a $SU(n)$ transformation of the basis (3.51) and the new basis after this transformation can be considered as n independent Dirac fermions. In this new basis, the branch-point twist fields can be written as a product of $U(1)$ twist fields acting on these independent Dirac fermions from different copies:

$$\mathcal{T}^{\eta}_{\rm Dirac} = \prod_k \sigma^{\eta}_{(k,\alpha_k)} \tag{3.54}$$

where $k = -\frac{n}{2}+1, \cdots, \frac{n}{2}$ represents the copy number and $\alpha_k = \frac{2k-1}{2n}$ is associated with the $U(1)$ element $e^{2\pi i \alpha_k}$, for $\alpha_k \in [0,1]$. It is worth mentioning that the relation (3.54) is only valid in the case of even n. For n odd, the dimension of the branch-point twist field constructed from this factorisation relation does not agree with (3.49) which is predicted by the conformal field theory. More details of the derivation of (3.54) can be found in Appendix B of [46]. On the other hand, from the point of view that the n-copy Dirac theory is a doubled n-copy Ising model, the branch-point twist field $\mathcal{T}^{\eta}_{\rm Dirac}$ has the relation

$$\mathcal{T}^{\eta}_{\rm Dirac} = \mathcal{T}^{\eta}_a \otimes \mathcal{T}^{\eta}_b \tag{3.55}$$

where $\mathcal{T}^{\eta}_a$ and $\mathcal{T}^{\eta}_b$ are the branch-point twist fields in the copies a and b of the n-copy Ising model, respectively.

Chapter 4

Liouville space and form factors in mixed states

In quantum field theory, correlation functions in mixed states can be described via a trace expression involving the density matrix ρ:

$$\langle \mathcal{O}(x,t)\cdots\rangle_\rho := \frac{\mathrm{Tr}\,(\rho\,\mathcal{O}(x,t)\cdots)}{\mathrm{Tr}\,(\rho)} \tag{4.1}$$

where $\mathcal{O}$ is a local field and $\cdots$ represents other local fields at different positions. The density matrix can have different forms according to different mixed states it describes. For instance, in thermal equilibrium with a Gibbs ensemble, the (un-normalized) density matrix is specialized as

$$\rho = \rho_\beta := e^{-\beta H}\,,$$

where β is the inverse temperature and H is the Hamiltonian. In the Ising model, the density matrix for a non-equilibrium steady state sustaining a constant energy flow, admits the form [39]

$$\rho = \rho_{\mathrm{ness}} := e^{-\beta_l \int_0^\infty d\theta\, m\cosh\theta\, a^\dagger(\theta)a(\theta) - \beta_r \int_{-\infty}^0 d\theta\, m\cosh\theta\, a^\dagger(\theta)a(\theta)} \tag{4.2}$$

where β_l^{-1} and β_r^{-1} are the left- and right-temperatures of the asymptotic baths driving the steady state, and where $a^\dagger(\theta)$ and $a(\theta)$ are asymptotic operators of creation and annihilation type. Further, it has been argued that after quantum quenches, the density matrix becomes the exponential of a linear combination of local conserved charges (generalized Gibbs ensemble) [31, 32], and this has been demonstrated in the Ising model [34, 35]. Since local conserved charges in the Ising model have the form $\int d\theta\, e^{s\theta}\, a^\dagger(\theta)a(\theta)$, the density matrix in this case is also the exponential of an integral over particle densities $a^\dagger(\theta)a(\theta)$.

Mixed-state correlation functions can be evaluated from the trace expression (4.1) by using the knowledge of the matrix elements of local fields in the vacuum. However, this method exhibits great difficulty, for instance, in the calculation of correlation functions for twist fields. This difficulty stems from two problems. One is that the re-summation of infinite number of states is required in order to gain the full temperature dependence. Another problem comes with the realization that summing explicitly over diagonal matrix elements generically suffers from divergencies at colliding rapidities. To overcome these two problems, we employ a novel approach based on the Liouville space and the mixed-state form factors.

4.1 Liouville space in general

Recall that at zero temperature, correlation functions are vacuum expectation values in the Hilbert space, and the large-distance form factor expansion (2.50) can be obtained by using the completeness of the basis of asymptotic states. Thanks to the Gelfand-Naimark-Segal (GNS) construction (see for instance the book [64]), a new Hilbert space can be constructed above a vacuum associated with a density matrix (or more precisely with a state, seen as a linear functional on a C^*-algebra), so that mixed-state correlation functions are vacuum expectation values in this space. The resulting Hilbert space is basically the space of operators (more precisely, a certain completion of a certain quotient of the C^*-algebra), and it is referred to as the Liouville space (sometimes referred to as the associated Hilbert space) [56–58]. In this fashion, mixed-state correlation functions can be obtained, using the resolution of the identity with respect to the Liouville space, in terms of a "form factor expansion". The idea of Liouville space arises from the theory of thermo-field dynamics. To our knowledge, it was in [21, 23] that the Liouville-space idea was applied for the first time to the form factor program in integrable quantum field theory. It is not our intention to go into the delicate details of how to construct a C^*-algebra in order to mathematically have the ingredients necessary for the application of the GNS construction. We will rather provide some basic principles in a slightly different but equivalent formulation in order to explain some of the subtleties involved in the process of obtaining a form factor expansion. Combining arguments in [23] and [55], we will provide in this section an introduction of the Liouville space for general mixed states with diagonal density matrices in general integrable models of quantum field theory and define the associated mixed-state form factors.

4.1.1 Formal structure

We consider the space of operators with basis formed by the product of a set of asymptotic operators of creation and annihilation type $V_{a_1}^{\epsilon_1}(\theta_1)\cdots V_{a_N}^{\epsilon_N}(\theta_N)$ with $\theta_1 > \ldots > \theta_N$, $\epsilon_j = \pm$, a_j representing particle types, and $N \in \mathbb{N}$. Here and below, we use

$$V_a^+(\theta) := V_a^\dagger(\theta), \quad V_a^-(\theta) := V(\theta).$$

As introduced in section 2.1, these asymptotic operators construct the basis of asymptotic states of Hilbert space and for general integrable models of QFT they satisfy Zamolodchikov's algebra

$$V_{a_i}(\theta_i)V_{a_j}(\theta_j) = \sum_{b_i,b_j} S_{a_i,a_j}^{b_i,b_j}(\theta_i-\theta_j)V_{b_j}(\theta_j)V_{b_i}(\theta_i)\,, \tag{4.3}$$

$$V_{a_i}^\dagger(\theta_i)V_{a_j}^\dagger(\theta_j) = \sum_{b_i,b_j} S_{a_i,a_j}^{b_i,b_j}(\theta_i-\theta_j)V_{b_j}^\dagger(\theta_j)V_{b_i}^\dagger(\theta_i), \tag{4.4}$$

$$V_{a_i}(\theta_i)V_{a_j}^\dagger(\theta_j) = \sum_{b_i,b_j} S_{a_j,b_i}^{b_j,a_j}(\theta_j-\theta_i)V_{b_j}^\dagger(\theta_j)V_{b_i}(\theta_i) + \delta_{a_i,a_j}\delta(\theta_i-\theta_j)\,. \tag{4.5}$$

This space is called Liouville space $\mathcal{L}_\rho$. It is an inner-product space based on $\mathrm{End}(\mathcal{H})$, with inner product specified by the density matrix ρ. With $A, B \in \mathrm{End}(\mathcal{H})$, we denote the corresponding Liouville states by $|A\rangle^\rho$, $|B\rangle^\rho$ respectively, and we set the inner product to be

$${}^\rho\langle A|B\rangle^\rho = \frac{\mathrm{Tr}\left(\rho\, A^\dagger B\right)}{\mathrm{Tr}\,(\rho)}. \tag{4.6}$$

We restrict ourselves to density matrices ρ which are diagonal on the asymptotic state basis:

$$\rho = \exp\left[-\int d\theta \sum_a W_a(\theta)\, V_a^\dagger(\theta)V_a(\theta)\right] \tag{4.7}$$

where the functions $W_a(\theta)$ are integrable on the real line and they guarantee a well-defined density matrix. These choices of density matrix include the usual Gibbs state at finite temperature or chemical potential, as well as the non-equilibrium steady state (4.2) and the generalized Gibbs ensembles. We will consider two cases: the *untwisted* and the *twisted* cases. In the untwisted construction, we consider the density matrix (4.7). In the twisted construction, we consider the density matrix $\rho^\sharp$ with the presence of an unitary operator which implement the $U(1)$ symmetry:

$$\begin{aligned}\rho^\sharp &= \exp\left[-\int d\theta \sum_a W_a^\sharp(\theta)\, V_a^\dagger(\theta) V_a(\theta)\right] \\ &= \exp\left[-\int d\theta \sum_a W_a(\theta)\, V_a^\dagger(\theta) V_a(\theta) - 2\pi i \alpha Q\right], \quad \alpha \in (0,1) \end{aligned} \tag{4.8}$$

where Q is the Hermitian conserved charge associated with the $U(1)$ symmetry, given by

$$Q = \int d\theta \sum_a u(a) V_a^\dagger(\theta) V_a(\theta), \tag{4.9}$$

with $u(a)$ the charge of the excitation a, and hence $W^\sharp(\theta) = W(\theta) + 2i\pi\alpha u(a)$. Note that we could actually employ in the twisted density matrix a more general operator which is associated with any global symmetry of QFT models. But, due to the limited models considered in this thesis, we will consider only the density matrix of the form (4.8) in the twisted case.

For convenience, we define the basis of states in the Liouville space in terms of the usual annihilation and creation operators, but with a particular normalization:

$$|\mathrm{vac}\rangle^\rho \equiv \mathbf{1}, \quad |\theta_1, \ldots, \theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \equiv Q^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1, \ldots, \theta_N)\, V_{a_1}^{\epsilon_1}(\theta_1) \cdots V_{a_N}^{\epsilon_N}(\theta_N), \tag{4.10}$$

where the normalization factors are given by

$$Q^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1, \ldots, \theta_N) := \prod_{i=1}^N \epsilon_i^{1-f_{a_i}} \left(1 - (-1)^{f_{a_i}} e^{-\epsilon_i W_{a_i}(\theta_i)}\right) \tag{4.11}$$

with f_a corresponding to the statistics of the particle of type a ($f_a = 1$ for the fermionic particle and $f_a = 0$ for the bosonic particle) and where we refer to a doublet (a, ϵ) as representing the type of a "Liouville particle" of rapidity θ. The Liouville space can be physically interpreted as the space of different types of particles and holes excitations created from the Liouville vacuum consisting of a finite density of particles with statistical distribution determined by ρ. In this sense, a basis element $|\theta_1, \ldots, \theta_N\rangle^\rho_{\epsilon_1,\ldots,\epsilon_N}$ represents the presence of N Liouville particles including particles $(a_j, \epsilon_j = +)$ or holes $(a_j, \epsilon_j = -)$ above the finite density. The choice of the normalization (4.96) leads to nice analytic properties which will be explained below. To avoid overcounting states with the same set of rapidities, we need to choose an ordering, for instance, $\theta_1 > \cdots > \theta_N$. In fact, it is convenient to define states with other orderings of rapidities by exchanging Liouville particles. It is worth mentioning that we define states with colliding rapidities as exactly zero:

$$|\theta_1,\ldots,\theta_N\rangle^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} = 0, \quad \theta_i=\theta_j \text{ for } i\neq j \text{ and } a_i=a_j\ . \tag{4.12}$$

This means that the exchange of Liouville particles will not give rise to any delta-function "contact terms" and the Liouville space is formed by a continuous basis without discrete (or delta-function) part at colliding rapidities. Then the resolution of the identity in Liouville space can be "symmetrized", and re-expressed through integrals over the full line,

$$\begin{aligned}\mathbf{1} = \sum_{N=0}^{\infty}\sum_{a_1,\ldots,a_N}\sum_{\epsilon_1,\ldots,\epsilon_N}\int_{-\infty}^{\infty}\frac{d\theta_1\cdots d\theta_N}{N!\,\prod_{i=1}^{N}\epsilon_i^{1-f_{a_i}}\left(1-(-1)^{f_{a_i}}e^{-\epsilon_i W_{a_i}(\theta_i)}\right)}\\ \times|\theta_1,\ldots,\theta_N\rangle^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}\ {}^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta_1,\ldots,\theta_N|\end{aligned} \tag{4.13}$$

where the colliding-rapidity submanifold of $\mathbb{R}^N$ has zero measure in a decomposition of the identity, and where the factor $N!$ in the denominator arises from overcounting states.

Generally, the basis of states are not orthonormal. However, in the case of diagonal scattering (and this is the only case which will be considered from now on)

$$S^{b_1,b_2}_{a_1,a_2}(\theta) = \delta^{b_1}_{a_1}\delta^{b_2}_{a_2}S_{a_1,a_2}(\theta) \tag{4.14}$$

(without summation over repeated indices), it is possible to calculate, using the cyclic property of the trace and Zamolodchikov's algebra, the following quantities

$$\begin{aligned}\langle V_a(\theta)V^\dagger_{a'}(\theta')\rangle_\rho &= e^{W_a(\theta)}\langle V^\dagger_{a'}(\theta')V_a(\theta)\rangle_\rho\\ &= e^{W_a(\theta)}\left(S_{a',a}(\theta'-\theta)\langle V_a(\theta)V^\dagger_{a'}(\theta')\rangle_\rho + (-1)^{1-f_a}\delta_{a,a'}\delta(\theta-\theta')\right)\\ &= \frac{\delta_{a,a'}\delta(\theta-\theta')}{1-(-1)^{f_a}e^{-W_a(\theta)}}\end{aligned} \tag{4.15}$$

and

$$\begin{aligned}\langle V^\dagger_a(\theta)V_{a'}(\theta')\rangle_\rho &= e^{-W_a(\theta)}\langle V_{a'}(\theta')V^\dagger_a(\theta)\rangle_\rho\\ &= e^{-W_a(\theta)}\left(S_{a,a'}(\theta-\theta')\langle V_a(\theta)V^\dagger_{a'}(\theta')\rangle_\rho + \delta_{a,a'}\delta(\theta-\theta')\right)\\ &= \frac{(-1)^{1-f_a}\delta_{a,a'}\delta(\theta-\theta')}{1-(-1)^{f_a}e^{W_a(\theta)}}\end{aligned} \tag{4.16}$$

where $(-1)^{f_a}\equiv S_{a,a}(0)=\pm$ and $(-1)^{1-f_a}=\mp$, corresponding to the statistics of the particle of type a. Following the same recipe, we can further write down the inner products of basis states

$$ {}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}^{\quad\rho}\langle\theta_1,\ldots,\theta_N|\theta'_1,\ldots,\theta'_N\rangle^{\rho}_{(a'_1,\epsilon'_1)\cdots(a'_N,\epsilon'_N)} = \prod_{i=1}^{N}\left[Q^{\rho}_{a_i,\epsilon_i}(\theta_i)\delta_{a_i,a'_i}\delta_{\epsilon_i,\epsilon'_i}\delta(\theta_i-\theta'_i)\right] \tag{4.17} $$

where we assume the ordering $\theta_1 > \cdots > \theta_N$ and $\theta'_1 > \cdots > \theta'_N$. In fact, this is equivalent with using Wick's theorem applied to traces of products of asymptotic operators with contractions given by

$$ \langle V^{\epsilon_1}_{a_1}(\theta_1)V^{\epsilon_2}_{a_2}(\theta_2)\rangle_{\rho} = \frac{\delta_{\epsilon_1,-\epsilon_2}\delta_{a_1,a_2}\delta(\theta_1-\theta_2)}{Q^{\rho}_{a_1,-\epsilon_1}(\theta_1)}. \tag{4.18} $$

It can seen from (4.17) that the states in Liouville space are not "canonically" normalized due to the existence of normalization factors.

In order to describe in a convenient way the states in Liouville space, we define Liouville operators $\mathbf{Z}_{a,\epsilon}(\theta)$ and its hermitian conjugate $\mathbf{Z}^{\dagger}_{a,\epsilon}(\theta)$ (sometimes referred to as "superoperators") such that

$$ \mathbf{Z}_{a,\epsilon}(\theta)|\text{vac}\rangle^{\rho} = 0, \quad |\theta_1,\ldots,\theta_N\rangle^{\rho}_{(a_1,\epsilon_1)\cdots,(a_N,\epsilon_N)} = \mathbf{Z}^{\dagger}_{a_1,\epsilon_1}(\theta_1)\cdots\mathbf{Z}^{\dagger}_{a_N,\epsilon_N}(\theta_N)|\text{vac}\rangle^{\rho}. \tag{4.19} $$

These operators satisfy the following exchange relations

$$ \mathbf{Z}_{a_i,\epsilon_i}(\theta_i)\mathbf{Z}_{a_j,\epsilon_j}(\theta_j) = S_{\gamma_i,\gamma_j}(\theta_i-\theta_j)\mathbf{Z}_{a_j,\epsilon_j}(\theta_j)\mathbf{Z}_{a_i,\epsilon_i}(\theta_i) \tag{4.20} $$

$$ \mathbf{Z}^{\dagger}_{a_i,\epsilon_i}(\theta_i)\mathbf{Z}^{\dagger}_{a_j,\epsilon_j}(\theta_j) = S_{\gamma_i,\gamma_j}(\theta_i-\theta_j)\mathbf{Z}^{\dagger}_{a_j,\epsilon_j}(\theta_j)\mathbf{Z}^{\dagger}_{a_i,\epsilon_i}(\theta_i) \tag{4.21} $$

$$ \mathbf{Z}_{a_i,\epsilon_i}(\theta_i)\mathbf{Z}^{\dagger}_{a_j,\epsilon_j}(\theta_j) = S_{\gamma_j,\gamma_i}(\theta_j-\theta_i)\mathbf{Z}^{\dagger}_{a_j,\epsilon_j}(\theta_j)\mathbf{Z}_{a_i,\epsilon_i}(\theta_i) + Q^{\rho}_{a_i,\epsilon_i}(\theta_i)\delta_{a_i,a_j}\delta\epsilon_1,\epsilon_2\,\delta(\theta_i-\theta_j) \tag{4.22} $$

where we denote the Liouville particle types by the couples $\gamma = (a,\epsilon)$ and where

$$ S_{\gamma_i,\gamma_j}(\theta_i-\theta_j) = \begin{cases} S_{a_i,a_j}(\theta_i-\theta_j) & (\epsilon_1=\epsilon_2) \\ S_{a_j,a_i}(\theta_j-\theta_i) & (\epsilon_1=-\epsilon_2). \end{cases} \tag{4.23} $$

It can be checked that this algebra is in agreement with the inner products (4.17). The space $\mathcal{L}_{\rho}$ can be identified with the Fock space over this algebra (it is sometimes referred to as a "Liouville-Fock" space [59]) and it is this algebra that describes the way how Liouville particles scatter.

4.1.2 Liouville left- and right-actions

To each operator $A \in \mathrm{End}(\mathcal{H})$, one can define two operators, $A^\ell \in \mathrm{End}(\mathcal{L}_\rho)$ and $A^r \in \mathrm{End}(\mathcal{L}_\rho)$, by left- and right-action of A, respectively:

$$A^\ell|B\rangle^\rho = |AB\rangle^\rho, \qquad A^r|B\rangle^\rho = |BA\rangle^\rho. \tag{4.24}$$

The left-action linear map $A \mapsto A^\ell$ is an algebra homomorphism, $(AB)^\ell = A^\ell B^\ell$; the right-action linear map $A \mapsto A^r$ is an algebra anti-homomorphism, $(AB)^r = B^r A^r$. Recalling the definition (4.6), we consider the quantity :

$$^\rho\langle B|A^\ell|C\rangle^\rho = {}^\rho\langle B|AC\rangle^\rho = \langle B^\dagger AC\rangle_\rho = {}^\rho\langle A^\dagger B|C\rangle^\rho \tag{4.25}$$

and this implies that left-action Liouville operators act on conjugate vectors as

$$^\rho\langle B|A^\ell = {}^\rho\langle A^\dagger B|. \tag{4.26}$$

Similarly, we consider the quantity:

$$^\rho\langle B|A^r|C\rangle^\rho = {}^\rho\langle B|CA\rangle^\rho = \langle B^\dagger CA\rangle_\rho = \langle \rho^{-1}A\rho B^\dagger C\rangle_\rho = {}^\rho\langle B\rho A^\dagger \rho^{-1}|C\rangle^\rho \tag{4.27}$$

and this implies that right-action Liouville operators act on conjugate vectors as

$$^\rho\langle B|A^r = {}^\rho\langle B\rho A^\dagger\rho^{-1}|. \tag{4.28}$$

It is obviously that left- and right-action Liouville operators commute with each other,

$$A^\ell B^r = B^r A^\ell. \tag{4.29}$$

The Hermitian conjugation of operators on $\mathcal{H}$ can also be translated by left- and right-action linear maps onto that of operators on $\mathcal{L}_\rho$. In particular, we find, by conjugating the equations (4.26) and (4.28), that, for every $A \in \mathrm{End}(\mathcal{H})$,

$$\left(A^\ell\right)^\dagger = \left(A^\dagger\right)^\ell, \qquad (A^r)^\dagger = \left(\rho A^\dagger \rho^{-1}\right)^r. \tag{4.30}$$

Hence, the Hermitian conjugation commutes with the left-action map, but not not with the right-action map. Specializing to operators $V_a^\epsilon(\theta)$ and using

$$\rho\, V_a^\epsilon(\theta)\, \rho^{-1} = e^{-\epsilon W_a(\theta)} V_a^\epsilon(\theta) \tag{4.31}$$

as well as linearity of the right-action map, we obtain

$$\left(V_a^{\epsilon}(\theta)^r\right)^\dagger = e^{\epsilon W_a(\theta)} V_a^{-\epsilon}(\theta)^r. \tag{4.32}$$

In the Liouville space, there should also be generators of symmetry transformations on $\mathcal{L}_\rho$ or Hermitian conserved charges. Let us consider left- and right-action of the Hermitian conserved charge Q (4.9) of $\mathcal{H}$: Q^ℓ and Q^r. We act Q^ℓ on a Liouville state with n particles:

$$\begin{aligned}
Q^\ell|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} &= \prod_{i=1}^N \left(1+e^{-\epsilon_i W_{a_i}(\theta_i)}\right) |Q V_{a_1}^{\epsilon_1}(\theta_1)\cdots V_{a_N}^{\epsilon_N}(\theta_N)\rangle^\rho \\
&= \prod_{i=1}^N \left(1+e^{-\epsilon_i W_{a_i}(\theta_i)}\right) |V_{a_1}^{\epsilon_1}(\theta_1)\cdots V_{a_N}^{\epsilon_N}(\theta_N) Q\rangle^\rho \\
&\quad + \sum_{i=1}^N \epsilon_i u(a_i) \prod_{i=1}^N \left(1+e^{-\epsilon_i W_{a_i}(\theta_i)}\right) |V_{a_1}^{\epsilon_1}(\theta_1)\cdots V_{a_N}^{\epsilon_N}(\theta_N)\rangle^\rho \\
&= Q^r|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \\
&\quad + \sum_{i=1}^N \epsilon_i u(a_i) |\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}
\end{aligned} \tag{4.33}$$

where we use $[Q, V_a^\epsilon(\theta)] = \epsilon\, u(a) V_a^\epsilon(\theta)$ in the second step. As we see, the Liouville operator $Q^\ell - Q^r$ is diagonal on Liouville eigenstates and has eighenvalue $\sum_{i=1}^N \epsilon_i u(a_i)$. It is then natural to define the Hermitian conserved charge on $\mathcal{L}_\rho$ as

$$\mathbf{Q} := Q^\ell - Q^r = \sum_a \sum_\epsilon \int d\theta\, \epsilon\, u(a)\, \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\mathbf{Z}_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)}. \tag{4.34}$$

The Hamiltonian and momentum can also be defined in this way:

$$\mathbf{H} := H^\ell - H^r, \quad \mathbf{H}|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} = \sum_{i=1}^N \epsilon_i m_{a_i} \cosh\theta_i |\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}$$

$$\mathbf{P} := P^\ell - P^r, \quad \mathbf{P}|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} = \sum_{i=1}^N \epsilon_i m_{a_i} \sinh\theta_i |\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}$$

In terms of the operators $Z_{a,\epsilon}(\theta)$ and $Z^\dagger_{a,\epsilon}(\theta)$, they read

$$\mathbf{H} := H^\ell - H^r = \sum_a \sum_\epsilon \int d\theta\, \epsilon\, m_a \cosh\theta\, \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\mathbf{Z}_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)} \tag{4.35}$$

$$\mathbf{P} := P^\ell - P^r = \sum_a \sum_\epsilon \int d\theta\, \epsilon\, m_a \sinh\theta\, \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\mathbf{Z}_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)}. \tag{4.36}$$

The Hamiltonian generates time evolution and the momentum generates spatial translation. For any $\mathbf{A} \in \mathcal{L}_\rho$, we have

$$\mathbf{A}(x,t) = e^{i\mathbf{H}t - i\mathbf{P}x}\mathbf{A}e^{-i\mathbf{H}t+i\mathbf{P}x}. \tag{4.37}$$

This is in agreement with the left- and right-action maps: with $A(x,t) = e^{iHt-iPx}Ae^{-iHt+iPx}$ we have

$$A(x,t)^\ell = A^\ell(x,t), \quad A(x,t)^r = A^r(x,t) \tag{4.38}$$

where we used the homomorphism and anti-homomorphism properties of the left- and right-action maps, respectively, as well as the fact that left- and right-action Liouville operators commute with each other. It is also in agreement with the correspondence between Hilbert space operators and Liouville space vectors:

$$|A(x,t)\rangle^\rho = e^{i\mathbf{H}t-i\mathbf{P}x}|A\rangle^\rho. \tag{4.39}$$

We can also construct the left- and right-actions of the Hilbert space creation and annihilation operators $V_a^\epsilon(\theta)$ by the operators $\mathbf{Z}_{a,\epsilon}(\theta)$ and $\mathbf{Z}^\dagger_{a,\epsilon}(\theta)$. Let $\mathbf{n} = \sum_a \sum_\epsilon \int d\theta\, \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\mathbf{Z}_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)} \in \mathrm{End}(\mathcal{L}_\rho)$ be the number operator of the Liouville space, with

$$\mathbf{n}|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} = N|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}. \tag{4.40}$$

Taking into account the action of the operator $V_a^\epsilon(\theta)^\ell$ on Liouville states and on their Hermitian conjugate

$$\begin{aligned} V_a^\epsilon(\theta)^\ell|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} &= \frac{1}{Q^\rho_{a,\epsilon}(\theta)}|\theta,\theta_1,\ldots,\theta_N\rangle^\rho_{(a,\epsilon)(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \\ &= \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)}|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \end{aligned}$$

and

$$\begin{aligned} {}^{\rho}_{(a_1\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta_1,\ldots,\theta_N|V_a^\epsilon(\theta)^\ell &= {}^{\rho}_{(a,-\epsilon)(a_1\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta,\theta_1,\ldots,\theta_N|\frac{1}{Q^\rho_{a,-\epsilon}(\theta)} \\ &= {}^{\rho}_{(a_1\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta_1,\ldots,\theta_N|\frac{\mathbf{Z}_{a,-\epsilon}(\theta)}{Q^\rho_{a,-\epsilon}(\theta)}, \end{aligned}$$

we have

$$V_a^\epsilon(\theta)^\ell = \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)} + \frac{\mathbf{Z}_{a,-\epsilon}(\theta)}{Q^\rho_{a,-\epsilon}(\theta)}. \tag{4.41}$$

Similarly, considering

$$\begin{aligned} V_a^\epsilon(\theta)^r|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} &= \frac{1}{Q^\rho_{a,\epsilon}(\theta)}|\theta_1,\ldots,\theta_N,\theta\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)(a,\epsilon)} \\ &= \frac{(-1)^N}{Q^\rho_{a,\epsilon}(\theta)}|\theta,\theta_1,\ldots,\theta_N\rangle^\rho_{(a,\epsilon)(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \\ &= \frac{\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\,(-1)^{\mathbf{n}}}{Q^\rho_{a,\epsilon}(\theta)}|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \end{aligned}$$

and

$$\begin{aligned} {}^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta_1,\ldots,\theta_N|V_a^\epsilon(\theta)^r &= {}^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)(a,-\epsilon)}\langle\theta_1,\ldots,\theta_N,\theta|\frac{e^{\epsilon W_a(\theta)}}{Q^\rho_{a,-\epsilon}(\theta)} \\ &= {}^{\rho}_{(a,-\epsilon)(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta,\theta_1,\ldots,\theta_N|\frac{(-1)^N}{Q^\rho_{a,\epsilon}(\theta)} \\ &= {}^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}\langle\theta_1,\ldots,\theta_N|\frac{(-1)^{\mathbf{n}}\mathbf{Z}_{a,-\epsilon}(\theta)}{Q^\rho_{a,\epsilon}(\theta)}, \end{aligned}$$

we have

$$V_a^\epsilon(\theta)^r = \frac{1}{Q^\rho_{a,\epsilon}(\theta)}\left(\mathbf{Z}^\dagger_{a,\epsilon}(\theta) - \mathbf{Z}_{a,-\epsilon}(\theta)\right)(-1)^{\mathbf{n}} \tag{4.42}$$

(the result for this operator, presented in [55] is incorrect). From Zamolodchikov's algebra, using the homomorphism and anti-homomorphism properties of the left- and right-map, we should have the following algebras

$$V_{a_i}^{\epsilon_i}(\theta_i)^\ell\, V_{a_j}^{\epsilon_j}(\theta_j)^\ell = S_{\gamma_i,\gamma_j}(\theta_i-\theta_j)V_{a_j}^{\epsilon_j}(\theta_j)^\ell\, V_{a_i}^{\epsilon_i}(\theta_i)^\ell + \epsilon_j^{1-f_{a_j}}\delta_{a_i,a_j}\delta_{\epsilon_i,-\epsilon_j}\delta(\theta_i-\theta_j) \tag{4.43}$$

$$V_{a_i}^{\epsilon_i}(\theta_i)^r\, V_{a_j}^{\epsilon_j}(\theta_j)^r = S_{\gamma_j,\gamma_i}(\theta_j-\theta_i)V_{a_j}^{\epsilon_j}(\theta_j)^r\, V_{a_i}^{\epsilon_i}(\theta_i)^r + \epsilon_i^{1-f_{a_j}}\delta_{a_i,a_j}\delta_{\epsilon_i,-\epsilon_j}\delta(\theta_i-\theta_j) \tag{4.44}$$

One can verify that operators defined by (4.41) and (4.42) indeed reproduce these algebras, using relations

$$S_{\gamma_i,\gamma_j}(\theta_i-\theta_j) = S_{\bar\gamma_i,\bar\gamma_j}(\theta_i-\theta_j) = S_{\bar\gamma_j,\gamma_i}(\theta_j-\theta_i) = S_{\gamma_j,\bar\gamma_i}(\theta_j-\theta_i) \tag{4.45}$$

where $\bar\gamma = (a,-\epsilon)$ for $\gamma = (a,\epsilon)$, and

$$\frac{1}{Q^\rho_{a,\epsilon}(\theta)} + \frac{(-1)^{1-f_a}}{Q^\rho_{a,-\epsilon}(\theta)} = \epsilon^{1-f_a}. \tag{4.46}$$

Moreover, they agree with relations

$$(V_a^\epsilon(\theta)^\ell)^\dagger = (V_a^{-\epsilon}(\theta))^\ell, \qquad (V_a^\epsilon(\theta)^r)^\dagger = e^{-\epsilon W_a(\theta)}(V_a^{-\epsilon}(\theta))^r$$

due to the fact that the number operator $\mathbf{n}$ is Hermitian

$$\mathbf{n}^\dagger = \mathbf{n}.$$

Note that we will consider from now on only the left-action, which is sufficient for the purpose of this thesis.

4.1.3 Mixed-state form factors

Definition

After the setup of the Liouville space, we can see that mixed-state correlation functions are then vacuum expectation values in $\mathcal{L}_\rho$:

$$\langle A\rangle_\rho = {}^\rho\langle \text{vac}|A^\ell|\text{vac}\rangle^\rho. \tag{4.47}$$

Hence, using (4.13), two-point functions should have a spectral decomposition on $\mathcal{L}_\rho$ where left-action matrix elements are involved. This gives rise to the definition of mixed-state form factors associated to ρ: they are the matrix elements

$$f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) := {}^\rho\langle\text{vac}|\mathcal{O}^\ell|\theta_1,\ldots,\theta_N\rangle^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)} \tag{4.48}$$

(or more precisely, analytic functions of the rapidities continued from the region $\theta_1 > \cdots > \theta_N$), where $\mathcal{O}$ is implicitly at the space-time point (0,0). These form factors satisfy, from the algebra (4.21), the relation

$$\begin{aligned} & f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_j,\epsilon_j)(a_{j+1},\epsilon_{j+1})\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N) \\ = \; & S_{a_j,a_{j+1}}(\theta_j-\theta_{j+1}) f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_{j+1},\epsilon_{j+1})(a_j,\epsilon_j)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_{j+1},\theta_j,\ldots,\theta_N) \end{aligned} \tag{4.49}$$

which allows us to extract the scattering matrix by analytic continuing mixed-state form factors to different orderings of the rapidities. In addition, it can be inferred from the cyclic property of the trace that

$${}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}^{\qquad\qquad\rho}\langle\theta_1,\ldots,\theta_N|\mathcal{O}^\ell|\text{vac}\rangle^\rho = f^{\rho;\mathcal{O}}_{(a_N,-\epsilon_N)\cdots(a_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1). \tag{4.50}$$

From this definition, mixed-state form factors can be considered as traces with insertions of operators $V_a^\epsilon(\theta)$, up to the overall factor $Q^\rho_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)$, and up to the subtraction of contact terms at colliding rapidities. Let us illustrate this more explicitly.

Traces of products of creation and annihilation operators are evaluated by using Wick's theorem (this follows from cyclicity of the trace and Zamolodchikov's algebra) with contractions given by

$$Q^{\rho}_{(a_1,\epsilon_1)(a_2,\epsilon_2)}(\theta_1,\theta_2)\langle V^{\epsilon_1}_{a_1}(\theta_1)V^{\epsilon_2}_{a_2}(\theta_2)\rangle_\rho = {}_{(a_2,-\epsilon_2)}^{\quad\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{(a_1,\epsilon_1)}.$$

Traces with a further insertion of a local field, expressed through creation and annihilation operators, are evaluated similarly. This leads in a standard way to a diagrammatic expression, associating a single vertex to the local field. In this sense, mixed-state form factors are obtained by summing over connected diagrams,

$$f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) = \left[Q^{\rho}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)\langle\mathcal{O}\,V^{\epsilon_1}_{a_1}(\theta_1)\cdots V^{\epsilon_N}_{a_N}(\theta_N)\rangle_\rho\right]_{\text{connected}}.$$

Take two-particle form factors for example,

$$f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)(a_2,\epsilon_2)}(\theta_1,\theta_2) = Q^{\rho}_{(a_1,\epsilon_1)(a_2,\epsilon_2)}(\theta_1,\theta_2)\langle\mathcal{O}\,V^{\epsilon_1}_{a_1}(\theta_1)V^{\epsilon_2}_{a_2}(\theta_2)\rangle_\rho - \langle\mathcal{O}\rangle_\rho\ {}_{(a_2,-\epsilon_2)}^{\quad\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{(a_1,\epsilon_1)}. \tag{4.51}$$

Using cyclicity of the trace, other matrix elements for two particles are written as

$$\begin{aligned}
{}_{(a_2,\epsilon_2)}^{\quad\rho}\langle\theta_2|\mathcal{O}^\ell|\theta_1\rangle^{\rho}_{(a_1,\epsilon_1)} &= f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)(a_2,-\epsilon_2)}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_\rho\ {}_{(a_2,\epsilon_2)}^{\quad\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{(a_1,\epsilon_1)} && (4.52)\\
{}_{(a_2,\epsilon_2)(a_1,\epsilon_1)}^{\qquad\quad\rho}\langle\theta_2,\theta_1|\mathcal{O}^\ell|\text{vac}\rangle^{\rho} &= f^{\rho;\mathcal{O}}_{(a_1,-\epsilon_1)(a_2,-\epsilon_2)}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_\rho\ {}_{(a_2,\epsilon_2)}^{\quad\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{(a_1,-\epsilon_1)} && (4.53)
\end{aligned}$$

Similar equations for many particles can be obtained in the same fashion.

Mixing phenomenon

Consider local fields $\mathcal{O}$ that are expressed through normal-ordered products of finitely many asymptotic operators $V_a(\theta)$ and $V_a^\dagger(\theta)$. It is a simple matter to evaluate the traces defining mixed-state form factors (or to evaluate any correlation functions) for such fields $\mathcal{O}$ by using Wick's theorem. For instance, one can check explicitly in the Ising model that for the Majorana fermionic field ψ we have

$$f^{\rho;\psi}_{\epsilon}(\theta) = \begin{cases} \langle\text{vac}|\psi|\theta\rangle & (\epsilon=+)\\ \langle\theta|\psi|\text{vac}\rangle & (\epsilon=-)\end{cases}$$

and that for the energy field $\varepsilon = i : \bar{\psi}\psi :$ we have

$$f_-^{\rho;\varepsilon}(-) = \frac{m}{\pi}\int_0^\infty d\theta \frac{1}{1+e^{W(\theta)}}, \quad f_{\epsilon_1,\epsilon_2}^{\rho;\varepsilon}(\theta_1,\theta_2) = \begin{cases} \langle \text{vac}|\varepsilon|\theta_1,\theta_2\rangle & (\epsilon_1=\epsilon_2=+) \\ \langle\theta_2|\varepsilon|\theta_1\rangle & (\epsilon_1=+,\ \epsilon_2=-) \\ \langle\theta_2,\theta_1|\varepsilon|\text{vac}\rangle & (\epsilon_1=\epsilon_2=-) \end{cases}$$

where the zero-particle form factor $f_-^{\rho;\varepsilon}(-)$ of the field ε is just its expectation value $\langle\varepsilon\rangle_\rho$ (all other form factors are zero). It can be inferred from these formulas that the mixed-state form factors of ψ and of ε are equal to matrix elements on $\mathcal{H}$ of the fields

$$\mathfrak{U}(\psi) := \psi, \quad \mathfrak{U}(\varepsilon) := \varepsilon + \langle\varepsilon\rangle_\rho \mathbf{1} \tag{4.54}$$

respectively, where the matrix element taken depends on the Liouville particle types. This is a *mixing* phenomenon and it is in fact completely general: a local field is transformed (or mixed) into a linear combination involving its "ascendants" under Zamolodchikov's algebra.

This phenomenon of mixing suggests the existence of a mixing map $\mathfrak{U}$ on the space of local fields such that the following relation holds:

$$\begin{aligned} & f^{\rho;\mathcal{O}}_{(a_1,+)\cdots(a_j,+)(a_{j+1},-)\cdots(a_N,-)}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N) \\ = \ & {}_{a_N,\ldots,a_{j+1}}\langle\theta_N,\ldots,\theta_{j+1}|\mathfrak{U}(\mathcal{O})|\theta_1,\ldots,\theta_j\rangle_{a_1,\ldots,a_j} \end{aligned} \tag{4.55}$$

where there are j plus signs and $N-j$ minus signs as indices on the left-hand side. Note that matrix elements of $\mathcal{O}$ on $\mathcal{H}$ can be considered as the pure-state limits $W_a(\theta)\to\infty$ (uniformly on θ) of mixed-state form factors. For instance, inside the trace defining the mixed-state form factor

$$f^{\rho;\mathcal{O}}_{(a_1,+)\cdots(a_j,+)(a_{j+1},-)\cdots(a_N,-)}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N),$$

we can bring all operators $V_a(\theta)$ to the left while keeping all operators $V_a^\dagger(\theta)$ to the right of $\mathcal{O}$, by using the cyclic property of the trace. After taking the pure-state limit, we see that

$$\begin{aligned} & \lim_{W_a\to\infty} f^{\rho;\mathcal{O}}_{(a_1,+)\cdots(a_j,+)(a_{j+1},-)\cdots(a_N,-)}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N) \\ = \ & {}_{a_N,\ldots,a_{j+1}}\langle\theta_N,\ldots,\theta_{j+1}|\mathcal{O}|\theta_1,\ldots,\theta_j\rangle_{a_1,\ldots,a_j}. \end{aligned} \tag{4.56}$$

This holds for all values of rapidities such that $\theta_j \neq \theta_k$ for $j \neq k$, and thanks to (4.49) we may evaluate the limit for other choices of signs. From this, we can denote by

$$f^{\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) := \lim_{W_a\to\infty} f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) \tag{4.57}$$

the matrix elements of $\mathcal{O}$ on $\mathcal{H}$ which are obtained from taking pure-state limits of general mixed-state form factors. In particular, the usual form factors, with excited states on the right and the vacuum on the left, are $f^{\mathcal{O}}_{(a_1,+)\ldots(a_N,+)}(\theta_1,\ldots,\theta_N)$. In this sense, the relation (4.55) can be generalized as

$$f^{\rho;\mathcal{O}}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) = f^{\mathfrak{U}(\mathcal{O})}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N). \tag{4.58}$$

It will be convenient to use a notation for form factors with general Liouville states $|A\rangle^\rho$:

$$f^{\rho;\mathcal{O}}\big[|A\rangle^\rho\big] = {}^\rho\langle\text{vac}|\mathcal{O}^\ell|A\rangle^\rho. \tag{4.59}$$

We will also use the notation

$$f^{\mathcal{O}}\big[|A\rangle^\rho\big] := \lim_{W_a\to\infty} f^{\rho;\mathcal{O}}\big[|A\rangle^\rho\big]. \tag{4.60}$$

With these notations, (4.58) becomes

$$f^{\rho;\mathcal{O}}\big[|A\rangle^\rho\big] = f^{\mathfrak{U}(\mathcal{O})}\big[|A\rangle^\rho\big] \tag{4.61}$$

One consequence of (4.61) is that mixed-state form factors of local fields which are normal-ordered products of finitely many asymptotic operators $V_a(\theta)$ and $V_a^\dagger(\theta)$ are entire functions of the rapidities, no matter the analytic properties of $W(\theta)$. This is the reason for our choice of the normalization factor (4.96). Note that this argument holds for non-interacting fields, for instance, local fields $\mathcal{O}$. But we expect that it is still true for interacting fields, including twist fields and general local fields in integrable interacting models.

We recall that when evaluating vacuum form factors of local fields given by normal-ordered products, there can be no contractions inside the field. However, when evaluating mixed-state form factors, there exist contractions

$$\langle V^\dagger_{a_1}(\theta_1)V_{a_2}(\theta_2)\rangle_\rho = \frac{\delta_{a_1,a_2}\delta(\theta_1-\theta_2)}{Q^\rho_{a_1,-}(\theta_1)}$$

inside such fields. It is the map $\mathfrak{U}$ which implements these additional "internal" contractions. Since a local field $\mathcal{O}$ can be seen as a state in Liouville space, $\mathfrak{U}$ is also a map that translates the Liouville state $|\mathcal{O}\rangle^\rho$ to another Liouville state which is a linear combination of $|\mathcal{O}\rangle^\rho$ and those with lower numbers of Liouville particles. From this point of view, the map $\mathfrak{U}$ can be represented conveniently by the action of an operator on the Liouville space

$$|\mathfrak{U}(\mathcal{O})\rangle^\rho = \mathbf{U}|\mathcal{O}\rangle^\rho. \tag{4.62}$$

It turns out that the operator $\mathbf{U}$ performing the necessary Wick contractions is given by

$$\mathbf{U} = \exp\left[\sum_a \int d\theta\, \frac{\mathbf{Z}_{a,-}(\theta)\mathbf{Z}_{a,+}(\theta)}{Q^\rho_{a,-}(\theta)}\right]. \tag{4.63}$$

We may also see these additional contractions as arising from a change of normal ordering, from one with respect to the usual vacuum $|\text{vac}\rangle$ to one with respect to the Liouville vacuum $|\text{vac}\rangle^\rho$. The normal ordering in Liouville space is defined to bring all operators $\mathbf{Z}^\dagger_{a,\epsilon}(\theta)$ to the right of all operators $\mathbf{Z}_{a,\epsilon}(\theta)$ without taking any delta-function term but taking all S-matrices involved in the exchanges. If we denote by $\circ\!\!\!\circ \cdot \circ\!\!\!\circ$ the normal ordering with respect to $|\text{vac}\rangle^\rho$, then we have

$$|\mathcal{O}\rangle^\rho = \mathbf{U} \,\circ\!\!\!\circ\, \mathcal{O}^\ell \,\circ\!\!\!\circ\, |\text{vac}\rangle^\rho. \tag{4.64}$$

In order to prove (4.64), let us consider the normal-ordered operators

$$\mathcal{O} = V^\dagger_{a_1}(\theta_1)\cdots V^\dagger_{a_k}(\theta_k)V_{a_{k+1}}(\theta_{k+1})\cdots V_{a_n}(\theta_n) := \prod_i V^{\epsilon_i}_{a_i}(\theta_i) \tag{4.65}$$

for fixed n and k, with all nonegative integers $n \geq k \geq 0$. Using (4.41) and the definition of normal-ordering $\circ\!\!\!\circ \cdot \circ\!\!\!\circ$, we have

$$\circ\!\!\!\circ\, \mathcal{O}^\ell \,\circ\!\!\!\circ\, |\text{vac}\rangle^\rho = \prod_i \frac{\mathbf{Z}^\dagger_{a_i,\epsilon_i}(\theta_i)}{Q^\rho_{\nu_i,\epsilon_i}(\theta_i)}|\text{vac}\rangle^\rho \tag{4.66}$$

$${}^\rho\langle\text{vac}|\,\circ\!\!\!\circ\, \mathcal{O}^\ell \,\circ\!\!\!\circ = {}^\rho\langle\text{vac}|\prod_i \frac{\mathbf{Z}_{a_i,-\epsilon_i}(\theta_i)}{Q^\rho_{\nu_i,-\epsilon_i}(\theta_i)} \tag{4.67}$$

Further, direct calculation shows that

$$\mathbf{U}\mathbf{Z}^\dagger_{a,\epsilon}(\theta)\mathbf{U}^{-1} = \mathbf{Z}^\dagger_{a,\epsilon}(\theta) + \epsilon^{f_a}\frac{Q^\rho_{a,\epsilon}(\theta)}{Q^\rho_{a,-}(\theta)}\mathbf{Z}_{a,-\epsilon}(\theta) \tag{4.68}$$

where we used Zamolodchikov's algebra and the property $S_{a_i,a_j}(\theta_i-\theta_j)S_{a_j,a_i}(\theta_j-\theta_i) = 1$. Then, on the right-hand side of (4.64), using (4.66), (4.68) and $\mathbf{U}^{-1}|\text{vac}\rangle^\rho = |\text{vac}\rangle^\rho$, we have

$$\mathbf{U}\,\circ\!\!\!\circ\, \mathcal{O}^\ell \,\circ\!\!\!\circ\, |\text{vac}\rangle^\rho \;=\; \prod_{i=1}^k \left(\frac{\mathbf{Z}^\dagger_{a_i,+}(\theta_i)}{Q^\rho_{a_i,+}(\theta_i)} + \frac{\mathbf{Z}_{a_i,-}(\theta_i)}{Q^\rho_{a_i,-}(\theta_i)}\right)\prod_{i=k+1}^n\left(\frac{\mathbf{Z}^\dagger_{a_i,-}(\theta_i)}{Q^\rho_{a_i,-}(\theta_i)} + \epsilon^{f_a}\frac{\mathbf{Z}_{a_i,+}(\theta_i)}{Q^\rho_{a_i,-}(\theta_i)}\right)|\text{vac}\rangle^\rho$$

and on the left-hand side, using (4.99), we have

$$\mathcal{O}^{\ell}|\mathrm{vac}\rangle^{\rho}=\prod_{i=1}^{k}\left(\frac{\mathbf{Z}^{\dagger}_{a_i,+}(\theta_i)}{Q^{\rho}_{a_i,-}(\theta_i)}+\frac{\mathbf{Z}_{a_i,-}(\theta_i)}{Q^{\rho}_{a_i,+}(\theta_i)}\right)\prod_{i=k+1}^{n}\left(\frac{\mathbf{Z}^{\dagger}_{a_i,-}(\theta_i)}{Q^{\rho}_{a_i,+}(\theta_i)}+\frac{\mathbf{Z}_{a_i,+}(\theta_i)}{Q^{\rho}_{a_i,-}(\theta_i)}\right)|\mathrm{vac}\rangle^{\rho}\,.$$

These are equal, in light of the fact that $\mathbf{Z}_{a_i,+}(\theta_i)\mathbf{Z}^{\dagger}_{a_j,-}(\theta_j)=S_{a_i,a_j}(\theta_i-\theta_j)\mathbf{Z}^{\dagger}_{a_j,-}(\theta_j)\mathbf{Z}_{a_i,+}(\theta_i)$ and that $\mathbf{Z}_{a,+}(\theta)|\mathrm{vac}\rangle^{\rho}=0$. The relation (4.64) plays an essential role in expressing mixed-state form factors of local fields $\mathcal{O}$ in terms of matrix elements on Hilbert space. On one hand, the Hermitian conjugation of (4.64) leads immediately to the relation

$$f^{\rho;\mathcal{O}}\big[|A\rangle^{\rho}\big]={}^{\rho}\langle\mathrm{vac}|{:}\mathcal{O}^{\ell}{:}\mathbf{U}^{\dagger}|A\rangle^{\rho}. \tag{4.69}$$

On the other hand, using (4.64) and the definition (4.62) of the map $\mathfrak{U}$, we obtain

$$\mathcal{O}^{\ell}|\mathrm{vac}\rangle^{\rho}={:}\mathfrak{U}(\mathcal{O})^{\ell}{:}|\mathrm{vac}\rangle^{\rho}. \tag{4.70}$$

Hermitian conjugating (4.70) and considering $\mathfrak{U}(\mathcal{O}^{\dagger})=\mathfrak{U}(\mathcal{O})^{\dagger}$, we have

$${}^{\rho}\langle\mathrm{vac}|\mathcal{O}^{\ell}={}^{\rho}\langle\mathrm{vac}|{:}\mathfrak{U}(\mathcal{O})^{\ell}{:}. \tag{4.71}$$

This gives rise to the relation

$$f^{\rho;\mathcal{O}}\big[|A\rangle^{\rho}\big]={}^{\rho}\langle\mathrm{vac}|{:}\,\mathfrak{U}(\mathcal{O})^{\ell}{:}|A\rangle^{\rho}. \tag{4.72}$$

Due to (4.67), (4.22) and (4.19), it can be shown that

$${}^{\rho}\langle\mathrm{vac}|{:}\mathcal{O}^{\ell}{:}|A\rangle^{\rho}=f^{\mathcal{O}}\big[|A\rangle^{\rho}\big]. \tag{4.73}$$

Putting together (4.69), (4.72) and (4.73), we finally obtain

$$f^{\rho;\mathcal{O}}\big[|A\rangle^{\rho}\big]=f^{\mathfrak{U}(\mathcal{O})}\big[|A\rangle^{\rho}\big]=f^{\mathcal{O}}\big[\mathbf{U}^{\dagger}|A\rangle^{\rho}\big]. \tag{4.74}$$

where (4.61) is reproduced and hence the existence of the map $\mathfrak{U}$ is confirmed.

From the above proof, we can see that the relation (4.64) indeed play an essential role. But (4.64) can not be proved without the equation (4.68). To make sure the equation (4.68) holds, the operator $\mathbf{U}$ is required to have the form $\mathbf{U}=e^{\mathbf{B}}$ with $\mathbf{B}$ satisfying

$$[\mathbf{B},\mathbf{Z}^{\dagger}_{a,\epsilon}(\theta)]=\epsilon^{f_a}\frac{Q^{\rho}_{a,\epsilon}(\theta)}{Q^{\rho}_{a,-}(\theta)}\mathbf{Z}_{a,-\epsilon}(\theta),\quad [\mathbf{B},\mathbf{Z}_{a,\epsilon}(\theta)]=0,\quad \mathbf{B}|\mathrm{vac}\rangle^{\rho}=0.$$

This is the reason why the operator $\mathbf{U}$ is written as (4.63).

Now we summarise the above arguments. First, we observe the mixing phenomenon that mixed-state form factors of local fields $\mathcal{O}$ can be obtained as the known vacuum form factors of a linear combination of $\mathcal{O}$ and their ascendants. Then we suspect there exists a map $\mathfrak{U}$ such that mixed-state form factors can be evaluated by the relation (4.58). We define the map $\mathfrak{U}$ through the relation (4.62) which involves an operator $\mathbf{U}$ acting on the Liouville space. With the explicite form of $\mathbf{U}$ (4.63), we prove the relation (4.58) and hence the existence of the map $\mathfrak{U}$. To sum up in a word, we find a map $\mathfrak{U}$ or an operator $\mathbf{U}$ which provides a way of evaluating mixed-state form factors from the known vacuum form factors.

Finally, as stated in [55], this technique of employing the map $\mathfrak{U}$ in order to evaluate mixed-state form factors is in parallel with the exponential conformal change of coordinates in conformal field theory, which is used to calculate finite-temperature correlation functions from zero-temperature correlation functions. However, we focus in this these on massive models with general density matrices of the form (4.7). It would be interesting to gain a fuller geometric or algebraic understanding of it.

4.2 Ising model

In the Ising model, the fundamental fields are free Majorana fermionic fields which are real, and the spectrum contains only one particle type. The untwisted density matrix describing mixed states for this model is given by

$$\rho = \exp\left[-\int d\theta\, W(\theta)\, a^{\dagger}(\theta)a(\theta)\right] \tag{4.75}$$

where $a(\theta)$ and $a^{\dagger}(\theta)$ are mode operators of free Majorana fields, and the integrable function $W(\theta)$ ensures that the result is a well-defined density matrix. In the twisted construction, the density matrix $\rho^{\sharp}$ involves the unitary operator Z (3.18) that implements the $\mathbb{Z}_2$ symmetry (2.64): $W^{\sharp}(\theta) = W(\theta) + i\pi$. We assume $W(\theta)$ in the twisted construction to be uniformly positive:

$$\inf_{\theta\in\mathbb{R}} W(\theta) > 0, \tag{4.76}$$

and we will explain this in section 7.1.1. The Liouville space associated to the Ising model is spanned by the basis that is the product of a set of creation and annihilation operators of the free Majorana fields:

$$|\mathrm{vac}\rangle^{\rho} \equiv \mathbf{1}, \quad |\theta_1,\ldots,\theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N} \equiv Q^{\rho}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)\, a^{\epsilon_1}(\theta_1)\cdots a^{\epsilon_N}(\theta_N), \tag{4.77}$$

where we use the notations

$$a^{+}(\theta) := a^{\dagger}(\theta),\; a^{-}(\theta) := a(\theta),$$

where the ordering of the rapidities is chosen as $\theta_1 < \cdots < \theta_N$, and where the normalization factors are simply related to the Fermi filling fractions,

$$Q^{\rho}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) := \prod_{i=1}^{N}\left(1+e^{-\epsilon_i W(\theta_i)}\right). \tag{4.78}$$

Since the Ising model is a free model with simple S-matrix $S(\theta) = -1$, these creation and annihilation operators satisfy the canonical anti-commutation relations (2.85). Using Wick's theorem on traces of products of mode operators $a^{\epsilon}(\theta)$, the inner product in $\mathcal{L}_\rho$ is evaluated as

$$_{\epsilon_1,\ldots,\epsilon_N}^{\;\;\;\;\;\;\;\;\rho}\langle\theta_1,\ldots,\theta_N|\theta'_1,\ldots,\theta'_N\rangle^{\rho}_{\epsilon'_1,\ldots,\epsilon'_N} = \prod_{i=1}^{N}\left[\left(1+e^{-\epsilon_i W(\theta_i)}\right)\delta_{\epsilon_i,\epsilon'_i}\delta(\theta_i-\theta'_i)\right] \tag{4.79}$$

where we assume the ordering $\theta_1 > \cdots > \theta_N$ and $\theta'_1 > \cdots > \theta'_N$. Thus, we have the resolution of the identity on $\mathcal{L}_\rho$:

$$\mathbf{1} = \sum_{N=0}^{\infty}\sum_{\epsilon_1,\ldots,\epsilon_N}\int_{-\infty}^{\infty}\frac{d\theta_1\cdots d\theta_N}{N!\prod_{j=1}^{N}\left(1+e^{-\epsilon_j W(\theta_j)}\right)}|\theta_1,\ldots,\theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N}\;{}_{\epsilon_1,\ldots,\epsilon_N}^{\;\;\;\;\;\;\;\;\rho}\langle\theta_1,\ldots,\theta_N|. \tag{4.80}$$

The associated Liouville operators of $a^{\epsilon}(\theta)$ can be obtained through Liouville left-action and they are written as

$$a^{\epsilon}(\theta)^{\ell} = \frac{\mathbf{Z}^{\dagger}_{\epsilon}(\theta)}{1+e^{-\epsilon W(\theta)}} + \frac{\mathbf{Z}^{\dagger}_{-\epsilon}(\theta)}{1+e^{\epsilon W(\theta)}} \tag{4.81}$$

where $\mathbf{Z}_{\epsilon}(\theta)$ and $\mathbf{Z}^{\dagger}_{\epsilon}(\theta)$ are Liouville mode operators and they are defined such that

$$\mathbf{Z}_{\epsilon}(\theta)|\text{vac}\rangle^{\rho} = 0, \quad |\theta_1,\ldots,\theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N} = \mathbf{Z}^{\dagger}_{\epsilon_1}(\theta_1)\cdots\mathbf{Z}^{\dagger}_{\epsilon_N}(\theta_N)|\text{vac}\rangle^{\rho}. \tag{4.82}$$

These Liouville mode operators obey anti-commutation relations

$$\{\mathbf{Z}_{\epsilon}(\theta),\mathbf{Z}_{\epsilon'}(\theta')\} = \{\mathbf{Z}^{\dagger}_{\epsilon}(\theta),\mathbf{Z}^{\dagger}_{\epsilon'}(\theta')\} = 0, \quad \{\mathbf{Z}_{\epsilon}(\theta),\mathbf{Z}^{\dagger}_{\epsilon'}(\theta')\} = \left(1+e^{-\epsilon W(\theta)}\right)\delta_{\epsilon,\epsilon'}\delta(\theta-\theta') \tag{4.83}$$

which gives the canonical anti-commutation relations

$$\left\{a^{\epsilon}(\theta)^{\ell}, a^{\epsilon'}(\theta')^{\ell}\right\} = \delta_{\epsilon,\epsilon'}\,\delta(\theta-\theta') \tag{4.84}$$

that Liouville operators $a^\epsilon(\theta)^\ell$ should satisfy. In terms of these operators, the Hamiltonian and momentum in $\mathcal{L}_\rho$ are given by

$$\begin{aligned}
\mathbf{H} &= \sum_\epsilon \int d\theta\, \epsilon\, E_\theta \,\frac{\mathbf{Z}^\dagger_\epsilon(\theta)\mathbf{Z}_\epsilon(\theta)}{1+e^{-\epsilon W(\theta)}} \\
\mathbf{P} &= \sum_\epsilon \int d\theta\, \epsilon\, p_\theta \,\frac{\mathbf{Z}^\dagger_\epsilon(\theta)\mathbf{Z}_\epsilon(\theta)}{1+e^{-\epsilon W(\theta)}}.
\end{aligned} \tag{4.85}$$

Averages in the density matrix ρ are vacuum expectation values in $\mathcal{L}_\rho$:

$$\langle A\rangle_\rho = {}^\rho\langle \mathrm{vac}|A^\ell|\mathrm{vac}\rangle^\rho. \tag{4.86}$$

Using the resolution of the identity (4.80), we can obtain form factor expansions for mixed-state two-point functions. The form factors in $\mathcal{L}_\rho$ are defined as the matrix elements of Liouville left-action operators:

$$f^{\rho;\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) := {}^\rho\langle \mathrm{vac}|\mathcal{O}^\ell|\theta_1,\ldots,\theta_N\rangle^\rho_{\epsilon_1,\ldots,\epsilon_N}. \tag{4.87}$$

Due to the algebra (4.83), these form factors satisfy the relation

$$f^{\rho;\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N) = -f^{\rho;\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_{j+1},\theta_j,\ldots,\theta_N), \tag{4.88}$$

and the cyclic property of the trace implies

$${}_{\epsilon_1,\ldots,\epsilon_N}^{\quad\rho}\langle \theta_1,\ldots,\theta_N|\mathcal{O}|\mathrm{vac}\rangle^\rho = f^{\rho;\mathcal{O}}_{-\epsilon_N,\ldots,-\epsilon_1}(\theta_N,\ldots,\theta_1). \tag{4.89}$$

Mixed-state form factors are traces with insertions of operators $a^\epsilon(\theta)$, up to the overall factor $Q^\rho_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)$, subtracting off all "external" contraction terms in terms of

$$Q^\rho_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)\langle a^{\epsilon_1}(\theta_1)a^{\epsilon_2}(\theta_2)\rangle_\rho = {}_{-\epsilon_2}^{\ \ \rho}\langle\theta_2|\theta_1\rangle^\rho_{\epsilon_1}.$$

at colliding rapidities, and they are obtained by summing over connected diagrams,

$$f^{\rho;\mathcal{O}}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) = \Big[Q^\rho_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)\langle \mathcal{O}\, a^{\epsilon_1}(\theta_1)\cdots a^{\epsilon_N}(\theta_N)\rangle_\rho\Big]_{\mathrm{connected}}. \tag{4.90}$$

For instance,

$$f^{\rho;\mathcal{O}}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) = Q^\rho_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)\langle \mathcal{O}\, a^{\epsilon_1}(\theta_1)a^{\epsilon_2}(\theta_2)\rangle_\rho - \langle\mathcal{O}\rangle_\rho\ {}_{-\epsilon_2}^{\ \ \rho}\langle\theta_2|\theta_1\rangle_{\epsilon_1}. \tag{4.91}$$

Using cyclicity of the trace, we obtain other matrix elements of $\mathcal{O}^\ell$ for two particles

$$
\begin{aligned}
{}_{\epsilon_2}^{\;\rho}\langle\theta_2|\mathcal{O}^\ell|\theta_1\rangle_{\epsilon_1}^{\rho} &= f^{\rho;\mathcal{O}}_{\epsilon_1,-\epsilon_2}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_\rho \; {}_{\epsilon_2}^{\;\rho}\langle\theta_2|\theta_1\rangle_{\epsilon_1}^{\rho}, \qquad (4.92)\\
{}_{\epsilon_2,\epsilon_1}^{\quad\;\rho}\langle\theta_2,\theta_1|\mathcal{O}^\ell|\mathrm{vac}\rangle^{\rho} &= f^{\rho;\mathcal{O}}_{-\epsilon_1,-\epsilon_2}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_\rho \; {}_{\epsilon_2}^{\;\rho}\langle\theta_2|\theta_1\rangle_{-\epsilon_1}^{\rho}. \qquad (4.93)
\end{aligned}
$$

Similar equations generalize (4.91)-(4.93) to matrix elements for many particles.

4.3 Dirac theory

The spectrum of the Dirac theory contains two particle types which we denote by $\nu = \pm$. The untwisted density matrix is

$$
\rho = \exp\left[-\int d\theta \sum_\nu W_\nu(\theta)\, D_\nu^\dagger(\theta) D_\nu(\theta)\right] \qquad (4.94)
$$

where $D_\nu(\theta)$ and $D_\nu^\dagger(\theta)$ are mode operators of free $U(1)$ Dirac fields, and $W_\nu(\theta)$ are integrable functions ensuring the density matrix is well-defined. In the twisted case, we consider there exist in $\rho^\sharp$ two extra unitary operators $e^{2\pi i\nu\alpha \int d\theta D_\nu^\dagger(\theta) D_\nu(\theta)}$ which implement the $U(1)$ symmetry and hence $W_\nu^\sharp(\theta) = W_\nu(\theta) + 2\pi i\nu\alpha$. It is worth attention that it is not necessary in the Dirac theory to impose a positive condition like (4.76) to functions $W_\nu(\theta)$. We will provide an argument for this in subsection 7.1.2. The Liouville space is spanned by a set of products of creation and annihilation operators in Hilbert space with some particular normalization:

$$
|\mathrm{vac}\rangle^\rho \equiv \mathbf{1}, \quad |\theta_1,\ldots,\theta_N\rangle^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)} \equiv Q^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)\, D_{\nu_1}^{\epsilon_1}(\theta_1)\cdots D_{\nu_N}^{\epsilon_N}(\theta_N), \qquad (4.95)
$$

with the ordering $\theta_1 > \cdots > \theta_N$, where

$$
Q^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) := \prod_{i=1}^{N}\left(1 + e^{-\epsilon_i W_{\nu_i}(\theta_i)}\right) \qquad (4.96)
$$

and we use notations

$$
D_\nu^+(\theta) := D_\nu^\dagger(\theta),\; D_\nu^-(\theta) := D_\nu(\theta).
$$

Applying Wick's theorem on traces of products of mode operators $D_\nu^\epsilon(\theta)$, the inner product of basis states can be deduced as

$$
{}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}^{\qquad\qquad\quad\rho}\langle\theta_1,\ldots,\theta_N|\theta_1',\ldots,\theta_N'\rangle^{\rho}_{(\nu_1',\epsilon_1')\cdots(\nu_N',\epsilon_N')} = \prod_{i=1}^{N}\left[\left(1+e^{-\epsilon_i W_{\nu_i}(\theta_i)}\right)\delta_{\nu_i,\nu_i'}\,\delta_{\epsilon_i,\epsilon_i'}\,\delta(\theta_i-\theta_i')\right] \qquad (4.97)
$$

with the ordering $\theta_1 > \cdots > \theta_N$ and $\theta'_1 > \cdots > \theta'_N$. The resolution of the identity on $\mathcal{L}_\rho$ is then given by

$$\begin{aligned}\mathbf{1} &= \sum_{N=0}^{\infty} \sum_{\nu_1,\ldots,\nu_N} \sum_{\epsilon_1,\ldots,\epsilon_N} \int_{-\infty}^{\infty} \left[\frac{d\theta_1 \cdots d\theta_N}{N! \prod_{j=1}^{N} \left(1 + e^{-\epsilon_j W_{\nu_i}(\theta_j)}\right)} \right. \\ &\quad \left. \times |\theta_1, \ldots, \theta_N\rangle^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)} \; {}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}^{\rho}\langle \theta_1, \ldots, \theta_N| \right]. \end{aligned} \tag{4.98}$$

To $D_\nu^\epsilon(\theta)$, their associated left-action Liouville operators are

$$D_\nu^\epsilon(\theta)^\ell = \frac{\mathbf{Z}^\dagger_{\nu,\epsilon}(\theta)}{1 + e^{-\epsilon W_\nu(\theta)}} + \frac{\mathbf{Z}_{\nu,-\epsilon}(\theta)}{1 + e^{\epsilon W_\nu(\theta)}} \tag{4.99}$$

where $\mathbf{Z}^\dagger_{\nu,\epsilon}(\theta)$ and $\mathbf{Z}_{\nu,\epsilon}(\theta)$ are both defined as Liouville mode operators satisfying anti-commutation relations

$$\{\mathbf{Z}_{\nu,\epsilon}(\theta), \mathbf{Z}^\dagger_{\nu',\epsilon'}(\theta')\} = \left(1 + e^{-\epsilon W_\nu(\theta)}\right) \delta_{\nu,\nu'}\, \delta_{\epsilon,\epsilon'}\, \delta(\theta - \theta') \tag{4.100}$$

$$\{\mathbf{Z}_{\nu,\epsilon}(\theta), \mathbf{Z}_{\nu',\epsilon'}(\theta')\} = \{\mathbf{Z}^\dagger_{\nu,\epsilon}(\theta), \mathbf{Z}^\dagger_{\nu',\epsilon'}(\theta')\} = 0\;. \tag{4.101}$$

The Liouville space can be seen as the Fock space over this algebra,

$$\mathbf{Z}_{\nu,\epsilon}(\theta)|\text{vac}\rangle^\rho = 0, \quad |\theta_1, \ldots, \theta_N\rangle^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)} = \mathbf{Z}^\dagger_{\nu_1,\epsilon_1}(\theta_1) \cdots \mathbf{Z}^\dagger_{\nu_N,\epsilon_N}(\theta_N)|\text{vac}\rangle^\rho. \tag{4.102}$$

With the definitions above, it is obvious to see that the mixed-state averages of operators on $\mathcal{H}$ are vacuum expectation value on $\mathcal{L}_\rho$:

$$\langle \mathcal{O} \rangle_\rho = {}^\rho\langle \text{vac}|\mathcal{O}^\ell|\text{vac}\rangle^\rho. \tag{4.103}$$

Using the resolution of the identity (4.98), two-point functions, such as

$$\langle \mathcal{O}(x,\tau)\mathcal{O}^\dagger(0)\rangle_\rho = {}^\rho\langle \text{vac}|\mathcal{O}(x,\tau)^\ell \mathcal{O}^\dagger(0,0)^\ell|\text{vac}\rangle^\rho$$

should have a spectral decomposition on $\mathcal{L}_\rho$, where we define the matrix elements of left-action operators in Liouville space as mixed-state form factors

$$f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1, \ldots, \theta_N) := {}^\rho\langle \text{vac}|\mathcal{O}(0,0)^\ell|\theta_1, \ldots, \theta_N\rangle^\rho_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}\;. \tag{4.104}$$

With the definition (4.104) and anti-commutation relations (4.101), the mixed-state form factors satisfy the relations

$$\begin{aligned} & f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)\cdots(\nu_j,\epsilon_j)(\nu_{j+1},\epsilon_{j+1})\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_j,\theta_{j+1},\ldots,\theta_N) \\ = & -f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)\cdots(\nu_{j+1},\epsilon_{j+1})(\nu_j,\epsilon_j)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_{j+1},\theta_j,\ldots,\theta_N)\,. \end{aligned} \tag{4.105}$$

The cyclicity of traces leads to the relation

$$_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}{}^{\rho}\langle\theta_1,\ldots,\theta_N|\mathcal{O}^{\ell}|\text{vac}\rangle^{\rho} = f^{\rho;\mathcal{O}}_{(\nu_N,-\epsilon_N)\cdots(\nu_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1). \tag{4.106}$$

The mixed-state form factors are essentially traces with insertion of operators $D^{\epsilon}_{\nu}(\theta)$, up to an overall factor $Q^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)$ and up to the subtraction of contact terms at colliding rapidities:

$$\begin{aligned} & f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) \\ = & \left[Q^{\rho}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)\langle\mathcal{O}\,D^{\epsilon_1}_{\nu_1}(\theta_1)\cdots D^{\epsilon_N}_{\nu_N}(\theta_N)\rangle_{\rho}\right]_{\text{connected}}. \end{aligned} \tag{4.107}$$

For example, two-particle mixed-state form factors can be written as

$$f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = Q^{\rho}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)\langle\mathcal{O}D^{\epsilon_1}_{\nu_1}(\theta_1)D^{\epsilon_2}_{\nu_2}(\theta_2)\rangle_{\rho} - {}_{(\nu_2,-\epsilon_2)}{}^{\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{(\nu_1,\epsilon_1)}\,\langle\mathcal{O}\rangle_{\rho}\,. \tag{4.108}$$

Again, using cyclicity of the trace, we have

$$_{\nu_2,\epsilon_2}{}^{\rho}\langle\theta_2|\mathcal{O}^{\ell}|\theta_1\rangle^{\rho}_{\nu_1,\epsilon_1} = f^{\rho;\mathcal{O}}_{(\nu_1,\epsilon_1)(\nu_2,-\epsilon_2)}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_{\rho}\ {}_{\nu_2,\epsilon_2}{}^{\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{\nu_1,\epsilon_1}, \tag{4.109}$$

$$_{(\nu_2,\epsilon_2)(\nu_1,\epsilon_1)}{}^{\rho}\langle\theta_2,\theta_1|\mathcal{O}^{\ell}|\text{vac}\rangle^{\rho} = f^{\rho;\mathcal{O}}_{(\nu_1,-\epsilon_1)(\nu_2,-\epsilon_2)}(\theta_1,\theta_2) + \langle\mathcal{O}\rangle_{\rho}\ {}_{\nu_2,\epsilon_2}{}^{\rho}\langle\theta_2|\theta_1\rangle^{\rho}_{\nu_1,-\epsilon_1} \tag{4.110}$$

More general matrix elements for many particles can be expressed in the same fashion.

Chapter 5

Form factors of twist fields at finite temperature

The Liouville-space method was first employed in [21, 23] for evaluating correlation functions at finite temperature in the Ising model. This can be seen as the starting point of the application of the Liouville-space method to integrable models of QFT. Although the main object of this thesis is to obtain correlation functions in general diagonal mixed-states, it is intuitive to start with a review of previous works on the subject of correlation functions at finite temperature in order to show the original motivation to develop the Liouville-space method.

At finite temperature, a correlation function is not simply the vacuum expectation value as at zero temperature but the Gibbs ensemble expectation value which is a statistical average of quantum averages:

$$\langle \mathcal{O}(x,t)\cdots\rangle_{\rho_\beta} = \frac{\mathrm{Tr}(e^{-\beta H}\mathcal{O}(x,t)\cdots)}{\mathrm{Tr}(e^{-\beta H})} \tag{5.1}$$

where β is the inverse temperature and H is the Hamiltonian. In the Matsubara imaginary-time formalism [6], the definition (5.1) leads to the Kubo-Martin-Schwinger (KMS) identity [137, 138]

$$\langle \mathcal{O}(x,\tau)\cdots\rangle_{\rho_\beta} = (-1)^f \langle \mathcal{O}(x,\tau+\beta)\cdots\rangle_{\rho_\beta} \tag{5.2}$$

where the value of f is determined by the statistics of operator $\mathcal{O}$ ($f = 1$ for fermionic operators and $f = 0$ for bosonic operators), τ is real time $\tau = it$ and the dots $(\cdots)$ represents local fields at time τ and different positions. In light of the KMS identity, finite-temperature correlation functions can be interpreted as correlation functions of a Hilbert space on an infinite cylinder with the spatial coordinate $-\tau$ running on a circle of radius $r = \beta$ and Euclidean time x going along the cylinder. In this picture, we have

$$\langle \mathcal{O}(x,\tau)\cdots\rangle_{\rho_\beta} = e^{i\pi s/2}{}_\beta\langle \text{vac}|\mathcal{O}_\beta(-\tau,x)\cdots|\text{vac}\rangle_\beta \tag{5.3}$$

where s is the spin of operator $\mathcal{O}$, $|\text{vac}\rangle_\beta$ is the vacuum state in the Hilbert space on the circle and $\mathcal{O}_\beta(-\tau,x)$ is the corresponding operator acting on the new Hilbert space. The phase factor arising on the righ-hand side of (5.3) stems from the requirement that the corresponding operator $\mathcal{O}_\beta$ is Hermitian in the Hilbert space on the circle, provided that the operator $\mathcal{O}$ is Hermitian in the Hilbert space on the line. In the quantization scheme on the circle, there are usually two type of sectors: one is called Neveu-Schwartz (NS) sector where the fundamental fields (bosonic or fermionic) are anti-periodic on the circle, and another is called Ramond (R) sector where the fundamental fields are periodic on the circle. In general, sectors in this quantization scheme are associated with the quasi-periodicity condition:

$$\langle \mathcal{O}_f(x,\tau)\cdots\rangle_{\rho_\beta} = (-1)^f e^{2\pi i\alpha}\langle \mathcal{O}_f(x,\tau+\beta)\cdots\rangle_{\rho_\beta} \tag{5.4}$$

where $\mathcal{O}$ is a fundamental field and $\alpha \in [0,1]$. In this thesis, we consider only fermionic models with fundamental fermion fields. For such models, the trace (5.1), operators which are local with respect to the fermion fields naturally correspond the NS sector due to the KMS identity (with $f = 1$).

Using the resolution of the identity on the Hilbert space H_β on the circle, the vacuum expectation value on the right-hand side of (5.3) can be written as

$$\begin{aligned}
&{}_\beta\langle \text{vac}|\mathcal{O}_\beta(x,\tau)\mathcal{O}_\beta(0,0)|\text{vac}\rangle_\beta \\
&= \sum_{k=0}^{\infty}\sum_{n_1,\ldots,n_k}\frac{e^{\sum_{j=1}^k n_j\frac{2\pi i x}{\beta} - E_{n_1\cdots n_k}\tau}}{k!} \\
&\times {}_\beta\langle \text{vac}|\mathcal{O}_\beta(0,0)|n_1,\ldots,n_k\rangle_\beta\, {}_\beta\langle n_1,\ldots,n_k|\mathcal{O}_\beta(0,0)|\text{vac}\rangle_\beta
\end{aligned} \tag{5.5}$$

where eigenvalues of the momentum are parameterized by discrete variables n_j, and energies $E_{n_1\cdots n_k}$ depends on n_j as well as additional discrete parameters including quantum numbers, particle types. The values of these discrete variables n_j are in accordance with the KMS identity or the quasi-periodicity condition.

The formula (5.5) is only valid for those fields which are local with respect to the fundamental fields. If any field which is non-local with respect to the fundamental fields, for instance, a twist field, is present inside the trace defining a finite-temperature correlation function, the formula (5.5) is no longer correct. This is because a twist field will affect the vacuum sector in the quantization on the circle. This can be interpreted by looking at quasi-periodicity conditions involving twist fields. We assume Ψ to the fundamental fermionic field and T to be the twist field with the twist property

$$\Psi(x)T^{+}(x') = (-1)^{f_\Psi f_{T^+}} e^{-2\pi i\alpha\Theta(x-x')}T^{+}(x')\Psi(x)\,, \quad x \neq x' \tag{5.6}$$

and

$$\Psi(x)T^{-}(x') = (-1)^{f_\Psi f_{T^-}} e^{2\pi i\alpha\Theta(x'-x)}T^{-}(x')\Psi(x)\,, \quad x \neq x'. \tag{5.7}$$

Using (5.6), (5.7) and the cyclic property of the trace, we have

$$\langle\Psi(x,\tau)T^{+}(x',\tau)\cdots\rangle_\beta = \begin{cases} -e^{-2\pi i\alpha}\langle\Psi(x,\tau+\beta)T^{+}(x',\tau)\cdots\rangle_\beta & (x>0) \\ -\langle\Psi(x,\tau+\beta)T^{+}(x',\tau)\cdots\rangle_\beta & (x<0) \end{cases} \tag{5.8}$$

and

$$\langle\Psi(x,\tau)T^{-}(x',\tau)\cdots\rangle_\beta = \begin{cases} -\langle\Psi(x,\tau+\beta)T^{-}(x',\tau)\cdots\rangle_\beta & (x>0) \\ -e^{2\pi i\alpha}\langle\Psi(x,\tau+\beta)T^{-}(x',\tau)\cdots\rangle_\beta & (x<0) \end{cases} \tag{5.9}$$

where the dots $(\cdots)$ represent fields which are local with respect to the fermion field Ψ, at time τ but different position from x. According to these quasi-periodicity conditions, one of the vacuum, corresponding to quantization on the circle, should be in a different sector. Denoting by $|\mathrm{vac}_\alpha\rangle_\beta$ the vacuum on the circle, which is associated to the quasi-periodicity condition (5.4), we have

$$\langle T^{+}(x,\tau)\cdots\rangle_\beta = (e^{i\pi s/2}\cdots)\,{}_\beta\langle\mathrm{vac}_{\frac{1}{2}+\alpha}|T^{+}_\beta(-\tau,x)\cdots|\mathrm{vac}_{\frac{1}{2}}\rangle_\beta \tag{5.10}$$

and

$$\langle T^{-}(x,\tau)\cdots\rangle_\beta = (e^{i\pi s/2}\cdots)\,{}_\beta\langle\mathrm{vac}_{\frac{1}{2}}|T^{-}_\beta(-\tau,x)\cdots|\mathrm{vac}_{\frac{1}{2}-\alpha}\rangle_\beta \tag{5.11}$$

where $|\mathrm{vac}_{\frac{1}{2}}\rangle_\beta$ is the NS vacuum. The energy of the vacuum varies among different sectors. For instance, in the Ising model, the vacuum energies for the NS sector and the R sector are given by [23]

$$\mathcal{E}_{\frac{1}{2}} = \varepsilon - \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\log\left(1+e^{-m\beta\cosh\theta}\right) \tag{5.12}$$

$$\mathcal{E}_{0} = \varepsilon - \int_{-\infty}^{\infty}\frac{d\theta}{2\pi}\cosh\theta\log\left(1-e^{-m\beta\cosh\theta}\right) \tag{5.13}$$

where we denote by $\mathcal{E}_\alpha$ the energy of the vacuum in the quantization on the circle and where ε is a common term to both vacuum energies. In this spirit, the insertion of a twist field inside the finite-temperature correlation function will produce a real exponential factor coming from the energy difference between the vacua. In another word, the one-point thermal function of the twist field is not translation invariant:

$$\langle T^{\eta}(x,\tau)\rangle_{\beta} = e^{\eta\left(\mathcal{E}_{\frac{1}{2}+\eta\alpha}-\mathcal{E}_{\frac{1}{2}}\right)x}\langle T^{\eta}(0,\tau)\rangle_{\beta} \tag{5.14}$$

Therefore, in the presence of twist fields, the vacuum expectation value (5.5) should be rewritten as

$$\begin{aligned}
&{}_{\beta}\langle \mathrm{vac}_{\alpha_1}|\mathcal{O}_{\beta}(x,\tau)\mathcal{O}_{\beta}(0,0)|\mathrm{vac}_{\alpha_2}\rangle_{\beta} \\
&= \sum_{k=0}^{\infty}\sum_{n_1,\ldots,n_k}\frac{e^{\sum_{j=1}^{k} n_j \frac{2\pi i x}{\beta}+(\Delta\mathcal{E}-E_{n_1\cdots n_k}\tau)}}{k!} \\
&\times {}_{\beta}\langle \mathrm{vac}_{\alpha_1}|\mathcal{O}_{\beta}(0,0)|n_1,\ldots,n_k\rangle_{\beta}\, {}_{\beta}\langle n_1,\ldots,n_k|\mathcal{O}_{\beta}(0,0)|\mathrm{vac}_{\alpha_2}\rangle_{\beta}
\end{aligned} \tag{5.15}$$

where $\Delta\mathcal{E}$ is the energy difference between the vacuum $|\mathrm{vac}_{\alpha_1}\rangle$ and the vacuum above which the states $|n_1,\ldots,n_k\rangle_{\beta}$ are constructed.

Using the relation with correlation functions on the circle and applying the form factor expansion technique is one of the ways to evaluate finite-temperature correlation functions in integrable QFT. The formula (5.15) works for any integrable QFT and can be generalized to multi-point correlation functions. However, this method does not seem quite practical, since the Hilbert space under this quantization scheme has a very complicated structure. The matrix elements ${}_{\beta}\langle \mathrm{vac}|\mathcal{O}_{\beta}(0,0)|n_1,\ldots,n_k\rangle_{\beta}$, namely form factors on the circle, and the energy $E_{n_1\cdots n_k}$ are not accessible in general. An exception is in the Ising with mass m and the exact forms of the energy levels are known:

$$E_{n_1\cdots n_k} = \sum_{j=1}^{k}\sqrt{m^2+\left(\frac{2\pi n_j}{\beta}\right)^2} \tag{5.16}$$

where $n_j \in \mathbb{Z}+\frac{1}{2}$ for the NS sector and $n_j \in \mathbb{Z}$ for the R sector. Also, matrix elements of the primary order and disorder fields in the lattice Ising model were deduced in [139–141], and in [20] using the free-fermion equations of motions and "double trick". One can then obtain, via the form factor expansion on the circle, exponentially decaying behaviour of static correlation functions in the quantum Ising chain.

Another way of obtain finite-temperature correlation functions, proposed by Leclair, Lesage, Sachdev and Saleur in [19], is to perform direct calculation of the trace in representation (5.1) by using general properties of matrix elements of local fields. For instance, the two-point function at finite temperature was determined as

$$\begin{aligned}
&\langle \mathcal{O}_1(x,\tau)\mathcal{O}_2^\dagger(0,0)\rangle_{\rho_\beta} \\
&= \sum_{k=0}^{\infty} \sum_{\epsilon_1,\ldots,\epsilon_k=\pm} \int \frac{d\theta_1\cdots d\theta_k\, e^{\sum_{j=1}^k \epsilon_j(imx\sinh\theta_j - m\tau\cosh\theta_j)}}{k!\prod_{j=1}^k(1+e^{-\epsilon_j mL\cosh\theta_j})} \\
&\times F^{\mathcal{O}_1}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k)\left(F^{\mathcal{O}_2}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k)\right)
\end{aligned} \tag{5.17}$$

where $F^{\mathcal{O}_1}_{\epsilon_1,\ldots,\epsilon_k}(\theta_1,\ldots,\theta_k) = \langle \mathrm{vac}|\mathcal{O}|\theta_1 - \tilde{\epsilon}_1 i\pi,\ldots,\theta_k - \tilde{\epsilon}_k i\pi\rangle$ with $\tilde{\epsilon} = (\epsilon-1)/2$ are matrix elements of local field $\mathcal{O}$ in the Hilbert space on the line. Compared to the result (5.5) following from the form factor expansion on the circle, this representation is better for studying dynamical correlation function in imaginary-time formalism. Leclair and Mussardo conjectured in [9] that this method can be easily generalized to interacting models. However, Saleur argued in [10] that this generalization might not be correct, and Castro-Alvaredo and Fring [13] verified this incorrectness by performing the numerical calculation in the scaling Lee-Yang model. Leclair and Mussardo also deduced in [9], following the same method, a formula for one-point functions of local fields in interacting models. Even though the results of [9] hold in various limits for some fields in the Dirac fermion model and in the Federbush model in [13], most of them are in fact incorrect due to the neglect of some singularities in the derivation of (5.17).

Different from the two methods mentioned above, our Liouville space approach applies the ideas of integrable Quantum field theory to thermo-field dynamics. This is a new way to obtain correlation functions in mixed states and the evaluation of finite-temperature correlation functions in the free Majorana theory, performed by Doyon in [21, 23], is the first step in this direction. Correlation functions at finite temperature can be expressed as vacuum expectation values on the Liouville space $\mathcal{L}_\rho$ with $\rho = \exp[-\beta H]$ and it is natural to evaluate these vacuum expectation values by performing form factor expansions with respect to $\mathcal{L}_\rho$. Hence, the main work is to determine the finite-temperature form factors defined within this Liouville space. As explained in 2.3, zero-temperature form factors of a local field can be obtained as the solutions to a set of consistency equations, namely a Riemann-Hilbert problem, which they have to satisfy. In the Liouville space for thermal Gibbs states, one can expect a similar Riemann-Hilbert problem for finite-temperature form factors. Thermal form factors of non-interacting fields (fields that are local with respect to the fundamental fermion or boson field) can be trivially computed by deriving and solving a simple Riemann-Hilbert problem. But, concerning twist fields which are interacting fields, the associated Riemann-Hilbert problems are more involved and the determination of their thermal form factors requires more considerations. In the first section of the present chapter, we will give the review of a Riemann-Hilbert problem derived in [21] for determining finite-temperature form factors of twist fields in Ising model. The second section is devoted to the generalization of this technique to the Dirac free fermion. We will show how to obtain the exact finite-temperature form factors of $U(1)$ twist fields by deriving and solving a similar Riemann-Hilbert problem [142].

5.1 Ising model

5.1.1 Riemann-Hilbert problem

Let us commence with finite-temperature one-particle form factors of the disorder fields μ^η in the Ising model [21]. In the Ising Gibbs thermal state with the untwisted density matrix $\rho_\beta = e^{-\beta H}$, we consider a two-point function in the imaginary-time formalism

$$g(x,\tau) = {}^{\rho_\beta}\langle \mathrm{vac}|\mu^\eta(0)^\ell \psi(x,\tau)^\ell|\mathrm{vac}\rangle^{\rho_\beta} \,. \tag{5.18}$$

Using free Majorana fermions' mode expansions (2.83), the two-point function $g(x,\tau)$ admits the form

$$g(x,\tau) = \frac{1}{2}\sqrt{\frac{m}{\pi}}\int d\theta\, e^{\theta/2}\left(\frac{f_+^{\rho_\beta;\mu^\eta}(\theta)}{1+e^{-\beta E_\theta}}e^{-ip_\theta x+E_\theta\tau} + \frac{f_-^{\rho_\beta;\mu^\eta}(\theta)}{1+e^{\beta E_\theta}}e^{ip_\theta x-E_\theta\tau}\right) \tag{5.19}$$

which should be convergent in the region $-\beta < \tau < 0$. The function $g(x,\tau)$ can also be written as

$$g(x,\tau) = \frac{\mathrm{Tr}\left(e^{-\beta H}\mu^\eta(0)\psi(x,\tau)\right)}{\mathrm{Tr}(e^{-\beta H})}$$

From this representation, using the cyclicity of the trace and translation invariance

$$e^{\beta H}\psi(x,\tau)e^{-\beta H} = \psi(x,\tau+\beta),$$

we have

$$\begin{aligned} g(x,\tau) &= \frac{\mathrm{Tr}\left(\mu^\eta(0)\psi(x,\tau)e^{-\beta H}\right)}{\mathrm{Tr}(e^{-\beta H})} \\ &= \frac{\mathrm{Tr}\left(\mu^\eta(0)e^{-\beta H}\psi(x,\tau+\beta)\right)}{\mathrm{Tr}(e^{-\beta H})} \\ &= \frac{\mathrm{Tr}\left(e^{-\beta H}\psi(x,\tau+\beta)\mu^\eta(0)\right)}{\mathrm{Tr}(e^{-\beta H})}. \end{aligned}$$

For $x > 0$, the fermion field $\psi(x, \tau+\beta)$ commutes with the twist field $\mu^+(0)$ but anti-commutes with the twist field $\mu^-(0)$. So we have

$$\begin{aligned} g(x,\tau) &= -(\delta_{\eta-} - \delta_{\eta+}) \frac{\mathrm{Tr}\left(e^{-\beta H}\mu^\eta(0)\psi(x,\tau+\beta)\right)}{\mathrm{Tr}(e^{-\beta H})} \\ &= -(\delta_{\eta-} - \delta_{\eta+})\, g(x,\tau+\beta) \end{aligned}$$

For $x < 0$, the fermion field $\psi(x, \tau+\beta)$ anti-commutes with the twist field $\mu^+(0)$ but commutes with the twist field $\mu^-(0)$. So we have

$$\begin{aligned} g(x,\tau) &= -(\delta_{\eta+} - \delta_{\eta-}) \frac{\mathrm{Tr}\left(e^{-\beta H}\mu^\eta(0)\psi(x,\tau+\beta)\right)}{\mathrm{Tr}(e^{-\beta H})} \\ &= -(\delta_{\eta+} - \delta_{\eta-})\, g(x,\tau+\beta) \end{aligned}$$

Thus, we obtain the quasi-periodicity equation, namely the KMS relation, for the function $g(x, \tau)$:

$$g(x,\tau+\beta) = -(\delta_{\eta-} - \delta_{\eta+})\, g(x,\tau) \qquad (x > 0) \tag{5.20}$$

$$g(x,\tau+\beta) = -(\delta_{\eta+} - \delta_{\eta-})\, g(x,\tau) \qquad (x < 0). \tag{5.21}$$

Taking into account this KMS relation and deforming the contours of the integrals in (5.19), we find that one-particle form factors $f_\epsilon^{\rho_\beta;\mu^\eta}(\theta)$ should satisfy the following requirements:

1. Analytic structure: $f_\epsilon^{\rho_\beta;\mu^\eta}(\theta)$ are analytic as functions of θ on the complex plane except at some simple poles. Analytic structure is specialized in the region $\mathrm{Im}(\theta) \in [-i\pi, i\pi]$:

   (a) Thermal poles and zeros:

   $f_\epsilon^{\rho_\beta;\mu^\eta}(\theta)$ has poles at

   $$\theta = \lambda_n - \eta\epsilon\frac{\pi i}{2}, \quad n \in \mathbb{Z}$$

   and zeroes at

   $$\theta = \lambda_n - \eta\epsilon\frac{\pi i}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}\,.$$

   (b) $f_+^{\rho_\beta;\mu^\eta}(\theta)$ and $f_-^{\rho_\beta;\mu^\eta}(\theta)$ are related by relations:

   $$f_+^{\rho_\beta;\mu^\eta}(\theta + i\pi/2) = i f_-^{\rho_\beta;\mu^\eta}(\theta - i\pi/2) \quad \text{for all real } \theta \text{ except } \theta = \lambda_n,\ n \in \mathbb{Z} + \tfrac{1}{2} \tag{5.22}$$

and

$$f_{+}^{\rho_\beta;\mu^\eta}(\theta - i\pi/2) = -i f_{-}^{\rho_\beta;\mu^\eta}(\theta + i\pi/2) \quad \text{for all } \theta \text{ except } \theta = \lambda_n,\ n \in \mathbb{Z} \tag{5.23}$$

2. Crossing symmetry:

$$f_{+}^{\rho_\beta;\mu^\eta}(\theta \pm i\pi) = \pm i f_{-}^{\rho_\beta;\mu^\eta}(\theta)\,. \tag{5.24}$$

This property can be used for obtaining one-particle form factors with different charges. The term "crossing symmetry" comes from the zero temperature case and this can be seen more clearly by rewriting (5.24) as

$$^{\rho_\beta}\langle \text{vac}|\mu^\eta(0)^\ell|\theta \pm i\pi\rangle_{+}^{\rho_\beta} = \pm i\, ^{\rho_\beta}\langle \theta|\mu^\eta(0)^\ell|\text{vac}\rangle^{\rho_\beta}$$

where we used the relation (4.89).

3. Quasi-periodicity:

$$f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta \pm 2i\pi) = -f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta)\,. \tag{5.25}$$

5.1.2 Solutions

The solutions to this Rimann-Hilbert problem can be completely fixed, up to a normalization, by the asymptotic behavior $f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta) \sim O(1)$ at $|\theta| \to \infty$, since the disorder field μ is a primary field of spin 0. The one-particle mixed-state form factors admit the representation [21, 23]:

$$f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta) = e^{\epsilon\frac{i\pi}{4}} \frac{1}{\sqrt{2\pi}} \exp\left[\epsilon\eta \int_{-\infty-\epsilon\eta i0^+}^{\infty-\epsilon\eta i0^+} \frac{d\theta'}{2\pi i} \frac{1}{\sinh(\theta - \theta')} \log\left(\tanh\frac{\beta E_\theta}{2}\right)\right] \langle\sigma^\eta\rangle_{\rho_\beta}\,. \tag{5.26}$$

It is a simple matter to check that the functions $f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta)$ have poles and zeros at the positions stated in the Riemann-Hilbert problem. By performing analytic continuation in θ, we can see $f_{\epsilon}^{\rho_\beta;\mu^\eta}(\theta)$ satisfy crossing symmetry and quasi-periodicity. The normalization $\langle\sigma^\eta\rangle_{\rho_\beta}$ has the same vale for both order fields with branch cut on the left and on the right, and it was computed in [143] as

$$m^{\frac{1}{8}} 2^{\frac{1}{12}} e^{-\frac{1}{8}} A^{\frac{3}{2}} \times$$
$$\exp\left[\frac{(m\beta)^2}{2} \int\int_{-\infty}^{\infty} \frac{d\theta_1 d\theta_2}{(2\pi)^2} \frac{\sinh\theta_1 \sinh\theta_2}{\sinh(m\beta\cosh\theta_1)\sinh(m\beta\cosh\theta_2)} \ln\left|\left(\coth\frac{\theta_1 - \theta_2}{2}\right)\right|\right]$$

where A is Glaisher's constant. Since $\langle\sigma^\eta\rangle_{\rho_\beta}$ is real, it is obvious that relations

$$\mathrm{Re}\left(e^{\pm i\pi/4} f_\pm^{\rho_\beta;\mu^+}(\theta \pm i\pi/2)\right) = 0 \quad \text{for all real } \theta \text{ except at } \theta = \lambda_n,\ n \in \mathbb{Z} + \tfrac{1}{2}$$

and

$$\mathrm{Re}\left(e^{\mp i\pi/4} f_\pm^{\rho_\beta;\mu^+}(\theta \mp i\pi/2)\right) = 0 \quad \text{for all real } \theta \text{ except at } \theta = \lambda_n,\ n \in \mathbb{Z}$$

are satisfied. In fact, we can see from (5.26) that the finite-temperature one-particle form factor of the disorder field is constructed by adjoining a θ-dependent function, which can be called as the "leg factor", with the zero-temperature normalized one-particle form factor, up to the normalization $\langle\sigma^\eta\rangle_{\rho_\beta}$:

$$f_\epsilon^{\rho_\beta;\mu^\eta}(\theta) = f_\epsilon^{(0)\eta}(\theta) h_\epsilon^\eta(\theta) \langle\sigma^\eta\rangle_{\rho_\beta} \tag{5.27}$$

where we define zero-temperature normalized one-particle form factors

$$f_\epsilon^{(0)\eta}(\theta) := \lim_{W\to\infty} \langle\sigma^\eta\rangle_\rho^{-1} f_\epsilon^{\rho;\mu^\eta}(\theta) \tag{5.28}$$

and where the leg factor $h_\epsilon^\eta(\theta)$ is of the form

$$h_\epsilon^\eta(\theta) = \exp\left[\epsilon\eta \int_{-\infty-\epsilon\eta i0^+}^{\infty-\epsilon\eta i0^+} \frac{d\theta'}{2\pi i} \frac{1}{\sinh(\theta-\theta')} \log\left(\tanh\frac{\beta E_\theta}{2}\right)\right]. \tag{5.29}$$

It is the analytic properties of the leg factor that make one-particle form factors $f_\epsilon^{\rho_\beta;\mu^\eta}(\theta)$ satisfy all conditions mentioned in the Riemann-Hilbert problem. In the same spirit, finite-temperature two-particle form factors of the order field is obtained as

$$f_{\epsilon_1,\epsilon 2}^{\rho_\beta;\sigma^\eta}(\theta_1,\theta_2) = f_{\epsilon_1,\epsilon_2}^{(0)\eta}(\theta_1,\theta_2) h_{\epsilon_1}^\eta(\theta_1) h_{\epsilon_2}^\eta(\theta_2) \langle\sigma^\eta\rangle_{\rho_\beta} \tag{5.30}$$

where we define zero-temperature normalized two-particle form factors:

$$f_{\epsilon_1,\epsilon_2}^{(0)\eta}(\theta_1,\theta_2) := \lim_{W\to\infty} \langle\sigma\rangle_\rho^{-1} f_{\epsilon_1,\epsilon_2}^{\rho;\sigma^\eta}(\theta_1,\theta_2) \tag{5.31}$$

and they are given by

$$f_{\epsilon_1,\epsilon_2}^{(0)\eta}(\theta_1,\theta_2) = \frac{i}{2\pi} e^{\epsilon_1 \frac{i\pi}{4}} e^{\epsilon_2 \frac{i\pi}{4}} \tanh\left(\frac{\theta_2-\theta_1+\eta i(\epsilon_2-\epsilon_1)}{2}\right)^{\epsilon_1\epsilon_2}. \tag{5.32}$$

5.1.3 Derivation of the Riemann-Hilbert problem associated to the twist field μ^+

Now we present the derivation of the Riemann-Hilbert problem associated to the twist field μ^+, which is originally from [21]. The Riemann-Hilbert problem associated to the twist field μ^- can be derived following similar arguments.

Analytic structure

We consider the function

$$g(x,\tau) = \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(\frac{f_+^{\rho_\beta;\mu^+}(\theta)}{1+e^{-\beta E_\theta}} e^{-ip_\theta x + E_\theta \tau} + \frac{f_-^{\rho_\beta;\mu^+}(\theta)}{1+e^{\beta E_\theta}} e^{ip_\theta x - E_\theta \tau} \right) \tag{5.33}$$

with conditions

$$g(x,\tau+\beta) = -g(x,\tau) \qquad (x<0) \tag{5.34}$$

$$g(x,\tau+\beta) = g(x,\tau) \qquad (x>0) \tag{5.35}$$

For $x<0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta + i\pi/2$ and in the term containing e^{ip_θ} as $\theta \to \theta - i\pi/2$ so that the form factor expansion of $g(x,\tau)$ is still convergent. When shifting the contours, we take residues of poles. By defining

$$g_+(\theta) = \frac{f_+^{\rho_\beta;\mu^+}(\theta)}{1+e^{-\beta E_\theta}}\,, \qquad g_-(\theta) = \frac{f_-^{\rho_\beta;\mu^+}(\theta)}{1+e^{\beta E_\theta}}\,, \tag{5.36}$$

we have

$$\begin{aligned} g(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(e^{i\pi/4} g_+(\theta+i\pi/2) + e^{-i\pi/4} g_-(\theta-i\pi/2) \right) e^{E_\theta x + ip_\theta \tau} \\ &\quad + \sum_n i\pi \,\mathrm{Res}\,(g_+(\theta), \lambda_n + i\pi/2)\, e^{\lambda_n/2 + i\pi/4} e^{E_{\lambda_n} x + ip_{\lambda_n}\tau} \\ &\quad - \sum_n i\pi \,\mathrm{Res}\,(g_-(\theta), \lambda_n - i\pi/2)\, e^{\lambda_n/2 - i\pi/4} e^{E_{\lambda_n} x + ip_{\lambda_n}\tau}\,. \end{aligned}$$

To meet the anti-periodic condition (5.34), we must have

$$\begin{aligned} &\frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(e^{i\pi/4} g_+(\theta+i\pi/2) + e^{-i\pi/4} g_-(\theta-i\pi/2) \right) e^{E_\theta x + ip_\theta \tau} \\ = &-\frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(e^{i\pi/4} g_+(\theta+i\pi/2) + e^{-i\pi/4} g_-(\theta-i\pi/2) \right) e^{E_\theta x + ip_\theta(\tau+\beta)} \end{aligned}$$

and

$$\begin{aligned}
&\sum_n i\pi \,\mathrm{Res}\,(g_+(\theta), \lambda_n + i\pi/2)\, e^{\lambda_n/2+i\pi/4} e^{E_{\lambda_n}x+ip_{\lambda_n}\tau} \\
&-\sum_n i\pi \,\mathrm{Res}\,(g_-(\theta), \lambda_n - i\pi/2)\, e^{\lambda_n/2-i\pi/4} e^{E_{\lambda_n}x+ip_{\lambda_n}\tau} \\
= \;&-\sum_n i\pi \,\mathrm{Res}\,(g_+(\theta), \lambda_n + i\pi/2)\, e^{\lambda_n/2+i\pi/4} e^{E_{\lambda_n}x+ip_{\lambda_n}(\tau+\beta)} \\
&+\sum_n i\pi \,\mathrm{Res}\,(g_-(\theta), \lambda_n - i\pi/2)\, e^{\lambda_n/2-i\pi/4} e^{E_{\lambda_n}x+ip_{\lambda_n}(\tau+\beta)},
\end{aligned}$$

which lead to the following requirements for $g_\pm(\theta)$:

- $e^{i\pi/4}g_+(\theta+i\pi/2)+e^{-i\pi/4}g_-(\theta-i\pi/2) = 0$ for all real θ except $\theta = \lambda_n$, $n \in \mathbb{Z}+\frac{1}{2}$;
- $g_+(\theta)$ has poles at $\theta = \lambda_n + i\pi/2$ and $g_-(\theta)$ has poles at $\theta = \lambda_n - i\pi/2$, where $\sinh \lambda_n = \frac{2\pi n}{m\beta}$ with $n \in \mathbb{Z} + 1/2$.

By recalling definition (5.36), using the $f_\pm^{\rho_\beta;\mu^+}(\theta) = \left(f_\mp^{\rho_\beta;\mu^+}(\theta)\right)^*$, considering the poles of functions $\frac{1}{1+e^{\pm\beta E_\theta}}$, and assuming that functions $f_\pm^{\rho_\beta;\mu^+}(\theta)$ have only simple poles, the requirements above can be written in the language of one-particle form factors $f_\pm^{\rho_\beta;\mu^+}(\theta)$ as

- $\mathrm{Re}\left(e^{i\pi/4}f_+^{\rho_\beta;\mu^+}(\theta+i\pi/2)\right) = 0$ and $\mathrm{Re}\left(e^{-i\pi/4}f_-^{\rho_\beta;\mu^+}(\theta-i\pi/2)\right) = 0$ for all real θ except $\theta = \lambda_n$, $n \in \mathbb{Z} + \frac{1}{2}$;
- $f_+^{\rho_\beta;\mu^+}(\theta)$ does not have poles at $\theta = \lambda_n + i\pi/2$ and $f_-^{\rho_\beta;\mu^+}(\theta)$ does not have poles at $\theta = \lambda_n - i\pi/2$.

For $x > 0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta - i\pi/2$ and in the term containing e^{ip_θ} as $\theta \to \theta + i\pi/2$. Again, we take the poles at appropriate values and we get:

$$\begin{aligned}
g(x,\tau) \;=\; &\frac{1}{2}\sqrt{\frac{m}{\pi}}\int d\theta\, e^{\theta/2}\left(e^{-i\pi/4}g_+(\theta-i\pi/2)+e^{i\pi/4}g_-(\theta+i\pi/2)\right)e^{-E_\theta x-ip_\theta\tau} \\
&-\sum_n i\pi \,\mathrm{Res}\,(g_+(\theta), \lambda_n - i\pi/2)\, e^{\lambda_n/2-i\pi/4} e^{-E_{\lambda_n}x-ip_{\lambda_n}\tau} \\
&+\sum_n i\pi \,\mathrm{Res}\,(g_-(\theta), \lambda_n + i\pi/2)\, e^{\lambda_n/2+i\pi/4} e^{-E_{\lambda_n}x-ip_{\lambda_n}\tau}\,.
\end{aligned}$$

To meet the periodic condition (5.35), the following requirements must be satisfied:

- $e^{-i\pi/4}g_+(\theta-i\pi/2)+e^{i\pi/4}g_-(\theta+i\pi/2) = 0$ for all real θ except $\theta = \lambda_n$, $n \in \mathbb{Z}$;
- $g_+(\theta)$ has poles at $\theta = \lambda_n - i\pi/2$ and $g_-(\theta)$ has poles at $\theta = \lambda_n + i\pi/2$, where $\sinh \lambda_n = \frac{2\pi n}{m\beta}$ with $n \in \mathbb{Z}$.

We then translate these requirements in terms of one-particle form factors $f_{\pm}^{\rho_\beta;\mu^+}(\theta)$:

- $\mathrm{Re}\left(e^{-i\pi/4} f_{+}^{\rho_\beta;\mu^+}(\theta - i\pi/2)\right) = 0$ and $\mathrm{Re}\left(e^{i\pi/4} f_{-}^{\rho_\beta;\mu^+}(\theta + i\pi/2)\right) = 0$ for all real θ except $\theta = \lambda_n$, $n \in \mathbb{Z}$;
- $f_{\pm}^{\rho_\beta;\mu^+}(\theta)$ have poles at $\theta = \lambda_n \mp i\pi/2$, $n \in \mathbb{Z}$ and have zeroes at $\theta = \lambda_n \mp i\pi/2$, $n \in \mathbb{Z} + 1/2$.

Crossing symmetry (5.24) **and quasi-periodicity** (5.25)

For $x < 0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta + i\pi$ and in the term containing e^{ip_θ} as $\theta \to \theta - i\pi$. By taking the poles at the lines of imaginary $\pm\pi/2$, we have

$$\begin{aligned} g(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(ig_+(\theta + i\pi)e^{ip_\theta x - E_\theta \tau} - ig_-(\theta - i\pi)e^{-ip_\theta x + E_\theta \tau}\right) \\ &+ \sum_{n\in\mathbb{Z}+1/2} 2i\pi\, \mathrm{Res}\left(g_+(\theta), \lambda_n + i\pi/2\right) e^{\lambda_n/2 + i\pi/4} e^{E_{\lambda_n} x + ip_{\lambda_n}\tau} \\ &- \sum_{n\in\mathbb{Z}+1/2} 2i\pi\, \mathrm{Res}\left(g_-(\theta), \lambda_n - i\pi/2\right) e^{\lambda_n/2 - i\pi/4} e^{E_{\lambda_n} x + ip_{\lambda_n}\tau}\,. \end{aligned}$$

By recognizing the sum of last two terms as $2g(x,\tau)$, we obtain again a representation of the two-point function $g(x,\tau)$:

$$g(x,\tau) = -\frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(\frac{if_+^{\rho_\beta;\mu^+}(\theta + i\pi)}{1 + e^{\beta E_\theta}} e^{ip_\theta x - E_\theta\tau} - \frac{if_-^{\rho_\beta;\mu^+}(\theta - i\pi)}{1 + e^{-\beta E_\theta}} e^{-ip_\theta x + E_\theta \tau}\right) \tag{5.37}$$

which is of the same form as (5.33) and is still valid in the region $-\beta < \tau < 0, x < 0$. Since a presentation of this form should be unique, comparing (5.33) and (5.37) gives

$$f_+^{\rho_\beta;\mu^+}(\theta + i\pi) = i f_-^{\rho_\beta;\mu^+}(\theta).$$

For $x > 0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta - i\pi$ and in the term containing e^{ip_θ} as $\theta \to \theta + i\pi$. By taking the poles at the lines of imaginary $\mp\pi/2$, we have

$$\begin{aligned} g(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(-ig_+(\theta - i\pi)e^{ip_\theta x - E_\theta \tau} + ig_-(\theta + i\pi)e^{-ip_\theta x + E_\theta \tau}\right) \\ &+ \sum_{n\in\mathbb{Z}} 2i\pi\, \mathrm{Res}\left(g_+(\theta), \lambda_n - i\pi/2\right) e^{\lambda_n/2 - i\pi/4} e^{-E_{\lambda_n} x - ip_{\lambda_n}\tau} \\ &- \sum_{n\in\mathbb{Z}} 2i\pi\, \mathrm{Res}\left(g_-(\theta), \lambda_n + i\pi/2\right) e^{\lambda_n/2 + i\pi/4} e^{-E_{\lambda_n} x - ip_{\lambda_n}\tau}\,. \end{aligned}$$

By recognizing the sum of last two terms as $2g(x,\tau)$, we obtain again a representation of the two-point function $g(x,\tau)$:

$$g(x,\tau) = \frac{1}{2}\sqrt{\frac{m}{\pi}} \int d\theta\, e^{\theta/2} \left(\frac{i f_+^{\rho_\beta;\mu^+}(\theta - i\pi)}{1+e^{\beta E_\theta}} e^{ip_\theta x - E_\theta \tau} - \frac{i f_-^{\rho_\beta;\mu^+}(\theta + i\pi)}{1+e^{-\beta E_\theta}} e^{-ip_\theta x + E_\theta \tau} \right) \tag{5.38}$$

which is valid in the region$-\beta < \tau < 0, x > 0$. Comparing (5.33) and (5.38) gives the other crossing relation

$$f_+^{\rho_\beta;\mu^+}(\theta - i\pi) = -i f_-^{\rho_\beta;\mu^+}(\theta)\,.$$

Quasi-periodicity can be obtained by performing crossing symmetry for two times.

5.2 Dirac model

The original results presented in this section are collected from the work [142]. Before we start, let us introduce some useful notations. We define the normalized mixed-state one-particle form factors of the $U(1)$ fermionic twist fields

$$f_{\nu,\epsilon}^{\eta}(\theta) \quad := \quad \langle \sigma_\alpha^\eta \rangle_\rho^{-1} f_{\nu,\epsilon}^{\rho;\mu^\eta}(\theta) \tag{5.39}$$

where $f_{\nu,\epsilon}^{\rho;\mu^\eta}(\theta) := \langle \sigma_\alpha^\eta \rangle_\rho^{-1} \left[\delta_{\nu,-\epsilon} f_{\nu,\epsilon}^{\rho;\sigma^\eta_{\alpha-1,\alpha}}(\theta) + \delta_{\nu,\epsilon} f_{\nu,\epsilon}^{\rho;\sigma^\eta_{\alpha+1,\alpha}}(\theta) \right]$ and $\langle \sigma_\alpha^\eta \rangle_\rho$ is the normalization, and the normalized mixed-state two-particle form factors of the $U(1)$ bosonic twist fields

$$f_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}^{\eta}(\theta_1,\theta_2) \quad := \quad \langle \sigma_\alpha^\eta \rangle_\rho^{-1} f_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}^{\rho;\sigma_\alpha^\eta}(\theta_1,\theta_2). \tag{5.40}$$

Their pure-state limits are denoted by

$$f_{\nu,\epsilon}^{(0)\eta}(\theta) \quad := \quad \lim_{W_\pm \to \infty} f_{\nu,\epsilon}^{\eta}(\theta)\,, \tag{5.41}$$

$$f_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}^{(0)\eta}(\theta_1,\theta_2) \quad := \quad \lim_{W_\pm \to \infty} f_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}^{\eta}(\theta_1,\theta_2)\,. \tag{5.42}$$

Using the relation (4.56) and the vacuum matrix elments of $U(1)$ twist fields in Hilbert space, we have

$$\begin{aligned} f_{\nu,\epsilon}^{(0)+}(\theta) &= (-i\epsilon\delta_{\nu,-} + \delta_{\nu,+}) \frac{e^{-i\pi\nu\alpha/2}}{\Gamma(1+\nu\epsilon\alpha)} m^{\nu\epsilon\alpha+1/2} e^{(\nu\epsilon\alpha+1/2)\theta}\,, \\ f_{\nu,\epsilon}^{(0)-}(\theta) &= f_{\nu,\epsilon}^{(0)+}(\theta) e^{2\pi i\nu\alpha\delta_{\epsilon,+}} \end{aligned} \tag{5.43}$$

and

$$f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = \delta_{\epsilon_1,\epsilon_2}\delta_{\nu_1,-\nu_2}\,\nu_1\epsilon_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}}{\cosh(\frac{\theta_1-\theta_2}{2})} + \delta_{\epsilon_1,-\epsilon_2}\delta_{\nu_1,\nu_2}\,i\nu_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}e^{-i\pi\eta\nu_1\alpha}}{\sinh\left(\frac{\theta_1-\theta_2+\eta i(\epsilon_1-\epsilon_2)0^+}{2}\right)}\,. \tag{5.44}$$

5.2.1 Riemann-Hilbert problem

We start again with finite-temperature one-particle form factors of the $U(1)$ fermionic twist fields in the Dirac theory. In the Dirac thermal Gibbs state with untwisted density matrix, where $W_\nu(\theta)=\beta E_\theta$, we consider a two-point function in imaginary-time formalism

$$g(x,\tau) = {}^{\rho}\langle\text{vac}|\sigma^{\eta}_{\alpha-\nu(-1),\alpha}(0)^{\ell}\,\Psi_R^{-\nu}(x,\tau)^{\ell}|\text{vac}\rangle^{\rho} \tag{5.45}$$

where we denote by $\Psi^{\mp}$ the fermion operator Ψ and its Hermitian Conjugation $\Psi^\dagger$ respectively. Using Dirac fermions' mode expansions (2.92), (2.93), and Liouville left-action, the two-point function $g(x,\tau)$ can be written as a finite-temperature form factor expansion:

$$g(x,\tau) = \sqrt{m}\int d\theta e^{\theta/2}\left[i\frac{f^{\rho;\mu^\eta}_{\nu+}(\theta)}{1+e^{-LE_\theta}}e^{\tau E_\theta - ixP_\theta} + \frac{f^{\rho;\mu^\eta}_{-\nu,-}(\theta)}{1+e^{LE_\theta}}e^{-\tau E_\theta+ixP_\theta}\right] \tag{5.46}$$

which should be convergent in the region $-\beta<\tau<0$. Following the same recipe for obtaining (5.20) and (5.21), we can derive the KMS relation for $g(x,\tau)$ (5.45):

$$g(x,\tau) = -\left(\delta_{\eta+}\,e^{-\eta\nu 2\pi i\alpha}+\delta_{\eta-}\right)g(x,\tau+\beta) \qquad (x>0) \tag{5.47}$$

$$g(x,\tau) = -\left(\delta_{\eta-}\,e^{-\eta\nu 2\pi i\alpha}+\delta_{\eta+}\right)g(x,\tau+\beta) \qquad (x<0)\,. \tag{5.48}$$

To make sense of KMS relations (5.47) and (5.48), one-particle form factor $f^{\rho;\mu^\eta}_{\nu+}(\theta)$ and $f^{\rho;\mu^\eta}_{-\nu,-}(\theta)$ should meet the following requirements:

1. Analytic structure: $f^{\rho;\mu^\eta}_{\nu+}(\theta)$ and $f^{\rho;\mu^\eta}_{-\nu,-}(\theta)$ are analytic as functions of θ on the complex plane except at some simple poles. Analytic structure is specialized in the region $\text{Im}(\theta)\in[-i\pi,i\pi]$:

   (a) Thermal poles and zeroes:

   $f^{\rho;\mu^\eta}_{\nu+}(\theta)$ has poles at

$$\theta = \gamma^\nu_n - \eta\frac{\pi i}{2}, \quad n\in\mathbb{Z}+\frac{1}{2}$$

and zeroes at

$$\theta = \lambda_n - \eta\frac{\pi i}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}\,;$$

$f^{\rho;\mu^\eta}_{-\nu,-}(\theta)$ has poles at

$$\theta = \gamma^\nu_n + \eta\frac{\pi i}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}$$

and zeroes at

$$\theta = \lambda_n + \eta\frac{\pi i}{2}, \quad n \in \mathbb{Z} + \frac{1}{2}$$

where

$$\sinh\gamma^\nu_n = \frac{2\pi(n+\nu\alpha)}{m\beta}\,, \quad \sinh\lambda_n = \frac{2\pi n}{m\beta}\,,$$

(b) $f^{\rho;\mu^\eta}_{\nu+}(\theta)$ and $f^{\rho;\mu^\eta}_{-\nu,-}(\theta)$ are related by relations

$$f^{\rho;\mu^\eta}_{\nu+}(\theta \pm i\pi/2) = \pm f^{\rho;\mu^\eta}_{-\nu,-}(\theta \mp i\pi/2) \tag{5.49}$$

for all θ except $\theta = \gamma^\nu_n$, $n \in \mathbb{Z} + \frac{1}{2}$.

2. Crossing symmetry:

$$f^{\rho;\mu^\eta}_{\nu+}(\theta \pm i\pi) = \pm f^{\rho;\mu^\eta}_{-\nu,-}(\theta)\,. \tag{5.50}$$

3. Quasi-periodicity:

$$f^{\rho;\mu^\eta}_{\epsilon\nu,\epsilon}(\theta \pm 2i\pi) = -f^{\rho;\mu^\eta}_{\epsilon\nu,\epsilon}(\theta)\,. \tag{5.51}$$

Taking into account the above Riemann-Hilbert problem as well as the fact that mixed-state form factors reproduce the ordinary form factors in Hilbert space under the limit $W_\pm(\theta) \to \infty$, and by analogy with thermal form factors of twist fields in Ising model (5.27), we conjecture that finite-temperature one-particle form factors of the fermionic twist fields are expressed again as a product of the leg factor and the vacuum one-particle form factor, up to the overall normalization $\langle\sigma^\eta_\alpha\rangle_\rho$:

$$f^{\rho;\mu^\eta}_{\nu\epsilon}(\theta) = f^\eta_{\nu\epsilon}(\theta)\langle\sigma^\eta_\alpha\rangle_\rho = f^{(0)\eta}_{\nu\epsilon}(\theta)h^\eta_{\nu\epsilon}(\theta)\langle\sigma^\eta_\alpha\rangle_\rho \tag{5.52}$$

with $h^\eta_{\nu\epsilon}(\theta)$ the leg-factor:

$$\begin{aligned} h^\eta_{\nu\epsilon}(\theta) \quad &= \quad \exp\left[\int \frac{d\theta'}{2\pi i}\frac{A^\eta_{\nu\epsilon}(\theta\theta')}{\cosh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-LE_{\theta'}}}{1+e^{2\pi i\eta\nu\alpha}e^{-LE_{\theta'}}}\right)\right. \\ &\left. + \int_{-\infty-\eta\epsilon i0^+}^{\infty-\eta\epsilon i0^+}\frac{d\theta'}{2\pi i}\frac{B^\eta_{\nu\epsilon}(\theta\theta')}{\sinh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-LE_{\theta'}}}{1+e^{-2\pi i\eta\nu\alpha}e^{-LE_{\theta'}}}\right)\right]. \end{aligned} \tag{5.53}$$

Factors $A^{\eta}_{\nu\epsilon}(\theta\theta')$ and $B^{\eta}_{\nu\epsilon}(\theta\theta')$, due to (5.69), (5.50) and (5.51), must satisfy a set of relations

1.

$$\begin{aligned} A^{\eta}_{\nu\epsilon}(\theta \pm \pi i/2, \theta \pm \pi i/2) &= B^{\eta}_{\nu\epsilon}(\theta \pm \pi i/2, \theta \pm \pi i/2) = -\eta\epsilon/2 \\ A^{\eta}_{\nu\epsilon}(\theta \pm \pi i/2, \theta') &= \pm i B^{\eta}_{-\nu,-\epsilon}(\theta \mp \pi i/2, \theta') \\ B^{\eta}_{\nu\epsilon}(\theta \pm \pi i/2, \theta') &= \pm i A^{\eta}_{-\nu,-\epsilon}(\theta \mp \pi i/2, \theta')\,. \end{aligned} \tag{5.54}$$

2.

$$\begin{aligned} A^{\eta}_{\nu\epsilon}(\theta \pm \pi i, \theta) &= \pm\eta\epsilon i/2 \\ B^{\eta}_{\nu\epsilon}(\theta \pm \pi i, \theta \pm \pi i) &= -\eta\epsilon/2 \\ A^{\eta}_{\nu\epsilon}(\theta \pm \pi i, \theta') &= \pm i B^{\eta}_{-\nu,-\epsilon}(\theta, \theta') \\ B^{\eta}_{\nu\epsilon}(\theta \pm \pi i, \theta') &= \pm i A^{\eta}_{-\nu,-\epsilon}(\theta, \theta')\,. \end{aligned} \tag{5.55}$$

3.

$$\begin{aligned} A^{\eta}_{\nu\epsilon}(\theta \pm 2\pi i, \theta \pm \pi i) &= \pm\eta\epsilon i/2 \\ B^{\eta}_{\nu\epsilon}(\theta \pm 2\pi i, \theta) &= \eta\epsilon/2\,. \end{aligned} \tag{5.56}$$

To fully determine factors $A^{\eta}_{\nu\epsilon}(\theta\theta')$ and $B^{\eta}_{\nu\epsilon}(\theta\theta')$, it is intuitive to exploit low-temperature expansions of finite-temperature form factors. From the trace definition of mixed-state form factors and factorisation of higher-particle twist field form factors, we can deduce low-temperature expansions of one- and two-particle normalized form factors for $U(1)$ twist fields:

$$\begin{aligned} f^{\eta}_{\nu,\epsilon}(\theta) &= f^{(0)\eta}_{\nu,\epsilon}(\theta) + \sum_{\nu'} \int d\theta' e^{-\beta E_{\theta'}} \Big[f^{(0)\eta}_{(\nu,\epsilon)(\nu',+)}(\theta,\theta') f^{(0)\eta}_{\nu',-}(\theta') \\ &\quad - f^{(0)\eta}_{(\nu,\epsilon)(\nu',-)}(\theta,\theta') f^{(0)\eta}_{\nu',+}(\theta') \Big] \\ &\quad + O(e^{-2m\beta}) \end{aligned} \tag{5.57}$$

$$\begin{aligned} f^{\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) &= f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \\ &\quad + \sum_{\nu} \int d\theta e^{-\beta E_{\theta}} \Big[f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu,-)}(\theta_1,\theta) f^{(0)\eta}_{(\nu_2,\epsilon_2)(\nu,+)}(\theta_2,\theta) \\ &\quad - f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu,+)}(\theta_1,\beta) f^{(0)\eta}_{(\nu_2,\epsilon_2)(\nu,-)}(\theta_2,\beta) \Big] + O(e^{-2m\beta}) \\ &\quad . \end{aligned} \tag{5.58}$$

Then, we turn our attention to the expression of $f^{\eta}_{\nu\epsilon}(\theta)$ in our conjecture (5.52). We taylor expand in $h^{\eta}_{\nu\epsilon}(\theta)$ two logarithmic terms as functions of $e^{-\beta E'_{\theta}}$ at the point $e^{-\beta E'_{\theta}} = 0$ in low-temperature limit,

$$\begin{aligned} f^{\eta}_{\nu\epsilon}(\theta) &= f^{(0)\eta}_{\nu,\epsilon}(\theta) + f^{(0)\eta}_{\nu,\epsilon}(\theta)\Bigg[\int \frac{d\theta'}{2\pi i}\frac{A^{\eta}_{\nu,\epsilon}(\theta,\theta')}{\cosh(\frac{\theta-\theta'}{2})}(1-e^{2\pi i\eta\nu\alpha})e^{-\beta E_{\theta'}} \\ &+ \int \frac{d\theta'}{2\pi i}\frac{B^{\eta}_{\nu,\epsilon}(\theta,\theta')}{\sinh(\frac{\theta-\theta'}{2})}(1-e^{-2\pi i\eta\nu\alpha})e^{-\beta E_{\theta'}}\Bigg] + O(e^{-2m\beta}) \end{aligned} \tag{5.59}$$

By comparing (5.57) and (5.59), we arrive at

$$A^{\eta}_{\nu,\epsilon}(\theta,\theta') = B^{\eta}_{\nu,\epsilon}(\theta,\theta') = -\eta\epsilon\,\frac{1}{2}e^{\frac{(\theta',-\theta)}{2}}\,. \tag{5.60}$$

which are in agreement with relations (5.54), (5.55) and (5.56). Thus, finite-temperature one-particle form factors of the fermionic $U(1)$ twist fields are fully obtained :

$$f^{\rho;\mu^{\eta}}_{\nu,\epsilon}(\theta) = f^{(0)\eta}_{\nu,\epsilon}(\theta)h^{\eta}_{\nu,\epsilon}(\theta)\langle\sigma^{\eta}_{\alpha}\rangle_{\rho} \tag{5.61}$$

with

$$\begin{aligned} h^{\eta}_{\nu,\epsilon}(\theta) &= \exp\Bigg[-\eta\epsilon\int\frac{d\theta'}{2\pi i}\frac{\frac{1}{2}e^{\frac{(\theta'-\theta)}{2}}}{\cosh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-\beta E_{\theta'}}}{1+e^{2\pi i\eta\nu\alpha}e^{-\beta E_{\theta'}}}\right) \\ &-\eta\epsilon\int_{-\infty-\eta\epsilon i0^+}^{\infty-\eta\epsilon i0^+}\frac{d\theta'}{2\pi i}\frac{\frac{1}{2}e^{\frac{(\theta'-\theta)}{2}}}{\sinh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-\beta E_{\theta'}}}{1+e^{-2\pi i\eta\nu\alpha}e^{-\beta E_{\theta'}}}\right)\Bigg]\,. \end{aligned} \tag{5.62}$$

Again, this solution is unique due to the asymptotic behavior $f^{\rho;\mu^{\eta}}_{\nu,\epsilon}(\theta) \sim O(1)$ at $|\theta| \to \infty$. Similarly, we postulate that finite-temperature two-particle form factors of the $U(1)$ bosonic twist fields admit the representation of the form:

$$f^{\rho;\sigma^{\eta}_{\alpha}}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)h^{\eta}_{\nu_1,\epsilon_1}(\theta_1)h^{\eta}_{\nu_2,\epsilon_2}(\theta_2)\langle\sigma^{\eta}_{\alpha}\rangle_{\rho} \tag{5.63}$$

which indeed reproduces the correct ordinary two-particle form factors in low-temperature limit.

5.2.2 Derivation of the Riemann-Hilbert problem associated to the fermionic $U(1)$ twist fields with $\eta = +$

Analytic structure

We consider the function

$$g(x,\tau)=\sqrt{m}\int d\theta e^{\theta/2}\left[i\frac{f^{\rho;\mu^+}_{\nu+}(\theta)}{1+e^{-LE_\theta}}e^{\tau E_\theta-ixP_\theta}+\frac{f^{\rho;\mu^+}_{-\nu,-}(\theta)}{1+e^{LE_\theta}}e^{-\tau E_\theta+ixP_\theta}\right]. \tag{5.64}$$

with conditions

$$g(x,\tau) = -e^{-\nu 2\pi i\alpha}g(x,\tau+\beta) \qquad (x>0) \tag{5.65}$$

$$g(x,\tau) = -g(x,\tau+\beta) \qquad (x<0)\,. \tag{5.66}$$

For $x<0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta\to\theta+i\pi/2$ and in the term containing e^{ip_θ} as $\theta\to\theta-i\pi/2$ so that the form factor expansion of $g(x,\tau)$ is still convergent. When shifting the contours, we take residues of poles. By defining

$$g_{\nu,+}(\theta)=\frac{f^{\rho_\beta;\mu^+}_{\nu,+}(\theta)}{1+e^{-\beta E_\theta}}\,,\qquad g_{-\nu,-}(\theta)=\frac{f^{\rho_\beta;\mu^+}_{-\nu,-}(\theta)}{1+e^{\beta E_\theta}}\,, \tag{5.67}$$

we have

$$\begin{aligned} g(x,\tau) \;=\; & \sqrt{m}\int d\theta\, e^{\theta/2}\left(ie^{i\pi/4}g_{\nu,+}(\theta+i\pi/2)+e^{-i\pi/4}g_{-\nu,-}(\theta-i\pi/2)\right)e^{E_\theta x+ip_\theta\tau}\\ &+\sum_n i\pi\,\mathrm{Res}\,(g_{\nu,+}(\theta),\lambda_n+i\pi/2)\,ie^{\lambda_n/2+i\pi/4}e^{E_{\lambda_n}x+ip_{\lambda_n}\tau}\\ &-\sum_n i\pi\,\mathrm{Res}\,(g_{-\nu,-}(\theta),\lambda_n-i\pi/2)\,e^{\lambda_n/2-i\pi/4}e^{E_{\lambda_n}x+ip_{\lambda_n}\tau}\,. \end{aligned}$$

To meet the anti-periodic condition (5.66), the following requirements must be satisfied:

- $ie^{i\pi/4}g_{\nu,+}(\theta+i\pi/2)+e^{-i\pi/4}g_{-\nu,-}(\theta-i\pi/2)=0$ for all real θ except $\theta=\lambda_n$, $n\in\mathbb{Z}+\frac{1}{2}$;
- $g_{\nu,+}(\theta)$ has poles at $\theta=\lambda_n+i\pi/2$ and $g_{-\nu,-}(\theta)$ has poles at $\theta=\lambda_n-i\pi/2$, where $\sinh\lambda_n=\frac{2\pi n}{m\beta}$ with $n\in\mathbb{Z}+1/2$.

By recalling definition (5.67), considering the poles of functions $\frac{1}{1+e^{\pm\beta E_\theta}}$, and assuming that functions $f^{\rho_\beta;\mu^+}_{\pm\nu,\pm}(\theta)$ have only simple poles, the requirements above can be written in the language of one-particle form factors $f^{\rho_\beta;\mu^+}_{\pm\nu,\pm}(\theta)$ as

- $f^{\rho;\mu^+}_{\nu+}(\theta)$ and $f^{\rho;\mu^+}_{-\nu,-}(\theta)$ are related by relations

$$f^{\rho;\mu^+}_{\nu+}(\theta+i\pi/2)=f^{\rho;\mu^+}_{-\nu,-}(\theta-i\pi/2) \tag{5.68}$$

  for all real θ except $\theta=\lambda_n$, $n\in\mathbb{Z}+\frac{1}{2}$;

- $f^{\rho_\beta;\mu^+}_{\nu,+}(\theta)$ does not have poles at $\theta = \lambda_n + i\pi/2$ and $f^{\rho_\beta;\mu^+}_{-\nu,-}(\theta)$ does not have poles at $\theta = \lambda_n - i\pi/2$, for $n \in \mathbb{Z} + \frac{1}{2}$.

For $x > 0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta - i\pi/2$ and in the term containing e^{ip_θ} as $\theta \to \theta + i\pi/2$. Again, we take the poles at appropriate values and we get:

$$\begin{aligned} g(x,\tau) &= \sqrt{m}\int d\theta\, e^{\theta/2}\left(ie^{-i\pi/4}g_{\nu,+}(\theta - i\pi/2) + e^{i\pi/4}g_{-\nu,-}(\theta + i\pi/2)\right)e^{-E_\theta x - ip_\theta \tau} \\ &\quad - \sum_n i\pi\,\mathrm{Res}\,(g_{\nu,+}(\theta), \gamma_n^\nu - i\pi/2)\, ie^{\gamma_n^\nu/2 - i\pi/4} e^{-xm\cosh\gamma_n^\nu - i\tau m \sinh\gamma_n^\nu} \\ &\quad + \sum_n i\pi\,\mathrm{Res}\,(g_{-\nu,-}(\theta), \gamma_n^\nu + i\pi/2)\, e^{\gamma_n^\nu/2 + i\pi/4} e^{-xm\cosh\gamma_n^\nu - i\tau m \sinh\gamma_n^\nu}\,. \end{aligned}$$

To meet the periodic condition (5.65), the following requirements must be satisfied:

- $ie^{-i\pi/4}g_{\nu,+}(\theta - i\pi/2) + e^{i\pi/4}g_{-\nu,-}(\theta + i\pi/2) = 0$ for all real θ except $\theta = \gamma_n^\nu$, $n \in \mathbb{Z} + \frac{1}{2}$;
- $g_{\nu,+}(\theta)$ has poles at $\theta = \gamma_n^\nu - i\pi/2$ and $g_{-\nu,-}(\theta)$ has poles at $\theta = \gamma_n^\nu + i\pi/2$, where $\sinh\gamma_n^\nu = \frac{2\pi(n+\nu\alpha)}{m\beta}$ with $n \in \mathbb{Z} + 1/2$.

We then translate these requirements in terms of one-particle form factors $f^{\rho_\beta;\mu^+}_{\pm\nu,\pm}(\theta)$:

- $f^{\rho;\mu^+}_{\nu+}(\theta)$ and $f^{\rho;\mu^+}_{-\nu,-}(\theta)$ are related by relations

$$f^{\rho;\mu^+}_{\nu+}(\theta - i\pi/2) = f^{\rho;\mu^+}_{-\nu,-}(\theta + i\pi/2) \tag{5.69}$$

  for all real θ except $\theta = \gamma_n^\nu$, $n \in \mathbb{Z} + \frac{1}{2}$;
- $f^{\rho_\beta;\mu^+}_{\pm\nu,\pm}(\theta)$ have poles at $\theta = \gamma_n^\nu \mp i\pi/2$ and $f^{\rho_\beta;\mu^+}_{-\nu,-}(\theta)$ have zeros at $\theta = \lambda_n \mp i\pi/2$, for $n \in \mathbb{Z} + \frac{1}{2}$.

Crossing symmetry (5.50) **and quasi-periodicity** (5.51)

For $x < 0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta \to \theta + i\pi$ and in the term containing e^{ip_θ} as $\theta \to \theta - i\pi$. By taking the poles at the lines of imaginary $\pm\pi/2$, we have

$$\begin{aligned} g(x,\tau) &= \sqrt{m}\int d\theta\, e^{\theta/2}\left(-g_{\nu,+}(\theta + i\pi)e^{ip_\theta x - E_\theta\tau} - ig_{-\nu,-}(\theta - i\pi)e^{-ip_\theta x + E_\theta \tau}\right) \\ &\quad + \sum_n 2i\pi\,\mathrm{Res}\,(g_{\nu,+}(\theta), \lambda_n + i\pi/2)\, e^{\lambda_n/2 + i\pi/4} e^{E_{\lambda_n}x + ip_{\lambda_n}\tau} \\ &\quad - \sum_n 2i\pi\,\mathrm{Res}\,(g_{-\nu,-}(\theta), \lambda_n - i\pi/2)\, e^{\lambda_n/2 - i\pi/4} e^{E_{\lambda_n}x + ip_{\lambda_n}\tau}\,. \end{aligned}$$

By recognizing the sum of last two terms as $2g(x,\tau)$, we obtain again a representation of the two-point function $g(x,\tau)$:

$$g(x,\tau) = \sqrt{m}\int d\theta\, e^{\theta/2}\left(\frac{f_{\nu,+}^{\rho_\beta;\mu^+}(\theta+i\pi)}{1+e^{\beta E_\theta}}e^{ip_\theta x-E_\theta\tau} + \frac{if_{-\nu,-}^{\rho_\beta;\mu^+}(\theta-i\pi)}{1+e^{-\beta E_\theta}}e^{-ip_\theta x+E_\theta\tau}\right) \tag{5.70}$$

which is of the same form as (5.64) and is still valid in the region$-\beta<\tau<0, x<0$. Since a presentation of this form should be unique, comparing (5.64) and (5.70) gives

$$f_{\nu,+}^{\rho_\beta;\mu^+}(\theta+i\pi) = f_{-\nu,-}^{\rho_\beta;\mu^+}(\theta).$$

For $x>0$, we shift the θ-contour in the term containing e^{-ip_θ} as $\theta\to\theta-i\pi$ and in the term containing e^{ip_θ} as $\theta\to\theta+i\pi$. By taking the poles at the lines of imaginary $\mp\pi/2$, we have

$$\begin{aligned} g(x,\tau) \;=\;& \sqrt{m}\int d\theta\, e^{\theta/2}\left(g_{\nu,+}(\theta-i\pi)e^{ip_\theta x-E_\theta\tau} + ig_{-\nu,-}(\theta+i\pi)e^{-ip_\theta x+E_\theta\tau}\right)\\ &- \sum_{n\in\mathbb{Z}+1/2} i\pi\,\mathrm{Res}\left(g_{\nu,+}(\theta),\gamma_n^\nu-i\pi/2\right) ie^{\gamma_n^\nu/2-i\pi/4}e^{-xm\cosh\gamma_n^\nu-i\tau m\sinh\gamma_n^\nu}\\ &+ \sum_{n\in\mathbb{Z}+1/2} i\pi\,\mathrm{Res}\left(g_{-\nu,-}(\theta),\gamma_n^\nu+i\pi/2\right) e^{\gamma_n^\nu/2+i\pi/4}e^{-xm\cosh\gamma_n^\nu-i\tau m\sinh\gamma_n^\nu}. \end{aligned}$$

By recognizing the sum of last two terms as $2g(x,\tau)$, we obtain again a representation of the two-point function $g(x,\tau)$:

$$g(x,\tau) = -\sqrt{m}\int d\theta\, e^{\theta/2}\left(\frac{f_{\nu,+}^{\rho_\beta;\mu^+}(\theta-i\pi)}{1+e^{\beta E_\theta}}e^{ip_\theta x-E_\theta\tau} + \frac{if_{-\nu,-}^{\rho_\beta;\mu^+}(\theta+i\pi)}{1+e^{-\beta E_\theta}}e^{-ip_\theta x+E_\theta\tau}\right) \tag{5.71}$$

which is valid in the region$-\beta<\tau<0, x>0$. Comparing (5.64) and (5.71) gives the other crossing relation

$$f_{\nu,+}^{\rho_\beta;\mu^+}(\theta-i\pi) = -f_{-\nu,-}^{\rho_\beta;\mu^+}(\theta).$$

Again, quasi-periodicity can be obtained by performing crossing symmetry for two times.

Derivation of the Riemann-Hilbert problem associated to the fermionic $U(1)$ twist fields with $\eta=-$ follows the same procedure.

5.2.3 Low temperature expansion

Let us start with one-particle form factors

$$f^{\eta}_{\nu,\epsilon}(\theta) = Q^{\rho}_{\nu,\epsilon}(\theta)\frac{\mathrm{Tr}\,(\rho\,\mu^{\eta}_{\alpha}\,D^{\epsilon}_{\nu}(\theta))}{\mathrm{Tr}\,(\rho\,\sigma^{\eta}_{\alpha})}\,. \tag{5.72}$$

We have

$$\begin{aligned}
&\mathrm{Tr}\,(\rho\,\mu^{\eta}_{\alpha}\,D^{\epsilon}_{\nu}(\theta))\\
= \;&\langle \mathrm{vac}|\rho\,\mu^{\eta}_{\alpha}D^{\epsilon}_{\nu}(\theta)|\mathrm{vac}\rangle + \sum_{\nu'}\int d\theta' e^{-\beta E_{\theta'}}\,{}_{\nu'}\langle\theta'|\mu^{\eta}_{\alpha}D^{\epsilon}_{\nu}(\theta)|\theta'\rangle_{\nu'} + O(e^{-2m\beta})\\
= \;& f^{(0)\eta}_{\nu,\epsilon}(\theta) + \sum_{\nu'}\int d\theta' e^{-\beta E_{\theta'}} f^{(0)\eta}_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\theta',\theta') + O(e^{-2m\beta})
\end{aligned} \tag{5.73}$$

and

$$\begin{aligned}
\mathrm{Tr}\,(\rho\sigma^{\eta}_{\alpha}) &= \langle \mathrm{vac}|\sigma^{\eta}_{\alpha}|\mathrm{vac}\rangle + \sum_{\nu'}\int d\theta' e^{-\beta E_{\theta'}}\,{}_{\nu'}\langle\theta'|\sigma^{\eta}_{\alpha}|\theta'\rangle_{\nu'} + +O(e^{-2m\beta})\\
&= \langle\sigma_{\alpha}\rangle + \sum_{\nu'}\int d\theta' e^{-\beta E_{\theta'}} f^{(0)\eta}_{(\nu',+)(\nu',-)}(\theta',\theta') + O(e^{-2m\beta})\,.
\end{aligned} \tag{5.74}$$

In the low-temperature limit, $1/\mathrm{Tr}\,(\rho\sigma_{\alpha})$ in (5.72) can be expanded as

$$\frac{1}{\mathrm{Tr}\,(\rho\sigma_{\alpha})} = \frac{1}{\langle\sigma_{\alpha}\rangle} - \frac{1}{\langle\sigma_{\alpha}\rangle^2}\sum_{\nu'}\int d\theta' e^{-\beta E_{\theta'}} f^{(0)\eta}_{(\nu',+)(\nu',-)}(\theta',\theta') + O(e^{-2m\beta})\,. \tag{5.75}$$

Substituting (5.73) and (5.75) into (5.72), and factorizing form factors $f^{(0)\eta}_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\theta',\theta')$ through Wick's theorem:

$$\begin{aligned}
&f^{(0)\eta}_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\theta',\theta')\\
&\quad= f^{(0)\eta}_{\nu,\epsilon}(\theta) f^{(0)\eta}_{(\nu',+)(\nu',-)}(\theta',\theta') + f^{(0)\eta}_{(\nu,\epsilon)(\nu',+)}(\theta,\theta') f^{(0)\eta}_{\nu',-}(\theta') - f^{(0)\eta}_{(\nu,\epsilon)(\nu',-)}(\theta,\theta') f^{(0)\eta}_{\nu',+}(\theta')\,,
\end{aligned}$$

yield the low-temperature expansion of normalized mixed-state one-particle form factors (5.57). Then, we consider two-particle form factors

$$f^{\eta}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = Q^{\rho}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)\frac{\mathrm{Tr}\left(\rho\,\sigma^{\eta}_{\alpha}\,D^{\epsilon_1}_{\nu_1}(\theta_1)D^{\epsilon_2}_{\nu_2}(\theta_2)\right)}{\mathrm{Tr}\,(\rho\,\sigma^{\eta}_{\alpha})}\,. \tag{5.76}$$

Likewise, we have

$$\begin{aligned}
&\mathrm{Tr}\left(\rho\,\sigma_\alpha^\eta\, D_{\nu_1}^{\epsilon_1}(\theta_1)D_{\nu_2}^{\epsilon_2}(\theta_2)\right)\\
=\;&\langle \mathrm{vac}|\rho\,\sigma_\alpha^\eta D_{\nu_1}^{\epsilon_1}(\theta_1)D_{\nu_2}^{\epsilon_2}(\theta_2)|\mathrm{vac}\rangle + \sum_\nu \int d\theta e^{-\beta E_\theta}\,{}_\nu\langle\theta|\sigma_\alpha^\eta D_{\nu_1}^{\epsilon_1}(\theta_1)D_{\nu_2}^{\epsilon_2}(\theta_2)|\theta\rangle_\nu + O(e^{-2m\beta})\\
=\;& f^{(0)\eta}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) + \sum_\nu \int d\theta e^{-\beta E_\theta} f^{(0)\eta}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)(\nu,+)(\nu,-)}(\theta_1,\theta_2,\theta,\theta) + O(e^{-2m\beta})
\end{aligned} \tag{5.77}$$

Using (5.77), (5.75), and the factorisation

$$\begin{aligned}
&f^{(0)\eta}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)(\nu,+)(\nu,-)}(\theta_1,\theta_2,\theta,\theta)\\
&\qquad = f^{(0)\eta}_{(\nu_1,\epsilon_2)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) f^{(0)\eta}_{(\nu,+)(\nu,-)}(\theta,\theta) - f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu,+)}(\theta_1,\theta) f^{(0)\eta}_{(\nu_2,\epsilon_2)(\nu,-)}(\theta_2,\theta)\\
&\qquad\quad + f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu,-)}(\theta_1,\theta) f^{(0)\eta}_{(\nu_2,\epsilon_2)(\nu,+)}(\theta_2,\theta),
\end{aligned}$$

we obtain the low-temperature expansion of normalized mixed-state two-particle form factors (5.58).

Chapter 6

Form factors of twist fields in mixed states

In this chapter, we present the exact result for mixed-state form factors of twist fields in the Ising model and in the Dirac theory. It has been shown in section 4.1.3 that the map $\mathfrak{U}$ in principle allows us to calculate mixed-state form factors in the Liouville space $\mathcal{L}_\rho$ from the known matrix elements on the associated Hilbert $\mathcal{H}$. However, this technique seems to break down for evaluating mixed-state form factors of the twist fields. This is because the evaluation involves an infinite re-summation: twist fields are infinite linear combinations of normal-ordered products (since they have nonzero matrix elements for arbitrary large number of particles), hence there are infinitely many internal contractions. This re-summation in principle gives rise to two effects: first, the overall normalization of mixed-state form factors, encoded into the mixed-state expectation value (mixed-state one-point function), is modified from its vacuum value; second, the dependence on the rapidities θ_j of the form factors is affected (Here we will only consider the dependence on the rapidities). On the other hand, we demonstrated in chapter 5 that in the case of thermal Gibbs state, we can formulate a set of equations and analytic conditions by setting up a Riemann-Hilbert problem derived from the finite-temperature KMS relation. The minimal solutions are finite-temperature one-particle form factors. But, when it comes to the case of general diagonal mixed-states, such techniques can not be employed, due to the fact that the analytic structure of the eigenvalue of the density matrix is in general not accessible.

In this chapter, we exploit a novel approach based on deriving and solving a system of non-linear functional differential equations. The derivation of this system of equations follows from the definition of mixed-state form factors and Wick's theorem. Using such technique of functional differential equations, we perform "automatically" the infinite resummation of contractions in order to obtain the exact rapidity dependence of twist-field form factors. As usual, this method will be applied both on the twist fields in the Ising model and in the Dirac theory. In the end of this chapter, we also present the general solution of this system of equations as the integral-operator kernel, which can be seen as an alternative representation of mixed-state form factors of twist fields. The results presented in this chapter can be found in [55, 142].

6.1 Exact form factors of twist fields in mixed states

As we discussed, for general diagonal mixed states, form factors of twist fields can not be obtained from either the Riemann-Hilbert problem technique or the the map $\mathfrak{U}$ technique. However, from the results (5.27), (5.30), (5.61) and (5.63) derived in chapter 5, we see that these finite-temperature form factors depend on the eigenvalue of the density matrix in a very trivial way. Then it is not unreasonable for us to conjecture that mixed-state form factors of twist fields admit a similar representation in terms of the leg factor. In this section, we will show the exact mixed-state form factors of twist fields in the Ising model [55] and in the Dirac theory [142]. We will also discuss the analytic properties of these form factors.

6.1.1 Ising model

In analogy with (5.27) and (5.30), it was conjectured in [55] that: the one- and two-particle mixed-state form factors of disorder and order fields are given, respectively, by

$$f_{\epsilon}^{\rho;\mu^\eta}(\theta) = f_{\epsilon}^{(0)\eta}(\theta) h_{\epsilon}^{\eta}(\theta) \langle \sigma^\eta \rangle_\rho \tag{6.1}$$

$$f_{\epsilon_1,\epsilon_2}^{\rho;\sigma^\eta}(\theta_1,\theta_2) = f_{\epsilon_1,\epsilon_2}^{(0)\eta}(\theta_1,\theta_2) h_{\epsilon_1}^{\eta}(\theta_1) h_{\epsilon_2}^{\eta}(\theta_2) \langle \sigma^\eta \rangle_\rho \tag{6.2}$$

where

$$h_{\epsilon}^{\eta}(\theta) = \exp\left[\epsilon\eta \int_{-\infty-\epsilon\eta i0^+}^{\infty-\epsilon\eta i0^+} \frac{d\theta'}{2\pi i} \frac{1}{\sinh(\theta-\theta')} \log\left(\tanh\frac{W(\theta')}{2}\right)\right] \tag{6.3}$$

$$f_{\epsilon}^{(0)\eta}(\theta) = \eta e^{\epsilon\frac{i\pi}{4}} \frac{1}{\sqrt{2\pi}} \tag{6.4}$$

$$f_{\epsilon_1,\epsilon_2}^{(0)\eta}(\theta_1\theta_2) = \frac{i}{2\pi} e^{\epsilon_1\frac{i\pi}{4}} e^{\epsilon_2\frac{i\pi}{4}} \tanh\left(\frac{\theta_2-\theta_1+\eta i(\epsilon_2-\epsilon_1)}{2}\right)^{\epsilon_1\epsilon_2}. \tag{6.5}$$

The above mixed-state form factors are defined for real rapidities, in general, as distributions obtained from boundary values of analytic functions. In light of the exponential form of twist fields (3.25), form factors with higher numbers of particles can be obtained by using Wick's theorem on the particles. The factorization of multi-particle form factors follows the rules: the overall normalization is $\langle\sigma^\eta\rangle_\rho$, the contraction of two particles (θ_1,ϵ_1) and (θ_2,ϵ_2) is given by the normalized two-particle form factor $f^{\rho;\sigma^\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)/\langle\sigma^\eta\rangle_\rho$, and the remaining single particle (θ,ϵ), if any, contributes a factor $f^{\rho;\mu^\eta}_{\epsilon}(\theta)/\langle\sigma^\eta\rangle_\rho$; further, a minus sign should be introduced for every crossing of contractions.

Leg factors $h^\eta_\epsilon(\theta)$ are analytic functions of θ in the strip

$$I^\eta_\epsilon := \left\{ \theta\in\mathbb{C} : \begin{array}{ll} \mathrm{Im}(\theta)\in(0,\pi) & (\epsilon\eta=+) \\ \mathrm{Im}(\theta)\in(-\pi,0) & (\epsilon\eta=-) \end{array} \right\}, \tag{6.6}$$

and for $\theta\in\mathbb{R}$ they are ordinary integrable functions obtained by continuous continuation from these analyticity regions. Hence, form factors of σ^η and μ^η are also analytic functions in the strip (6.6), except for possible "kinematic poles" coming from form factors $f^{(0)\eta}_{\epsilon_i,\epsilon_j}(\theta_i,\theta_j)$ of two particles with equal rapidities $(\theta_i=\theta_j)$ but opposite charges $(\epsilon_i=-\epsilon_j)$.

It can be checked that the above form factors in terms of leg factors of the form (6.3) are in agreement with (4.50), using the relation ${}_{\epsilon_1,\ldots,\epsilon_N}^{\;\;\rho}\langle\theta_1,\ldots,\theta_N|\mathcal{O}|\mathrm{vac}\rangle^\rho = \left(f^{\rho;\mathcal{O}^\dagger}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)\right)^*$, $(h^\eta_\epsilon(\theta))^* = h^\eta_{-\epsilon}(\theta)$, and (3.24). Further, the function $h^\eta_\epsilon(\theta)$ may be analytically continued from the strip (6.6) where it is analytic, to an extended region by extending on both sides of the strip. The extended region depends on the analytic properties of $W(\theta)$ around the real line. Let us assume that $W(\theta)$ is analytic on a neighborhood of some parts of the real line. If θ lies in this region, and either θ or $\theta+\eta\epsilon i\pi$ lies in I^η_ϵ, then the analytic continuation is obtained from

$$h^\eta_\epsilon(\theta)h^\eta_\epsilon(\theta+\eta\epsilon i\pi) = \coth\frac{W(\theta)}{2}. \tag{6.7}$$

Also, leg-factors with different values of ϵ are related to each other:

$$h^\eta_+(\theta)h^\eta_-(\theta) = \eta\coth\frac{W(\theta)}{2}, \tag{6.8}$$

this being valid for all $\theta\in\mathbb{R}$ in addition to all values of θ in the analyticity region of $W(\theta)$. This along with (6.7) implies that

$$h^\eta_-(\theta) = \eta h^\eta_+(\theta+\eta i\pi) \tag{6.9}$$

whenever the arguments lie in the analytic region of the leg factors. In turn, this leads to the crossing symmetry of mixed-state form factors

$$f^{\rho;\omega^\eta}_{\epsilon_1,\epsilon_2,\ldots,\epsilon_N}(\theta_1+\eta\epsilon_1 i\pi,\theta_2,\ldots,\theta_N)=\eta\epsilon_1 i\, f^{\rho;\omega^\eta}_{-\epsilon_1,\epsilon_2,\ldots,\epsilon_N}(\theta_1,\theta_2,\ldots,\theta_N). \tag{6.10}$$

which in the case of $W(\theta)=\beta E_\theta$ reproduces (5.24). Finally, we see that the form factors given above indeed reproduce reproduce the correct vacuum form factors at $e^{-W(\theta)}=0$.

6.1.2 Dirac theory

In the same spirit, we can generalized the results (5.61) and (5.63) obtained in the Dirac theory in the case of thermal Gibbs state to general diagonal mixed-states. This has been done in the original work [142].

Since the spectrum of the Dirac theory consists of two particle types, the terms βE_θ in the leg factor should be replaced by $W_+(\theta)$ and $W_-(\theta)$, respectively. We conjecture that the diagonal mixed-state one- and two-particle form factors of $U(1)$ twist fields are given by

$$f^{\rho;\mu^\eta}_{\nu,\epsilon}(\theta) = f^{(0)\eta}_{\nu,\epsilon}(\theta)h^\eta_{\nu,\epsilon}(\theta)\langle\sigma^\eta_\alpha\rangle_\rho \tag{6.11}$$

$$f^{\rho;\sigma^\eta_\alpha}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)h^\eta_{\nu_1,\epsilon_1}(\theta_1)h^\eta_{\nu_2,\epsilon_2}(\theta_2)\langle\sigma^\eta_\alpha\rangle_\rho \tag{6.12}$$

where

$$\begin{aligned} h^\eta_{\nu,\epsilon}(\theta) = \exp\Bigg[&-\eta\epsilon\int\frac{d\theta'}{2\pi i}\frac{\frac{1}{2}e^{\frac{(\theta'-\theta)}{2}}}{\cosh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-W_{-\nu}(\theta')}}{1+e^{2\pi i\eta\nu\alpha}e^{-W_{-\nu}(\theta')}}\right)\\ &-\eta\epsilon\int_{-\infty-\eta\epsilon i0^+}^{\infty-\eta\epsilon i0^+}\frac{d\theta'}{2\pi i}\frac{\frac{1}{2}e^{\frac{(\theta'-\theta)}{2}}}{\sinh(\frac{\theta-\theta'}{2})}\log\left(\frac{1+e^{-W_\nu(\theta')}}{1+e^{-2\pi i\eta\nu\alpha}e^{-W_\nu(\theta')}}\right)\Bigg] \end{aligned} \tag{6.13}$$

and

$$\begin{aligned} f^{(0)+}_{\nu,\epsilon}(\theta) &= (-i\epsilon\delta_{\nu,-}+\delta_{\nu,+})\frac{e^{-i\pi\nu\alpha/2}}{\Gamma(1+\nu\epsilon\alpha)}m^{\nu\epsilon\alpha+1/2}e^{(\nu\epsilon\alpha+1/2)\theta},\\ f^{(0)-}_{\nu,\epsilon}(\theta) &= f^{(0)+}_{\nu,\epsilon}(\theta)e^{2\pi i\nu\alpha\delta_{\epsilon,+}}, \end{aligned} \tag{6.14}$$

$$f^{(0)\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = \delta_{\epsilon_1,\epsilon_2}\delta_{\nu_1,-\nu_2}\,\nu_1\epsilon_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}}{\cosh(\frac{\theta_1-\theta_2}{2})} + \delta_{\epsilon_1,-\epsilon_2}\delta_{\nu_1,\nu_2}\,i\nu_1\frac{\sin(\pi\alpha)}{2\pi i}\frac{e^{\nu_1\epsilon_1\alpha(\theta_1-\theta_2)}e^{-i\pi\eta\nu_1\alpha}}{\sinh\left(\frac{\theta_1-\theta_2+\eta i(\epsilon_1-\epsilon_2)0^+}{2}\right)}\,. \tag{6.15}$$

Higher-particle form factors can be evaluated by using Wick's theorem on the particles. The overall normalization is $\langle\sigma^\eta_\alpha\rangle_\rho$, the contraction of two particles $(\theta_1,\nu_1,\epsilon_1)$ and $(\theta_2,\nu_2,\epsilon_2)$ is given by the normalized two-particle form factor $f^\eta_{(\nu_1\epsilon_1)(\nu_2\epsilon_2)}(\theta_1\theta_2)$, and the remaining single particle (θ,ν,ϵ), if any, gives a factor $f^\eta_{\nu\epsilon}(\theta)$; further, there is a minus sign for every crossing of contractions.

The analyticity regions of the leg factor $h^\eta_{\nu\epsilon}(\theta)$ are still the strip I^η_ϵ (6.6). Form factors of $U(1)$ twist fields are also analytic functions in this strip, except for possible kinematic poles coming from two-particle form factors $f^{(0)\eta}_{(\nu_1\epsilon_1)(\nu_2\epsilon_2)}(\theta_1\theta_2)$ with $(\epsilon_1=-\epsilon_2,\nu_1=\nu_2)$ at colliding rapidities. Leg-factors $h^\eta_{\nu\epsilon}(\theta)$, as functions of $\theta\in\mathbb{R}$, are ordinary integrable functions obtained by continuous continuation from these analyticity regions, and they satisfy

$$h^\eta_{\nu\epsilon}(\theta)h^\eta_{\nu,-\epsilon}(\theta) = \frac{1+e^{-W_\nu(\theta)}}{1+e^{-2\pi i\eta\nu\alpha}e^{-W_\nu(\theta)}}\,. \tag{6.16}$$

It is a simple matter to check that mixed-state form factors above do agree with (4.105) considering (6.14),(6.15), and with (4.106) using (3.27), (3.37), and complex conjugation. Concerning the normalization $\langle\sigma^\eta_\alpha\rangle_\rho$, it has not been exactly calculated so far. However, in analogy to the computation of c_α in [49], we can obtain a recursion relation for the normalization [142]

$$\frac{\langle\sigma^\eta_{\alpha+1}\rangle_\rho}{\langle\sigma^\eta_\alpha\rangle_\rho} = \frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)}m^{2\alpha+1}, \tag{6.17}$$

by considering mixed-state one-particle form factors of the fermionic primary twist fields. The derivation is as follow. By similar arguments to those leading to mixed-state form factors of twist fields $\sigma^\eta_{\alpha\pm1,\alpha}$, we can also deduce mixed-state form factors of $\sigma^\eta_{\alpha,\alpha\pm1}$. For instance, we have

$${}^\rho\langle\text{vac}|\sigma^\eta_{\alpha-1,\alpha}(0)|\theta\rangle^\rho_{+,+} = i\frac{e^{-i\pi\alpha/2}}{\Gamma(1-\alpha)}e^{2\pi i\alpha\delta_{\eta,-}}m^{-\alpha+1/2}e^{(\alpha-1/2)\theta}\,h^\eta_{+,+}(\theta)\langle\sigma^\eta_\alpha\rangle_\rho\,. \tag{6.18}$$

After a shift $\alpha\mapsto\alpha+1$, we arrive at

$${}^\rho\langle\text{vac}|\sigma^\eta_{\alpha,\alpha+1}(0)|\theta\rangle^\rho_{+,+} = -\frac{e^{-i\pi\alpha/2}}{\Gamma(-\alpha)}e^{2\pi i\alpha\delta_{\eta,-}}m^{-\alpha-1/2}e^{(\alpha+1/2)\theta}\,h^\eta_{+,+}(\theta)\langle\sigma^\eta_{\alpha+1}\rangle_\rho\,. \tag{6.19}$$

Notice that the leg factor $h^{\eta}_{+,+}(\theta)$ is invariant under the shift $\alpha \mapsto \alpha+1$. Then, comparison with the mixed-state one-particle form factor of $\sigma^{\eta}_{\alpha,\alpha+1}$ given by (6.11)

$$^{\rho}\langle \mathrm{vac}|\sigma^{\eta}_{\alpha,\alpha+1}(0)|\theta\rangle^{\rho}_{+,+} = -\frac{e^{-i\pi\alpha/2}}{\Gamma(1+\alpha)} e^{2\pi i\alpha\delta_{\eta,-}} m^{\alpha+1/2} e^{(\alpha+1/2)\theta}\, h^{\eta}_{+,+}(\theta)\langle\sigma^{\eta}_{\alpha}\rangle_{\rho}$$

leads to the recursion relation (6.17) which, in the pure-state limit, is in agreement with the result of [49] obtaind in the $U(1)$ Dirac model at zero temperature

$$\frac{\langle\sigma_{\alpha+1}\rangle}{\langle\sigma_{\alpha}\rangle} = \frac{\Gamma(-\alpha)}{\Gamma(1+\alpha)} m^{2\alpha+1}\,.$$

To fully determine the normalization $\langle\sigma^{\eta}_{\alpha}\rangle_{\rho}$, one needs to find the initial condition of (6.17), which involves the $W_{\nu}(\theta)$ function indicating the mixed states. Once the normalization for $\alpha \in [0, 1/2]$ is known, it is known for $\alpha \in [-1/2, 1/2]$ by conjugation, and then known for all α thanks to this recursion relation.

6.2 Non-linear functional differential system of equations

This section is devoted to the proof of the mixed-state form factors of twist fields (6.1) (6.2) in the Ising model and (6.11) (6.12) in the Dirac theory. The verification is based on a system of non-linear functional differential equations involving one- and two-particle mixed-state form factors of twist fields. We provide the derivation of these equations and discuss the uniqueness of their solutions. Finally, we verify our form factors by substituting them into these equations. Note that this novel methods provide an alternative proof of the known expression for finite-temperature form factors of twist fields, which, contrary to analytic-property methods, does not require “minimality” assumptions. The main results presented in this section are collected from the works [55] and [142].

6.2.1 Ising model

We recall that the twist fields σ and μ can be expresses as normal-ordered exponentials of bilinear expressions in the creation / annihilation operators. For instance the order field σ_+ admit a representation of the form

$$\sigma^{\eta} = \langle\sigma\rangle : \exp\left[\sum_{\epsilon_1,\epsilon_2}\int d\theta_1 d\theta_2\, F^{\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) a^{\epsilon_1}(\theta_1) a^{\epsilon_2}(\theta_2)\right] : \tag{6.20}$$

where following the notation (4.57), $F^{\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)$ can be rewritten as

$$F^{\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) = -\frac{1}{2\langle\sigma\rangle} f^{\sigma^\eta}_{-\epsilon_1,-\epsilon_2}(\theta_1,\theta_2) \tag{6.21}$$

As a consequence, Wick's theorem can be applied to factorize higher-number mixed-state form factors of these fields into products of one-and two-particle mixed-state form factors. Based on Wick's theorem and the trace definition (4.90), we will establish a system of functional differential equations for the one- and two-particle form factors of the field μ^η and σ^η [55]. These equations are nonlinear first-order differential equations and can be seen as functionals of the function $W : \theta \mapsto W(\theta)$ which characterizes the density matrix. With the initial condition at $W(\theta) = \infty$, given by the matrix elements of the twist fields on the Hilbert space, the solution is unique. Indeed, these matrix elements fully characterize the kernel $F^{\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)$ in the bilinear expression (6.20).

Derivation [55]

We start with defining notations

$$\tilde{f}^{\eta}_{\epsilon}(\theta) := Q^{\rho}_{\epsilon}(\theta)\frac{\mathrm{Tr}\,(\rho\,\mu^\eta\,a^\epsilon(\theta))}{\mathrm{Tr}\,(\rho\,\sigma^\eta)}, \quad \tilde{f}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) := Q^{\rho}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)\frac{\mathrm{Tr}\,(\rho\,\sigma^\eta\,a^{\epsilon_1}(\theta_1)a^{\epsilon_2}(\theta_2))}{\mathrm{Tr}\,(\rho\,\sigma^\eta)}. \tag{6.22}$$

Notations for insertions of higher numbers of creation and annihilation operators are defined in a similar way. Using $\delta\rho/\delta W(\beta) = -a^\dagger(\beta)a(\beta)\,\rho$, we find

$$\frac{\delta}{\delta W(\beta)}\frac{\mathrm{Tr}\,(\rho\,\mu^\eta\,a^\epsilon(\theta))}{\mathrm{Tr}\,(\rho\,\sigma^\eta)} = -\frac{\mathrm{Tr}\left(\rho\,\mu^\eta\,a^\epsilon(\theta)a^\dagger(\beta)a(\beta)\right)}{\mathrm{Tr}\,(\rho\,\sigma^\eta)} + \frac{\mathrm{Tr}\,(\rho\,\mu^\eta\,a^\epsilon(\theta))}{\mathrm{Tr}\,(\rho\,\sigma^\eta)}\frac{\mathrm{Tr}\left(\rho\,\sigma^\eta\,a^\dagger(\beta)a(\beta)\right)}{\mathrm{Tr}\,(\rho\,\sigma^\eta)}.$$

The right-hand side can be simplified via Wick's theorem,

$$\tilde{f}^{\eta}_{\epsilon}(\theta)\tilde{f}^{\eta}_{+,-}(\beta,\beta) - \tilde{f}^{\eta}_{\epsilon,+,-}(\theta,\beta,\beta) = \tilde{f}^{\eta}_{+}(\beta)\tilde{f}^{\eta}_{\epsilon,-}(\theta,\beta) - \tilde{f}^{\eta}_{-}(\beta)\tilde{f}^{\eta}_{\epsilon,+}(\theta,\beta),$$

this implies, from the definition (4.78),

$$\left(\frac{\delta}{\delta W(\beta)} + \frac{\epsilon\delta(\beta-\theta)}{1+e^{\epsilon W(\beta)}}\right)\tilde{f}^{\eta}_{\epsilon}(\theta) = \frac{\tilde{f}^{\eta}_{+}(\beta)\tilde{f}^{\eta}_{\epsilon,-}(\theta,\beta) - \tilde{f}^{\eta}_{-}(\beta)\tilde{f}^{\eta}_{\epsilon,+}(\theta,\beta)}{4\cosh^2\frac{W(\beta)}{2}}. \tag{6.23}$$

In the same recipe, differentiating $\mathrm{Tr}\left(\rho\,\sigma^\eta\, a^{\epsilon_1}(\theta_1)a^{\epsilon_2}(\theta_2)\right)/\mathrm{Tr}\left(\rho\,\sigma^\eta\right)$ gives

$$\left(\frac{\delta}{\delta W(\beta)} + \frac{\epsilon_1\delta(\beta-\theta_1)}{1+e^{\epsilon_1 W(\beta)}} + \frac{\epsilon_2\delta(\beta-\theta_2)}{1+e^{\epsilon_2 W(\beta)}}\right)\tilde{f}^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)$$
$$= \frac{\tilde{f}^\eta_{\epsilon_1,+}(\theta_1,\beta)\tilde{f}^\eta_{\epsilon_2,-}(\theta_2,\beta) - \tilde{f}^\eta_{\epsilon_1,-}(\theta_1,\beta)\tilde{f}^\eta_{\epsilon_2,+}(\theta_2,\beta)}{4\cosh^2\frac{W(\beta)}{2}}. \tag{6.24}$$

Introducing the notations

$$f^\eta_\epsilon(\theta) := \langle\sigma\rangle_\rho^{-1} f^{\rho;\mu^\eta}_\epsilon(\theta), \qquad f^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) := \langle\sigma\rangle_\rho^{-1} f^{\rho;\sigma^\eta}_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2), \tag{6.25}$$

and recalling (4.90), we then have

$$\tilde{f}^\eta_\epsilon(\theta) = f^\eta_\epsilon(\theta), \quad \tilde{f}^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) = f^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) + \left(1+e^{-\epsilon_1 W(\theta_1)}\right)\delta_{\epsilon_1+\epsilon_2,0}\,\delta(\theta_1-\theta_2). \tag{6.26}$$

Thus, the system of differential equations can be translated into a system for the mixed-state form factors themselves using (6.26),

$$\frac{\delta f^\eta_\epsilon(\theta)}{\delta W(\beta)} = \frac{f^\eta_+(\beta)f^\eta_{\epsilon,-}(\theta,\beta) - f^\eta_-(\beta)f^\eta_{\epsilon,+}(\theta,\beta)}{4\cosh^2\frac{W(\beta)}{2}} \tag{6.27}$$

$$\frac{\delta f^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)}{\delta W(\beta)} = \frac{f^\eta_{\epsilon_1,+}(\theta_1,\beta)f^\eta_{\epsilon_2,-}(\theta_2,\beta) - f^\eta_{\epsilon_1,-}(\theta_1,\beta)f^\eta_{\epsilon_2,+}(\theta_2,\beta)}{4\cosh^2\frac{W(\beta)}{2}}. \tag{6.28}$$

where the delta-function terms are vanishing.

As we see, equation (6.28) serves as a continuous family of non-linear first-order functional differential equations for the W-functionals $f^\eta_{+,+}(\theta_1,\theta_2)$, $f^\eta_{+,-}(\theta_1,\theta_2)$, $f_{-,+}(\theta_1,\theta_2)$ and $f^\eta_{-,-}(\theta_1,\theta_2)$ $(\theta_1,\theta_2\in\mathbb{R})$. Once this system of equations is solved, the solution can be fed into (6.27) to provide a continuous family of linear first-order differential equations for the W-functionals $f^\eta_+(\theta)$ and $f^\eta_-(\theta)$ for all $\theta\in\mathbb{R}$.

Uniqueness

According to (4.74), for the disorder field μ^η, we have

$$\begin{aligned} f^{\rho;\mu^\eta}_\epsilon(\theta) &= {}^\rho\langle\mathrm{vac}|(\mu^\eta)^\ell|\theta\rangle^\rho_\epsilon = {}^\rho\langle\mathrm{vac}|\circ(\mu^\eta)^\ell\circ\mathbf{U}^\dagger|\theta\rangle^\rho_\epsilon \\ &= {}^\rho\langle\mathrm{vac}|\circ(\mu^\eta)^\ell\circ|\theta\rangle^\rho_\epsilon + \int d\beta\frac{1}{1+e^{W(\beta)}}\,{}^\rho\langle\mathrm{vac}|\circ(\mu^\eta)^\ell\circ\mathbf{a}^\dagger_-(\beta)\mathbf{a}^\dagger_+(\beta)|\theta\rangle^\rho_\epsilon + \ldots \\ &= f^{\mu^\eta}_\epsilon(\theta) + \int d\beta\,\frac{1}{1+e^{W(\beta)}}\left[f^{\sigma^\eta}_{\epsilon,+}(\theta,\beta)f^{\mu^\eta}_-(\beta) - f^{\sigma^\eta}_{\epsilon,-}(\theta,\beta)f^{\mu^\eta}_+(\beta)\right] + \ldots \end{aligned}$$

where

$$f_\epsilon^{\mu^\eta}(\theta) = \lim_{W\to\infty} f_\epsilon^{\rho;\mu^\eta}(\theta), \qquad f_{\epsilon_1,\epsilon_2}^{\sigma^\eta}(\theta_1,\theta_2) = \lim_{W\to\infty} f_{\epsilon_1,\epsilon_2}^{\rho;\sigma^\eta}(\theta_1,\theta_2).$$

Then

$$\begin{aligned} f_\epsilon^\eta(\theta) &= f_\epsilon^{\rho;\mu^\eta}(\theta)/\langle\sigma\rangle_\rho \\ &= \frac{1}{\langle\sigma\rangle_\rho}\left[f_\epsilon^{\mu^\eta}(\theta) + \int d\beta\,\frac{1}{1+e^{W(\beta)}}\left[f_{\epsilon,+}^{\sigma^\eta}(\theta,\beta)f_-^{\mu^\eta}(\beta) - f_{\epsilon,-}^{\sigma^\eta}(\theta,\beta)f_+^{\mu^\eta}(\beta)\right] + \ldots\right]. \end{aligned} \tag{6.29}$$

Following the same recipe, we can obtain

$$\begin{aligned} f_{\epsilon_1,\epsilon_2}^\eta(\theta_1,\theta_2) &= \frac{1}{\langle\sigma\rangle_\rho}\Bigg[f_{\epsilon_1,\epsilon_2}^{\sigma^\eta}(\theta_1,\theta_2) \\ &\quad + \int d\theta\frac{1}{1+e^{W(\theta)}}\left[f_{\epsilon_1,-}^{\sigma^\eta}(\theta_1,\theta)f_{\epsilon_2,+}^{\sigma^\eta}(\theta_2,\theta) - f_{\epsilon_1,+}^{\sigma^\eta}(\theta_1,\theta)f_{\epsilon_2,-}^{\sigma^\eta}(\theta_2,\theta)\right] + \ldots\Bigg]. \end{aligned} \tag{6.30}$$

From (6.29) and (6.30), we see that $f_\epsilon^\eta(\theta)$ and $f_{\epsilon_1,\epsilon_2}^\eta(\theta_1,\theta_2)$ can be expressed in terms of ordinary form factors which have been fixed. Hence, the solutions to the system of functional differential equations are unique. Since mixed-state form factors (6.1) and (6.2) reproduce the vacuum form factors at $W(\theta) = \infty$, then we only have to verify that they satisfy the system of equations (6.27), (6.28) in order to prove that they are correct.

Solution

Considering the system of equations (6.27), (6.28) is analytic, it is sufficient for us to focus on the analyticity region of the corresponding mixed-state form factors, in order to verify the validity of (6.1) and (6.2). For convenience, we give a list of basic terms involved in (6.27) and (6.28).

$$\begin{aligned} f_\epsilon^\eta(\theta) &= \eta\frac{e^{\frac{\epsilon i\pi}{4}}}{\sqrt{2\pi}}h_\epsilon^\eta(\theta) \\ f_{\epsilon_1,\epsilon_2}^\eta(\theta_1,\theta_2) &= h_{\epsilon_1}^\eta(\theta_1)h_{\epsilon_2}^\eta(\theta_2)\,\frac{i}{2\pi}e^{\epsilon_1\frac{i\pi}{4}}e^{\epsilon_2\frac{i\pi}{4}}\left(\tanh\frac{\theta_2-\theta_1}{2}\right)^{\epsilon_1\epsilon_2} \\ \frac{\delta}{\delta W(\beta)}h_\epsilon^\eta(\theta) &= \frac{\eta\epsilon}{2\pi i}\frac{1}{\sinh W(\beta)}\frac{1}{\sinh(\theta-\beta)}h_\epsilon^\eta(\theta). \end{aligned} \tag{6.31}$$

Let us start with considering (6.27). The left-hand side is

$$\frac{\epsilon}{2\pi i}\frac{1}{\sinh W(\beta)}\frac{1}{\sinh(\theta-\beta)}\frac{e^{\frac{\epsilon i\pi}{4}}}{\sqrt{2\pi}}h_\epsilon^\eta(\theta)$$

and the right-hand side, in virtue of (6.8), can be shown to be

$$-\frac{1}{4\pi i}\frac{1}{\sinh W(\beta)}\left(\left(\tanh\frac{\beta-\theta}{2}\right)^{-\epsilon}-\left(\tanh\frac{\beta-\theta}{2}\right)^{\epsilon}\right)\frac{e^{\frac{\epsilon i\pi}{4}}}{\sqrt{2\pi}}h^{\eta}_{\epsilon}(\theta).$$

These are equal thanks to the relation

$$\frac{1}{\sinh x}=\frac{1}{2}\left(\coth\frac{x}{2}-\tanh\frac{x}{2}\right).$$

We then consider (6.28). On the left-hand side, we find

$$\frac{1}{4\pi^2}\frac{1}{\sinh W(\beta)}\left(\frac{\eta\epsilon_1}{\sinh(\theta_1-\beta)}+\frac{\eta\epsilon_2}{\sinh(\theta_2-\beta)}\right)\left(\tanh\frac{\theta_2-\theta_1}{2}\right)^{\epsilon_1\epsilon_2}$$
$$\times e^{\frac{\epsilon_1 i\pi}{4}}e^{\frac{\epsilon_2 i\pi}{4}}h^{\eta}_{\epsilon_1}(\theta_1)h^{\eta}_{\epsilon_2}(\theta_2)$$

whereas on the right-hand side, again using (6.8),

$$-\frac{1}{8\pi^2}\frac{\eta}{\sinh W(\beta)}\times$$
$$\times\left(\left(\tanh\frac{\beta-\theta_1}{2}\right)^{\epsilon_1}\left(\tanh\frac{\beta-\theta_2}{2}\right)^{-\epsilon_2}-\left(\tanh\frac{\beta-\theta_1}{2}\right)^{-\epsilon_1}\left(\tanh\frac{\beta-\theta_2}{2}\right)^{\epsilon_2}\right)\times$$
$$\times\, e^{\frac{\epsilon_1 i\pi}{4}}e^{\frac{\epsilon_2 i\pi}{4}}h^{\eta}_{\epsilon_1}(\theta_1)h^{\eta}_{\epsilon_2}(\theta_2).$$

Again, these are equal thanks to the relations

$$\left(\frac{1}{\sinh(\theta_1-\beta)}+\frac{1}{\sinh(\theta_2-\beta)}\right)\tanh\frac{\theta_2-\theta_1}{2}$$
$$=\frac{1}{2}\left(\tanh\frac{\beta-\theta_2}{2}\coth\frac{\beta-\theta_1}{2}-\tanh\frac{\beta-\theta_1}{2}\coth\frac{\beta-\theta_2}{2}\right)$$
$$\left(\frac{1}{\sinh(\theta_1-\beta)}-\frac{1}{\sinh(\theta_2-\beta)}\right)\coth\frac{\theta_2-\theta_1}{2}$$
$$=\frac{1}{2}\left(\coth\frac{\beta-\theta_2}{2}\coth\frac{\beta-\theta_1}{2}-\tanh\frac{\beta-\theta_1}{2}\tanh\frac{\beta-\theta_2}{2}\right)$$

These two relations can be verified by analyzing pole structures of both sides as analytic functions of β. In the first relation, both sides have poles at $\beta=\theta_1$ and $\beta=\theta_2$ with residues $\tanh\frac{\theta_1-\theta_2}{2}$, and in the second the residues are $\pm\coth\frac{\theta_1-\theta_2}{2}$ respectively. In each relation, both sides change sign under $\beta\mapsto\beta+i\pi$ and no other poles than those mentioned are found in any strip of width $i\pi$.

6.2.2 Dirac theory

As we recall, $U(1)$ twist fields are also in the form of normal-ordered exponential of bilinear combinations. For instance, the twist field σ_α^η is given by

$$\sigma_\alpha^\eta = \langle\sigma_\alpha\rangle \left(: \exp\left[\sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)} \int d\theta_1 d\theta_2 F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) D^{\epsilon_1}_{\nu_1}(\theta_1) D^{\epsilon_2}_{\nu_2}(\theta_2) \right] : \right) \tag{6.32}$$

with $F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) = -\frac{1}{2} f^{(0)\eta}_{(\nu_1,-\epsilon_1)(\nu_2,-\epsilon_2)}(\theta_1,\theta_2)$. Thus, mimicking arguments in the previous subsection, we deduce, from the trace definition and Wick's theorem, a system of non-linear functional differential equations for $U(1)$ twist fields mixed-state form factors as functions of $W_\nu(\theta)$. This work is collected from the original paper [142].

Derivation

We denote

$$\tilde{f}^\eta_{\nu,\epsilon}(\theta) \;:=\; Q^\rho_{\nu,\epsilon}(\theta) \frac{\mathrm{Tr}\left(\rho\mu_\alpha D^\epsilon_\nu(\theta)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} \tag{6.33}$$

$$\tilde{f}^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \;:=\; Q^\rho_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \frac{\mathrm{Tr}\left(\rho\sigma_\alpha D^{\epsilon_1}_{\nu_1}(\theta_1) D^{\epsilon_2}_{\nu_2}(\theta_2)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} \tag{6.34}$$

and the notations are similar for higher numbers of insertions of creation and annihilation operators. Using

$$\frac{\partial\rho}{\partial W_\nu(\theta)} = -\rho D^+_\nu(\theta) D_\nu(\theta)\,, \tag{6.35}$$

we have

$$\begin{aligned}
&\frac{\partial}{\partial W_{\nu'}(\beta)} \frac{\mathrm{Tr}\left(\rho\mu_\alpha D^\epsilon_\nu(\theta)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} \\
&= -\frac{\mathrm{Tr}\left(\rho\mu_\alpha D^\epsilon_\nu(\theta) D^+_{\nu'}(\beta) D_{\nu'}(\beta)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} + \frac{\mathrm{Tr}\left(\rho\mu_\alpha D^\epsilon_\nu(\theta)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} \frac{\mathrm{Tr}\left(\rho\sigma_\alpha D^+_{\nu'}(\beta) D_{\nu'}(\beta)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)} \\
&= \frac{\tilde{f}^\eta_{\nu,\epsilon}(\theta)\tilde{f}^\eta_{(\nu',+)(\nu',-)}(\beta,\beta) - \tilde{f}^\eta_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\beta,\beta)}{Q^\rho_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\beta,\beta)} \\
&= \frac{\tilde{f}^\eta_{\nu',+}(\beta)\tilde{f}^\eta_{(\nu,\epsilon)(\nu',-)}(\theta,\beta) - \tilde{f}^\eta_{\nu',-}(\beta)\tilde{f}^\eta_{(\nu,\epsilon)(\nu',+)}(\theta,\beta)}{Q^\rho_{(\nu,\epsilon)(\nu',+)(\nu',-)}(\theta,\beta,\beta)}
\end{aligned} \tag{6.36}$$

where we use Wick's theorem in the last step. By recalling the definition (4.107), we find

$$\left(\frac{\partial}{\partial W_{\nu'}(\beta)} + \frac{\epsilon\delta_{\nu,\nu'}\delta(\theta-\beta)}{Q^{\rho}_{\nu,-\epsilon}(\theta)}\right)\tilde{f}^{\eta}_{\nu,\epsilon}(\theta) = \frac{\tilde{f}^{\eta}_{\nu',+}(\beta)\tilde{f}^{\eta}_{(\nu,\epsilon)(\nu',-)}(\theta,\beta) - \tilde{f}^{\eta}_{\nu',-}(\beta)\tilde{f}^{\eta}_{(\nu,\epsilon)(\nu',+)}(\theta,\beta)}{4\cosh^2\left(\frac{W_{\nu'}(\beta)}{2}\right)}. \tag{6.37}$$

We then differentiate

$$\frac{\mathrm{Tr}\left(\rho\sigma_\alpha D^{\epsilon_1}_{\nu_1}(\theta_1)D^{\epsilon_2}_{\nu_2}(\theta_2)\right)}{\mathrm{Tr}\left(\rho\sigma_\alpha\right)}$$

with respect to $W_{\nu'}(\beta)$, and we find, following the same lines,

$$\left(\frac{\partial}{\partial W_{\nu}(\beta)} + \frac{\epsilon_1\delta_{\nu,\nu_1}\delta(\beta-\theta_1)}{1+e^{\epsilon_1 W_{\nu_1}(\theta_1)}} + \frac{\epsilon_2\delta_{\nu,\nu_2}\delta(\beta-\theta_2)}{1+e^{\epsilon_2 W_{\nu_2}(\theta_2)}}\right)\tilde{f}^{\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)$$
$$= \frac{\tilde{f}^{\eta}_{(\nu_1,\epsilon_1)(\nu,+)}(\theta_1,\beta)\tilde{f}^{\eta}_{(\nu_2,\epsilon_2)(\nu,-)}(\theta_2,\beta) - \tilde{f}^{\eta}_{(\nu_1,\epsilon_1)(\nu,-)}(\theta_1,\beta)\tilde{f}^{\eta}_{(\nu_2,\epsilon_2)(\nu,+)}(\theta_2,\beta)}{4\cosh^2\left(\frac{W_{\nu'}(\beta)}{2}\right)}. \tag{6.38}$$

Finally, we obtain a system of functional differential equations for the mixed-state form factors of $U(1)$ twist fields:

$$\frac{\partial f^{\eta}_{\nu,\epsilon}(\theta)}{\partial W_{\nu'}(\beta)} = \frac{f^{\eta}_{\nu',+}(\beta)f^{\eta}_{(\nu,\epsilon)(\nu',-)}(\theta,\beta) - f^{\eta}_{\nu',-}(\beta)f^{\eta}_{(\nu,\epsilon)(\nu',+)}(\theta,\beta)}{4\cosh^2\left(\frac{W_{\nu'}(\beta)}{2}\right)} \tag{6.39}$$

$$\frac{\partial f^{\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)}{\partial W_{\nu}(\beta)} = \frac{f^{\eta}_{(\nu_1,\epsilon_1)(\nu,+)}(\theta_1,\beta)f^{\eta}_{(\nu_2,\epsilon_2)(\nu,-)}(\theta_2,\beta) - f^{\eta}_{(\nu_1,\epsilon_1)(\nu,-)}(\theta_1,\beta)f^{\eta}_{(\nu_2,\epsilon_2)(\nu,+)}(\theta_2,\beta)}{4\cosh^2\left(\frac{W_{\nu}(\beta)}{2}\right)}. \tag{6.40}$$

This system of functional differential equations enjoys the virtue that it does not require the analytic structure of $W_\nu(\theta)$ and it works for general $W_\nu(\theta)$.

Uniqueness

Again, from (4.74), we can deduce:

$$f^{\eta}_{\nu,\epsilon}(\theta) = \frac{1}{\langle\sigma\rangle_\rho}\left[f^{\mu^\eta}_{\nu,\epsilon}(\theta) + \sum_{\nu'}\int d\beta\,\frac{1}{1+e^{W_{\nu'}(\beta)}}\left[f^{\sigma^\eta}_{(\nu,\epsilon)(\nu',+)}(\theta,\beta)f^{\mu^\eta}_{\nu',-}(\beta)\right.\right.$$
$$\left.\left.-f^{\sigma^\eta}_{(\nu,\epsilon)(\nu',-)}(\theta,\beta)f^{\mu^\eta}_{\nu',+}(\beta)\right] + \ldots\right] \tag{6.41}$$

and

$$
\begin{aligned}
f^{\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \quad = \quad & \frac{1}{\langle\sigma_\alpha\rangle_\rho}\Bigg[f^{\sigma^\eta}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \\
& + \sum_\nu \int d\theta \frac{1}{1+e^{W_\nu(\theta)}}\Big[f^{\sigma^\eta}_{(\nu_1,\epsilon_1)(\nu,-)}(\theta_1,\theta)f^{\sigma^\eta}_{(\nu_2,\epsilon_2)(\nu,+)}(\theta_2,\theta) \\
& -f^{\sigma^\eta}_{(\nu_1,\epsilon_1)(\nu,+)}(\theta_1,\theta)f^{\sigma^\eta}_{(\nu_2,\epsilon_2)(\nu,-)}(\theta_2,\theta)\Big] + \dots\Bigg], \qquad (6.42)
\end{aligned}
$$

leading to the uniqueness of the solutions to functional differential equations (6.39) and (6.40). It is trivial to see that mixed-state form factors (6.11) and (6.12) in the large-$W_\nu(\theta)$ limit do reproduce the correct vacuum form factors. So it is sufficient to prove if they are the solution of functional differential equations (6.39) and (6.40).

Solution

Consider (6.39) first. On the left-hand side, we find

$$
\frac{\eta\epsilon f^{+}_{\nu,\epsilon}(\theta)}{4\pi i}\frac{1-e^{-2\pi i\eta\nu'\alpha}}{1+e^{-2\pi i\eta\nu'\alpha}e^{-W_{\nu'}(\beta)}}\frac{e^{\frac{\beta-\theta}{2}}}{1+e^{W_{\nu'}(\beta)}}\left(\frac{\delta_{-\nu,\nu'}}{\cosh(\frac{\theta-\beta}{2})}+\frac{\delta_{\nu,\nu'}}{\sinh(\frac{\theta-\beta}{2})}\right)
$$

and on the right-hand side, using

$$
\begin{aligned}
f^{(0)\eta}_{-\nu,-\epsilon}(\beta) \quad &= \quad \epsilon i e^{(\nu\epsilon\alpha+1/2)(\beta-\theta)}e^{i\pi\eta\nu\alpha}f^{(0)}_{\nu,\epsilon}(\beta)\,, \\
h^{\eta}_{\nu',+}(\beta)h^{\eta}_{\nu',-}(\beta) \quad &= \quad \frac{1+e^{-W_{\nu'}(\beta)}}{1+e^{-2\pi i\eta\nu'\alpha}e^{-W_{\nu'}(\beta)}}\,,
\end{aligned}
$$

we find

$$
\frac{\eta\epsilon f^{\omega}_{\nu,\epsilon}(\theta)}{2\pi}\frac{\sin(\eta\nu'\alpha)e^{-i\pi\eta\nu'\alpha}}{1+e^{-2\pi i\eta\nu'\alpha}e^{-W_{\nu'}(\beta)}}\frac{e^{\frac{\beta-\theta}{2}}}{1+e^{W_{\nu'}(\beta)}}\left(\frac{\delta_{-\nu,\nu'}}{\cosh(\frac{\theta-\beta}{2})}+\frac{\delta_{\nu,\nu'}}{\sinh(\frac{\theta-\beta}{2})}\right).
$$

These are equal thanks to the relation

$$
\sin x = \frac{e^{ix}-e^{-ix}}{2i}\,. \qquad (6.43)
$$

Then, we consider (6.40). On the left-hand side, we find

$$
\begin{aligned}
&\eta\nu_1\tfrac{\sin(\pi\alpha)}{8\pi^2}\kappa(1-e^{-2\pi i\eta\nu\alpha})\Bigg[\delta_{\epsilon_1,\epsilon_2}\delta_{\nu_1,-\nu_2}\left(\frac{\delta_{\nu_1,\nu}}{\sinh\left(\frac{\theta_1-\beta}{2}\right)\cosh\left(\frac{\theta_2-\beta}{2}\right)}+\frac{\delta_{-\nu_1,\nu}}{\cosh\left(\frac{\theta_1-\beta}{2}\right)\sinh\left(\frac{\theta_2-\beta}{2}\right)}\right) \\
&\quad -i\epsilon_1 e^{-i\pi\eta\nu_1\alpha}\delta_{\epsilon_1,-\epsilon_2}\delta_{\nu_1,\nu_2}\left(\frac{\delta_{\nu_1,\nu}}{\sinh\left(\frac{\theta_1-\beta}{2}\right)\sinh\left(\frac{\theta_2-\beta}{2}\right)}+\frac{\delta_{-\nu_1,\nu}}{\cosh\left(\frac{\theta_1-\beta}{2}\right)\cosh\left(\frac{\theta_2-\beta}{2}\right)}\right)\Bigg]
\end{aligned}
$$

where

$$\kappa := \frac{e^{\epsilon_1 \nu_1 (\theta_1 - \theta_2)\alpha}}{(1 + e^{-2\pi i \eta \nu \alpha} e^{-W_\nu(\beta)})(1 + e^{W_\nu(\beta)})}$$

and on the right-hand side, we find

$$\frac{\sin^2(\pi\alpha)}{-4\pi^2}\kappa \left[i e^{-i\pi\eta\nu\alpha} \delta_{\epsilon_1,\epsilon_2} \delta_{\nu_1,-\nu_2} \left(\frac{\delta_{\nu_1,\nu}}{\sinh\left(\frac{\theta_1-\beta}{2}\right)\cosh\left(\frac{\theta_2-\beta}{2}\right)} - \frac{\delta_{-\nu_1,\nu}}{\cosh\left(\frac{\theta_1-\beta}{2}\right)\sinh\left(\frac{\theta_2-\beta}{2}\right)} \right) \right.$$
$$\left. -\epsilon_1 \delta_{\epsilon_1,-\epsilon_2} \delta_{\nu_1,\nu_2} \left(\frac{\delta_{\nu_1,\nu} e^{-2\pi i \eta \nu_1 \alpha}}{\sinh\left(\frac{\theta_1-\beta}{2}\right)\sinh\left(\frac{\theta_2-\beta}{2}\right)} - \frac{\delta_{-\nu_1,\nu}}{\cosh\left(\frac{\theta_1-\beta}{2}\right)\cosh\left(\frac{\theta_2-\beta}{2}\right)} \right) \right] .$$

These are equal thanks to the relation (6.43). It is worth notion that this system of non-linear functional differential equations provides an alternative check of our proposed finite-temperature form factors of $U(1)$ twist fields (5.61) and (5.63).

6.3 General solution as integral-operator kernel

In previous sections, we provided the exact results for form factors of twist fields and showed they are correct by deriving and solving a system of non-linear functional differential equations. It is worth stressing that these differential equations hold for any local field that can be expressed as normal-ordered exponential of bilinear forms in fermion operators. The results we presented are just the solution when form factors involved in the differential equations are specialized as those of twist fields. It is not known that a general solution will also possess the leg-factor structure found.

Nevertheless, we can obtain a general solution which is expressed in terms of integral-operator kernels. As we illustrated before, the map $\mathfrak{U}$ does not provide a very efficient way to calculate mixed-state form factors of twist fields due to the complicated expression of twist fields. However, the operator U, describing this map, can be used to deduce the general solution to those differential equations, which can be seen as an alternative representation for mixed-state form factors of twist fields. We present in this section the derivation of this general solution in the Ising model [55] and in the Dirac theory [142], respectively.

6.3.1 Ising model

Let us start in the Ising model. We consider, without loss of generality, the normalized two-particle mixed-state form factors of the order field σ^η with the two-particle state on the left. By introducing the short hand notation

$$S^\eta = \sum_{\epsilon_1,\epsilon_2} \int d\theta_1 d\theta_2 \, F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) a^{\epsilon_1}(\theta_1) a^{\epsilon_2}(\theta_2), \tag{6.44}$$

the normalized two-particle form factors are written as

$${}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|(:e^{S^\eta}:)^\ell|\text{vac}\rangle^\rho. \tag{6.45}$$

To simplify the action on the vacuum, we employ the relation (4.64). Recalling the definition of $a^\epsilon(\theta)^\ell$ (4.81) and the Liouville-space normal ordering, we then have

$$\begin{aligned} {}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|(:e^{S^\eta}:)^\ell|\text{vac}\rangle^\rho &= {}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|\mathbf{U}\left(\circ\!\!\!\circ\, e^{S^\eta} \circ\!\!\!\circ\right)^\ell|\text{vac}\rangle^\rho \\ &= {}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|\mathbf{U}e^{\tilde{S}^\eta}|\text{vac}\rangle^\rho \end{aligned} \tag{6.46}$$

where we denote

$$\tilde{S}^\eta = \sum_{\epsilon_1,\epsilon_2} \int \frac{d\theta_1 d\theta_2}{Q^\rho_{-\epsilon_1}(\theta_1) Q^\rho_{-\epsilon_2}(\theta_2)} F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)\, \mathbf{Z}^\dagger_{\epsilon_1}(\theta_1)\mathbf{Z}^\dagger_{\epsilon_2}(\theta_2). \tag{6.47}$$

Further, noting that $\mathbf{U}|\text{vac}\rangle^\rho = 0$, we rewrite (6.46) as

$${}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|(:e^{S^\eta}:)^\ell|\text{vac}\rangle^\rho = {}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|e^{\mathbf{U}\tilde{S}^\eta\mathbf{U}^{-1}}|\text{vac}\rangle^\rho.$$

Finally, using the relation (4.68) for $f_a = 1$, we arrive at

$${}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|(:e^{S^\eta}:)^\ell|\text{vac}\rangle^\rho = {}^{\rho}_{\epsilon_1,\epsilon_2}\langle\theta_1,\theta_2|e^{D^\eta}|\text{vac}\rangle^\rho \tag{6.48}$$

where

$$D^\eta = \sum_{\epsilon_1,\epsilon_2} \int d\theta_1 d\theta_2 \, F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) \left(\frac{\mathbf{Z}^\dagger_{\epsilon_1}(\theta_1)}{Q^\rho_{-\epsilon_1}(\theta_1)} + \frac{\epsilon_1\, \mathbf{Z}_{-\epsilon_1}(\theta_1)}{Q^\rho_+(\theta_1)} \right) \left(\frac{\mathbf{Z}^\dagger_{\epsilon_2}(\theta_2)}{Q^\rho_{-\epsilon_2}(\theta_2)} + \frac{\epsilon_2\, \mathbf{Z}_{-\epsilon_2}(\theta_2)}{Q^\rho_+(\theta_2)} \right). \tag{6.49}$$

From (6.48), the problem of evaluating the normalized two-particle mixed-state form factor (6.45) has been reduced to the computation of the matrix element of a pure exponential. Hence, we may use standard Bogoliubov-transformation techniques.

One way to implement such techniques is as follows. We construct basis $b_j, b_j^\dagger$ with discrete indices $j = 1, 2, \ldots, n$, which satisfies canonical anti-commutation relations

$$\{b_j, b_k^\dagger\} = \delta_{jk}, \qquad \{b_j, b_k\} = \{b_j^\dagger, b_k^\dagger\} = 0 \tag{6.50}$$

We define the vacuum as $|0\rangle$ and the column vector V as

$$V = \left(b_1, \ldots, b_n; b_1^\dagger, \ldots, b_n^\dagger\right)^T, \quad V^\dagger = \left(b_1^\dagger, \ldots, b_n^\dagger; b_1, \ldots, b_n\right).$$

Taking into account the form of the matrix element on the right-hand side of (6.48), we write

$$\langle 0|VV^\dagger e^{V^\dagger JV}|0\rangle = \lim_{\beta\to 0} \mathrm{Tr}(e^{-\beta N} VV^\dagger e^{V^\dagger JV}), \quad N = \frac{1}{2}V^\dagger \sigma_z V \tag{6.51}$$

where σ_z is the pauli matrix (recall (2.52)) and J is a general 2-block by 2-block matrix. When evaluating the trace in (6.51), we can move V along one cycle by using cyclic property of the trace and commutation relations

$$[N, V] = -\sigma_z V, \quad [V^\dagger JV, V] = MV, \quad M = \sigma_x J^T \sigma_x - J, \quad \{V, V^\dagger\} = I \tag{6.52}$$

where σ_x is the pauli matrix and I represents the identity matrix. Then we get the relation

$$\left(1 + e^{\beta\sigma_z} e^M\right) \mathrm{Tr}\left(e^{-\beta N} VV^\dagger e^{V^\dagger JV}\right) = e^{\beta\sigma_z} e^M \mathrm{Tr}\left(e^{-\beta N} e^{V^\dagger JV}\right). \tag{6.53}$$

Take the limit $\beta \to \infty$ and consider only the divergent terms proportional to e^β. We have

$$\begin{aligned} &\begin{pmatrix} (e^M)_{11} & (e^M)_{12} \\ 0 & 0 \end{pmatrix} \langle 0| \begin{pmatrix} (VV^\dagger)_{11} & (VV^\dagger)_{12} \\ (VV^\dagger)_{21} & (VV^\dagger)_{22} \end{pmatrix} e^{V^\dagger JV}|0\rangle \\ = &\begin{pmatrix} (e^M)_{11} & (e^M)_{12} \\ 0 & 0 \end{pmatrix} \langle 0|e^{V^\dagger JV}|0\rangle. \end{aligned}$$

Using rules of matrix product and $\langle 0|(VV^\dagger)_{22} e^{V^\dagger JV}|0\rangle = 0$, we find the matrix equation

$$\frac{\langle 0|(VV^\dagger)_{12} e^{V^\dagger JV}|0\rangle}{\langle 0|e^{V^\dagger JV}|0\rangle} = \left((e^M)_{11}\right)^{-1} (e^M)_{12} \tag{6.54}$$

where the matrix elements on the left-hand side can be considered as normalized two-particle form factors of the field $e^{V^\dagger JV}$.

Now, we can apply these techniques to our two-particle form factor problem. Since these techniques are only valid for Fock space based on canonical anti-commutation algebra, we have to define new Liouville mode operators

$$\mathbf{b}_\epsilon^\dagger(\theta) = \frac{\sqrt{Q_+^\rho(\theta)}}{Q_{-\epsilon}^\rho(\theta)} \mathbf{Z}_\epsilon^\dagger(\theta), \quad \mathbf{b}_\epsilon(\theta) = \frac{1}{\sqrt{Q_+^\rho(\theta)}} \mathbf{Z}_\epsilon(\theta).$$

We then see that

$$D^\eta = \sum_{\epsilon_1,\epsilon_2} \int \frac{d\theta_1 d\theta_2}{\sqrt{C_+(\theta_1)C_+(\theta_2)}} F^\eta_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2) \left(\mathbf{b}^\dagger_{\epsilon_1}(\theta_1) + \epsilon_1\, \mathbf{b}_{-\epsilon_1}(\theta_1)\right) \left(\mathbf{b}^\dagger_{\epsilon_2}(\theta_2) + \epsilon_2\, \mathbf{b}_{-\epsilon_2}(\theta_2)\right). \tag{6.55}$$

The matrix of integral operators J can be obtained by identifying $D^\eta = V^* J V$, and consequently we can write down the matrix M in the 4 by 4 form taking into account the blocks discussed above as well as the internal particle-type ϵ block structure

$$M = \begin{pmatrix} x & -y & -y & -x \\ z & x^t & x^t & -z \\ -z & -x^t & -x^t & z \\ x & -y & -y & -x \end{pmatrix} \tag{6.56}$$

where the integral operators x, y, z have kernels

$$\begin{aligned} x(\theta_1,\theta_2) &= \frac{F^\eta_{+-}(\theta_1,\theta_2) - F^\eta_{-+}(\theta_2,\theta_1)}{\sqrt{Q^\rho_+(\theta_1)Q^\rho_+(\theta_2)}} \\ y(\theta_1,\theta_2) &= \frac{F^\eta_{++}(\theta_1,\theta_2)}{\sqrt{Q^\rho_+(\theta_1)Q^\rho_+(\theta_2)}} \\ z(\theta_1,\theta_2) &= \frac{F^\eta_{--}(\theta_1,\theta_2)}{\sqrt{Q^\rho_+(\theta_1)Q^\rho_+(\theta_2)}} \end{aligned}$$

and t represent matrix transpose. Thanks to the notion that M is nilpotent, namely $M^2 = 0$, we can express its exponential simply as

$$e^M = 1 + M\,.$$

Direct calculations show that

$$R := W_{12} = \begin{pmatrix} -(1+q)^{-1}\, y\, (1+x^t)^{-1} & -q\,(1+q)^{-1} \\ -q^t\,(1+q^t)^{-1} & (1+q^t)^{-1}\, z\, (1+x)^{-1} \end{pmatrix}, \quad q = x + y\,(1+x^t)^{-1}\, z.$$

We are thus led to conclude that the normalized two-particle form factor can be given via the kernel of the integral-operator R,

$$_{\epsilon_1,\epsilon_2}^{\ \ \rho}\langle\theta_1,\theta_2|(:e^{S^\eta}:)^\ell|\mathrm{vac}\rangle^\rho = \sqrt{Q^\rho_+(\theta_1)Q^\rho_+(\theta_2)}\, R_{\epsilon_1,\epsilon_2}(\theta_1,\theta_2)$$

where $\epsilon_{1,2} = +$ is on the first row / column and $\epsilon_{1,2} = -$ is on the second.

6.3.2 Dirac theory

In the same spirit, we present an integral-operator expression for mixed-state form factors of $U(1)$ twist fields. Again, we consider the twist field σ_α^η (6.32). For convenience, we define

$$S^\eta := \sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)} \int d\theta_1 d\theta_2 F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) D^{\epsilon_1}_{\nu_1}(\theta_1) D^{\epsilon_2}_{\nu_2}(\theta_2) \tag{6.57}$$

and it follows that

$$\sigma_\alpha^\eta = \langle \sigma_\alpha \rangle \left(: e^{S^\eta} : \right) . \tag{6.58}$$

Now, we focus on normalized mixed-state two-particle form factors

$$_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1, \theta_2 | \left(: e^{S^\eta} : \right)^\ell |\text{vac}\rangle^\rho .$$

Using (4.64), the definition (4.99), the property of Liouville normal-ordering and the fact that $\mathbf{U}|\text{vac}\rangle^\rho = |\text{vac}\rangle^\rho$, we have

$$\begin{aligned} {}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1, \theta_2 | \left(: e^{S^\eta} : \right)^\ell |\text{vac}\rangle^\rho &= {}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1, \theta_2 | \mathbf{U} \left(\circ\!\!\!\circ\, e^{S^\eta} \circ\!\!\!\circ \right)^\ell |\text{vac}\rangle^\rho \\ & {}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1, \theta_2 | \mathbf{U} e^{\tilde{S}^\eta} |\text{vac}\rangle^\rho \\ & {}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1, \theta_2 | e^{\mathbf{U}\tilde{S}^\eta \mathbf{U}^{-1}} |\text{vac}\rangle^\rho \end{aligned}$$

where

$$\tilde{S}^\eta = \sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)} \int d\theta_1 d\theta_2 \frac{F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2)}{Q^\rho_{\nu_1,-\epsilon_1}(\theta_2) Q^\rho_{\nu_2,-\epsilon_2}(\theta_2)} \mathbf{Z}^\dagger_{\nu_1,\epsilon_1}(\theta_1) \mathbf{Z}^\dagger_{\nu_2,\epsilon_2}(\theta_2) . \tag{6.59}$$

Thanks to (4.68), we then have

$$_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1 \theta_2 | \left(: e^{S^\eta} : \right)^\ell |\text{vac}\rangle^\rho = {}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}{}^{\rho}\langle \theta_1 \theta_2 | e^{G^\eta} |\text{vac}\rangle^\rho \tag{6.60}$$

where

$$\begin{aligned} G^\eta &= \sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)} \int d\theta_1 d\theta_2 \, F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \\ & \times \left(\frac{\mathbf{Z}^\dagger_{\nu_1,\epsilon_1}(\theta_1)}{Q^\rho_{\nu_1,-\epsilon_1}(\theta_1)} + \frac{\epsilon_1 \mathbf{Z}_{\nu_1,-\epsilon_1}(\theta_1)}{Q^\rho_{\nu_1,+}(\theta_1)} \right) \left(\frac{\mathbf{Z}^\dagger_{\nu_2,\epsilon_2}(\theta_2)}{Q^\rho_{\nu_2,-\epsilon_2}(\theta_2)} + \frac{\epsilon_2 \mathbf{Z}_{\nu_2,-\epsilon_2}(\theta_2)}{Q^\rho_{\nu_2,+}(\theta_2)} \right) . \end{aligned} \tag{6.61}$$

Again, the matrix element of a pure exponential on the right-hand side of (6.60) suggests employing standard Bogoliubov-transformation techniques as we used in the previous subsection.

Considering the spectrum of the Dirac theory involves two particle types, we construct basis $b_j, b_j^\dagger, c_j, c_j^\dagger$ with discrete indices $j = 1, 2, \ldots, n$, which satisfies canonical anti-commutation relations

$$\{b_j, b_k^\dagger\} = \{c_j, c_k^\dagger\} = \delta_{jk} \tag{6.62}$$

with other anti-commutators being zero. We define the vacuum as $|0\rangle$ and the column vector V as

$$\begin{aligned} V &= \left(b_1, \ldots, b_n, c_1, \ldots, c_n; b_1^\dagger, \ldots, b_n^\dagger, c_1^\dagger, \ldots, c_n^\dagger\right)^T , \\ V^\dagger &= \left(b_1^\dagger, \ldots, b_n^\dagger, c_1^\dagger, \ldots, c_n^\dagger; b_1, \ldots, b_n, c_1, \ldots, c_n\right) . \end{aligned} \tag{6.63}$$

Following the same procedure, we have

$$\frac{\langle 0|(VV^\dagger)_{12} e^{V^\dagger J V}|0\rangle}{\langle 0|e^{V^\dagger J V}|0\rangle} = \left((e^M)_{11}\right)^{-1} (e^M)_{12} \tag{6.64}$$

where J is a general 2-block by 2-block matrix and $M = \sigma_x J^T \sigma_x - J$.

By defining new Liouville mode operators

$$\mathbf{b}^\dagger_{\nu,\epsilon}(\theta) = \frac{\sqrt{Q^\rho_{\nu,+}(\theta)}}{Q^\rho_{\nu,-\epsilon}(\theta)} \mathbf{Z}^\dagger_{\nu\epsilon}(\theta) , \quad \mathbf{b}_{\nu,\epsilon}(\theta) = \frac{1}{\sqrt{Q^\rho_{\nu,+}(\theta)}} \mathbf{Z}_{\nu,\epsilon}(\theta) \tag{6.65}$$

which satisfy the canonical anti-commutation relation

$$\begin{aligned} \{\mathbf{b}_{\nu,\epsilon}(\theta), \mathbf{b}^\dagger_{\nu',\epsilon'}(\theta')\} &= \delta_{\nu,\nu'}\delta_{\epsilon,\epsilon'}\delta(\theta-\theta') , \\ \{\mathbf{b}_{\nu,\epsilon}(\theta), \mathbf{b}_{\nu',\epsilon'}(\theta')\} &= \{\mathbf{b}^\dagger_{\nu,\epsilon}(\theta), \mathbf{b}^\dagger_{\nu',\epsilon'}(\theta')\} = 0 . \end{aligned}$$

we find

$$\begin{aligned} G^\eta &= \sum_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)} \int \frac{d\theta_1 d\theta_2}{\sqrt{Q^\rho_{\nu_1,+}(\theta_1) Q^\rho_{\nu_2,+}(\theta_2)}} F^\eta_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \\ &\quad \times \left(\mathbf{b}^\dagger_{\nu_1,\epsilon_1}(\theta_1) + \epsilon_1 \mathbf{b}_{\nu_1,-\epsilon_1}(\theta_1)\right) \left(\mathbf{b}^\dagger_{\nu_2,\epsilon_2}(\theta_2) + \epsilon_2 \mathbf{b}_{\nu_2,-\epsilon_2}(\theta_2)\right) . \end{aligned} \tag{6.66}$$

Identifying $G^\eta = V^\dagger J V$, considering the index (ν, ϵ), we write down the matrix M in the 8 by 8 form

$$M = \begin{pmatrix} x_+ & 0 & 0 & -y & 0 & -x_+ & -y & 0 \\ 0 & x_+^t & h & 0 & x_+^t & 0 & 0 & -h \\ 0 & y^t & x_- & 0 & y^t & 0 & 0 & -x_b \\ -h^t & 0 & 0 & x_-^t & 0 & h^t & x_-^t & 0 \\ 0 & -x_+^t & -h & 0 & -x_+^t & 0 & 0 & h \\ x_+ & 0 & 0 & -y & 0 & -x_+ & y & 0 \\ h^t & 0 & 0 & -x_-^t & 0 & -h^t & -x_-^t & 0 \\ 0 & y^t & x_- & 0 & y^t & 0 & 0 & -x_- \end{pmatrix} \tag{6.67}$$

where the integral operators x_+, x_-, y and z have kernels

$$\begin{aligned} x_+(\theta_1, \theta_2) &= \frac{2F^\eta_{(+,+)(+,-)}(\theta_1, \theta_2)}{\sqrt{Q^\rho_{+,+}(\theta_1) Q^\rho_{+,+}(\theta_2)}} \\ x_-(\theta_1, \theta_2) &= \frac{2F^\eta_{(-,+)(-,-)}(\theta_1, \theta_2)}{\sqrt{Q^\rho_{-,+}(\theta_1) Q^\rho_{-,+}(\theta_2)}} \\ y(\theta_1, \theta_2) &= \frac{2F^\eta_{(+,+)(-,+)}(\theta_1, \theta_2)}{\sqrt{Q^\rho_{+,+}(\theta_1) Q^\rho_{-,+}(\theta_2)}} \\ h(\theta_1, \theta_2) &= \frac{2F^\eta_{(+,-)(-,-)}(\theta_1, \theta_2)}{\sqrt{Q^\rho_{+,+}(\theta_1) Q^\rho_{-,+}(\theta_2)}} . \end{aligned}$$

It can be checked that M is again nilpotent, namely $M^2 = 0$. Hence, the exponential of it reads

$$e^M = 1 + M .$$

Direct calculations show that

$$R := \left((e^M)_{11} \right)^{-1} (e^M)_{12} = \begin{pmatrix} 0 & g & k & 0 \\ -g^t & 0 & 0 & l \\ -k^t & 0 & 0 & m \\ 0 & -l^t & -m^t & 0 \end{pmatrix} \tag{6.68}$$

where

$$
\begin{aligned}
g &= \left[x_+ + I - y\left(x_-^t + I\right)^{-1} h^t\right]^{-1} - I\,, \quad k = \left[h^t - \left(x_-^t + I\right) y^{-1}(x_+ + I)\right]^{-1} \\
m &= \left[x_- + I - y^t\left(x_+^t + I\right)^{-1} h\right]^{-1} - I\,, \quad l = \left[y^t - (x_- + I)\, h^{-1}(x_+^t + I)\right]^{-1}\,.
\end{aligned}
$$

Finally, the normalized two-particle mixed-state form factor is

$$
{}_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}^{\qquad\qquad\rho}\langle\theta_1,\theta_2|\left(:e^{S^\eta}:\right)^\ell|\mathrm{vac}\rangle^\rho = \sqrt{Q_{\nu_1+}(\theta_1)Q_{\nu_2+}(\theta_2)}\,R_{(\nu_1,\epsilon_1)(\nu_2,\epsilon_2)}(\theta_1,\theta_2) \tag{6.69}
$$

where $(++)$, $(+-)$ are on the first two rows/columns and $(-+)$, $(--)$ are on the second.

Chapter 7

Applications

In the previous chapter, we obtained the exact mixed-state form factors of twist fields by deriving and solving a system of non-linear functional differential equations, in the Ising model and in the Dirac theory. With these results, we will in this chapter enter the analysis of mixed-state correlation functions of twist fields. In the first section, we will present representations of correlation functions for twist fields in general diagonal mixed states, in the Ising model [55] and in the Dirac theory [142], respectively. Following these general results, we will turn to correlation functions of twist fields in three particular mixed states including the thermal Gibbs state [21, 23], non-equilibrium stead state [55], and generalized Gibbs ensemble [55]. Finally, we will end up with the deduction of the Rényi entropy for integer n as an application of the results for mixed-state correlation functions of $U(1)$ twist fields [142].

7.1 Mixed-state two-point correlation functions of twist fields

Using the resolution of the identity (4.13) on the Liouville space, we can obtain a series expression for two-point functions in terms of form factors (4.48), leading to an expansion similar to (2.50). Taking into account the state normalization (4.17), the resolution of the identity gives

$$\begin{aligned}&\langle\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle_\rho\\ =\ &\sum_{N=0}^{\infty}\sum_{a_1,\ldots,a_N}\sum_{\epsilon_1,\ldots,\epsilon_N}\int\frac{d\theta_1\cdots d\theta_N}{N!}\left[\frac{e^{\sum_{j=1}^N\left(i\epsilon_j m_{a_j}\sinh\theta_j x-i\epsilon_j m_{a_j}\cosh\theta_j t\right)}}{\prod_{j=1}^N\epsilon_j^{1-f_{a_j}}\left(1-(-1)^{f_{a_j}}e^{-\epsilon_j W_{a_j}(\theta_j)}\right)}\right.\\ &\left. f^{\rho;\mathcal{O}_1}_{(a_1,\epsilon_1)\cdots(a_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)f^{\rho;\mathcal{O}_2}_{(a_N,-\epsilon_N)\cdots(a_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1)\right].\end{aligned}\tag{7.1}$$

This expression is expected to hold for any non-interacting field $\mathcal{O}$ whose form factors are zero for large enough numbers of particles so that the above series truncates. Local fermion multilinears, for instance the fermion fields ψ and $\bar{\psi}$ or the field ε, are such examples of $\mathcal{O}$.

The integrals in (7.1), however, require some analysis. We consider for now free-fermion models. From the arguments in section 4.1.3, form factors of fermion multilinears are entire functions of the rapidities. If the value of $W_{a_j}(\theta_j)$ increases as $|\theta_j| \to \infty$, then the integral over θ_j is convergent for $\epsilon_j = -$. However, the integral over θ_j is in general not convergent for $\epsilon_j = +$. In analogy with the standard $i0^+$ prescription for correlation functions in the context of QFT, assuming $W_{a_j}(\theta_j)$ grows like, or faster than $e^{\alpha \cosh \theta_j}$ for some $\alpha > 0$ as $|\theta_j| \to \infty$, we can make both cases $\epsilon_j = \pm$ convergent by replacing t with $t - i0^+$. With this prescription, the correlation function is seen as the boundary value, at $t \in \mathbb{R}$, of a function of t analytic on some neighborhood of $\mathbb{R}$ in the region $\mathrm{Im}(t) < 0$.

In fact, we can make this boundary value finite at space-like distances ($x^2 > t^2$) for any $W_{a_j}(\theta_j)$ as long as $W_{a_j}(\theta_j)$ is analytic on neighborhoods of (K, ∞) and $(-\infty, -K)$, for $K > 0$ large enough. To see this, assuming without loss of generality that $x > 0$, we shift the contours as $\theta_j \mapsto \theta_j + \epsilon_j i0^+$ in the region $|\mathrm{Re}(\theta_j)| > K$, so that θ_j remains in the analyticity region of $W_a(\theta_j)$. It turns out that this boundary value with integrals on the shifted contours is indeed finite at space-like distances ($x^2 > t^2$).

For twist fields, the form factor expansion is infinite, as these fields have non-zero form factors for arbitrary large numbers of particles. However, the resulting infinite series (7.1) is not the correct representation of the two-point function. The form factor expansion is modified in various ways, because of the branch cuts emanating from twist fields as expressed in the twist condition. In this section, we will provide intuitive arguments for the modifications required [55, 142] and present a conjecture for the exact series expansion. Throughout we take again without loss of generality $x > 0$, and we concentrate solely on the space-like region $x^2 > t^2$

7.1.1 Ising model

Specializing the mixed-state two-point correlation function (7.1) to the case of the Ising model, we have

$$\langle \mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle_\rho$$
$$= \sum_{N=0}^{\infty}\sum_{\epsilon_1,\ldots,\epsilon_N}\int \frac{d\theta_1\cdots d\theta_N}{N!}\left[\frac{e^{\sum_{j=1}^{N}(i\epsilon_j m\sinh\theta_j x - i\epsilon_j m\cosh\theta_j t)}}{\prod_{j=1}^{N}\left(1+e^{-\epsilon_j W(\theta_j)}\right)}\right.$$
$$\left. f^{\rho;\mathcal{O}_1}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)\, f^{\rho;\mathcal{O}_2}_{-\epsilon_N,\ldots,-\epsilon_1}(\theta_N,\ldots,\theta_1)\right].$$

If $\mathcal{O}_1$ and $\mathcal{O}_2$ are both order or disorder fields, three modifications are required.

Three modifications

First, if the general expansion $\sum_s {}^\rho\langle \text{vac}|\mathcal{O}_1(x,t)^\ell|s\rangle^\rho\, {}^\rho\langle s|\mathcal{O}_2(0,0)^\ell|\text{vac}\rangle^\rho$ is a large distance expansion, then $|s\rangle^\rho\, {}^\rho\langle s|$ should be interpreted as intermediate states over the region between the fields $\mathcal{O}_1(x,t)^\ell$ and $\mathcal{O}_2(0,0)^\ell$, and the vacuum states ${}^\rho\langle \text{vac}|$ and $|\text{vac}\rangle^\rho$ should represent what is happening on the far right and left respectively. It has been argued in [21, 23], via the comparison between finite-temperature form factors and form factors on the circle, that the intermediate states must lie in a region which is not affected by the branch cuts of twist fields. Although this argument is based on the thermal Gibbs state, it is expected to be valid for general mixed states. Indeed, the function ${}^\rho\langle \text{vac}|\mathcal{O}_1(x,t)^\ell|s\rangle^\rho$ has no "knowledge" of the operator $\mathcal{O}_2(0,0)^\ell$, so it should not be affected by the cut from $\mathcal{O}_2(0,0)^\ell$. This means that we have to obtain form factor expansions where no cut is present in the region between 0 and x, for instance,

$$\langle \sigma^+(x,t)\sigma^-(0,0)\rangle_\rho, \qquad \langle \mu^+(x,t)\mu^-(0,0)\rangle_\rho. \tag{7.2}$$

On the other hand, as we recall, the free Majorana field theory is the scaling limit of the Ising lattice model. According to (2.70), mixed-state correlation functions of spin operators, under the scaling limit, give rise to mixed-state correlation functions $\langle \sigma_+(x,t)\sigma_+(0,0)\rangle_\rho$ in the ordered regime or $\langle \mu^+(x,t)\mu^+(0,0)\rangle_\rho$ in the disordered regime. Here the choice of a direction for the branch cut must be kept the same for each twist fields inside the correlation function, due to the Jordan-Wigner transformation, which the Pauli spin matrices are written as infinite products of fermion operators starting at the matrix's site and going in a fixed direction. As a consequence, a cut can be found *between* 0 and x, and our form factors, according to our arguments above, cannot directly be used.

Fortunately, the unitary operator Z can help us obtain correlation function of the type (7.2) in which the branch cut is absent in the region between 0 and x:

$$\langle\sigma^{+}(x,t)\sigma^{-}(0,0)\rangle_{\rho^\sharp}, \quad \langle\mu^{+}(x,t)\mu^{-}(0,0)\rangle_{\rho^\sharp} \tag{7.3}$$

where we recall $\rho^\sharp = z^{-1}\rho$ with $W^\sharp(\theta) = W(\theta) + \pi i$.

Second, as we illustrated in the beginning of chapter 5, the insertion of a twist field inside finite-temperature correlation functions or traces will affect one of the vacuum sectors in the correspondence to vacuum expectation values in the quantization on the circle and hence gives rise to the free energy difference between the sector where a cut lies and the sector where no cut lies. This statement is assumed to hold for general mixed states anad can be expressed via

$${}^{\rho}\langle\text{vac}|\omega^{\eta}(x,t)|\theta_1,\ldots,\theta_N\rangle^{\rho}_{\epsilon_1,\ldots,\epsilon_N} = e^{\eta x\mathcal{E}} e^{\sum_{j=1}^{N}\left(i\epsilon_j p_{\theta_j}x - i\epsilon_j E_{\theta_j}t\right)} f^{\rho;\omega^\eta}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) \tag{7.4}$$

for both order and disorder twist fields $\omega = \sigma$ and $\omega = \mu$, with $\mathcal{E}$ the *free energy deficit*

$$\mathcal{E} = \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m\cosh\theta \log\left(\coth\frac{W(\theta)}{2}\right) \tag{7.5}$$

which is generalized from (5.12) and (5.13). Denoting $\mathcal{E}^\sharp$ the free energy deficit associated to $W^\sharp$, we have

$$\mathcal{E}^\sharp = -\mathcal{E}. \tag{7.6}$$

Finally, mixed-state form factors of twist fields are not entire functions of the rapidities but distributions defined as boundary values of analytic functions with no colliding rapidities. So we have to shift the contour towards the analytic region for the purpose of obtaining a well-defined form factor expansion. Hence, we need to further require that $W(\theta)$ be analytic on a neighborhood of $\mathbb{R}$.

Mixed-state form factor expansion

Putting these subtleties together, we obtain the mixed-state correlation functions for both order and disorder fields in the Ising model:

$$\begin{aligned} &\langle\omega^+(x,t)\omega^+(0,0)\rangle_\rho \\ &= e^{-x\mathcal{E}}\sum_{N=0}^{\infty}\sum_{\epsilon_1,\ldots,\epsilon_N}\int\frac{d\theta_1\cdots d\theta_N}{N!}\frac{e^{\sum_{j=1}^N\left(i\epsilon_j p_{\theta_j}x-i\epsilon_j E_{\theta_j}t\right)}}{\prod_{j=1}^N\left(1-e^{-\epsilon_j W(\theta_j)}\right)} \\ &\quad\times f^{\rho^\sharp;\omega^+}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)f^{\rho^\sharp;\omega^-}_{-\epsilon_N,\ldots,-\epsilon_1}(\theta_N,\ldots,\theta_1)\end{aligned} \tag{7.7}$$

where $\omega=\sigma$ or $\omega=\mu$, and where $W(\theta)$ is assumed to be analytic on a neighborhood of $\theta\in\mathbb{R}$. As we see from (7.7), it is possible for the factor $\frac{1}{1-e^{-\epsilon_j W(\theta_j)}}$ to have a pole at some value of θ_j where $W(\theta_j)=0$. This is the reason why we have to impose the positivity condition (4.76) for the twisted case.

As in the case of fermion multilinears, in order to obtain a large-distance expansion from the conditionally convergent integrals, we shift the θ_j contour in (7.7) by $\epsilon i_j 0^+$ for $\eta>0$ so that θ_j remains in the analyticity region of $W(\theta_j)$ and . Note that these shifts keep the rapidities in the the form factors involved.

If we shift the contours further, we may come across singularities of the function $(1-e^{\pm W(\theta)})^{-1}$. It is these singularities that determine the large-distance asymptotic behavior of the two-point function. Among these singularities, we denote by $\theta^\star$ the one which makes $|\mathrm{Im}(\sinh\theta)|$ minimum. We then have

$$\langle\sigma^+(x,t)\sigma^+(0,0)\rangle_\rho = \langle\sigma_+\rangle_\rho\langle\sigma_-\rangle_\rho e^{-x\mathcal{E}}\left(1+O\left(e^{-2mx|\mathrm{Im}(\sinh\theta^\star)|}\right)\right) \tag{7.8}$$

$$\langle\mu^+(x,t)\mu^+(0,0)\rangle_\rho = O\left(e^{-mx(\mathcal{E}+|\mathrm{Im}(\sinh\theta^\star)|)}\right) \tag{7.9}$$

Here, the exponential decay includes possible algebraic or other non-exponential factors in mx, which depend on the type of singularities. We will provide examples of this in our investigation of two specific mixed states: the generalized Gibbs ensemble in section 7.4, and the non-equilibrium steady state in subsection 7.3.3.

7.1.2 Dirac theory

From (7.1), mixed-state two-point correlation functions are given by

$$\begin{aligned}&\langle\mathcal{O}_1(x,t)\mathcal{O}_2(0,0)\rangle_\rho\\ =\ &\sum_{N=0}^{\infty}\sum_{\nu_1,\ldots,\nu_N}\sum_{\epsilon_1,\ldots,\epsilon_N}\int\frac{d\theta_1\cdots d\theta_N}{N!}\left[\frac{e^{\sum_{j=1}^N\left(i\epsilon_j p_{\theta_j}x-i\epsilon_j E_{\theta_j}t\right)}}{\prod_{j=1}^N\left(1+e^{-\epsilon_j W_{\nu_j}(\theta_j)}\right)}\right.\\ &\left.f^{\rho;\mathcal{O}_1}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)f^{\rho;\mathcal{O}_2}_{(\nu_N,-\epsilon_N)\cdots(\nu_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1)\right].\end{aligned}$$

Again, for $U(1)$ twist fields, this representation needs to be modified from three aspects.

Three modifications

First, in order to obtain form factor expansions where no cut is present in the region between 0 and x, we can use the relation $\sigma^+_{\alpha'}=\sigma^-_{\alpha'}Z^{-1}$ and we then have

$$\begin{aligned}\langle\sigma^+_\alpha(x,t)\sigma^+_{\alpha'}(0,0)\rangle_\rho &= \langle\sigma^+_\alpha(x,t)\sigma^-_{\alpha'}(0,0)\rangle_{\rho^\sharp}\\ \langle\sigma^+_{\alpha\pm1,\alpha}(x,t)\sigma^+_{\alpha'\mp1,\alpha'}(0,0)\rangle_\rho &= \langle\sigma^+_{\alpha\pm1,\alpha}(x,t)\sigma^-_{\alpha'\mp1,\alpha'}(0,0)\rangle_{\rho^\sharp}\end{aligned}\tag{7.10}$$

where $\rho^\sharp:=z^{-1}\rho$ is the twisted density matrix with $W^\sharp_\pm(\theta)=W_\pm(\theta)\pm2\pi i\alpha'$. Note that two-point functions

$$\langle\sigma^+_{\alpha\pm1,\alpha}(x,t)\sigma^+_{\alpha'\pm1,\alpha'}(0,0)\rangle_\rho$$

are not considered, as they are all zero, due to the fact that twist fields $\sigma^\eta_{\alpha+1,\alpha}$ has non-zero one-particle form factors $f^{\rho;\sigma^\eta_{\alpha+1,\alpha}}_{\nu\epsilon}(\theta)$ only for $\nu=\epsilon$ while twist fields $\sigma^\eta_{\alpha-1,\alpha}$ has non-zero one-particle form factors $f^{\rho;\sigma^\eta_{\alpha-1,\alpha}}_{\nu\epsilon}(\theta)$ only for $\nu=-\epsilon$.

Second, the branch cut of a twist field inside the mixed-state correlation function changes one of the vacuum sectors and affect the translation property of the x-dependent matrix element via

$$^\rho\langle\text{vac}|\omega^\eta(x,t)|\theta_1,\ldots,\theta_N\rangle^\rho_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}=e^{\eta x\mathcal{E}}e^{\sum_{j=1}^N\left(i\epsilon_j p_{\theta_j}x-i\epsilon_j E_{\theta_j}t\right)}f^{\rho;\omega^\eta}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N)\tag{7.11}$$

for both twist fields $\omega=\sigma_\alpha$ and $\omega=\sigma_{\alpha\pm1,\alpha}$, with $\mathcal{E}$ the free energy deficit:

$$\mathcal{E}=\sum_{\nu=\pm}\int\frac{d\theta}{2\pi}m\cosh\theta\log\left(\frac{1+e^{-W_\nu(\theta)}}{1+e^{-2\pi i\nu\alpha}e^{-W_\nu(\theta)}}\right)\tag{7.12}$$

which is generalized from (7.5). We denote by $\mathcal{E}^\sharp$ the free energy deficit associated to $W^\sharp$ and we have

$$\mathcal{E}^\sharp = -\mathcal{E}\,. \tag{7.13}$$

Finally, assuming $W_\nu(\theta)$ to be analytic around the real line, we shift the contour towards the analyticity region of mixed-state form factors of twist fields for the purpose of obtaining a well-defined form factor expansion.

Mixed-state form factor expansion

According to the modifications stipulated above, we propose the followings:

Proposition 7.1.1. With $W_\nu(\theta)$ analytic on a neighborhood of $\theta \in \mathbb{R}$, we have

$$\begin{aligned} &\langle \sigma^+_\alpha(x,t)\sigma^+_{\alpha'}(0,0)\rangle_\rho \\ = \; & e^{-x\mathcal{E}} \sum_{N=0}^\infty \sum_{\nu_1,\ldots,\nu_N} \sum_{\epsilon_1,\ldots,\epsilon_N} \int \frac{d\theta_1\cdots d\theta_N}{N!} \left[\frac{e^{\sum_{j=1}^N \left(i\epsilon_j p_{\theta_j} x - i\epsilon_j E_{\theta_j} t\right)}}{\prod_{j=1}^N \left(1+e^{-\epsilon_j \nu_j 2\pi i\alpha} e^{-\epsilon_j W_{\nu_j}(\theta_j)}\right)} \right. \\ & \left. f^{\rho^\sharp;\sigma^+_\alpha}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) f^{\rho^\sharp;\sigma^-_{\alpha'}}_{(\nu_N,-\epsilon_N)\cdots(\nu_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1)\right] \end{aligned} \tag{7.14}$$

and

$$\begin{aligned} &\langle \sigma^+_{\alpha\pm1,\alpha}(x,t)\sigma^+_{\alpha'\mp1,\alpha'}(0,0)\rangle_\rho \\ = \; & e^{-x\mathcal{E}} \sum_{N=0}^\infty \sum_{\nu_1,\ldots,\nu_N} \sum_{\epsilon_1,\ldots,\epsilon_N} \int \frac{d\theta_1\cdots d\theta_N}{N!} \left[\frac{e^{\sum_{j=1}^N \left(i\epsilon_j p_{\theta_j} x - i\epsilon_j E_{\theta_j} t\right)}}{\prod_{j=1}^N \left(1+e^{-\epsilon_j \nu_j 2\pi i\alpha} e^{-\epsilon_j W_{\nu_j}(\theta_j)}\right)} \right. \\ & \left. f^{\rho^\sharp;\sigma^+_{\alpha\pm1,\alpha}}_{(\nu_1,\epsilon_1)\cdots(\nu_N,\epsilon_N)}(\theta_1,\ldots,\theta_N) f^{\rho^\sharp;\sigma^-_{\alpha'\mp1,\alpha'}}_{(\nu_N,-\epsilon_N)\cdots(\nu_1,-\epsilon_1)}(\theta_N,\ldots,\theta_1)\right]\,. \end{aligned} \tag{7.15}$$

Form factor expansions (7.14) and (7.15) are infinite since twist fields have non-zero mixed-state form factors for arbitrarily large number of particles. In order to have convergent integrals in our expansions, we shift the θ_j contours in (7.14) and (7.15) by $i\epsilon_j\zeta$ for $\zeta > 0$ small enough in such a way that θ_j remains in the analyticity region of $W_\nu(\theta_j)$ and of the form factors involved. It is worth attention that a positivity condition like (4.76) is not necessary in the twisted case, since $e^{-\epsilon\nu 2\pi\alpha}$ is generally not equal to -1 for generic α.

The large distance leading behaviour of the two-point functions (7.14) and (7.15) are determined by singularities of the function $\left(1+e^{-\epsilon\nu 2\pi i\alpha}e^{-\epsilon W_\nu(\theta)}\right)^{-1}$. Among these singularities, we denote by $\theta^\star$ the one making $|\mathrm{Im}(\sinh\theta)|$ minimum. we have

$$\begin{aligned}
\langle\sigma^+_\alpha(x,t)\sigma^+_{\alpha'}(0,0)\rangle_\rho &= \langle\sigma^+_\alpha\rangle_{\rho^\sharp}\,\langle\sigma^-_{\alpha'}\rangle_{\rho^\sharp}\,e^{-x\mathcal{E}}\left(1+O\left(e^{-mx|\mathrm{Im}(\sinh\theta_1^\star)|-mx|\mathrm{Im}(\sinh\theta_2^\star)|}\right)\right)\\
\langle\sigma^+_{\alpha\pm1,\alpha}(x,t)\sigma^+_{\alpha'\mp1,\alpha'}(0,0)\rangle_\rho &= O\left(e^{-x\mathcal{E}-mx|\mathrm{Im}(\sinh\theta^\star)|}\right) \qquad (7.16)
\end{aligned}$$

where exponential decays include possible algebraic or other non-exponential factors in mx determined by the type of singularities.

7.2 Thermal Gibbs state

The first application of the general results presented in the previous section can be made to the finite-temperature correlation functions. As was mentioned before, finite-temperature correlation functions are related to vacuum correlation functions on the circle. In this section, we will restrict ourselves to the Ising model and show the finite-temperature correlation function obtained from (7.7) indeed reproduce a form factor expansion of the vacuum expectation values on the circle, which can be seen as a verification of our Liouville-space method.

In case of the thermal Gibbs state, we rewrite (7.7) in the imaginary-time formalism by replacing t with $-i\tau$ and specialize in (7.7) $W(\theta)=\beta E_\theta$:

$$\begin{aligned}
\langle\omega^+(x,\tau)\omega^+(0,0)\rangle_{\rho_\beta} &= e^{-x\mathcal{E}_\beta}\sum_{N=0}^\infty\sum_{\epsilon_1,\ldots,\epsilon_N}\int\frac{d\theta_1\cdots d\theta_N}{N!}\frac{e^{\sum_{j=1}^N\left(i\epsilon_j p_{\theta_j}x-\epsilon_j E_{\theta_j}\tau\right)}}{\prod_{j=1}^N\left(1-e^{-\epsilon_j\beta E_{\theta_j}}\right)}\\
&\quad\times f^{\rho_\beta^\sharp;\omega^+}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N)f^{\rho_\beta^\sharp;\omega^-}_{-\epsilon_N,\ldots,-\epsilon_1}(\theta_N,\ldots,\theta_1) \qquad (7.17)
\end{aligned}$$

where the thermal "free energy deficit" $\mathcal{E}_\beta$ is given by

$$\mathcal{E}_\beta=\int_{-\infty}^\infty\frac{d\theta}{2\pi}m\cosh\theta\log\left(\coth\frac{\beta E_\theta}{2}\right). \qquad (7.18)$$

By re-arranging the rapidities and exchanging particles, we can rewrite the expansion (7.17) in the form

$$\begin{aligned}
&\langle\omega^{+}(x,\tau)\omega^{+}(0,0)\rangle_{\rho_\beta}\\
=\ & e^{-x\mathcal{E}_\beta}\sum_{N=0}^{\infty}\sum_{K=0}^{N}\int\frac{d\theta_1\cdots d\theta_N}{K!(N-K)!}\\
&\times\frac{e^{\sum_{j=1}^{K}\left(ip_{\theta_j}x-E_{\theta_j}\tau\right)-\sum_{K+1}^{N}\left(ip_{\theta_j}x-E_{\theta_j}\tau\right)}}{\prod_{j=1}^{K}\left(1-e^{-\beta E_{\theta_j}}\right)\prod_{j=K+1}^{N}\left(1-e^{\beta E_{\theta_j}}\right)}\\
&\times f^{\rho_\beta^{\sharp};\omega^{+}}_{+,\ldots,+,-,\ldots,-}(\theta_1,\ldots,\theta_K,\theta_{K+1},\ldots,\theta_N)\\
&\times f^{\rho_\beta^{\sharp};\omega^{-}}_{+,\ldots,+,-,\ldots,-}(\theta_N,\ldots,\theta_{K+1},\theta_K,\ldots,\theta_1)
\end{aligned}\tag{7.19}$$

where K represents the number of rapidities with positive charges. In order to obtain a convergent large-distance expansion, we can shift in (7.19) every θ_j-contour associated to the rapidity with $\epsilon_j=+$, towards the positive imaginary direction by $i\pi$, and we will not shift all the θ_k-contours associated to the rapidities with $\epsilon_k=-$. Using crossing symmetry, the integrand on the shifted contours in (7.19) becomes

$$(-1)^K\frac{e^{\sum_{j=1}^{N}\left(-ip_{\theta_j}x+E_{\theta_j}\tau\right)}}{\prod_{j=1}^{N}\left(1-e^{\beta E_{\theta_j}}\right)}f^{\rho_\beta^{\sharp};\omega^{+}}_{-,\ldots,-}(\theta_1\cdots\theta_N)f^{\rho_\beta^{\sharp};\omega^{-}}_{+,\ldots,+}(\theta_N\cdots\theta_1)$$

where $(-1)^K$ comes from shifting K rapidity variables by $i\pi$. In addition, when shifting the contours, we have to take into account residue contributions from the poles of the factors $\left(1-e^{-\beta E_{\theta_j}}\right)^{-1}$. Hence, we may replace every θ_j-integral by

$$\int d\theta_j\mapsto\begin{cases}\int d\theta_j & (\epsilon_j=-)\\ -\int d\theta_j+\text{residues} & (\epsilon_j=+).\end{cases}$$

Consider the terms which involve contributions from taking residues for P particles, from integrals with shifted contours for $K-P$ particles, and from integrals with non-shifted contours for $N-K$, with fixed N. By re-labeling rapidities and exchanging particles, the sum of these terms over K are given by

$$\sum_{K=P}^{N}\frac{1}{K!(N-K)!}\frac{K!}{(K-P)!P!}\sum_{n_1,\ldots,n_p\in\mathbb{Z}}(2\pi)^N\prod_{j=1}^{p}\left(\frac{e^{-mx\cosh\lambda_{n_j}-im\tau\sinh\lambda_{n_j}}}{m\beta\cosh\lambda_{n_j}}\right)$$
$$\times(-1)^{K-P}\int d\theta_{P+1}\cdots d\theta_N\frac{e^{-\sum_{j=P+1}^{N}(imx\sinh\theta_j-m\tau\cosh\theta_j)}}{\prod_{j=P+1}^{N}\left(1+e^{\beta E_{\theta_j}}\right)}$$
$$\times f^{\rho^\sharp_\beta;\omega^+}_{+,\ldots,+,-,\ldots,-}(\lambda_{n_1}+i\pi/2,\ldots,\lambda_{n_P}+i\pi/2,\theta_{P+1},\ldots,\theta_N)$$
$$\times f^{\rho^\sharp_\beta;\omega^-}_{+,\ldots,+,-,\ldots,-}(\theta_N,\ldots,\theta_{P+1},\lambda_{n_P}+i\pi/2,\ldots,\lambda_{n_1}+i\pi/2)$$

where

$$\lambda_n=\text{arcsinh}\left(\frac{2\pi n}{\beta m}\right), \tag{7.20}$$

where we evaluate the residues of poles at position $\theta_j=\lambda_{n_j}$ for all $n_j\in\mathbb{Z}$, $j=1,\ldots,P$, where the factor $(-1)^{K-P}$ comes from shifting $K-P$ rapidities and using the crossing relation, where on the third line, there are P positive charges and $N-P$ negative charges, and where on the fourth line, there are $N-P$ positive charges and P negative charges. It turns out that the sum over K vanishes whenever $P\neq N$, namely only the residues contributions are left. Thus, we obtain the large-distance expansion

$$\begin{aligned}&\langle\omega^+(x,\tau)\omega^+(0,0)\rangle_{\rho_\beta}\\ &= e^{-x\mathcal{E}_\beta}\sum_{N}^{\infty}\frac{1}{N!}\sum_{n_1,\ldots,n_p\in\mathbb{Z}}e^{\sum_{j=1}^{N}\left(-mx\cosh\lambda_{n_j}-n_j\frac{2\pi i\tau}{\beta}\right)}\prod_{j=1}^{N}\left(\frac{2\pi}{m\beta\cosh\lambda_{n_j}}\right)\\ &\quad\times f^{\rho^\sharp_\beta;\omega^+}_{+,\ldots,+}(\lambda_{n_1}+i\pi/2,\ldots,\lambda_{n_N}+i\pi/2)f^{\rho^\sharp_\beta;\omega^-}_{-,\ldots,-}(\lambda_{n_N}+i\pi/2,\ldots,\lambda_{n_1}+i\pi/2)\end{aligned} \tag{7.21}$$

for $\omega=\sigma$ or $\omega=\mu$. As we see, this large-distance expansion is in the same form with that of the correlation function on the circle (5.15). In light of the relations between finite-temperature form factors and form factors on the circle [23],

$$_\beta\langle\text{vac}_{\frac{1}{2}}|\omega^+_\beta(0)|n_1,\ldots,n_N\rangle_\beta=e^{\frac{i\pi s}{2}}\prod_{j=1}^{N}\sqrt{\frac{2\pi}{m\beta\cosh\lambda_{n_j}}}f^{\rho^\sharp_\beta;\omega^+}_{+,\ldots,+}(\lambda_{n_1}+i\pi/2,\ldots,\lambda_{n_N}+i\pi/2) \tag{7.22}$$

and

$$_\beta\langle n_1,\ldots,n_N|\omega^-_\beta(0)|\text{vac}_{\frac{1}{2}}\rangle_\beta=e^{\frac{i\pi s}{2}}\prod_{j=1}^{N}\sqrt{\frac{2\pi}{m\beta\cosh\lambda_{n_j}}}f^{\rho^\sharp_\beta;\omega^-}_{-,\ldots,-}(\lambda_{n_1}+i\pi/2,\ldots,\lambda_{n_N}+i\pi/2) \tag{7.23}$$

where $|\text{vac}_{\frac{1}{2}}\rangle_\beta$ represents the vacuum in the NS sector and the states $|n_1,\ldots,n_N\rangle_\beta$ are in the R sector, and where $\sinh \lambda_{n_j} = \frac{2\pi n_j}{m\beta}$ for $n_j \in \mathbb{Z}$, we can explicitly show that (7.21) indeed reproduce the uncontroversial large-distance correlation function on the circle:

$$\begin{aligned}
&\langle \omega^+(x,\tau)\omega^+(0,0)\rangle_{\rho_\beta} \\
&= e^{i\pi s} e^{-x\mathcal{E}_\beta} \sum_{N}^{\infty} \frac{1}{N!} \sum_{n_1,\ldots,n_p \in \mathbb{Z}} e^{\sum_{j=1}^{N}\left(-mx\cosh\lambda_{n_j} - n_j \frac{2\pi i \tau}{\beta}\right)} \prod_{j=1}^{N} \left(\frac{2\pi}{m\beta\cosh\lambda_{n_j}}\right) \\
&\quad \times {}_\beta\langle \mathrm{vac}_{\frac{1}{2}}|\omega_\beta^+(0)|n_1,\ldots,n_N\rangle_{\beta\beta}\langle n_1,\ldots,n_N|\omega_\beta^-(0)|\mathrm{vac}_{\frac{1}{2}}\rangle_\beta. \qquad (7.24)
\end{aligned}$$

From this, we see that the finite-temperature correlation function $\langle \omega^+(x,\tau)\omega^+(0,0)\rangle_{\rho_\beta}$ can be interpreted as corresponding to the correlation function in the quantization on the circle, where the natural sector is not the NS sector as usual but the R sector, and where the insertion of two twist fields with cuts in opposite directions change the sectors of both vacua to the NS sector without affecting the excited states. In this sense, the Z operator can be seen as playing the role in not only changing the twist field ω^+ to ω^- but also changing the natural sector given by the trace from the NS sector to the R sector. Finally, it is worth mentioning that the result (7.24) serves as a strong verification of our Liouville-space method.

7.3 Non-equilibrium steady state

A non-equilibrium steady state (NESS) can be seen as a state in which the dynamics of the model are time-independent but characterized by the presence of flows of energy, particles, charge, etc with constant rate. Non-equilibrium steady states can be obtained by a long-time unitary evolution of two semi-infinite halves of a system initially separately thermalized at different temperatures β_l^{-1} and β_r^{-1}. The investigation of this state has been carried out in critical systems [37, 38] using conformal field theory, systems near to criticality [39] using general massive quantum field theory, and in [69] using integrability, and in the XY quantum chain (Ising model) in [36], [40]. The density matrix describing the non-equilibrium steady state in the Ising model admits a factorized form. In terms of the notations in this thesis, this result can be presented by specializing the function $W(\theta)$ as

$$W_{\mathrm{ness}}(\theta) := \beta_l E_\theta \Theta(\theta) + \beta_r E_\theta \Theta(-\theta) \qquad (7.25)$$

where $\Theta(x)$ is the Heavyside step function (we denote by ρ_{ness} the associated density matrix). Associating an inverse temperature β_l to right-moving particles, and an inverse temperature β_r to left-moving ones, the result (7.25) can be justified by the physical situation that right-moving particles, which was thermalized at temperature β_l^{-1}, come from the far left, and vice versa.

7.3.1 Analytic properties of non-equilibrium form factors

In chapter 5, we deduced a set of analytic properties of the finite-temperature form factors of twist fields by using the KMS relations. In the case of non-equilibrium steady state, we can expect a Riemann-Hilbert problem for non-equilibrium form factors of twist fields in the same spirit as the one for the finite-temperature form factors of twist fields.

We focus for now the Ising model. We derive first the KMS relations for our non-equilibrium steady state. Consider the two-point function in imaginary-time formalism in the Ising model

$$g_{\text{ness}}(x,\tau) = {}^{\rho_{\text{ness}}}\langle\text{vac}|\,\mu^{+}(0,0)^{\ell}\psi(x,\tau)^{\ell}|\text{vac}\rangle^{\rho_{\text{ness}}} = \text{Tr}\left(\rho_{\text{ness}}\,\mu^{+}(0,0)\psi(x,\tau)\right).$$

where we concentrate, without loss of generality, on the twist fields with branch cuts on the right. By defining non-local fermion operators

$$\begin{aligned}
\psi^{l,r}(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}}\int_{\theta\gtrless 0} d\theta\, e^{\theta/2}\left(a(\theta)\,e^{ip_\theta x - E_\theta\tau} + a^\dagger(\theta)\,e^{-ip_\theta x + E_\theta \tau}\right) \\
\bar\psi^{l,r}(x,\tau) &= -\frac{i}{2}\sqrt{\frac{m}{\pi}}\int_{\theta\gtrless 0} d\theta\, e^{-\theta/2}\left(a(\theta)\,e^{ip_\theta x - E_\theta\tau} - a^\dagger(\theta)\,e^{-ip_\theta x + E_\theta \tau}\right), \qquad (7.26)
\end{aligned}$$

we have

$$\rho_{\text{ness}}\psi(x,\tau)\rho_{\text{ness}}^{-1} = \psi^{l}(x,\tau-\beta_l) + \psi^{r}(x,\tau-\beta_r)\,. \qquad (7.27)$$

Using the exchange relations (3.21) with disorder field μ^+, (7.27), and the cyclic property of the trace, we obtain the "non-equilibrium KMS relation" or the "generalized KMS relation"

$$\begin{aligned}
g_{\text{ness}}(x,\tau) &= -\left(g^{l}_{\text{ness}}(x,\tau-\beta_l) + g^{r}_{\text{ness}}(x,\tau-\beta_r)\right) \qquad (x<0) \qquad (7.28)\\
g_{\text{ness}}(x,\tau) &= \left(g^{l}_{\text{ness}}(x,\tau-\beta_l) + g^{r}_{\text{ness}}(x,\tau-\beta_r)\right) \qquad (x>0) \qquad (7.29)
\end{aligned}$$

where we denote $g^{l,r}_{\text{ness}}(x,\tau) = {}^{\rho_{\text{ness}}}\langle\text{vac}|\,\mu^{+}(0,0)^{\ell}\psi^{l,r}(x,\tau)^{\ell}|\text{vac}\rangle^{\rho_{\text{ness}}}$. Although (7.28) and (7.29) are a non-local relations, they can still be used to establish analytic properties of form factors. To see this, we expand the fermion fields in $g_{\text{ness}}(x,\tau)$ by recalling the mode expansion (2.83),

$$\begin{aligned}
g_{\text{ness}}(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}}\left[\int_{-\infty}^{0} d\theta\, e^{\frac{\theta}{2}}\frac{f_{+}^{\rho_{\text{ness}};\mu^+}(\theta)}{1+e^{-\beta_r E_\theta}}e^{-ip_\theta x + E_\theta\tau} + \int_{0}^{\infty} d\theta\, e^{\frac{\theta}{2}}\frac{f_{+}^{\rho_{\text{ness}};\mu^+}(\theta)}{1+e^{-\beta_l E_\theta}}e^{-ip_\theta x + E_\theta\tau}\right.\\
&\left.+\int_{-\infty}^{0} d\theta\, e^{\frac{\theta}{2}}\frac{f_{-}^{\rho_{\text{ness}};\mu^+}(\theta)}{1+e^{\beta_r E_\theta}}e^{ip_\theta x - E_\theta\tau} + \int_{0}^{\infty} d\theta\, e^{\frac{\theta}{2}}\frac{f_{-}^{\rho_{\text{ness}};\mu^+}(\theta)}{1+e^{\beta_l E_\theta}}e^{ip_\theta x - E_\theta\tau}\right], \quad (7.30)
\end{aligned}$$

and then the non-local fermion fields in $g^l_{\rm ness}(x,\tau-\beta_l)+g^r_{\rm ness}(x,\tau-\beta_r)$ by the definition (7.26),

$$\begin{aligned}
&g^l_{\rm ness}(x,\tau-\beta_l)+g^r_{\rm ness}(x,\tau-\beta_r)\\
&= \frac{1}{2}\sqrt{\frac{m}{\pi}}\left[\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}\frac{f_+^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{-\beta_r E_\theta}}e^{-ip_\theta x+E_\theta(\tau-\beta_r)}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}\frac{f_+^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{-\beta_l E_\theta}}e^{-ip_\theta x+E_\theta(\tau-\beta_l)}\right.\\
&\quad\left.+\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}\frac{f_-^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{\beta_r E_\theta}}e^{ip_\theta x-E_\theta(\tau-\beta_r)}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}\frac{f_-^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{\beta_l E_\theta}}e^{ip_\theta x-E_\theta(\tau-\beta_l)}\right]. \qquad (7.31)
\end{aligned}$$

Let us define

$$g^r_\epsilon(\theta)=\frac{f_\epsilon^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{-\epsilon\beta_r E_\theta}} \text{ for } \theta<0, \qquad g^l_\epsilon(\theta)=\frac{f_\epsilon^{\rho_{\rm ness};\mu^+}(\theta)}{1+e^{-\epsilon\beta_l E_\theta}} \text{ for } \theta>0. \qquad (7.32)$$

Then (7.30) and (7.31) become

$$\begin{aligned}
g_{\rm ness}(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}}\left[\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}g^r_+(\theta)e^{-ip_\theta x+E_\theta\tau}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}g^l_+(\theta)e^{-ip_\theta x+E_\theta\tau}\right.\\
&\quad\left.+\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}g^r_-(\theta)e^{ip_\theta x-E_\theta\tau}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}g^l_-(\theta)e^{ip_\theta x-E_\theta\tau}\right] \qquad (7.33)
\end{aligned}$$

and

$$\begin{aligned}
&g^l_{\rm ness}(x,\tau-\beta_l)+g^r_{\rm ness}(x,\tau-\beta_r)\\
&= \frac{1}{2}\sqrt{\frac{m}{\pi}}\left[\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}g^r_+(\theta)e^{-ip_\theta x+E_\theta(\tau-\beta_r)}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}g^l_+(\theta)e^{-ip_\theta x+E_\theta(\tau-\beta_l)}\right.\\
&\quad\left.+\int_{-\infty}^0 d\theta\, e^{\frac{\theta}{2}}g^r_-(\theta)e^{ip_\theta x-E_\theta(\tau-\beta_r)}+\int_0^\infty d\theta\, e^{\frac{\theta}{2}}g^l_-(\theta)e^{ip_\theta x-E_\theta(\tau-\beta_l)}\right], \qquad (7.34)
\end{aligned}$$

respectively. For $x<0$, we shift, in (7.33) and (7.34), the θ-contours in the terms containing e^{-ip_θ} as $\theta\to\theta+i\pi/2$ and in the terms containing e^{ip_θ} as $\theta\to\theta-i\pi/2$. By assuming functions $g^r_\epsilon(\theta)$ have poles at $\theta=\lambda_{n_r}+\epsilon i\pi/2$ and $g^l_\epsilon(\theta)$ have poles at $\theta=\lambda_{n_l}+\epsilon i\pi/2$ and taking care of the contours surrounding the segment from 0 to $\pm i\pi/2$, we have

$$
\begin{aligned}
g_{\rm ness}(x,\tau) &= \frac{1}{2}\sqrt{\frac{m}{\pi}}\sum_\epsilon\left[\int_{-\infty}^0 d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^r(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta\tau}\right.\\
&\quad+\int_0^\infty d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^l(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta\tau}\\
&\quad+\int_{\epsilon i\pi/2}^0 d\theta\, e^{\theta/2}\left[g_\epsilon^r(\theta-0^+)-g_\epsilon^l(\theta+0^+)\right] e^{-\epsilon ip_\theta x+\epsilon E_\theta\tau}\\
&\quad+\epsilon\sum_n i\pi\,{\rm Res}\left(g_\epsilon^r(\theta),\lambda_n^r+\epsilon i\pi/2\right) e^{\lambda_n^r/2+\epsilon i\pi/4} e^{m\cosh\lambda_{n_l}x+i\sinh\lambda_{n_l}\tau}\\
&\quad\left.+\epsilon\sum_n i\pi\,{\rm Res}\left(g_\epsilon^l(\theta),\lambda_n^l+\epsilon i\pi/2\right) e^{\lambda_n^l/2+\epsilon i\pi/4} e^{m\cosh\lambda_{n_l}x+i\sinh\lambda_{n_l}\tau}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&g^l_{\rm ness}(x,\tau-\beta_l)+g^r_{\rm ness}(x,\tau-\beta_r)\\
&= \frac{1}{2}\sqrt{\frac{m}{\pi}}\sum_\epsilon\left[\int_{-\infty}^0 d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^r(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta(\tau-\beta_r)}\right.\\
&\quad+\int_0^\infty d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^l(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta(\tau-\beta_l)}\\
&\quad+\int_{\epsilon i\pi/2}^0 d\theta\, e^{\theta/2}\left[g_\epsilon^r(\theta-0^+)e^{-\epsilon E_\theta\beta_r}-g_\epsilon^l(\theta+0^+)e^{-\epsilon E_\theta\beta_l}\right] e^{-\epsilon ip_\theta x+\epsilon E_\theta\tau}\\
&\quad+\epsilon\sum_n i\pi\,{\rm Res}\left(g_\epsilon^r(\theta),\lambda_n^r+\epsilon i\pi/2\right) e^{\lambda_n^r/2+\epsilon i\pi/4} e^{m\cosh\lambda_{n_r}x+i\sinh\lambda_{n_r}(\tau-\beta_r)}\\
&\quad\left.+\epsilon\sum_n i\pi\,{\rm Res}\left(g_\epsilon^l(\theta),\lambda_n^l+\epsilon i\pi/2\right) e^{\lambda_n^l/2+\epsilon i\pi/4} e^{m\cosh\lambda_{n_l}x+i\sinh\lambda_{n_l}(\tau-\beta_l)}\right].
\end{aligned}
$$

The equality in the non-equilibrium KMS relation (7.28) requires that functions $g_\epsilon^r(\theta)$ have poles at $\theta=\lambda_{n_r}+\epsilon i\pi/2$ for $n_r\in\mathbb{Z}+1/2$, $n_r<0$ and $g_\epsilon^l(\theta)$ have poles at $\theta=\lambda_{n_l}+\epsilon i\pi/2$ for $n_l\in\mathbb{Z}+1/2$, $n_l>0$, where

$$
\sinh\lambda_{n_r}=\frac{2\pi n_r}{m\beta_r},\quad \sinh\lambda_{n_l}=\frac{2\pi n_l}{m\beta_l}. \tag{7.35}
$$

Further, the equality in (7.28) requires the shifted contours to cancel each other:

$$
\begin{aligned}
&\sum_\epsilon\left[\int_{-\infty}^0 d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^r(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta\tau}\right]\\
&= -\sum_\epsilon\left[\int_{-\infty}^0 d\theta\, e^{\theta/2+\epsilon i\pi/4} g_\epsilon^r(\theta+\epsilon i\pi/2) e^{E_\theta x+ip_\theta(\tau-\beta_r)}\right]
\end{aligned}
$$

and

$$\begin{aligned}&\sum_\epsilon \left[\int_0^\infty d\theta\, e^{\theta/2+\epsilon i\pi/4} g^l_\epsilon(\theta+\epsilon i\pi/2) e^{E_\theta x + i p_\theta \tau}\right] \\ = &-\sum_\epsilon \left[\int_0^\infty d\theta\, e^{\theta/2+\epsilon i\pi/4} g^l_\epsilon(\theta+\epsilon i\pi/2) e^{E_\theta x + i p_\theta (\tau-\beta_l)}\right].\end{aligned}$$

This leads to the condition

$$e^{i\pi/4} g_+^{r,l}\left(\theta + \frac{i\pi}{2}\right) + e^{-i\pi/4} g_-^{r,l}\left(\theta - \frac{i\pi}{2}\right) = 0 \tag{7.36}$$

except for the poles at $\theta = \lambda_{n_{r,l}}$. Finally, the equality requires the contours surrounding the segment from $\epsilon i\pi/2$ to 0 to cancel each other

$$\begin{aligned}&\int_{\epsilon i\pi/2}^0 d\theta\, e^{\theta/2} \left[g^r_\epsilon(\theta - 0^+) - g^l_\epsilon(\theta+0^+)\right] e^{-\epsilon i p_\theta x + \epsilon E_\theta \tau} \\ = &-\int_{\epsilon i\pi/2}^0 d\theta\, e^{\theta/2} \left[g^r_\epsilon(\theta - 0^+) e^{-\epsilon E_\theta \beta_r} - g^l_\epsilon(\theta+0^+) e^{-\epsilon E_\theta \beta_l}\right] e^{-\epsilon i p_\theta x + \epsilon E_\theta \tau}.\end{aligned}$$

This gives

$$g^r_\epsilon(\theta - 0^+) - g^l_\epsilon(\theta + 0^+) = -g^r_\epsilon(\theta - 0^+) e^{-\epsilon E_\theta \beta_r} + g^l_\epsilon(\theta+0^+) e^{-\epsilon E_\theta \beta_l}$$

which implies

$$\frac{g^r_\epsilon(\theta - 0^+)}{g^l_\epsilon(\theta+0^+)} = \frac{1 + e^{-\epsilon E_\theta \beta_l}}{1 + e^{-\epsilon E_\theta \beta_r}}. \tag{7.37}$$

with $\theta \in [0, \epsilon i\pi/2]$. For $x > 0$, we shift, in (7.33) and (7.34), the θ-contours in the terms containing e^{-ip_θ} as $\theta \to \theta - i\pi/2$ and in the terms containing e^{ip_θ} as $\theta \to \theta + i\pi/2$. By taking residues of poles and taking care of the contours surrounding the segment from 0 to $\pm i\pi/2$, we have

$$\begin{aligned} g_{\text{ness}}(x,\tau) = \frac{1}{2}\sqrt{\frac{m}{\pi}} \sum_\epsilon \Bigg[&\int_{-\infty}^0 d\theta\, e^{\theta/2+\epsilon i\pi/4} g^r_{-\epsilon}(\theta + \epsilon i\pi/2) e^{-E_\theta x - i p_\theta \tau} \\ &+ \int_0^\infty d\theta\, e^{\theta/2+\epsilon i\pi/4} g^l_{-\epsilon}(\theta+\epsilon i\pi/2) e^{-E_\theta x - i p_\theta \tau} \\ &+ \int_{\epsilon i\pi/2}^0 d\theta\, e^{\theta/2}\left[g^r_{-\epsilon}(\theta - 0^+) - g^l_{-\epsilon}(\theta+0^+)\right] e^{\epsilon i p_\theta x - \epsilon E_\theta \tau} \\ &+ \epsilon \sum_n i\pi\, \text{Res}\left(g^r_{-\epsilon}(\theta), \lambda^r_n + \epsilon i\pi/2\right) e^{\lambda^r_n/2 + \epsilon i\pi/4} e^{-m\cosh\lambda_{n_l} x - im\sinh\lambda_{n_l}\tau} \\ &+ \epsilon \sum_n i\pi\, \text{Res}\left(g^l_{-\epsilon}(\theta), \lambda^l_n + \epsilon i\pi/2\right) e^{\lambda^l_n/2 + \epsilon i\pi/4} e^{-m\cosh\lambda_{n_l} x - im\sinh\lambda_{n_l}\tau}\Bigg]\end{aligned}$$

and

$$
\begin{aligned}
&g^{l}_{\text{ness}}(x,\tau-\beta_l)+g^{r}_{\text{ness}}(x,\tau-\beta_r)\\
&= \frac{1}{2}\sqrt{\frac{m}{\pi}}\sum_{\epsilon}\Bigg[\int_{-\infty}^{0} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{r}_{\epsilon}(\theta+\epsilon i\pi/2)e^{E_\theta x+ip_\theta(\tau-\beta_r)}\\
&\qquad +\int_{0}^{\infty} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{l}_{\epsilon}(\theta+\epsilon i\pi/2)e^{E_\theta x+ip_\theta(\tau-\beta_l)}\\
&\qquad +\int_{\epsilon i\pi/2}^{0} d\theta\, e^{\theta/2}\left[g^{r}_{-\epsilon}(\theta-0^+)e^{\epsilon E_\theta\beta_r}-g^{l}_{-\epsilon}(\theta+0^+)e^{\epsilon E_\theta \beta_l}\right]e^{\epsilon i p_\theta x-\epsilon E_\theta \tau}\\
&\qquad +\epsilon\sum_{n} i\pi\,\text{Res}\left(g^{r}_{-\epsilon}(\theta),\lambda_{n_r}+\epsilon i\pi/2\right)e^{\lambda^{r}_{n}/2+\epsilon i\pi/4}e^{-m\cosh\lambda_{n_r}x-im\sinh\lambda_{n_r}(\tau-\beta_r)}\\
&\qquad +\epsilon\sum_{n} i\pi\,\text{Res}\left(g^{l}_{-\epsilon}(\theta),\lambda_{n_l}+\epsilon i\pi/2\right)e^{\lambda_{n_l}/2+\epsilon i\pi/4}e^{-m\cosh\lambda_{n_l}x-im\sinh\lambda_{n_l}(\tau-\beta_l)}\Bigg].
\end{aligned}
$$

Likewise, the equality in the non-equilibrium KMS relation (7.29) requires that functions $g^{r}_{\epsilon}(\theta)$ have poles at $\theta=\lambda_{n_r}-\epsilon i\pi/2$ for $n_r\in\mathbb{Z}$, $n_r<0$ and $g^{l}_{\epsilon}(\theta)$ have poles at $\theta=\lambda_{n_l}-\epsilon i\pi/2$ for $n_l\in\mathbb{Z}$, $n_l>0$. Further, the equality in (7.29) requires the shifted contours to cancel each other:

$$
\begin{aligned}
&\sum_{\epsilon}\left[\int_{-\infty}^{0} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{r}_{-\epsilon}(\theta+\epsilon i\pi/2)e^{-E_\theta x-ip_\theta \tau}\right]\\
=\;&\sum_{\epsilon}\left[\int_{-\infty}^{0} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{r}_{-\epsilon}(\theta+\epsilon i\pi/2)e^{-E_\theta x-ip_\theta(\tau-\beta_r)}\right]
\end{aligned}
$$

and

$$
\begin{aligned}
&\sum_{\epsilon}\left[\int_{0}^{\infty} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{l}_{-\epsilon}(\theta+\epsilon i\pi/2)e^{-E_\theta x-ip_\theta \tau}\right]\\
=\;&\sum_{\epsilon}\left[\int_{0}^{\infty} d\theta\, e^{\theta/2+\epsilon i\pi/4} g^{l}_{-\epsilon}(\theta+\epsilon i\pi/2)e^{-E_\theta x-ip_\theta(\tau-\beta_l)}\right].
\end{aligned}
$$

This leads to the condition

$$
e^{-i\pi/4}g^{r,l}_{+}\left(\theta-\frac{i\pi}{2}\right)+e^{i\pi/4}g^{r,l}_{-}\left(\theta+\frac{i\pi}{2}\right)=0 \tag{7.38}
$$

except for the poles at $\theta=\lambda_{n_{r,l}}$. Finally, the equality requires the contours surrounding the segment from $\epsilon i\pi/2$ to 0 to cancel each other

$$
\begin{aligned}
&\int_{\epsilon i\pi/2}^{0} d\theta\, e^{\theta/2}\left[g^{r}_{-\epsilon}(\theta-0^+)-g^{l}_{-\epsilon}(\theta+0^+)\right]e^{\epsilon i p_\theta x-\epsilon E_\theta\tau}\\
=\;&\int_{\epsilon i\pi/2}^{0} d\theta\, e^{\theta/2}\left[g^{r}_{-\epsilon}(\theta-0^+)e^{\epsilon E_\theta\beta_r}-g^{l}_{-\epsilon}(\theta+0^+)e^{\epsilon E_\theta\beta_l}\right]e^{\epsilon i p_\theta x-\epsilon E_\theta\tau}.
\end{aligned}
$$

This gives

$$g^r_\epsilon(\theta - 0^+) - g^l_\epsilon(\theta + 0^+) = g^r_\epsilon(\theta - 0^+)e^{-\epsilon E_\theta \beta_r} - g^l_\epsilon(\theta + 0^+)e^{-\epsilon E_\theta \beta_l}$$

which implies

$$\frac{g^r_\epsilon(\theta - 0^+)}{g^l_\epsilon(\theta + 0^+)} = \frac{1 - e^{-\epsilon E_\theta \beta_l}}{1 - e^{-\epsilon E_\theta \beta_r}}. \tag{7.39}$$

with $\theta \in [0, -\epsilon i\pi/2]$.

By recalling the definition (7.32), we obtain a set of analytic conditions for the non-equilibrium one-particle form factors of the disorder field μ^+:

- In the region $\text{Im} \in [0, \epsilon i\pi]$, $f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta)$ has no poles; in the region $\text{Im} \in [0, -\epsilon i\pi]$, $f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta)$ has poles at $\theta = \lambda_{n_r} - \epsilon i\pi/2$ for $n_r \in \mathbb{Z}$, $n_r < 0$ and at $\theta = \lambda_{n_l} - \epsilon i\pi/2$ for $n_l \in \mathbb{Z}$, $n_l > 0$, and has zeros at $\theta = \lambda_{n_r} - \epsilon i\pi/2$ for $n_r \in \mathbb{Z} + 1/2$, $n_r < 0$ and at $\theta = \lambda_{n_l} - \epsilon i\pi/2$ for $n_l \in \mathbb{Z} + 1/2$, $n_l > 0$, where $\sinh \lambda_{n_r} = \frac{2\pi n_r}{m\beta_r}$ and $\sinh \lambda_{n_l} = \frac{2\pi n_l}{m\beta_l}$.

- $\frac{f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta - 0^+)}{f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta + 0^+)} = 1$ for $\theta \in [0, \epsilon i\pi/2]$, which imply that $f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta)$ are continuous through the segment $[0, \epsilon i\pi/2]$; $\frac{f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta - 0^+)}{f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta + 0^+)} = \left(\frac{1+e^{-\beta_r E_\theta}}{1-e^{-\beta_r E_\theta}}\right)\left(\frac{1-e^{-\beta_l E_\theta}}{1+e^{-\beta_l E_\theta}}\right)$ for $\theta \in [0, -\epsilon i\pi/2]$, which imply that $f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta)$ are discontinuous through the segment $[0, -\epsilon i\pi/2]$ and they have branch cuts running from 0 to $-\epsilon i\pi/2$.

- $\text{Re}\left(e^{i\pi/4} f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta + i\pi/2)\right) = 0$ except at $\theta = \lambda_{n_r}, \lambda_{n_l}$ for $n_r, n_l \in \mathbb{Z} + 1/2$, $n_r < 0, n_l > 0$ and $\text{Re}\left(e^{-i\pi/4} f^{\rho_{\text{ness}};\mu^+}_\epsilon(\theta - i\pi/2)\right) = 0$ except at $\theta = \lambda_{n_r}, \lambda_{n_l}$ for $n_r, n_l \in \mathbb{Z}$, $n_r < 0, n_l > 0$.

7.3.2 Large-distance expansion of two-point correlation functions

In case of non-equilibrium steady state, we specialize in (7.7) $W(\theta) = W_{\text{ness}}(\theta)$:

$$\begin{aligned}\langle \omega^+(x,t)\omega^+(0,0)\rangle_{\rho_{\text{ness}}} &= e^{-x\mathcal{E}_{\text{ness}}} \sum_{N=0}^{\infty} \sum_{\epsilon_1,\ldots,\epsilon_N} \int \frac{d\theta_1 \cdots d\theta_N}{N!} \\ &\quad \times \frac{e^{\sum_{j=1}^N \left(i\epsilon_j p_{\theta_j} x - \epsilon_j E_{\theta_j} \tau\right)}}{\prod_{j=1}^N \left(1 - e^{-\epsilon_j W_{\text{ness}}(\theta_j)}\right)} \\ &\quad \times f^{\rho^\sharp_{\text{ness}};\omega^+}_{\epsilon_1,\ldots,\epsilon_N}(\theta_1,\ldots,\theta_N) f^{\rho^\sharp_{\text{ness}};\omega^-}_{-\epsilon_N,\ldots,-\epsilon_1}(\theta_N,\ldots,\theta_1)\end{aligned} \tag{7.40}$$

where the non-equilibrium "free energy deficit" $\mathcal{E}_{\text{ness}}$ can be written as the average of the free energy deficits $\mathcal{E}_{\beta_{r,l}}$ associated to equilibrium thermal density matrices $\rho_{\beta_{r,l}}$,

$$\mathcal{E}_{\text{ness}} = \frac{1}{2}\left(\mathcal{E}_{\beta_l} + \mathcal{E}_{\beta_r}\right), \quad \mathcal{E}_{\beta_{r,l}} := \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} m \cosh\theta \log\left(\coth \frac{m\beta_{r,l}\cosh(\theta)}{2}\right). \tag{7.41}$$

By re-arranging the rapidities and exchanging particles, we can bring the expansion (7.40) in the form

$$\begin{aligned}
&\langle \omega^+(x,t)\omega^+(0,0)\rangle_{\rho_{\text{ness}}} \\
= \; & e^{-x\mathcal{E}_{\text{ness}}} \sum_{N=0}^{\infty}\sum_{K=0}^{N} \int \frac{d\theta_1\cdots d\theta_N}{K!(N-K)!} \\
&\times \frac{e^{\sum_{j=1}^{K}\left(ip_{\theta_j}x - E_{\theta_j}\tau\right) - \sum_{K+1}^{N}\left(ip_{\theta_j}x - E_{\theta_j}\tau\right)}}{\prod_{j=1}^{K}\left(1 - e^{-W_{\text{ness}}(\theta_j)}\right)\prod_{j=K+1}^{N}\left(1 - e^{W_{\text{ness}}(\theta_j)}\right)} \\
&\times f^{\rho^\sharp_{\text{ness}};\omega^+}_{+,\ldots,+,-,\ldots,-}(\theta_1,\ldots,\theta_K,\theta_{K+1},\ldots,\theta_N) \\
&\times f^{\rho^\sharp_{\text{ness}};\omega^-}_{+,\ldots,+,-,\ldots,-}(\theta_N,\ldots,\theta_{K+1},\theta_K,\ldots,\theta_1)
\end{aligned} \tag{7.42}$$

where K represents the number of rapidities with positive charges. In order to obtain a convergent large-distance expansion, we can shift in (7.42) every θ_j-contour associated to the rapidity with $\epsilon_j = +$, towards the positive imaginary direction by $i\pi$, and we will not shift all the θ_k-contours associated to the rapidities with $\epsilon_k = -$. Using crossing symmetry, the integrand on the shifted contours in (7.42) becomes

$$(-1)^K \frac{e^{\sum_{j=1}^{N}\left(ip_{\theta_j}x - E_{\theta_j}\tau\right)}}{\prod_{j=1}^{N}\left(1 - e^{-W_{\text{ness}}(\theta_j)}\right)} f^{\rho^\sharp_{\text{ness}};\omega^+}_{-,\ldots,-}(\theta_1\cdots\theta_N) f^{\rho^\sharp_{\text{ness}};\omega^-}_{+,\ldots,+}(\theta_N\cdots\theta_1)$$

where $(-1)^K$ comes from shifting K rapidity variables by $i\pi$. Further, when shifting the contours, we have to take into account not only residue contributions from the poles of the factors $\left(1 - e^{-W_{\text{ness}}(\theta_j)}\right)^{-1}$, but also the integrals running on both sides of the segment $\theta \in [0, i\pi]$ due to discontinuity of $W_{\text{ness}}(\theta_j)$ at the point $\theta_j = 0$. Hence, we may replace every θ_j-integral by

$$\int d\theta_j \mapsto \begin{cases} \displaystyle\int d\theta_j & (\epsilon_j = -) \\ \displaystyle -\int d\theta_j + \text{residues} + \text{imaginary segment} & (\epsilon_j = +). \end{cases}$$

Consider the terms which involve contributions from taking residues for P particles, from integrating on imaginary segments for M particles, from integrals with shifted contours for $K - P - M$ particles, and from integrals with non-shifted contours for $N - K$, with fixed N. By re-labeling rapidities and exchanging particles, the sum of these terms over K are given by

$$
\begin{aligned}
&\sum_{K=P+M}^{N} \frac{1}{K!(N-K)!}\frac{K!}{(K-P)!P!}\frac{(K-P)!}{M!(K-P-M)!} \\
&\quad \times \sum_{n_1,\dots,n_p\in\mathbb{Z}/\{0\}} (2\pi)^N \prod_{j=1}^{p}\left(\frac{e^{-mx\cosh\lambda_{n_j}-im\tau\sinh\lambda_{n_j}}}{m\beta(n_j)\cosh\lambda_{n_j}}\right) \\
&\times \int_0^{i\pi} d\theta_{P+1}\cdots d\theta_{P+M} \prod_{j=P+1}^{P+M}\left(e^{imx\sinh\theta_j-m\tau\cosh\theta_j}\right) \\
&\quad \times \left(\frac{1}{1-e^{-m\beta_l\cosh(\theta+0^+)}}-\frac{1}{1-e^{-m\beta_r\cosh(\theta-0^+)}}\right) \\
&\times(-1)^{K-P-M}\int d\theta_{P+M+1}\cdots d\theta_N \frac{e^{-\sum_{j=P+M+1}^{N}(imx\sinh\theta_j-m\tau\cosh\theta_j)}}{\prod_{j=P+M+1}^{N}\left(1+e^{W_{\rm ness}(\theta_j)}\right)} \\
&\times f^{\rho^\sharp_{\rm ness};\omega^+}_{+,\dots,+,-,\dots,-}(\lambda_{n_1}+i\pi/2,\dots,\lambda_{n_P}+i\pi/2,\theta_{P+1},\dots,\theta_{P+M},\theta_{P+M+1},\dots,\theta_N) \\
&\times f^{\rho^\sharp_{\rm ness};\omega^-}_{+,\dots,+,-,\dots,-}(\theta_N,\dots,\theta_{P+M+1},\theta_{P+M},\dots,\theta_{P+1},\lambda_{n_P}+i\pi/2,\dots,\lambda_{n_1}+i\pi/2)
\end{aligned}
$$

where

$$
\lambda_n = \mathrm{arcsinh}\left(\frac{2\pi n}{\beta(n)m}\right), \quad \beta(n)=\begin{cases}\beta_l & (n>0)\\ \beta_r & (n<0)\end{cases}, \tag{7.43}
$$

where we evaluate the residues of poles at position $\theta_j=\lambda_{n_j}$ for all $n_j\in\mathbb{Z}/0$, $j=1,\dots,P$, where the factor $(-1)^{K-P-M}$ comes from shifting $K-P-M$ rapidities and using the crossing relation, where on the sixth line, there are $P+M$ positive charges and $N-P-M$ negative charges, and where on the seventh line, there are $N-P-M$ positive charges and $P+M$ negative charges. It turns out that the sum over K vanishes whenever $P+M\neq N$, namely only the residues and imaginary segments contributions are left. After changing the imaginary segments contributions to integrations over real invariables, we exchange particles, putting the extra combinatorial factor $N!/(P!M!)$, and sum these terms over N,

M, $\{n_1, \ldots, n_P\}$. This gives the large-distance expansion

$$\begin{aligned}
&\langle \omega^+(x,t)\omega^+(0,0)\rangle_{\rho_{\text{ness}}} \\
&= e^{-x\mathcal{E}_{\text{ness}}} \sum_{P,M=0}^{\infty} \frac{1}{P!M!} \sum_{n_1,\ldots,n_p \in \mathbb{Z}/\{0\}} (2\pi)^N \prod_{j=1}^{p} \left(\frac{e^{-mx\cosh\lambda_{n_j} - im\tau\sinh\lambda_{n_j}}}{m\beta(n_j)\cosh\lambda_{n_j}} \right) \\
&\quad \times i^M \int_0^\pi d\theta_1 \cdots d\theta_M \prod_{j=1}^{M} \left(e^{imx\sinh\theta_j - m\tau\cosh\theta_j} \right) \\
&\quad \times \left(\frac{1}{1 - e^{-m\beta_l\cosh(\theta_j + 0^+)}} - \frac{1}{1 - e^{-m\beta_r\cosh(\theta_j - 0^+)}} \right) \\
&\quad \times f^{\rho^\sharp_{\text{ness}};\omega^+}_{+,\ldots,+}(\lambda_{n_1} + i\pi/2, \ldots, \lambda_{n_P} + i\pi/2, i\theta_1, \ldots, i\theta_M) \\
&\quad \times f^{\rho^\sharp_{\text{ness}};\omega^-}_{-,\ldots,-}(i\theta_M, \ldots, i\theta_1, \lambda_{n_P} + i\pi/2, \ldots, \lambda_{n_1} + i\pi/2)
\end{aligned} \tag{7.44}$$

for $\omega = \sigma$ or $\omega = \mu$.

It can be seen that in NESS correlation functions (7.44) there also exist the standard Matsubara frequencies which have been shown in previous section to appear in finite-temperature correlation functions (7.21). But here there are two frequencies which are associated to two temperatures and depend on the sign of n_j. As we said, the Matsubara frequencies admit an interpretation as coming from the quantization of the momentum in a quantum system on the circle, since finite-temperature correlation functions correspond to vacuum expectation values on the circle by a (quasi-)periodicity condition in imaginary time. We see that this formalism partly survives in our non-equilibrium steady state: there are two different circumferences, for right- and left-moving particles, respectively. Separate (quasi-)periodicity conditions in imaginary time for right- and left-movers, for instance, (7.28) and (7.29) do not make full sense in the massive theory, as right- and left-movers do not separate (local fields do not factorize). This is the reason for the presence of extra integrals on finite intervals and we can interpret these integrals as providing the bridge for right- and left-moving modes to jump from the circle of circumference β_l to that of circumference β_r and vice versa.

It is worth mentioning that the result of [43] obtained in the (anisotropic) XY model out of equilibrium, under the scaling limit, will reproduce the leading exponential decay $e^{-x\mathcal{E}_{\text{ness}}}$ in (7.44).

7.3.3 Leading large-distance behavior of two-point correlation functions

Concerning the large-distance behavior of two-point function (7.44), it is a non-trivial task to determine the form of the leading or subleading terms multiplying this exponential decay. The main difficulty we have to deal with is to analyze the relative strength of the terms with different particle numbers. This difficulty is in the root of the non-analyticity of $W(\theta)$ at $\theta = 0$. Conjecturing that the one- and two-particle contributions give the correct form, perhaps up to normalizations, we will carry out the following analysis.

The terms for $N \neq 0$ involve exponential decaying factors $e^{-mx\cosh(\lambda_{n_j})}$ and hence are subleading compared to the terms for $N = 0$ with no such exponentials. Due to this notion, we neglect all exponentially decaying terms coming from higher values of N. To evaluate the leading large-mx behavior (setting $\mathrm{t} = 0$ for simplicity) for the disorder two-point function, we consider $N = 0$ and $M = 1$, and only the part of the integral near 0 and π is sufficient. Hence we consider

$$\frac{\langle\sigma\rangle_{\rho_{\text{ness}}}}{2}\, e^{-x\mathcal{E}_{\text{ness}}} \int_0^\pi d\theta\, e^{-mx\sin\theta} \frac{\sinh\left(\frac{\beta_l-\beta_r}{2}\cos\theta\right)}{\sinh\left(\frac{\beta_l}{2}\cos\theta\right)\sinh\left(\frac{\beta_r}{2}\cos\theta\right)} h^{\sharp}{}^{+}_{+}(i\theta)\, h^{\sharp}{}^{-}_{-}(i\theta). \tag{7.45}$$

The leading large-mx behavior is evaluated by expanding for θ near to 0 and π. In fact, one can evaluate the leading small-θ behavior of leg factors by extracting the pole $1/(\theta-\theta')$ from the factor $1/\sinh(\theta-\theta')$ in (6.3), thus obtaining

$$h^\eta_\epsilon(\theta) \propto \begin{cases} \theta^{-\eta i\epsilon\gamma} & (\text{for } W_{\text{ness}}) \\ \theta^{\eta i\epsilon\gamma} & (\text{for } W^\sharp_{\text{ness}}) \end{cases}, \quad \gamma := \frac{1}{2\pi}\log\left(\coth\frac{m\beta_r}{2}\,\tanh\frac{m\beta_l}{2}\right). \tag{7.46}$$

Using (7.46) and (6.9), we find $h^{\sharp}{}^{+}_{+}(i\theta)\, h^{\sharp}{}^{-}_{-}(i\theta) \propto (i\theta)^{2i\gamma}$ near to $\theta = 0$, and similarly $\propto (i\pi - i\theta)^{-2i\gamma}$ near to $\theta = \pi$. Omitting the overall finite, real (temperature-dependent) factor, we then obtain, asymptotically,

$$e^{iB}\int_0^\infty d\theta\, e^{-mx\theta}\theta^{2i\gamma} + c.c. \propto \frac{A}{mx}\cos(2\gamma\log(mx) + B)$$

for some phase e^{iB}. A more careful calculation gives (7.48) with

$$\begin{aligned} A &= 2\frac{\sinh\frac{(\beta_l-\beta_r)m}{2}}{\sinh\frac{\beta_l m}{2}\sinh\frac{\beta_r m}{2}}|\Gamma(1+2i\gamma)| \qquad (7.47)\\ B &= \arg\big(\Gamma(1+2i\gamma)\big) + \\ &\quad + \frac{1}{2\pi}\int_{|\theta|>1} d\theta\,\frac{1}{\sinh\theta}\log\coth\frac{W_{\text{ness}}(\theta)}{2} + \frac{1}{2\pi}\int_{|\theta|<1} d\theta\left(\frac{1}{\sinh\theta}-\frac{1}{\theta}\right)\log\coth\frac{W_{\text{ness}}(\theta)}{2}. \end{aligned}$$

Thus, we have

$$\langle\mu^+(x,0)\mu^+(0,0)\rangle_{\rho_{\text{ness}}} \propto e^{-x\mathcal{E}_{\text{ness}}}\langle\sigma^+\rangle_{\rho_{\text{ness}}}\langle\sigma^-\rangle_{\rho_{\text{ness}}}\left(\frac{A}{mx}\cos(2\gamma\log(mx)+B) + \ldots\right). \tag{7.48}$$

The omitted part in (7.48) comes from higher values of N and M, and contain terms which admit the exactly same form as the first subleading term in (7.48), but with different constants A and B, with A possibly logarithmically divergent. Indeed, for terms with $M > 1$ and $N = 0$, we expect integrals of the type

$$\int_0^\pi d\theta_1 d\theta_2 \left(\tan\frac{\theta_1-\theta_2}{2}\right)^2 e^{-mx(\sin\theta_1+\sin\theta_2)} \times \text{leg factors.}$$

For $\theta_1 \sim 0$ and $\theta_2 \sim \pi$, and vice versa, the integrand has a second-order pole, leading to logarithmic divergences. We do not have a clear interpretation of the potential logarithmic divergence of the constant A, but it is likely to have connection with the large-time limit taken to generate the steady state. Nevertheless, the oscillating form obtained from the $M = 1$ analysis should be correct, and the possible divergences may be expected to be re-absorbed into the normalization of the field. These subleading terms with $M > 1$ are in the form with higher oscillating frequency in $\log(mx)$ and with higher power of $1/(mx)$.

A similar analysis can be applied to the two-point function of the order field. Obviously, the leading term of the large-distance behavior is $e^{-x\mathcal{E}_{\text{ness}}}\langle\sigma^+\rangle_{\rho_{\text{ness}}}\langle\sigma^-\rangle_{\rho_{\text{ness}}}$. From the arguments leading to the conclusion that the next subleading term in the disorder-field case are of the same form as the first subleading term, the subleading term of the large-distance behavior, in the case of the order field, is $e^{-x\mathcal{E}_{\text{ness}}}\langle\sigma^+\rangle_{\rho_{\text{ness}}}\langle\sigma^-\rangle_{\rho_{\text{ness}}}O(1)$. This implies that the vacuum expectation value $\langle\sigma^\pm\rangle_{\rho_{\text{ness}}}$ may not not reproduce the correct normalization for the leading exponential decay of the two-point function, since $\langle\sigma^+(x,0)\sigma^+(0,0)\rangle_{\rho_{\text{ness}}} \sim C\, e^{-x\mathcal{E}_{\text{ness}}}$ with in general $C \neq \langle\sigma^+\rangle_{\rho_{\text{ness}}}\langle\sigma^-\rangle_{\rho_{\text{ness}}}$. In this case, the full physical meaning, beyond its involvement in the form factor expansion (7.44), of the "expectation value" $\langle\sigma^\pm\rangle_{\rho_{\text{ness}}}$ would not be fully understood, considering such expectation values are usually obtained through the large-distance asymptotic under the condition of conformal normalization at small distances.

In spite of these problems, we still have our important finding: the large-distance behavior of two-point correlation function, both for the order and disordered regimes, contains oscillatory terms in $\log(mx)$ with frequencies that are multiple of 2γ.

7.4 Quantum quenches

Consider a quantum system initially prepared in the ground state $|\Psi_0\rangle$ of a given Hamiltonian $H(g_0)$, where g_0 is the coupling constant. At time $t = 0$, the coupling constant is suddenly changed from g_0 to a different value g. This process is called a quantum quench.

The sudden change of the Hamiltonian drives the system out of equilibrium and the system will evolve unitarily by means of the new Hamiltonian $H(g)$. The main problem that has been widely investigated is whether the system relaxes to a stationary state after a large evolution time, and if it does, what is the characteristics of this stationary state. Some experimental progress following this direction has been made by a groundbreaking "quantum Newton's cradle" experiment [144], which focuses on the relaxation towards a stationary state in systems of ultra-cold atoms Kinoshita. This experiment demonstrates the crucial role played by dimensionality and conservation laws in many-body quantum dynamics out of equilibrium. It has been shown in this experiment that three dimensional condensates reach quickly a stationary state characterized by an effective temperature, which is a process of "thermalization", whereas quasi one-dimensional systems exhibit a slow relaxation towards unusual non-thermal distribution. This difference arises from the existence of additional local conservation laws in quasi one-dimensional systems, which in turn poses the question whether quantum integrability has a influence on stationary behaviour of non-equilibrium evolution after a quantum quench. A tremendous amount of works have been sparked to address this issue and it is a common observation that thermalization indeed occurs in generic non-integrable systems but it is not obtained for integrable systems due to the restriction of the infinite number of conserved quantities. In integrable systems, the final stationary state is rather described in terms of a so-called generalized Gibbs ensemble (GGE) [31, 32]. The density matrix describing a GGE is expected to be given by

$$\rho_{\rm GGE} = e^{-\sum_{n=1}^{\infty} \beta_n H_n} \tag{7.49}$$

where H_n are the local conserved quantities, and β_n are the associated generalized inverse temperatures which are determined by the requirement that the averages of the conserved densities in the GGE be equal to those in the initial state $|\Psi_0\rangle$:

$$\frac{{\rm Tr}(\rho_{\rm GGE}\, H_n)}{{\rm Tr}\rho_{\rm GGE}} = \langle\Psi_0|H_n|\Psi_0\rangle. \tag{7.50}$$

In GGEs, only conserved quantities that are bounded from below are admitted to appear in (7.50) and the series in the exponential in (7.50) is assumed to be convergent. Moreover, the state described by a GGE does not allow for flow of energy, particles, etc, and hence it is in a natural "generalized" equilibrium due to the absence of entropy production, although it is not strictly at equilibrium which is characterized by a standard Gibbs' ensemble.

In the Ising model, a series of studies [34, 35] have confirmed the occurrence of the GGE. Our results can be applied directly to GGEs in the Ising model thanks to the trivial observation that in the Ising model, conserved quantities are linear combinations (integrals) of $a^\dagger(\theta)a(\theta)$ (a similar statement holds for general integrable QFT, using appropriate asymptotic-state creation and annihilation operators). One may construct generalized hamiltonians $H_n = (Q_n + Q_{-n})/2$ using charges Q_n defined as

$$Q_n = \int d\theta\, e^{n\theta}\, a^\dagger(\theta)a(\theta) \tag{7.51}$$

(and in particular H_1 is the usual hamiltonian), and one has, according to (4.75),

$$W(\theta) = \sum_{n=1}^{\infty} \beta_n \cosh(n\theta). \tag{7.52}$$

But more general functions $W(\theta)$ are allowed by our formalism. Hence our results (7.1) when specialized in the Ising model (for fermion fields and all their descendants) and (7.7) (for order and disorder fields) give, in principle, the full large-distance expansion of correlation functions in the Ising model in any GGE.

Refs [34, 35, 41] considered a particular quench corresponding to an instantaneous change of the transverse field in (2.57) at time $t = 0$ from h_0 to h. In these works, analytic results have been obtained, by employing methods based on the form factor approach and the determinant representation of correlation functions for free fermion theories, for the full asymptotic time and distance dependence of one- and two-point order parameter correlation functions in the thermodynamic limit after this quench within the ordered and disordered phase. In the scaling limit, the quench mentioned above corresponds to a quench in the fermion mass of the related free Majorana field theory at time $t = 0$ from m_0 to m. In [145], the long time behaviour of the order parameter (one-point function) after a quench of the mass within the ordered phase was studied using form factors, reproducing the scaling limit of the corresponding results in [34]. Here we consider instead the two-point function, directly in the steady state described by a GGE. The result of the works [34, 35, 41] can be expressed, in the scaling limit, by the following choice of $W(\theta)$:

$$\tanh\frac{W(\theta)}{2} = \frac{\sinh^2\theta + \kappa\, m_0/m}{\cosh\theta\sqrt{\sinh^2\theta + m_0^2/m^2}} \tag{7.53}$$

where $\kappa = +$ corresponds to a quench from ferromagnetic to ferromagnetic or from anti-ferromagnetic to anti-ferromagnetic regimes, and $\kappa = -$ corresponds to the other cases. We see that for $\kappa = -$ this is out of the context that we considered, since $W(\theta) \leq 0$ for small enough values of θ. However for $\kappa = +$ this can be treated with the present formalism. In particular, we find that

$$\frac{1}{1-e^{-W}} = \frac{1}{2}(1+U), \quad \frac{1}{1-e^{W}} = \frac{1}{2}(1-U) \tag{7.54}$$

where

$$U(\theta) := \frac{\cosh\theta\sqrt{\sinh^2\theta + m_0^2/m^2}}{\sinh^2\theta + m_0/m} \tag{7.55}$$

The function $U(\theta)$ has poles at

$$\sinh\theta = \pm i\,\sqrt{m_0/m}$$

and branch points at

$$\sinh\theta = \pm i\, m_0/m.$$

This implies that the expansion (7.7) provides a full large-distance expansion for correlation functions of ordered and disordered fields in the universal stationary regime occurring after such magnetic-field quenches, and in particular that the large-distance behavior is of the form (7.8) with exponential decay controlled by

$$|\mathrm{Im}(\sinh\theta^\star)| = \sqrt{m_0/m} \quad \text{or} \quad m_0/m.$$

Let us first consider the leading large-distance behavior in the disordered (anti-ferromagnetic) regime. The spin-spin correlation function for the disorder fields has an exponentially decaying factor $e^{-x\mathcal{E}}$ controlled by $\mathcal{E}$ (7.5), with a possible algebraic factor. In order to compute the possible algebraic factor, we look at the one-particle contribution

$$\sum_\epsilon \frac{1}{2}\int d\theta\, e^{i\epsilon p_\theta x}(1+\epsilon U(\theta)) f_\epsilon^{\rho^\sharp;\mu^+}(\theta) f_{-\epsilon}^{\rho^\sharp;\mu^-}(\theta)$$

where U is given by (7.55). If $m_0 > m$, then $\sqrt{m_0/m} < m_0/m$, hence the poles determine the least-decaying behavior. In this case, we deform the θ contours away from the real line, in the direction $\mathrm{sign}(\mathrm{Im}(\theta)) = \epsilon$ in which the form factors are analytic, all the way to $\mathrm{Im}(\theta) = \epsilon i\pi/2$. There, poles and branch points are found respectively at

$$\theta_p = \epsilon i\pi/2 \pm \alpha_p \tag{7.56}$$

where

$$\cosh\alpha_p = \sqrt{m_0/m}, \quad \alpha_p > 0, \tag{7.57}$$

and at

$$\theta_b = \epsilon i\pi/2 \pm \alpha_b \tag{7.58}$$

where

$$\cosh\alpha_b = m_0/m, \quad \alpha_b > 0, \tag{7.59}$$

with $\alpha_p < \alpha_b$. In light of crossing symmetry (6.10), we have

$$f_+^{\rho^\sharp;\mu_+}(\alpha + i\pi/2) f_-^{\rho^\sharp;\mu_-}(\alpha + i\pi/2) = -f_-^{\rho^\sharp;\mu_+}(\alpha - i\pi/2) f_+^{\rho^\sharp;\mu_-}(\alpha - i\pi/2), \tag{7.60}$$

and direct calculation shows that

$$U(\alpha + i\pi/2) = -U(\alpha - i\pi/2) \tag{7.61}$$

for α between $-\alpha_b$ and α_b (that is, away from the branch cut). Using (7.60) and (7.61), we find

$$\begin{aligned}&(1 + U(\alpha + i\pi/2)) f_+^{\rho^\sharp;\mu_+}(\alpha + i\pi/2) f_-^{\rho^\sharp;\mu_-}(\alpha + i\pi/2) \\ &= -(1 - U(\alpha - i\pi/2)) f_-^{\rho^\sharp;\mu_+}(\alpha - i\pi/2) f_+^{\rho^\sharp;\mu_-}(\alpha - i\pi/2)\end{aligned} \tag{7.62}$$

and this allows us to cancel out the integrals with $\epsilon = \pm$ on $\alpha \in [-\alpha_b, \alpha_b]$. Thus, there only remain residue contributions at the poles $\alpha = \pm\alpha_p$, and integrals along branch cuts $[\alpha_b, \infty)$ and $[-\alpha_b, -\infty)$. Since $\alpha_p < \alpha_b$, exponential decaying factors $e^{-mx\cosh\alpha}$ involved in the integrand in the latter integrals are subleading compared to the exponential factors $e^{-mx\cosh\alpha_p}$ coming from the residues taken at the poles. The leading behavior is then determined by these residues,

$$\begin{aligned} I_\pm &= \pi i e^{-\sqrt{mm_0}x} f_+^{\rho^\sharp;\mu_+}(\pi i/2 + \theta_p) f_-^{\rho^\sharp;\mu_-}(\pi i/2 + \theta_p) \mathrm{Res}\,(U(z))\,|_{z=\pi i/2 \pm \alpha^\star} + \\ &\quad \pi i e^{-\sqrt{mm_0}x} f_-^{\rho^\sharp;\mu^+}(-\pi i/2 + \theta_p) f_+^{\rho^\sharp;\mu^-}(-\pi i/2 + \theta_p) \mathrm{Res}\,(U(z))\,|_{z=-\pi i/2 \pm \alpha^\star} \end{aligned}$$

Using crossing symmetry again, we get:

$$I_\pm = e^{-\sqrt{mm_0}x} 4\pi \sqrt{m_0/m - 1} f_+^{\rho^\sharp;\mu^+}(\pi i/2 + \theta_p) f_-^{\rho^\sharp;\mu^-}(\pi i/2 \pm \alpha^\star)$$

Therefore, the leading behaviour is

$$O\left(e^{-(\mathcal{E} + \sqrt{mm_0})x}\right) \quad (m_0 > m). \tag{7.63}$$

On the other hand, if $m_0 < m$, then $\sqrt{m_0/m} > m_0/m$, hence the branch points determine the least-decaying behavior. In this case, both poles and branch points which are closest to the real line are found on the imaginary axis below the $\mathrm{Im}(\theta) = \pi/2$ line and above the $\mathrm{Im}(\theta) = -\pi/2$ line. The branch points are at $\theta_b = \pm i\kappa_b$ with

$$\sin\kappa_b = m_0/m \tag{7.64}$$

and the poles at $\pm i\kappa_p$ with

$$\sin\kappa_p = \sqrt{m_0/m} > \sin\kappa_b. \tag{7.65}$$

We choose the branch cuts as the horizontal lines emanating from $\theta = \pm i\kappa_b$ to $\theta = \pm i\kappa_b + \infty$. In this case, we shift the contours in the direction $\text{sign}(\text{Im}(\theta)) = \epsilon$ up to the position of the poles at $\text{Im}(\theta) = \epsilon i\kappa_p$ (or a little bit before it), going around the branch cut $[\epsilon i\kappa_b, \epsilon i\kappa_b + \infty]$. We first consider the one-particle contribution from the integrals along the branch cuts. With the change of variable $\theta = i\epsilon\kappa_b + \beta$, $\beta \in \mathbb{R}$, we have

$$\sum_\epsilon \frac{1}{2}\int_0^\infty d\beta \left(e^{-m_0x\cosh\beta + i\epsilon\sqrt{m^2-m_0^2}x\sinh\beta}\right)\left(-U(i\epsilon\kappa_b + i0^+ + \beta) + U(i\epsilon\kappa_b - i0^+ + \beta)\right)$$
$$\times f_\epsilon^{\rho^\sharp;\mu^+}(i\epsilon\kappa_b + \beta) f_{-\epsilon}^{\rho^\sharp;\mu^-}(i\epsilon\kappa_b + \beta) \tag{7.66}$$

where we used (7.64). Taking the principal square root form of the U functions leads to

$$-\sum_\epsilon \int_0^\infty d\beta \left(e^{-m_0x\cosh\beta + i\epsilon\sqrt{m^2-m_0^2}x\sinh\beta}\right)\sqrt{\sinh\beta}\,T_\epsilon(\beta) f_\epsilon^{\rho^\sharp;\mu^+}(i\epsilon\kappa_b + \beta) f_{-\epsilon}^{\rho^\sharp;\mu^-}(i\epsilon\kappa_b + \beta)$$

where

$$T_\epsilon(\beta) = \frac{\sqrt{1-m_0^2/m^2}\cosh\beta + i\epsilon m_0/m\sinh\beta}{(1-2m_0^2/m^2)\sinh^2\beta + i\epsilon 2m_0/m\sqrt{1-m_0^2/m^2}\sinh\beta\cosh\beta - m_0^2/m^2\cosh^2\beta + m_0/m}$$
$$\times\sqrt{(1-2m_0^2/m^2)\sinh\beta + i\epsilon 2m_0/m\sqrt{1-m_0^2/m^2}\cosh\beta - m_0^2/m^2\sinh\beta}.$$

By Taylor expanding $\sinh\beta$ and $\cosh\beta$, changing the variable β to $\beta/(mx)$, we obtain the leading term of (7.67) in the large-x limit:

$$-e^{-m_0x}(mx)^{-3/2}\left[T_+(0) f_+^{\rho^\sharp;\mu^+}(i\kappa_b) f_-^{\rho^\sharp;\mu^-}(i\kappa_b)\int_0^\infty d\beta e^{i\beta\sqrt{1-m_0^2/m^2}}\right.$$
$$\left.+T_-(0) f_-^{\rho^\sharp;\mu^+}(-i\kappa_b) f_+^{\rho^\sharp;\mu^-}(-i\kappa_b)\int_0^\infty d\beta e^{-i\beta\sqrt{1-m_0^2/m^2}}\right]. \tag{7.67}$$

Following similar lines, we find that the one-particle contribution coming from the integrals on the lines $\theta = \epsilon i\kappa_p + \beta$ is of order $e^{-m_0x}(mx)^{-1}$, which is subleading compared to the one-particle contribution coming from the integrals along the branch cuts. Thus, in this case, the leading large-distance behavior is

$$O\left(x^{-3/2}e^{-(\mathcal{E}+m_0)x}\right) \quad (m_0 < m). \tag{7.68}$$

In the ordered regime the leading behavior is simply determined by $e^{-\varepsilon x}$ coming from the zero-particle contribution. The first subleading asymptotic terms are obtained by considering the two-particle form factor contributions,

$$\sum_{\epsilon_1\epsilon_2} \frac{\epsilon_1\epsilon_2}{4} \int \frac{d\theta_1 d\theta_2}{2!} e^{\sum_{j=1}^2 i\epsilon_j p_{\theta_j} x} U(\theta_1)U(\theta_2) f_{\epsilon_1\epsilon_2}^{\rho^\sharp;\sigma^+}(\theta_1,\theta_2) f_{-\epsilon_2-\epsilon_1}^{\rho^\sharp;\sigma^-}(\theta_2,\theta_1).$$

Consider first $m_0 > m$. Again, we deform the θ_1 and θ_2 contours following the same way as we did before. We first deform the θ_1 contour up to $\mathrm{Im}(\theta_1) = \epsilon_1 i\pi/2$. Using the crossing symmetry relation (6.10), the integrals for $\epsilon_1 = +$ and $\epsilon_1 = -$ cancel out (for any ϵ_2 and θ_2), except for the residues of the poles at $\epsilon_1 i\pi/2 \pm \alpha_p$ (7.57) and for the branch cuts further away from the imaginary axis, on $|\mathrm{Re}(\theta)| > \alpha_b$. The residues contribute the part of the leading contribution, $O\left(e^{-\sqrt{mm_0}x}\right)$. We then deform the θ_2 contour. The remaining integrals, after taking the θ_1 residues at $\epsilon_1 i\pi/2 + s\alpha_p$ (for $s = \pm$), are proportional to

$$\sum_{\epsilon_2} \epsilon_2 \int d\theta_2\, e^{i\epsilon_2 p_{\theta_2} x} U(\theta_2) h_{\epsilon_2}^{\sharp+}(\theta_2) h_{-\epsilon_2}^{\sharp-}(\theta_2) \tanh\left(\frac{\theta_2 - (i\pi/2 + s\alpha_p) + i(\epsilon_2 - 1)\mathbf{0}}{2}\right)^{2\epsilon_2}.$$

The integrals for $\epsilon_2 = +$ and $\epsilon_2 = -$ can be cancelled out by shifting to $\mathrm{Im}(\theta_2) = \epsilon_2 i\pi/2$, except for the poles at $\theta_2 = \epsilon_2 i\pi/2 \pm \alpha_p$ and the branch cuts further away from the imaginary axis. For $\epsilon_2 = +$, the pole of $U(\theta_2)$ at $\theta_2 = i\pi/2 + s\alpha_p$ is cancelled by the zero in the tanh factor (similarly for the mirror pole with $\epsilon_2 = -$). As a result, only the pole at $\theta_2 = i\pi/2 - s\alpha_p$ contributes. This means that, overall, the leading large-distance behavior of the two-particle form factor contribution is given by the product of the residues taken at $\theta_1 = \epsilon_1 i\pi/2 + s\alpha_p$ and $\theta_2 = \epsilon_2 i\pi/2 - s\alpha_p$, with $s = +$ and $s = -$. It turns out that the product of residues is independent of s due to invariance under the exchange $\theta_1 \leftrightarrow \theta_2$. This then provides an overall decay $O\left(e^{-2\sqrt{mm_0}s}\right)$. Thus the leading behaviour is

$$e^{-\mathcal{E}x}\left(1 + O\left(e^{-2\sqrt{mm_0}x}\right)\right) \quad (m_0 > m).$$

Following the same recipe, in the case $m_0 > m$, we find that the branch cuts determine the leading exponential decaying. Hence, the leading behavior of the correlation function is

$$e^{-\mathcal{E}x}\left(1 + O\left(x^{-3}e^{-2m_0x}\right)\right) \quad (m_0 < m).$$

7.5 Rényi entropy

The Rényi entropy can be used to evaluate the bipartite entanglement entropy which is a measure of quantum entanglement [146]. For the definition of the entanglement entropy, consider a composite quantum system with Hilbert space $\mathcal{H} = \mathcal{H}_A \otimes \mathcal{H}_B$ in a ground state $|\mathrm{gs}\rangle$. The entanglement entropy S_A is the von Neumann entropy associated to the reduced density matrix ρ_A of the subsystem A:

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}\left(\rho_A \log(\rho_A)\right), \quad \rho_A = \mathrm{Tr}_{\mathcal{H}_B}\left(|\mathrm{gs}\rangle\langle\mathrm{gs}|\right). \tag{7.69}$$

Using the "replica trick" [45, 147], the bipartite entanglement entropy can be obtained as the limit $n \to 1$ of the Rényi entropy for positive real n:

$$S_A = \lim_{n\to 1} S_A^{(n)}, \quad S_A^{(n)} = \frac{1}{1-n} \log \mathrm{Tr}_{\mathcal{H}_A} \rho_A^n. \tag{7.70}$$

It has been demonstrated in [147] that the Rényi entropy for integer n is related to the partition function on a multi-sheeted Riemann surface with branch points. The authors in [46, 127] introduced the so-called branch-point twist fields which correspond to branch points so that their correlation functions are the partition functions on multi-sheeted Riemann surfaces. Therefore, the problem of calculating the biparticle entanglement entropy is reduced to the evaluation of the two-point correlation function of the branch-point twist fields. In [46], the two-point correlation function of the branch-point twist fields were computed by exploiting the factorized scattering method for integrable models of QFT.

In this section, we consider the composite system mentioned above in mixed states in the Ising model. In this case, the Rényi entropy for integer n is related with the mixed-state two-point correlation function of the branch-point twist fields in the n-copy Ising model. Thanks to the relation (3.54), our result (7.14) can be applied directly to the evaluation of the mixed-state two-point function of the branch-point twist fields in the n-copy Ising model. Taking analytic continuation in n of the Rényi entropy and computing the result at $n = 1$ could in principle give the mixed-state bipartite entanglement entropy. However, this is beyond the scope of this thesis.

Mixed-state correlation function and Rényi entropy for integer n

According to the arguments in [46], the Rényi entropy for integer n in the Ising model can be written in terms of the mixed-state two-point correlation function of the branch-point twist fields (without loss of generality we consider only the branch-point twist fields with cuts going towards the right):

$$S_A^{(n)} = \frac{1}{1-n} \log\left[\varepsilon^{2d_n} Z_n \langle \mathcal{T}(x,0)\tilde{\mathcal{T}}(0,0)\rangle_{\rho_{\mathrm{I}}^{(n)}}\right] \tag{7.71}$$

where Z_n is an n-independent non-universal normalisation constant with $Z_1 = 1$, ε is a short-distance cutoff which is chosen in such a way that $dZ_n/dn = 1$, d_n is the scaling dimension (3.49), and $\rho_{\mathrm{I}}^{(n)}$ represents the density matrix of the n-copy Ising model. Now we evaluate the two-point function $\langle \mathcal{T}(x,0)\tilde{\mathcal{T}}(0,0)\rangle_{\rho_{\rm I}^{(n)}}$. We first define the density matrix $\rho_{\rm D}^{(n)}$ for the n-copy Dirac theory:

$$\rho_{\rm D}^{(n)} = \rho_{\rm D}^{1} \otimes \cdots \otimes \rho_{\rm D}^{n} \tag{7.72}$$

where $\rho_{\rm D}^{i}$ is the density matrix in the $i^{\rm th}$-copy of the model. Then, in light of (3.54), the two-point correlation function of branch-point twist fields $\mathcal{T}_{\rm Dirac}(x,0)$ and $\tilde{\mathcal{T}}_{\rm Dirac}(0,0)$ in the n-copy Dirac theory in mixed states can be written as

$$\begin{aligned}
&\langle \mathcal{T}^{+}_{\rm Dirac}(x,0)\tilde{\mathcal{T}}^{+}_{\rm Dirac}(0,0)\rangle_{\rho_{\rm D}^{(n)}} \\
=\;& \frac{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{(n)}}\left(\rho_{\rm D}^{(n)} \prod_{m=1}^{n} \sigma^{+}_{(m,\alpha_m)}(x,0) \prod_{p=1}^{n} \sigma^{+}_{(p,-\alpha_p)}(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{(n)}}\left(\rho_{\rm D}^{(n)}\right)} \\
=\;& \frac{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{1}}\left(\rho_{\rm D}^{1}\, \sigma^{+}_{(1,\alpha_1)}(x,0)\sigma^{+}_{(1,-\alpha_1)}(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{1}}\left(\rho_{\rm D}^{1}\right)} \cdots \frac{\mathrm{Tr}_{\mathcal{H}_{n}}\left(\rho_{\rm D}^{n}\, \sigma^{+}_{(n,\alpha_n)}(x,0)\sigma^{+}_{(n,-\alpha_n)}(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{n}}\left(\rho_{\rm D}^{n}\right)} \\
=\;& \langle \sigma^{+}_{(1,\alpha_1)}(x,0)\sigma^{+}_{(1,-\alpha_1)}(0,0)\rangle_{\rho_{\rm D}^{1}} \cdots \langle \sigma^{+}_{(n,\alpha_n)}(x,0)\sigma^{+}_{(n,-\alpha_n)}(0,0)\rangle_{\rho_{\rm D}^{n}}
\end{aligned} \tag{7.73}$$

where $\mathcal{H}_{\rm D}^{(n)}$ is the Hilbert space of the n-copy Dirac theory and $\mathcal{H}_{\rm D}^{i}$ is the Hilbert space of the $i^{\rm th}$-copy of it, and where in the first step we used relations (3.48) and (3.27). On the other hand, we can define the density matrix $\rho_{\rm D}^{(n)}$ in another way:

$$\rho_{\rm D}^{(n)} = \rho_{{\rm I}_a}^{(n)} \otimes \rho_{{\rm I}_b}^{(n)} \tag{7.74}$$

where $\rho_{{\rm I}_a}^{(n)}$ and $\rho_{{\rm I}_b}^{(n)}$ are the density matrices in the copies a and b of the n-copy Ising model, respectively. Using the relation (3.55), we have

$$\begin{aligned}
&\langle \mathcal{T}^{+}_{\rm Dirac}(x,0)\tilde{\mathcal{T}}^{+}_{\rm Dirac}(0,0)\rangle_{\rho_{\rm D}^{(n)}} \\
=\;& \frac{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{(n)}}\left(\rho_{\rm D}^{(n)} \mathcal{T}_a^{+}(x,0)\mathcal{T}_b^{+}(x,0)\tilde{\mathcal{T}}_a(0,0)\tilde{\mathcal{T}}_b(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{\rm D}^{(n)}}\left(\rho_{\rm D}^{(n)}\right)} \\
=\;& \frac{\mathrm{Tr}_{\mathcal{H}_{{\rm I}_a}^{(n)}}\left(\rho_{{\rm I}_a}^{(n)}\, \mathcal{T}_a^{+}(x,0)\tilde{\mathcal{T}}_a(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{{\rm I}_a}^{(n)}}\left(\rho_{{\rm I}_a}^{(n)}\right)} \frac{\mathrm{Tr}_{\mathcal{H}_{{\rm I}_b}^{(n)}}\left(\rho_{{\rm I}_b}^{(n)}\, \mathcal{T}_b^{+}(x,0)\tilde{\mathcal{T}}_b(0,0)\right)}{\mathrm{Tr}_{\mathcal{H}_{{\rm I}_b}^{(n)}}\left(\rho_{{\rm I}_b}^{(n)}\right)} \\
=\;& \left(\langle \mathcal{T}(x,0)\tilde{\mathcal{T}}(0,0)\rangle_{\rho_{\rm I}^{(n)}}\right)^2
\end{aligned} \tag{7.75}$$

where $\mathcal{H}_{{\rm I}_a}^{(n)}$ and $\mathcal{H}_{{\rm I}_b}^{(n)}$ are the Hilbert spaces of the copies a and b of the n-copy Ising model, respectively. By comparing (7.73) and (7.75), we see that the mixed-state two-point correlation function of the branch-point twist fields in the n-copy Ising model admits the representation of the form

$$\begin{aligned} & \left\langle \mathcal{T}(x,0)\tilde{\mathcal{T}}(0,0)\right\rangle_{\rho_{\rm D}^{(n)}} \\ = & \left(\left\langle \sigma^{+}_{(1,\alpha_1)}(x,0)\sigma^{+}_{(1,-\alpha_1)}(0,0)\right\rangle_{\rho^1_{\rm D}}\cdots\left\langle \sigma^{+}_{(n,\alpha_n)}(x,0)\sigma^{+}_{(n,-\alpha_n)}(0,0)\right\rangle_{\rho^n_{\rm D}}\right)^{1/2} \end{aligned} \tag{7.76}$$

where the mixed-state two-point functions of $U(1)$ twist fields are known from (7.14). Finally, we obtain the Rényi entropy for integer n:

$$\begin{aligned} S_n = & \frac{1}{1-n}\times \\ & \log\left[\varepsilon^{\frac{1}{6}(n-\frac{1}{n})}Z_n\left(\left\langle \sigma^{+}_{(1,\alpha_1)}(x,0)\sigma^{+}_{(1,-\alpha_1)}(0,0)\right\rangle_{\rho^1_{\rm D}}\cdots\left\langle \sigma^{+}_{(n,\alpha_n)}(x,0)\sigma^{+}_{(n,-\alpha_n)}(0,0)\right\rangle_{\rho^n_{\rm D}}\right)^{\frac{1}{2}}\right]. \end{aligned} \tag{7.77}$$

Chapter 8

Conclusion

8.1 Work done

This thesis has contributed to the investigation of correlation functions in general mixed states with diagonal density matrices in the context of integrable models of quantum field theory, using the Liouville-space method [21, 23, 55]. Generalizing works in [21, 23, 55], we have constructed the Liouville space for general integrable models of quantum field theory, including interacting models, and we have defined the associated mixed-state form factors. Our method is based on the GNS construction of C^*-algebras and is a departure from other methods which has been widely applied on massive integrable models of QFT. We do not directly calculate the trace, defining the two-point mixed-state correlation function, through the knowledge of vacuum form factors and matrix elements in the Hilbert space. Instead, we identify the mixed-state two-point function as the vacuum expectation values with respect to the Liouville space. Hence, the two-point function can be obtained from form factor expansions with respect to the Liouville space, in parallel with zero-temperature form factor expansion technique. Our Liouville-space method avoids resummations of partition-divergencies, which are required in the explicit calculation of the traces defining mixed-state two-point functions. These resummations have been automatically performed in our defined mixed-state form factors and resulting form factor expansions

Following arguments in [21, 23], we have derived a Riemann-Hilbert problem for finite-temperature one-particle form factors of $U(1)$ twist fields in the free Dirac fermion theory and solved it by additionally making use of low temperature expansions [142]. We found that these finite-temperature form factors have the same structure as those obtained in the Ising model [21, 23].

Thanks to the observation that finite-temperature form factors of twist fields, in both Ising model and Dirac model, depend on the eigenvalues of the density matrices, in trivial ways, we conjectured the same form for form factors of twist fields in general mixed states with only the replacement of βE_θ by $W(\theta)$ for the Ising model and by $W_a(\theta)$ for the Dirac model. We have deduced for these mixed-state form factors a system of non-linear functional differential equations [55, 142]. Once vacuum form factors are known, mixed-state form factors can be uniquely determined by these equations. We demonstrated that our exact mixed-state form factors obey this system of equations, hence confirming their validity. Deriving and solving these equations is a novel approach to evaluate mixed-state form factors, which is different from the standard methods based on solving a Riemann-Hilbert problem for zero-temperature or finite-temperature form factors . It is more powerful as it does not rely on any strong analytic property for the eigenvalues of the density matrix. This approach looks similar to techniques used in classical integrable models in order to obtain bilinear differential equations for tau-functions, which are however usually associated to correlation functions instead of form factors [55] (see for instance [52]). But we do not know yet if there is a full technical equivalence.

With the exact mixed-state form factors at hand, we have presented the general results for mixed-state two-point correlation functions of twist fields, in terms of form factors expansions [55, 142]. We have given detailed explanations of three subtleties involved in these representations, which arise from the presence of the cuts emanating from twist fields and the fact that form factors of twist fields are not entire functions of rapidities. We have demonstrated the application of our form factor expansions to some specific mixed states: thermal-flow non-equilibrium steady states [55] and generalized Gibbs ensembles after quantum quenches [55], both in the Ising model. In particular, we have discovered oscillating terms $\log(mx)$ appearing in the leading large-distance behavior of order and disorder non-equilibrium correlation functions [55]. In addition, using the relation (3.54) between the branch-point twist fields in the n-copy Ising model and the $U(1)$ twist fields in the n-copy Dirac theory , and the result (7.14), we have derived the Rényi entropy for integer n, which can be used for the evaluation of the bipartite entanglement entropy.

8.2 Future developments

In spite of the series of results we have achieved in this thesis, there are still several open problems remain on the technical level and left for further investigation.

First, both application to quantum quenches and to the non-equilibrium energy-flow steady state need to be further developed. In order to achieve this, we have to extend our formalism to more general functions $W(\theta)$, including with discontinuities and with regions of negativity. Indeed, for instance, the validity of our form factor expansion (7.7) is based on the assumption that $W(\theta)$ is analytic on a neighborhood of $\theta \in \mathbb{R}$. However, the function $W_{\rm ness}(\theta)$ associated to non-equilibrium states is not analytic at $\theta = 0$ and does not satisfy this assumption. Moreover, we recall that, in the Ising model, for the twisted construction, the positivity condition (4.76) is required in order to avoid the possibility that the factor $\frac{1}{1-e^{-\epsilon W(\theta)}}$ in (7.7) has some pole on the real line. Due to this, in the application to quantum quenches in the Ising model, our formalism is not adapted to the case of $W(\theta)$ in (7.53) with $\kappa = -$. Nevertheless, thanks to the notion that the positivity condition for $W_\nu(\theta)$ with generic α is not necessary in the Dirac theory, we expect this problem could be solved by calculating the associated quantum quench in the Dirac theory with $\alpha \neq 1/2$ and then take the limit $\alpha = 1/2$ after the calculation.

Second, it would be very interesting to explore what generalizes the "quantization on the circle" correspondence of finite-temperature correlation functions, and the Matsubara frequencies, in general integrable models of QFT. As was argued in [21, 23], thermal correlation functions are related to vacuum expectation values in the quantization scheme on the circle. We expect that there exists also an alternative quantization where vacuum expectation values reproduce mixed-state correlation functions. From our investigation of the non-equilibrium steady state and quantum quench applications, a study of the singularity structure of the filling factors (for boson or fermion models)

$$\frac{1}{\epsilon_j^{1-f_{a_j}}\left(1-(-1)^{f_{a_j}}e^{-\epsilon_j W_{a_j}(\theta_j)}\right)}$$

would shed light on the determination of the generalized quantization scheme we are seeking for.

Third, the normalization of the mixed-state form factors of twist fields, both in the Ising model and the Dirac theory, needs to be evaluated exactly, although we have obtained a recursion relation (6.17) in the Dirac theory.

Fourth, it would be interesting to further develop the Liouville space. For instance, at zero temperature, as was stated in section 2.3, in integrable QFT, there exists a set of consistency equations constraining the vacuum form factors in the Hilbert space. These equations have their origin on very general principles of QFT. At finite temperature, as we showed in chapter 5, these exists also a Riemann-Hilbert problem, from which we obtained the exact thermal form factors. However, this set of constraints is directly derived from the KMS relation. It is expected that a deeper understanding of the Liouville space would lead us to a set of more "natural" properties for thermal or non-equilibrium form factors, which are based on general principles of QFT and thermofield dynamics theory.

Finally, with the success made both in the Ising model and Dirac model, our Liouville-space method definitely deserves further investigation on mixed-state correlation functions in more integrable models, particularly in interacting models with non-trivial scattering matrix. Although using Wick's theorem is not quite efficient in integrable models, we can still expect a system of nonlinear functional differential equations which are not closed equations but an infinite set of equations relating form factors with more and more particles. In analogy with the form factor equations for vacuum form factors, these equations may lead to solutions or efficient expansions like low-temperature expansions.

Appendix A

Extra work: high- and low-temperature limit of the exact current in non-equilibrium steady states in integrable QFT

A.1 Introduction

In recent years, the thermodynamics of quantum systems out of equilibrium has been a subject of intense investigation. On the one hand, this is stimulated by the recent experimental progress, opening the possibility to drive quantum systems away from equilibrium in a controlled way and to study their non-equilibrium properties (see for instance [148–153]). On the other hand, this is also due to the theoretical progress with the discovery of several classes of fluctuation theorems (also known as fluctuation relations), which generalize the fluctuation-dissipation theorem [154] to systems far from equilibrium (see [155, 156]) and describe some universal properties of non-equilibrium fluctuations.

Situations of particular interest are those where steady currents of local quantities exist: steady flows of energy, charge, particles, etc. In these situations, although external forces, if any, are time-independent, the system is not at equilibrium due to a permanent creation of entropy. Two ingredients at the heart of this situation are fluctuations of the currents, and their scaled cumulant generating function (SCGF) which are related to the large-deviation functions by a Legendre transform. Fluctuations are one of the most fundamental concepts arising in statistical physics. The SCGF fully characterises the fluctuations of these constant flows at large times and encodes for many properties of their non-equilibrium statistics. Symmetry properties of the SCGF are described by fluctuation theorems. (Exact evaluation of the SCGFs and their associated fluctuation theorems play a essential role in developing the general theory of non-equilibrium steady states.)

In integrable quantum systems, many quantities, such as all energy eigenstates and eigenvalues, and correlation functions of local operators, can in principle be obtained non-perturbatively, thanks to integrability. The study of the equilibrium thermodynamic properties of integrable models has been carried out by a very successful method which mixes factorized scattering theory and Bethe ansatz ideas. This method is the well-known thermodynamic Bethe ansatz (TBA) and was initially proposed by Al.B. Zamolodchikov [157]. Other exact methods includes those based on free-fermion techniques (Clifford algebras). For the universal regime of exactly gapless quantum models, which is described by conformal field theory (CFT), there also exist a series of powerful approaches [158, 159].

In light of these methods, many exact results have been obtained for SCGF with respect to charge flows in non-equilibrium integrable quantum systems. In the case of charge flows, the SCGF is referred to as the "full-counting statistics" (FCS). Proposals [160, 161] by Lesovik and Levitov constitute the first presentation of the exact result for the FCS in free fermion models and triggered a series of further investigations [162–165]. Other exact results were obtained in Luttinger liquids (critical free boson systems) in [166] with the help of non-equilibrium bosonization technique ,and in the low-temperature universal regime of quantum critical models in [38] using general CFT. In certain integrable interacting impurity models, the SCGF was obtained in [167, 168], using TBA-like methods. Exact charge current and shot noise (zero-temperature second cumulant) have also been analyzed in a diversity of integrable models, see for instance [169–181].

However, concerning energy flows, exact results for currents in interacting models and for SCGF in general are not reported until recently. The first exact result for the SCGF was obtained, to our knowledge, in the case of a quantum chain of harmonic oscillators in [182]. Exact results for the energy current and SCGF in the universal non-equilibrium regime at low temperatures were derived in general quantum critical models in [37, 38] by using CFT techniques, for the quantum Ising chain in a magnetic field in [40] by employing free fermion techniques and for any integrable model of relativistic quantum field theory with diagonal scattering in [69] by generalizing TBA to non-equilibrium steady states. (star configuration) It is worth mentioning that it has been demonstrated in [183] that a condition of pure transmission leads to the "extended" fluctuation relations, which allow one to exactly evaluate the SCGF from the current alone.

In this appendix, we present the high- and low- temperature limit of the exact energy current obtained in [69]. Our setup consists in preparing two hamiltonian reservoirs at different temperatures and connecting them at a contact point. After the contact is established, energy or charge will transfers from one reservoir to another. At large time, the flow between these two reservoirs becomes constant in time and the system reaches a non-equilibrium steady state. We then consider the scaling limit: the quantum system is assumed to be in the universal regime near a quantum critical point (with unit dynamical exponent), with a mass gap and the driving temperatures assumed to be much smaller than any microscopic energy scale. This regime is described by massive quantum field theory.

A.2 Physical situation

A.2.1 Physical description

Consider a homogeneous open quantum chain of length L with local interactions. We assume it to be initially cut into two identical parts , each of which is of length $L/2$. These two halves do not interact with each other and we denote by H^l and H^r (left and right) the Hamiltonian of them, respectively. We prepare these two halves at temperature $T_l = \beta_l^{-1}$ and $T_r = \beta_r^{-1}$ respectively. Hence, the initial density matrix describing the system is $\rho_0 = e^{-\beta_l H_l^L - \beta_r H_r^L}$. Then, we unitarily evolve this density matrix with the full Hamiltonian $H^L = H_l^L + H_r^L + \delta H$:

$$\rho(t) = e^{-iHt}\rho_0 e^{iHt}, \tag{A.1}$$

by connecting these two halves at a contact point. The term δH arises from the connection energy of the few links connecting both halves and it is assumed to be independent of the system size L.

Averages of observables at time t are given by

$$\langle \mathcal{O} \rangle (L;t) = \frac{\mathrm{Tr}\,(\rho(t)\,\mathcal{O})}{\mathrm{Tr}\,(\rho_0)}. \tag{A.2}$$

The so-called steady state limit is defined by the limits $L \to \infty$ and then $t \to \infty$ in that order. If the expectation value (A.2) under the steady state limit exist, the system is said to reach a steady state with respect to the observable $\mathcal{O}$:

$$\langle \mathcal{O} \rangle_{\mathrm{stat}} := \lim_{t\to\infty} \lim_{L\to\infty} \langle \mathcal{O} \rangle (L;t). \tag{A.3}$$

Taking into account the locality of the Hamiltonians, we expect the existence of the steady state limit of any local observable. This admits a physical interpretation: in the infinite size limit, halves of the system serves as effective thermal reservoirs, each at its own temperature $T_{l,r}$, and they can absorb and emit independent thermalized excitations unboundedly for all times $t \ll L/v$ where v is a propagation velocity; indeed within such times excitations do not have time to bounce off the extreme left and right parts in order to create non-thermal correlations or re-emit absorbed excitations. In particular, the energy current observable

$$\mathcal{J} := \frac{\mathrm{i}}{2}[H^L, H_r^L - H_l^L] = \frac{\mathrm{i}}{2}[\delta H, H_r^L - H_l^L], \tag{A.4}$$

which is independent of L, is local, due to the locality of δH. Hence, the system is expected to reach a steady state with respect to the energy current.

A.2.2 Steady state in massive QFT

We assume the quantum chain mentioned in the previous section to have a parameter h (for instance, an external magnetic field) so that the system has a quantum critical point at $h = h_c$ with unite dynamical exponent. As was stated in section 2.4, at $h = h_c$, the energy gap Δ vanishes. The system, at $h = h_c$ and in the low-temperature regime, is described by CFT. The low-temperature, low-gap region near the quantum criticality is obtained by taking the scaling limit $h \to h_c$ and $T_{l,r} \propto \Delta$and the system under this limit is described by massive QFT. Employing general QFT arguments, the literatures [37, 39] proposed an exact representation of the non-equilibrium density matrix ρ_{stat} which describes the the steady state limit of the previous section in the scaling limit

$$\langle \mathcal{O} \rangle_{\text{stat}} = \frac{\mathrm{Tr}\,(\rho_{\text{stat}}\,\mathcal{O})}{\mathrm{Tr}\,(\rho_{\text{stat}})}, \tag{A.5}$$

Consider a model of relativistic QFT with a spectrum of ℓ particle types, with masses $m_1, \ldots, m_\ell$. The vacuum is defined as $|\text{vac}\rangle$ and multi-particle asymptotic states are given by

$$|\theta_1, \ldots, \theta_n\rangle_{i_1,\ldots,i_n} \ : \ \theta_1 > \ldots > \theta_n, \quad i_1, \ldots, i_n \in \{1, 2, \ldots, \ell\}. \tag{A.6}$$

The density matrix ρ_{stat}, describing the energy-flow non-equilibrium steady state, is diagonalised on the basis of asymptotic states:

$$\rho_{\text{stat}}|\theta_1, \ldots, \theta_n\rangle_{i_1,\ldots,i_n} = e^{-\sum_k W_{i_k}(\theta_k)}|\theta_1, \ldots, \theta_n\rangle_{i_1,\ldots,i_n} \tag{A.7}$$

where

$$W_i(\theta) := (\beta_l \Theta(\theta) + \beta_r \Theta(-\theta))\, m_i \cosh\theta \tag{A.8}$$

with $\Theta(\theta)$ Heavyside's step function.

The density matrix (A.7) holds for the mathematical description of the energy-flow steady state in the XY model in [36] and is in agreement with the one obtained in the Ising model in [40].

A.3 The non-equilibrium steady state TBA equations

In this section, we will present the exact energy current obtained in [69]. We consider a general integrable model of relativistic QFT with diagonal scattering matrices.

In QFT, the energy current operator $\mathcal{J}$ is the momentum density $p(x)$. Using (A.5), we can express the average energy current J as

$$J = \frac{\mathrm{Tr}\left(\rho_{\mathrm{stat}}\, p(0)\right)}{\mathrm{Tr}\left(\rho_{\mathrm{stat}}\right)}. \tag{A.9}$$

where we consider $p(x)$ at an arbitrary point, say $x = 0$. The evaluation of this trace requires two ingredients. One is the matrix elements of $p(0)$. The other one is the knowledge of how to perform the trace. This is due to the fact that in states with nonzero densities of particles (i.e. infinitely many particles), the interaction between particles becomes important and this affects the way the trace is defined. In this sense, the information of the "local structure" (the full scattering and the set of local observables) is encoded not only into the matrix elements of $p(0)$ but also into the way the trace is performed.

In order to implement these two elements of information, the authors in [69] employ Al. B. Zamolodchikov's TBA arguments [157] which are based on the factorized scattering theory and the Bethe ansatz method. We consider the model defined on a finite, periodic space of circumference L. Integrable models of QFT on the finite space, as stated in [157], have similar description with that of Bethe ansatz integrable systems. Each state is described by a wavefunction which obeys the well-known Bethe ansatz equation. The quasi-particles in these states possess momenta p_k and energies e_k whose sums give the total momentum and energy of the states, with relativistic dispersion relation. Then, there is a natural finite-length extrapolation ρ^L_{stat} of ρ_{stat}, defined by its action on each state $|v\rangle$:

$$\rho^L_{\mathrm{stat}}|v\rangle = e^{-\sum_k W_k}|v\rangle, \quad W_k = \left(\beta_l \Theta(p_k) + \beta_r \Theta(-p_k)\right) e_k. \tag{A.10}$$

The finite-L extrapolation of the average current is the average of $p(0)$ with respect to ρ^L_{stat}:

$$J_L = \frac{\mathrm{Tr}_L\left[\rho^L_{\mathrm{stat}} p(0)\right]}{\mathrm{Tr}_L[\rho^L_{\mathrm{stat}}]} \tag{A.11}$$

with the relation $J = \lim_{L\to\infty} J_L$. Using translation invariance, we then have

$$J_L = L^{-1} \frac{\mathrm{Tr}_L \left[\rho_{\mathrm{stat}}^L P_L\right]}{\mathrm{Tr}_L[\rho_{\mathrm{stat}}^L]} \tag{A.12}$$

where $P = \int_{-L/2}^{L/2} dx\, p(x)$ is the total momentum. Introducing a generating parameter a associated to the momentum, we rewrite the current as

$$J = -\lim_{L\to\infty} \frac{d}{da} \log \mathrm{Tr}_L \left(\rho_{\mathrm{stat}}^L\, e^{-aP_L}\right)\Big|_{a=0} \tag{A.13}$$

We may also define the "free energy" f^a and evaluate the current from it:

$$f^a := -\lim_{L\to\infty} L^{-1} \log \mathrm{Tr}_L \left(\rho_{\mathrm{stat}}^L\, e^{-aP_L}\right), \quad J = \frac{d}{da} f^a\Big|_{a=0}. \tag{A.14}$$

It has been demonstrated in [69] that the free energy f^a can be evaluated by generalizing Al. B. Zamolodchikov's TBA arguments [157] to the non-equilibrium steady state (an extension of TBA arguments beyond Gibb's equilibrium was first carried out, to our best knowledge, for quantum quenches in the Lieb-Liniger model [184]), and the result leads to the exact non-equilibrium steady state TBA (NESSTBA) equation for the energy current in general integrable relativistic QFT with diagonal scatterings [69] :

$$\begin{aligned}
J &= \sum_{i=1}^{\ell} \int_{-\infty}^{\infty} \frac{d\theta}{2\pi} \frac{m_i \cosh\theta\, x_i(\theta)}{1+e^{\epsilon_i(\theta)}} \\
x_i(\theta) &= m_i \sinh\theta + \sum_{j=1}^{\ell} \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi} \frac{\varphi_{ij}(\theta-\gamma)\, x_j(\gamma)}{1+e^{\epsilon_j(\gamma)}} \\
\epsilon_i(\theta) &= W_i(\theta) - \sum_{j=1}^{\ell} \int_{-\infty}^{\infty} \frac{d\gamma}{2\pi}\, \varphi_{ij}(\theta-\gamma)\, \log(1+e^{-\epsilon_j(\gamma)})
\end{aligned} \tag{A.15}$$

where $\varphi_{ij}(\theta) = -\mathrm{i}\partial_\theta \log S_{ij}(\theta)$ and $S_{ij}(\theta)$ is the two-particle scattering matrix, and where $W_i(\theta)$ is defined in (A.8). It is conventional to define the L-functions

$$L_i(\theta) = \log(1+e^{-\epsilon_i(\theta)}), \tag{A.16}$$

as these functions possess interesting and perhaps clearer features than the pseudo-energies $\epsilon_i(\theta)$. Note that the psedo-energies, consequently the L-function, are not continuous at $\theta = 0$

$$\epsilon_i(+0) - \epsilon_i(-0) = m(\beta_l - \beta_r), \tag{A.17}$$

which is contrary to the equilibrium case.

A.4 High-temperature limit of the current

In this appendix we will consider the expression for the current presented in (A.15) in terms of thermodynamic quantities and carry out its high temperature limit. The concrete analysis we present in this chapter collects the results reported in [69]. Our computation is based on arguments in [157] where a similar work was carried out for determining the effective central charge $c_{\text{eff}}(r)$.

In an ℓ-particle system, the free energy in (A.14) is given by

$$f^a = -\sum_{i=1}^{\ell} \int \frac{d\theta}{2\pi} m_i \cosh\theta \log(1 + e^{-\epsilon_i(\theta)}). \tag{A.18}$$

Here, the pseudo-energies $\epsilon_i(\theta)$ satisfy thermodynamic bethe ansatz (TBA) equations:

$$-W_i(\theta) - am_i \sinh\theta + \varepsilon_i(\theta) + \sum_{j=1}^{\ell} \int \frac{d\theta'}{2\pi} \varphi_{ij}(\theta - \theta') \log(1 + e^{-\epsilon_j(\theta')}) = 0 \tag{A.19}$$

where

$$W_i(\theta) = \beta_L m_i \cosh\theta\, \Theta(\theta) + \beta_R m_i \cosh\theta\, \Theta(-\theta). \tag{A.20}$$

We can rewrite (A.18) as a sum of two integrals:

$$f^a = -\sum_{i=1}^{\ell} \int_{-\infty}^{0} \frac{d\theta}{2\pi} \frac{r_{ir}}{\beta_r} \cosh\theta L_i(\theta) - \sum_{i=1}^{\ell} \int_{0}^{\infty} \frac{d\theta}{2\pi} \frac{r_{il}}{\beta_l} \cosh\theta L_i(\theta) := f_r^a + f_l^a \tag{A.21}$$

where $r_{ir} = m_i\beta_r$ and $r_{il} = m_i\beta_l$. We can obtain the steady-state current in CFT by taking the limits $r_{ir} \to 0$ and $r_{il} \to 0$. In this high temperature limit, $\epsilon_i(\theta)$ and consequently $L_i(\theta)$ are constants with limiting values $\epsilon_i(0), L_i(0)$ in the central region $-\log(2/r_{ir}) \ll \theta \ll \log(2/r_{il})$ and it goes to infinity at the two edges. Therefore, $L_i(\theta)$ exhibits a typical plateau behaviour in the central region and has a double exponential falloff outside this region (we have seen many graphical examples of this in previous sections). As r_{il} and r_{ir} go to zero, the plateaux become wider and the form of their two edges tends to some universal pattern. The limiting form of the left edge is determined by the "kink" solution $L_{ik-}(\theta)$ which satisfies the equation

$$-\left(1 - \frac{a}{\beta_r}\right) e^{-\theta} + \epsilon_{ik-}(\theta) + \sum_{j=1}^{n} \int \frac{d\theta'}{2\pi} \varphi_{ij}(\theta - \theta') L_{jk-}(\theta') = 0 \tag{A.22}$$

where $\epsilon_{ik-}(\theta) \equiv \epsilon_i[\theta - \log(2/r_{ir})]$ and $L_{ik-}(\theta) \equiv L_i[\theta - \log(2/r_{ir})]$. Likewise, the limiting form of the right edge is determined by the function $L_{ak+}(\theta)$ which satisfies the equation

$$-\left(1+\frac{a}{\beta_l}\right)e^{\theta} + \epsilon_{ik+}(\theta) + \sum_{j=1}^{n}\int \frac{d\theta'}{2\pi}\varphi_{ij}(\theta-\theta')L_{jk+}(\theta') = 0 \tag{A.23}$$

where $\epsilon_{ik+}(\theta) \equiv \epsilon_i[\theta + \log(2/r_{il})]$ and $L_{ik+}(\theta) \equiv L_i[\theta + \log(2/r_{il})]$. These equations follow from (A.19) by performing the indicated rapidity shifts. Thus, the high temperature limit of the free energies $f^a_{r,l}$ can be written in terms of these kink solutions:

$$f^a_r = -\frac{1}{\beta_r}\sum_{i=1}^{\ell}\int \frac{d\theta}{2\pi}e^{-\theta}L_{ik-}(\theta), \qquad f^a_l = -\frac{1}{\beta_l}\sum_{i=1}^{\ell}\int \frac{d\theta}{2\pi}e^{\theta}L_{ik+}(\theta) \tag{A.24}$$

Differentiating (A.22) and (A.23) with respect to θ we have

$$\left(1-\frac{a}{\beta_r}\right)e^{-\theta} + \frac{\partial \epsilon_{ik-}(\theta)}{\partial\theta} + \sum_{j=1}^{\ell}\int \frac{dL_{jk-}(\theta')}{2\pi}\varphi_{ij}(\theta-\theta') = 0 \tag{A.25}$$

and

$$-\left(1+\frac{a}{\beta_l}\right)e^{\theta} + \frac{\partial \epsilon_{ik+}(\theta)}{\partial\theta} + \sum_{j=1}^{\ell}\int \frac{dL_{jk+}(\theta')}{2\pi}\varphi_{ij}(\theta-\theta') = 0 \tag{A.26}$$

Solving these equations for $e^{\pm\theta}$ and substituting $e^{-\theta}$ in f^a_r and e^{θ} in f^a_l we obtain

$$f^a_r = \frac{1}{\beta_r - a}\sum_{i=1}^{\ell}\int \frac{d\theta}{2\pi}\left(\frac{\partial \epsilon_{ik-}(\theta)}{\partial\theta} + \sum_{j=1}^{\ell}\int \frac{dL_{jk-}(\theta')}{2\pi}\varphi_{ij}(\theta-\theta')\right)L_{ik-}(\theta) \tag{A.27}$$

and

$$f^a_l = -\frac{1}{\beta_l + a}\sum_{i=1}^{\ell}\int \frac{d\theta}{2\pi}\left(\frac{\partial \epsilon_{ik+}(\theta)}{\partial\theta} + \sum_{j=1}^{\ell}\int \frac{dL_{jk+}(\theta')}{2\pi}\varphi_{ij}(\theta-\theta')\right)L_{ik+}(\theta) \tag{A.28}$$

Let us consider f^a_r in more detail. When $r_{ir}, r_{il} \to 0$ we can rewrite it as

$$f^a_r = \frac{1}{\beta_r - a}\frac{1}{2\pi}\left[\sum_{i=1}^{\ell}\int_{\epsilon_i(0)}^{\infty} d\epsilon \log(1+e^{-\epsilon}) - \sum_{i,j=1}^{\ell}\int_{\epsilon_i(0)}^{\infty}\frac{d\epsilon_{ik-}}{1+e^{\epsilon_{ik-}}}\varphi_{ij} * L_{jk-}(\theta)\right] \tag{A.29}$$

where we assumed there is parity invariance $\varphi_{ij}(\theta) = \varphi_{ji}(\theta)$ and we used $dL = -d\epsilon/(1+e^{\epsilon})$. By substituting the convolution $\varphi_{ij} * L_{jk-}(\theta)$ by its expression from equation (A.22), we have

$$\begin{aligned}\int_{\epsilon_i(0)}^{\infty}\frac{d\epsilon_{ik-}}{1+e^{-\epsilon_{ik-}}}\varphi_{ij}*L_{jk-}(\theta) &= -\int_{\epsilon_i(0)}^{\infty}\frac{\epsilon_{ik-}d\epsilon_{ik-}}{1+e^{\epsilon_{ik-}}}+\frac{\beta_r-a}{\beta_r}\int_{L_i(0)}^{\infty}e^{-\theta}dL_{ik-}\\ &= -\int_{\epsilon_i(0)}^{\infty}\frac{\epsilon_{ik-}d\epsilon_{ik-}}{1+e^{\epsilon_{ik-}}}+\frac{\beta_r-a}{\beta_r}\int_{-\log(\frac{r_{ir}}{2})}^{\infty}e^{-\theta}L_{ik-}d\theta\end{aligned} \tag{A.30}$$

where in the last line we used integration by parts. Thanks to the observation that the last term is (up to constants and summing up in i) nothing but the original function f_r^a, substituting (A.30) into (A.29) leads to

$$f_r^a=\frac{1}{4\pi}\left(\frac{1}{\beta_r-a}\right)\sum_{i=1}^{\ell}\int_{\epsilon_i(0)}^{\infty}d\epsilon\left(\log(1+e^{-\epsilon})+\frac{\epsilon}{1+e^{\epsilon}}\right). \tag{A.31}$$

Following the same recipe, we can deduce f_l^a as

$$f_l^a=-\frac{1}{4\pi}\left(\frac{1}{\beta_l+a}\right)\sum_{i=1}^{\ell}\int_{\epsilon_i(0)}^{\infty}d\epsilon\left(\log(1+e^{-\epsilon})+\frac{\epsilon}{1+e^{\epsilon}}\right). \tag{A.32}$$

Putting together (A.31) and (A.32), we have:

$$f^a=\frac{1}{4\pi}\left(\frac{1}{\beta_r-a}-\frac{1}{\beta_l+a}\right)\sum_{i=1}^{\ell}\int_{\epsilon_i(0)}^{\infty}d\epsilon\left(\log(1+e^{-\epsilon})+\frac{\epsilon}{1+e^{\epsilon}}\right). \tag{A.33}$$

Differentiating (A.33) with respect to a at $a=0$, we finally obtain

$$J(\beta_l,\beta_r)=\frac{1}{4\pi}\left[\frac{1}{\beta_l^2}-\frac{1}{\beta_r^2}\right]\sum_{i}^{\ell}\int_{\epsilon_i(0)}^{\infty}d\epsilon\left(\log(1+e^{-\epsilon})+\frac{\epsilon}{1+e^{\epsilon}}\right) \tag{A.34}$$

which is a representation in terms of the Roger's dilogarithm function. To determine the values $\epsilon_i(0)$, we should make use of the property that the TBA kernel $\psi_{ij}(\theta)$ is usually picked about $\theta=0$, and the property that the function $L_j(\theta)$ in the limits $r_{ir}\to 0$ and $r_{il}\to 0$ is constant around $\theta=0$. This implies that the convolution $\varphi_{ij}*L_j(\theta)$ may be approximated by

$$\varphi_{ij}*L_j(\theta)\approx N_{ij}\log(1+e^{-\epsilon_i}), \tag{A.35}$$

with $N_{ij}=-\frac{1}{2\pi}\int\varphi_{ij}(\theta)d\theta$. The original (both at and out of equilibrium) TBA equations then become the constant TBA equations:

$$\epsilon_i(0)=\sum_{j=1}^{n}N_{ij}\log(1+e^{-\epsilon_j(0)}) \tag{A.36}$$

which were introduced in [185]. As we can see, the result (A.34) exactly reproduces the CFT prediction given in [37]

$$J_{\rm CFT} = \frac{c\pi}{12}(T_l^2 - T_r^2) \tag{A.37}$$

where c is the central charge, with the identification that

$$c = \frac{3}{\pi^2}\sum_i^{\ell}\int_{\epsilon_i(0)}^{\infty} d\epsilon\left(\log(1+e^{-\epsilon}) + \frac{\epsilon}{1+e^{\epsilon}}\right). \tag{A.38}$$

The connection of the Roger's dilogarithm function with the central charge has been discussed in detail in the classic literature on the subject [157, 186].

We now consider another interesting physical situation in which one temperature is kept constant and low while the other temperature becomes very large. Thus we have r_{ir} = const. and $r_{il} \to 0$. In order to study the current in this case we start again with the separation of the ground state energy into the two contributions (A.21). In this situation the functions $\epsilon_i(\theta)$ are not continuous any more at $\theta = 0$ due to the presence of $W(\theta)$ in the TBA equations and they have two different limiting values at $\theta = 0$ with the relation:

$$\epsilon_{i-}(0) = \epsilon_{i+}(0) + r_{ir}, \tag{A.39}$$

where we use the notation $\epsilon_{i+}(\theta)$ and $\epsilon_{i-}(\theta)$ to represent $\epsilon(\theta)$ in the regions $\theta > 0$ and $\theta < 0$, respectively. We also have to note that the left-right asymmetry of the L-function has been broken. The right part of L-function remains the same: it has a plateau at $\log(1 + e^{-\epsilon_{i+}(0)})$ in the region $0 \leq \theta \ll \log(2/r_{il})$. But the left part has only a quick exponential falloff without exhibiting any plateau behaviour, since $\epsilon_{i-}(\theta) \approx r_{ir}\cosh\theta$ for $\theta < 0$ and $r_{ir} \geq 1$. As a result, f_r^a is simply defined by the first term in (A.21) and can not be further simplified, while f_l^a can be computed, following same lines, as the second term in (A.33). Finally, differentiating with respect to a and setting $a = 0$ yield the result:

$$\begin{aligned} J(\beta_l,\beta_r) &= \sum_{i=1}^{n}\left[\int_{-\infty}^{0}\frac{d\theta}{2\pi}m_i\cosh\theta\frac{x_{i-}(\theta)}{1+e^{\varepsilon_{i-}(\theta)}} + \frac{1}{4\pi\beta_l^2}\int_{\epsilon_{i+}(0)}^{\infty}d\epsilon\left(\log(1+e^{-\epsilon}) + \frac{\epsilon}{1+e^{\epsilon}}\right)\right] \\ &= \sum_{i=1}^{n}\int_{-\infty}^{0}\frac{d\theta}{2\pi}m_i\cosh\theta\frac{x_{i-}(\theta)}{1+e^{\varepsilon_{i-}(\theta)}} + \frac{\pi c}{12\beta_l^2}. \end{aligned} \tag{A.40}$$

where $x_{i-}(\theta)$ can be obtained from (A.19) as $x_{i-}(\theta) = \left.\frac{d\epsilon_{i-}(\theta)}{da}\right|_{a=0}$ and where in the last line we used the relation (A.38).

A.5 Low-temperature expansion of the current

Although the TBA equations generally need to be solved numerically, at low temperatures a perturbative expansion may be used leading to analytic results. In particular, these analytic results will contribute to the rigorous proof of the non-additivity of the current as discussed in [69]. Following the finite-volume regularization methods of Pozsgay and Takacs [16], we will deduce the low-temperature expansion of the energy current. This deduction can be found in [69].

We again assume a single particle spectrum for simplicity. For convention, the finite-volume multi-particle states can be denoted as

$$|\theta_1, ..., \theta_n\rangle_L$$

The corresponding energy levels are determined by the Bethe ansatz equations

$$Q_k(\theta_1, ..., \theta_n) = mL \sinh\theta_k + \sum_{l\neq k} \delta(\theta_k - \theta_l) = 2\pi I_k \quad , \quad k = 1, ..., n \tag{A.41}$$

where I_k are momentum quantum numbers and $\delta(\theta) = -i \log S(\theta)$ is the two-particle scattering phase-shift. The density of multi-particle states can be obtained by

$$\rho(\theta_1, ..., \theta_n) = \det \mathcal{J}^{(n)} \quad , \quad \mathcal{J}^{(n)}_{kl} = \frac{\partial Q_k(\theta_1, ..., \theta_n)}{\partial \theta_l} \quad , \quad k, l = 1, ..., n \tag{A.42}$$

Let us expand the traces in (A.12):

$$\begin{aligned} \mathrm{Tr}_L(\rho^L_{\mathrm{stat}} P) &= \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m \sinh\theta^{(1)} + \frac{1}{2} \sideset{}{'}\sum_{\theta^{(2)}_1 \theta^{(2)}_2} e^{-\sum_{i=1}^{2} W(\theta^{(2)}_i)} \sum_{i=1}^{2} m \sinh\theta^{(2)}_i \\ &\quad + \frac{1}{6} \sideset{}{'}\sum_{\theta^{(3)}_1 \theta^{(3)}_2 \theta^{(3)}_3} e^{-\sum_{i=1}^{3} W(\theta^{(3)}_i)} \sum_{i=1}^{3} m \sinh\theta^{(3)}_i + O(e^{-4W}) \end{aligned} \tag{A.43}$$

and

$$\begin{aligned} \mathrm{Tr}_L(\rho^L_{\mathrm{stat}}) &= 1 + \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} + \frac{1}{2} \sideset{}{'}\sum_{\theta^{(2)}_1 \theta^{(2)}_2} e^{-\sum_{i=1}^{2} W(\theta^{(2)}_i)} \\ &\quad + \frac{1}{6} \sideset{}{'}\sum_{\theta^{(3)}_1 \theta^{(3)}_2 \theta^{(3)}_3} e^{-\sum_{i=1}^{3} W(\theta^{(3)}_i)} + O(e^{-4W}) \end{aligned} \tag{A.44}$$

At low temperature, we have the expansion

$$\begin{aligned}\frac{1}{\mathrm{Tr}_L(\rho^L_{\mathrm{stat}})} &= 1-\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}+\left(\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\right)^2-\frac{1}{2}{\sum_{\theta_1^{(2)}\theta_2^{(2)}}}' e^{-\sum_{i=1}^2 W(\theta_i^{(2)})}\\ &-\left(\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\right)^3+\left(\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\right){\sum_{\theta_1^{(2)}\theta_2^{(2)}}}' e^{-\sum_{i=1}^2 W(\theta_i^{(2)})}\\ &-\frac{1}{6}{\sum_{\theta_1^{(3)}\theta_2^{(3)}\theta_3^{(3)}}}' e^{-\sum_{i=1}^3 W(\theta_i^{(3)})}+O(e^{-4W}).\end{aligned} \tag{A.45}$$

The prefactors $1/n!$ for every multi-particle sum account for overcounted states with different ordering of the same set of rapidities. The prime in the multi-particle sum indicates all quantum numbers (rapidities) for the state are different. The upper indices of the rapidities and W represent the number of particles in the state.

With (A.43) and (A.45), we can now obtain the current up to exponential corrections at finite volume L. In the limit $L\to\infty$, the low temperature expansion of the current should be recovered. Here, we present the calculation of the current up to the first three orders.

- First order

  At finite volume L, we have the first-order contribution to the current:

$$J_1^L=\frac{1}{L}\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m\sinh\theta^{(1)}. \tag{A.46}$$

  We then take the limit $L\to\infty$ by replacing the sum over rapidities by an integral over the states in the rapidity space, namely $\sum_{\theta^{(1)}}\to\int\frac{d\theta}{2\pi}\rho_1(\theta)$. The density of one-particle states $\rho_1(\theta)$ is obtained by

$$\rho_1(\theta)=L\frac{d\,p(\theta)}{d\theta}=\frac{d\,m\sinh\theta}{d\theta}=mL\cosh\theta. \tag{A.47}$$

  Therefore, the first order term of the current is

$$J_1=\lim_{L\to\infty}J_1^L=m^2\int\frac{d\theta}{2\pi}e^{-W(\theta)}\sinh\theta\cosh\theta. \tag{A.48}$$

- Second order

A similar computation can be performed for second order terms. We find the second-order term of the current at finite volume L:

$$
\begin{aligned}
J_2^L &= \frac{1}{L}\left[-\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m\sinh\theta^{(1)} \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\right. \\
&\quad \left. + \frac{1}{2}\sum_{\theta_1^{(2)}\theta_2^{(2)}}{}^{'} e^{-\sum_{i=1}^{2} W(\theta_i^{(2)})} \sum_{i=1}^{2} m\sinh\theta_i^{(2)}\right] \\
&= \frac{1}{L}\left[-\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m\sinh\theta^{(1)} \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\right. \\
&\quad + \frac{1}{2}\sum_{\theta_1^{(2)}\theta_2^{(2)}} e^{-\sum_{i=1}^{2} W(\theta_i^{(2)})} \sum_{i=1}^{2} m\sinh\theta_i^{(2)} \\
&\quad \left. - \frac{1}{2}\sum_{\theta_1^{(2)}=\theta_2^{(2)}} e^{-2W(\theta_1^{(2)})} 2m\sinh\theta_1^{(2)}\right]
\end{aligned}
$$

where the last term corresponds to a two-particle state with equal quantum numbers ($\theta_1^{(2)} = \theta_2^{(2)}$) of the two particles. In this case, the two-particle Bethe ansatz equations degenerate to a one-particle equation, which means that density of this two-particle state is again ρ_1. In the large L limit, we may replace the sums with integrals as

$$
\sum_{\theta^{(1)}} \to \int \frac{d\theta}{2\pi}\rho_1(\theta), \qquad \sum_{\theta_1^{(2)}=\theta_2^{(2)}} \to \int \frac{d\theta}{2\pi}\rho_1(\theta), \qquad \sum_{\theta_1^{(2)}\theta_2^{(2)}} \to \int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\rho_2(\theta_1,\theta_2).
$$

The Bethe ansatz equations for a two-particle state are

$$
\begin{aligned}
mL\sinh\theta_1 + \delta(\theta_1-\theta_2) &= Q_1(\theta_1\theta_2) \\
mL\sinh\theta_2 + \delta(\theta_2-\theta_1) &= Q_2(\theta_1\theta_2),
\end{aligned} \tag{A.49}
$$

and hence the relevant density of two-particle states is given by

$$
\rho_2(\theta_1,\theta_2) = \det\begin{pmatrix} mL\cosh\theta_1 + \varphi(\theta_1-\theta_2) & -\varphi(\theta_1-\theta_2) \\ -\varphi(\theta_1-\theta_2) & mL\sinh\theta_2 + \varphi(\theta_1-\theta_2) \end{pmatrix}.
$$

Taking into account the fact that $S(\theta)S(-\theta) = 1$, we find that $\varphi(\theta) = \varphi(-\theta)$. Using this property, relabeling integration variables and exchanging the order of integration, we obtain

$$
\begin{aligned}
J_2 &= \lim_{L\to\infty} J_2^L \\
&= \frac{1}{L}\Bigg[-\int (\frac{1}{2\pi})^2 d\theta_1 d\theta_2 e^{-W(\theta_1)-W(\theta_2)} m\sinh\theta_1 (mL)^2 \cosh\theta_1 \cosh\theta_2 \\
&+\frac{1}{2}\int (\frac{1}{2\pi})^2 d\theta_1 d\theta_2 e^{-W(\theta_1)-W(\theta_2)} \Big(m\sinh\theta_1 + m\sinh\theta_2 \Big) \times \\
&\Big(m^2L^2\cosh\theta_1\cosh\theta_2 + mL\cosh\theta_1\varphi(\theta_1-\theta_2) + mL\cosh\theta_2\varphi(\theta_1-\theta_2) \Big) \\
&- \int d\theta e^{-2W(\theta)} m\sinh\theta mL\cosh\theta \Bigg] \\
&= m^2 \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \cosh\theta_1 (\sinh\theta_1 + \sinh\theta_2)\varphi(\theta_1-\theta_2) e^{-W(\theta_1)-W(\theta_2)} \\
&-m^2 \int \frac{d\theta}{2\pi} \cosh\theta \sinh\theta e^{-2W(\theta)}. \qquad \text{(A.50)}
\end{aligned}
$$

- Third order

Finally, we look at third-order contributions

$$
\begin{aligned}
J_3^L &= \frac{1}{L}\Bigg[\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m\sinh\theta^{(1)} \Big(\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} \Big)^2 \\
&-\frac{1}{2}\sum_{\theta^{(1)}} m\sinh\theta^{(1)} \sum_{\theta_1^{(2)}\theta_2^{(2)}}{}' e^{-\sum_{i=1}^2 W(\theta_i^{(2)})} \\
&-\frac{1}{2}\sum_{\theta_1^{(2)}\theta_2^{(2)}}{}' e^{-\sum_{i=1}^2 W(\theta_i^{(2)})} \sum_{i=1}^{2} m\sinh\theta_i^{(2)} \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} \\
&+\frac{1}{6}\sum_{\theta_1^{(3)}\theta_2^{(3)}\theta_3^{(3)}}{}' e^{-\sum_{i=1}^3 W(\theta_i^{(3)})} \sum_{i=1}^{3} m\sinh\theta_i^{(3)} \Bigg]. \qquad \text{(A.51)}
\end{aligned}
$$

By using the relations

$$
\begin{aligned}
\sum_{\theta_1^{(2)}\theta_2^{(2)}}{}' &= \sum_{\theta_1^{(2)}\theta_2^{(2)}} - \sum_{\theta_1^{(2)}=\theta_2^{(2)}} \\
\sum_{\theta_1^{(3)}\theta_2^{(3)}\theta_3^{(3)}}{}' &= \sum_{\theta_1^{(3)}\theta_2^{(3)}\theta_3^{(3)}} -3 \sum_{\theta_1^{(3)},\theta_2^{(3)}=\theta_3^{(3)}} +2 \sum_{\theta_1^{(3)}=\theta_2^{(3)}=\theta_3^{(3)}}
\end{aligned}
$$

we can rewrite

$$
\begin{aligned}
J_3^L \quad = \quad & \frac{1}{L}\Bigg[\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} m\sinh\theta^{(1)} \Bigg(\sum_{\theta^{(1)}} e^{-W(\theta^{(1)})}\Bigg)^2 \\
& -\frac{1}{2}\sum_{\theta^{(1)}} m\sinh\theta^{(1)} \sum_{\theta_1^{(2)}\theta_2^{(2)}} e^{-\sum_{i=1}^{2} W(\theta_i^{(2)})} \\
& +\frac{1}{2}\sum_{\theta^{(1)}} m\sinh\theta^{(1)} \sum_{\theta_1^{(2)}=\theta_2^{(2)}} e^{-2W(\theta_1^{(2)})} \\
& -\frac{1}{2}\sum_{\theta_1^{(2)}\theta_2^{(2)}} e^{-\sum_{i=1}^{2} W(\theta_i^{(2)})} \sum_{i=1}^{2} m\sinh\theta_i^{(2)} \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} \\
& +\frac{1}{2}\sum_{\theta_1^{(2)}=\theta_2^{(2)}} e^{-2W(\theta_1^{(2)})} 2m\sinh\theta_1^{(2)} \sum_{\theta^{(1)}} e^{-W(\theta^{(1)})} \\
& +\frac{1}{6}\sum_{\theta_1^{(3)}\theta_2^{(3)}\theta_3^{(3)}} e^{-\sum_{i=1}^{3} W(\theta_i^{(3)})} \sum_{i=1}^{3} m\sinh\theta_i^{(3)} \\
& -\frac{1}{2}\sum_{\theta_1^{(3)},\theta_2^{(3)}=\theta_3^{(3)}} e^{-W(\theta_1^{(3)})-2W(\theta_2^{(3)})} (m\sinh\theta_1^{(3)} + 2m\sinh\theta_2^{(3)}) \\
& +\frac{1}{3}\sum_{\theta_1^{(3)}=\theta_2^{(3)}=\theta_3^{(3)}} e^{-3W(\theta_1^{(3)})} 3m\sinh\theta_1^{(3)}\Bigg]. \qquad (A.52)
\end{aligned}
$$

In the large L limit, we replace the sums with integrals as in previous cases with the addition of

$$
\begin{aligned}
\sum_{\theta_1^{(2)}\theta_2^{(2)}\theta_1^{(3)}} \quad & \rightarrow \quad \int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\frac{d\theta_3}{2\pi}\rho_3(\theta_1,\theta_2,\theta_3), \\
\sum_{\theta_1^{(2)},\theta_2^{(2)}=\theta_1^{(3)}} \quad & \rightarrow \quad \int \frac{d\theta_1}{2\pi}\frac{d\theta_2}{2\pi}\rho_3(\theta_1,\theta_2=\theta_3), \\
\sum_{\theta_1^{(2)}=\theta_2^{(2)}=\theta_1^{(3)}} \quad & \rightarrow \quad \int \frac{d\theta}{2\pi}\rho_1(\theta). \qquad (A.53)
\end{aligned}
$$

The density $\rho_3(\theta_1,\theta_2=\theta_3)$ can be obtained from the Bethe ansatz equations for a three-particle state with two equal quantum numbers

$$
\begin{aligned}
mL\sinh\theta_1 + 2\delta(\theta_1-\theta_2) \quad & = \quad Q_1(\theta_1,\theta_2) \\
mL\sinh\theta_2 + \delta(\theta_2-\theta_1) \quad & = \quad Q_2(\theta_1,\theta_2). \qquad (A.54)
\end{aligned}
$$

The relevant density is given by

$$\rho_3(\theta_1, \theta_2 = \theta_3) = \det \begin{pmatrix} mL\cosh\theta_1 + 2\varphi(\theta_1 - \theta_2) & -2\varphi(\theta_1 - \theta_2) \\ -\varphi(\theta_1 - \theta_2) & mL\sinh\theta_2 + \varphi(\theta_1 - \theta_2) \end{pmatrix}$$

Similarly, we can obtain $\rho_3(\theta_1, \theta_2, \theta_3)$ from the Bethe ansatz equations for a three-particle state with three distinct quantum numbers

$$\begin{aligned} mL\sinh\theta_1 + \delta(\theta_1 - \theta_2) + \delta(\theta_1 - \theta_3) &= Q_1(\theta_1, \theta_2, \theta_3) \\ mL\sinh\theta_2 + \delta(\theta_2 - \theta_1) + \delta(\theta_2 - \theta_3) &= Q_2(\theta_1, \theta_2, \theta_3) \\ mL\sinh\theta_3 + \delta(\theta_3 - \theta_1) + \delta(\theta_3 - \theta_2) &= Q_3(\theta_1, \theta_2, \theta_3), \end{aligned} \tag{A.55}$$

as

$$\det \begin{pmatrix} E_1L + \varphi(\theta_{12}) + \varphi_(\theta_{13}) & -\varphi(\theta_{12}) & -\varphi(\theta_{13}) \\ -\varphi(\theta_{12}) & E_2L + \varphi(\theta_{12}) + \varphi(\theta_{23}) & -\varphi(\theta_{23}) \\ -\varphi(\theta_{13}) & -\varphi(\theta_{23}) & E_3L + \varphi(\theta_{13}) + \varphi(\theta_{23}) \end{pmatrix}, \tag{A.56}$$

where for convenience we used the notation $E_i \equiv m\cosh\theta_i$ and $\varphi(\theta_{ij}) \equiv \varphi(\theta_i - \theta_j)$. Therefore, by performing a similar but more tedious computation, we arrive at the third order term of the current

$$\begin{aligned} J_3 \;=\; & -\frac{1}{2}m^2 \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \cosh\theta_1(\sinh\theta_1 + 2\sinh\theta_2)\varphi(\theta_1 - \theta_2)e^{-W(\theta_1)-2W(\theta_2)} \\ & + m^2 \int \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3} \cosh\theta_1 \sum_{i=1}^{3}(\sinh\theta_i)\varphi(\theta_1 - \theta_2)\varphi(\theta_2 - \theta_3)e^{-\sum_{i=1}^{3} W(\theta_i)} \\ & + \frac{1}{2}m^2 \int \frac{d\theta_1 d\theta_2 d\theta_3}{(2\pi)^3} \cosh\theta_1 \sum_{i=1}^{3}(\sinh\theta_i)\varphi(\theta_1 - \theta_2)\varphi(\theta_1 - \theta_3)e^{-\sum_{i=1}^{3} W(\theta_i)} \\ & - m^2 \int \frac{d\theta_1 d\theta_2}{(2\pi)^2} \cosh\theta_1(2\sinh\theta_1 + \sinh\theta_2)\varphi(\theta_1 - \theta_2)e^{-2W(\theta_1)-W(\theta_2)} \\ & + m^2 \int \frac{d\theta}{2\pi} \cosh\theta \sinh\theta e^{-3W(\theta)} \end{aligned} \tag{A.57}$$

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