Selberg Integral and Gauge/Toda Duality

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Doctoral Dissertation

Selberg Integral and Gauge/Toda Duality
（Selberg積分とゲージ／戸田双対性）

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Abstract

We work on the proof of Gauge/Toda Duality(AGT conjecture) through two different ways.First we generalized A. Mironov et. al.'s idea to the much more complicated SU(N) case. We calculated the conformal block in the form of Dotsenko-Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials. We found a formula for such Selberg average which satisfies some nontrivial consistency conditions and showed that it reproduces the SU(N) version of AGT conjecture. Besides, we worked out many technical details, including proofs of lemmas lacked in A. Mironov et. al.'s paper, which are essential to bring Selberg average into the form of Yang-Mills partition function. The other approach is based on recursion relations. We derive an infinite set of recursion formulae for Nekrasov instanton partition function for linear quiver U(N) supersymmetric gauge theories in 4D. They have a structure of a deformed version of $\mathcal{W}_{1+\infty}$ algebra which is called $\mathbf{SH}^c$ algebra (or degenerate dual affine Hecke algebra) in the literature. The algebra contains W_N algebra with general central charge defined by a parameter β, which gives the Ω background in Nekrasov's analysis. Some parts of the formulae are identified with the conformal Ward identity for the conformal block function for Toda field theory. The SU(N) constraints give a direct support for AGT conjecture for general quiver gauge theories.

Contents

1 Introduction

This thesis features on the author's efforts to prove AGT conjecture[1]. AGT conjecture surprises the academics as an amazing identification between two seemingly totally independent subjects: the partition function of 4 dimensional gauge theory and the correlation function of 2 dimensional conformal field theory. These two subjects have been respectively important research objects in both physics and mathematics, but even mathematicians did not notice the relation between them. Thus AGT conjecture achieves great concern from both fields and is highly evaluated as a new territory of mathematical physics, and many related researches are inspired since its publication.

Historically, string duality is crucial in the understanding of different types of string theories and their counterparts like quantum gravity and gauge theories, and serves as a powerful tool for studying strongly-coupled theories. For years most of this kind of research has been limited to AdS/CFT, but in 2002, Nekrsov performed a technique called Ω deformation in the reduction from 6D $\mathcal{N}=1$ gauge theory to 4D $\mathcal{N}=2$ gauge theory, and implied its connection with 2D conformal theory[2]. He found exact formulae of the partition function (Nekrasov partition function) of the $\mathcal{N}=2$ gauge theory, and showed that it reproduces the prepotential as determined by the Seiberg-Witten curve[3, 4]. Later in 2009, some news attracts people ' s attention. Alday, Gaiotto and Tachikawa presented an interesting observation that the Nekrasov partition functions of certain class of N=2 SU(2) gauge theories seem to coincide with the correlation function of 2D Liouville field theory (AGT conjecture). Soon later, Wyllard [5] and others [6, 7] have presented a generalization to SU(N) case.

In their observations, the correlation functions of Liouville (Toda) field theories[8, 9] are identified with the integral of the Nekrasov partition function $Z_{\rm Nek}$, where the instanton part $Z_{\rm inst}$ in the gauge theory written in a form [10] is identical to the conformal blocks, and the perturbative part $Z_{\rm 1loop}$ corresponds to the (product of) three point functions. AGT conjecture is illuminating in showing a correspondence between 4D Yang-Mills and 2D integrable models and will be fundamental in the understanding of the duality of gauge theories. It will also be relevant to understanding strong coupling physics of multiple M5-branes. In this respect, it will be important to explore to what extent and how this conjecture holds. Especially, since the coincidence was found through the first few orders in the instanton expansion, the exact computation of conformal block is needed in the Liouville(Toda) side.

Since then there were many attempts on the interpretation of AGT conjecture, but no complete proof had been achieved. In 2011, A. Mironov et. al. had embarked on an interesting step toward this direction[11, 12]. They used the Dotsenko-Fateev method [13] to calculate the conformal blocks. They analyzed the simplest example $SU(2)$, $N_f=4$ and proved the AGT relation for a special choice of the Ω deformation parameter $\beta=-\epsilon_1/\epsilon_2=1$. The key step in their analysis is the reduction of the Dotsenko-Fateev (DF) formula to Selberg average with one or two Jack polynomial(s) which was computed explicitly by Kadell [14].This work is of great importance as the first direct proof, but still rigorous analysis is in demand. Recently, O. Schiffmann and E. Vasserot successfully introduced an algebra called $\mathbf{SH}^c$ to prove the AGT conjecture[15], yet their work is limited to pure super Yang-Mills theory.

We have been working on the proof of AGT conjecture through two different ways.

First we generalized A. Mironov et. al.'s idea to the much more complicated SU(N) case[16]. We calculated the conformal block in the form of Dotsenko-Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials. The old Dotsenko-Fateev integral and the choice of paths of the screening operators play key roles in this correspondence, and their significance to Matrix models and conformal blocks are pointed out in the work[17]. In [18], Itoyama and Oota calculated the SU(2) case of the reduction from DF integral to Selberg-Jack integral, and we performed the SU(N) version's study.

Selberg integral is an n-dimensional generalization of the Euler beta integral, and Jack polynomial is a kind of symmetric polynomial labeled by Young diagram. Selberg integral showed its prominence, evidenced by its central role in random matrix theory, Calogero-Sutherland quantum many body systems, Knizhnik-Zamolodchikov equations, and multivariable orthogonal polynomial theory. By q-deformation, Jack polynomial will upgrade to MacDonald polynomial, whose application in five dimensions is in anticipation. These two subjects have long histories and are wide applied in both mathematical and physical fields. Yet the interaction of them has been a forbidding issue.

Surely, if we want to achieve a full direct proof for SU(N) case, the exact expression of SU(N) Selberg Jack integral will be required. No such formula is available in the mathematics literature, so we need to calculate this kind of integral by ourselves. Fortunately there are still some materials that for us to refer to. For SU(2) case, the relevant Selberg averages for one and two Jack polynomials were obtained by Kadell[14], and The one-Jack Selberg integral for SU(N) could be calculated by the formula offered by Warnaar[19, 20]. There works serve as a good hint for our calculation. Furthermore, another advantage we own is that, we already more or less know the deserved form of the Selberg Jack integral, from the expectation of AGT conjecture.

Though the actual process is much more complicated than expected, we manage to found a formula for such Selberg average which satisfies some nontrivial consistency conditions and showed that it reproduces the SU(N) version of AGT conjecture. Besides, we work out many technical details, including proofs of lemmas lacked in A. Mironov et. al. ' s paper, which are essential to bring Selberg average into the form of Yang-Mills partition function. This work is the first direct approach of SU(N) AGT conjecture with $\beta = 1$.

Our recent method is based on recursion relations. We derive an infinite set of recursion formulae for Nekrasov instanton partition function for linear quiver U(N) supersymmetric gauge theories in 4D. They have a structure of a deformed version of $\mathcal{W}_{1+\infty}$ algebra which is called $\mathbf{SH}^c$ algebra in the literature. The algebra contains W_N algebra with general central charge defined by a parameter β, which gives the Ω background in Nekrasov ' s analysis. Some parts of the formulae are identified with the conformal Ward identity for the conformal block function for Toda field theory. The SU(N) constraints give a direct support for AGT conjecture for general quiver gauge theories.

In detail, the instanton partition function for linear quiver gauge theories is decomposed into matrix like product with a factor $Z_{\vec{Y},\vec{W}}$ which depends on two sets of Young diagrams (28). Here the Young diagrams $\vec{Y} = (Y_1, \cdots, Y_N)$ represent the fixed points of $U(N)$ instanton moduli space under localization. $Z_{\vec{Y},\vec{W}}$ consists of contribution from one bifundamental hypermultiplet and vectormultiplets. We find that the building block $Z_{\vec{Y},\vec{W}}$ satisfies an infinite series of recursion relations,

$$\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0\,, \tag{1}$$

where $\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}}$ represents a sum of the Nekrasov partition function with instanton number larger or less than $Z_{\vec{Y},\vec{W}}$ by one with appropriate coefficients and $U_{\pm 1,n}$ are polynomials of parameters such as mass of bifundamental matter or VEV of gauge multilets. The subscript n takes any non-negative integer. The detailed form of the recursion formula and its derivation is done in the section 8. The recursion formula is derived by a complicated but straightforward calculation from the definition of the factor $Z_{\vec{Y},\vec{W}}$. We note that a classical limit of the such relations was recently explored in [21].

Then we give an interpretation of (1). We show that the variation in (1) can be understood as an action of an infinite-dimensional extended conformal algebra. It is defined in [15] and named $\mathbf{SH}^c$ algebra.[1] For this

[1] This name of the algebra appears only in [15]. Degenerate double affine Hecke algebra, or DDAHA in short, may be more appropriate. We thank Y. Tachikawa for informing us of the relevance of [15].

purpose, we construct an explicit representation where the basis of the Hilbert space is labeled by sets of N Young diagrams. Physically, it can be understood that these states correspond to instantons characterized by the same set of Young diagrams. In our previous paper [22], we showed a similar form of recursion formula under self-dual Ω-background ($\epsilon_1 + \epsilon_2 = 0$) and discussed that it can be interpreted in terms of $\mathcal{W}_{1+\infty}$ algebra. The analysis here is a natural generalization to any Ω-deformation. SH^c algebra contains a parameter β, which is related to Ω-deformation parameters by $\beta = -\epsilon_1/\epsilon_2$. When we take $\beta = 1$, (1) reduces to that in [22] and the action of SH^c algebra can be identified with the $\mathcal{W}_{1+\infty}$ algebra. We will also see SH^c algebra contains Heisenberg$\times$Virasoro subalgebra and its central charge is the same as that of Heisenberg$\times W_N$ algebra with background charge $Q = \sqrt{\beta} - 1/\sqrt{\beta}$. The combination of Heisenberg algebra with W_N appears in [23, 24, 25], where the authors formally construct a basis of Hilbert space of Heisenberg$\times W_N$ algebra which reproduces the factorized form of Nekrasov partition function. Such observation implies that one may regard the formula (1) as the conformal Ward identities which characterize the conformal block function.

We mention that there is another one parameter deformation of $\mathcal{W}_{1+\infty}$ algebra [26], $W_\infty[\mu]$ in the context of higher spin supergravity. SH^c and $W_\infty[\mu]$ share a property that they are generated by infinite higher spin generators and contains W_N algebra with general β as their reduction. Here we use SH^c since their action on a basis parametrized by sets of Young diagram is already known. It is natural to expect that these two algebras are identical although their appearances are very different. It should be also noted that the introduction of further deformation parameter is possible [27, 28, 29] and was applied to a generalization of AGT conjecture [30].

As we will see later, it is tempting to speculate that identities from SH^c algebra fully reproduce the conformal block function. Because of a technical difficulty to characterize the vertex operator in SH^c, explicit demonstration of the relation is limited to the Heisenberg and Virasoro subalgebra. For these cases, the recursion for $n = 0, 1$ can be indeed interpreted as Ward identities. The algebra SH^c was introduced in [15] to prove the AGT conjecture for pure super Yang-Mills theory. Our analysis shows that it may be applied to linear quiver gauge theories as well. For the recent development toward such direction, see also [31].

This thesis is orgnazied as follows. In section 2 we review the Nekrasov partiion function, including its origin from the Ω deformation of $\mathcal{N} = 2$ SUSY gauge theories, and focus on its application on linear quiver gauge groups explicitly. In section 3, we give a brief introduction to Liouville field theory and its SU(N) generalization, the Toda field theory. DOZZ formula, conformal blocks, and the basic properties of the boson fields are provided, which are of importance in later discussions.In section 4 we introduce the famous AGT conjecture, i.e., the duality between 4D SUSY gauge theories and 2D conformal theories.The CFT correlation functions can be regarded as the integral of the Nekrasov partition function Z_{Nek}, with three-point functions (DOZZ formula) correspond to the one-loop part of Z_{Nek}, and conformal blocks to the intanton part. In section 5, we study the Dotsenko and Fatteev's method of screening operators and transform the four point correlation function in Toda field theories into the Dotsenko-Fatteev integral.

Section 6 and 7 are one of the main results of this thesis. In section 6, after a short review of Selberg integral, we perform some calculation to reduce the Dotsenko-Fatteev integral to a Selberg average of Jack polynomials. Then based on limited known results, we conjecture a formula of SU(N) Selberg-Jack integral, and show that it clears several nontrivial consistency checks. In section 7, we present a direct approach on AGT conjecture, mainly with the Ω deformation parameter $\beta = 1$. After the proof of several important lemmas and evaluation of parameters, we manage to identify the Selberg-Jack integral with the instanton part of the Nekrasov partition function. Section 8 and 9 illustrate another substantial approach to the proof of AGT conjecture, using a recursion method. In section 8, we construct the recursion formula for Nekrasov partition function, with the help of some mathematical formulae. Further we construct the basis using the symmetry algebra $\mathbf{SH}^c$, which contains Heisenberg and Virasoro algebra as its subalgebra, and can also be reduced to $\mathcal{W}_{1+\infty}$ algebra. In section 9, By using a modified vertex operator, we successfully obtain the ward identities for $U(1)$ currents and Virasoro generators, which serve as strong supports to the AGT conjecture. In the appendix, substantial mathematical oriented proof and calculations are provided, which are the main contributions of the author in his cooperation projects[16, 32].

2 Nekrasov partition function

In both physical and mathematical fields, it has been a puzzle that why Donaldson invariants are related to periods of Seiberg-Witten curves [33] . Nekrasov's discovery on the relation between Nekrasov partion function and Seiberg-Witten prepotential is considered as a first step towards the understanding of the mysterious relation.

For pure $\mathcal{N}=2$ supersymmetric Yang-Mills theory with the gauge group $U(N)$ and its maximal torus $\mathbf{T}=U(1)^N$, The action is given by the integral over the superspace [34]:

$$S=\frac{1}{8\pi h^\vee}\int d^4xd^4\theta\,\mathrm{Im}\left(\frac{\tau}{2}\,\mathrm{Tr}_{\mathrm{adj}}\,\Psi^2\right), \tag{2}$$

where the fields contain: A_μ, ψ^A_α and ϕ, where A_μ is a vector boson, ψ^A_α, $A=1,2$ are two Weyl spinors and ϕ is a complex scalar. Since vector bosons are usually associated with a gauge symmetry, A_μ is supposed to be a gauge boson corresponding to a gauge group G. It follows that it transforms in the adjoint representation of G. To maintain the $\mathcal{N}=2$ supersymmetry ψ^A_α and ϕ should also transform in the adjoint representation.Here $\mathrm{Tr}_{\mathrm{adj}}$ means that the trace is taken over the adjoint representation. These fields form the ($\mathcal{N}=2$) chiral multiplet (sometimes called the gauge or the vector multiplet).

The most natural superfield representation for the chiral multiplet is given in the extended superspace, which has the coordinates $x^\mu, \theta^\alpha_A, \bar{\theta}^A_{\dot\alpha}$, $A=1,2$. Then we have

$$\Psi(x,\theta,\bar\theta)=\phi(x)+\sqrt{2}\theta^\alpha_A\psi^A_\alpha(x)-\frac{i}{\sqrt{2}}\epsilon_{AB}\theta^{\alpha A}\theta^{\beta B}\sigma^{\mu\nu}_{\alpha\beta}F_{\mu\nu}(x)+\dots \tag{3}$$

Besides, we have

$$\tau=\frac{4\pi i}{g^2}+\frac{\vartheta}{2\pi}, \tag{4}$$

with g^2 being the Yang-Mills coupling constant (and the Plank constant as well) and ϑ is the instanton angle. Its contribution to the action is given by the topological term, ϑk where k is the instanton number:

$$k=-\frac{1}{16\pi^2h^\vee}\int\mathrm{Tr}_{\mathrm{adj}}(F\wedge F), \tag{5}$$

where the curvature $F=\frac{1}{2}F_{\mu\nu}\mathrm{d}x^\mu\wedge\mathrm{d}x^\nu=\mathrm{d}A+A\wedge A$.

In the low energy limit, when the $\mathcal{N}=2$ supersymmetry is unbroken, the most general effective action can be obtained by the following generalization of (2):

$$S_{\mathrm{eff}}=\frac{1}{8\pi h^\vee}\int d^4xd^4\theta\,\mathrm{Im}\Big[\mathcal{F}(\Psi,\Lambda)\Big]$$

where $\mathcal{F}(a,\Lambda)$ is a holomorphic gauge-invariant function called the prepotential. Its classical expression can be read from (2): $\mathcal{F}_{\mathrm{class}}(a)=\dfrac{\tau}{2}a^2$. All perturbative correction are contained in the 1-loop term which is equal to

$$\mathcal{F}_{\mathrm{pert}}(a,\Lambda)=-\sum_{\alpha\in\Delta^+}(\alpha\cdot a)^2\left(\ln\left|\frac{\alpha\cdot a}{\Lambda}\right|-\frac{3}{2}\right) \tag{6}$$

where Λ is the dynamically generated scale. In this formula the highest root is supposed to have length 2.

The action (2) can be considered as a 5 + 1 dimensional $\mathcal{N} = \infty$ supersymmetric Yang-Mills theory in the Ω-background and compactified on the two dimensional torus. Actually the easiest way to construct the action of the 4D super Yang-Mills theory with extended supersymmetry is to apply dimensional reduction from higher dimensional minimal supersymmetric theories [35]. Consider lifting the $\mathcal{N} = 2$ four dimensional theory to $\mathcal{N} = (1, 0)$ six dimensional theory, and then compactify the six dimensional $\mathcal{N} = 1$ susy gauge theory on the manifold with the topology $\mathbf{T}^2 \times \mathbf{R}^4$ with the metric :

$$ds^2 = r^2 dz d\bar{z} + g_{\mu\nu} \left(dx^\mu + V^\mu dz + \bar{V}^\mu d\bar{z}\right) \left(dx^\nu + V^\nu dz + \bar{V}^\nu d\bar{z}\right), \tag{7}$$

where $V^\mu = \Omega^\mu_\nu x^\nu,\ \bar{V}^\mu = \bar{\Omega}^\mu_\nu x^\nu$, and

$$\Omega^{\mu\nu} = \begin{pmatrix} 0 & \epsilon_1 & 0 & 0 \\ -\epsilon_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \epsilon_2 \\ 0 & 0 & -\epsilon_2 & 0 \end{pmatrix}, \quad \bar{\Omega}^{\mu\nu} = \begin{pmatrix} 0 & \bar{\epsilon}_1 & 0 & 0 \\ -\bar{\epsilon}_1 & 0 & 0 & 0 \\ 0 & 0 & 0 & \bar{\epsilon}_2 \\ 0 & 0 & -\bar{\epsilon}_2 & 0 \end{pmatrix} \tag{8}$$

Here $\Omega^{\mu\nu} = G^{\nu\kappa}\Omega^\mu_\kappa$ etc.The area r^2 of the torus is to be sent to zero. For $[\Omega, \bar{\Omega}] = 0$ the metric is flat.

The action of the four dimensional theory in the limit $r \to 0$ is not that of the pure supersymmetric Yang-Mills theory on $\mathbf{R}^4$. Rather, it is a deformation of the latter by the Ω, $\bar{\Omega}$-dependent terms. We shall write down here only the terms with bosonic fields (for simplicity, we have set $\vartheta_0 = 0$):

$$S(\Omega)^{bos} = -\frac{1}{2g_0^2}\mathrm{Tr}\left(\frac{1}{2}F_{\mu\nu}^2 + (D_\mu\Phi - \Omega^\nu_\lambda x^\lambda F_{\mu\nu})(D_\mu\bar{\Phi} - \bar{\Omega}^\nu_\lambda x^\lambda F_{\mu\nu}) + [\Phi, \bar{\Phi}]^2\right) \tag{9}$$

We shall call the theory (9) an $\mathcal{N} = 2$ theory in the Ω-background.

With his idea of the Ω-background, Nekrasov calculated the following partition function

$$Z(\tau, a, m, \epsilon) = \int_{\phi(\infty)=a} D\Phi DAD\lambda \dots e^{-S(\Omega)} \tag{10}$$

of the $\mathcal{N} = 2$ susy gauge theory with all the higher couplings on the background with the fixed asymptotics of the Higgs field at infinity. We take the limit $\bar{\tau}_0 \to \infty$, and the partition function becomes the sum over the instanton charges of the integrals over the moduli spaces $\mathcal{M}$ of instantons of the measure, obtained by the developing the path integral perturbation expansion around instanton solutions.

The detailed calculation can be found in, for example, [36]. The general idea is:
Split the configuration A_μ as

$$A_\mu = A_\mu^{\mathrm{ASD}} + \delta A_\mu \tag{11}$$

with the anti self-dual part A_μ^{ASD} and its deviation δA_μ. When δA_μ is small, the action becomes

$$S = \frac{8\pi^2 k}{g^2} + \int d^4x(\text{2nd order of } \delta A_\mu) + (\text{higher terms}), \tag{12}$$

so that the partition function can also be divided in the form

$$Z = \int [DA_\mu] e^{-S} = \sum_k \int [DA_\mu^{\mathrm{ASD}}] \int [\delta A_\mu] e^{-\frac{8\pi^2 k}{g^2} + \cdots} \tag{13}$$

the exact investigation using this expansion is first done in [37], but due to the complexity of dealing δA_μ , we consider the simple model

$$Z^{\text{instanton}} = \sum_k q^k \int [DA_\mu^{\text{SD}}] = \sum_k q^k \int_{\mathcal{M}_{N,k}} d\text{vol}、 \tag{14}$$

where $d$vol is the natural volume form on $\mathcal{M}_{N,k}$,and $q = \exp(-8\pi/g^2)$ is the chemical potential of the instanton number.The apparent divergence can be controlled by the insertion of some Gauss-like factor.

The important property of this partition function (10) is that it gives the prepotential of the theory in the limit $\epsilon_1 = -\epsilon_2 = \hbar \to 0$

$$F(\tau, a, m) = \lim_{\hbar\to 0} \hbar^2 \log Z(\tau, a, m; \hbar, -\hbar). \tag{15}$$

This limit was evaluated for a number of $\mathcal{N} = 2$ theories, and reproduced the prepotential as determined by the Seiberg-Witten curve.

Nekrasov partition function is applied in various matter contents. In this paper, we focous on the partition function for $G = U(N_1) \times \cdots \times U(N_n)$ linear quiver gauge theory:

$$Z_{\text{full}}(q; a, m; \epsilon) = Z_{\text{tree}} Z_{\text{1loop}} Z_{\text{inst}}, \quad Z_{\text{inst}}(q; a, m; \epsilon) = \sum_{\mathbf{Y}} \mathbf{q}^{\mathbf{Y}} z(\mathbf{Y}, a, m), \tag{16}$$

where the instanton is labeled by a N-tuple of Young diagrams: $\mathbf{Y} := (\vec{Y}^{(1)}, \cdots, \vec{Y}^{(n)})$, (Fig. 1). The parameter a (resp. m) represents the diagonalized VEV of vector multiplets (resp. mass of hypermultiplets) whereas $q_i = e^{\pi i \tau_i}$ is the instanton expansion parameter for ith gauge group $SU(N_i)$, $\mathbf{q}^{\mathbf{Y}} := \prod_{i=1}^n q_i^{|\vec{Y}^{(i)}|}$. The total partition function is decomposed into a product of the contributions of the perturbative parts Z_{tree}, $Z_{1-\text{loop}}$ and non-perturbative instanton correction Z_{inst}. The latter is further decomposed into a sum of sets of Young diagrams. $\vec{Y}^{(i)} = (Y_1^{(i)}, \cdots, Y_{N_i}^{(i)})$ is a collection of N_i Young diagram which parameterizes the fixed points of instanton moduli space for i th gauge group $U(N_i)$.

We will mainly focus on the instanton part. The coefficient $z(\mathbf{Y}, a, m)$ is described as a product of the contributions

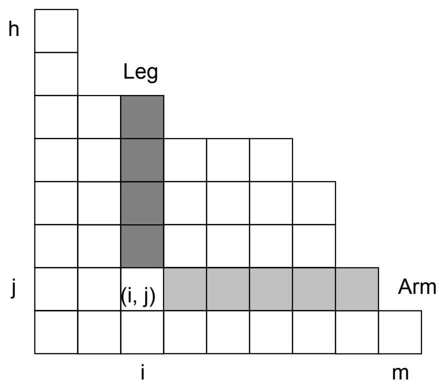

Figure 1: Young diagrams are very useful in representing conjugacy classes in group theory. The above is a Young diagram Y of (8,6,6,5,5,5,4,2,1). The ith column is named as Y_i. $h = Y_1$ is the height of Y, while $m = Y_1'$ is called the length of Y, where Y' stands for the transposed Young diagram.The arm-length and leg-length of the box (i, j) in the tableaux Y are denoted by $\text{Arm}_Y(i,j)$ and $\text{Leg}_Y(i,j)$ defined separately as $\text{Arm}_Y(i,j) = Y_j' - i, \quad \text{Leg}_Y(i,j) = Y_i - j$. For the box $(i, j) = (3, 2)$, the arm-length and leg-length are 5 and 4,respectively.

of the gauge- and hyper multiplets which describes the system:

$$z(\mathbf{Y}, a, m) = \prod_{i=1}^{n} z_{\text{vect}}(a^{(i)}, \vec{Y}^{(i)}) \prod_R z_R(\vec{Y}, a, m) \ , \tag{17}$$

where R is the representation for each hypermultiplets (we set $\beta = -\epsilon_1/\epsilon_2$):

$$
\begin{aligned}
z_{\rm bifund}(a,\vec{Y};b,\vec{W};m) &= \prod_{t}^{N_1}\prod_{s=1}^{N_2} G_{Y_t,W_s}(a_t-b_s-m)G_{W_s,Y_t}(b_s-a_t+m+1-\beta)\ , && (18)\\
z_{\rm fund}(a,\vec{Y};m) &= \prod_{s=1}^{N} f_{Y_s}(a_s-m-1+\beta)\ , && (19)\\
z_{\rm afd}(a,\vec{Y};m) &= z_{\rm fund}(a,\vec{Y},-1+\beta-m)\ , && (20)\\
z_{\rm adj}(a,\vec{Y};m) &= z_{\rm bifund}(a,\vec{Y},a,\vec{Y},m)\ , && (21)\\
z_{\rm vect}(a,\vec{Y}) &= 1/z_{\rm adj}(a,\vec{Y},0)\ . && (22)
\end{aligned}
$$

In eq.(18), the hypermultiplet is supposed to transform as bifundamental associated with gauge group $U(N_1)\times U(N_2)$. Similarly, in eq.(19), the fundamental representation is associated with $U(N)$. The function G in eq.(18) is a function with respect to the tableau Y's arm-length and leg-length

$$G_{Y,W}(x) = \prod_{(i,j)\in Y}\Big(x+\beta(Y'_j-i)+(W_i-j)+\beta\Big)\ , \tag{23}$$

and the function f in (19) is defined as

$$f_Y(z) = \prod_{(i,j)\in Y}(z+\beta(i-1)-(j-1))\ . \tag{24}$$

Single gauge group case First we focus on the simplest case, $G = SU(N)$, with $N_f = 2N$ hypermultiplets in fundamental representation. In this specific example, the partition function is written as

$$Z_{\rm full}(q;a,\mu;\epsilon) = Z_{\rm tree}Z_{\rm 1loop}Z_{\rm inst},\quad Z_{\rm inst}(q;a,m;\epsilon) = \sum_{\vec{Y}} q^{|\vec{Y}|}N^{\rm inst}_{\vec{Y}}(a,\mu), \tag{25}$$

$$N^{\rm inst}_{\vec{Y}}(a,\mu) = z_{\rm vect}(\vec{Y},a)\prod_{i=1}^{2N} z_{\rm fund}(\vec{Y},\mu_i) = \frac{\prod_{s=1}^{N}\prod_{k=1}^{2N} f_{Y_s}(\mu_k+a_s)}{\prod_{t,s=1}^{N} g_{Y_t,Y_s}(a_t-a_s)}, \tag{26}$$

with

$$g_{Y,W}(x) := G_{Y,W}(x)G_{Y,W}(x+1-\beta). \tag{27}$$

μ_i $(i = 1,\cdots,2N)$ are mass parameters of hypermultiplets with fundamental representation.

Linear quiver case For four-dimensional $\mathcal{N} = 2$ superconformal linear quiver gauge theory with $U(N)\times U(N)\times\cdots\times U(N)$ gauge group, we make a different choice of $z_{\rm vect}$, but with the total contribution remains the same. The instanton partition function of $N = 2$ gauge theories can be written in the following form

$$Z^{\rm Nek}_{\rm inst} = \sum_{\vec{Y}^{(1)},\cdots,\vec{Y}^{(n)}} q_i^{|\vec{Y}^{(i)}|}\bar{V}_{\vec{Y}^{(1)}}\cdot Z_{\vec{Y}^{(1)}\vec{Y}^{(2)}}\cdots Z_{\vec{Y}^{(n-1)}\vec{Y}^{(n)}}\cdot V_{\vec{Y}^{(n)}}\ . \tag{28}$$

$$
\begin{aligned}
Z_{\vec{Y}^{(i)}\vec{Y}^{(i+1)}} &= Z(\vec{a}^{(i)},\vec{Y}^{(i)};\vec{a}^{(i+1)},\vec{Y}^{(i+1)};\mu^{(i)}), && (29)\\
\bar{V}_{\vec{Y}^{(1)}} &= Z(\vec{\lambda},\vec{\emptyset};\vec{a}^{(1)},\vec{Y}^{(1)};\mu^{(0)}), && (30)\\
V_{\vec{Y}^{(n)}} &= Z(\vec{a}^{(n)},\vec{Y}^{(n)};\vec{\lambda}',\vec{\emptyset};\mu^{(n)}), && (31)
\end{aligned}
$$

where $q_i = \exp(2\pi i\tau_i)$ represents the complexified coupling constant τ_i of i-th $U(N)$ gauge group, $\vec{Y}^{(i)}$ is a set of N Young diagram characterizing fixed points of localization in the instanton moduli space of the i-th $U(N)$. $\vec{a}^{(i)}$

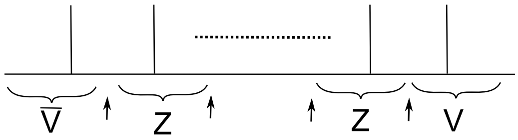

Figure 2: Decomposition of Nekrasov function

is the VEV for an adjoint scalar field in the vector multiplet of i-th $U(N)$ and $\mu^{(i)}$ is the mass parameter for the bifundamental matter field which interpolates i^{th} and $i+1^{\text{th}}$ gauge groups. We write $\vec{\emptyset}$ to represent a set of null Young diagrams $(\emptyset, \cdots, \emptyset)$.

The building block reads,

$$Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) = \frac{z_{\text{bf}}}{\tilde{z}_{\text{vect}}} = \frac{\prod_{p,q=1}^{N} \tilde{g}_{Y_p W_q}(a_p - b_q - \mu)}{\left(\prod_{p,q} \tilde{g}_{Y_p Y_q}(a_p - a_q)\tilde{g}_{W_p W_q}(b_p - b_q)\right)^{1/2}} \tag{32}$$

The function $\tilde{g}_{YW}$ is

$$\tilde{g}_{Y,W}(x) \;=\; \prod_{(i,j)\in Y} (x + \beta(Y'_j - i + 1) + W_i - j) \prod_{(i,j)\in W} (-x + \beta(W'_j - i) + Y_i - j + 1)\,. \tag{33}$$

The decomposition of the form (28) seems to be natural if we recall the pants decomposition of multi-point function on sphere and the dictionary of AGT relation; A bifundamental and a vector multiplet correspond to a vertex operator insertion and an internal line respectively (see Fig.2).

3 Toda field theory

The conformal properties of two-dimensional surface can be understood with the help of quantum Liouville filed theory[38], which can be transformed to a two-dimensional conformal field theory. Its algebra of generators of the conformal symmetry coincides with the Virasoro algebra, which can be obtained from affine $\mathfrak{sl}(2)$ algebra. Generalize the $\mathfrak{sl}(2)$ algebra to a general affine simple Lie algebra $\mathfrak{g}$, one obtains associative algebra (W algebra) as a direct extension of the Virasoro algebra[39]. The associated theory generalizing Liouville filed theory with simple Lie algebra $\mathfrak{g}$ is called Toda Field theory.

In subsection 3.1, the simpler and special case − Liouville theory is studied, with the concepts DOZZ formula and conformal blocks explained[1, 40, 41]. Then in subsection 3.2, we discuss the general properties of Toda Field theories[8, 9, 42].

3.1 Liouville theory: DOZZ formula and Conformal blocks

Let us begin with the action of free scalar field(free boson) $\varphi(z,\bar{z})$[43],

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g}\left(g^{ab}\partial_a\varphi\partial_b\varphi\right) \tag{34}$$

The two-point function of φ reads,

$$\langle\varphi(z,\bar{z})\varphi(w,\bar{w})\rangle = \log|z-w|^2. \tag{35}$$

Here $\log|z-w|^2$ is the Green function of the Lagrangian $\partial_z\partial_{\bar{z}}$, which is equivalent to the point charge potential of 2D electromechanics. Thus the free boson system is sometimes called Coulomb Gas.

As a result of the Equation of motion $\partial_z\partial_{\bar{z}}\varphi = 0$, $\varphi(z,\bar{z})$ can be divided as the sum of the holomophic and antiholomophic parts

$$\varphi(z,\bar{z}) = \phi(z) + \bar{\phi}(\bar{z}). \tag{36}$$

We will concentrate on the holomophic field $\phi(z)$ in the discussion of Lioubille and Toda theory.

The central charge c of the free boson CFT is fixed as $c = 1$. In order to have more variations, the usual method is to combine the scalar field with the world sheet curvature. Under the general 2D metric g_{ab}, the action becomes

$$S = \frac{1}{8\pi}\int d^2x\sqrt{g}\left(g^{ab}\partial_a\varphi\partial_b\varphi - iQR\varphi\right), \tag{37}$$

with $g = \mathrm{det(g_{ab})}$, R the scalar curvature, and Q a constant. When the metric is given by

$$ds^2 = \sigma dz d\bar{z}, \quad g_{z\bar{z}} = \frac{1}{2}\sigma, \quad g_{zz} = g_{\bar{z}\bar{z}} = 0, \tag{38}$$

the scalar curvature R has the form

$$R = \sigma^{-1}(-4\partial_z\partial_{\bar{z}}\log\sigma), \tag{39}$$

thus

$$\sqrt{g}R = -2\partial_z\partial_{\bar{z}}\log\sigma. \tag{40}$$

Apparently the complex plane metric $ds^2 = dzd\bar{z}$ leads to $\sqrt{g}R = 0$. Yet on the other hand, at $z = \infty$ under the coordinate transformation $w = \frac{1}{z}$, the metric becomes $ds^2 = w^{-2}\bar{w}^{-2}dwd\bar{w}$, so $\sqrt{g}R \sim \delta^2(0)$. This means that on the Riemann sphere, there is a charge iQ at $z = \infty$. Thus Q is named as the background charge.

Now we are ready to move to the Lioubille field theory[40] , with the action

$$S = \frac{1}{4\pi}\int d^2z\sqrt{\hat{g}}\left(\hat{g}^{ab}\partial_a\phi\partial_b\phi + QR\phi + 4\pi\mu e^{2b\phi}\right). \tag{41}$$

The parameters follow the definitions above: the background charge Q=i b-i/b, b the dimensionless coupling constant, R the scalar curvature of the background metric $\hat{g}$, and μ called the cosmological constant.
The boson field ϕ has the mode expansion

$$\phi(z) = \phi_0 + a_0\log z - \sum_{n\neq 0}\frac{a_n}{n}z^{-n} \tag{42}$$

with the commutation relations

$$[a_n, a_m] = n\delta_{n+m,0}, \qquad [a_n, \phi_0] = \delta_{n,0} \tag{43}$$

Then the Fock vacuum $|0\rangle$ is constructed through

$$a_n|0\rangle = 0, \qquad n \geq 0, \qquad \langle 0|\phi_0 = \langle 0|a_n = 0, \qquad n < 0 \tag{44}$$

Define the correlator of operator $\mathcal{O}$ to be

$$\langle\mathcal{O}\rangle = \langle 0|\mathcal{O}|0\rangle \tag{45}$$

It can be checked that

$$\langle\phi(z)\phi(w)\rangle = \log(z-w) \tag{46}$$

The corresponding energy momentum tensor is

$$T(z) = \frac{1}{2} :\partial\phi(z)^2: - \frac{iQ}{\sqrt{2}}\partial^2\phi(z) \equiv \sum_{n\in\mathbb{Z}} L_n z^{-n-2} \tag{47}$$

with the central charge

$$c = 1 + 6Q^2 \tag{48}$$

DOZZ formula [44, 45, 46]

In conformal field theory we often encounter the three point correlation function with arbitrary vectors β

$$\langle V_{\beta_1}(z_1,\bar{z}_1)V_{\beta_2}(z_2,\bar{z}_2)V_{\beta_3}(z_3,\bar{z}_3)\rangle = \frac{C(\beta_1,\beta_2,\beta_3)}{|z_{12}|^{2(\Delta_1+\Delta_2-\Delta_3)}|z_{13}|^{2(\Delta_1+\Delta_3-\Delta_2)}|z_{23}|^{2(\Delta_2+\Delta_3-\Delta_1)}}. \tag{49}$$

in the Liouville theory case, this formular has the name of DOZZ formula(Dorn, Otto, Zamolodchikov and Zamolodchikov). Under the $\mathfrak{sl}(2)$ condition, C is calculable that we have

$$\begin{aligned} C(\beta_1,\beta_2,\beta_3) = &\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{(Q-\beta_1-\beta_2-\beta_3)/b} \\ &\times \frac{\Upsilon'(0)\Upsilon(2\beta_1)\Upsilon(2\beta_2)\Upsilon(2\beta_3)}{\Upsilon(\beta_1+\beta_2+\beta_3-Q)\Upsilon(\beta_1+\beta_2-\beta_3)\Upsilon(\beta_1-\beta_2+\beta_3)\Upsilon(-\beta_1+\beta_2+\beta_3)} \end{aligned} \tag{50}$$

where

$$\gamma(x) = \Gamma(x)/\Gamma(1-x). \tag{51}$$

and

$$\Upsilon(x) = \frac{1}{\Gamma_2(x|b,b^{-1})\Gamma_2(Q-x|b,b^{-1})}. \tag{52}$$

With $\Gamma_2(x|\epsilon_1,\epsilon_2)$ the Barnes' double gamma functions.

Conformal blocks [47, 48, 49, 50]

In Liouville theory, a certain order correlation function of primary field $\mathcal{O}$ can be built from lower order ones:

$$\langle \mathcal{O}_1\cdots\mathcal{O}_j\mathcal{O}_{j+1}\cdots\mathcal{O}_n\rangle = \sum_{i,M,N}\langle\mathcal{O}_1\cdots\mathcal{O}_j\mathcal{L}_{-M}\mathcal{O}_i\rangle_M K^{-1}_{MN}\langle\mathcal{L}_{-N}\mathcal{O}_i\mathcal{O}_{j+1}\cdots\mathcal{O}_n\rangle_N \tag{53}$$

Where $\mathcal{L}_{-M,-N}$ stand for the descendants of the primary field $\mathcal{O}_i$

$$\mathcal{L}_{-N}\mathcal{O}_i = L_{-n_1}L_{-n_2}\cdots L_{-n_N}\mathcal{O}_i \tag{54}$$

with L_n the generators of the Virasoro algebra introduced in(3.1), and $k=\sum_{i=1}^N n_i$ the level of the descendant. The matrix K (called Gram matrix) at level k, are given by the inner product of $L_{-N}|\mathcal{O}_i\rangle$, the corresponding descendants of the primary field $\mathcal{O}_i$.
For example, at level two,

$$K = \begin{pmatrix} \langle\mathcal{O}_i|L_2L_{-2}|\mathcal{O}_i\rangle & \langle\mathcal{O}_i|L_1^2L_{-2}|\mathcal{O}_i\rangle \\ \langle\mathcal{O}_i|L_2L_{-1}^2|\mathcal{O}_i\rangle & \langle\mathcal{O}_i|L_1^2L_{-1}^2|\mathcal{O}_i\rangle \end{pmatrix} = \begin{pmatrix} 4h+c/2 & 6h \\ 6h & 4h(1+2h) \end{pmatrix} \tag{55}$$

where h is the conformal dimension of $\mathcal{O}_i$. Then the elementary building blocks can be expressed of the form

$$\begin{aligned} R_M(h_1,h_2,h_3) &= \frac{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{L}_{-M}\mathcal{O}_3\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle} \\ S_{M,N}(h_1,h_2,h_3) &= \frac{\langle\mathcal{L}_{-M}\mathcal{O}_1\mathcal{O}_2\mathcal{L}_{-N}\mathcal{O}_3\rangle}{\langle\mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\rangle} \end{aligned} \tag{56}$$

together with $K^{-1}(h)$ representing the propagator, we can construct conformal blocks of higher order.
Such as, four-point correlation function on the sphere

$$R(h_4,h_3,h)K^{-1}(h)R(h,h_2,h_1), \tag{57}$$

for five-point, it is

$$R(h_4,h_5,h_b)K^{-1}(h_b)S(h_b,h_3,h_a)K^{-1}(h_a)R(h_a,h_2,h_1) \tag{58}$$

one point conformal blocks on the torus

$$Tr\left(K^{-1}(h)S(h,h_1,h)\right), \tag{59}$$

and for two-point case

$$Tr\left(K^{-1}(h_a)S(h_a,h_2,h_b)K^{-1}(h_b)S(h_b,h_1,h_a)\right) \tag{60}$$

The full conformal block is obtained by multiplying each contribution with $q_1^{k_1}\cdots q_n^{k_n}$ and adding up all contributions, where the level of each contribution is fixed by the level of its internal propagators, $k_1,k_2,\dots,k_n$. For example, for the five-point conformal block on the sphere and the two-point conformal block on the torus, we have

$$\begin{aligned}\mathcal{F}^{5pt}_{g=0} &= 1+\tfrac{(-h_1+h_2+h_a)(h_3+h_a-h_b)}{2h_a}q_1+\tfrac{(-h_4+h_5+h_b)(h_3-h_a+h_b)}{2h_b}q_2+\cdots\\ \mathcal{F}^{2pt}_{g=1} &= 1+\tfrac{(h_1+h_a-h_b)(h_2+h_a-h_b)}{2h_a}q_1+\tfrac{(h_1-h_a+h_b)(h_2-h_a+h_b)}{2h_b}q_2+\cdots\end{aligned} \tag{61}$$

3.2 General Toda theory properties

The Lagrangian of the $\mathfrak{sl}(n)$ conformal Toda Field theory is given by

$$\mathcal{L}=\frac{1}{8\pi}(\partial_a\phi)^2+\mu\sum_{k=1}^{n-1}e^{b(\alpha_k,\phi)} \tag{62}$$

Let $\phi(z)=(\phi_1(z),\cdots,\phi_N(z))$ be free bosons which satisfies the operator-product expansion: $\phi_j(z)\phi_k(0)\sim\delta^{jk}\log(z)$. μ is the cosmological constant, b is the dimensionless coupling constant. Denote $\mathfrak{h}$ as the Cartan subalgebra of a Lie algebra $\mathfrak{g}$ (here it is $\mathfrak{sl}(n)$),and $\mathfrak{h}^*$ its dual. α_k are the simple roots of $\mathfrak{sl}(n)$, ρ is the weyl vector (half of the sum of all positive roots),and $(\cdot,\cdot)$ denotes the scalar product, $\langle\cdot,\cdot\rangle$ denotes the pairings between $\mathfrak{h}$ and $\mathfrak{h}^*$. The Cartan matrix $C_{ij}=(\alpha_i,\alpha_j)$ is

$$C_{ij}=\begin{pmatrix} 2 & -1 & 0 & \cdots\cdots & 0\\ -1 & 2 & -1 & \cdots\cdots & 0\\ 0 & -1 & \cdots\cdots & \cdots & \cdots\\ \cdots\cdots & \cdots & \cdots & -1 & 0\\ 0 & \cdots\cdots & -1 & 2 & -1\\ 0 & \cdots\cdots & 0 & -1 & 2\end{pmatrix}. \tag{63}$$

The action is obtained by integrating the Lagrangian in reference metric $\hat{g}^{ab}$ on a surface with curvature R,

$$S=\int d^2\sigma\sqrt{\hat{g}}\left(\frac{1}{8\pi}\hat{g}^{ab}(\partial_a\phi,\partial_b\phi)-\frac{(\rho,\phi)}{4\pi}iQ\hat{R}+\mu\sum_{k=1}^{n-1}e^{b(\alpha_k,\phi)}\right) \tag{64}$$

The chiral field ϕ has the mode expansion

$$\phi(z)=\phi_0+a_0\log z-\sum_{n\neq0}\frac{a_n}{n}z^{-n}\in\mathfrak{h} \tag{65}$$

The correlator in $SU(N)$ Toda field theory is given as the conformal block for W_N algebra which consists of the operator algebra chiral operators $W^{(s)}(z)$ with spin $s=2,\cdots,N$. It has a free boson representation [51].

$$R_N=:\prod_{m=1}^{N}\left(Q\frac{d}{dz}-i(h_m,\partial_z\phi)\right):=\sum_k W^{(k)}(z)\left(Q\frac{d}{dz}\right)^{N-k}. \tag{66}$$

h_m are vectors in $\mathbb{R}^N$ and defined by $(h_j)_k = \delta_{jk} - \frac{1}{N}$. Since it satisfies $\sum_{m=1}^{N}(h_j)_m = 0$, a component of ϕ is decoupled. The definition (66) gives $W^{(0)}(z) = 1$ and $W^{(1)}(z) = 0$. The Virasoro generator is

$$W^{(2)}(z) = \frac{1}{2} : (\partial_z \phi)^2 : -iQ(\rho, \partial_z^2 \phi), \quad \rho = \sum_{i=1}^{N-1} \omega_i = (\frac{N-1}{2}, \frac{N-3}{2}, \cdots, -\frac{N-1}{2}), \tag{67}$$

which has the central charge $c = (N-1)(1 + N(N+1)Q^2)$.

Spinless primary fields field parameterized by $(n-1)$ component vector parameter β

$$V_\beta = e^{(\beta,\phi)} \tag{68}$$

are the essential objects of Toda Theory. Their multipoint correlation functions

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \dots V_{\beta_l}(z_l, \bar{z}_l) \rangle = \int [\mathcal{D}\phi] e^{-S} V_{\beta_1}(z_1, \bar{z}_1) \dots V_{\beta_l}(z_l, \bar{z}_l) \tag{69}$$

are one of the most important problems in Toda field theory.This problem is nontrivial due to the exponential interaction term in the Lagrangian. However, if pertubatively expanded in in cosmological constant μ, correlation functions are equal to zero unless the on shell condition is satisfied

$$\sum_{j=1}^{l} \beta_j + b \sum_{k=1}^{n-1} s_k \alpha_k = 2Q\rho \tag{70}$$

with s_k some non-negative integer.
After performing the zero mode integral[52], we arrive at

$$\langle V_{\beta_1}(z_1, \bar{z}_1) \dots V_{\beta_l}(z_l, \bar{z}_l) \rangle = \frac{1}{b^{n-1}} \int [\mathcal{D}\tilde{\phi}] e^{-S_0} \left[\prod_{k=1}^{n-1} \Gamma(-s_k) \left(\mu \int e^{b(\alpha_k, \tilde{\phi})} \right)^{s_k} \right] V_{\beta_1}(z_1, \bar{z}_1) \dots V_{\beta_l}(z_l, \bar{z}_l) \tag{71}$$

with

$$s_k = \frac{(2Q\rho - \sum \beta_j, \omega_k)}{b} \tag{72}$$

ω_k being the fundamental weights of $\mathfrak{sl}(n)$, and the integration is performed over the free massless fields

$$S_0 = \frac{1}{8\pi} \int (\partial_a \phi)^2 d^2x. \tag{73}$$

4 AGT conjecture

In [1] Alday, Gaiotto and Tachikawa pointed out that Nekrasov partition function is identical to the correlation functions of Liouville theory when the gauge group is $SU(2)$. It takes the form (here we give example of n-point function on sphere):

$$\begin{aligned} &\langle V_n(\infty) V_{n-1}(1) V_{n-2}(q_1) \cdots V_2(q_1 \cdots q_{n-3}) V_1(0) \rangle \\ &= \sum_{\psi_1, \cdots, \psi_{n-3}} C_{V_1 V_2 V_1} \cdots C_{V_{n-3} V_{n-1} V_{n-1}} |\mathcal{F}_{V_1 V_2 U_1 \cdots U_{n-3} V_{n-1} V_n}(z_1, \cdots, z_n)|^2 . \end{aligned} \tag{74}$$

Here the product of the constants $C_{V_1 V_2 U_1}$ etc. are from the 3-point functions. For Liouville case, it is given by DOZZ formula [44, 45, 46, 53, 54, 55]. The function $\mathcal{F}$ carries the coordinate (q) dependence and reflects the contributions of the conformal descendants. It is the conformal block we mentioned before.

In order to give the identification of partition function with the correlator, we need some identification of parameters: $a, m \leftrightarrow \alpha$ and the coordinate q in CFT is identified with the coupling constant $q = e^{\pi i \tau}$ in Yang-Mills.

Here $\alpha \in \mathbf{R}^N$ is a parameter which appears in the exponential of the vertex operator $V_\alpha = e^{i(\alpha,\phi)}$ inserted in the correlator.

Namely, on the one hand, The Nekrasov' partition function is

$$Z_{\text{full}}(\tau, a, m; \epsilon_i) = Z_{\text{classical}} Z_{\text{1-loop}} Z_{\text{instanton}}. \tag{75}$$

On the other hand,

$$\text{Liouville correlators} = (\text{Three}-\text{point functions}) \times (\text{Conformal blocks}) \tag{76}$$

We will see that $Z_{\text{instanton}}$ corresponds to conformal blocks, while the integration of Z_{full} corresponds to Liouville correlation functions.

4.1 Instanton sums and conformal blocks correspondence

Sphere with four punctures We have encountered the general expansion of Z_{inst}. Now for $U(2)$ theory with $N_f = 4$ flavors

$$Z_{\text{inst}}^{U(2),N_f=4} = \sum_{\vec{Y}} q^{|\vec{Y}|} z_{\text{vector}}(\vec{a},\vec{Y}) z_{\text{antifund}}(\vec{a},\vec{Y},\mu_1) z_{\text{antifund}}(\vec{a},\vec{Y},\mu_2) z_{\text{fund}}(\vec{a},\vec{Y},\mu_3) z_{\text{fund}}(\vec{a},\vec{Y},\mu_4). \tag{77}$$

$\vec{a} = (a_1, a_2)$ is the adjoint vev of the $U(2)$ gauge multiplet, $\mu_{1,2}$ are the masses of two hypermultiplets in the anti-fundamental, and $\mu_{3,4}$ are those of the fundamentals.
Redefine:

$$\mu_1 = m_0 + \tilde{m}_0, \qquad \mu_2 = m_0 - \tilde{m}_0, \qquad \mu_3 = m_1 + \tilde{m}_1, \qquad \mu_4 = m_1 - \tilde{m}_1. \tag{78}$$

And decouple the $U(1)$ part

$$Z_{\text{inst}}^{U(2),N_f=4}(a, m_0, \tilde{m}_0, m_1, \tilde{m}_1) = (1-q)^{2m_0(Q-m_1)} \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q) \tag{79}$$

Surprisingly, it is Checked [2] that $\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)$ is the conformal block of a virasoro algebra with central charge $c = 1 + 6Q^2$ at position $\infty, 1, q, 0$, and an intermediate state operators of dimension

$$\begin{aligned} \Delta_1 &= \beta_0(Q-\beta_0), \qquad \Delta_2 = m_0(Q-m_0), \\ \Delta_3 &= m_1(Q-m_1), \qquad \Delta_4 = \beta_1(Q-\beta_1), \\ \Delta &= \beta(Q-\beta). \end{aligned} \tag{80}$$

Sphere and torus with multiple punctures For the multi-punctured sphere, again let us first redefine the masses,

$$\mu_1 = m_0 + \tilde{m}_0, \quad \mu_2 = m_0 - \tilde{m}_0, \quad \mu_3 = m_n + \tilde{m}_1, \quad \mu_4 = m_n - \tilde{m}_1. \tag{81}$$

Decouple the $U(1)$ factor

$$Z_{\text{inst}}^{U(2)\text{ linear quiver}}(q_i; a_i; m_i; \tilde{m}_i) = Z^{U(1)\text{ linear}}(q_i; m_i) \mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta_1}{}^{m_1} \cdots {}_{\beta_n}{}^{m_n}{}_{\beta_{n+1}}(q_1, q_2, \ldots, q_n) \tag{82}$$

Here $\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta_1}{}^{m_1} \cdots {}_{\beta_n}{}^{m_n}{}_{\beta_{n+1}}(q_1, q_2, \ldots, q_n)$ is supposed to be the conformal block of Virasoro algebra with central charge $c = 1 + 6Q^2$ for a sphere with $n+3$ punctures at $\infty,\ 1,\ q_1,\ q_1q_2,\ \ldots, q_1q_2\cdots q_n,\ 0.$,
the corresponding operator dimensions are respectively

$$\beta_0(Q-\beta_0),\ m_0(Q-m_0),\ \ldots,\ m_n(Q-m_n),\ \beta_{n+1}(Q-\beta_{n+1}) \tag{83}$$

[2]Up to order q^{11} in [1].

and for the i-th intermediate channel, it is $\beta_i(Q-\beta_i)$.

Likewise, for the torus with multiple punctures, we have

$$Z_{\text{inst}}^{U(2)\text{ necklace quiver}}(q_i;a_i;m_i)=Z^{U(1)\text{ necklace}}(q_i;m_i)\mathcal{F}_{\beta_1}{}^{m_1}\cdots_{\beta_n}{}^{m_n}(q_1,q_2,\ldots,q_n) \tag{84}$$

$\mathcal{F}_{\beta_1}{}^{m_1}\cdots_{\beta_n}{}^{m_n}(q_1,q_2,\ldots,q_n)$ is the conformal block of Virasoro algebra with central charge $c=1+6Q^2$ for a torus with multiple punctures at
$1,\ q_1,\ q_1q_2,\ \ldots,\ q_1q_2\cdots q_{n-1}$. The operator at the i-th puncture has dimension $m_i(Q-m_i)$, while for the i-th intermediate channel it is $\beta_i(Q-\beta_i)$.

4.2 Liouville correlators and the full partition function

Sphere with four punctures For a Liouville theory on a sphere, the four-point correlation function of V at positions $\infty,1,q,0$ is[53, 55]

$$\langle V_{\beta_0}(\infty)V_{m_0}(1)V_{m_1}(q)V_{\beta_1}(0)\rangle=\int\frac{d\beta}{2\pi}C(\beta_0^*,m_0,\beta)C(\beta^*,m_1,\beta_1)\left|q^{\Delta_\beta-\Delta_{m_1}-\Delta_{\beta_1}}\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\right|^2. \tag{85}$$

Where $C(\beta_1,\beta_2,\beta_3)$ is the three point function given by the DOZZ formula we have seen.

$$r.h.s=f(\beta_0^*)f(m_0)f(m_1)f(\beta_1)\left|q^{Q^2/4-\Delta_{m_1}-\Delta_{\beta_1}}\right|^2\int a^2da\left|Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\right|^2 \tag{86}$$

where

$$f(\beta)=\left[\pi\mu\gamma(b^2)b^{2-2b^2}\right]^{-\beta/b}\Upsilon(2\beta) \tag{87}$$

and

$$Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)=q^{-a^2}\frac{\prod\Gamma_2(\hat{m}_0\pm\tilde{m}_0\pm a+Q/2)\prod\Gamma_2(\hat{m}_1\pm\tilde{m}_1\pm a+Q/2)}{\Gamma_2(2a+b)\Gamma_2(2a+1/b)}\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q). \tag{88}$$

The above equation can be transformed into

$$\begin{aligned}Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)&=q^{-a^2}\times z^{\text{1-loop}}_{\text{vector}}(a)z^{\text{1-loop}}_{\text{antifund}}(a,\mu_1)z^{\text{1-loop}}_{\text{antifund}}(a,\mu_2)z^{\text{1-loop}}_{\text{fund}}(a,\mu_3)z^{\text{1-loop}}_{\text{fund}}(a,\mu_4)\mathcal{F}_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\\&=Z_{\text{classical}}Z_{\text{1-loop}}Z_{\text{instanton}}\\&=Z_{\text{full}}\end{aligned} \tag{89}$$

As a result, the four-point function of Liouville theory is proportional to the integration of Nekrasov' partition function[10].

$$\langle V_{\beta_0}(\infty)V_{m_0}(1)V_{m_1}(q)V_{\beta_1}(0)\rangle\propto\int a^2da\left|Z_{\beta_0}{}^{m_0}{}_{\beta}{}^{m_1}{}_{\beta_1}(q)\right|^2 \tag{90}$$

Sphere with multiple punctures More generally, the multipoint correlation function in Liouville theory has its gauge field interpretation through a sphere with multiple punctures

$$\begin{aligned}&\langle V_{\beta_0}(\infty)V_{m_0}(1)V_{m_1}(q_1)\cdots V_{m_n}(q_1\cdots q_n)V_{\beta_{n+1}}(0)\rangle=\\&\qquad cf(\beta_0)f(\beta_{n+1})\prod f(m_i)\int\prod(a_i^2da_i)\left|Z_{\beta_0}{}^{m_0}{}_{\beta_1}{}^{m_1}\cdots_{\beta_n}{}^{m_n}{}_{\beta_{n+1}}(q_i)\right|^2\end{aligned} \tag{91}$$

with $Z_{\beta_0}{}^{m_0}{}_{\beta_1}{}^{m_1}\cdots_{\beta_n}{}^{m_n}{}_{\beta_{n+1}}(q_i)$ being the Nekrasov's full partition function. The torus case can be solved in a similar way.

4.3 SU(N) generalization

To be more explicit, for the specific example of $SU(N)$ gauge theory with $N_f = 2N$ fundamental matter, the relevant Toda correlator is written in the form

$$\langle V_{\alpha_4}(\infty)V_{\alpha_3}(1)V_{\alpha_2}(q)V_{\alpha_1}(0)\rangle \,, \tag{92}$$

where the insertion of screening operators is necessary for the charge conservation. The conformal block of this correlation function is written in the form,

$$\mathcal{F}_{\alpha_4,\alpha_3,\alpha_2,\alpha_1}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} N^{\rm Toda}_{\vec{Y}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4)\,. \tag{93}$$

It is known that the four point function of Toda theory can be obtained for special choice of parameters [8, 9], namely the two of the vertex operator momentum (say α_2 and α_3) should be proportional to either ω_1 or ω_{N-1} where ω_i $(i = 1, \cdots, N-1)$ is the fundamental weight of A_{N-1}.

AGT conjecture for $SU(N)$ [5, 6] implies that partition function and the correlator are the same. In particular it implies,

$$N^{\rm inst}_{\vec{Y}}(a,\mu) = N^{\rm Toda}_{\vec{Y}}(\alpha_1,\alpha_2,\alpha_3,\alpha_4), \tag{94}$$

if we identify the parameters,

$$a = \alpha\ ; \qquad \mu = -\alpha_1 - (1-\beta)\rho, \quad \tilde{\mu} = -\alpha_4 - (1-\beta)\rho\,; \tag{95}$$

where $\mu = (\mu_1, \cdots, \mu_N)$ and $\tilde{\mu} = (\mu_{N+1}, \cdots, \mu_{2N})$ are mass parameters of vector multiplets. $\alpha = \alpha_1 + \alpha_2 + \beta\sum_a N_a e_a + (1-\beta) = -(\alpha_4 + \alpha_3 + \beta\sum_a \tilde{N}_a e_a + (1-\beta))$ is the momentum which appears in the intermediate channel (N_a and $\tilde{N}_a$ are the numbers of screening charges and e_a is the simple root of A_{N-1}). Weyl vector $\rho = \sum_{i=1}^{N-1}\omega_i$ shows up to represent the corrections of the background charge. As explained, we choose α_2 and α_3 to be proportional to ω_1.

We focus on this "identity" in the following.

5 Dotsenko-Fateev integral

To understand Dotsenko and Fateev's idea on screening operators, we first recall the simple action of curvature coupled scalar field (37). Through the shift of the scalar field $\varphi \to \varphi + \varphi_0$, the variation of action is [43]

$$\delta S = \frac{iQ\varphi_0}{8\pi}\int d^2x\sqrt{g}R. \tag{96}$$

According to Gauss-Bonnet's theorem, for a 2D Riemann surface with genus h,

$$\int d^2x\sqrt{g}R = 8\pi(1-h) \tag{97}$$

In the case of sphere, $h = 0$, so

$$\delta S = iQ\varphi_0. \tag{98}$$

Since the action $S(\varphi)$ and the functional integral measure $D\varphi$ is invariant under the shift of φ, it is easy to find that the correlation function satisfies

$$\langle e^{\alpha_1\varphi(z_1,\bar{z}_1)}\cdots e^{\alpha_N\varphi(z_N,\bar{z}_N)}\rangle = e^{(\sum_{i=1}^N \alpha_i - iQ\varphi_0)}\langle e^{\alpha_1\varphi(z_1,\bar{z}_1)}\cdots e^{\alpha_N\varphi(z_N,\bar{z}_N)}\rangle, \tag{99}$$

it will become zero unless

$$\sum_{i=1}^{N} \alpha_i = iQ. \tag{100}$$

For later convenience, let us set $Q = i2\alpha_0$, so that the central charge is $c = 1 - 24\alpha_0^2$ and the conformal weight is $\Delta = \alpha^2 - 2\alpha\alpha_0$. We construct an nonzero correlator

$$\langle V_{\alpha_1} V_{\alpha_2} V_{\alpha_3} V_{\alpha_4} \rangle \ , \tag{101}$$

with $\sum \alpha_i = 2\alpha_0$. we want all the operators to have the same conformal dimension Δ, if they are to be identified as a single physical operator.So we are offered with the choice between V_α and $V_{2\alpha_0 - \alpha}$. If $\alpha_0 = 0$, such a function can be easily found

$$\langle V_\alpha V_\alpha V_{-\alpha} V_{-\alpha} \rangle \ , \tag{102}$$

in the case of $\alpha_0 \neq 0$, even if we turn to functions like

$$\langle V_\alpha V_\alpha V_{2\alpha_0 - \alpha} V_{2\alpha_0 - \alpha} \rangle \ , \tag{103}$$

$$\langle V_\alpha V_\alpha V_\alpha V_{2\alpha_0 - \alpha} \rangle \ , \tag{104}$$

apprently $\sum \alpha_i = 2\alpha_0$ cannot be satisfied. However there is a method to make the above correlators nonzero. There are two nontrivial operators which can "screen" additional charges. Such screening operators should have conformal dimension $\Delta = 0$ so that they do not change the conformal properties of the correlator. A local operator with $\Delta = 0$ is an identity operator of the algebra. Here it has two representatives, $V_{\alpha_0}(z)$ and $V_{2\alpha_0}(z)$. But neither of these could provide the necessary screening. There remains the possibility of the integral operators like

$$Q = \int \frac{dz}{2\pi i} \mathcal{O}(z). \tag{105}$$

For the operator Q to be conformal invariant, the operator $\mathcal{O}(z)$ must have $\Delta = 1$. We take $\mathcal{O}(z) = V_\alpha(z) =: e^{\alpha\phi(z)}$: , with the conformal dimension

$$\Delta_\alpha = \alpha^2 - 2\alpha\alpha_0 = 1. \tag{106}$$

Thus its OPE with the stress tensor becomes

$$T(z)\mathcal{O}(w) = \frac{1}{(z-w)^2}\mathcal{O}(w) + \frac{1}{z-w}\partial_w \mathcal{O}(w) + \cdots = \partial_w\left(\frac{1}{z-w}\mathcal{O}(w)\right) + \cdots \tag{107}$$

So as long as the boundary contribution is ignored, we have

$$[T(z), \int \frac{dz}{2\pi i} \mathcal{O}(z)] = 0. \tag{108}$$

This means Q commutes with the Virasoro algebra L_n. There are two solutions to (106),

$$\alpha_\pm = \alpha_0 \pm \sqrt{\alpha_0^2 + 1} \tag{109}$$

So there are two screening operators

$$Q_\pm = \int \frac{dz}{2\pi i} \mathcal{O}_\pm(z), \qquad \mathcal{O}_\pm(z) = V_{\alpha_\pm}(z). \tag{110}$$

In the case of (104), we have $\sum \alpha_i - 2\alpha_0 = 2\alpha$, which can be canceled by adding $Q_\pm$, with α quantized:

$$2\alpha = -\tilde{n}\alpha_- - \tilde{m}\alpha_+. \tag{111}$$

In general, the 4-point function will have the following form

$$\begin{aligned}&\langle \phi_{n,m}(z_1)\phi_{n,m}(z_2)\phi_{n,m}(z_3)\phi_{n,m}(z_4)\rangle \\ &= \int \frac{du_1}{2\pi i} \cdots \int \frac{du_{n-1}}{2\pi i} \int \frac{dv_1}{2\pi i} \cdots \int \frac{dv_{m-1}}{2\pi i} \langle V_{n,m}(z_1)V_{n,m}(z_2)V_{n,m}(z_3)V_{n,m}(z_4)\mathcal{O}_-(u_1)\cdots\mathcal{O}_-(u_{n-1})\mathcal{O}_+(v_1)\cdots\mathcal{O}_+(v_{m-1})\rangle\end{aligned} \tag{112}$$

This is the famous Dotsenko-Fatteev integral.

5.1 Application on $SU(N)$ Toda field theory

The primary operator of W_N algebra is given as the vertex operators:

$$V_{\vec{\alpha}}(z) =: e^{(\alpha,\phi(z))} : , \tag{113}$$

which has the OPE with the W_N generators as

$$W_k(z)V_\alpha(0) = \frac{w_k(\alpha)}{z^k}V_\alpha(0) + O(z^{-k+1}) , \tag{114}$$

with

$$w_2(\alpha) = \Delta(\alpha) = \frac{1}{2}(\alpha,\alpha) + iQ(\rho,\alpha) , \tag{115}$$

$$w_k(\alpha) = (-1)^k \sum_{1\le i_1\le\cdots\le i_k\le n} \prod_{m=1}^{k} (Q(k-m) + i(h_{i_m},\alpha)) . \tag{116}$$

As discussed above, in order to derive non-vanishing correlation function of the form $\langle V_{\vec{\alpha}_1}(z_1)\cdots V_{\vec{\alpha}_M}(z_M)\rangle$, we have freedom to insert screening operators,

$$Q_j^{(\pm)} = \int \frac{dz}{2\pi i} V_j^{(\pm)}(z) = \int \frac{dz}{2\pi i} : e^{\alpha_\pm(e_j,\phi(z))} : . \tag{117}$$

By the requirement of conformal invariance, $w_2(\alpha) = 1$, we need to put $w_2(\alpha_\pm e_j) = 1$. By writing $Q = ib - i/b$, the two solutions are $\alpha_+ = b$, $\alpha_- = -1/b$.

For the computation of four point functions $\langle V_{\vec{\alpha}_4}(\infty)V_{\vec{\alpha}_3}(1)V_{\vec{\alpha}_2}(q)V_{\vec{\alpha}_1}(0)\rangle$ we insert N_a screening currents integrated along $[0,q]$ and $\tilde{N}_a$ currents integrated $[1,\infty]$. This is a useful prescription to see the connection with the Selberg formula [18]. For simplicity, we assume we need only the screening operators $Q^{(+)}$ in the correlator. It gives the Dotsenko-Fateev integral [13] for the four point functions,

$$\begin{aligned}&Z_{\rm DF}(q) = \\ &\left\langle\!\!\left\langle : e^{(\tilde{\alpha_1},\phi(0))} :: e^{(\tilde{\alpha_2},\phi(q))} :: e^{(\tilde{\alpha_3},\phi(1))} :: e^{(\tilde{\alpha_4},\phi(\infty))} : \prod_{a=1}^{N-1}\left(\int_0^q : e^{b(e_a,\phi(z))} : dz\right)^{N_a}\left(\int_1^\infty : e^{b(e_a,\phi(z))} : dz\right)^{\tilde{N}_a} \right\rangle\!\!\right\rangle .\end{aligned} \tag{118}$$

For the charge conservation, this correlator has nonvanishing norm only when

$$\tilde{\alpha_1} + \tilde{\alpha_2} + \tilde{\alpha_3} + \tilde{\alpha_4} + b\sum_a (N_a + \tilde{N}_a)e_a + 2iQ\rho = 0 . \tag{119}$$

We apply Wick's theorem to evaluate the correlator

$$\left\langle\left\langle : e^{(\tilde{\alpha}_1,\phi(z_1))} : \; \dots \; : e^{(\tilde{\alpha}_n,\phi(z_n))} : \right\rangle\right\rangle = \prod_{1\le i<j\le n} (z_j - z_i)^{(\tilde{\alpha}_i,\tilde{\alpha}_j)} \,, \tag{120}$$

where e_a are the simple roots of $SU(N)$, and $(,)$ the bilinear symmetric form on the space dual to the Cartan subalgebra. To be consistent with the parameters introduced in the last section, defining $\tilde{\alpha}_i = \alpha_i/b$, $\beta = b^2$, (118) becomes

$$\begin{aligned} Z_{\mathrm{DF}}(q) &= q^{(\alpha_1,\alpha_2)/\beta}(1-q)^{(\alpha_2,\alpha_3)/\beta} \prod_{a=1}^{N-1}\prod_{I=1}^{N_a}\int_0^q dz_I^{(a)} \prod_{J=N_a+1}^{N_a+\tilde{N}_a}\int_1^\infty dz_J^{(a)} \prod_{i<j}^{N_a+\tilde{N}_a}(z_j^{(a)}-z_i^{(a)})^{2\beta} \quad \times \\ &\times \prod_{i}^{N_a+\tilde{N}_a}(z_i^{(a)})^{(\alpha_1,e_a)}(z_i^{(a)}-q)^{(\alpha_2,e_a)}(z_i^{(a)}-1)^{(\alpha_3,e_a)} \prod_{a=1}^{N-2}\prod_{i}^{N_a+\tilde{N}_a}\prod_{j}^{N_{a+1}+\tilde{N}_{a+1}}(z_j^{(a+1)}-z_i^{(a)})^{-\beta} \quad . \end{aligned} \tag{121}$$

Noticing that we do not include the 3-point functions in the correlator, so this expression should be compared with the instanton contribution of Yang-Mills partition functions in AGT conjecture [17].

6 Selberg integral

In 1944 Selberg find a proof of a noteworthy multiple integral which now plays the role as one of the most fundamental hypergeometric integrals [19, 20].

Selberg integral

$$\int_{[0,1]^k} |\Delta(x)|^{2\gamma} \prod_{i=1}^k x_i^{\alpha-1}(1-x_i)^{\beta-1}\,\mathrm{d}x = \prod_{i=1}^k \frac{\Gamma(\alpha+(i-1)\gamma)\Gamma(\beta+(i-1)\gamma)\Gamma(i\gamma+1)}{\Gamma(\alpha+\beta+(i+k-2)\gamma)\Gamma(\gamma+1)}. \tag{122}$$

When $k=1$ the Selberg integral simplifies to the Euler beta integral [56]

$$\int_0^1 x^{\alpha-1}(1-x)^{\beta-1}\,\mathrm{d}x = \frac{\Gamma(\alpha)\Gamma(\beta)}{\Gamma(\alpha+\beta)}, \qquad \Re(\alpha)>0,\ \Re(\beta)>0, \tag{123}$$

Where k is a positive integer, $x=(x_1,\dots,x_k)$, $\mathrm{d}x = \mathrm{d}x_1\cdots\mathrm{d}x_k$, and

$$\Delta(x) = \prod_{1\le i<j\le k}(x_i - x_j) \tag{124}$$

the Vandermonde product.

For $\alpha,\beta,\gamma\in\mathbb{C}$ such that

$$\Re(\alpha)>0,\ \Re(\beta)>0,\ \Re(\gamma)>-\min\{1/k, \Re(\alpha)/(k-1), \Re(\beta)/(k-1)\} \tag{125}$$

which reduces to the standard definition of the gamma function

$$\Gamma(\alpha) = \int_0^\infty x^{\alpha-1}\mathrm{e}^{-x}\mathrm{d}x, \qquad \Re(\alpha)>0 \tag{126}$$

upon taking $(\beta,x)\to(\zeta,x/\zeta)$ (with $\zeta\in\mathbb{R}$) and letting $\zeta\to\infty$.

Though the Selberg integral was largely overlooked at the time of its publication, now, nearly 70 years later, it has been widely regarded as one of the most fundamental and important hypergeometric integrals. It has connections

and applications to orthogonal polynomials, random matrices, finite reflection groups, hyperplane arrangements, Knizhnik–Zamolodchikov equations and more, check e.g., [57, 58, 59, 60, 61, 62, 63, 64].
Here we consider its A_{N-1} extension [19] (A_{N-1} Selberg integral):

$$S_{\vec{u},\vec{v},\beta} = \int dx \prod_{a=1}^{N-1} \left[\left|\Delta\left(x^{(a)}\right)\right|^{2\beta} \prod_{i=1}^{N_a} \left(x_i^{(a)}\right)^{u_a} \left(1 - x_i^{(a)}\right)^{v_a} \right] \prod_{a=1}^{N-2} \left|\Delta\left(x^{(a)}, x^{(a+1)}\right)\right|^{-\beta} , \tag{127}$$

where $\int dx := \int_0^1 dx^{(1)} \dots \int_0^1 dx^{(N-1)}$. As indicated, the integral contains parameters $\vec{u} = (u_1, \cdots, u_{N-1})$, $\vec{v} = (v_1, \cdots, v_{N-1})$ and β. Similarly, A_{N-1} Selberg average is the integration with the Selberg integration kernel,

$$\left\langle f \right\rangle_{\vec{u},\vec{v},\beta} = \frac{1}{S_{\vec{u},\vec{v},\beta}} \int dx \prod_{a=1}^{N-1} \left[\left|\Delta\left(x^{(a)}\right)\right|^{2\beta} \times \prod_{i=1}^{N_a} \left(x_i^{(a)}\right)^{u_a} \left(1 - x_i^{(a)}\right)^{v_a} \right] \times \prod_{a=1}^{N-2} \left|\Delta\left(x^{(a)}, x^{(a+1)}\right)\right|^{-\beta} f(x) . \tag{128}$$

6.1 Reduction to Selberg integral

In this subsection, we rewrite the Dotsenko-Fateev integral in the form of A_{N-1} Selberg average for the product of N Jack polynomials (see appendix A for a summary of relevant material and [65, 66] for further mathematical details). In physics literature, Jack polynomial is the eigenfunction of quantum Calogero-Sutherland model and relevant to the representation theory of W_N algebra. See for example [67, 68]. The appearance of the product of N Jack polynomials reminds us of another line of recent developments [69, 25, 70, 24, 77] for the computation of conformal block where the convenient basis for the Hilbert space is expressed in terms of Jack polynomial. In particular for $\beta = 1$, it is expressed as product of N Schur polynomial. While the mathematical origin of the appearance of Jack polynomial is different, there should be a good hint to be learned from each other.

Proposition 1 *The integral (121) can be written in the following form (up to $U(1)$ factor),*

$$Z_{DF}(q) = \sum_{\vec{Y}} q^{|\vec{Y}|} \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}\left(-r_k^{(a)} - \frac{v'_{a+}}{\beta}\right) \right\rangle_+ \left\langle \prod_{a=1}^{N} j_{Y_a}^{(\beta)}\left(\tilde{r}_k^{(a)} + \frac{v'_{a-}}{\beta}\right) \right\rangle_- . \tag{129}$$

Here we have to explain some notations. $\vec{Y}$ is a collection of N Young diagrams, $j_Y^{(\beta)}$ is normalized Jack symmetric polynomial. We introduced new parameters $v_{a\pm}$ and $u_{a\pm}$ by

$$v_{a+} = (\alpha_2, e_a), \quad v_{a-} = (\alpha_3, e_a), \quad u_{a+} = (\alpha_1, e_a), \quad u_{a-} = (\alpha_4, e_a), \tag{130}$$

where we use a relation

$$u_{a+} + u_{a-} + v_{a+} + v_{a-} + \beta \sum_b C_{ab}(N_b + \tilde{N}_b) = 2\beta - 2 \tag{131}$$

implied by Eq.(119) to define u_{a-}. The Selberg average $\langle \cdots \rangle_\pm$ is taken with respect to these parameters, $\langle \cdots \rangle_\pm := \langle \cdots \rangle_{\vec{u}_\pm, \vec{v}_\pm, \beta}$. $r_k^{(a)}$ and $\tilde{r}_k^{(a)}$ is related to the integration variables $x_i^{(a)}$ and $y_i^{(a)}$ through

$$r_k^{(a)} := p_k^{(a)} - p_k^{(a-1)}, \quad p_k^{(a)} := \sum_i (x_i^{(a)})^k \quad \text{and} \quad \tilde{r}_k^{(a)} := \tilde{p}_k^{(a)} - \tilde{p}_k^{(a-1)}, \quad \tilde{p}_k := \sum_i (y_i^{(a)})^k , \tag{132}$$

with $p_k^{(0)} = p_k^{(N)} = \tilde{p}_k^{(0)} = \tilde{p}_k^{(N)} = 0$. Finally $v'_{a-} := -\sum_{s=1}^{a-1} v_{s-}$, and $v'_{(N-a)+} := \sum_{s=1}^{a} v_{(N-s)+}$.

In particular, when $N = 2$, the above reduce to (notice that $v'_{1-} = v'_{2+} = 0$)

$$Z_{DF}(q) = \sum_{A,B} q^{|A|+|B|} \left\langle j_A^{(\beta)}\left(-p_k - \frac{v_+}{\beta}\right) j_B^{(\beta)}(p_k) \right\rangle_+ \left\langle j_A^{(\beta)}(\tilde{p}_k) j_B^{(\beta)}\left(-\tilde{p}_k - \frac{v_-}{\beta}\right) \right\rangle_- , \tag{133}$$

which was used in [11]. The proposition is a generalization of their result.

Proof: Let us derive the proposition in the rest of this subsection. Following the procedure in [18] for $SU(2)$, we rename the integration variables in (121) $z_I =: qx_I$, $1 \le I \le N_a$ and $z_J =: \frac{1}{y_J}$, $N_a + 1 \le J \le N_a + \tilde{N}_a$. Then Eq.(121) is rewritten as a double average[3],

$$\left\langle\!\!\left\langle \prod_{a=1}^{N-1} \left\{ \prod_{i=1}^{N_a} (1 - qx_i^{(a)})^{v_{a-}} \prod_{j=1}^{\tilde{N}_a} (1 - qy_j^{(a)})^{v_{a+}} \right\} \prod_{a=1}^{N-1} \prod_{b=1}^{N-1} \left\{ \prod_{i=1}^{N_a} \prod_{j=1}^{\tilde{N}_b} (1 - qx_i^{(a)} y_j^{(b)})^{C_{ab}\beta} \right\} \right\rangle_+ \right\rangle_- , \tag{134}$$

where C_{ab} is A_{N-1} Cartan matrix,

$$C_{ab} = \begin{cases} 2 & a = b \\ -1 & a = b \pm 1 \\ 0 & |a - b| > 1 \ , \end{cases}$$

and the Selberg average $\langle \cdots \rangle_+$ (resp. $\langle \cdots \rangle_-$) is taken over the variables $x_i^{(a)}$ (resp. $y_i^{(a)}$) with parameters $\vec{u}_+, \vec{v}_+$ (resp. $\vec{u}_-, \vec{v}_-$).

We change the second product in the integral (134) into exponential form

$$\begin{aligned} \prod_{a,b=1}^{N-1} \prod_{i=1}^{N_a} \prod_{j=1}^{\tilde{N}_b} (1 - qx_i^{(a)} y_j^{(b)})^{C_{ab}\beta} &= \exp\left\{ \beta \sum_{a,b=1}^{N-1} C_{ab} \sum_{i,j} \ln(1 - qx_i^{(a)} y_j^{(b)}) \right\} \\ &= \exp\left\{ -\beta \sum_{a,b=1}^{N-1} C_{ab} \sum_{k=1}^{\infty} \frac{q^k}{k} p_k^{(a)} \tilde{p}_k^{(b)} \right\} \\ &= \exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \left[2 \sum_{a=1}^{N-1} p_k^{(a)} \tilde{p}_k^{(a)} - \sum_{a=2}^{N-1} p_k^{(a)} \tilde{p}_k^{(a-1)} - \sum_{a=1}^{N-2} p_k^{(a)} \tilde{p}_k^{(a+1)} \right] \right\} \\ &= \exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} r_k^{(a)} \tilde{r}_k^{(a)} \right\} . \end{aligned} \tag{135}$$

In the second line, we performed Taylor expansion and rewrite the variables x, y by $p_k^{(a)}$ and $\tilde{p}_k^{(b)}$. In the last line, we rewrite $p_k, \tilde{p}_k$ by $r_k^{(a)}, \tilde{r}_k^{(a)}$.

Likewise, we rewrite

$$\prod_{a=1}^{N-1} \prod_{i=1}^{N_a} (1 - qx_i^{(a)})^{v_{a-}} = \exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N-1} p_k^{(a)} \frac{v_{a-}}{\beta} \right\} \equiv \exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} r_k^{(a)} \frac{v'_{a-}}{\beta} \right\} . \tag{136}$$

In the second equivalence we change the basis from $p_k^{(a)}$ to $r_k^{(a)}$. The coefficients v'_{a-} are determined from v_{a-} with an additional condition $v'_{1-} := 0$ which is somewhat arbitrary. Similarly,

$$\prod_{a=1}^{N-1} \prod_{j=1}^{\tilde{N}_a} (1 - qy_j^{(a)})^{v_{a+}} = \exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} \tilde{r}_k^{(a)} \frac{v'_{a+}}{\beta} \right\} . \tag{137}$$

This time we define v'_{a+} from another condition $v'_{N+} = 0$ for the convenience of later arguments.

Combining the above factors together, the integrand in (134) takes the form

$$\begin{aligned} &\exp\left\{ -\beta \sum_{k=1}^{\infty} \frac{q^k}{k} \sum_{a=1}^{N} \left[(r_k^{(a)} + \frac{v'_{a+}}{\beta})(\tilde{r}_k^{(a)} + \frac{v'_{a-}}{\beta}) - \frac{v'_{a+}}{\beta} \frac{v'_{a-}}{\beta} \right] \right\} \\ &\qquad = \prod_{a=1}^{N} (1 - q)^{v'_{a+} v'_{a-}/\beta} \sum_{\vec{Y}} \prod_{a=1}^{N} q^{|\vec{Y}|} j_{Y_a}(-r_k^{(a)} - \frac{v'_{a+}}{\beta}) j_{Y_a}(\tilde{r}_k^{(a)} + \frac{v'_{a-}}{\beta}) , \end{aligned} \tag{138}$$

[3] The $U(1)$ prefactors are omitted for its irrelevance to the Nekrasov function.

where we have made use of the Cauchy-Stanley identity (269) for the Jack polynomial in the second line

$$\exp(\beta\sum_{k=1}^{\infty}\frac{1}{k}p_kp'_k)=\sum_R j_R^{(\beta)}(p)j_R^{(\beta)}(p')\,. \tag{139}$$

So the conformal blocks (118) finally becomes

$$\prod_{a=1}^{N}(1-q)^{v'_{a+}v'_{a-}/\beta}\sum_{\vec{Y}}q^{|\vec{Y}|}\left\langle\prod_{a=1}^{N}j_{Y_a}^{(\beta)}(-r_k^{(a)}-\frac{v'_{a+}}{\beta})\right\rangle_{+}\left\langle\prod_{a=1}^{N}j_{Y_a}^{(\beta)}(\tilde{r}_k^{(a)}+\frac{v'_{a-}}{\beta})\right\rangle_{-}\,. \tag{140}$$

Absorbing the prefactor into the $U(1)$ part of the product, we arrive at (129). **QED**

6.2 Known results and a conjecture on Selberg average

The Dotzenko-Fateev integral is now reduced to the evaluation of Selberg average of N Jack polynomials. Before evaluating that, let us first summarize the known results on Selberg average in the literature.

$SU(2)$ **case:** The relevant Selberg averages for one and two Jack polynomials were obtained by Kadell [14],

$$\left\langle J_Y^{(\beta)}(p)\right\rangle_{u,v,\beta}^{SU(2)}=\frac{[N\beta]_Y[u+N\beta+1-\beta]_Y}{\prod_{(i,j)\in Y}\Big(\beta(Y'_j-i)+(Y_i-j)+\beta\Big)[u+v+2N\beta+2-2\beta]_Y}\,, \tag{141}$$

$$\begin{aligned}\left\langle J_A^{(\beta)}(p+w)J_B^{(\beta)}(p)\right\rangle^{SU(2)} &= \frac{[v+N\beta+1-\beta]_A[u+N\beta+1-\beta]_B}{[N\beta]_A[u+v+N\beta+2-2\beta]_B}\times \\ &\times\frac{\prod_{i<j}^{N}\Big(A_i-A_j+(j-i)\beta\Big)_\beta\prod_{i<j}^{N}\Big(B_i-B_j+(j-i)\beta\Big)_\beta}{\prod_{i,j}^{N}\Big(\,u+v+2\beta N+2+A_i+B_j-(1+i+j)\beta\,\Big)_\beta}\times\frac{\prod_{i,j}^{N}\Big(\,u+v+2\beta N+2-(1+i+j)\beta\,\Big)_\beta}{\prod_{i<j}^{N}\Big((j-i)\beta\Big)_\beta\prod_{i<j}^{N}\Big((j-i)\beta\Big)_\beta}\,,\end{aligned} \tag{142}$$

where we have used the following notation

$$[x]_A=\prod_{(i,j)\in A}(x-\beta(i-1)+j-1)=(-1)^{|A|}f_A(-x)\,, \tag{143}$$

and Pochhammer symbol

$$(x)_k=\frac{\Gamma(x+k)}{\Gamma(x)}=x(x+1)\dots(x+k-1)\,. \tag{144}$$

$J_Y^{(\beta)}$, the Jack polynomial, is related to normalized one $j_Y^{(\beta)}$ as (267). Inclusion of a shift w of the argument for the two Jack case was conjectured in [11]. Together with the identity $j_A^{(\beta)}(-p/\beta)=(-1)^{|A|}j_{A'}^{(1/\beta)}(p)$ and an identification of parameter $w=(v+1-\beta)/\beta$, these are sufficient to evaluate (129) for $SU(2)$ case [11].

$SU(n+1)$ **case:** The one-Jack Selberg integral for $SU(n+1)$ could be calculated by the formula offered by Warnaar [19]. To perform the integral, we need to restrict the parameter v as,

$$v_2=\cdots=v_n=0,\quad\text{and}\quad v_1=v. \tag{145}$$

As already explained, in Toda field theory, this condition is necessary to solve conformal Ward identity for the W-algebra [5, 8]. The formula by Warnaar is,

$$\begin{aligned}\left\langle J_B^{(\beta)}(p_k^{(n)})\right\rangle_{\vec{u},\vec{v},\beta}^{SU(n+1)} &= \prod_{1\le i<j\le N_n}\frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}}\times \\ &\times\prod_{a=1}^{n}\prod_{i=1}^{N_n}\frac{(u_{n-a+1}+\cdots+u_n+a+(N_n-a-i+1)\beta)_{B_i}}{(v_{n-a+1}+u_{n-a+1}+\cdots+u_n+a+1+(N_n+N_{n-a+1}-N_{n-a}-a-i)\beta)_{B_i}}\,.\end{aligned} \tag{146}$$

To evaluate (129), we need Selberg average of $(n+1)$ Jack polynomials. While we do not perform the integration so far, we find a formula for $\beta = 1$ which reproduces known results and satisfies consistency conditions[4]. As explained in appendix A, the Jack polynomial for $\beta = 1$ is called Schur polynomial and we write $J_Y^{(\beta)}|_{\beta=1} = \chi_Y$.

Conjecture *We propose the following formula of Selberg average for $n+1$ Schur polynomials,*

$$\begin{aligned}
&\left\langle \chi_{Y_1}(-p_k^{(1)} - v_1') \dots \chi_{Y_r}(p_k^{(r-1)} - p_k^{(r)} - v_r') \dots \chi_{Y_{n+1}}(p_k^{(n)}) \right\rangle^{SU(n+1)}_{\vec{u},\vec{v},\beta=1} \\
&= \prod_{s=1}^{n} \left\{ (-1)^{|Y_s|} \times \frac{[v_s + N_s - N_{s-1}]_{Y_s'}}{[N_s + N_{s-1}]_{Y_s'}} \times \prod_{1 \le i < j \le N_{s-1}+N_s} \frac{(j-i+1)_{Y'_{si}-Y'_{sj}}}{(j-i)_{Y'_{si}-Y'_{sj}}} \right\} \times \prod_{1\le i<j\le N_n} \frac{(j-i+1)_{Y_{(n+1)i}-Y_{(n+1)j}}}{(j-i)_{Y_{(n+1)i}-Y_{(n+1)j}}} \\
&\times \prod_{1 \le t < s \le n+1} \left\{ \frac{[v_t + u_t + \dots + u_{s-1} + N_t - N_{t-1}]_{Y_t'}}{[v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s]_{Y_t'}} \times \frac{[-v_s + u_t + \dots + u_{s-1} - N_s + N_{s-1}]_{Y_s}}{[v_t - v_s + u_t + \dots + u_{s-1} - N_{t-1} - N_s + N_{s-1}]_{Y_s}} \right. \\
&\quad \left. \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{s-1}} \frac{v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 - (i+j)}{v_t - v_s + u_t + \dots + u_{s-1} + N_t - N_{t-1} - N_s + N_{s-1} + 1 + Y'_{ti} + Y_{sj} - (i+j)} \right\},
\end{aligned} \tag{147}$$

with $v_r' := \sum_{a=r}^{n} v_a = v\delta_{r1}$ after imposing the constraint (145).

As we wrote, this formula seems reasonable since

- It reproduces the AGT relation as we will see in the next section.
- It is reduced to the known results for $\beta = 1$ with the help of (322),
  - (a) For $Y_1 = \dots = Y_n = \emptyset$, and $Y_{n+1} = B$, the above reduce to the A_n one Jack integral (146).
  - (b) For $n = 1$, $Y_1 = A$ and $Y_2 = B$, it coincides with the A_1 two Jack integral (142).
  - (c) For $n = 2$, $Y_1 = R$, $Y_2 = \emptyset$, and $Y_3 = B$, the above is consistent with the A_2 two Jack integral (274) given by Warnaar [20].
  - (d) For $N_n = 0$, $u_n = 0$ and $Y_{n+1} = \emptyset$, the above reduces to the formula for A_{n-1}.

Another type of consistency conditions is also considered. For the simplest case, we start from multiplying a trivial zero factor $v + (-p_1^{(1)} - v) + (p_1^{(1)} - p_1^{(2)}) + \dots + (p_1^{(n-1)} - p_1^{(n)}) + p_1^{(n)} = 0$ in the integrand of (147). We then apply to each term a property of Schur polynomial,

$$p_1 \chi_R(p_k) = \sum_{\tilde{R}} \chi_{\tilde{R}}(p_k) , \tag{148}$$

where the summation is over all possible Young diagrams which can be obtained from R by adding one cell. This gives rise to a consistency condition for any combination $(Y_1, \cdots, Y_{n+1})$;

$$\begin{aligned}
&v \left\langle \chi_{Y_1}(-p_k^{(1)} - v_1') \dots \chi_{Y_r}(p_k^{(r-1)} - p_k^{(r)} - v_r') \dots \chi_{Y_{n+1}}(p_k^{(n)}) \right\rangle^{SU(n+1)}_{\vec{u},\vec{v},\beta=1} \\
&+ \sum_{r=1}^{n+1} \sum_{\tilde{Y}_r} \left\langle \chi_{Y_1}(-p_k^{(1)} - v_1') \dots \chi_{\tilde{Y}_r}(p_k^{(r-1)} - p_k^{(r)} - v_r') \dots \chi_{Y_{n+1}}(p_k^{(n)}) \right\rangle^{SU(n+1)}_{\vec{u},\vec{v},\beta=1} = 0 .
\end{aligned} \tag{149}$$

While this looks trivial, the cancellation becomes rather nontrivial. We give a detailed computation for the simpler cases, $n = 2$ ($SU(3)$) with Y_1, Y_2, Y_3 being rectangle Young diagrams, in appendix C.

[4] Actually we could guess a formula for general β (see appendix B) which reproduces the known results. While the formula looks quite reasonable, it does not pass one of the consistency checks. It seems that some modifications up to the terms proportional to $1 - \beta$ are needed.

We may write easily some generalizations of (148) such as,

$$\chi_{[n]}(p_k)\chi_R(p_k) = \sum_{\tilde{R}} \chi_{\tilde{R}}(p_k) \ , \tag{150}$$

where $\tilde{R}/R$ is $[n]$. We hope that such series of consistency conditions may serve as a proof of the formula (147) in the future.

7 Direct approach on AGT conjecture

7.1 AGT conjecture from Selberg integral

In the following, we present a 'proof' of AGT conjecture for $SU(n+1)$ case by using the postulated formulae for Selberg average in 6.2. It is a generalization of the proof for $SU(2)$ case in [11, 12]. As we already mentioned, what we need to see is the coincidence of partition function,

$$Z_{\text{inst}}(q) = Z_{\text{DF}}(q) \ , \tag{151}$$

up to $U(1)$ factor but we would like to see the stronger condition, namely the coefficient N^{Inst} in the instanton partition function (26) with the similar coefficient N^{Toda} in (129)

$$N^{\text{inst}}_{\vec{Y}} = N^{\text{Toda}}_{\vec{Y}} . \tag{152}$$

We show that this stronger identity holds at $\beta = 1$.

We note that both coefficients have the factorized form:

$$N^{\text{inst}}_{\vec{Y}} \equiv N^{\text{inst}}_{\vec{Y}+} N^{\text{inst}}_{\vec{Y}-}, \quad N^{\text{Toda}}_{\vec{Y}} \equiv N^{\text{Toda}}_{\vec{Y}+} N^{\text{Toda}}_{\vec{Y}-}, \tag{153}$$

with

$$\begin{aligned} N^{\text{inst}}_{\vec{Y}+} &\equiv \frac{\prod_{s=1}^{n+1}\prod_{k=1}^{n+1} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{n+1} G_{Y_t,Y_s}(a_t - a_s)} \prod_{s=1}^{n+1} \left\{ (-1)^{|Y_s|} \sqrt{\frac{G_{Y_s,Y_s}(0)}{G_{Y_s,Y_s}(1-\beta)}} \right\} , \\ N^{\text{inst}}_{\vec{Y}-} &\equiv \frac{\prod_{s=1}^{n+1}\prod_{k=n+2}^{2n+2} f_{Y_s}(\mu_k + a_s)}{\prod_{t,s=1}^{n+1} G_{Y_t,Y_s}(a_t - a_s + 1 - \beta)} \prod_{s=1}^{n+1} \left\{ (-1)^{|Y_s|} \sqrt{\frac{G_{Y_s,Y_s}(1-\beta)}{G_{Y_s,Y_s}(0)}} \right\} , \end{aligned} \tag{154}$$

and

$$N^{\text{Toda}}_{\vec{Y}\pm} \equiv \left\langle \prod_{a=1}^{n+1} j^{(\beta)}_{Y_a}(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}) \right\rangle_{\pm} = \prod_{a=1}^{n+1} \sqrt{\frac{G_{Y_a,Y_a}(0)}{G_{Y_a,Y_a}(1-\beta)}} \left\langle \prod_{a=1}^{n+1} J^{(\beta)}_{Y_a}(-r^{(a)}_k - \frac{v'_{a\pm}}{\beta}) \right\rangle_{\pm} . \tag{155}$$

We remind that $r^{(a)}_k \equiv p^{(a)}_k - p^{(a-1)}_k$, $v'_{a-} = -\sum_{s=1}^{a-1} v_{s-}$ and $v'_{(N-a)+} = \sum_{s=1}^{a} v_{(N-s)+}$. Therefore, the problem left is to figure out whether the (n+1)-Jack Selberg integral has the same form with its Nekrasov counterpart for $\beta = 1$,

$$N^{\text{Toda}}_{\vec{Y}\pm} = N^{\text{inst}}_{\vec{Y}\pm} \quad . \tag{156}$$

7.2 Special case: $\vec{Y} = (\emptyset, \cdots, \emptyset, B)$, arbitrary β

In the following, we prove (156) for '+' part. Proof for '$-$' is similar. We will omit the lower index"+" in $v_{a+}$and u_{a+} as long as there are no misunderstanding.

We start from the simplest case, when $Y_1 = \cdots = Y_n = \emptyset,\ Y_{n+1} = B$. In this case, the Selberg integral is already proved by Warnaar for arbitrary β. So our proof for this case is exact and holds without the restriction of β.

In the instanton part, we have,

$$N^{\text{inst}}_{(\emptyset,\ldots,\emptyset,B)+} = \frac{(-1)^{|B|}\prod_{k=1}^{n+1} f_B(\mu_k + a_{n+1})}{\sqrt{G_{B,B}(0)G_{B,B}(1-\beta)}\prod_{m=1}^{n} G_{B,\emptyset}(a_{n+1}-a_m)}\,. \tag{157}$$

On the other hand, the one-Jack Selberg integral is given in (146)

$$\begin{aligned} N^{\text{Toda}}_{(\emptyset,\ldots,\emptyset,B)+} &= \left\langle j_B^{(\beta)}(p_k^{(n)})\right\rangle_+^{SU(n+1)} \\ &= \sqrt{\frac{G_{B,B}(0)}{G_{B,B}(1-\beta)}} \times \left\langle J_B(p_k^{(n)})\right\rangle_+^{SU(n+1)} \\ &= \sqrt{\frac{G_{B,B}(0)}{G_{B,B}(1-\beta)}} \times \prod_{1\le i<j\le N_n} \frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} \\ &\quad \times \prod_{a=1}^{n}\prod_{i=1}^{N_n} \frac{(u_{n-a+1}+\cdots+u_n+a+(N_n-a-i+1)\beta)_{B_i}}{(v_{n-a+1}+u_{n-a+1}+\cdots+u_n+a+1+(N_n+N_{n-a+1}-N_{n-a}-a-i)\beta)_{B_i}}\,. \end{aligned} \tag{158}$$

To see the equivalence, first we note that the function $f_B(x)$ in N^{inst} is linked to the notation $[x]_B$ by (143). Then we need to rewrite G_{AB} in terms of $(x)_B$ in (158). For this purpose, we need the following lemmas which will be proved in appendix:

Lemma 1

$$\prod_{1\le i<j\le N} \frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} = \frac{[N\beta]_B}{G_{B,B}(0)} \tag{159}$$

Lemma 2

$$\prod_{i=1}^{N}(x-i\beta)_{B_i} = \Big[x-\beta\Big]_B \tag{160}$$

Lemma 3

$$[x]_B = (-1)^{|B|}G_{B,\emptyset}(-x+1-\beta) \tag{161}$$

With the help of these formulae, we arrive at the results

$$\begin{aligned} &N^{\text{Toda}}_{(\emptyset,\ldots,\emptyset,B)} = \left\langle j_B^{(\beta)}(p_k^{(n)})\right\rangle = \\ &= \frac{[N_n\beta]_B}{\sqrt{G_{B,B}(0)G_{B,B}(1-\beta)}} \times \prod_{a=1}^{n} \frac{(-1)^{|B|}[u_{n-a+1}+\cdots+u_n+N_n\beta+a-a\beta]_B}{G_{B,\emptyset}(-(v_{n-a+1}+u_{n-a+1}+\cdots+u_n+N_n\beta+N_{n-a+1}\beta-N_{n-a}\beta+a-a\beta))}\,. \end{aligned} \tag{162}$$

This is equivalent to (157), with the identifications of parameters (where we have omitted the lower index"+" in v_{a+}

and u_{a+}) [5]

$$
\begin{aligned}
&\mu_{n+1}+a_{n+1}=-N_n\beta\ ,\\
&\vdots\\
&\mu_s+a_{n+1}=-\big(u_s+\cdots+u_n+N_n\beta+(n-s+1)(1-\beta)\big)\ ,\\
&\vdots\\
&\mu_1+a_{n+1}=-\big(u_1+\cdots+u_n+N_n\beta+n(1-\beta)\big)\ ,\\
&a_n-a_{n+1}=v_n+u_n+2N_n\beta-N_{n-1}\beta+1-\beta\ ,\\
&\vdots\\
&a_s-a_{n+1}=v_s+u_s+\cdots+u_n+N_n\beta+N_s\beta-N_{s-1}\beta+(n-s+1)(1-\beta)\ ,\\
&\vdots\\
&a_1-a_{n+1}=v_1+u_1+\cdots+u_n+N_n\beta+N_1\beta+n(1-\beta)\ ,
\end{aligned}
\tag{163}
$$

with the restriction $v_2=\cdots=v_n=0$ and $v_1=v$. While this looks complicated, it is simplified in the vector notation in $\mathbf{R}^{n+1}$,

$$
a=\alpha_1+\alpha_2+\beta\sum_a N_a e_a+(1-\beta)\rho,\quad \mu=-\alpha_1-(1-\beta)\rho\,, \tag{164}
$$

where $a=\sum_{i=1}^{N+1}a_ih_i$ and $\mu=\sum_{i=1}^{N+1}\mu_ih_i$. We note that a thus written can be identified with the momentum of the vertex in the intermediate channel. This gives (95). Eq.(164) is the desired identification of parameters in $SU(N+1)$ AGT conjecture [5, 6]. We note that this holds for arbitrary β.

7.3 General case: arbitrary $\vec{Y}$, $\beta=1$

By interpolation method, we have derived that the $(N+1)$-Schur Selberg integral has the form of (147):
At $\beta=1$,

$$
\begin{aligned}
&N^{\text{Toda}}_{\vec{Y}+}\\
&=\left\langle \chi_{Y_1}(-p_k^{(1)}-(v_1+\cdots+v_n))\ldots\chi_{Y_r}(p_k^{(r-1)}-p_k^{(r)}-\frac{v_r+\cdots+v_n}{\beta})\ldots\chi_{Y_{n+1}}(p_k^{(n)})\right\rangle^{SU(n+1)}_{\vec{u},\vec{v},\beta}\\
&=\prod_{s=1}^{n}\left\{(-1)^{|Y_s|}\times\frac{[v_s+N_s-N_{s-1}]_{Y'_s}}{[N_s+N_{s-1}]_{Y'_s}}\times\prod_{1\le i<j\le N_{s-1}+N_s}\frac{(j-i+1)_{Y'_{si}-Y'_{sj}}}{(j-i)_{Y'_{si}-Y'_{sj}}}\right\}\times\prod_{1\le i<j\le N_n}\frac{(j-i+1)_{Y_{(n+1)i}-Y_{(n+1)j}}}{(j-i)_{Y_{(n+1)i}-Y_{(n+1)j}}}\\
&\times\prod_{1\le t<s\le n+1}\left\{\frac{[v_t+u_t+\cdots+u_{s-1}+N_t-N_{t-1}]_{Y'_t}}{[v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s]_{Y'_t}}\times\frac{[-v_s+u_t+\cdots+u_{s-1}-N_s+N_{s-1}]_{Y_s}}{[v_t-v_s+u_t+\cdots+u_{s-1}-N_{t-1}-N_s+N_{s-1}]_{Y_s}}\right.\\
&\qquad\left.\times\prod_{i=1}^{N_t}\prod_{j=1}^{N_{s-1}}\frac{v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s+N_{s-1}+1-(i+j)}{v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s+N_{s-1}+1+Y'_{ti}+Y_{sj}-(i+j)}\right\}.
\end{aligned}
\tag{165}
$$

Then with the lemmas (159) to (161) introduced in the last section and a new assistant (which only holds at $\beta=1$),[6]

Lemma 4

$$
\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{\big(x+1-(i+j)\beta\big)_\beta}{\big(x+1+A'_i+B_j-(i+j)\beta\big)_\beta}=\frac{(-1)^{|B|}[x-N_2\beta+1-\beta]_{A'}[x-N_1\beta+1-\beta]_B}{G_{A,B}(x)G_{B,A}(-x)} \tag{166}
$$

[5] There is some degree of freedom to choose the possible identifications.
[6] Check the appendix for the proof.

Equation (165) transforms to

$$\begin{aligned}
&\Big\langle \chi_{Y_1}(-p_k^{(1)}-(v_1+\cdots+v_n))\ldots\chi_{Y_r}(p_k^{(r-1)}-p_k^{(r)}-\frac{v_r+\cdots+v_n}{\beta})\ldots\chi_{Y_{n+1}}(p_k^{(n)})\Big\rangle_{\vec{u},\vec{v},\beta}^{SU(n+1)}\\
&=\prod_{s=1}^{n}\left\{(-1)^{|Y_s|}\times\frac{[v_s+N_s-N_{s-1}]_{Y_s'}}{G_{Y_s',Y_s'}(0)}\right\}\times\frac{[N_n]_{Y_{n+1}}}{G_{Y_{n+1},Y_{n+1}}(0)}\times\\
&\times\prod_{1\le t<s\le n+1}\left\{\frac{[v_t+u_t+\cdots+u_{s-1}+N_t-N_{t-1}]_{Y_t'}}{1}\times\frac{[-\big(v_s-u_t-\cdots-u_{s-1}+N_s-N_{s-1}\big)]_{Y_s}}{1}\times\right.\\
&\quad\times\frac{1}{G_{Y_t,Y_s}\big(v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s+N_{s-1}\big)}\times\\
&\quad\left.\times\frac{(-1)^{|Y_s|}}{G_{Y_s,Y_t}\big(-(v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s+N_{s-1})\big)}\right\}.
\end{aligned}\tag{167}$$

Further notice that for $\beta=1$,

$$[x]_{A'}=(-1)^{|A|}[-x]_A=f_A(x),\quad G_{A',A'}(x)=G_{A,A}(x)\tag{168}$$

(167) is equivalent to its Nekrasov counterpart (154) $N^{\text{inst}}_{\vec{Y}+}$ at $\beta=1$ with the identifications(163) and the following (where we have again omitted the lower index"+" in $v_{a+}$and u_{a+})

$$\begin{aligned}
&a_t-a_s=v_t-v_s+u_t+\cdots+u_{s-1}+N_t-N_{t-1}-N_s+N_{s-1}\ ,\\
&\mu_s+a_t=v_t+u_t+\cdots+u_{s-1}+N_t-N_{t-1}\ ,\\
&\mu_t+a_s=v_s-u_t-\cdots-u_{s-1}+N_s-N_{s-1}\ ,\\
&\mu_s+a_s=v_s+N_s-N_{s-1}\ ,
\end{aligned}\tag{169}$$

where $1\le t<s\le n$. The above are of course in accordance with (163) and (164).

This implies AGT relation for $SU(n+1)$ at $\beta=1$.

8 Recursive approach for general beta case

In the above sections we found a formula for such Selberg average and show that it reproduces the SU(N) version of AGT conjecture with $\beta=1$. The only pity is that, our formulae for Selberg average are not based on explicit evaluation but determined by consistency. So now we would like to turn to another approach, using the recursive method.

8.1 Recursion formula for Nekrasov partition function

In this section, we present the accurate form of the formula (1) and then derive it from the definition (32). For this purpose, we need to introduce some notations. We decompose Y,W into rectangles $Y=(r_1,\cdots,r_f;s_1,\cdots,s_f)$ (with $0<r_1<\cdots<r_f$, $s_1>\cdots>s_f>0$, see Figure 3 for the parametrization). We use f_p (resp. $\bar{f}_p$) to represent the number of rectangles of Y_p (resp W_p). Furthermore, we write (with $r_0=s_{k+1}=0$);

$$A_k(Y)\ =\ \beta r_{k-1}-s_k-\xi,\quad(k=1,\cdots,f+1)\,,\tag{170}$$

$$B_k(Y)\ =\ \beta r_k-s_k,\quad(k=1,\cdots,f)\,,\tag{171}$$

where $\xi:=1-\beta$. $A_k(Y)$ (resp. $B_k(Y)$) represents the k^{th} location where a box may be added to (resp. deleted from) Young diagram Y (Figure 4) composed with a map from location to $\mathbf{C}$. $\nu\in\mathbf{C}$ is an arbitrary constant.

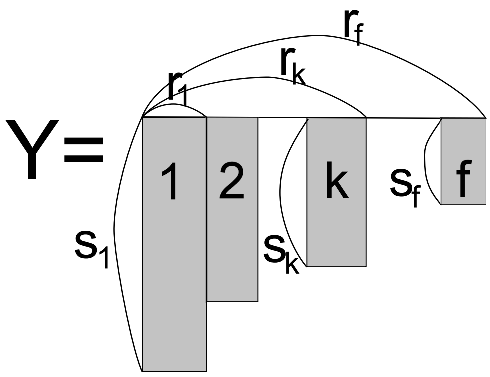

Figure 3: Decomposition of Young diagram by rectangles. For later convenience, in this and the next section we choose the notation where the diagram is upside-down compared to before

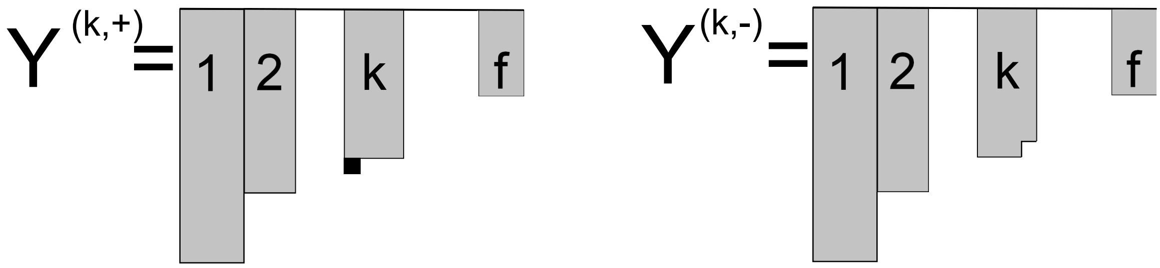

Figure 4: Locations of boxes

We denote $Y^{(k,+)}$ (resp. $Y^{(k,-)}$) as the Young diagram obtained from Y by adding (resp. deleting) a box at $(r_{k-1}+1, s_k+1)$ (resp. (r_k, s_k)). Similarly we use notation $\vec{Y}^{(k\pm),p} = (Y_1, \cdots, Y_p^{(k,\pm)}, \cdots, Y_N)$ to represent the variation of a Young diagram in a set of Young tables $\vec{Y}$.

One can write the schematic relation (1) more explicitly. We define,

$$\begin{aligned}
\delta_{-1,n} Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) &= \sum_{p=1}^{N} \left(\sum_{k=1}^{f_p+1} (a_p + \nu + A_k(Y_p))^n \Lambda_p^{(k,+)}(\vec{a}, \vec{Y}) Z(\vec{a}, \vec{Y}^{(k,+),p}; \vec{b}, \vec{W}; \mu) \right. \\
&\quad \left. - \sum_{k=1}^{\tilde{f}_p} (b_p + \mu + \nu + B_k(W_p))^n \Lambda_p^{(k,-)}(\vec{b}, \vec{W}) Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}^{(k,-),p}; \mu) \right), \qquad (172) \\
\delta_{1,n} Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) &= \sum_{p=1}^{N} \left(- \sum_{k=1}^{f_p} (a_p + \nu + B_k(Y_p))^n \Lambda_p^{(k,-)}(\vec{a}, \vec{Y}) Z(\vec{a}, \vec{Y}^{(k,-),p}; \vec{b}, \vec{W}; \mu) \right. \\
&\quad \left. + \sum_{k=1}^{\tilde{f}_p} (b_p + \nu + \mu + A_k(W_p) + \xi)^n \Lambda_p^{(k,+)}(\vec{b}, \vec{W}) Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}^{(k,+),p}; \mu) \right), \qquad (173)
\end{aligned}$$

where we introduced coefficients Λ:

$$\Lambda_p^{(k,+)}(\vec{a}, \vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q} \frac{a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + A_k(Y_p) - B_\ell(Y_q)} {\prod}'^{f_q+1}_{\ell=1} \frac{a_p - a_q + A_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + A_k(Y_p) - A_\ell(Y_q)} \right) \right)^{1/2} \qquad (174)$$

$$\Lambda_p^{(k,-)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N}\left(\prod_{\ell=1}^{f_q+1}\frac{a_p-a_q+B_k(Y_p)-A_\ell(Y_q)-\xi}{a_p-a_q+B_k(p)-A_\ell(q)}{\prod}'^{f_q}_{\ell=1}\frac{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)+\xi}{a_p-a_q+B_k(Y_p)-B_\ell(Y_q)}\right)\right)^{1/2} \quad (175)$$

Prime in the product symbol ($\prod'$) represents that $(\ell,q)=(k,p)$ is excluded in the product. A parameter ν is arbitrary.

In order to define the polynomial $U_{\pm 1,n}$, we introduce a generating functional for multi-variables, $x_1,\cdots,x_{\mathcal{N}}$, $y_1,\cdots,y_{\mathcal{N}}$, (the expansion around $\zeta=\infty$),

$$\prod_{I=1}^{\mathcal{N}}\frac{\zeta-y_I}{\zeta-x_I}=1+\sum_{n=1}^{\infty}q_n(x,y)\zeta^{-n}\,. \quad (176)$$

which gives the order n polynomial q_n in variables x_I and y_I. $U_{\pm 1,n}$ is written in terms of q_n as

$$U_{-1,n}=\beta^{-1/2}q_{n+1}(x,y),\qquad U_{1,n}=\beta^{-1/2}q_{n+1}(x,y)\,. \quad (177)$$

where we need make replacements of variables:

$$x_I \to \{\nu+A_k(Y_p),\nu+\mu+B_k(W_p)\},\ y_I\to\{\nu+\mu+A_k(W_p)+\xi,\nu+B_k(Y_p)-\xi\} \quad \text{for} \quad U_{-1,n}\,, \quad (178)$$
$$x_I \to \{\nu+\mu+A_k(W_p)+\xi,\nu+B_k(Y_p)\},\ y_I\to\{\nu+A_k(Y_p)+\xi,\nu+\mu+B_k(W_p)\} \quad \text{for} \quad U_{1,n}\,. \quad (179)$$

Here k,p run over all possible values and the number of variables is $\mathcal{N}=N+\sum_{p=1}^{N}(f_p+\bar{f}_p)$.

We note that the right hand side of (176) is written as

$$\exp\left(\sum_{n=1}^{\mathcal{N}}\frac{\zeta^{-n}}{n}p_n(x,y)\right),\quad p_n(x,y):=\sum_{I=1}^{\mathcal{N}}({x_I}^n-{y_I}^n)\,. \quad (180)$$

In terms of p_n, the function q_n is written as,

$$q_1=p_1,\quad q_2=\frac{1}{2}(p_2+p_1^2),\cdots \quad (181)$$

and so on. In general it takes the form of Schur polynomial for single row Young diagram (n) written in terms of power sum polynomial.

Let us give a proof of the recursion relation (1). It is based on a direct evaluation of the variations of Nekrasov partition function which is given in the appendix.

By the formulae (334–341), the left hand side of (172,173) are written in the form,

$$\beta^{-1/2}\sum_{I=1}^{\mathcal{N}}(x_I)^n\frac{\prod_{J=1}^{\mathcal{N}}(x_I-y_J)}{\prod'_J(x_I-x_J)} \quad (182)$$

with the replacements (178,179). We rewrite this expression in the form of the generating functional,

$$\sum_{I=1}^{\mathcal{N}}\left(\sum_{n=0}^{\infty}\frac{x_I^n}{\zeta^{n+1}}\right)\frac{\prod_{J=1}^{\mathcal{N}}(x_I-y_J)}{\prod'_J(x_I-x_J)}=\sum_{I=1}^{\mathcal{N}}\frac{1}{\zeta-x_I}\frac{\prod_{J=1}^{\mathcal{N}}(x_I-y_J)}{\prod'_J(x_I-x_J)}=\prod_{I=1}^{\mathcal{N}}\frac{\zeta-y_I}{\zeta-x_I}-1\,. \quad (183)$$

From the second to the third term, we need use a nontrivial identity [22] which can be proved by comparing the locations of poles and the residue on both hand side. The third term takes the form of the left hand side of (172,173). Comparing the coefficients of $\zeta^{-(n+1)}$, we arrive at the recursion formula (1).

8.2 Symmetry algebra $\mathbf{SH}^c$

In this section, we show that the structure of the one box variations in (1) has a nonlinear algebra which is denoted as SH^c in the literature [15]. It has generators $D_{r,s}$ with $r \in \mathbf{Z}$ and $s \in \mathbf{Z}_{\geq 0}$. We call the first index r as degree and the second index s as order of generator. The commutation relation for degree $\pm 1, 0$ generators is defined by,

$$\begin{aligned} [D_{0,l}, D_{1,k}] &= D_{1,l+k-1}, \quad l \geq 1, && (184)\\ [D_{0,l}, D_{-1,k}] &= -D_{-1,l+k-1}, \quad l \geq 1, && (185)\\ [D_{-1,k}, D_{1,l}] &= E_{k+l} \quad l, k \geq 1, && (186)\\ [D_{0,l}, D_{0,k}] &= 0, \ k, l \geq 0, && (187) \end{aligned}$$

where E_k is a nonlinear combination of $D_{0,k}$ determined by

$$1 + (1-\beta) \sum_{l \geq 0} E_l s^{l+1} = \exp(\sum_{l \geq 0} (-1)^{l+1} c_l \pi_l(s)) \exp(\sum_{l \geq 0} D_{0,l+1} \omega_l(s)), \tag{188}$$

where

$$\begin{aligned} &\pi_l(s) = s^l G_l(1 + (1-\beta)s), && (189)\\ &\omega_l(s) = \sum_{q=1,-\beta,\beta-1} s^l (G_l(1-qs) - G_l(1+qs)), && (190)\\ &G_0(s) = -\log(s), \quad G_l(s) = (s^{-l} - 1)/l \quad l \geq 1. && (191) \end{aligned}$$

The parameters c_l $(l \geq 0)$ are central charges. First few E_l can be computed more explicitly as,

$$\begin{aligned} E_0 &= c_0, && (192)\\ E_1 &= -c_1 + c_0(c_0 - 1)\xi/2, && (193)\\ E_2 &= c_2 + c_1(1-c_0)\xi + c_0(c_0-1)(c_0-2)\xi^2/6 + 2\beta D_{0,1}, && (194)\\ E_3 &= 6\beta D_{0,2} + 2c_0\beta\xi D_{0,1} + \cdots, && (195)\\ E_4 &= 12\beta D_{0,3} + 6c_0\beta\xi D_{0,2} + (-c_0\beta\xi^2 + c_0^2\beta\xi^2 - 2c_1\beta\xi + 2 - 4\xi + 4\xi^2 - 2\xi^3) D_{0,1} + \cdots. && (196) \end{aligned}$$

where $\cdots$ are terms which does not contain $D_{0,l}$.

Other generators are defined recursively by,

$$\begin{aligned} D_{l+1,0} &= \frac{1}{l} [D_{1,1}, D_{l,0}], \qquad & D_{-l-1,0} &= \frac{1}{l} [D_{-l,0}, D_{-1,1}], && (197)\\ D_{r,l} &= [D_{0,l+1}, D_{r,0}] \qquad & D_{-r,l} &= [D_{-r,0}, D_{0,l+1}]. && (198) \end{aligned}$$

for $l \geq 0, r > 0$.

Some of the basic properties of SH^c [15] are following:

- The algebra has a natural action on the fixed points of localization in the moduli space of $SU(N)$ instantons.
- It can be derived as a singular limit of double affine Hecke algebra (DAHA) [27].
- When $\beta \to 1$, the algebra reduces to much simpler algebra $W_{1+\infty}$.
- For general β, the algebra contains W_N algebra when the representation is constructed out of N Young diagrams.
- It has close relation with the recursion relation among Jack polynomials.

8.3 Introduction of the basis

The $SU(N)$ generalization of AGT conjecture implies that the partition function (28) can be written as the conformal block of $n+3$ point function of $SU(N)$ Toda field theory where the Hilbert space $\mathcal{H}$ is described by chiral W_n algebra with $U(1)$ factor.

The conformal block can also be illustrated by Figure 2. It can be reduced to the multiplication of three point functions by inserting a complete basis of the Hilbert space at the intermediate channel. In Figure 2, insertion points of such operators are depicted by arrows. In $W_n + U(1)$ system, the basis of the Hilbert space is labeled by N Young tables $\vec{Y}$. Then it may be possible to choose such basis such that the factor $Z_{\vec{Y},\vec{W}}$ in the previous section may be rewritten as $Z_{\vec{Y},\vec{W}} \sim \langle \vec{Y}|V(1)|\vec{W}\rangle$ with some vertex operator V. The existence of such basis was formally claimed in [23, 24] for general β in terms of Jack polynomial, but the explicit form was not given except for some simple examples.

An exceptional case occurs when $\beta = 1$ and the system is described by N pairs of free fermions (appendix G). In this case, there is a reasonable guess on the explicit form of $|\vec{a}, \vec{Y}\rangle$ [25, 70] as a product of Schur polynomials, namely $|\vec{Y}\rangle \sim \prod_{p=1}^{N} \chi_{Y^{(p)}}$. (See also [77] for a similar analysis.)

Now for the general β case, to see the relation with (1), we introduce a Hilbert space $\mathcal{H}_{\vec{a}}$ spanned by an basis $|\vec{a}, \vec{Y}\rangle$ where $\vec{a} \in \mathbf{C}^N$ and $\vec{Y} = (Y_1, \cdots, Y_N)$ is a set of N Young tables. The dual basis $\langle \vec{a}, \vec{Y}|$ is defined such that

$$\langle \vec{a}, \vec{Y}|\vec{b}, \vec{W}\rangle = \delta_{\vec{Y},\vec{W}}\delta(\vec{a} - \vec{b})\,. \tag{199}$$

Inspired from $\beta = 1$ case, we DEFINE the action of $D_{\pm 1,l}, D_{0,l}$ on the ket and bra basis as,

$$D_{-1,l}|\vec{b}, \vec{W} > \;=\; (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q} (b_q + B_t(W_q))^l \Lambda_q^{(t,-)}(\vec{W})|\vec{b}, \vec{W}^{(t,-),q} >\,, \tag{200}$$

$$D_{1,l}|\vec{b}, \vec{W} > \;=\; (-1)^l \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q+1} (b_q + A_t(W_q))^l \Lambda_q^{(t,+)}(\vec{W})|\vec{b}, \vec{W}^{(t,+),q} >\,, \tag{201}$$

$$D_{0,l+1}|\vec{b}, \vec{W} > \;=\; (-1)^l \sum_{q=1}^{N} \sum_{\mu \in W_q} (b_q + c(\mu))^l |\vec{b}, \vec{W} >\,, \tag{202}$$

$$\langle \vec{a}, \vec{Y}|D_{-1,l} \;=\; (-1)^l \sum_{p=1}^{N} \sum_{t=1}^{f+1} (a_p + A_t(Y_p))^l \Lambda_p^{(t,+)}(\vec{Y})\langle \vec{a}, \vec{Y}^{(t,+),p}|\,, \tag{203}$$

$$\langle \vec{a}, \vec{Y}|D_{1,l} \;=\; (-1)^l \sum_{p=1}^{N} \sum_{t=1}^{f} (a_p + B_t(Y_p))^l \Lambda_p^{(t,-)}(\vec{Y})\langle \vec{a}, \vec{Y}^{(t,-),p}|\,, \tag{204}$$

$$\langle \vec{a}, \vec{Y}|D_{0,l+1} \;=\; (-1)^l \sum_{p=1}^{N} \sum_{\mu \in Y_p} (a_p + c(\mu))^l \langle \vec{a}, \vec{Y}|\,, \tag{205}$$

where $c(\mu) = \beta i - j$ for $\mu = (i, j)$.

With such definitions, we claim that the action of $D_{a,l}$ on the ket and bra basis satisfies SH^c algebra with central charges

$$c_l = \begin{cases} \sum_{q=1}^{N}(b_q - \xi)^l & \text{(for ket)} \\ \sum_{p=1}^{N}(a_q - \xi)^l & \text{(for bra)} \end{cases}\,. \tag{206}$$

We note that the "central charges" depend on the label $\vec{a}, \vec{b}$ in bra and ket state in general except for $c_0 = N$. Of course, when the inner product between them becomes nonvanishing ($\vec{a} = \vec{b}$), they coincide.

Up to overall sign and shift of parameters $a_p \to a_p + \nu$ and $b_p \to b_p + \mu + \nu$, the coefficients which define $D_{\pm 1,l}$ are identical to the variations $\delta_{\pm 1,l}$ in (172,173). This observation suggests that the partition function may be written as an inner product of the basis $\langle \vec{a} + \nu\vec{e}, \vec{Y}|$ and $|\vec{b} + (\nu + \mu)\vec{e}, \vec{W}\rangle$ ($\vec{e} := (1, \cdots, 1)$) with some operator insertions, and the recursion formula should be regarded as the Ward identity for the symmetry algebra SH^c. We will pursue this idea in the following.

Actually there exists a small mismatch in the above observation. The coefficient appearing in (173) is shifted from the coefficient in (201) by ξ. As we see later, this factor will be canceled by slightly modifying the vertex operator inserted between two basis. Namely the vertex operator is not primary due to the $U(1)$ factor.

We need to perform a lengthy computation to confirm that the action of $D_{\pm 1,l}$ indeed gives a representation of SH^c. See appendix H for some detail.

8.4 Comparison with $\mathcal{W}_{1+\infty}$

For general value of β, SH^c is a complicated nonlinear algebra. Simplification occurs when we choose parameter $\beta = 1$. In this case, the nonlinear algebra reduces to a linear algebra $\mathcal{W}_{1+\infty}$. It is an algebra of higher order differential operator $z^n D^m$ ($n \in \mathbf{Z}$, $m = 0, 1, 2, \cdots$, $D = z\partial_z$). Then a quantum generator $\mathcal{W}(z^n D^m)$ is assigned to each differential operator (say $z^n D^m$) and satisfies the algebra with a central extension,

$$[\mathcal{W}(z^n e^{xD}), \mathcal{W}(z^m e^{yD})] = (e^{mx} - e^{ny})\mathcal{W}(z^{n+m} e^{(x+y)D}) - C\frac{e^{mx} - e^{ny}}{e^{x+y} - 1}\delta_{n+m,0} \,. \tag{207}$$

The connection between SH^c and $\mathcal{W}_{1+\infty}$ was already explained in appendix F in [15]. In our previous paper [22], we use the explicit action of $\mathcal{W}_{1+\infty}$ generators on the free fermion Fock space and have shown that Nekrasov partition function satisfies a recursion formula associated with the symmetry.

Here we make a direct comparison of the action of $\mathcal{W}_{1+\infty}$ algebra on the free fermion Fock space in [22] with the corresponding action of SH^c (200–202). For simplicity, we consider $N = 1$ case.

$$\mathcal{W}(zD^l)|a, Y\rangle = (-1)^l \sum_{i=1}^{f} (a + B_i(Y) - 1)^l |a, Y^{(i,-)}\rangle, \tag{208}$$

$$\mathcal{W}(z^{-1}D^l)|a, Y\rangle = (-1)^l \sum_{i=1}^{f+1} (a + A_i(Y))^l |a, Y^{(i,+)}\rangle \,. \tag{209}$$

We need rewrite λ in [22] with $-a$ here. This implies the correspondence in the $\beta \to 1$ limit:

$$D_{-1,l} \leftrightarrow \mathcal{W}(z(D+1)^l) = \mathcal{W}(D^l z), \tag{210}$$

$$D_{1,l} \leftrightarrow \mathcal{W}(z^{-1}D^l). \tag{211}$$

One may proceed to see the correspondence between the generators in $\mathcal{W}_{1+\infty}$ with those in SH^c. The recursion formulae and the Ward identity obtained in [22] can be obtained from the corresponding formulae in this paper by taking the limit $\beta \to 1$.

8.5 Heisenberg and Virasoro algebra in SH^c

In the following, we focus on the important subalgebra in SH^c, namely the Heisenberg (or $U(1)$ current) and Virasoro algebra. They are important since we can make the explicit evaluation of Ward identity because the higher generators in general have nonlinear commutation relation with the vertex operator.

Generators of Heisenberg (J_l) and Virasoro algebra (L_l) are embedded in SH^c as [15],

$$J_l = (-\sqrt{\beta})^{-l} D_{-l,0}, \quad J_{-l} = (-\sqrt{\beta})^{-l} D_{l,0}, \quad J_0 = E_1/\beta, \tag{212}$$
$$L_l = (-\sqrt{\beta})^{-l} D_{-l,1}/l + (1-l)c_0\xi J_l/2\,,$$
$$L_{-l} = (-\sqrt{\beta})^{-l} D_{l,1}/l + (1-l)c_0\xi J_{-l}/2\,,$$
$$L_0 = [L_1, L_{-1}]/2 = D_{0,1} + \frac{1}{2\beta}\left(c_2 + c_1(1-c_0)\xi + \frac{\xi^2}{6}c_0(c_0-1)(c_0-2)\right). \tag{213}$$

The commutation relations among these generators are the standard ones,

$$[J_n, J_m] = \frac{nN}{\beta}\delta_{n+m,0}, \tag{214}$$
$$[L_n, J_m] = -mJ_{n+m}, \tag{215}$$
$$[L_n, L_m] = (n-m)L_{n+m} + \frac{c}{12}(n^3-n)\delta_{n+m,0}\,. \tag{216}$$

The derivation of these simple formulae from SH^c commutator is nontrivial since in the commutation relation of SH^c, we have generators with degree$= \pm 1, 0$ while J_n, L_n have degree n. Proof of the first line is given in [15] . We need derive the commutation relation among them recursively. The confirmation of Virasoro algebra is much more tedious but we give the explicit computation of $[L_2, L_{-2}]$ in appendix I. This particular commutation relation is important since it implies the central charge of Virasoro algebra is related to those in SH^c as,

$$c = \frac{1}{\beta}\left(-c_0^3\xi^2 + c_0 - c_0\xi + c_0\xi^2\right) = 1 + (N-1)(1 - Q^2(N^2+N))\,, \quad Q := \sqrt{\beta} - \sqrt{\beta}^{-1} = -\beta^{-1/2}\xi\,. \tag{217}$$

This is the central charge for a combined system of W_N algebra and a free scalar field. It motivate us to propose a free field representation,

$$J(z) = \sum_n J_n z^{-n-1} = \beta^{-1/2}\sum_{i=1}^N \partial_z\varphi^{(i)}(z)\,, \tag{218}$$
$$T(z) = \sum_{i=1}^N \left(\frac{1}{2}(\partial\varphi^{(i)}(z))^2 - Q\rho_i\partial^2\varphi^{(i)}(z)\right) \tag{219}$$

with

$$\varphi^{(i)}(z) = q^{(i)} + \alpha_0^{(i)}\log z - \sum_{n\neq 0}\frac{\alpha_n^{(i)}}{n}z^{-n}\,, \tag{220}$$
$$[\alpha_n^{(i)}, \alpha_m^{(j)}] = n\delta_{n+m,0}\delta_{ij}\,, \quad [\alpha_m^{(i)}, q^{(j)}] = \delta_{m,0}\delta_{ij}\,. \tag{221}$$

Eqs.(212, 213) imply

$$J_0|\vec{a},\vec{Y}\rangle = \frac{1}{\beta}\left(-\sum_i(a_i-\xi) + \frac{\xi N(N-1)}{2}\right)|\vec{a},\vec{Y}\rangle, \tag{222}$$
$$L_0|\vec{a},\vec{Y}\rangle = \left(|\vec{Y}| + \frac{1}{2\beta}\left(\sum_i(a_i-\xi)^2 + (1-N)\xi\sum_i(a_i-\xi) + \frac{\xi^2}{6}N(N-1)(N-2)\right)\right)|\vec{a},\vec{Y}\rangle\,. \tag{223}$$

We assign the eigenvalue of $\alpha_0^{(i)}$ on the state $|\vec{a},\vec{Y}\rangle$ as

$$\alpha_0^{(i)}|\vec{a},\vec{Y}\rangle = p_i|\vec{a},\vec{Y}\rangle\,, \quad p_i := -\frac{a_i}{\sqrt{\beta}} - Qi\,, \quad i = 1,\cdots,N\,. \tag{224}$$

With such assignments, we can rewrite (222, 223) in the more familiar form,

$$J_0|\vec{a},\vec{Y}\rangle = \frac{1}{\sqrt{\beta}}(\vec{p}\cdot\vec{e})\,|\vec{a},\vec{Y}\rangle, \quad L_0|\vec{a},\vec{Y}\rangle = \left(|\vec{Y}| + \Delta(\vec{p})\right)|\vec{a},\vec{Y}\rangle, \qquad \Delta(\vec{p}) := \frac{\vec{p}\cdot(\vec{p}-2Q\vec{\rho})}{2}\,, \tag{225}$$

where $\rho_i = \frac{N+1}{2} - i$. $\Delta(\vec{p})$ is the conformal dimension of a vertex operator $:e^{\vec{p}\vec{\varphi}}:$.

9 Nekrasov partition function as a correlator and Heisenberg-Virasoro constraints

In the previous sections, we have seen that the recursion formulae for Nekrasov partition function takes a form of the representation of SH^c algebra in terms of the orthonormal basis. We have also seen that SH^c algebra contains Heisenberg and Virasoro algebras as its subalgebras.

As mentioned in section 8.3 , we observe that AGT conjecture can be proved once we prove the relation

$$Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)=\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle, \tag{226}$$

with the orthonormal basis $|\vec{a},\vec{Y}\rangle$ defined in previous sections and a vertex operator V. Existence of such basis was formally proved in [24]. The vertex operator is factorized as $V=\tilde{V}^H V^W$ where V^W is the vertex operator for W_N algebra and $\tilde{V}^H$ describes the contribution of $U(1)$ factor. Furthermore it is known that the correlator of Toda theory is calculable only for the special momenta.

$$\vec{p}=-\kappa\vec{e}_1 \quad \text{or} \quad \vec{p}=-\kappa\vec{e}_N, \quad \vec{e}_1=(1,0,\cdots,0), \quad \vec{e}_N=(0,\cdots,0,1)\,. \tag{227}$$

The new parameter κ is to be determined later. For the convenience of the computation, we take the latter choice. $\tilde{V}^H$ and V^W in the decomposition should be written as,

$$\tilde{V}^H_\kappa=e^{-\frac{\kappa}{N}\vec{e}\cdot\vec{\varphi}}, \quad V^W_\kappa=e^{-\kappa(\vec{e}_N-\frac{\vec{e}}{N})\vec{\varphi}}\,, \tag{228}$$

for $\vec{p}$ taking the second value in (227). This form of W_N vertex operator is also important in the context of AGT conjecture. V^W_κ is a vertex operator corresponding to the so-called simple puncture. As we see, we need modify $\tilde{V}^H$ to meet the behavior of $U(1)$ factor in AGT conjecture.

The relation (226) can be established once one proves that the partition function Z satisfies the recursion relation which defines the right hand side [22] . Namely,

$$\begin{aligned}0= \quad & (\langle\vec{a}+\nu\vec{e},\vec{Y}|D_{n,m})V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\\ &-\langle\vec{a}+\nu\vec{e},\vec{Y}|\,[D_{n,m},V(1)]\,|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle-\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)(D_{n,m}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle)\,. \end{aligned}\tag{229}$$

The right hand side gives the Ward identity for the conformal block. One may translate such relation into a recursion relation which Z should satisfy if we use the relation (226). It may sound strange to use the relation to be proved. Here we use it as the assumption in the inductive method. It is obvious that the relation (226) holds for the trivial case $\vec{Y}=\vec{W}=\vec{\emptyset}$ with a proper definition of the inner product. General relation (226) will be obtained through the Ward identities by induction.

As we have seen, the recursion relation for Z exists for $n=\pm1$ and arbitrary $m\geq 0$. Other relations should be derived from them. On the right hand side of (229), we have already defined the action of $D_{n,m}$ on the basis. A problem is that the commutation relation with the vertex operator cannot be written in the closed form except for Heisenberg and Virasoro generators. Thus we focus on these cases in the following though it is not sufficient to complete the inductive proof.

9.1 Modified vertex operator for $U(1)$ factor

While the definition of the vertex operator for W_N algebra is well-known, those for U(1) factor V^H is somewhat tricky [1, 24, 74].[7] We give a brief account on the construction.

[7] We thank V. Pasquier to point out this important fact.

The free boson field which describes the $U(1)$ part is given by the operators J_n defined in the previous section. With

$$\alpha_n = \sqrt{\beta/N} J_n, \tag{230}$$

we define a free boson field as,

$$\phi(z) = q + \alpha_0 \log z - \sum_{n\neq 0} \frac{\alpha_n}{n} z^{-n} = \frac{\vec{e}\cdot\vec{\varphi}}{\sqrt{N}}\,. \tag{231}$$

We modify the vertex operator $\tilde{V}^H$ for the $U(1)$ factor as,

$$V^H_\kappa(z) = e^{\frac{1}{\sqrt{N}}(NQ-\kappa)\phi_-} e^{\frac{-1}{\sqrt{N}}\kappa\phi_+}\,, \tag{232}$$

$$\phi_+ = \alpha_0 \log z - \sum_{n=1}^{\infty} \frac{\alpha_n}{n} z^{-n}\,, \quad \phi_- = q + \sum_{n=1}^{\infty} \frac{\alpha_{-n}}{n} z^n\,. \tag{233}$$

Such definition of modified vertex operator is needed to reproduce the contribution of $U(1)$ factor in the correlator [8],

$$\langle V^H_{\kappa_1}(z_1)\cdots V^H_{\kappa_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{\frac{-\kappa_i(NQ-\kappa_j)}{N}}\,. \tag{234}$$

Due to the modification, the commutation relation with $U(1)$ current (Heisenberg generator) becomes asymmetric,

$$[\alpha_m, V^H_\kappa(z)] = \frac{1}{\sqrt{N}}(NQ-\kappa) z^m V^H_\kappa(z), \quad [\alpha_{-n}, V^H_\kappa(z)] = \frac{-1}{\sqrt{N}}\kappa z^{-n} V^H_\kappa(z)\,, \tag{235}$$

for $m \geq 0$, $n > 0$.

Unlike the standard definition of the vertex operator $V =: e^{\kappa\phi} :$, the conformal property of the modified vertex becomes rather complicated. It is, however, helpful to understand the recursion relations (1) which has some anomaly as well. We define the Virasoro generator for the $U(1)$ factor as,

$$L^H_n = \frac{1}{2}\sum_m : \alpha_{n-m}\alpha_m :\,, \tag{236}$$

which has $c = 1$. The commutator of the total Virasoro genrators $L_n = L^H_n + L^W_n$ with the vertex $V_\kappa(z) = V^H_\kappa(z) V^W_\kappa(z)$ becomes,

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{(NQ-\kappa)^2}{2N}(n+1)z^n V_\kappa(z) + \sqrt{N}Q\sum_{m=0}^{n} z^{n-m} V_\kappa(z)\alpha_m + (n+1)z^n \Delta_W V_\kappa(z),\ \ n\geq 0 \tag{237}$$

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{\kappa^2}{2N}(n+1)z^n V_\kappa(z) - \sqrt{N}Q\sum_{m=1}^{|n|} z^{n+m} \alpha_{-m} V_\kappa(z) + (n+1)z^n \Delta_W V_\kappa(z),\ \ n<0\,, \tag{238}$$

where $\Delta_W = \frac{\kappa(\kappa-Q(N-1))}{2} - \frac{\kappa^2}{2N}$ is the conformal dimension of W_N vertex operator V^W_κ with Toda momenta $\vec{p} = -\kappa(\vec{e}_N - \frac{\vec{e}}{N})$ as in (228). The anomaly due to the modification of $U(1)$ vertex manifests itself through the third term on the right hand side. We write the commutator for the special cases $n = \pm 1, 0$ for the convenience of later calculation.

$$[L_1, V_\kappa(z)] = z^2\partial_z V_\kappa(z) + \frac{(NQ-\kappa)^2}{N} z V_\kappa(z) + \sqrt{N}Qz V_\kappa(z)\alpha_0 + \sqrt{N}Q V_\kappa(z)\alpha_1 + 2z\Delta_W V_\kappa(z)\,, \tag{239}$$

$$[L_0, V_\kappa(z)] = z\partial_z V_\kappa(z) + \frac{(NQ-\kappa)^2}{2N} V_\kappa(z) + \sqrt{N}Q V_\kappa(z)\alpha_0 + \Delta_W V_\kappa(z)\,, \tag{240}$$

$$[L_{-1}, V_\kappa(z)] = \partial_z V_\kappa(z)\,. \tag{241}$$

In the following, we examine the relation (229) for Heisenberg ($U(1)$) and Virasoro generators for $D_{n,m}$.

[8] Compared with the reference [24], we included the zero mode to modify the commutator with the Virasoro generator.

9.2 Ward identities for $U(1)$ currents

We start from examining the case $n=0$ which can be interpreted as the Ward identity for $J_{\pm 1}$,

$$
\begin{aligned}
&(\langle \vec{a}+\nu\vec{e},\vec{Y}|J_{\pm1})V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle - \langle \vec{a}+\nu\vec{e},\vec{Y}|V(1)(J_{\pm1}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle) \\
&= \langle \vec{a}+\nu\vec{e},\vec{Y}|\,[J_{\pm1},V(1)]\,|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\,.
\end{aligned}
\tag{242}
$$

By the definition of the representation of SH^c algebra (212, **??**, 200) and the vertex operator (235), the action of J_1 on the bra and ket basis and the commutator with the vertex operator are given as,

$$
\langle \vec{a}+\nu\vec{e},\vec{Y}|J_1 = (-\sqrt{\beta})^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p+1}\langle \vec{a}+\nu\vec{e},\vec{Y}^{(k,+),p}|\Lambda_p^{(k,+)}(\vec{Y}),
\tag{243}
$$

$$
J_1|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle = (-\sqrt{\beta})^{-1}\sum_{q=1}^{N}\sum_{\ell=1}^{\tilde{f}_p}\Lambda_q^{(\ell,-)}(\vec{W})|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}^{(\ell,-),q}\rangle\,,
\tag{244}
$$

$$
[J_1,V_\kappa(1)] = \frac{1}{\sqrt{\beta}}(NQ-\kappa)V_\kappa(1)\,.
\tag{245}
$$

Plugging them into (242) gives,

$$
\begin{aligned}
&(-\sqrt{\beta})^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p+1}\Lambda_p^{(k,+)}(\vec{Y})\langle \vec{a}+\nu\vec{e},\vec{Y}^{(k,+),p}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle \\
&-(-\sqrt{\beta})^{-1}\sum_{q=1}^{N}\sum_{\ell=1}^{\tilde{f}_p}\Lambda_q^{(\ell,-)}(\vec{W})\langle \vec{a}+\nu\vec{e},\vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}^{(\ell,-),q}\rangle \\
&\qquad = \frac{1}{\sqrt{\beta}}(NQ-\kappa)\langle \vec{a}+\nu\vec{e},\vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle)\,.
\end{aligned}
\tag{246}
$$

Using the assumption (226), the left hand side of (246) becomes

$$
\sqrt{\beta}^{-1}\delta_{-1,0}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,.
\tag{247}
$$

On the other hand, taking account of U(1) charge conservation condition, which is derived from the action of J_0,

$$
\kappa = -\beta^{-1/2}\sum_{p=1}^{N}(a_p-b_p-\mu)\,,
\tag{248}
$$

the right hand side of (246) becomes

$$
\frac{1}{\sqrt{\beta}}(NQ-\kappa)Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \beta^{-1}\sum_{p=1}^{N}(a_p-b_p-\mu-\xi)Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sqrt{\beta}^{-1}U_{-1,0}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,.
\tag{249}
$$

Thus the Ward identity for J_1 is proved since it is identified with the recursion formula $\delta_{-1,0}Z_{\vec{Y},\vec{W}} - U_{-1,0}Z_{\vec{Y},\vec{W}} = 0$.

Derivation of the identity for J_{-1} can be performed similarly. The actions of J_{-1} are given by

$$
\langle \vec{a}+\nu\vec{e},\vec{Y}|J_{-1} = (-\sqrt{\beta})^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p}\langle \vec{a}+\nu\vec{e},\vec{Y}^{(k,-),p}|\Lambda_p^{(k,-)}(\vec{Y}),
\tag{250}
$$

$$
J_{-1}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle = (-\sqrt{\beta})^{-1}\sum_{q=1}^{N}\sum_{\ell=1}^{\tilde{f}_p+1}\Lambda_q^{(\ell,+)}(\vec{W})|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}^{(\ell+-),q}\rangle\,,
\tag{251}
$$

$$
[J_{-1},V_\kappa(1)] = -\frac{1}{\sqrt{\beta}}\kappa V_\kappa(1)\,.
\tag{252}
$$

By the assumption (226), we have

$$\begin{aligned}\langle \vec{a}+\nu\vec{e},\vec{Y}|J_{-1}V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle &- \langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(1)J_{-1}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle \\ &= -\sqrt{\beta}^{-1}\delta_{1,0}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,, \end{aligned} \tag{253}$$

$$\langle \vec{a}+\nu\vec{e},\vec{Y}|[J_{-1},V_\kappa(1)]|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle = -\beta^{-1/2}\kappa Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,. \tag{254}$$

In the last equality in (254), we use $U(1)$ charge conservation (248). It shows the equivalence between the recursion formula $\delta_{1,0}Z_{\vec{Y},\vec{W}} - U_{1,0}Z_{\vec{Y},\vec{W}} = 0$ and the Ward identity for J_{-1}. We note that the modification of the vertex operator is necessary to produce the Ward identities for $U(1)$ currents.

9.3 Ward identities for Virasoro generators

We proceed to examine the equivalence of the Ward identity for Virasoro generators and the recursion formula. The actions of L_1 on the basis and the vertex operator are evaluated by (213, 200–205, 237),

$$\langle \vec{a}+\nu\vec{e},\vec{Y}|L_1 = \sqrt{\beta}^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p}\langle \vec{a}+\nu\vec{e},\vec{Y}^{(k,+),p}|(a_p+\nu+A_k(Y_p))\Lambda^{(k,+),p}(\vec{Y}),$$

$$L_1|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle = \sqrt{\beta}^{-1}\sum_{q=1}^{N}\sum_{\ell=1}^{f_p}\Lambda^{(\ell,-),q}(\vec{W})(b_q+\nu+\mu+B_\ell(W_q)+\xi)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}^{(\ell,-),q}\rangle\,,$$

$$[L_1,V_\kappa(1)] = \partial V_\kappa(1) + \frac{(NQ-\kappa)^2}{N}V_\kappa(1) + \sqrt{N}QV_\kappa(1)\alpha_0 + \sqrt{N}QV_\kappa(1)\alpha_1 + 2\Delta_W V_\kappa(1)\,.$$

As we see from the derivative term in the commutator, in order to evaluate the Virasoro Ward identities, we need to evaluate $\langle \vec{a}+\nu\vec{e},\vec{Y}|\partial V(1)|\vec{b}+(\nu+\mu+\xi)\vec{e},\vec{W}\rangle$. Since the modified vertex operator is not a primary operator, the correlator does not have the standard dependence on the position of the vertex operator. We can, however, derive it through the Ward identity of L_0.

According to the actions of L_0 on the basis (223), we have

$$\begin{aligned}&\frac{\langle \vec{a}+\nu\vec{e},\vec{Y}|L_0V_\kappa(z)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle - \langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(z)L_0|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(z)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle} \\ &= \Delta\left(-\frac{\vec{a}+\nu\vec{e}}{\sqrt{\beta}} - Q\vec{\rho} + Q\frac{N+1}{2}\vec{e}\right) + |\vec{Y}| - \Delta\left(-\frac{\vec{b}+(\nu+\mu)\vec{e}}{\sqrt{\beta}} - Q\vec{\rho} + Q\frac{N+1}{2}\vec{e}\right) - |\vec{W}|\,. \end{aligned}\tag{255}$$

On the other hand, from the commutator between L_0 and vertex operator (240), we obtain

$$\begin{aligned}&\left.\frac{\langle \vec{a}+\nu\vec{e},\vec{Y}|[L_0,V_\kappa(z)]|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(z)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}\right|_{z=1} = \left.\frac{\langle \vec{a}+\nu\vec{e},\vec{Y}|z\partial_z V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}\right|_{z=1} \\ &- \left.\frac{\langle \vec{a}+\nu\vec{e},\vec{Y}|\sqrt{N}QV_\kappa(z)\alpha_0|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(z)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}\right|_{z=1} - \frac{(NQ-\kappa)^2}{2N} - \Delta_W\,. \end{aligned}\tag{256}$$

Since (255) is identical with (256) by the Ward identity for L_0, the derivative term can be evaluated as follows,

$$\begin{aligned}&\frac{\langle \vec{a}+\nu\vec{e},\vec{Y}|\partial_z V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle \vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle} \\ =&\Delta\left(-\frac{\vec{a}+\nu\vec{e}}{\sqrt{\beta}} - Q\vec{\rho} + Q\frac{N+1}{2}\vec{e}\right) + |\vec{Y}| - \Delta\left(-\frac{\vec{b}+(\nu+\mu)\vec{e}}{\sqrt{\beta}} - Q\vec{\rho} + Q\frac{N+1}{2}\vec{e}\right) - |\vec{W}| \\ &-\frac{\xi}{\beta}\Bigg(-\sum_{p=1}^{N}(b_p+\nu+\mu) + N(N-1)\xi/2\Bigg) - \frac{(NQ-\kappa)^2}{2N} - \frac{\kappa(\kappa-Q(N-1))}{2} + \frac{\kappa^2}{2N}\,. \end{aligned}$$

Now we are ready to check the recursion relation for Virasoro generators. Applying (226), we obtain

$$\begin{aligned}&\langle\vec{a}+\nu\vec{e},\vec{Y}|L_1V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle-\langle\vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(1)L_1|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\\&=\sqrt{\beta}^{-1}\delta_{-1,1}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)-Q\sum_{q=1}^{N}\sum_{\ell=1}^{f_p}\Lambda^{(\ell,-),q}(\vec{W})Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(\ell,-),q};\mu)\,.\end{aligned}\tag{257}$$

Unlike in the J_1 case, an additional term appears because the action of SH^c algebra on the ket space is slightly different from the action of $\delta_{-1,1}$ on $Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)$ as we have explained previously. The commutator part becomes

$$\begin{aligned}&\langle\vec{a}+\nu\vec{e},\vec{Y}|[L_1,V_\kappa(1)]|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\\&=\Bigg\{\Delta\left(-\frac{\vec{a}+\nu\vec{e}}{\sqrt{\beta}}-Q\vec{\rho}+Q\frac{N+1}{2}\vec{e}\right)+|\vec{Y}|-\Delta\left(-\frac{\vec{b}+(\nu+\mu)\vec{e}}{\sqrt{\beta}}-Q\vec{\rho}+Q\frac{N+1}{2}\vec{e}\right)-|\vec{W}|\\&+\frac{(NQ-\kappa)^2}{2N}+\frac{\kappa(\kappa-Q(N-1))}{2}-\frac{\kappa^2}{2N}\Bigg\}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)-Q\sum_{q=1}^{N}\sum_{\ell=1}^{f_p}\Lambda^{(\ell,-),q}(\vec{W})Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(\ell,-),q};\mu)\\&=\sqrt{\beta}^{-1}U_{-1,1}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)-Q\sum_{q=1}^{N}\sum_{\ell=1}^{f_p}\Lambda^{(\ell,-),q}(\vec{W})Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(\ell,-),q};\mu)\,.\end{aligned}\tag{258}$$

In the last equality we use (248). This also have an anomalous term since the modified vertex is not primary operator and its commutator with L_1 has the $V_\kappa J_1$ term. However, the anomalies in (257) and (258) are identical and the Ward identity for L_1 is reduced to the recursion relation $\delta_{-1,1}Z_{\vec{Y},\vec{W}}-U_{-1,1}Z_{\vec{Y},\vec{W}}=0$ which is already proved. We note that the identity holds only when we have the special value for the vertex momentum (227).

In the same way, for L_{-1}, we have

$$\begin{aligned}&\langle\vec{a}+\nu\vec{e},\vec{Y}|L_{-1}V_\kappa(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle-\langle\vec{a}+\nu\vec{e},\vec{Y}|V_\kappa(1)L_{-1}|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\\&=\sqrt{\beta}^{-1}\delta_{1,1}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,,\end{aligned}\tag{259}$$

$$\begin{aligned}&\langle\vec{a}+\nu\vec{e},\vec{Y}|[L_{-1},V_\kappa(1)]|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle\\&=\Bigg\{\Delta\left(-\frac{\vec{a}+\nu\vec{e}}{\sqrt{\beta}}-Q\vec{\rho}+\frac{N+1}{2}\vec{e}\right)+|\vec{Y}|-\Delta\left(-\frac{\vec{b}+(\nu+\mu)\vec{e}}{\sqrt{\beta}}-Q\vec{\rho}+\frac{N+1}{2}\vec{e}\right)-|\vec{W}|\\&-\frac{\xi}{\beta}\Bigg(-\sum_{p=1}^{N}(b_p+\nu+\mu)+N(N-1)\xi/2\Bigg)-\frac{(NQ-\kappa)^2}{2N}-\frac{\kappa(\kappa-Q(N-1))}{2}+\frac{\kappa^2}{2N}\Bigg\}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\\&=\sqrt{\beta}^{-1}U_{1,1}Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu)\,.\end{aligned}\tag{260}$$

Again, we use (248) to derive the last equality in (260). Thus, the recursion formula $\delta_{1,1}Z_{\vec{Y},\vec{W}}-U_{1,1}Z_{\vec{Y},\vec{W}}=0$ can be identified with the Ward identity. These two consistency conditions are highly nontrivial and strongly suggest that the identify (1) are a part of the Ward identities for the extended conformal symmetry.

10 Conclusion

we calculated the conformal block in the form of Dotsenko-Fateev integral and reduce it in the form of Selberg integral of N Jack polynomials. We found a formula for such Selberg average and show that it reproduces the SU(N) version of AGT conjecture with $\beta=1$. The only pity is that, our formulae for Selberg average are not based on explicit evaluation but determined by consistency. Thus as long as we want to pursue a direct and complete proof of this crucial string duality, the full evaluation of Selberg-Jack integral and W-algebra will be necessary.

Further we chose another method and showed that Nekrasov instanton partition function for SU(N) gauge theories satisfies recursion relations in the form of U(1)+Virasoro constraints,with β to be chosen arbitrarily. This case is much more complicated than before and we introduce many new methods to solve the issues caused by the arbitrary β. For example, we have to choose a modified vertex operator to satisfy the commuting feature; and to define the basis, we need the help of $\mathbf{SH}^c$ algebra, etc. These make the calculation rather tedious, and also call for more precise physical interpretations. The constraints give a strong support for AGT conjecture for general quiver gauge theories. One remarkable feature is that the proof is not restricted by the number of boxes of Young diagrams but holds in all orders analytically. However, due to its recursive nature, this method provides an indirect testification, still not enough for a complete proof.

We would like to mention some recent papers which are relevant to this work. In [75], large N limit (N is the size of Young tableaux) is taken to relate AGT conjecture to matrix model. There should be similar limit in our recursion formula where the computation becomes much simpler and the relation with Nekrasov-Shatashvili limit [76] will be clearer. In [77] , the correlator of primary fields is defined in terms of null state condition of W_N algebra which in tern related to Calogero-Sutherland system. Since the symmetry of Jack polynomial is identified with SH^c, there should be interesting connection with the current work. In [78], an M-theoretic approach to AGT relation was explored.

We note that the two parameter extension of $\mathcal{W}_{1+\infty}$ [27], there are some important progress in terms of AGT relation [30]. It is, however, nontrivial to derive AGT from the results from DAHA since the degeneration limit is singular. We hope to come back to this issue in our future work.

Moreover, SH^c seems to have interesting applications to quantum Hall effects or higher spin theories [79]. This may be also interesting directions.

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A Jack polynomials

Jack polynomials $J_Y^{(\beta)}[z_1,\cdots,z_M]$ are a kind of symmetric polynomials of variables $z_1,\cdots,z_M$ labeled by a Young diagram Y. Detailed properties of Jack polynomial is given in [65]. they are characterized by the fact that they are the eigenfunctions of Calogero-Sutherland Hamiltonian written in the form,

$$\mathcal{H}=\sum_{i=1}^{M} D_i^2+\beta\sum_{i<h}\frac{z_i+z_j}{z_i-z_j}(D_i-D_j),\quad D_i:=z_i\frac{\partial}{\partial z_i}\,. \tag{261}$$

Sometimes they are written as functions of power sum $p_k(z)=\sum_i z_i^k$. In the text, we write the Jack polynomial in terms of them, $J_Y^{(\beta)}(p_1,p_2,\cdots)\equiv J_Y^{(\beta)}(p_k):=J_Y^{(\beta)}[z_1,\cdots,z_M]$. The explicit form of low level ones are listed below;

$$\begin{aligned}
&J_{[1]}^{(\beta)}(p_k)=p_1\ ,\\
&J_{[2]}^{(\beta)}(p_k)=\frac{p_2+\beta p_1^2}{\beta+1},\quad J_{[11]}^{(\beta)}(p_k)=\frac{1}{2}\left(p_1^2-p_2\right)\ ,\\
&J_{[3]}^{(\beta)}(p_k)=\frac{2p_3+3\beta p_1p_2+\beta^2p_1^3}{(\beta+1)(\beta+2)},\ J_{[21]}^{(\beta)}(p_k)=\frac{(1-\beta)p_1p_2-p_3+\beta p_1^3}{(\beta+1)(\beta+2)},\ J_{[111]}^{(\beta)}(p_k)=\frac{1}{6}p_1^3-\frac{1}{2}p_1p_2+\frac{1}{3}p_3\ .
\end{aligned} \tag{262}$$

Jack polynomials are orthogonal with each other $(J_{Y_1},J_{Y_2})\propto\delta_{Y_1Y_2}$. There are two inner products defined for the symmetric polynomial which has such property. One is defined in terms of products of power sum,

$$\langle p_1^{k_1}\cdots p_n^{\ell_n},p_1^{\ell_1}\cdots p_n^{\ell_n}\rangle_\beta=\delta_{\vec{k},\vec{\ell}}\beta^{-\sum_i k_i}\prod_{i=1}^{n} i^{k_i}k_i!\,. \tag{263}$$

We write the norm for this inner product as $\langle J_Y,J_Y\rangle_\beta=||J_Y||^2$. The explicit form of the norm is given in the literature[65]

$$||J_A^{(\beta)}||^2=\frac{Q_Y}{P_Y}\ , \tag{264}$$

with P_Y and Q_Y given by

$$P_Y=\prod_{(i,j)\in Y}\Big(\beta(Y'_j-i)+(Y_i-j)+\beta\Big)=G_{Y,Y}(0)\ , \tag{265}$$

$$Q_Y=\prod_{(i,j)\in Y}\Big(\beta(Y'_j-i)+(Y_i-j)+1\Big)=G_{Y,Y}(1-\beta)\ . \tag{266}$$

In this paper, we denote the normalized Jack polynomials as,

$$j_Y^{(\beta)}(p):=\frac{J_Y^{(\beta)}(p)}{||J_Y^{(\beta)}||}=\sqrt{\frac{G_{Y,Y}(0)}{G_{Y,Y}(1-\beta)}}J_Y^{(\beta)}(p)\ . \tag{267}$$

Especially, at $\beta=1$, Jack polynomials reduced to Schur polynomials χ_Y :

$$j_Y^{(\beta)}|_{\beta=1}=J_Y^{(\beta)}|_{\beta=1}=\chi_Y\ . \tag{268}$$

The relation between Jack polynomial and Toda theory is that Jack polynomial is characterized as the null states of W-algebra, as discussed, for example, in [67]. In particular, the Calogero-Sutherland Hamiltonian (261) is written in in terms of Virasoro and W-generators (see, for example, eq.(52) of [67]).

The relevance of Jack polynomial in Selberg integral is through the Cauchy-Riemann relations,

$$\prod_{i,j}(1-x_iy_j)^{-\beta}=\sum_Y J_Y^{(\beta)}[x]J_Y^{(\beta)}[y]||J_Y||^{-2},\quad \prod_{i,j}(1+x_iy_j)=\sum_Y J_{Y'}^{(1/\beta)}[x]J_Y^{(\beta)}[y]\,. \tag{269}$$

The first property was essentially used in the text.

B Formula for general β

Here we write a formula of A_n Selberg average for product of $n+1$ Jack polynomials which generalizes (147). While some modifications on the terms proportional to $1-\beta$ are required to meet the constraints (149), it survives other constraints which are quite nontrivial. We write this formula since it may give a useful hints in the future development, though some modifications are necessary.

The formula for $n+1$ Jack polynomials should be close to the following,

$$
\begin{aligned}
&\left\langle J^{(\beta)}_{Y_1}(-p^{(1)}_k-\frac{v_1+\cdots+v_n}{\beta})\dots J^{(\beta)}_{Y_r}(p^{(r-1)}_k-p^{(r)}_k-\frac{v_r+\cdots+v_n}{\beta})\dots J^{(\beta)}_{Y_{n+1}}(p^{(n)}_k)\right\rangle^{SU(n+1)}\\
&=\prod_{s=1}^{n}\left\{(-1)^{|Y_s|}\frac{[v_s+N_s\beta-N_{s-1}\beta]_{Y'_s}}{[N_s\beta+N_{s-1}\beta]_{Y'_s}}\prod_{1\le i<j\le N_{s-1}+N_s}\frac{((j-i+1)\beta)_{Y'_{si}-Y'_{sj}}}{((j-i)\beta)_{Y'_{si}-Y'_{sj}}}\right\}\times\prod_{1\le i<j\le N_n}\frac{((j-i+1)\beta)_{Y_{(n+1)i}-Y_{(n+1)j}}}{((j-i)\beta)_{Y_{(n+1)i}-Y_{(n+1)j}}}\\
&\times\prod_{1\le t<s\le n}\left\{\frac{[v_t+u_t+\cdots+u_{s-1}+N_t\beta-N_{t-1}\beta+(s-t+1)(1-\beta)]_{Y'_t}}{[v_t-v_s+u_t+\cdots+u_{s-1}+N_t\beta-N_{t-1}\beta-N_s\beta+(s-t+1)(1-\beta)]_{Y'_t}}\right.\times\\
&\times\frac{[-v_s+u_t+\cdots+u_{s-1}-N_s\beta+N_{s-1}\beta+(s-t)(1-\beta)]_{Y_s}}{[v_t-v_s+u_t+\cdots+u_{s-1}-N_{t-1}\beta-N_s\beta+N_{s-1}\beta+(s-t+1)(1-\beta)]_{Y_s}}\times\\
&\times\left.\prod_{i=1}^{N_t}\prod_{j=1}^{N_{s-1}}\frac{\left(v_t-v_s+u_t+\cdots+u_{s-1}+N_t\beta-N_{t-1}\beta-N_s\beta+N_{s-1}\beta+(s-t)(1-\beta)+1-(i+j)\beta\right)_\beta}{\left(v_t-v_s+u_t+\cdots+u_{s-1}+N_t\beta-N_{t-1}\beta-N_s\beta+N_{s-1}\beta+(s-t)(1-\beta)+1+Y'_{ti}+Y_{sj}-(i+j)\beta\right)_\beta}\right\}\\
&\times\prod_{1\le s\le n}\left\{\frac{[u_s+\cdots+u_n+N_n\beta+(n-s+1)(1-\beta)]_{Y_{n+1}}}{[v_s+u_s+\cdots+u_n+N_n\beta-N_{s-1}\beta+(n-s+2)(1-\beta)]_{Y_{n+1}}}\right.\times\\
&\times\left.\prod_{i=1}^{N_s}\prod_{j=1}^{N_n}\frac{\left(v_s+u_s+\cdots+u_n+N_n\beta+N_s\beta-N_{s-1}\beta+(n-s+1)(1-\beta)+1-(i+j)\beta\right)_\beta}{\left(v_s+u_s+\cdots+u_n+N_n\beta+N_s\beta-N_{s-1}\beta+(n-s+1)(1-\beta)+1+Y'_{si}+Y_{(n+1)j}-(i+j)\beta\right)_\beta}\right\}\ .
\end{aligned}
\tag{270}
$$

It satisfies consistency conditions with the known results:

(a) For $Y_1=\cdots=Y_n=\emptyset$, and $Y_{n+1}=B$, with the help of (322) the above reduce to the A_n one Jack integral (146). The proof of this statement is obvious.

(b) For $n=1$, $Y_1=A$ and $Y_2=B$, the above reduce to

$$
\begin{aligned}
\left\langle J^{(\beta)}_A(-p^{(1)}_k-\frac{v_1}{\beta})J^{(\beta)}_B(p^{(1)}_k)\right\rangle^{SU(2)}_{u,v,\beta}&=(-1)^{|A|}\times\frac{[v+N\beta]_{A'}[u+N\beta+1-\beta]_B}{[N\beta]_{A'}[u+v+N\beta+2-2\beta]_B}\times\\
&\times\prod_{1\le i<j\le N}\frac{(A'_i-A'_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\prod_{1\le i<j\le N}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\\
&\times\prod_{i,j=1}^{N}\frac{\left(u+v+2N\beta+1-\beta+1-(i+j)\beta\right)_\beta}{\left(u+v+2N\beta+A'_i+B_j+1-\beta+1-(i+j)\beta\right)_\beta}\ ,
\end{aligned}
\tag{271}
$$

which is consistent with the A_1 two Jack integral (142) by considering

$$
j^{(\beta)}_A(-p/\beta)=(-1)^{|A|}j^{(1/\beta)}_{A'}(p)\ . \tag{272}
$$

(c) For $n=2$, $Y_1=R$, $Y_2=\emptyset$, and $Y_3=B$, with the help of (322) the above reduce to

$$\begin{aligned}
&\Big\langle J_R^{(\beta)}(-p_k^{(1)}-\frac{v_{1+}+v_{(2)+}}{\beta})J_B^{(\beta)}(p_k^{(2)})\Big\rangle_+^{SU(3)} \\
&=(-1)^{|R|}\times\prod_{1\le i<j\le N_1}\frac{((j-i+1)\beta)_{R'_i-R'_j}}{((j-i)\beta)_{R'_i-R'_j}}\times\prod_{1\le i<j\le N_2}\frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} \\
&\times\frac{1}{[v_1-v_2+u_1+2N_1\beta-N_2\beta+2(1-\beta)]_{R'}} \\
&\quad\times\frac{[v_1+u_1+N_1\beta+2-2\beta]_{R'}}{1}\quad\times 1 \\
&\times\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{\big(v_1+u_1+u_2+N_2\beta+N_1\beta+2(1-\beta)+1-(i+j)\beta\big)_\beta}{\big(v_1+u_1+u_2+N_2\beta+N_1\beta+2(1-\beta)+1+R'_i+B_j-(i+j)\beta\big)_\beta} \\
&\quad\times\frac{[u_1+u_2+N_2\beta+2(1-\beta)]_B}{[v_1+u_1+u_2+N_2\beta+3(1-\beta)]_B}\quad\times\frac{[v_1+N_1\beta]_{R'}}{[N_1\beta]_{R'}} \\
&\times\frac{1}{[v_2+u_2+2N_2\beta-N_1\beta+2(1-\beta)]_B} \\
&\quad\times\frac{[u_2+N_2\beta+(1-\beta)]_B}{1}\quad\times 1\,.
\end{aligned}\tag{273}$$

Notice the shift in j_R's argument, and the restrictions $v_2=0,\quad v_1=v,\quad v_1+v_2=\beta-1$ (this last restrictions is only claimed by Warnaar's A_2 two Jack integral), the above is consistent with the A_2 two Jack integral given by Warnaar[20] as below

$$\begin{aligned}
&\Big\langle J_R^{(\beta)}(p_k^{(1)})J_B^{(\beta)}(p_k^{(2)})\Big\rangle_{u,v,\beta}^{SU(3)} \\
&=\prod_{1\le i<j\le N_1}\frac{((j-i+1)\beta)_{R_i-R_j}}{((j-i)\beta)_{R_i-R_j}}\prod_{1\le i<j\le N_2}\frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} \\
&\times\ \frac{[u_1+N_1\beta+1-\beta]_R}{[v_1+u_1+2N_1\beta-N_2\beta+2-2\beta]_R}\times\frac{[u_2+N_2\beta+1-\beta]_B}{[v_2+u_2+2N_2\beta-N_1\beta+2-2\beta]_B}\times \\
&\times\prod_{i=1}^{N_1}\prod_{j=1}^{N_2}\frac{\big(u_1+u_2+N_1\beta+N_2\beta+1-\beta+1-(i+j)\beta\big)_\beta}{\big(u_1+u_2+N_1\beta+N_2\beta+R_i+B_j+1-\beta+1-(i+j)\beta\big)_\beta}\ .
\end{aligned}$$

(d) For $N_n = 0$ (so that $u_n = v_n = 0$, and $Y_{n+1} = \emptyset$), the above reduce to

$$\left\langle J^{(\beta)}_{Y_1}(-p^{(1)}_k - \frac{v_1 + \cdots + v_n}{\beta}) \ldots J^{(\beta)}_{Y_r}(p^{(r-1)}_k - p^{(r)}_k - \frac{v_r + \cdots + v_n}{\beta}) \ldots J^{(\beta)}_{Y_{n+1}}(p^{(n)}_k) \right\rangle^{SU(n+1)}$$
$$= \prod_{s=1}^{n-1} \left\{ (-1)^{|Y_s|} \frac{[v_s + N_s\beta - N_{s-1}\beta]_{Y'_s}}{[N_s\beta + N_{s-1}\beta]_{Y'_s}} \prod_{1\le i<j\le N_{s-1}+N_s} \frac{((j-i+1)\beta)_{Y'_{si}-Y'_{sj}}}{((j-i)\beta)_{Y'_{si}-Y'_{sj}}} \right\} \times \prod_{1\le i<j\le N_{n-1}} \frac{((j-i+1)\beta)_{Y_{(n)i}-Y_{(n)j}}}{((j-i)\beta)_{Y_{(n)i}-Y_{(n)j}}}$$
$$\times \prod_{1\le t<s\le n-1} \left\{ \frac{[v_t + u_t + \cdots + u_{s-1} + N_t\beta - N_{t-1}\beta + (s-t+1)(1-\beta)]_{Y'_t}}{[v_t - v_s + u_t + \cdots + u_{s-1} + N_t\beta - N_{t-1}\beta - N_s\beta + (s-t+1)(1-\beta)]_{Y'_t}} \times \right.$$
$$\times \frac{[-v_s + u_t + \cdots + u_{s-1} - N_s\beta + N_{s-1}\beta + (s-t)(1-\beta)]_{Y_s}}{[v_t - v_s + u_t + \cdots + u_{s-1} - N_{t-1}\beta - N_s\beta + N_{s-1}\beta + (s-t+1)(1-\beta)]_{Y_s}} \times$$
$$\left. \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{s-1}} \frac{\left(v_t - v_s + u_t + \cdots + u_{s-1} + N_t\beta - N_{t-1}\beta - N_s\beta + N_{s-1}\beta + (s-t)(1-\beta) + 1 - (i+j)\beta\right)_\beta}{\left(v_t - v_s + u_t + \cdots + u_{s-1} + N_t\beta - N_{t-1}\beta - N_s\beta + N_{s-1}\beta + (s-t)(1-\beta) + 1 + Y'_{ti} + Y_{sj} - (i+j)\beta\right)_\beta} \right\}$$
$$\times \prod_{1\le t\le n-1} \left\{ 1 \times \frac{[u_t + \cdots + u_{n-1} + N_{n-1}\beta + (n-s)(1-\beta)]_{Y_n}}{[v_t + u_t + \cdots + u_{n-1} + N_{n-1}\beta - N_{t-1}\beta + (n-t+1)(1-\beta)]_{Y_n}} \times \right.$$
$$\left. \times \prod_{i=1}^{N_t} \prod_{j=1}^{N_{n-1}} \frac{\left(v_t + u_t + \cdots + u_{n-1} + N_{n-1}\beta + N_t\beta - N_{t-1}\beta + (n-t)(1-\beta) + 1 - (i+j)\beta\right)_\beta}{\left(v_t + u_t + \cdots + u_{n-1} + N_{n-1}\beta + N_t\beta - N_{t-1}\beta + (n-t)(1-\beta) + 1 + Y'_{si} + Y_{(n+1)j} - (i+j)\beta\right)_\beta} \right\}$$
$$\times 1 \,. \tag{274}$$

This is just the expression of

$$\left\langle J_{Y_1}(-p^{(1)}_k - \frac{v_1 + \cdots + v_{(n-1)}}{\beta}) \ldots J_{Y_r}(p^{(r-1)}_k - p^{(r)}_k - \frac{v_r + \cdots + v_{(n-1)}}{\beta}) \ldots J_{Y_n}(p^{(n-1)}_k) \right\rangle^{A_{n-1}} .$$

In addition, we would like to attach the consistence checks which our formula fails to pass, according to some deviations proportional to $\beta - 1$. Those are expected to be modified in our future work.

Check 1.

$$\left\langle J_1(p^{(1)}_k - p^{(2)}_k) \right\rangle^{SU(3)}_+ = \left\langle J_1(p^{(1)}_k) \right\rangle^{SU(3)}_{\vec{u},\vec{v},\beta} - \left\langle J_1(p^{(2)}_k) \right\rangle^{SU(3)}_{\vec{u},\vec{v},\beta} \tag{275}$$

Using $[x]_1 = x$, and $G_{1,1}(0) = \beta$.

Check 2.

$$\left\langle J_1(p^{(1)}_k - p^{(2)}_k) J_1(p^{(2)}_k) \right\rangle^{SU(3)}_+ = \left\langle J_1(p^{(1)}_k) J_1(p^{(2)}_k) \right\rangle^{SU(3)}_+ - \left\langle J_2(p^{(2)}_k) \right\rangle^{SU(3)}_+ - \frac{2}{1+\beta} \left\langle J_{11}(p^{(2)}_k) \right\rangle^{SU(3)}_+ \tag{276}$$

Using $[x]_2 = x(x+1)$, $[x]_{11} = x(x-\beta)$, $G_{2,2}(0) = \beta(\beta+1)$ and $G_{11,11}(0) = 2\beta^2$.

C Proof of consistency relations

Here we present the detailed computation of the second sets of consistency conditions (149) in the text.
When $n = 2$ ($SU(3)$ case), making use of (168), and setting $Y_1 = R$, $Y_2 = A$, $Y_3 = B$, the conjecture (147) becomes

$$\begin{aligned}
&\left\langle \chi_R(-p_k^{(1)} - v)\chi_A(p_k^{(1)} - p_k^{(2)})\chi_B(p_k^{(2)}) \right\rangle^{SU(3)} \\
&= \frac{[-v_1 - N_1]_R}{G_{R,R}(0)} \times \frac{[N_1 - N_2]_A}{G_{A,A}(0)} \times \frac{[N_2]_B}{G_{B,B}(0)} \\
&\times \frac{[-v - u_1 - N_1]_R}{[-v - u_1 - N_1 + N_2]_R} \times \frac{[u_1 + N_1 - N_2]_A}{[v + u_1 + N_1 - N_2]_A} \times \frac{[u_1 + u_2 + N_2]_B}{[v + u_1 + u_2 + N_2]_B} \times \frac{[u_2 + N_2]_B}{[u_2 - N_1 + N_2]_B} \\
&\times \prod_{j=1}^{N_1}\prod_{i=1}^{N_1} \frac{v + u_1 + 2N_1 - N_2 + 1 - (i+j)}{v + u_1 + 2N_1 - N_2 + 1 + R'_j + A_i - (i+j)} \\
&\times \prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{v + u_1 + u_2 + N_1 + N_2 + 1 - (i+j)}{v + u_1 + u_2 + N_1 + N_2 + 1 + R'_j + B_i - (i+j)} \\
&\times \prod_{j=1}^{N_2}\prod_{i=1}^{N_2} \frac{u_2 - N_1 + 2N_2 + 1 - (i+j)}{u_2 - N_1 + 2N_2 + 1 + A'_j + B_i - (i+j)} \quad ,
\end{aligned} \tag{277}$$

where we have switched the name of i and j in the last three lines.

For simplicity, we consider the case with R, A, B being rectangle Young diagrams, when (148) reduce to

$$p_1\chi_A(p_k) = \chi_{\hat{A}}(p_k) + \chi_{\breve{A}}(p_k) \ , \tag{278}$$

as illustrated in Figure 5.

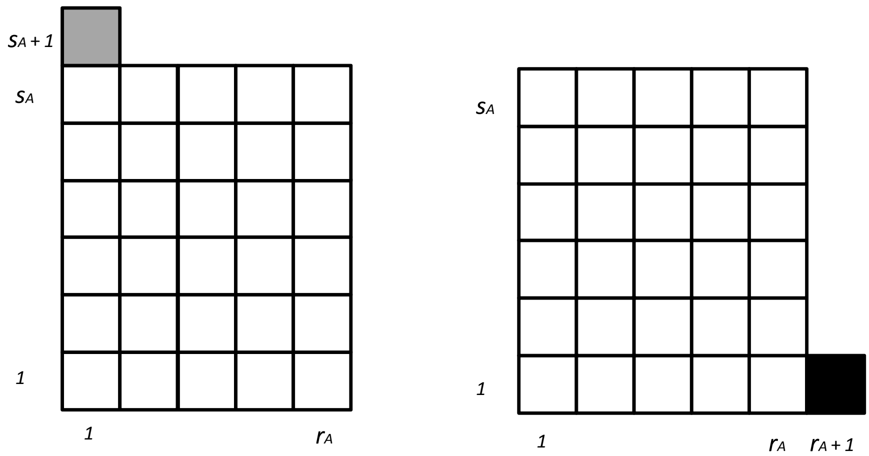

Figure 5: The white cells stands for A, with length r_A and height s_A. the left is the diagram of $\hat{A}$, with an extra grey cell compared to A; the right is the diagram of $\breve{A}$, with an extra black cell compared to A. $A_i = s_A$, $A'_j = r_A$, $\hat{A}_1 = s_A + 1$, and $\breve{A}'_1 = r_A + 1$.

Now at $\beta = 1$, there are

$$[x]_A = \prod_{i=1}^{r_A}\prod_{j=1}^{s_A}(x - i + j) \quad , \quad G_{A,A}(0) = \prod_{i=1}^{r_A}\prod_{j=1}^{s_A}(r_A + s_A - i - j + 1) \quad . \tag{279}$$

Furthermore with the information given in Figure 5, we find several lemmas shown below

$$\frac{[x]_{\hat{A}}}{[x]_A} = x + s_A \quad , \quad \frac{[x]_{\breve{A}}}{[x]_A} = x - r_A \quad , \tag{280}$$

$$\frac{G_{A,A}(0)}{G_{\hat{A},\hat{A}}(0)} = \prod_{j=1}^{s_A} \frac{r_A + s_A - j}{r_A + s_A - j + 1} = \frac{r_A}{r_A + s_A} \quad , \quad \frac{G_{A,A}(0)}{G_{\check{A},\check{A}}(0)} = \prod_{i=1}^{r_A} \frac{r_A + s_A - i}{r_A + s_A - i + 1} = \frac{s_A}{r_A + s_A} \quad , \tag{281}$$

$$\begin{aligned} &\prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{x + 1 + A'_j + B_i - (i+j)}{x + 1 + \hat{A}'_j + B_i - (i+j)} \\ &= \prod_{i=1}^{N_2} \frac{x + 1 + 0 + B_i - i - (s_A + 1)}{x + 1 + 1 + B_i - i - (s_A + 1)} = \prod_{i=1}^{r_B} \frac{x + s_B - s_A - i}{x + s_B - s_A - i + 1} \times \prod_{i=r_B+1}^{N_2} \frac{x - s_A - i}{x - s_A - i + 1} \\ &= \frac{x + s_B - s_A - r_B}{x + s_B - s_A} \times \frac{x - s_A - N_2}{x - s_A - r_B} \quad , \end{aligned} \tag{282}$$

$$\begin{aligned} &\prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{x + 1 + A'_j + B_i - (i+j)}{x + 1 + \check{A}'_j + B_i - (i+j)} \\ &= \prod_{i=1}^{N_2} \frac{x + 1 + r_A + B_i - i - 1}{x + 1 + r_A + 1 + B_i - i - 1} = \prod_{i=1}^{r_B} \frac{x + s_B + r_A - i}{x + s_B + r_A - i + 1} \times \prod_{i=r_B+1}^{N_2} \frac{x + r_A - i}{x + r_A - i + 1} \\ &= \frac{x + s_B + r_A - r_B}{x + s_B + r_A} \times \frac{x + r_A - N_2}{x + r_A - r_B} \quad , \end{aligned} \tag{283}$$

$$\begin{aligned} &\prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{x + 1 + A'_j + B_i - (i+j)}{x + 1 + A'_j + \hat{B}_i - (i+j)} \\ &= \prod_{j=1}^{N_1} \frac{x + 1 + A'_j + s_B - 1 - j}{x + 1 + A'_j + s_B + 1 - 1 - j} = \prod_{j=1}^{s_A} \frac{x + r_A + s_B - j}{x + r_A + s_B - j + 1} \times \prod_{j=s_A+1}^{N_1} \frac{x + s_B - j}{x + s_B - j + 1} \\ &= \frac{x + r_A + s_B - s_A}{x + r_A + s_B} \times \frac{x + s_B - N_1}{x + s_B - s_A} \quad , \end{aligned} \tag{284}$$

and

$$\begin{aligned} &\prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{x + 1 + A'_j + B_i - (i+j)}{x + 1 + A'_j + \check{B}_i - (i+j)} \\ &= \prod_{j=1}^{N_1} \frac{x + 1 + A'_j + 0 - (r_B + 1) - j}{x + 1 + A'_j + 1 - (r_B + 1) - j} = \prod_{j=1}^{s_A} \frac{x + r_A - r_B - j}{x + r_A - r_B - j + 1} \times \prod_{j=s_A+1}^{N_1} \frac{x - r_B - j}{x - r_B - j + 1} \\ &= \frac{x + r_A - r_B - s_A}{x + r_A - r_B} \times \frac{x - r_B - N_1}{x - r_B - s_A} \quad . \end{aligned} \tag{285}$$

With the help of the above lemmas, we can calculate that

$$
\begin{aligned}
&\frac{\left\langle \chi_{\hat{R}}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} \\
&= \frac{[-v_1-N_1]_{\hat{R}}}{[-v_1-N_1]_R} \times \frac{G_{R,R}(0)}{G_{\hat{R},\hat{R}}(0)} \times \frac{[-v-u_1-N_1]_{\hat{R}}}{[-v-u_1-N_1]_R} \times \frac{[-v-u_1-N_1+N_2]_R}{[-v-u_1-N_1+N_2]_{\hat{R}}} \\
&\quad\times \prod_{j=1}^{N_1}\prod_{i=1}^{N_1} \frac{v+u_1+2N_1-N_2+1+R'_j+A_i-(i+j)}{v+u_1+2N_1-N_2+1+\hat{R}'_j+A_i-(i+j)} \\
&\quad\times \prod_{j=1}^{N_1}\prod_{i=1}^{N_2} \frac{v+u_1+u_2+N_1+N_2+1+R'_j+B_i-(i+j)}{v+u_1+u_2+N_1+N_2+1+\hat{R}'_j+B_i-(i+j)} \\
&= (-v-N_1+s_R) \times \frac{r_R}{r_R+s_R} \times \frac{-v-u_1-N_1+s_R}{-v-u_1-N_1+N_2+s_R} \times \\
&\quad\times \frac{v+u_1+2N_1-N_2+s_A-s_R-r_A}{v+u_1+2N_1-N_2+s_A-s_R} \times \frac{v+u_1+N_1-N_2-s_R}{v+u_1+2N_1-N_2-s_R-r_A} \times \\
&\quad\times \frac{v+u_1+u_2+N_1+N_2+s_B-s_R-r_B}{v+u_1+u_2+N_1+N_2+s_B-s_R} \times \frac{v+u_1+u_2+N_1-s_R}{v+u_1+u_2+N_1+N_2-s_R-r_B} \quad .
\end{aligned}
\tag{286}
$$

Likewise, we have

$$
\begin{aligned}
&\frac{\left\langle \chi_{\check{R}}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} \\
&= (-v-N_1-r_R) \times \frac{s_R}{r_R+s_R} \times \frac{-v-u_1-N_1-r_R}{-v-u_1-N_1+N_2-r_R} \times \\
&\quad\times \frac{v+u_1+2N_1-N_2+s_A+r_R-r_A}{v+u_1+2N_1-N_2+s_A+r_R} \times \frac{v+u_1+N_1-N_2+r_R}{v+u_1+2N_1-N_2+r_R-r_A} \times \\
&\quad\times \frac{v+u_1+u_2+N_1+N_2+s_B+r_R-r_B}{v+u_1+u_2+N_1+N_2+s_B+r_R} \times \frac{v+u_1+u_2+N_1+r_R}{v+u_1+u_2+N_1+N_2+r_R-r_B} \quad ,
\end{aligned}
\tag{287}
$$

$$
\begin{aligned}
&\frac{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_{\hat{A}}(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} \\
&= (N_1-N_2+s_A) \times \frac{r_A}{r_A+s_A} \times \frac{u_1+N_1-N_2+s_A}{v+u_1+N_1-N_2+s_A} \times \\
&\quad\times \frac{v+u_1+2N_1-N_2+r_R+s_A-s_R}{v+u_1+2N_1-N_2+r_R+s_A} \times \frac{v+u_1+N_1-N_2+s_A}{v+u_1+2N_1-N_2+s_A-s_R} \times \\
&\quad\times \frac{u_2-N_1+2N_2+s_B-s_A-r_B}{u_2-N_1+2N_2+s_B-s_A} \times \frac{u_2-N_1+N_2-s_A}{u_2-N_1+2N_2-s_A-r_B} \quad ,
\end{aligned}
\tag{288}
$$

$$
\begin{aligned}
&\frac{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_{\check{A}}(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_{R}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} \\
&= (N_1-N_2-r_A) \times \frac{s_A}{r_A+s_A} \times \frac{u_1+N_1-N_2-r_A}{v+u_1+N_1-N_2-r_A} \times \\
&\quad\times \frac{v+u_1+2N_1-N_2+r_R-r_A-s_R}{v+u_1+2N_1-N_2+r_R-r_A} \times \frac{v+u_1+N_1-N_2-r_A}{v+u_1+2N_1-N_2-r_A-s_R} \times \\
&\quad\times \frac{u_2-N_1+2N_2+s_B+r_A-r_B}{u_2-N_1+2N_2+s_B+r_A} \times \frac{u_2-N_1+N_2+r_A}{u_2-N_1+2N_2+r_A-r_B} \quad ,
\end{aligned}
\tag{289}
$$

$$\frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_{\hat{B}}(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}$$
$$\begin{aligned}
&= (N_2+s_B)\times\frac{r_B}{r_B+s_B}\times\frac{u_1+u_2+N_2+s_B}{v+u_1+u_2+N_2+s_B}\times\frac{u_2+N_2+s_B}{u_2-N_1+N_2+s_B}\\
&\quad\times\frac{v+u_1+u_2+N_1+N_2+r_R+s_B-s_R}{v+u_1+u_2+N_1+N_2+r_R+s_B}\times\frac{v+u_1+u_2+N_2+s_B}{v+u_1+u_2+N_1+N_2+s_B-s_R}\times\\
&\quad\times\frac{u_2-N_1+2N_2+r_A+s_B-s_A}{u_2-N_1+2N_2+r_A+s_B}\times\frac{u_2-N_1+N_2+s_B}{u_2-N_1+2N_2+s_B-s_A}\quad ,
\end{aligned} \tag{290}$$

and

$$\frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_{\check{B}}(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}$$
$$\begin{aligned}
&= (N_2-r_B)\times\frac{s_B}{r_B+s_B}\times\frac{u_1+u_2+N_2-r_B}{v+u_1+u_2+N_2-r_B}\times\frac{u_2+N_2-r_B}{u_2-N_1+N_2-r_B}\\
&\quad\times\frac{v+u_1+u_2+N_1+N_2+r_R-r_B-s_R}{v+u_1+u_2+N_1+N_2+r_R-r_B}\times\frac{v+u_1+u_2+N_2-r_B}{v+u_1+u_2+N_1+N_2-r_B-s_R}\times\\
&\quad\times\frac{u_2-N_1+2N_2+r_A-r_B-s_A}{u_2-N_1+2N_2+r_A-r_B}\times\frac{u_2-N_1+N_2-r_B}{u_2-N_1+2N_2-r_B-s_A}\quad .
\end{aligned} \tag{291}$$

Summing v and the above six expressions together, we obtain

$$\begin{aligned}
v &+ \frac{\left\langle \chi_{\hat{R}}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} + \frac{\left\langle \chi_{\check{R}}(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}\\
&+ \frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_{\hat{A}}(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} + \frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_{\check{A}}(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle}\\
&+ \frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_{\hat{B}}(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} + \frac{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_{\check{B}}(p_k^{(2)})\right\rangle}{\left\langle \chi_R(-p_k^{(1)}-v)\chi_A(p_k^{(1)}-p_k^{(2)})\chi_B(p_k^{(2)})\right\rangle} = 0 \quad .
\end{aligned} \tag{292}$$

This reproduces (149), which serves as a quite nontrivial check of our conjecture (147).

D Proof of the lemmas

Lemma 1

$$\prod_{1\le i<j\le N}\frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} = \frac{[N\beta]_B}{G_{B,B}(0)} \tag{293}$$

Proof: Since $(x)_k = \frac{\Gamma(x+k)}{\Gamma(x)}$, we obtain

$$\frac{((j-i+1)\beta)_{B_i-B_j}}{((j-i)\beta)_{B_i-B_j}} = \frac{\Gamma((j-i+1)\beta+B_i-B_j)}{\Gamma((j-i+1)\beta)}\times\frac{\Gamma((j-i)\beta)}{\Gamma((j-i)\beta+B_i-B_j)} = \frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\ . \tag{294}$$

So we only need to prove the following

$$\prod_{1\le i<j\le N}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}=\frac{\prod\limits_{(i,j)\in B}(N\beta-\beta(i-1)+j-1)}{\prod\limits_{(i,j)\in B}\left[\beta(B'_j-i)+(B_i-j)+\beta\right]}=\frac{[N\beta]_B}{G_{B,B}(0)}\,. \tag{295}$$

Suppose the length of B to be m, The left hand side can be expressed as

$$\prod_{1\le i<j\le N}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}=\prod_{i=1}^{m}\prod_{j=m+1}^{N}\frac{(B_i+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\times\prod_{i=1}^{m-1}\prod_{j=i+1}^{m}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\,, \tag{296}$$

where

$$\begin{aligned}
&\prod_{i=1}^{m}\prod_{j=m+1}^{N}\frac{(B_i+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\\
&=\prod_{i=1}^{m}\prod_{j=m+1}^{N}\frac{(1+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}\frac{(2+(j-i)\beta)_\beta}{(1+(j-i)\beta)_\beta}\cdots\frac{(B_i+(j-i)\beta)_\beta}{(B_i-1+(j-i)\beta)_\beta}\\
&=\prod_{i=1}^{m}\prod_{j=m+1}^{N}\frac{(j-i+1)\beta}{(j-i)\beta}\frac{1+(j-i+1)\beta}{1+(j-i)\beta}\cdots\frac{B_i-1+(j-i+1)\beta}{B_i-1+(j-i)\beta}\\
&=\prod_{i=1}^{m}\prod_{j=m+1}^{N}\prod_{k=1}^{B_i}\frac{k-1+(j-i+1)\beta}{k-1+(j-i)\beta}=\prod_{i=1}^{m}\prod_{j=1}^{B_i}\prod_{k=m+1}^{N}\frac{j-1+(k-i+1)\beta}{j-1+(k-i)\beta}=\\
&=\prod_{(i,j)\in B}\prod_{k=m+1}^{N}\frac{j-1+(k-i+1)\beta}{j-1+(k-i)\beta}=\prod_{(i,j)\in B}\frac{j-1+(N-i+1)\beta}{j-1+(m-i+1)\beta}\,.
\end{aligned} \tag{297}$$

So what is left is to prove the following equation:

$$\prod_{i=1}^{m-1}\prod_{j=i+1}^{m}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}=\prod_{i=1}^{m}\prod_{j=1}^{B_i}\frac{(m\beta-\beta(i-1)+j-1)}{\left[\beta(B'_j-i)+(B_i-j)+\beta\right]}\,. \tag{298}$$

Notice when $1\le j\le B_m$, we have $B'_j=m$,

$$\prod_{j=1}^{B_m}\frac{(m\beta-\beta(m-1)+j-1)}{\left[\beta(m-m)+(B_m-j)+\beta\right]}=1\,. \tag{299}$$

Thus the sufficient condition of (298) is

$$\prod_{j=i+1}^{m}\frac{(B_i-B_j+(j-i)\beta)_\beta}{((j-i)\beta)_\beta}=\prod_{j=1}^{B_i}\frac{(m\beta-\beta(i-1)+j-1)}{\left[\beta(B'_j-i)+(B_i-j)+\beta\right]}\,, \tag{300}$$

which becomes our new goal.

In Figure 6, we have

$$B'_j=\begin{cases} m_1 & B_{m_2}+1\le j\le B_{m_1}\\ m_2 & B_{m_3}+1\le j\le B_{m_2}\\ \vdots & \vdots\\ m_n & 1\le j\le B_{m_n}\end{cases}$$

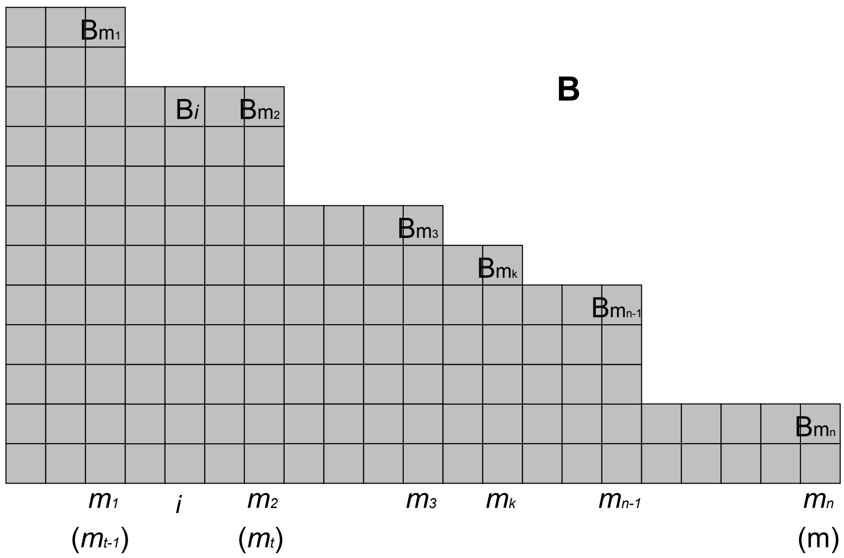

Figure 6:

and if $m_{t-1}+1 \le i \le m_t$, we have $B_i = B_{m_t}$. Besides, We define $B_{m_{n+1}} = 0$.

Now the denominator on the right hand side of (300) is

$$R_1 = \prod_{j=1}^{B_i} \Big[\beta(B'_j - i) + (B_i - j) + \beta\Big] = \prod_{k=t}^{n} \prod_{j=B_{m_{k+1}}+1}^{B_{m_k}} \Big[(B_i - j) + \beta(m_k - i + 1)\Big] , \tag{301}$$

and the left hand side of (300) is

$$\begin{aligned} L &= \prod_{j=i+1}^{m} \frac{(B_i - B_j + (j-i)\beta)_\beta}{((j-i)\beta)_\beta} \\ &= \prod_{j=m_t+1}^{m} \frac{(B_i - B_j + (j-i)\beta)_\beta}{((j-i)\beta)_\beta} \\ &= \prod_{j=m_t+1}^{m} \frac{(j-i+1)\beta}{(j-i)\beta} \, \frac{1+(j-i+1)\beta}{1+(j-i)\beta} \cdots \frac{B_i - B_j - 1 + (j-i+1)\beta}{B_i - B_j - 1 + (j-i)\beta} \\ &= \frac{\prod_{j=m_t+1}^{m} [(j-i+1)\beta]\,[(1+(j-i+1)\beta)] \cdots [(B_i - B_j - 1 + (j-i+1)\beta)]}{\prod_{k=m_t}^{m-1} [(k-i+1)\beta]\,[(1+(k-i+1)\beta)] \cdots [(B_i - B_{k+1} - 1 + (k-i+1)\beta)]} \\ &= \frac{[(m-i+1)\beta]\,[(1+(m-i+1)\beta)] \cdots [(B_i - B_m - 1 + (m-i+1)\beta)]}{[(m_t-i+1)\beta]\,[(1+(m_t-i+1)\beta)] \cdots [(B_i - B_{m_t+1} - 1 + (m_t-i+1)\beta)]} \times \\ &\times \prod_{j=m_t+1}^{m-1} \frac{[(j-i+1)\beta]\,[(1+(j-i+1)\beta)] \cdots [(B_i - B_j - 1 + (j-i+1)\beta)]}{[(j-i+1)\beta]\,[(1+(j-i+1)\beta)] \cdots [(B_i - B_{j+1} - 1 + (j-i+1)\beta)]} \,. \end{aligned} \tag{302}$$

Name the term in the last line to be H, we see $H = 1$ unless $B_j \ne B_{j+1}$, (i.e.,primary rows $j = m_k$). And notice that $B_{m_k+1} = B_{m_{k+1}}$, we can count only over the primary rows.

As a result, we find

$$H = \prod_{k=t+1}^{n-1} \frac{1}{[(B_i - B_{m_{k+1}} - 1 + (m_k - i + 1)\beta)]} \frac{1}{[(B_i - B_{m_{k+1}} + (m_k - i + 1)\beta)]} \cdots \frac{1}{[(B_i - B_{m_k} + (m_k - i + 1)\beta)]} \tag{303}$$
$$= \prod_{k=t+1}^{n-1} \prod_{j=B_{m_{k+1}}+1}^{B_{m_k}} \frac{1}{\left[(B_i - j) + \beta(m_k - i + 1)\right]} .$$

Combine the above three equations, we obtain

$$\begin{aligned} & R_1 \times L = \\ & \frac{[(m-i+1)\beta]\,[(1+(m-i+1)\beta)] \cdots [(B_i - B_m - 1 + (m-i+1)\beta)]}{[(m_t - i+1)\beta]\,[(1+(m_t-i+1)\beta)] \cdots [(B_i - B_{m_t+1} - 1 + (m_t - i+1)\beta)]} \times \\ & \times \prod_{j=1}^{B_m} \left[(B_i - j) + \beta(m - i + 1)\right] \times \prod_{j=B_{m_{t+1}}+1}^{B_{m_t}} \left[(B_i - j) + \beta(m_t - i + 1)\right] \\ & = \prod_{j=1}^{B_i} [(B_i - j) + \beta(m - i + 1)] = \prod_{j=1}^{B_i} [(m\beta - \beta(i-1) + j - 1)] . \end{aligned} \tag{304}$$

This is equivalent to (300), thus complete the proof of lemma 1.

Lemma 2

$$\prod_{i=1}^{N} (x - i\beta)_{B_i} = \left[x - \beta\right]_B \tag{305}$$

Proof: Use (144), we find

$$\begin{aligned} & \prod_{i=1}^{N} (x - i\beta)_{B_i} \\ & = \prod_{i=1}^{N} \frac{\Gamma(x - i\beta + B_i)}{\Gamma(x - i\beta)} = \prod_{i=1}^{m} \frac{\Gamma(x - i\beta + B_i)}{\Gamma(x - i\beta)} = \prod_{i=1}^{m} (x - i\beta)(x - i\beta + 1) \ldots (x - i\beta + B_i - 1) = \\ & = \prod_{i=1}^{m} \prod_{j=1}^{B_i} (x - i\beta + j - 1) = \prod_{(i,j)\in B} (x - \beta - \beta(i-1) + j - 1) = \left[x - \beta\right]_B , \end{aligned} \tag{306}$$

where m is the length of B.

Lemma 3

$$[x]_B = (-1)^{|B|} G_{B,\emptyset}(-x + 1 - \beta) \tag{307}$$

Proof:

$$[x]_B = \prod_{j=1}^{B_1} \prod_{i=1}^{B'_j} (x - \beta(i-1) + j - 1) = \prod_{j=1}^{B_1} \prod_{i=1}^{B'_j} (x - \beta(B'_j - i) + j - 1) = (-1)^{|B|} G_{B,\emptyset}(-x + 1 - \beta) . \tag{308}$$

The second equivalence is based on the fact that when j is fixed, both $i - 1$ and $B'_j - i$ count from 0 to $B'_j - 1$.

Lemma 4a

$$\prod_{i=1}^{N_2}\prod_{j=1}^{N_1}\frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)+B_i-j}=\prod_{(i,j)\in A}\frac{x+\beta(i-N_2)-j}{x+\beta(A'_j-i+1)+B_i-j}\prod_{(i,j)\in B}\frac{x+\beta(1-i)-N_1+j-1}{x-\beta(B'_j-i)-A_i+j-1}\,. \tag{309}$$

Proof: **Step 1:** Proof for $B=\emptyset$.
The left hand side of (309) is,

$$\begin{aligned}L_0&=\prod_{i=1}^{N_2}\prod_{j=1}^{N_1}\frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)-j}=\prod_{i=1}^{N_2}\prod_{j=1}^{h}\frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)-j}=\\&=\prod_{i=1}^{N_2}\prod_{j=1}^{h}\prod_{k=1}^{A'_j}\frac{x+\beta(-i+k)-j}{x+\beta(k-i+1)-j}=\prod_{j=1}^{h}\prod_{k=1}^{A'_j}\frac{x+\beta(-N_2+k)-j}{x+\beta k-j}=\prod_{(i,j)\in A}\frac{x+\beta(-N_2+i)-j}{x+\beta i-j}\,,\end{aligned} \tag{310}$$

where h is the hight of A.

On the other hand, the right hand side of (309) becomes,

$$R_0=\prod_{(i,j)\in A}\frac{x+\beta(i-N_2)-j}{x+\beta(A'_j-i+1)-j}=\prod_{(i,j)\in A}\frac{x+\beta(-N_2+i)-j}{x+\beta i-j}\,. \tag{311}$$

We see $L_0=R_0$, the equation (309) holds with $B=\emptyset$.

Step 2: Induction for other cases. Suppose (309) is valid for B. As shown in Figure 7, let us construct C which has only one cell difference from B: $C_m=B_m+1$, $B'_{B_m+1}=m-1$, $C'_{B_m+1}=m$, with m the length of B. (Notice that the special case $B_m=0$ means C_m starts from a new column, thus we can build any diagram from zero).

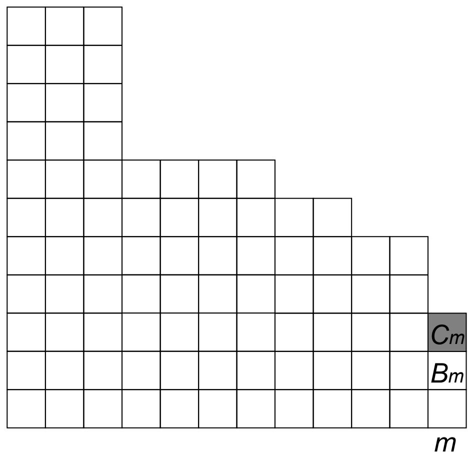

Figure 7: Construction of C. The white cells stands for B , while C has one extra cell (marked in black) than B in the last column.

so we just need to prove that

$$\prod_{i=1}^{N_2}\prod_{j=1}^{N_1}\frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)+C_i-j}=\prod_{(i,j)\in A}\frac{x+\beta(i-N_2)-j}{x+\beta(A'_j-i+1)+C_i-j}\prod_{(i,j)\in C}\frac{x+\beta(1-i)-N_1+j-1}{x-\beta(C'_j-i)-A_i+j-1}\,. \tag{312}$$

The left hand side of (312) is

$$\begin{aligned} L &= \prod_{i=1}^{N_2}\prod_{j=1}^{N_1} \frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)+C_i-j} \\ &= \prod_{i=1}^{N_2}\prod_{j=1}^{N_1} \frac{x+\beta(-i+1)-j}{x+\beta(A'_j-i+1)+B_i-j} \prod_{j=1}^{N_1} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j}\ . \end{aligned} \tag{313}$$

The first term on the right hand side of (312) is

$$\begin{aligned} R_1 &= \prod_{(i,j)\in A} \frac{x+\beta(i-N_2)-j}{x+\beta(A'_j-i+1)+C_i-j} \\ &= \prod_{(i,j)\in A} \frac{x+\beta(i-N_2)-j}{x+\beta(A'_j-i+1)+B_i-j} \prod_{j=1}^{A_m} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j}\ . \end{aligned} \tag{314}$$

And the second term becomes

$$\begin{aligned} R_2 &= \prod_{(i,j)\in C} \frac{x+\beta(1-i)-N_1+j-1}{x-\beta(C'_j-i)-A_i+j-1} \\ &= \prod_{(i,j)\in B} \frac{x+\beta(1-i)-N_1+j-1}{x-\beta(C'_j-i)-A_i+j-1} \times \frac{x+\beta(1-m)-N_1+B_m}{x-A_m+B_m} \\ &= \frac{x+\beta(1-m)-N_1+B_m}{x-A_m+B_m} \times \prod_{(i,j)\in B} \frac{x+\beta(1-i)-N_1+j-1}{x-\beta(B'_j-i)-A_i+j-1} \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)-A_i+B_m}{x-\beta(m-i)-A_i+B_m}\ . \end{aligned} \tag{315}$$

Since we have assumed the equation (309)is correct for B, we only need to proof

$$\begin{aligned} &\prod_{j=1}^{N_1} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j} \\ &= \prod_{j=1}^{A_m} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j} \\ &\times \frac{x+\beta(1-m)-N_1+B_m}{x-A_m+B_m} \times \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)-A_i+B_m}{x-\beta(m-i)-A_i+B_m}\ , \end{aligned} \tag{316}$$

which is equivalent to

$$\begin{aligned} &\prod_{j=A_m+1}^{N_1} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j} \\ &= \frac{x+\beta(1-m)-N_1+B_m}{x-A_m+B_m} \times \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)-A_i+B_m}{x-\beta(m-i)-A_i+B_m}\ . \end{aligned} \tag{317}$$

The left hand side of the above transforms to

$$\begin{aligned} L' &= \prod_{j=A_m+1}^{N_1} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j} \\ &= \prod_{j=h+1}^{N_1} \frac{x+\beta(-m+1)+B_m-j}{x+\beta(-m+1)+B_m+1-j} \prod_{j=A_m+1}^{h} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j} \\ &= \frac{x+\beta(-m+1)+B_m-N_1}{x+\beta(-m+1)+B_m-h} \prod_{j=A_m+1}^{h} \frac{x+\beta(A'_j-m+1)+B_m-j}{x+\beta(A'_j-m+1)+B_m+1-j}\ . \end{aligned} \tag{318}$$

Here h is again the hight of A. Name the second term of the last line as L_1',

$$\begin{aligned}
L_1' &= \prod_{j=A_m+1}^{h} \frac{x+\beta(A_j'-m+1)+B_m-j}{x+\beta(A_j'-m+1)+B_m+1-j} \\
&= \prod_{j=A_m+1}^{h} \left(\frac{x+\beta(-m+1)+B_m-j}{x+\beta(-m+1)+B_m+1-j} \frac{x+\beta(A_j'-m+1)+B_m-j}{x+\beta(-m+1)+B_m-j} \frac{x+\beta(-m+1)+B_m+1-j}{x+\beta(A_j'-m+1)+B_m+1-j} \right) \\
&= \prod_{j=A_m+1}^{h} \frac{x+\beta(-m+1)+B_m-j}{x+\beta(-m+1)+B_m+1-j} \times \\
&\times \prod_{j=A_m+1}^{h} \prod_{i=1}^{A_j'} \left(\frac{x+\beta(i-m+1)+B_m-j}{x+\beta(i-m)+B_m-j} \frac{x+\beta(i-m)+B_m+1-j}{x+\beta(i-m+1)+B_m+1-j} \right),
\end{aligned} \tag{319}$$

This time we call the last term of the last line as L_3.

The second term of the right hand side of (317) has the form

$$\begin{aligned}
R_2' &= \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)-A_i+B_m}{x-\beta(m-i)-A_i+B_m} \\
&= \prod_{i=1}^{m-1} \left(\frac{x-\beta(m-1-i)+B_m}{x-\beta(m-i)+B_m} \frac{x-\beta(m-1-i)-A_i+B_m}{x-\beta(m-1-i)+B_m} \frac{x-\beta(m-i)+B_m}{x-\beta(m-i)-A_i+B_m} \right) \\
&= \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)+B_m}{x-\beta(m-i)+B_m} \times \\
&\times \prod_{i=1}^{m-1} \prod_{j=1}^{A_i} \left(\frac{x+\beta(i-m+1)+B_m-j}{x+\beta(i-m+1)+B_m+1-j} \frac{x+\beta(i-m)+B_m+1-j}{x+\beta(i-m)+B_m-j} \right),
\end{aligned} \tag{320}$$

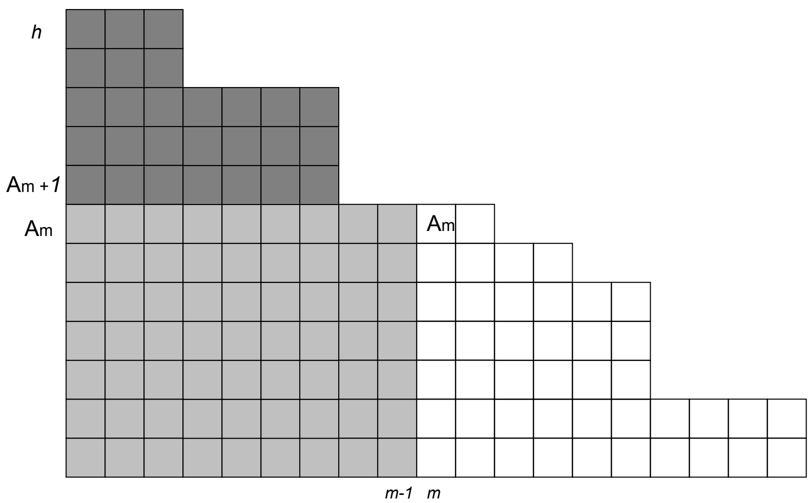

Figure 8: $\prod_{i=1}^{m-1}\prod_{j=1}^{A_i}$ is represented by the area marked by grey and black, while $\prod_{j=A_m+1}^{h}\prod_{i=1}^{A_j'}$ is represented only by the black cells. Their difference, the grey cells, stands for $\prod_{i=1}^{m-1}\prod_{j=1}^{A_m}$, which leads to the following equation.

so we find (see Figure 8)

$$\begin{aligned}
\frac{R_2'}{L_3} &= \prod_{i=1}^{m-1} \frac{x-\beta(m-1-i)+B_m}{x-\beta(m-i)+B_m} \times \\
&\times \prod_{i=1}^{m-1}\prod_{j=1}^{A_m} \left(\frac{x+\beta(i-m+1)+B_m-j}{x+\beta(i-m+1)+B_m+1-j} \frac{x+\beta(i-m)+B_m+1-j}{x+\beta(i-m)+B_m-j} \right) \\
&= \prod_{i=1}^{m-1} \frac{x+\beta(i-m+1)+B_m-A_m}{x+\beta(i-m)+B_m-A_m} \\
&= \frac{x+B_m-A_m}{x+\beta(1-m)+B_m-A_m} \,.
\end{aligned} \tag{321}$$

Combine (318), (319) and (321), it is straightforward to find that (317) is tenable, thus complete the proof.

Lemma 5

$$\begin{aligned}
\prod_{i=1}^{N_1}\prod_{j=1}^{N_2} \frac{\big(x+1-(i+j)\beta\big)_\beta}{\big(x+1+B_j-(i+j)\beta\big)_\beta} &= \frac{[x-N_1\beta+1-\beta]_B}{[x+1-\beta]_B} \,, \\
\prod_{i=1}^{N_1}\prod_{j=1}^{N_2} \frac{\big(x+1-(i+j)\beta\big)_\beta}{\big(x+1+A_i'-(i+j)\beta\big)_\beta} &= \frac{[x-N_2\beta+1-\beta]_{A'}}{[x+1-\beta]_{A'}}
\end{aligned} \tag{322}$$

These are actually the special case of Lemma 4, but hold for arbitrary β.

Proof: For the first statement, we have

$$\begin{aligned}
L &= \prod_{i=1}^{N_1}\prod_{j=1}^{N_2} \frac{\big(x+1-(i+j)\beta\big)_\beta}{\big(x+1+B_j-(i+j)\beta\big)_\beta} \\
&= \prod_{i=1}^{N_1}\prod_{j=1}^{m} \frac{\big(x+1-(i+j)\beta\big)\big(x+2-(i+j)\beta\big)\ldots\big(x-(i+j-1)\beta\big)}{\big(x+1+B_j-(i+j)\beta\big)\big(x+2+B_j-(i+j)\beta\big)\ldots\big(x+B_j-(i+j-1)\beta\big)} \\
&= \prod_{i=1}^{N_1}\prod_{j=1}^{m}\prod_{k=1}^{B_j} \frac{x+k-(i+j)\beta}{x+k-(i+j-1)\beta} = \prod_{j=1}^{m}\prod_{k=1}^{B_j} \frac{x-N_1\beta+k-j\beta}{x+k-j\beta} = \\
&= \prod_{(i,j)\in B} \frac{x-N_1\beta-i\beta+j}{x-i\beta+j} = \frac{[x-N_1\beta+1-\beta]_B}{[x+1-\beta]_B} = R \,,
\end{aligned} \tag{323}$$

where m is the length of B.

The second statement can be proved in totally the same way.

E More details about Eq.(183)

Let x_I $(I=1,\cdots,N)$ be arbitrary complex numbers. We first observe,

$$\sum_{I=1}^{N}\prod_{J(\neq I)}^{N} \frac{1}{x_I-x_J} = 0 \,. \tag{324}$$

If we apply it to a set of variables $\{x_1, \cdots, x_n, \xi\}$, $(\xi = x_{N+1})$ one derives,

$$\sum_{I=1}^{N} \frac{1}{\xi - x_I} \prod_{J(\neq I)} \frac{1}{x_I - x_J} = -\sum_{I=1}^{N} \prod_{J(\neq I)}^{N+1} \frac{1}{x_I - x_J} = \prod_{J=1}^{N} \frac{1}{\xi - x_J} =: \frac{1}{\xi^N} \sum_{n=0}^{\infty} \frac{b_n(x)}{\xi^n} . \tag{325}$$

The function $b_n(x)$ defined in the last line can be written as,

$$b_n(x) = \sum_{I_1 \leq \cdots \leq I_n} x_{I_1} \cdots x_{I_n} . \tag{326}$$

The first part of this equality can be expanded as, $\sum_{n=0}^{\infty} \sum_I \frac{(x_I)^n}{\xi^{n+1}} \prod_{J(\neq I)} \frac{1}{x_I - x_J}$. So we derived,

$$\sum_{I=1}^{N} (x_I)^n \prod_{J(\neq I)}^{N} \frac{1}{x_I - x_J} = \begin{cases} 0 & n = 0, \cdots, N-2 \\ b_{n-N+1}(x) & n \geq N-1 \end{cases} . \tag{327}$$

If we write $\prod_{I=1}^{M} (\xi + y_J) = \sum_{n=0}^{M} \xi^n f_{M-n}(y)$ with

$$f_n(x) = \sum_{I_1 < \cdots < I_n} x_{I_1} \cdots x_{I_n} , \tag{328}$$

Then we obtain,

$$\sum_{I=1}^{N} \frac{\prod_{I=1}^{M} (x_I + y_J)}{\prod_{J(\neq I)}^{N} (x_I - x_J)} = \sum_{n=0}^{M} f_{M-n}(y) \sum_{I=1}^{N} \frac{(x_I)^n}{\prod_{J(\neq I)}^{N} (x_I - x_J)} = \sum_{n=N-1}^{M} f_{M-n}(y) b_{n-N+1}(x) . \tag{329}$$

It is not difficult to show that the last quantity is the coefficient of ζ^{M-N+1} of the function $\frac{\prod_{J=1}^{\mathcal{M}} (\zeta + y_J)}{\prod_{I=1}^{N} (\zeta - x_I)}$.

F Variations of Nekrasov formula

We decompose Y, W into rectangles $Y = (r_1, \cdots, r_f; s_1, \cdots, s_f)$ and $W = (t_1, \cdots, t_{\tilde{f}}; u_1, \cdots, u_{\tilde{f}})$. Also we use the same notation such as $Y^{(k,\pm)}$ and $W^{(k,\pm)}$. For the variation of Y (resp. W), P_2, P_3 (resp. P_1, P_3) remain the same. Variation of P_1 (resp. P_2) produces a term which cancel the N_2 (resp. N_1) dependent term in the variation of Q. We also uses a notation $r_0 = s_{f+1} = t_0 = u_{\tilde{f}+1} = 0$. After some computation, we obtain,

$$\frac{\tilde{g}_{Y^{(k,+)},W}(x)}{\tilde{g}_{Y,W}(x)} = \frac{\prod_{\ell=1}^{\tilde{f}+1} (x + \beta(r_{k-1} - t_{\ell-1} + 1) + u_\ell - s_k - 1)}{\prod_{\ell=1}^{\tilde{f}} (x + \beta(r_{k-1} - t_\ell + 1) + u_\ell - s_k - 1)} , \tag{330}$$

$$\frac{\tilde{g}_{Y^{(k,-)},W}(x)}{\tilde{g}_{Y,W}(x)} = \frac{\prod_{\ell=1}^{\tilde{f}} (x + \beta(r_k - t_\ell) + u_\ell - s_k)}{\prod_{\ell=1}^{\tilde{f}+1} (x + \beta(r_k - t_{\ell-1}) + u_\ell - s_k)} , \tag{331}$$

$$\frac{\tilde{g}_{Y,W^{(\ell,+)}}(x)}{\tilde{g}_{Y,W}(x)} = \frac{\prod_{k=0}^{f} (-x + \beta(t_{\ell-1} - r_k) - u_\ell + s_{k+1})}{\prod_{k=1}^{f} (-x + \beta(t_{\ell-1} - r_k) - u_\ell + s_k)} , \tag{332}$$

$$\frac{\tilde{g}_{Y,W^{(\ell,-)}}(x)}{\tilde{g}_{Y,W}(x)} = \frac{\prod_{k=1}^{f} (-x + \beta(t_\ell - 1 - r_k) - u_\ell + s_k + 1)}{\prod_{k=0}^{f} (-x + \beta(t_\ell - r_k - 1) - u_\ell + s_{k+1} + 1)} . \tag{333}$$

These expressions becomes more compact by the use of the notation $A_k(Y_p), B_k(Y_p)$ in (170,171),

$$\frac{\tilde{g}_{Y_p^{(k,+)}W_q}(a_p-b_q-\mu)}{\tilde{g}_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{\ell=1}^{\tilde{f}_q+1}(a_p-b_q-\mu+A_k(Y_p)-A_\ell(W_q)-\xi)}{\prod_{\ell=1}^{\tilde{f}_q}(a_p-b_q-\mu+A_k(Y_p)-B_\ell(W_q))}, \tag{334}$$

$$\frac{\tilde{g}_{Y_p^{(k,-)}W_q}(a_p-b_q-\mu)}{\tilde{g}_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{\ell=1}^{\tilde{f}_q}(a_p-b_q-\mu+B_k(Y_p)-B_\ell(W_q))}{\prod_{\ell=1}^{\tilde{f}_q+1}(a_p-b_q-\mu+B_k(Y_p)-A_\ell(W_q)-\xi)}, \tag{335}$$

$$\frac{\tilde{g}_{Y_pW_q^{(\ell,+)}}(a_p-b_q-\mu)}{\tilde{g}_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+A_\ell(W_q)-A_k(Y_p))}{\prod_{k=1}^{f_p}(b_q-a_p+\mu+A_\ell(W_q)-B_k(Y_p)+\xi)}, \tag{336}$$

$$\frac{\tilde{g}_{Y_pW_q^{(\ell,-)}}(a_p-b_q-\mu)}{\tilde{g}_{Y_pW_q}(a_p-b_q-\mu)} = \frac{\prod_{k=1}^{f_p}(b_q-a_p+\mu+B_\ell(W_q)-B_k(Y_p)+\xi)}{\prod_{k=1}^{f_p+1}(b_q-a_p+\mu+B_\ell(W_q)-A_k(Y_p))}. \tag{337}$$

These are sufficient to calculate variation of z_{bf} in (32).

To derive the variation of z_{vect} for $p \neq q$, we need following formula which is obtained by putting $W_q \to Y_q$,

$$\frac{\tilde{g}_{Y_pY_q}(a_p-a_q)\tilde{g}_{Y_qY_p}(a_q-a_p)}{\tilde{g}_{Y_p^{(k,+)}Y_q}(a_p-a_q)\tilde{g}_{Y_qY_p^{(k,+)}}(a_q-a_p)} = \frac{\prod_{\ell=1}^{f_q}(a_p-a_q+A_k(Y_p)-B_\ell(Y_q))(a_p-a_q+A_k(Y_p)-B_\ell(Y_q)+\xi)}{\prod_{\ell=1}^{f_q+1}(a_p-a_q+A_k(Y_p)-A_\ell(Y_q)-\xi)(a_p-a_q+A_k(Y_p)-A_\ell(Y_q))}, \tag{338}$$

$$\frac{\tilde{g}_{Y_pY_q}(a_p-a_q)\tilde{g}_{Y_qY_p}(a_q-a_p)}{\tilde{g}_{Y_p^{(k,-)}Y_q}(a_p-a_q)\tilde{g}_{Y_qY_p^{(k,-)}}(a_q-a_p)} = \frac{\prod_{\ell=1}^{f_q+1}(a_p-a_q+B_k(Y_p)-A_\ell(Y_q)-\xi)(a_p-a_q+B_k(Y_p)-A_\ell(Y_q))}{\prod_{\ell=1}^{f_q}(a_p-a_q+B_k(Y_p)-B_\ell(Y_q))(a_p-a_q+B_k(Y_p)-B_\ell(Y_q)+\xi)}. \tag{339}$$

For the case $p = q$, we obtain,

$$\frac{\tilde{g}_{Y_p,Y_p}(0)}{\tilde{g}_{Y_p^{(k+)}Y_p^{(k+)}}(0)} = \frac{1}{\beta}\frac{\prod_{\ell=1}^{f_q}(A_k(Y_p)-B_\ell(Y_p))(A_k(Y_p)-B_\ell(Y_p)+\xi)}{\prod_{\ell=1,(\neq k)}^{f_q+1}(A_k(Y_p)-A_\ell(Y_p)-\xi)(A_k(Y_p)-A_\ell(Y_p))}, \tag{340}$$

$$\frac{\tilde{g}_{Y_p,Y_p}(0)}{\tilde{g}_{Y_p^{(k-)}Y_p^{(k-)}}(0)} = \frac{1}{\beta}\frac{\prod_{\ell=1}^{f_p+1}(B_k(Y_p)-A_\ell(Y_p)-\xi)(B_k(Y_p)-A_\ell(Y_p))}{\prod_{\ell=1}^{f_q}(B_k(Y_p)-B_\ell(Y_p))(B_k(Y_p)-B_\ell(Y_p)+\xi)}. \tag{341}$$

These formulae are sufficient to derive the recursion relation (1).

G Recursive construction under $\beta = 1$ limit

G.1 Free fermions

We start from the definition of fermions,

$$\bar{\psi}^{(p)}(z) = \sum_{n\in\mathbf{Z}} \bar{\psi}_n^{(p)} z^{-n-\lambda_p-1}, \quad \psi^{(p)}(z) = \sum_{n\in\mathbf{Z}} \psi_n^{(p)} z^{-n+\lambda_p}, \quad p = 1,\cdots,N, \quad z \in \mathbf{C} \tag{342}$$

with anti-commutation relation, $\{\bar{\psi}_n^{(p)}, \psi_m^{(q)}\} = \delta_{p,q}\delta_{n+m,0}$. We note that there are extra parameters $\vec{\lambda} \in \mathbf{R}^N$ which represent the shift of the usual mode expansion of fermion. We define the vacuum as, $|\vec{\lambda}\rangle = \otimes_{p=1}^N |\lambda^{(p)}\rangle$,

$$\bar{\psi}_n^{(p)}|\vec{\lambda}\rangle = \psi_m^{(p)}|\vec{\lambda}\rangle = 0 \quad (n \geq 0, m > 0), \quad \vec{\lambda} = (\lambda^{(1)},\cdots,\lambda^{(N)}). \tag{343}$$

The parameters $\vec{\lambda}$ represent the fermion sea levels. Similarly, the bra vacuum $\langle\vec{\lambda}| = \otimes_{p=1}^N \langle\lambda^{(p)}|$ is defined by

$$\langle\vec{\lambda}|\bar{\psi}_n^{(p)} = \langle\vec{\lambda}|\psi_m^{(p)} = 0 \quad (n < 0, m \leq 0), \quad \vec{\lambda} = (\lambda^{(1)},\cdots,\lambda^{(N)}). \tag{344}$$

In formula (226), the bra state has different sea level (say $\vec{\mu}$) in general. In such cases, we need redefine fermion mode expansion as $\psi^{(p)}(z) = \sum_{n\in\mathbf{Z}} \psi_n^{(p)} z^{-n+\lambda_p} = \sum_{n\in\mathbf{Z}} \tilde{\psi}_n^{(p)} z^{-n+\mu_p}$ and define the bra vacuum in terms of $\tilde{\psi}$. The

Hermitian conjugate is defined as $(|\vec{\lambda}\rangle)^\dagger = \langle\vec{\lambda}|$ and $\psi_n^\dagger = \bar{\psi}_{-n}$. This is consistent with the shift of label by the change of vacuum.

With this preparation, the fermion basis is,

$$|\vec{\lambda},\vec{Y}\rangle = \otimes_{p=1}^{N}\left(\bar{\psi}^{(p)}_{-\bar{r}_1^{(p)}}\bar{\psi}^{(p)}_{-\bar{r}_2^{(p)}}\cdots\bar{\psi}^{(p)}_{-\bar{r}_{s_1}^{(p)}}|\lambda^{(p)},-s_1\rangle\right),\quad |\lambda^{(p)},-s_1\rangle = \psi^{(p)}_{-s_1+1}\cdots\psi^{(p)}_{-1}\psi^{(p)}_0|\lambda^{(p)}\rangle \tag{345}$$

$$= (-1)^{|\vec{Y}|}\otimes_{p=1}^{N}\left(\psi^{(p)}_{-\bar{s}_1^{(p)}}\psi^{(p)}_{-\bar{s}_2^{(p)}}\cdots\psi^{(p)}_{-\bar{s}_{r_1}^{(p)}}|\lambda^{(p)},r_p\rangle\right),\quad |\lambda^{(p)},r_1\rangle = \bar{\psi}^{(p)}_{-r_1}\cdots\bar{\psi}^{(p)}_{-1}|\lambda^{(p)}\rangle \tag{346}$$

$$\langle\vec{\lambda},\vec{Y}| = (|\vec{Y},\vec{\lambda}\rangle)^\dagger \tag{347}$$

Here we represent a Young diagram Y_p by the number of each row $r_\sigma^{(p)} = ({}^{\mathrm{T}}Y_p)_\sigma$ or the number of each columns $s_\sigma^{(p)} = (Y_p)_\sigma$. The parameters with bar are $\bar{r}_\sigma^{(p)} = r_\sigma^{(p)} - \sigma + 1$ and $\bar{s}_\sigma^{(p)} = s_\sigma^{(p)} - \sigma$. These states give a natural basis of the Hilbert space with fixed fermion number. By construction, they are orthonormal $\langle\vec{Y},\vec{a}|\vec{W},\vec{b}\rangle = \delta_{\vec{Y},\vec{W}}\delta_{\vec{a},\vec{b}}$.

We define the vertex operator V_κ in (226) by standard bozonization technique. We write,

$$\psi^{(p)}(z) =: e^{-\phi_p(z)} :,\quad \bar{\psi}^{(p)}(z) =: e^{\phi_p(z)} :, \tag{348}$$

with

$$\phi_p(z) = x_p + a_0\log z - \sum_{n\neq 0}\frac{a_n^{(p)}}{n}z^{-n},\quad [a_n^{(p)},a_m^{(q)}] = n\delta_{p,q}\delta_{n+m,0},\ [x_p,a_0^{(q)}] = \delta_{p,q}. \tag{349}$$

The vacuum and the fermionic basis (345) is written in a form,

$$|\vec{\lambda}\rangle = \lim_{z\to 0} : e^{-\sum_p\lambda_p\phi_p(z)} : |\vec{0}\rangle,\quad |\vec{Y},\vec{\lambda}\rangle = \prod_p \chi_{Y^{(p)}}(a_{-n}^{(p)})|\vec{\lambda}\rangle. \tag{350}$$

Here $\chi_{Y^{(p)}}(a_{-n}^{(p)})$ is Schur polynomial expressed in terms of power sum $\mathbf{p}_n = \sum_i (x_i)^n$ and each $\mathbf{p}_n$ is replaced by $a_{-n}^{(p)}$. While the second expression is not used in the following, it is this expression that appeared in the literature [24, 25, 70].

G.2 Action on bra and ket basis for $\beta = 1$ case

In order to evaluate the action of $W(z^n e^{xD})$ $(n\neq 0)$ on $|\vec{\lambda},\vec{Y}\rangle$, a graphic representation (Maya diagram) of $|\vec{\lambda},\vec{Y}\rangle$ is useful. For the simplicity of argument, we take $N=1$ and remove the the index p in (345,346). We take the first expression (345) and rewrite it as,

$$|\lambda,Y\rangle = \bar{\psi}_{-\bar{r}_1}\bar{\psi}_{-\bar{r}_2}\cdots\bar{\psi}_{-\bar{r}_s}\bar{\psi}_s\bar{\psi}_{s+1}\cdots\bar{\psi}_L|-L,\lambda\rangle. \tag{351}$$

and take $L\to\infty$ limit. From this representation, we associate a Young diagram Y with a semi-inifinite sequence of integers $S_Y = \{\bar{r}_1,\bar{r}_2,\cdots,\bar{r}_s,-s,-s-1,\cdots\}$. We prepare an infinite strip of boxes with integer label and fill the boxes with the integer in S_Y (Figure 9 left). It represents the occupation of fermion in each level. To understand the correspondence with the Young diagram Y, we associate each black box with vertical up arrow and white box with horizontal right arrow. We connect these arrows for each box from the left on S_Y. Then the Young diagram shows up in the up/left corner (Figure 9 right). The generator $\mathcal{W}(z^n e^{xD}) = \sum_\ell e^{x(\ell+\lambda)} : \bar{\psi}_{\ell+n}\psi_{-\ell}$: flips one black box at ℓ to white and one white box at $-\ell-n$ to black (if wrong color was filled at each place, it vanishes). It amounts to flipping vertical arrow by horizontal one and vice versa. By analyzing the effect of such flipping, the action of $\mathcal{W}(z^n e^{xD})$ on $|\lambda,Y\rangle$ can be summarized as,

- For $n>0$ it erases a hook of length n and multiply $(-1)^{v(h)-1}e^{x(\ell+\lambda)}$ where $v(h)$ is the height of the hook. (Figure 10 up) If there are some hooks of length n, we sum over all such possibilities.

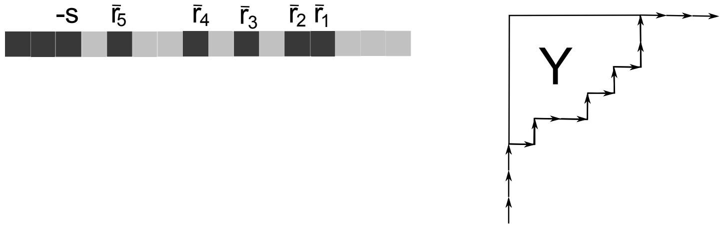

Figure 9: Young diagram and fermion state

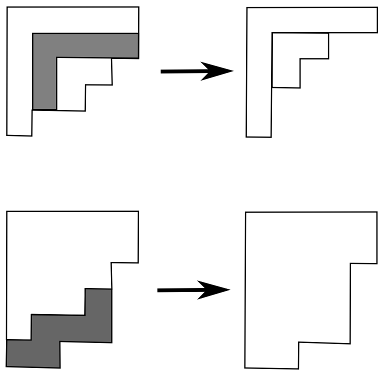

Figure 10: Action of $\mathcal{W}$ on $|Y\rangle$

- For $n < 0$ it adds a strip of length $|n|$ and multiply $(-1)^{v(h)-1}e^{x(\ell+\lambda)}$ where $v(h)$ is the height of the strip. (Figure 10 down) As in $n > 0$ case, if there are some possibility, we need to add them.

H Derivation of commutation relations of SH^c algebra

First we notice that

$$[D_{-1,k}, D_{1,l}]|\vec{b},\vec{W}> \; = \; (-1)^{k+l}\sum_{q=1}^{N}\left\{\sum_{t=1}^{\tilde{f}+1}(b_q + A_t(W_q))^{k+l}(\Lambda_q^{(t,+)}(\vec{b},\vec{W}))^2 - \sum_{t=1}^{\tilde{f}}(b_q + B_t(W_q))^{k+l}(\Lambda_q^{(t,-)}(\vec{b},\vec{W}))^2\right\}|\vec{b},\vec{W}> . \tag{352}$$

We have to be careful that the off-diagonal terms, where the two generator modifies different Young diagrams or two different box in the same Young diagram, cancels with each other. This can be checked as below.

H.1 Cancellation of off-diagonal terms

First, for a Yong diagram with one box removed $W^{(k,-)}$ (or added), we find the relation between $A_t(W^{(k,-)})$, $B_t(W^{(k,-)})$ and their counterparts of the orignal yong diagram W.

$$A_t(W^{(k,-)}) = \begin{cases} A_t(W) & 1 \le t \le k \\ B_k(W) & t = k+1 \\ A_{t-1}(W) & k+2 \le t \le \tilde{f}+2 \end{cases} , \quad B_t(W^{(k,-)}) = \begin{cases} B_t(W) & 1 \le t \le k-1 \\ B_k(W) - \beta & t = k \\ B_k(W) + 1 & t = k+1 \\ B_{t-1}(W) & k+2 \le t \le \tilde{f}+1 \end{cases} ,$$

$$A_s(W^{(k,+)}) = \begin{cases} A_s(W) & 1 \le s \le k-1 \\ A_k(W) - 1 & s = k \\ A_k(W) + \beta & s = k+1 \\ A_{s-1}(W) & k+2 \le s \le \tilde{f}+2 \end{cases}, \quad B_s(W^{(k,+)}) = \begin{cases} B_s(W) & 1 \le s \le k-1 \\ A_k(W) & s = k \\ B_{s-1}(W) & k+1 \le t \le \tilde{f}+1 \end{cases}.$$

With the above relations, we obtain that (For simplicity, in the following we do not write b_q explicitly, which always comes together with $A_t(W_q)$ and $B_t(W_q)$. This choice do not effect the proof at all),

$$D_{-1,k} D_{1,l} |\vec{b}, \vec{W} > \;=\; \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q^{(u,+),\gamma}} (B_t(\vec{W}_q^{(u,+),\gamma}))^k \Lambda_q^{(t,-)}(\vec{W}^{(u,+),\gamma}) \sum_{\gamma=1}^{N} \sum_{u=1}^{\tilde{f}_\gamma+1} (A_u(W_\gamma))^l \Lambda_\gamma^{(u,+)}(\vec{W}) |\vec{b}, \vec{W}^{(u,+),\gamma}_{(t,-),q} >, \quad (353)$$

$$D_{1,l} D_{-1,k} |\vec{b}, \vec{W} > \;=\; \sum_{\gamma=1}^{N} \sum_{u=1}^{\tilde{f}_\gamma^{(t,-),q}+1} (A_u(\vec{W}_\gamma^{(t,-),q}))^l \Lambda_\gamma^{(u,+)}(\vec{W}^{(t,-),q}) \sum_{q=1}^{N} \sum_{t=1}^{\tilde{f}_q} (B_t(W_k))^k \Lambda_q^{(t,-)}(\vec{W}) |\vec{b}, \vec{W}^{(t,-),q}_{(u,+),\gamma} > . \quad (354)$$

For $q = \gamma$, $t \ge u$,

$$(A_\mu(\vec{W}_\gamma^{(t,-),q}))^l \left(-\prod_{\delta=1}^{N} \left(\prod_{v=1}^{\tilde{f}_\delta^{(t,-),q}} \frac{A_u(\vec{W}_\gamma^{(t,-),q}) - B_v(\vec{W}_\delta^{(t,-),q} + \xi}{A_u(\vec{W}_\gamma^{(t,-),q}) - B_v(\vec{W}_\delta^{(t,-),q})} {\prod}'^{\tilde{f}_\delta^{(t,-),q}+1}_{\delta=1} \frac{A_u(\vec{W}_\gamma^{(t,-),q}) - A_v(\vec{W}_\delta^{(t,-),q} - \xi}{A_u(\vec{W}_\gamma^{(t,-),q}) - A_v(\vec{W}_\delta^{(t,-),q}} \right) \right)^{1/2}$$
$$= \text{common terms} \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) - 1)}{A_u(W_\gamma) - (B_t(W_\gamma) - \beta)} \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) + \beta)}{A_u(W_\gamma) - (B_t(W_\gamma) + 1)} \times \frac{A_u(W_\gamma) - (B_t(W_\gamma) + \xi)}{A_u(W_\gamma) - B_t(W_\gamma)}, \quad (355)$$

$$(B_{t+1}(\vec{W}_q^{(u,+),\gamma}))^k \left(-\prod_{p=1}^{N} \left(\prod_{s=1}^{\tilde{f}_p^{(u,+),\gamma}+1} \frac{B_{t+1}(\vec{W}_q^{(u,+),\gamma}) - A_s(\vec{W}_p^{(u,+),\gamma}) - \xi}{B_{t+1}(\vec{W}_q^{(u,+),\gamma}) - A_s(\vec{W}_p^{(u,+),\gamma})} {\prod}'^{\tilde{f}_p^{(u,+),\gamma}}_{s=1} \frac{B_{t+1}(\vec{W}_q^{(u,+),\gamma}) - B_s(\vec{W}_p^{(u,+),\gamma}) + \xi}{B_{t+1}(\vec{W}_q^{(u,+),\gamma}) - B_s(\vec{W}_p^{(u,+),\gamma})} \right) \right)^{1/2}$$
$$= \text{common terms} \times \frac{B_t(W_q) - (A_u(W_q) - \beta)}{B_t(W_q) - (A_u(W_q) - 1)} \times \frac{B_t(W_q) - (A_u(W_q) + 1)}{B_t(W_q) - (A_u(W_q) + \beta)} \times \frac{B_t(W_q) - (A_u(W_q) - \xi)}{B_t(W_q) - A_u(W_q)}. \quad (356)$$

Thus we find that $\sum_{(\gamma),u}^{(t,-),\gamma}$ cancels with $\sum_{(\gamma),t+1}^{(u,+),\gamma}$. For $q = \gamma$, $t \le u-2$, we have the same result. For $q = \gamma$, $t = u-1$, we have the direct sum. For $q \ne \gamma$, using a similar method, we also find that $\sum_q \sum_\gamma$ cancels with $\sum_\gamma \sum_q$. In total we show that all the off-diagonal terms are gone.

H.2 Evaluation of diagonal terms

Since the right hand side of (352) only depends on $k+l$, we have $[D_{-1,k}, D_{1,l}] = [D_{-1,0}, D_{1,l+k}]$. We need to define the action of $D_{0,l}$. For this purpose, we consider

$$X(s) = < \vec{b}, \vec{W} | \sum_{l \ge 0} [D_{-1,0}, D_{1,l}] s^l |\vec{b}, \vec{W} > . \quad (357)$$

Then from the definition of algebra, we obtain

$$
\begin{aligned}
s\xi X(s) &= \sum_{q=1}^{N}\left\{\sum_{t=1}^{\tilde f+1}\frac{s\xi}{1+s(b_q+A_t(W_q))}(\Lambda_q^{(t,+)}(\vec b,\vec W))^2-\sum_{t=1}^{\tilde f}\frac{s\xi}{1+s(b_q+B_t(W_q))}(\Lambda_q^{(t,-)}(\vec b,\vec W))^2\right\}\\
&=\sum_{q=1}^{N}\Bigg\{\sum_{t=1}^{\tilde f+1}\frac{s\xi}{1+s(b_q+A_t(W_q))}\prod_{p=1}^{N}\{\prod_{k=1}^{\tilde f}\frac{b_q-b_p+A_t(W_q)-B_k(W_p)+\xi}{b_q-b_p+A_t(W_q)-B_k(W_p)}\prod_{k\neq t}^{\tilde f+1}\frac{b_q-b_p+A_t(W_q)-A_k(W_p)-\xi}{b_q-b_p+A_t(W_q)-A_k(W_p)}\}\\
&\quad-\sum_{t=1}^{\tilde f}\frac{s\xi}{1+s(b_q+B_t(W_q))}\prod_{p=1}^{N}\{\prod_{(p),k\neq(q),t}^{\tilde f}\frac{b_q-b_p+B_t(W_q)-B_k(W_p)+\xi}{b_q-b_p+B_t(W_q)-B_k(W_p)}\prod_{k=1}^{\tilde f+1}\frac{b_q-b_p+B_t(W_q)-A_k(W_p)-\xi}{b_q-b_p+B_t(W_q)-A_k(W_p)}\}\Bigg\}\\
&=-1+\prod_{q=1}^{N}\prod_{t=1}^{\tilde f}\frac{1+s(b_q+B_t(W_q)-\xi)}{1+s(b_q+B_t(W_q))}\prod_{t=1}^{\tilde f+1}\frac{1+s(b_q+A_t(W_q)+\xi)}{1+s(b_q+A_t(W_q))}.
\end{aligned}
\tag{358}
$$

The last equality holds because the both sides (i) are degree 0 rational function in s, (ii) have the same simple poles and residues at $s=-1/(b_q+B_t(W_q)),-1/(b_q+A_t(W_q))$ and (iii) vanish at $s=0$. We can rewrite (358) as

$$
\begin{aligned}
1+s\xi X(s) &= \prod_{q=1}^{N}\prod_{t=1}^{\tilde f}\frac{1+s(b_q+B_t(W_q)-\xi)}{1+s(b_q+B_t(W_q))}\prod_{t=1}^{\tilde f+1}\frac{1+s(b_q+A_t(W_q)+\xi)}{1+s(b_q+A_t(W_q))}\\
&= \exp\left\{\sum_{q=1}^{N}\sum_{l=1}^{\infty}\frac{(-1)^l s^l}{l}(\sum_{t=1}^{\tilde f}(p_l(b_q+B_t(W_q))-p_l(b_q+B_t(W_q)-\xi)+\sum_{t=1}^{\tilde f+1}(p_l(b_q+A_t(W_q))-p_l(b_q+A_t(W_q)+\xi))\right\},
\end{aligned}
\tag{359}
$$

where $p_l(x_I)=\sum_I x^l$.

We define $H_l(W_q):=\sum_{t=1}^{\tilde f}(p_l(b_q+B_t(W_q))-p_l(b_q+B_t(W_q)-\xi)+\sum_{t=1}^{\tilde f+1}(p_l(b_q+A_t(W_q))-p_l(b_q+A_t(W_q)+\xi))$. Then we use a formula,

$$
H_l(W_q) = (b_q-\xi)^l-(b_q)^l-\sum_{\mu\in W_q}\sigma_l(c_q(\mu))\,, \tag{360}
$$

where $\sigma_l(x)=(x+1)^l-(x-1)^l+(x-\beta)^l-(x+\beta)^l+(x+\beta-1)^l-(x+1-\beta)^l$ and $c_q(\mu)=b_q+\beta i-j$ for $\mu=(i,j)$. It was proved in appendix B of [15]. Thus we can proceed as

$$
\begin{aligned}
1+\xi sX(s) &= \exp\left\{\sum_{q=1}^{N}\sum_{l=1}^{\infty}(-1)^l\frac{s^l}{l}((b_q-\xi)^l-(b_q)^l)-\sum_{q=1}^{N}\sum_{\mu\in W_q}\sum_{l=1}^{\infty}(-1)^l\frac{s^l}{l}\sigma_l((c_q(\mu))\right\}\\
&= \exp\left\{\sum_{q=1}^{N}\sum_{l=0}^{\infty}(-1)^{l+1}(b_q-\xi)^l\pi_l(s)\right\}\exp\left\{\sum_{q=1}^{N}\sum_{l=0}^{\infty}(\sum_{\mu\in W_q}(-1)^l c_q(\mu)^l)\omega_l(s)\right\}.
\end{aligned}
\tag{361}
$$

In the last equality of (361), we use the following formula

$$
\sum_{l=1}^{\infty}(-1)^{l+1}\frac{s^l}{l}\{(a+b)^l-a^l\}=\sum_{l=0}^{\infty}(-1)^{l+1}a^l s^l G_l(1+bs)\,, \tag{362}
$$

which can be proved by some computation. Comparing (361) with (188), the algebra (186) is proved once we set (202, 206). The proof of the algebra for the action on the bra state is similar.

I Derivation of Virasoro algebra from SH^c

Here we give a sample computation to give the Virasoro algebra from the definition of SH^c (184–187) and (213). We focus on the relation

$$
[L_2,L_{-2}]=4L_0+\frac{c}{2} \tag{363}
$$

since it gives the simplest commutator to give the Virasoro central charge.

The definition of generators gives

$$[L_2, L_{-2}] = \frac{1}{4\beta^2}\left\{[D_{-2,1}, D_{2,1}] - c_0\xi[D_{-2,1}, D_{2,0}] - c_0\xi[D_{-2,0}, D_{2,1}] + (c_0\xi)^2[D_{-2,0}, D_{2,0}]\right\} .$$

We express degree 2 generator as the commutator of degree 1 generator

$$D_{2,0} = [D_{1,1}, D_{1,0}]\,, \qquad D_{-2,0} = [D_{-1,0}, D_{-1,1}]\,,$$
$$D_{2,1} = [D_{1,2}, D_{1,0}]\,, \qquad D_{-2,1} = [D_{-1,0}, D_{-1,2}]\,. \tag{364}$$

The commutation relation between degree two operators is reduced to those for degree one operator. After some computation we arrive at

$$[D_{-2,1}, D_{2,1}] = 8\beta E_2 + 6c_0\beta\xi E_1 - c_0^2\beta\xi^2 + c_0^3\beta\xi^2 - 2c_0c_1\beta\xi + 2c_0\beta - 2c_0\beta\xi + 2c_0\beta\xi^2\,, \tag{365}$$
$$[D_{-2,0}, D_{2,1}] = -4c_1\beta + 4c_0^2\beta\xi - 2c_0\beta\xi\,, \tag{366}$$
$$[D_{-2,1}, D_{2,0}] = -4c_1\beta + 4c_0^2\beta\xi - 2c_0\beta\xi\,, \tag{367}$$
$$[D_{-2,0}, D_{2,0}] = 2c_0\beta\,. \tag{368}$$

It gives

$$[L_2, L_{-2}] = \frac{4}{2\beta}E_2 + \frac{1}{2\beta}\left\{-c_0^3\xi^2 + c_0 - c_0\xi + c_0\xi^2\right\} . \tag{369}$$

After identifying $L_0 = \frac{1}{2\beta}E_2$, we can identify the Virasoro central charge (217).

J Calculation of the $U(1)$ and Virasoro constraints

Calculation for $U(1)$ The formulas offered in appendix F enable us to calculate the following,

$$\frac{\langle \vec{a}+\nu\vec{e}, \vec{Y}|J_1 V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle}{\langle \vec{a}+\nu\vec{e}, \vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle} = (-\sqrt{\beta})^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p+1}\frac{\langle \vec{a}+\nu\vec{e}, \vec{Y}^{(k,+),p}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle}{\langle \vec{a}+\nu\vec{e}, \vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle}\Lambda_p^{(k,+)}(\vec{a}+\nu\vec{e}, \vec{Y})$$
$$= (-\sqrt{\beta})^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p+1}\left\{\prod_{q=1}^{M}\frac{\tilde{g}_{Y_p^{(k,+)}W_q}(a_p-b_q-\mu)}{\tilde{g}_{Y_pW_q}(a_p-b_q-\mu)} \times \left(\prod_{q\neq p}^{N}\frac{\tilde{g}_{Y_pY_q}(a_p-a_q)\tilde{g}_{Y_qY_p}(a_q-a_p)}{\tilde{g}_{Y_p^{(k,+)}Y_q}(a_p-a_q)\tilde{g}_{Y_qY_p^{(k,+)}}(a_q-a_p)}\right)^{1/2}\right.$$
$$\left.\times\left(\frac{\tilde{g}_{Y,Y}(0)}{\tilde{g}_{Y^{(k+)}Y^{(k+)}}}\right)^{1/2} \times \Lambda_p^{(k,+)}(\vec{a}+\nu\vec{e}, \vec{Y})\right\}$$
$$= \beta^{-1}\sum_{p=1}^{N}\sum_{k=1}^{f_p+1}\left\{\frac{\prod_{q=1}^{N}\prod_{\ell=1}^{\tilde{f}_q+1}(a_p-b_q-\mu+A_k(Y_p)-A_\ell(W_q)-\xi)}{\prod_{q=1}^{N}\prod_{\ell=1}^{\tilde{f}_q}(a_p-b_q-\mu+A_k(Y_p)-B_\ell(W_q))} \times \frac{\prod_{q=1}^{N}\prod_{\ell=1}^{\tilde{f}_q}(a_p-a_q+A_k(Y_p)-B_\ell(Y_q)+\xi)}{\prod_{q=1}^{N}\prod_{\ell=1}^{\prime f_q+1}(a_p-a_q+A_k(Y_p)-A_\ell(Y_q)}\right\}. \tag{370}$$

In the first equivalence we use the action of J_{-1} on the basis, and in the second equivalence we use the definition of Nekrasov partition function.While in the third equivalence, the total plus sign comes from the sign choice inside the square root, in accordance with $\beta = 1$ case.

Similarly, the variation on the ket side is evaluated as below,

$$\frac{\langle \vec{a}+\nu\vec{e}, \vec{Y}|V(1)J_1|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle}{\langle \vec{a}+\nu\vec{e}, \vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e}, \vec{W}\rangle}$$
$$= -\beta^{-1}\sum_{q=1}^{N}\sum_{\ell=1}^{\tilde{f}_p}\left\{\prod_{p=1}^{N}\frac{\prod_{k=1}^{f_p}(b_q+\mu-a_p+B_\ell(W_q)-B_k(Y_p)+\xi)}{\prod_{k=1}^{f_p+1}(b_q+\mu-a_p+B_\ell(W_q)-A_k(Y_p))} \times \prod_{p=1}^{N}\frac{\prod_{k=1}^{\tilde{f}_q+1}(b_q-b_p+B_\ell(W_q)-A_k(W_p)-\xi)}{\prod_{k=1}^{\prime f_q}(b_q-b_p+B_\ell(W_q)-B_k(W_p))}\right\}. \tag{371}$$

Then, by setting,

$$a_p+\nu+A_k(Y_p),\quad b_q+\nu+\mu+B_\ell(W_q)\equiv x_I,\quad a_p+\nu+B_k(Y_p)-\xi,\quad b_q+\nu+\mu+A_\ell(W_q)+\xi\equiv -y_J\,, \tag{372}$$

we find that the difference of the above two equations satisfy the formula(182) we mentioned in section 3, thus

$$\begin{aligned}&\frac{\langle\vec{a}+\nu\vec{e},\vec{Y}|J_1V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle-\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)J_1|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}{\langle\vec{a}+\nu\vec{e},\vec{Y}|V(1)|\vec{b}+(\xi+\nu+\mu)\vec{e},\vec{W}\rangle}\\&=\beta^{-1}\sum_{I=1}^{\mathcal{N}}\frac{\prod_{J=1}^{\mathcal{N}}(x_I-y_J)}{\prod_J'(x_I-x_J)}=\beta^{-1}\sum_{I=1}^{\mathcal{N}}(x_I+y_I)=\beta^{-1}\sum_{p=1}^{N}(a_p-b_p-\mu-\xi)\,.\end{aligned} \tag{373}$$

The J_{-1} and $L_{\pm1}$ cases can be illustrated in parallel.

Calculation for L_1 The quantity to be evaluated is,

$$\sum_{I=1}^{\mathcal{N}}x_I\frac{\prod_{J=1}^{\mathcal{N}}(x_I+y_J)}{\prod_{J\neq I}^{\mathcal{N}}(x_I-x_J)}=\sum_{I<J}^{\mathcal{N}}y_Iy_J+\sum_{I,J}^{\mathcal{N}}y_Ix_J+\sum_{I<J}^{\mathcal{N}}x_Ix_J+\sum_{I}^{\mathcal{N}}x_I^2\,. \tag{374}$$

We rewrite it explicitly (here t, u are the conterparts of r s in the Young diagram W),

$$\begin{aligned}\sum_{I,J}^{\mathcal{N}}y_Ix_J=&\sum_p^N\sum_q^N\sum_k^{f_p}\sum_l^{f_q+1}-(a_p-\xi+\nu-s_k^{(p)}+\beta r_k^{(p)})(a_q-\xi+\nu-s_l^{(q)}+\beta r_{l-1}^{(q)})\\&+\sum_p^N\sum_q^N\sum_k^{f_p}\sum_l^{\bar f_q}-(a_p-\xi+\nu-s_k^{(p)}+\beta r_k^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_l^{(q)})\\&+\sum_p^N\sum_q^N\sum_k^{\bar f_p+1}\sum_l^{f_q+1}-(b_p+\nu+\mu-u_k^{(p)}+\beta t_{k-1}^{(p)})(a_q-\xi+\nu-s_l^{(q)}+\beta r_{l-1}^{(q)})\\&+\sum_p^N\sum_q^N\sum_k^{\bar f_p+1}\sum_l^{\bar f_q}-(b_p+\nu+\mu-u_k^{(p)}+\beta t_{k-1}^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_l^{(q)})\,,\end{aligned} \tag{375}$$

$$\begin{aligned}\sum_{I<J}^{\mathcal{N}}x_Ix_J=&\sum_{p<q}^N\sum_k^{f_p+1}\sum_l^{f_q+1}(a_p-\xi+\nu-s_k^{(p)}+\beta r_{k-1}^{(p)})(a_q-\xi+\nu-s_l^{(q)}+\beta r_{l-1}^{(q)})\\&+\sum_p^N\sum_{k<l}^{f_p+1}(a_p-\xi+\nu-s_k^{(p)}+\beta r_{k-1}^{(p)})(a_p-\xi+\nu-s_l^{(p)}+\beta r_{l-1}^{(p)})\\&+\sum_p^N\sum_q^N\sum_k^{f_p+1}\sum_l^{\bar f_q}(a_p-\xi+\nu-s_k^{(p)}+\beta r_{k-1}^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_l^{(q)})\\&+\sum_{p<q}^N\sum_k^{\bar f_p}\sum_l^{\bar f_q}(b_p+\nu+\mu-u_k^{(p)}+\beta t_k^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_l^{(q)})\\&+\sum_p^N\sum_{k<l}^{\bar f_p}(b_p+\nu+\mu-u_k^{(p)}+\beta t_k^{(p)})(b_p+\nu+\mu-u_l^{(p)}+\beta t_l^{(p)})\,,\end{aligned} \tag{376}$$

$$\begin{aligned}
\sum_{I<J}^{\mathcal{N}} y_I y_J =& \sum_{p<q}^{N}\sum_{k}^{f_p}\sum_{l}^{f_q}(a_p-\xi+\nu-s_k^{(p)}+\beta r_k^{(p)})(a_q-\xi+\nu-s_l^{(q)}+\beta r_l^{(q)})\\
&+\sum_{p}^{N}\sum_{k<l}^{f_p}(a_p-\xi+\nu-s_k^{(p)}+\beta r_k^{(p)})(a_p-\xi+\nu-s_l^{(p)}+\beta r_l^{(p)})\\
&+\sum_{p}^{N}\sum_{q}^{N}\sum_{k}^{f_p}\sum_{l}^{\bar{f}_q+1}(a_p-\xi+\nu-s_k^{(p)}+\beta r_k^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_{l-1}^{(q)})\\
&+\sum_{p<q}^{N}\sum_{k}^{\bar{f}_p+1}\sum_{l}^{\bar{f}_q+1}(b_p+\nu+\mu-u_k^{(p)}+\beta t_{k-1}^{(p)})(b_q+\nu+\mu-u_l^{(q)}+\beta t_{l-1}^{(q)})\\
&+\sum_{p}^{N}\sum_{k<l}^{\bar{f}_p+1}(b_p+\nu+\mu-u_k^{(p)}+\beta t_{k-1}^{(p)})(b_p+\nu+\mu-u_l^{(p)}+\beta t_{l-1}^{(p)})\,,
\end{aligned} \tag{377}$$

$$\sum_{I}^{\mathcal{N}} x_I^2 = \sum_{p}^{N}\sum_{k}^{f_p+1}(a_p-\xi+\nu-s_k^{(p)}+\beta r_{k-1}^{(p)})^2+\sum_{p}^{N}\sum_{k}^{\bar{f}_p}(b_p+\nu+\mu-u_k^{(p)}+\beta t_k^{(p)})^2\,. \tag{378}$$

Sum the above four equations together, we find most of the cross terms cancel with each other, and the remaining is

$$\begin{aligned}
&\sum_{p}^{N}(a_p-\xi+\nu)^2+\sum_{p<q}^{N}(a_p-\xi+\nu)(a_q-\xi+\nu)-\sum_{p,q}^{N}(a_p-\xi+\nu)(b_q+\nu+\mu)\\
&+\sum_{p<q}^{N}(b_p+\nu+\mu)(b_q+\nu+\mu)+\sum_{p}^{N}\sum_{k}^{f_p}s_k^{(p)}(\beta r_k^{(p)}-\beta r_{k-1}^{(p)})-\sum_{p}^{N}\sum_{k}^{\bar{f}_p}u_k^{(p)}(\beta t_k^{(p)}-\beta t_{k-1}^{(p)})\\
&=\frac{1}{2}\sum_{p=1}^{N}(a_p+\nu-\xi)^2-\frac{1}{2}\sum_{p=1}^{N}(b_p+\nu+\mu)^2+\frac{1}{2}\Big(\sum_{p=1}^{N}(a_p-\xi-b_p-\mu)\Big)^2+\beta|\vec{Y}|-\beta|\vec{W}|\,,
\end{aligned} \tag{379}$$

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