Aspects of symmetries throughout the String Theory Moduli Space

Roberta Angius

Programa de Doctorado en Física Teórica

Instituto de Física Teórica UAM-CSIC

Departamento de Física Teórica

Facultad de Ciencias

Madrid, 2025

Memoria de Tesis Doctoral realizada por

Roberta Angius

presentada ante el Departamento de Física Teórica
de la Universidad Autónoma de Madrid
para optar al Título de Doctor en Física Teórica

Tesis Doctoral dirigida por **Dr. Ángel M. Uranga Urteaga**,
Profesor de Investigación del Instituto de Física Teórica UAM-CSIC

The following papers were published by the candidate while this thesis was being developed:

1. *End of the world boundaries for chiral quantum gravity theories*,
   R. Angius, Angel M. Uranga, Chuying Wang
   JHEP 03 (2025) 064 arXiv:2410.07322 - Inspire

2. *End of the world brane networks for infinite distance limits in CY moduli space*,
   R. Angius
   JHEP 09 (2024) 178 arXiv:2404.14486 - Inspire

3. *Topological defects in K3 sigma models*,
   R. Angius, Stefano Giaccari, Roberto Volpato
   JHEP 07 (2024) 111 arXiv:2402.08719 - Inspire

4. *Intersecting end of the world branes*,
   R. Angius, Andriana Makridou, Angel M. Uranga
   JHEP 03 (2023) 110 arXiv:2312.16286 - Inspire

5. *Small black hole explosions*,
   R. Angius, Jesús Huertas, Angel M. Uranga
   JHEP 06 (2023) 070 arXiv:2303.15903 - Inspire

6. *Dynamical Cobordism and the beginning of time: supercritical strings and tachyon condensation*,
   R. Angius, Matilda Delgado, Angel M. Uranga
   JHEP 08 (2022) 285 arXiv:2207.13108 - Inspire

7. *At the end of the world: Local Dynamical Cobordism*,
   R. Angius, José Calderón-Infante, Matilda Delgado, Jesús Huertas, Angel M. Uranga
   JHEP 06 (2022) 142 arXiv:2203.11240 - Inspire

The following articles were announced on arXiv by the candidate while this thesis was being developed and are currently under review for official publication in specialized journals:

8. *Relative Quantum Gravity: Localized Gravity and the Swampland*,
   Edoardo Anastasi, **R. Angius**, Jesús Huertas, Angel M. Uranga, Chuying Wang
   arXiv:2501.03310 - Inspire

9. *Non-invertible defects from the Conway SCFT to K3 sigma models I: general results*,
   R. Angius, Stefano Giaccari, Sarah M. Harrison, Roberto Volpato
   arXiv:2504.18619 - Inspire

Abstract

Presented as a compendium of articles, this thesis aims to explore various corners of the string theory moduli spaces, using *symmetries* as a guiding principle to uncover properties of the corresponding theories and their implications for spacetime dynamics.

Symmetries have long been a foundational principle in theoretical physics, offering profound insights in our understanding from classical mechanics to quantum field theories and, more recently, quantum gravity. Within the framework of the Swampland Program, the absence of global symmetries in any consistent quantum gravity theory stands as one of its most widely accepted conjecture, from which numerous constrains follow. A powerful reformulation of this principle, in terms of topological constraints on allowed compactification backgrounds in string theory, leads to the Cobordism Conjecture. This conjecture states that any consistent quantum gravity configuration must allow for a spacetime boundary, thereby predicting, when additional cobordism charges are present, the existence of stringy defects capable of trivializing these charges and enabling spacetime to end. The first five articles of this thesis focus on implementing this conjecture at the level of effective field theory via *Dynamical Cobordism* solutions. These configurations realize codimension-1 end of the world (ETW) boundaries as singularities at finite spacetime distance at which scalars diverge toward infinite field space distance. Such solutions provide a concrete probe of infinite distance limits in field/moduli spaces and offer a concrete framework to analyze associated Swampland constraints, such as the Distance Conjecture. We construct explicit examples involving one or multiple scalars diverging simultaneously, allowing the exploration of different regions of the moduli space through single ETW branes or networks of intersecting ETW branes. For instance, we use these tools to investigate the network of infinite distance singularities in the complex structure moduli space of Calabi-Yau fourfolds flux compactifications in M-theory. These singularities are described in terms of intersecting normal divisors, classified by asymptotic Hodge theory, and are realized in spacetime via intersecting ETW branes labeled by specific critical exponents encapsulating the relevant information of the mathematical classification.

In recent decades, the concept of symmetry has evolved beyond traditional group-theoretic frameworks to encompass generalized symmetries. These are characterized by topological operators supported on submanifolds of various codimensions, which may exhibit non-invertible fusion rules. In the final article of this thesis, we investigate the category of topological defects commuting with the spectral flow and the whole $\mathcal{N} = (4\ 4)$ superconformal symmetry in two dimensional *non-linear sigma models on K3*, across their moduli space. By studying their fusion with boundary states, we argue that while for certain K3 models infinitely many simple defects, and even a continuum, can occur, at generic points in the moduli space the category is actually trivial.

After a brief introduction and motivation for string theory, we present a summary of key aspects of the Swampland Program, which serves as a motivating thread throughout this work. This is followed by an overview of the string theory moduli space, with particular focus on the moduli space of K3 models. The thesis concludes with a summary of the main results obtained in the included articles and a discussion of possible future directions.

Resumen

Presentada como un compendio de artículos, esta tesis tiene como objetivo explorar diversas regiones del espacio de módulos de la teoría de cuerdas, utilizando las *simetrías* como principio rector para descubrir propiedades de las teorías correspondientes y sus implicaciones para la dinámica del espaciotiempo.

Las simetrías han sido durante mucho tiempo un principio fundamental en la física teórica, proporcionando ideas profundas para nuestra comprensión, desde la mecánica clásica hasta las teorías cuánticas de campos y, más recientemente, la gravedad cuántica. En el marco del Programa Swampland, la ausencia de simetrías globales en cualquier teoría consistente de gravedad cuántica constituye una de sus conjeturas más ampliamente aceptadas, de la cual se derivan numerosas restricciones. Una reformulación poderosa de este principio, en términos de restricciones topológicas sobre los espacios de compactificación permitidos en teoría de cuerdas, conduce a la Conjetura de Cobordismo. Esta conjetura afirma que cualquier configuración consistente de gravedad cuántica debe admitir un borde espaciotemporal, prediciendo efectivamente la existencia de defectos cuerdis que trivialicen posibles cargas adicionales de cobordismo. Los primeros cinco artículos de esta tesis se centran en la implementación de esta conjetura a nivel de teoría efectiva de campos mediante soluciones de *Cobordismo Dinámico*. Estas configuraciones realizan branas de fin del mundo (ETW) de codimensión uno como singularidades a distancia finita en el espaciotiempo, donde los escalares divergen hacia distancia infinita en el espacio de campos. Estas soluciones constituyen sondas concretas de los límites a distancia infinita en los espacios de campos o de módulos, y ofrecen un marco útil para analizar las restricciones del Swampland asociadas, como la Conjetura de la Distancia. Construimos ejemplos explícitos en los que uno o varios escalares divergen simultáneamente, lo que permite explorar distintas regiones del espacio de módulos mediante branas ETW individuales o redes de branas ETW que se intersecan. Como aplicación, utilizamos estas herramientas para estudiar la red de singularidades a distancia infinita en el espacio de módulos de estructuras complejas de compactificaciones con flujo de cuatro-forma en teoría M sobre variedades Calabi-Yau de cuatro dimensiones. Estas singularidades se describen mediante divisores normales que se intersecan, clasificados por la teoría de Hodge asintótica, y se realizan en el espaciotiempo como redes de branas ETW intersecantes etiquetadas por exponentes críticos específicos que encapsulan la información relevante de dicha clasificación matemática.

En las últimas décadas, el concepto de simetría ha evolucionado más allá de los marcos tradicionales de la teoría de grupos, abarcando las llamadas simetrías generalizadas. Estas se caracterizan por operadores topológicos soportados sobre subvariedades de diversas codimensiones, que pueden presentar reglas de fusión no invertibles. En el artículo final de esta tesis, investigamos la categoría de defectos topológicos que conmutan con el flujo espectral y con toda la simetría superconforme $\mathcal{N} = (4\ 4)$ en *modelos sigma no lineales bidimensionales sobre K3*, a lo largo de su espacio de módulos. Al estudiar su fusión con estados de frontera, argumentamos que, si bien en ciertos modelos de K3 pueden aparecer infinitos defectos simples, e incluso un continuo, en puntos genéricos del espacio de módulos la categoría resulta ser trivial.

Después de una breve introducción y motivación sobre la teoría de cuerdas, se presenta un resumen de los aspectos clave del Programa Swampland, que sirve de hilo conductor para gran parte de este trabajo. A continuación, se ofrece una visión general del espacio de módulos en teoría de cuerdas, con un enfoque particular en el espacio de módulos de los

modelos de K3. La tesis concluye con un resumen de los principales resultados obtenidos en los artículos incluidos, así como una discusión de posibles direcciones futuras.

A Guglielmo e Tommaso,
per non avermi mai lasciato

Acknowledgments

First and foremost, I want to profoundly thank my supervisor Ángel, without whom none of this would have been possible. I thank him for all the physics he has taught me, for all the ideas he has shared, and for the passion with which he transmitted them. I am grateful for every discussion, every correction, and every time he pointed me in the right direction. I thank him for the example he set, as a researcher, as a teacher, and as a person. I am truly grateful to him for giving me the opportunity to initiate this path, for his patience, for supporting and encouraging me throughout the entire process, for trusting me and believing in me at a time in my life when I had stopped believing in myself. It has been an honor to have him as my supervisor, and I hope that one day I can become, at least in part, like him.

I want to sincerely thank Roberto, who was my first mentor and model in the world of physics research, and who continues to be a constant point of reference and an example of how to approach physics. I thank him for continuing to teach me physics and mathematics with patience, passion and absolute precision, and for sharing projects and ideas with me throughout these years.

I would like to thank all my collaborators, Andriana, Anthony, Chuying, Edoardo, Jesús, José, Matilda, Sarah, Sergio and Stefano, for all the discussions and all their ideas, for all the times we disagreed and they made me angry, for all the physics they taught me, but above all, for working together with me.

I also want to thank all the members of the string theory group at the IFT, for creating such a wonderful and stimulating working environment, and for all their comments and discussions during the journal clubs, which inspired my curiosity toward new horizons I would not have imagined on my own.

Finally, I would like to thank Dieter Lüst and the Max Planck Institute in Munich, as well as Irene Valenzuela and the CERN Theory Department for their kind hospitality during part of my PhD.

Ai miei genitori, le mie stelle nel buio di questa vita.

"Ha la mente già tesa all'impresa sull'oceano profondo
Caravelle e una ciurma ha concesso, per quel viaggio tremendo
Per cercare di un mondo lontano ed incerto che non sa se ci sia
Ma è già l'alba e sul molo l'abbraccia una raffica di nostalgia

E naviga, naviga via
Verso un mondo impensabile ancora da ogni teoria
Naviga, naviga via
Nel suo cuore la Niña, la Pinta e la Santa Maria

È da un mese che naviga a vuoto quell'Atlantico amaro
Ma continua a puntare l'ignoto con lo sguardo corsaro
Sarà forse un'assurda battaglia, ma ignorare non puoi
Che l'Assurdo ci sfida per spingerci ad essere fieri di noi."
[*Cristoforo Colombo*] - F. Guccini

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1
General Introduction: Motivations and Context

One of the central challenges of modern physics is to bridge the gap between *General Relativity*, which describes gravity and the large-scale structure of spacetime, and *Quantum Mechanics*, which governs the behavior of particles and the microscopic world. This unification demands a theoretical framework capable of treating gravity at the quantum level. Currently, *String Theory* represents the most promising candidate to provide such a description.

Over the years, a number of different approaches have been pursued to investigate the underlying structure of a quantum gravity (QG) theory. On the one hand, we have a microscopic approach that explores the quantum structure of spacetime and matter by studying its key ingredients, such as worldsheet quantization, string compactification and exact computations of non-perturbative contributions. On the other hand, we have a completely orthogonal approach that tries to extract information from the low energy theory in order to isolate underlying principles that would be the basis of its quantum gravity completion. At low energies, whatever quantum gravity is, it should look like a gravitational effective field theory (EFT). To consider all the possible consistent EFTs and study the conditions they must satisfy in order to admit a consistent QG completion in the UV is the main goal of the *Swampland program* [1]. Many quantum gravity constraints investigate the behavior of the theory across its moduli space, the set of vacua parameterized by the vacuum expectation values (VEVs) of its scalar fields.

Despite the apparent differences in frameworks and techniques, the two approaches ultimately pursue the same goal: achieving a complete description of quantum gravity and identifying its fundamental rules. It is therefore not surprising that they are governed by the same underlying foundations. Among these, *symmetries* play a central role. The core objective of this work is to explore aspects of this principle within both approaches. In particular, the guiding idea is the one to use symmetry as a tool to probe distinct regions of the string theory moduli space. In the microscopic approach, we study the emergence of, standard and generalized, symmetries throughout the bulk of the moduli space of two-dimensional non-linear sigma models on 3 which arises as the worldsheet description of perturbative type II string theory compactified on a K3 surface. In the bottom-up approach, we examine how the absence of exact global symmetries in quantum gravity, more precisely through its reformulation in terms of the Cobordism Conjecture that restricts the admissible compactification backgrounds in string theory, require the existence of special loci in the corresponding moduli space where (certain) internal directions degenerate. In particular, we provide explicit realizations of these loci through the construction of spacetime boundaries via special classes of solutions of the corresponding effective field theories that implement Dynamical Cobordisms. Such solutions provide concrete probes of infinite distance limits in the field/moduli spaces of the low-energy effective actions obtained by flux compactification in string theory.

To make the manuscript self-contained, the remainder of this chapter offers a concise introduction to the Swampland Program, with a main emphasize on the Cobordism Conjecture, and a brief overview of Calabi–Yau moduli spaces, with particular focus on K3 surfaces. This material provides the minimal background for the reader to follow the articles in this compendium.

1.1 The Swampland Program

Our current description of fundamental physics is based on two complementary theories. On one hand, the *Standard Model* (SM) unifies the strong, electroweak, and Higgs-mediated interactions under the gauge group

$$= \quad (3) \quad\quad (2) \quad (1) \tag{1.1}$$

On the other hand, *General Relativity* (GR) provides a classical description of gravitational interaction as manifestation of the geometry of the spacetime. The relation between the gravity and the rest of the Universe is given by the action:

$$= \quad \frac{}{} \quad \frac{}{2} + \mathcal{L} \tag{1.2}$$

where is the reduced Plank mass, is the determinant of the space-time metric, is the Ricci-scalar and $\mathcal{L}$ contain all information on the remaining Universe but the gravity. In order to achieve a unified treatment of all fundamental interactions, this classical theory must be quantized. Since string theory emerges as the most viable candidate for this purpose, we will henceforth refer to it as the Quantum Gravity (QG) theory in these notes, unless explicitly stated otherwise.
One of the main features of string theory is the presence of extra dimensions beyond the four spacetime dimensions observed in General Relativity and Quantum Field Theory. In particular, consistency with Lorentz-invariance requires the bosonic string to propagate in 26 dimensions, while superstring theory is defined in 10 spacetime dimensions. The connection with observable four-dimensional physics is achieved through the compactification of the extra dimensions on an internal manifold, whose geometry can become increasingly intricate

$$\mathcal{M} = \mathcal{M} \quad \mathcal{M}$$

Naturally, different choices of the internal manifold $\mathcal{M}$ lead to distinct low-energy phenomenologies in the non-compact spacetime dimensions which constitute the string theory *Landscape*. Imposing constraints on the admissible internal geometries that yield realistic low-energy phenomenology has long been one of the major challenges for string theorists. This question also lies at the heart of the top-down motivations behind the Swampland Program. Low energy EFTs coming from these compactifications feature scalar fields whose vacuum expectation values (VEVs) parameterize a moduli space, a manifold representing the set of possible vacua. Even when a scalar potential is present, as long as it remains below the EFT's cutoff, the concept of moduli space remains meaningful. This moduli space plays a crucial role in understanding the behavior of the theory across different vacua, and many Swampland conjectures impose constraints on the behavior of theories as one moves through it [2].

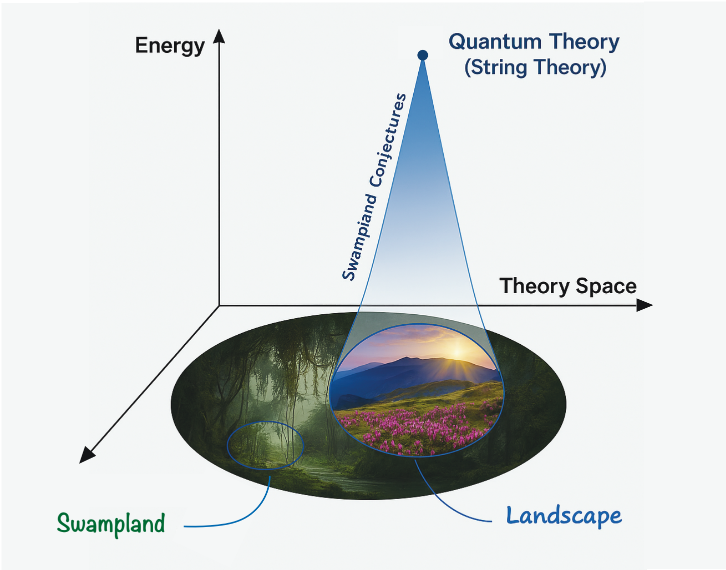

Figure 1.1: The Swampland and the Landscape of EFTs: the boundary between these two regions becomes increasingly restrictive at higher energies. In the extreme ultraviolet limit, one expects a unique theory of quantum gravity.

Now, let us examine the same situation from the perspective of these EFTs. Consider EFTs that are weakly coupled to gravity and appear to be self-consistent. By self-consistent, we mean that the theory is anomaly free, for instance it does not exhibit anomaly when a global symmetry is gauged by the coupling with a dynamical gauge field, nor does it display gravitational anomalies by the coupling with gravity. Starting from this setup, and interpreting these EFTs as potential low-energy limits of a complete theory of QG, we can formulate the following two questions:

(1) *Is it always possible to find a UV completion for such theories that yields to a consistent theory of quantum gravity?*

(2) *What are the conditions that these theories must satisfy in order to admit such a consistent completion?*

Providing answers to these two questions and especially formulating the criteria relevant to the second, is the central objective of the *Swampland Program* [1]. The name derives from the fact that the answer to the first question is negative: not all EFTs weakly coupled to gravity at low energies can be consistently completed into a quantum gravity theory in the UV. Specifically, those that do admit such a completion are said to belong to the *Landscape*, while the remaining theories lie in the *Swampland*. The criteria posed by the second question aim to delineate the boundary between these two regions. Ideally, they should be formulated using only the properties of the low energy theory, without reference to the UV completion. Finding such criteria is challenging, and in general, we do not have a proof of them from microscopic physics. For this reason, they are formulated as conjectures, known as the *Swampland conjectures*. The expectation is that these conditions

become progressively weaker, and eventually disappear in the limit $M_P \mapsto \infty$, where gravity decouples, see Figure 1.1. Based on this requirement, and the general preference in physics for continuity, we anticipate that as the energy approaches the Planck scale, the Swampland constraints become increasingly stringent. Consequently, the boundary of the Swampland is not only scale-dependent, but also becomes more restrictive at higher energies. In the extreme limit, one expects a unique quantum gravity theory. This idea is closely related to the still open question on *String Universality*, which concerns the existence of a unique quantum gravity theory in the UV.
An important point to emphasize for people approaching the subject for the first time concerns the methods used to identify these criteria. One approach involves starting with string theory and analyzing a huge number of its vacua to check whether they satisfy specific patterns. In general we distinguish between two classes of vacua: *string derived vacua*, which have a well-defined world-sheet description (such as orbifold geometries), and s*string inspired vacua*, which cannot be rigorously constructed from string theory but they own additional ingredients. The first class provides a weak test for these conjectures, while the second offers a stronger evidence. Another method involves using quantum gravity arguments directly on the low energy theory. The study of black holes being from these theories is a key component of this approach. Finally, the third method draws from the study of microscopic physics, offering insights based on fundamental principles.

Although the set of conjectures formulated so far is large (a subset of them is shown in Figure 1.2), and despite many of them initially seeming unrelated, a significant web of connections has emerged. This suggests that they may be different formulations of a few, perhaps even a single, fundamental quantum gravity principle that leaves its imprint in the infrared.

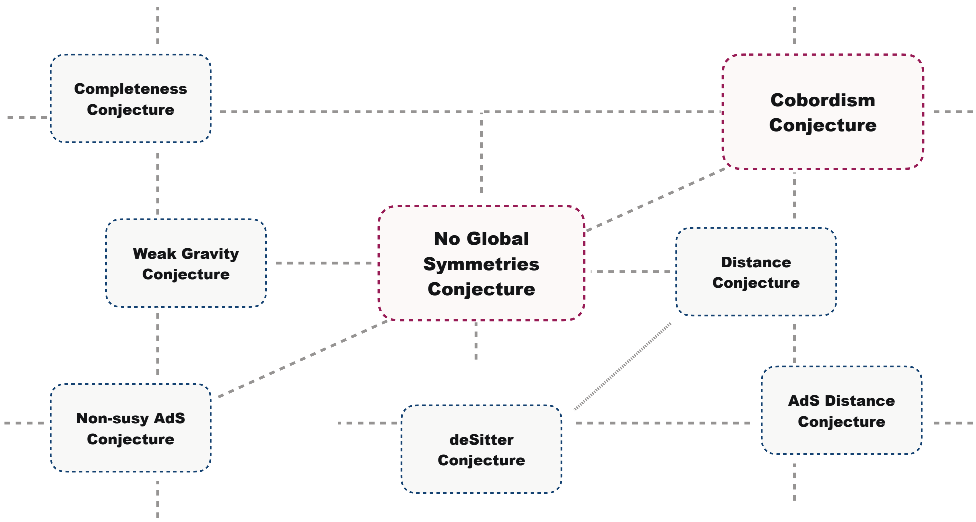

Figure 1.2: Web of Swampland Conjectures

We refer the reader to the literature, see [3–5] for a detailed and extensive introduction to the main conjectures that populate this network. In the following two subsections, we will provide a review of only two of them, the most relevant for the subsequent articles.

1.1.1 The No-Global Symmetry conjecture

Symmetries are fundamental tools to read the physics governing natural phenomena. Under a symmetry transformation, a physical theory remains invariant, and its ingredients, such as fields and operators, are organized into specific representations of the symmetry group In the standard language of quantum field theory, symmetries are generally divided into two categories:

- *Global symmetries* are transformations whose parameters do not depend on the spacetime position. They can be further divided in two classes: we have *internal* symmetries, which act on the internal degrees of freedom of the fields without affecting spacetime coordinates (examples include conservation of lepton number, baryon number, and flavor symmetries); and *spacetime* symmetries, which involve transformations of the spacetime coordinates themselves (for instance Lorentz transformations).
- *Local symmetries* are transformations whose parameters vary with the spacetime position. These include internal symmetries (gauge symmetries), which require the introduction of gauge fields to maintain invariance under local transformations, and spacetime symmetries (gravity).

A useful formalization of the concept of global symmetry in the modern language of quantum field theory employs the framework of topological operators.

Definition 1 (Global Symmetry) *Let $\mathcal{Q}$ be a QFT defined over a dimensional spacetime $\mathcal{M}$. A set of extended operators ( ), for each , supported on codimension-1 submanifolds of $\mathcal{M}$ defines a standard global symmetry on $\mathcal{Q}$ if:*

(i) They satisfy the group law of : ( ) ( ) = ( );

(ii) They map local operators to local operators: ( ) : $\mathcal{O}$( ) $\mathcal{O}$ ( );

(iii) They act non-trivially on the Hilbert space of the theory;

(iv) They are topological, meaning their correlation functions are invariant under continuous deformations of their support that do not cross other operator insertions.

The final condition ensures local energy conservation in the quantum field theory, implying that the stress-energy tensor remains invariant under the action of the symmetry.

Thus far, our discussion has focused on global symmetries within quantum field theories that exclude dynamical gravity, where numerous examples of such symmetries are well-established. The next step in our discussion is to understand the implications of introducing gravity into the framework. This problem has been considered for the first time in 1957 by Misner and Wheeler [6] and discussed in many papers [7–10] before to enter to be part of the web of swampland conjecture. The statement is the following.

No Global Symmetry Conjecture [7, 8]
A theory coupled to gravity can have no exact global symmetries.

As with the other Swampland conjectures, a complete proof of this statement remains elusive. Nonetheless, it stands as one of the most widely accepted conjectures, validated

by numerous compelling arguments. From perturbative string theory a motivation is that global symmetries on the worldsheet are always gauged in the target space [7]. Further evidence is furnished by considerations in black hole physics [8, 9, 11]. Consider a scenario where a particle carrying a global (1) charge is absorbed by an uncharged black hole. As the black hole undergoes Hawking radiation, it emits particles in a manner that is insensitive to global charges. Assuming charge conservation, when the evaporation terminates, we end with a charge remnant of Planckian size. Since the process holds for any arbitrarily large global charge and BHs in any representation, starting with a single charged particle we can end with an infinite spectrum of possible remnants corresponding to different charge values. Such an infinite set of remnants, which can be reasonably considered inconsistent [12], would violate the covariant entropy bound [13, 14], which limits the entropy, and thus the number of distinguishable states, within a given region of spacetime. This inconsistency of the EFT suggests that exact global symmetries cannot exist in a consistent theory of quantum gravity. However, if the symmetry is gauged, the associated gauge fields interact with the black hole, allowing the charge to be radiated away during evaporation. This process prevents the formation of problematic remnants and aligns with the entropy bounds, indicating that gauged symmetries are compatible with quantum gravity, whereas global symmetries are not. A potential criticism to the black hole argument against global symmetries is its apparent limitation to continuous global symmetries, seemingly excluding finite discrete ones. However, further evidence from the AdS/CFT correspondence suggests that even finite discrete global symmetries are incompatible with consistent quantum gravity theories. Specifically, Harlow and Ooguri [15,16] demonstrated, under certain assumptions, that any global symmetry, continuous or discrete, in the bulk would lead to inconsistencies in the boundary conformal field theory, implying that such symmetries cannot exist in the bulk of AdS/CFT frameworks.
Collectively, these considerations support the statement of the conjecture: while global charges may be acceptable within low-energy effective theories, they must be either broken or gauged when the theory is coupled to quantum gravity.

At this point, let us return to the realm of quantum field theories (QFTs), excluding gravity. The definition of symmetries in terms of topological operators has, over the past decades, driven research towards formulating a notion of generalized symmetries by relaxing properties *(i)-(iii)* outlined in Definition 1. In particular, this has led to defining symmetries as categories of topological defects supported on submanifolds of arbitrary codimension [17], which may obey non-invertible fusion rules [18–21] (see Section 2 of the sixth article of this compendium for an introduction to the topic).
Although there is substantial evidence for the presence of these symmetries in QFTs, an interesting question arises regarding the statement of the No Global Symmetry Conjecture in relation to them. The expectation is that they should also be gauged or broken when the energy increases and the coupling with gravity can no longer be neglected. However, despite significant progress in investigating this question, a complete picture is still lacking. A standard approach to thoroughly explore this question would involve obtaining a clear understanding of these generalized symmetries in the worldsheet conformal field theory (CFT) and then examining their counterparts in the target spacetime, as is done for standard symmetries. Despite considerable advancements in the discussion of these generalized symmetries, especially in two-dimensional rational CFTs where many general techniques have been developed [18, 22–30], the domain of non-rational theories, [31–33] remains largely unexplored. In the fifth article of this compendium, we focus on the initial

step of this exploration by initiating the study of the category of generalized symmetries in non-linear sigma models with a K3 target space that preserve the full $\mathcal{N} = (4\ \ 4)$ superconformal symmetry at central charge $\ = \ = 6$, and that are invariant under spectral flow transformations. K3 models offer the simplest examples of Calabi–Yau compactifications in type II string theory. However, the generic model is not a rational conformal field theory (CFT) and cannot be solved exactly. Nevertheless, due to the extensive spacetime and worldsheet supersymmetries, many general results about these models are known (such as the geometry of the moduli space, the elliptic genus, the spectrum, and the finite groups of symmetries at each point in the moduli space [34–36]) making them an ideal framework to initiate the exploration of topological defects in non-rational CFTs.

Although the No Global Symmetries Conjecture is a cornerstone of the Swampland program, its predictive power in bottom-up approaches is limited, as approximate global symmetries are allowed in the IR. To enhance its predictive power, we turn in the next section to its generalized reformulation in terms of the Cobordism Conjecture. We focus on this particular conjecture because it allows a direct connection with the exploration of the asymptotic limits in moduli space (or scalar field spaces in general). As one approaches infinite-distance limits in moduli space, certain scalar fields can traverse large distances, leading to the emergence of light towers of states, as described by the Swampland Distance Conjecture. These asymptotic regions often correspond to regimes where the theory admits a weakly coupled description, allowing for a bottom-up effective field theory (EFT) analysis. In this context, the interplay between the Cobordism Conjecture and the behavior of scalar fields in moduli space provides a framework to derive more stringent constraints on EFTs, enhancing the predictive capacity of the Swampland program: by examining the dynamics of scalar fields near the boundaries of moduli space, we can identify patterns and constraints that any consistent EFT must satisfy.

1.1.2 The Cobordism conjecture

A natural generalization of the No Global Symmetries Conjecture, discussed in the previous section, involves the absence of topological global charges. This leads to an immediate connection with the mathematical concept of cobordism, which establishes an equivalence relation among smooth manifolds that is more general than diffeomorphism.

Before to delve into the technical details, let us present two motivations for exploring this generalization beyond the Swampland program. First, string theory admits a vast landscape of vacua when compactified to lower dimensions. These vacua correspond to effective field theories (EFTs) that are weakly coupled to gravity, and the study of these vacua can help to identify selection rules for EFTs compatible with quantum gravity, offering a top-down perspective aligned with the Swampland program purpose. Moreover, from a phenomenological point of view, we aim to select vacua that yield realistic low-energy physics. Understanding the interconnections among different vacua could provide valuable insights to distinguish between realistic and non-realistic scenarios. Second, while we have several consistent string theory models, such as *Type I, Type IIA, Type IIB* and *Heterotic* for the superstring case, the aspiration is to uncover a unique, underlying theory of quantum gravity. The existence of dualities connecting these different string theories, such as T-duality between *Type IIA* and *Type IIB* in 9 dimensions, suggests that all the string theory models might be interconnected when compactified on fully compact spaces, leaving only the time direction non-compact.

To formalize this notion, we propose that any two distinct string theories should be connected in the following sense:

(1) They correspond to different regions in the moduli space of their compactifications;

(2) The energy required to transition from one region to another is finite.

From the perspective of the lower-dimensional theories, obtained upon compactification, this connection manifests as a *domain wall* linking two different vacua or EFTs with a finite energy cost. It is important to note that this requirement of connectedness is stronger than that implied by T-duality. Indeed, T-duality relates regions of moduli space that are infinitely distant from each other, suggesting an infinite energy cost for transitions between them.

The *Swampland Cobordism Conjecture* aims to formalize this concept of connectedness. We state that two quantum gravity theories in dimensions, $\mathcal{T}$ and $\mathcal{T}$, are equivalent if there exists a finite-energy domain wall connecting them. We denote this equivalence as $\mathcal{T}$ $\mathcal{T}$. Considering the set of such equivalence classes, we define the *Quantum Gravity Cobordism Group* in dimensions as

$$= \text{ d-dimensional QG theories}$$

Up to this point, we had not been concerned with whether the -dimensional theory arises from the compactification of a higher-dimensional one. Moreover, the set does not yet possess a group structure; to endow it with one, we must define an internal operation, an inverse element for each element, and an identity element. Before proceeding, let us motivate the term *cobordism* in this context by relating it to the concept of cobordism in mathematics. To understand the link we have to start considering -dimensional quantum gravity theories that result from compactifications of -dimensional string theories on -dimensional compact manifolds, such that $= +$. In lower dimensions, different -dimensional theories correspond to different choices of compactification manifolds $\mathcal{M}$. If we restrict our attention to smooth, compact manifolds, the previously defined equivalence classes under the relation align with the mathematical notion of cobordism equivalence classes . Specifically, we state the following definition.

Definition 2 (cobordism) *Two smooth, closed, unoriented manifolds $\mathcal{M}$ and $\mathcal{M}$ of real dimension are cobordant, i.e. $\mathcal{M}$ $\mathcal{M}$, if there exist a third smooth manifold of real dimension $+1$, such that:*

$$\mathcal{M} \quad \mathcal{M} = \tag{1.3}$$

The equivalence classes of manifolds equipped with this union operation form a group, called the *cobordism group*:

$$= \text{ Compact, closed k-dimensional manifolds}$$

For reasons of simplicity, in this definition of cobordism, we deal with smooth and unoriented manifolds. However, in physics, there are no restrictions against considering singular spaces or, more generally, non-geometrical backgrounds for compactifications. Therefore, while this basic notion of cobordism is enough to provide a basic understanding of how manifolds can be related via (smooth) deformations, it is insufficient when addressing manifolds endowed with additional structures, such as orientations. To accommodate such structures, the concept of tangential structures is necessary, see [37] for an introduction to the topic. These structures are formalized through maps from the manifold to supplementary spaces encoding the properties of the additional ingredients. An example of supplementary spaces are principal G-bundles, denoted as BG, associated to any Lie group G. In this context, a manifold equipped with a tangential structure corresponds to a map from the manifold to the appropriate supplementary space, encoding the additional geometric or topological data. Two such structured manifolds are considered cobordant if there exists a higher-dimensional manifold interpolating between them, along with a map extending the structures over this bordism.

Having highlighted the possibility to add these supplementary ingredients, we can now define the group structure and utilize it to provide a geometric perspective on the argument. Let Ω_d^{QG} be the set of classes of equivalent backgrounds for a $D = d + k$ dimensional String Theory.

- *Composition Law*
  For any pair of elements $[\mathcal{M}_1]$and $[\mathcal{M}_2] \in \Omega_d^{QG}$, we can associate a third element $[\mathcal{M}_1 \sqcup \mathcal{M}_2]$ such that:
  $$[\mathcal{M}_1] + [\mathcal{M}_2] = [\mathcal{M}_1 \sqcup \mathcal{M}_2] \tag{1.4}$$
  This operation is commutative due to the commutativity of the disjoint union, making Ω_k^{QG} an *Abelian* group. The equivalence class $[\mathcal{M}_1 \sqcup \mathcal{M}_2]$ may also contain a connected background, where "connectedness" refers to arcwise connectedness. This indicates that the group structure encapsulates the possibility of two separate backgrounds merging into a single connected background.

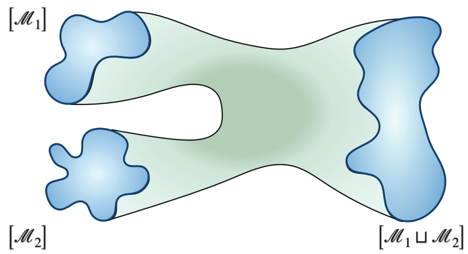

Figure 1.3: Geometric view of composition low. The central part of the picture represents the dynamically allowed processes on the D dimensional theory to connect different backgrounds. From the d dimensional perspective this process manifests as the Domain Wall.

- *Identity Element*
  The identity element in the group is represented by the class [0], satisfying
  $$[\mathcal{M}] + [0] = [0] + [\mathcal{M}] = [\mathcal{M}] \ , \quad \forall \ [\mathcal{M}] \in \Omega_d^{QG}$$
  The class [0] corresponds to an empty background. From a higher-dimensional perspective, it represents the class of compact k-dimensional manifolds that by itself are the boundary of an higher $(k + 1)$-dimensional manifold W:
  $$\mathcal{M} \in [0] \longleftrightarrow \mathcal{M} = \partial W.$$

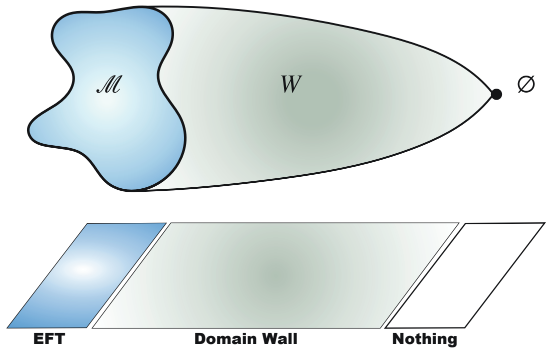

Figure 1.4: Cobordism to Nothing.

- *Inverse Element*
  For each element $[\mathcal{M}] \in \Omega_d^{QG}$ there exists an element $\left[\overline{\mathcal{M}}\right] \in \Omega_d^{QG}$, called the *inverse*, such that
  $$[\mathcal{M}] + \left[\overline{\mathcal{M}}\right] = [0]$$
  In the case of smooth orientable manifolds, the inverse element is the same manifold with the opposite orientation.

Now that we have summarized the fundamental notions of cobordism from both mathematical and physical perspectives, we can proceed to formulate the new conjecture. The underlying point is that different cobordism classes can be labeled by certain topological invariant quantities. For example, in the case of bordisms between orientable manifolds, the relevant characteristic classes are the Pontryagin and Stiefel–Whitney classes, which give rise to the Pontryagin and Stiefel–Whitney numbers. For manifolds endowed with internal fluxes F_k, there can be non-trivial cobordism classes characterized by the charge $\int_{\mathcal{M}_k} F_k$. Interpreting these invariants as topological global charges, and knowing that two manifolds are bordant (with respect to a given tangential structure) if and only if their corresponding characteristic numbers are the same, we arrive at the following conclusion. Since we believe that all theories of quantum gravity arising from string compactification are free from global symmetries (as posited by the No-Global-Symmetry Conjecture), we state [1]

Cobordism Conjecture [38]
Any consistent d-dimensional QG theory has to belong to the trivial cobordism class:
$$\Omega_d^{QG} = [0]$$

Now, suppose to consider a class of compactification backgrounds such that the corresponding cobordism group is non-trivial, namely $\Omega_d^{QG} \neq [0]$. Since we interpret this non-triviality as the presence of global symmetries, the backgrounds corresponding to these non-trivial classes give rise to EFTs in the Swampland. To cure the theories and rehabilitate them into the Landscape we have two possibilities:

[1] In sufficiently low dimensions global symmetries seems allowed, for example string worldsheets viewed as 2d quantum gravitational theories. In this case there exist same invariants to label different cobordism classes.

(i) *Breaking the symmetry*, by introducing charged objects in the complete theory capable of absorbing the extra cobordism charge and trivializing the corresponding backgrounds:

(1.5)

(ii) *Gauging the symmetry*, by coupling the corresponding global form symmetry with a gauge field (e.g. by adding Chern-Simons terms to the action). In this case, we correct the group by taking the quotient

(1.6)

All the preceding discussion of the cobordism conjecture has been framed at the topological level. A key insight is that, to construct realistic low-energy EFTs, we often require compactification backgrounds endowed with additional structures, such as fluxes, orientifold planes, or specific gauge bundles, to incorporate the necessary ingredients. These structures typically render the corresponding cobordism groups non-trivial. In this situation, the powerful prediction of the conjecture requires the existence of defects in the full complete theory able to trivialize the corresponding cobordism class (e.g., if we put RR fluxes in the compact manifolds, the conjecture predicts the existence of D-branes). One of the main goals of this thesis, to which the first four papers of this compendium are dedicated, has been the one to translate this topological description and its consequences in terms of predictivity of defects into the language of EFTs. The work, initiated by Buratti, Calderón-Infante, Delgado, and Uranga in their first works [39,40], expanded in many subsequent articles [41–52], marked the beginning of a novel research line in the Swampland Program, aiming to probe the implications of the Cobordism Conjecture in EFTs using Dynamical Cobordisms. The main implication of the Cobordism Conjecture exploited to develop this language has been the realization of the ETW configuration connecting a generic compactification background to the empty set [0] (see Figure 1.4). In particular, Dynamical Cobordisms represent effective realizations of these configurations through specific classes of solutions of the effective action that include spacetime boundaries. These solutions satisfy three main properties

(1) They are spacetime dependent solutions

(2) They show a Ricci singularity in the metric at finite distance in spacetime;

(3) They explore infinite distance regions of the corresponding scalar field space.

At the location of the singularity, the EFT predicts the presence of a codimension-1 defect in spacetime, playing the role of an End-of-the-World (ETW) brane. In the regime close to this ETW brane, Dynamical Cobordism solutions exhibit a universal behavior, allowing us to classify these objects in terms of a single real parameter that we call the *critical exponent*. Naturally, this classification remains valid only at the level of the EFT, which lacks sufficient information to determine the precise microscopic nature of the corresponding defect in the full UV theory. Nevertheless, it constitutes a robust prediction of its existence. A detailed introduction to the topic is presented in Section 2 of the first paper of this compendium, with generalizations to configurations involving multiple intersecting ETW branes discussed in the Section 3 of the third article. Crucially, property (3) indicates that these configurations explore corners of moduli or field space of the associated

EFT field theory located at infinite distance. This observation not only ties the framework to other Swampland conjectures, such as the Distance Conjecture and its convex hull formulation, but also shows that ETW brane solutions provide spacetime realizations of such infinite-distance loci. This motivates the second paper of the compendium, where small black holes act as ETW branes probing specific infinite-distance limits in the moduli space of Calabi–Yau threefolds. The third article extends this perspective by employing networks of intersecting ETW branes to explore infinite-distance limits of the moduli/field space where multiple scalars diverge simultaneously. In the fourth article these thecniques are applied to study the infinite-distance limits in the complex structure moduli space of flux compactifications on Calabi–Yau fourfolds. Finally, in the fifth article, we describe the construction of broad classes of explicit string theory backgrounds corresponding to 6d and 4d chiral theories with ETW boundaries.

1.2 The Calabi-Yau moduli space

To move beyond the bottom-up perspective adopted in the previous sections, it becomes necessary to turn to the full machinery of string theory. In particular, worldsheet conformal field theory (CFT) provides a powerful framework to probe not only the asymptotic limits but also the deep interior of moduli space, which is typically inaccessible with EFTs. A natural arena where this formalism applies is that of Calabi–Yau compactifications, where the internal geometry admits a worldsheet description as a two-dimensional non-linear sigma model (NLSM). The simplest non-trivial example to implement the study are K3 surfaces. Compactifying string theory on K3 provides a rich yet tractable setup, where the full moduli space, including regions far from weak coupling regimes, can be investigated using exact CFT techniques. The purpose of this section is to provide an introduction to geometrical aspects of the moduli space of Calabi-Yau manifolds with a main focus on 3 compactifications.

Compactification is required both to reconcile the critical dimension $= 10$ of superstring theory with the observed $= 4$ spacetime dimensions and to yield a viable low-energy description resembling the Standard Model. Let us consider the spacetime manifold to be of the form

$$\mathcal{M} = \qquad \mathcal{K} \tag{1.7}$$

Compactifying $\mathcal{N} = 1 \;\, 2$ supersymmetric theories on a torus $\mathcal{K} =$ preserves all supersymmetries, leading to $\mathcal{N} = 4 \;\, 8$ theories in four dimensions. This results from decomposing the Weyl spinor representation of $(1 \;\, 9)$ under

$$(1 \;\, 9) \qquad\qquad (1 \;\, 3) \qquad (6) \tag{1.8}$$

yielding

$$= (\quad) + (\quad \bar{} \;) \tag{1.9}$$

The large amount of supersymmetry makes toroidal compactifications of superstrings phenomenologically unrealistic. To match the supersymmetric extension of the Standard Model, we require $\mathcal{N} = 1$ in four dimensions, leading to the necessity of compactifications breaking more supersymmetry.
The requirement of unbroken supersymmetry after compactification implies the existence

of Killing spinors ( ), which parametrize the supersymmetry transformations and satisfy the Killing condition:

$$0 \quad 0 = 0 \tag{1.10}$$

where is the covariant derivative including the spin connection. This condition imposes both topological and differential constraints on the internal geometry. For compactifications to 4 dimensions, the manifold $\mathcal{K}$ must be Ricci-flat [53] (a necessary condition) and have a holonomy group contained in (3) [54](a sufficient condition). The decomposition of the spinor representations and ¯ of (6) under the holonomy group (3) tells us the number of singlet spinors that remain invariant under parallel transport. Since represents the set of possible transformations of a vector transported along a closed curve in $\mathcal{K}$, the Killing spinors, which must remain unchanged under parallel transport, are singlets under . If the holonomy group is exactly = (3), then the spinor representation decomposes as:

$$\tag{1.11}$$

indicating the presence of one covariantly constant spinor for each chirality. Starting from a ten-dimensional theory with $\mathcal{N} = 1$ supersymmetry, the decomposition of leads to a single (1 3) Majorana spinor in four dimensions, thereby preserving a single $\mathcal{N} = 1$ supersymmetry. If instead = (2), we have , indicating two covariantly constant spinors for each chirality, yielding $\mathcal{N} = 2$ supersymmetry in four dimensions. Powerful mathematical theorems guarantee the existence of large classes of manifolds admitting metrics with ( ) holonomy. These manifolds satisfy the so-called Calabi–Yau conditions and are therefore referred to as Calabi–Yau (CY) manifolds.

A Calabi–Yau -fold, denoted , is defined as a compact Kähler manifold of dimension 2 with vanishing first Chern class = 0. This condition is equivalent to the existence of a Ricci-flat Kähler metric, as established by Yau's proof of Calabi's conjecture. Such manifolds admit a nowhere-vanishing holomorphic (0)-form and possess a holonomy group contained within ( ), ensuring the presence of covariantly constant spinors. For = 1, the only Calabi–Yau onefold is the two-torus , for = 2, there are two main classes: and 3 surfaces (the only class with holonomy (2)). For 3, a vast landscape of topologically distinct Calabi–Yau manifolds emerges, making the study of their moduli spaces both rich and essential for string compactifications.

Kähler manifolds are complex manifolds whose Kähler form, obtained lowering one index of the complex structure, is closed:

$$= \quad - \quad = 0 \tag{1.12}$$

Starting from the pair ( ), where is a manifold and its Kähler form, we aim to construct the corresponding moduli space by studying deformations that preserve the Calabi–Yau conditions. There are two types of such deformations: those changing the complex structure of and those modifying the Kähler class, i.e. the cohomology class of . Yau's theorem guarantees that for a fixed complex structure and a given Kähler class, there exists a unique Ricci-flat Kähler metric on . Consequently, the space of deformations of the pair ( ) can be viewed as the parameter space of Ricci-flat Kähler metrics. The number of independent parameters in this space is determined in terms of the topological invariant Hodge numbers. In particular:

- *Kähler deformations* correspond to variation of the closed, non-exact (1 1) form . These deformations are parametrized by a number = of independent parameters, which, from a geometric point of view, control the size of the internal even-dimensional cycles, whose volumes are measured by integrating the 2 -form over them.

- *Complex structure deformations* correspond to the choice of a (1 1) form obtained by contracting the unique (0) form with the complex structure tensor . This means that we have = free parameters to fix this structure. From a geometric point of view they control the size of the internal cycles, whose volume is measured with .

Together, the Kähler and the complex structure parameters form the geometric moduli space of Calabi-Yau manifolds, which can be locally expressed as the product:

$$\mathcal{M} \quad = \mathcal{M}_{Kähler} \quad \mathcal{M} \tag{1.13}$$

Both factors of this expression are special Kähler manifolds. For an extended discussion about geometry and properties of $\mathcal{M}$ we refer the reader to the literature [53, 55–57]. In the Section 2 of the fourth article of this compendium we provide a review of the complex structure sector of the moduli space, $\mathcal{M}$,and its associated Hodge structure, particularly focusing on the characterization of its infinite distance limits.

We now examine the physical interpretation of these moduli within the lower- dimensional effective field theory resulting from Calabi–Yau compactifications. The spectrum of massless fields in this lower-dimensional theory is in one-to-one correspondence with the harmonic forms on the compact manifold, then it is related to its cohomology. Fluctuations of higher-dimensional fields, such as the metric, the gauge potentials, and the scalars, are expanded in a basis of these harmonic forms. The zero modes (harmonic representatives) persist as massless fields in the non-compact spacetime, with their multiplicities determined by the Hodge numbers of the Calabi–Yau manifold. Specifically, the geometric moduli arise from fluctuations of the internal metric.

From the worldsheet perspective, a Calabi–Yau compactification is described by a two-dimensional $\mathcal{N} = (2\ 2)$ nonlinear sigma model (NLSM) with central charge $= 2$ for a 2 -dimensional Calabi–Yau target space. This SCFT possesses two (1) R-symmetries under which operators are graded. Exactly marginal operators of the SCFT correspond to deformations of the action that preserve superconformal invariance to all orders. Infinitesimal complex structure deformations of the Calabi–Yau manifold are in one-to-one correspondence with elements of , which manifest in the SCFT as chiral-chiral primary operators with (1) charges (+1 +1). Deformations of the Kähler class, combined with the NS-NS two-form, yield complexified Kähler moduli. On the worldsheet, these appear as chiral-anti-chiral primary operators with (1) charges (1 +1).

1.2.1 The K3 moduli space

In this final section, we focus the discussion on K3 surfaces and their moduli space (see [34] for a complete discussion).

A K3 surface is a compact complex Kähler manifold of complex dimension two, i.e. a surface S such that:

(i) ( ) = ( ) = 0 ;

(ii) ( ) = 0;

A remarkable fact about K3 surfaces is the following theorem:

Theorem 1

Two K3 surfaces are always diffeomorphic.

Therefore, all K3 surfaces share the same topological invariants. This allows us to choose a simple representative and perform computations on it. For instance, the Euler characteristic produces the following result:

$$(\quad 3) = \quad (\quad 1) \quad = 24 \tag{1.14}$$

where denotes the th Betti number. Furthermore, the Lefschetz hyperplane theorem implies that K3 surfaces are simply connected, yielding a trivial fundamental group:

$$(\quad) = 0 \tag{1.15}$$

Through these results and using Poincaré duality, we can deduce the Hodge numbers of a K3 surface:

$$= 1 \quad = \quad = 0 \quad = \quad = 1 \quad = 20 \quad = \quad = 0 \quad = 1 \tag{1.16}$$

These invariants are fundamental in understanding the geometry and moduli space of K3 surfaces.

To construct the moduli space, let us begin by considering the K3 surface purely as an object in algebraic geometry and focus on the complex structure sector of its moduli space. To characterize the complex structure, we seach for simple quantities that depend on it. These are for example the periods, defined as the integrals of the holomorphic 2-form , over integral 2-cycles of the K3. Since the second homology group ( ) is torsion-free and it has rank is ( ) = 22, it is possible to prove the group isomorphism (3) . In addition to this group structure one can also introduce an inner product:

$$= (\quad \backslash \quad) \qquad (\quad 3 \quad) \tag{1.17}$$

where counts the number of oriented intersections between the involved cycles. This abelian group structure, combined with the inner product defined in equation (1.17), endows (3) with the structure of lattice. The signature of this lattice, determined via the Hodge index theorem, is (3 19). Moreover, Poincaré duality ensures that ( ) is a self-dual (unimodular) lattice. Consequently, the lattice of integral cohomology (3) (forms) is isomorphic to lattice of integral homology (3) (cycles). The next fact is that the lattice (3) is even, that is:

$$(\quad) \qquad 2 \tag{1.18}$$

The classification of even self-dual lattice is known. In particular, the lattice , of signature (3 19), is unique up to isometries. Since we can always choose a lattice basis such that the inner product is represented by the matrix:

where is the Cartan matrix associate to lattice , while U represents the hyperbolic plane:

$$= \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

In order to write the periods $^{-}$ = in terms of the lattice we must embed the basis of 2-cycles on : the choice is called *marking* of K3 surface, formally defined as:

$$: \quad (\quad) \tag{1.19}$$

A natural embedding is:

$$(\; 3 \quad) \qquad (\; 3 \quad) \tag{1.20}$$

The two-form ( ) span a space-like 2 plane in . The choice of a complex structure on a K3 surfaces corresponds to determine a vector space , which contains an even self-dual lattice and an oriented 2-plane . Variations in the complex structure correspond to rotations of this plane with respect to . The moduli space of complex structures on a marked K3 surface is the space of all possible oriented 2-planes in respect to fixed lattice . This space is called *Grassmannian*:

$$\frac{(3 \;\; 19)}{(2) \qquad (1 \;\; 19)} \tag{1.21}$$

the sign + denotes that we are using the subgroup of (3 19) preserving orientation on the space-like directions.
3 surfaces defined by 2-plane , which contains a light-like direction, will be near the boundary of our moduli space. As we approach this boundary, we expect the 3 surfaces to degenerate in some way. Moreover, there may be points within the moduli space, away from the boundary, that correspond to additional degenerations. They are orbifold points and they must necessarily be included in our moduli space.
To eliminate the dependence on the marking, we consider the moduli space of complex structures on K3 surfaces (including orbifold points) as the quotient:

$$\mathcal{M} \qquad \frac{\qquad}{\qquad} \tag{1.22}$$

where is the modular group inducing an isometry on ( ) and a diffeomorphism on the K3 surface. The resulting space $\mathcal{M}$ is not actually a Hausdorff space, and this non-Hausdorff nature is reflected in string theory under particular pathological circumstances. However, this is not the exact moduli space that appears in string theory; the relevant moduli spaces that do arise are, in fact, Hausdorff.

Then we try to modify the previous result, considering the moduli space of Einstein metrics on a K3 surface. We will assume that the metric is Kähler, and we define *Einstein metric* a (pseudo-)Riemannian metric on a (pseudo-)Riemannian manifold whose Ricci curvature is proportional to the metric. Then for a K3 surface this imply that the metric is Ricci-flat.
In a K3 surface, using Hodge duality we can decompose the cohomology ( ) into the cohomology of the the self-dual and anti-self-dual 2-forms $\mathcal{H}$:

$$(\mathcal{M} \quad) = \mathcal{H} \qquad \mathcal{H} \tag{1.23}$$

where $\mathcal{H} = 3$ and $\mathcal{H} = 19$.
We focus our attention on the space $\mathcal{H}$ viewed as a subspace of $(\mathcal{M})$. It is possible to show that $\mathcal{H}$ is spanned by the previous plane and by the Kähler forms direction (space-like and perpendicular to) represented by . As previously stated, Yau's theorem ensures that, once and are fixed, there exists a unique Ricci-flat Kähler metric on the K3 surface. An important aspect of this theorem is that rotations within $\mathcal{H}$ can alter the decomposition between and , thereby changing the complex structure and Kähler form of the K3 surface. However, these rotations do not affect the underlying Riemannian metric. Consequently, the family of complex structures compatible with a fixed metric is parametrized by the sphere of possible decompositions of $\mathcal{H}$ into and . This property comes from the fact that a K3 surface admits a hyperkähler structure.

Theorem 2
The moduli space of Einstein metrics $\mathcal{M}$ *for a K3 surface (including orbifold points) is given by the Grassmannian of oriented 3-planes within the space modulo the effects of diffeomorphisms acting on the lattice ( ):*

$$\mathcal{M} \quad (\quad) \quad (3\ 19)\ (\quad (3) \quad (19)) \tag{1.24}$$

where denote the volume of the K3 surface defined through:

$$(\ 3) = \qquad 0 \tag{1.25}$$

This is isomorphic to the space:

$$\mathcal{M} \quad (\quad) \quad (3\ 19)\ (\ (3) \quad (19)) \tag{1.26}$$

which is actually a Hausdorff space of dimension 58.

We now turn to the analysis of the moduli space of string theories on K3 surfaces from the worldsheet perspective. This approach effectively captures perturbative aspects of string theory but does not account for non-perturbative effects, which are more naturally described in the target space framework.
In the worldsheet formulation, string propagation is described by a two-dimensional field theory, where the string worldsheet is mapped into a target space via:

$$:$$

In the conformal gauge, the action for this non-linear sigma model (NLSM) is given by:

$$= \frac{}{4} \qquad \overline{} \qquad 2 \qquad + \tag{1.27}$$

Here, is the target space metric, is the antisymmetric B-field, is the dilaton, and is the scalar curvature of the worldsheet .
In order to obtain a consistent string theory we require that the two dimensional theory be conformally invariant with a specific value of central charge. This determines specifics constraints on the various parameters of the theory. One way of require the conformal invariance is to set constant, B closed and Ricci-flat.
Imposing supersymmetry on the NLSM leads to an interesting connection between the

model and Kähler structure of the target manifold. A world-sheet supersymmetry transformation takes the form:

$$= ^{-} \tag{1.28}$$

For $\mathcal{N} = 1$, we must have = . However, when $\mathcal{N}$ 1, the additional components (with = +) introduce a complex structure, resulting in a Kähler manifold for $\mathcal{N} = 2$ and a hyperkähler manifold for $\mathcal{N} = 4$. Conformal invariance in two-dimensional field theory leads to a natural separation into holomorphic and anti-holomorphic components, which in turn results in the decomposition of supersymmetry into left-moving and right-moving sectors.

When is a smooth K3 manifold, the corresponding worldsheet theory exhibits $\mathcal{N} = (4\ 4)$ superconformal symmetry, at least to leading order in . In this context, it has been demonstrated that the Ricci-flat metric on K3 surfaces remains exact at all orders in perturbation theory, with no corrections arising beyond the leading term.

In the action (1.27) three independent fields serve as couplings. Their variations, which preserve supersymmetry and conformal invariance, span the moduli space of NLSM on a K3 surface.

A complete analysis of this moduli space is rather complex; therefore, we make specific assumptions regarding the variations of the parameters.

We assume that any generic deformation of the Ricci-flat metric provides another inequivalent Ricci-flat metric corresponding to a distinct conformal field theory. We have 58 free parameters for this deformation (the dimension of moduli space of Einstein metrics on K3 surfaces (1.26)). The closed 2-form B contributes 22 parameters to the moduli space. We neglect its exact part, as it is irrelevant under the map into the target space. Finally we consider the dilaton. Since is constant over , the contribution of the dilaton term in (1.27) is a sum over the genera of , given by 2 (1). Since we want consider the conformal field theory on the fixed surface, we can ignore this sum, thus we can ignore the dilaton for the moment.

The resulting moduli space is described in terms of $58 + 22 = 80$ free parameters.

Examining the moduli space of Einstein metrics on K3 surfaces, we observe that the holonomy group contains a factor isomorphic to (3) (2),arising from the of complex structures. This (2) symmetry must be present also in the NLSM. Given the conformal invariance of the theory, supersymmetry decomposes into left- and right-moving sectors. Consequently, the resulting $\mathcal{N} = (4\ 4)$ superconformal symmetry includes two independent (2) factors (2) (2) (4), leading to an enhanced (4) holonomy for our moduli space of conformal field theories.

The Berger and Simons theorem states that any smooth neighborhood of the moduli space of conformally invariant NLSMs with a K3 target space is isomorphic to an open subset of:

$$\mathcal{T} = \frac{(4\ 20)}{(4) \qquad (20)} \tag{1.29}$$

where $\mathcal{T}$ is referred to as the *Teichmüller space*. To understand the global structure of the moduli space, we assume that it is a Hausdorff space and can be expressed as a quotient: $\mathcal{M}$ $\mathcal{T}$, where is some discrete group.

The space $\mathcal{T}$ is the Grassmannian space parametrizes space-like 4-planes in , and must contain the Grassmannian of space-like 3-planes in as subspace, corresponding to the moduli space of Einstein metrics on K3 surfaces.

Introducing the even self-dual lattice , we note that it plays a role analogous to in the context of Einstein metrics. The discrete group can thus be identified with the automorphism group ( ).
Consequently, the global moduli space of conformally invariant NLSMs on a K3 surface is given by:

$$\mathcal{M} = (\quad) \quad (4\ 20) \quad (4) \quad (20) \tag{1.30}$$

where the Teichmüller space can be to decomposed as:

$$\mathcal{T} = \frac{(4\ 20)}{(4) \quad (20)} \quad \frac{(3\ 19)}{(3) \quad (19)} \tag{1.31}$$

The first factor in this decomposition corresponds to the Teichmüller space of Einstein metrics on K3 surfaces, the second to the moduli of the B-field (excluding its exact part), and the third to the overall volume modulus.

In the sixth article of this collection, we begin the exploration of the category of topological defects (generalized symmetries) that preserve the $\mathcal{N} = (4\ 4)$ algebra and spectral flow across this moduli space. Using their fusion with boundary states, we derive a number of general results for the category of such defects. We argue that while for certain K3 models infinitely many simple defects, and even a continuum, can occur, at generic points in the moduli space the category is actually trivial, i.e. it is generated by the identity defect. Furthermore, we show that if a K3 model is at the attractor point for some BPS configuration of D-branes, then all topological defects have integral quantum dimension. We also conjecture that a continuum of topological defects arises if and only if the K3 model is a (possibly generalized) orbifold of a torus model.

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2
Dynamical Cobordism

This chapter contains the articles:

- *Dynamical Cobordism and the beginning of time: supercritical strings and tachyon condensation*,
  R. Angius, Matilda Delgado, Angel M. Uranga
  JHEP 08 (2022) 285 arXiv:2207.13108 - Inspire
- *Small black hole explosions*,
  R. Angius, Jesús Huertas, Angel M. Uranga
  JHEP 06 (2023) 070 arXiv:2303.15903 - Inspire
- *Intersecting end of the world branes*,
  R. Angius, Andriana Makridou, Angel M. Uranga
  JHEP 03 (2023) 110 arXiv:2312.16286 - Inspire
- *End of the world brane networks for infinite distance limits in CY moduli space*,
  R. Angius
  JHEP 09 (2024) 178 arXiv:2404.14486 - Inspire
- *End of the world boundaries for chiral quantum gravity theories*,
  R. Angius, Angel M. Uranga, Chuying Wang
  JHEP 03 (2025) 064 arXiv:2410.07322 - Inspire

Published for SISSA by Springer

Received: *August 2, 2022*
Accepted: *August 8, 2022*
Published: *August 29, 2022*

Dynamical Cobordism and the beginning of time: supercritical strings and tachyon condensation

Roberta Angius, Matilda Delgado and Angel M. Uranga

Instituto de Física Teórica IFT-UAM/CSIC,
C/ Nicolás Cabrera 13–15, Campus de Cantoblanco, 28049 Madrid, Spain

E-mail: roberta.angius@csic.es, matilda.delgado@uam.es, angel.uranga@csic.es

Abstract: We describe timelike linear dilaton backgrounds of supercritical string theories as time-dependent Dynamical Cobordisms in string theory, with their spacelike singularity as a boundary defining the beginning of time. We propose and provide compelling evidence that its microscopic interpretation corresponds to a region of (a strong coupling version of) closed tachyon condensation. We argue that this beginning of time is closely related to (and shares the same scaling behaviour as) the bubbles of nothing obtained in a weakly coupled background with lightlike tachyon condensation. As an intermediate result, we also provide the description of the latter as lightlike Dynamical Cobordism.

Keywords: String and Brane Phenomenology, Tachyon Condensation, Bosonic Strings

ArXiv ePrint: 2207.13108

https://doi.org/10.1007/JHEP08(2022)285

Contents

1 Introduction

One of the outstanding questions in string theory is the understanding of time-dependent backgrounds and in particular the resolution of cosmological (i.e. spacelike) singularities (see [1, 2] for reviews). On general grounds, and in analogy with timelike singularities, one may expect that stringy effects smooth out the singularity, thus providing a microscopic description of the beginning of time.

This is a natural proposal from the perspective of the Swampland Cobordism Conjecture [3], which states that any consistent theory of quantum gravity should admit configurations ending spacetime, namely boundaries or general cobordism defects leading to walls of nothing.[1] This also resonates with (a Lorentzian version of) the no-boundary proposal for the Hartle-Hawking wavefunction of the universe [9].

[1] A prototype is the infinite volume limit of bubbles of nothing in higher-dimensional compactifications in which the internal space shrinks to zero size [4] (see [5–8] for some recent work).

From this perspective, such cosmological solutions would correspond to dynamical time-dependent configurations with a beginning of time given by a cobordism defect extending in the spatial directions. This appealing picture is however hampered by the general lack of understanding of the microscopic structure of spacelike singularities.

The cobordism conjecture has been exploited at the topological level with interesting results, see e.g. [6, 10–15]. On the other hand, there is substantial progress in understanding the implications of the cobordism conjecture at the dynamical level. The configurations dubbed Dynamical Cobordisms in [16–18] (see also [19])[2] describe spacetime dependent solutions in which the fields run until they hit a real-codimension 1 singularity at finite distance in spacetime, at which certain scalars run off to infinite distance in field space. In several examples of such spatially varying solutions, the timelike singularities had a known string theory UV description, which displayed an end of spacetime. Remarkably, [18] showed that in the effective theory description these singularities (dubbed end-of-the-world (ETW) branes) follow universal scaling laws, and are characterized by a single critical exponent.

In this paper we take the natural next step of starting the study of time-dependent Dynamical Cobordism with spacelike singularities and of shedding some light on their resolution. The particular arena to explore these ideas are timelike linear dilaton backgrounds in supercritical bosonic string theory.

Supercritical string theories provide consistent versions of string theory in a general number D of spacetime dimensions, provided a suitable timelike linear dilaton background is turned on [30–33]. They provide an excellent testing ground for general features of string theory (see [34] for a recent example). In particular, and as will be relevant to our discussion, they constitute a setup in which closed string tachyon physics has been subject to quantitative analysis (see e.g. [35–41]).[3] For our purposes, the main property of these theories is that the timelike linear dilaton background makes them one of the simplest time-dependent setups in string theory.

We express these backgrounds as time-dependent Dynamical Cobordisms, exhibiting their beginning of time singularity and characterizing it as an ETW brane, with a precise critical exponent. We moreover propose, providing non-trivial support for it, that the stringy resolution of the singularity involves a region of (the strong coupling version of) bulk tachyon condensation. This is a realization of the mechanism in [45] in a different setup which, as promised, provides a stringy analogue of the Hartle-Hawking proposal.

Our approach is based on the realization that the beginning of time singularity, and the walls of nothing described via lightlike tachyon condensation in [35] (see also [36–38, 46–48] for related results) admit an ETW brane description in the effective theory with exactly the same critical exponent. Moreover, we show that these configurations, which seemingly contain two intersecting ETW walls, actually contain a single recombined one with two

[2] For the related topic of solutions in theories with dynamical tadpoles, see [20–23] for early work and [24–29] for related recent developments.

[3] See [42–44] and references therein for discussion of the fate of localized closed tachyons and related instabilities.

different asymptotic regions, a lightlike one corresponding to tachyon condensation at weak coupling and a spacelike one at strong coupling corresponding to the beginning of time.

A potential caveat to our analysis is the use of effective theories to describe tachyon condensation phenomena, which involve stringy scales and are not fully understood for closed tachyons (see [49] and references therein for further discussion). We however encounter that the main feature of the ETW wall, the critical exponent, is surprisingly robust under corrections of the effective action. This suggest that the main results may survive beyond the validity of the tools used to extract it in the present work. The same considerations apply to the study of the beginning of time singularity, which lies at strong coupling.

The paper is organized as follows. In section 2 we recall the Dynamical Cobordisms of [16–18], and the structure of the ETW branes in terms of their critical exponent. In section 3 we discuss the timelike linear dilaton background as a time-dependent solution: in section 3.1 we express it as a Dynamical Cobordism with a beginning of time; in section 3.2 we describe the singularity at the beginning of time as an ETW brane; and in section 3.3 we explore its UV description in terms of a timelike tachyon condensate. In section 4 we discuss walls of nothing arising in lightlike tachyon condensation and show that they correspond to Dynamical Cobordisms with a lightlike ETW brane: in section 4.1 we recall the worldsheet description, and in section 4.2 we provide their spacetime description and characterize their ETW brane and critical exponent. In section 5 we combine results and formulate our proposal that the UV description of the beginning of time in the linear dilaton background is (a strong coupling version of) closed tachyon condensation. In section 6 we offer some final thoughts. In appendix A we mention that the dimension quenching mechanism in [36, 37] can be described as a dynamical cobordism describing an interpolating wall [17] between theories of different dimension. Some calculational details have been postponed to appendices B, C.

2 Overview of Dynamical Cobordisms

In a series of papers [16–18] the analysis of dynamical spacetime-dependent solutions realizing cobordisms to nothing was initiated (see also [19]). Such solutions, from the perspective of the lower-dimensional effective field theory, present universal features that allow them to be described in a general framework as follows.

Consider the lower-dimensional EFT to be D-dimensional[4] Einstein gravity coupled to a scalar with arbitrary potential (in $M_{Pl} = 1$ units):

$$S = \int d^D x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right) . \tag{2.1}$$

We consider solutions in which the metric and scalar vary along one coordinate, denoted by y. The ansatz for the metric is

$$ds^2 = e^{-2\sigma(y)} ds^2_{D-1} + dy^2 . \tag{2.2}$$

[4] We use D for the spacetime dimension and $\mathcal{D}$ for the field space distance.

Here we follow earlier references and considered space-dependent running solutions. The sign flips necessary for time-dependent ones will be taken into account in the concrete examples of later sections.

In the following we take a flat metric for the $D-1$ dimensional slices. All solutions that describe a cobordism to nothing present a spacetime singularity at finite spacetime distance Δ where the scalars explore an infinite distance $\mathcal{D}$ in field space, this is the location of the ETW wall. Indeed, the solution does not extend beyond this point which, without loss of generality, we choose to be $y=0$.

One of the highlights of the analysis in [18] is that the solutions near ETW branes behave in a simple way. We quote some of the main expressions encapsulating this

$$\phi(y) \simeq -\frac{2}{\delta}\log y \quad , \quad \sigma(y) \simeq -\frac{4}{(D-2)\delta^2}\log y \quad , \quad |R| \simeq \frac{1}{y^2}\,. \tag{2.3}$$

with δ a scaling coefficient which characterizes the local solution near the ETW brane, and $|R|$ is the spacetime scalar curvature. Although [18] focused on space-dependent running solutions, it is straightforward to extend the discussion to time-dependent ones, and recover the same scaling laws.

From the above profiles, all solutions describing ETW walls present universal scaling relations between Δ, $\mathcal{D}$ and the spacetime scalar curvature $|R|$, as follows

$$\Delta \sim e^{-\frac{\delta}{2}\mathcal{D}} \quad , \quad |R| \sim e^{\delta\mathcal{D}}\,. \tag{2.4}$$

We also get that the scalar potential behaves as[5]

$$V(\phi) \simeq -a\,c\,e^{\delta\,\phi}\,, \tag{2.5}$$

for a constant $a<1$ related to δ by

$$\delta = 2\sqrt{\frac{D-1}{D-2}\,(1-a)}\,. \tag{2.6}$$

3 Supercritical strings as time-dependent Dynamical Cobordism

In this section we discuss the maximally symmetric configuration of supercritial strings, and interpret the necessary linear dilaton background as a running solution which satisfies the properties of a time-dependent Dynamical Cobordism. The local behaviour is hence that of an ETW brane.

In this work we focus on the supercritical bosonic theory. We expect similar ideas to apply to other supercritical theories, including supercritical type 0 or heterotic superstrings [36, 37].

[5]Note that if y is timelike, the overall sign of the potential changes. The quantity c is a positive constant related to the boundary condition used when solving the equations of motion. Subleading corrections to the potential can be included by promoting c to a function $c=c(\phi)$ with slower growth than an exponential.

3.1 Linear timelike dilaton as Dynamical Cobordism

Consider bosonic string theory in D-dimensional Minkowski space (in the string frame). In order to satisfy the central charge constraint for the theory, there is a linear dilaton background

$$\Phi = v_M X^M \, , \tag{3.1}$$

with

$$v \cdot v = -\frac{D-26}{6\alpha'} \, , \tag{3.2}$$

with contractions defined with respect to the flat Minkowski metric.

Hence, supercritical strings require a timelike dilaton gradient, whereas subcritical strings require an spacelike one. The critical $D = 26$ theory does not require a dilaton profile for consistency, but does admit a lightlike dilaton background. We thus expect our discussion to extend this background of the critical theory as well.

These linear dilaton theories define conformal theories exactly in α', which implies that they satisfy the equations of motion of the spacetime (string frame) action

$$S_{\text{str.}} = \frac{1}{2} \int d^D x \sqrt{-G_{(s)}} e^{-2\Phi} \left[-\frac{2(D-26)}{3\alpha'} + R_{(s)} + 4(\partial\Phi)^2 \right] . \tag{3.3}$$

We explicitly denote string frame quantities with an s subindex, while quantities with no subindex are implicitly defined in the Einstein frame.

In the following we focus on supercritical strings and timelike dilaton background

$$\Phi = -qX^0 \, , \tag{3.4}$$

where $q \equiv v^0$. Here we have absorbed a possible additive constant by shifting time, so that the dilaton vanishes at $X^0 = 0$. From the two solutions of (3.2) we choose the one leading to weak coupling $g_s = e^\Phi$ in the future $X^0 \to \infty$, namely

$$q = \sqrt{\frac{D-26}{6\alpha'}} \, . \tag{3.5}$$

In the next section, we reinterpret this linear dilaton background as a running solution with an ETW wall at the origin of time.

3.2 The ETW brane at the beginning of time

The spacetime physics of the singularity was considered from a cosmological perspective in [35]. Here we instead study it from the perspective of the ETW branes of dynamical cobordism in section 2.

To discuss the spacetime physics, we focus on the Einstein frame, so the metric reads

$$ds^2 = \exp\left(\frac{4qX^0}{D-2}\right) \eta_{MN} dx^M dx^N \, . \tag{3.6}$$

We see that at $X^0 \to -\infty$ the warp factor goes to zero, and we hit a singularity. We can introduce a time coordinate y giving the invariant interval to the singularity as

$$y = \frac{D-2}{2q} \exp\left(\frac{2qX^0}{D-2}\right) , \tag{3.7}$$

in terms of which (3.6) is recast as the time-dependent version of (2.2):

$$ds^2 = -dy^2 + \frac{4q^2y^2}{(D-2)^2}\, dx^m dx^m , \tag{3.8}$$

for $m = 1, \ldots, D-1$. We thus obtain

$$\sigma(y) = -\log y . \tag{3.9}$$

Comparing this to (2.3) gives:

$$\delta = \frac{2}{\sqrt{D-2}} . \tag{3.10}$$

Expressing the dilaton in terms of a scalar ϕ with canonical kinetic term in the Einstein frame as

$$\phi = \frac{2}{\sqrt{D-2}}\, \Phi \sim -\sqrt{D-2}\, \log y . \tag{3.11}$$

This is precisely the scaling relation for the scalar (2.3) for the value of δ in (3.10).

We also get the expected scaling of the potential. The Einstein frame action gives

$$S = \frac{1}{2} \int d^D x \sqrt{-G} \left[R - \frac{4}{D-2} (\partial\Phi)^2 - \frac{2(D-26)}{3\alpha'} \exp\left(\frac{4\Phi}{D-2}\right) \right] , \tag{3.12}$$

which, comparing with (2.5) and using the normalized dilaton (3.11) yields the precise value of δ in (3.10).

To conclude, we recover the scaling relations (2.4), which state that the configuration hits an ETW singularity at finite time in the past at which the scalar runs off to infinite distance in field space. According to the cobordism interpretation of such singularities in [17, 18], it defines a beginning of time, a boundary in the time direction, for this solution.

The microscopic description of the ETW brane requires some understanding of spacelike defects in string theory, which remains mostly *terra incognita*.[6] In our particular example, this is even more so since it lies at strong coupling.[7] Despite these difficulties, we find compelling evidence that the microscopic description of our spacelike ETW brane is the strong coupling avatar of tachyon condensation. We propose a direct approach to this proposal in the next section, and a further indirect, but quantitatively more reliable, route to support this picture in section 4.

[6] See e.g. [50] for attemps invoking S-branes.

[7] For the critical type IIA with a lightlike dilaton background, a microscopic description for the analogous singularity was proposed in [51], based on M(atrix) theory [52]. Such a description does not seem feasible in our case.

3.3 The timelike tachyon case

The resolution of spacelike singularities in a tractable worldsheet approach was addressed in [45] in a setup with a shrinking 1-cycle, in terms of the condensation of a closed string tachyon in the winding sector (see [53, 54] for proposed higher-genus generalizations). In short, the regime near the singularity was proposed to be coated by a longer duration region in which the tachyon condenses with an exponential profile. The latter describes an effective Liouville wall in the time direction, beyond which no string excitation can propagate. This was argued to be a stringy definition of the *nothing* in the Hartle-Hawking description of the wavefunction of the universe [9]. In this picture, spacetime emerges smoothly as the tachyon turns off. In our terms, it describes a cobordism to nothing in the time direction.

In this section we explore a similar interpretation for the spacelike singularity encountered in our timelike linear dilaton setup. The idea is to consider an exponential profile for the closed string tachyon of supercritical bosonic theory. The tachyon couples to the worldsheet as a 2d potential. The condition for this deformation to be marginal, to linear order in conformal perturbation theory, or equivalently, the linearized spacetime equation of motion for the tachyon is

$$\partial^2 T(X) - 2\, v^M\, \partial_M T(X) + \frac{4}{\alpha'} T(X) = 0\ . \tag{3.13}$$

We will discuss corrections to this later on.

For a general tachyon exponential profile

$$T(X^M) = \mu\, \exp(\beta_M\, X^M)\ . \tag{3.14}$$

we obtain a condition on β:

$$\beta \cdot \beta - 2v \cdot \beta + \frac{4}{\alpha'} = 0\,. \tag{3.15}$$

We now focus on a timelike tachyon profile

$$T = \mu \exp(-\beta^0 X^0)\,, \tag{3.16}$$

with the condition

$$-(\beta^0)^2 + 2q\beta^0 + \frac{4}{\alpha'} = 0\,. \tag{3.17}$$

There are two solutions to this quadratic equation. A possibility is to choose $\beta^0 < 0$, so that the tachyon grows for late times $X^0 \to \infty$. This is a good strategy to study the process of tachyon condensation in a weakly coupled regime, see e.g. [38]. In fact, it is closely related to our approach (albeit for lightlike tachyons) in section 4.

Here, instead, we are interested in having tachyon condensation at the beginning of time, to provide a resolution of the spacelike ETW brane at $y = 0$, hence we need the tachyon to grow in the past $X^0 \to -\infty$, we thus require $\beta^0 > 0$. Using (3.5) we have

$$\beta^0 = \frac{\sqrt{D-26} + \sqrt{D-2}}{\sqrt{6\alpha'}}\,. \tag{3.18}$$

We may now compare the relative growth of the string coupling $g_s = e^\Phi$ and of the tachyon as $X^0 \to -\infty$ to assess if the tachyon condensation could be studied using worldsheet techniques. We have

$$T/g_s = \mu \exp\left(-\sqrt{\frac{D-2}{6\alpha'}}X^0\right) . \tag{3.19}$$

This shows that the tachyon grows parametrically faster than the string coupling as $X^0 \to -\infty$. This leads to the expectation that the worldsheet analysis provides a reliable description of the physics at early times. In analogy with [45], and based on the extensive analysis in [35–37], the presence of the worldsheet potential creates a Liouville wall expelling all string excitations, providing a microscopic definition of an ETW brane in time.

The drawback of this approach is that it relied on trusting the linearized deformation approximation, which is expected to experience strong higher order corrections.[8] Therefore the scenario can be at most regarded as a qualitative description. In the next section we turn to a different approach, involving α' exact solutions.

4 Lightlike tachyon condensation

We are thus led to consider solutions under better control. In this section we consider an α'-exact solution of the supercritical linear dilaton theory with tachyon profile along a lightlike direction. As established in [35, 36] for the bosonic theory, at late times this leads to a wall of nothing moving at the speed of light, analogous to the asymptotic behaviour of a bubble of nothing. After recalling the argument, we carry out a new spacetime analysis that shows that at late times the background corresponds to a lightlike ETW brane, and show that its critical exponent is exactly the same as for the beginning of time ETW brane of the previous section. This tantalizing relation is a strong support for our interpretion of the beginning of time is (a strongly coupled version of) a closed string tachyon condensation phase, discussed in section 5.

4.1 Lightlike tachyon in the worldsheet description

Consider introducing an exponential tachyon background (3.14) along a lightlike direction $X^+ = (X^0 + X^1)/\sqrt{2}$

$$T = \mu \exp(\beta X^+) . \tag{4.1}$$

The linearized tachyon marginality condition (3.13) is satisfied for

$$\beta = \frac{2\sqrt{2}}{q\alpha'} . \tag{4.2}$$

At late times $X^0 \to \infty$, the string coupling is small and one may perform a reliable worldsheet analysis. As shown in [35] and contrary to the timelike tachyon case, the

[8] For the $\beta^0 < 0$ solution, corrections are interestingly expected to be suppressed in a large D approximation, as exploited in [55]. Although large D could be interesting in our $\beta^0 > 0$ case to increase the hierarchy in (3.19), it does not lead to a similar suppression of corrections.

deformation by the operator (4.1) is exact, as higher order corrections in the perturbation vanish, since the lightlike nature of the insertions prevent the existence of non-trivial Wick contractions. Furthermore, in light-cone coordinates the propagator of the $X^{+/-}$ fields is oriented from X^+ to X^- and we know that all interaction vertices introduced by the tachyon potential only depend on X^+. These two facts combined show that there are no possible Feynman diagrams beyond tree-level, which implies the solution is exact in α'. One can thus conclude that the linearized tree-level solution (4.1) is exactly conformally invariant.

The tachyon couples as a worldsheet potential, which grows infinitely at $X^+ \to \infty$. This 2d potential prevents any string modes from entering the corresponding region, which thus becomes a region of nothing. The physical interpretation of this is that the tachyon configuration describes a wall of nothing propagating at the speed of light, which effectively ends spacetime at an effective value of X^+.

The finite range in X^+ can be estimated by e.g. cutting off X^+ when the $T = 1$. This gives

$$\Delta X^+ = -\beta^{-1} \log \mu/\mu^* \,, \tag{4.3}$$

where μ^* defines a reference position from which we measure the range to the wall. A more precise derivation follows from the *gedanken* experiment of solving the motion of classical strings incoming into the tachyon wall [35]. The initial speed reduces to zero at a turning point, after which the string is pushed back by the tachyon wall and its speed asymptotes to that of light. The turning point position in X^+ in the formulas in [35] gives back the result (4.3).

Given the importance of the notion of finiteness on the location of the tachyon wall, we provide an alternative derivation, carried out by adapting the techniques in [45]. We briefly sketch the results here and give more computational details in appendix B. Decomposing the field $X^+(\tau,\sigma)$ into its zero and nonzero modes $X^+(\tau,\sigma) = X_0^+ + \widehat{X}^+(\tau,\sigma)$ and performing a Wick rotation, the Euclidean partition function reads:

$$Z(\mu) = \int dX_0^+ \int \mathcal{D}\widehat{X}^+ \mathcal{D}X^- \mathcal{D}X^i \mathcal{D}g \mathcal{D}(\text{ghosts}) e^{-S_E^{\text{deformed}}} \,, \tag{4.4}$$

with the Euclidean action:

$$\begin{aligned} S_E^{\text{deformed}} =& \frac{1}{2\pi\alpha'} \int d^2\sigma_E \sqrt{g} \left[\partial_{\sigma^0} \widehat{X}^+ \partial_{\sigma^0} X^- + \partial_{\sigma^1} X^- \partial_{\sigma^1} \widehat{X}^+ + \partial_\alpha X^i \partial_\alpha X^i \right] \\ &+ \frac{1}{2\pi} \int d^2\sigma_E \sqrt{g} R_2 \Phi(X) + \frac{i\mu_E}{2\pi} \int d^2\sigma_E \sqrt{g} e^{\beta X^+} \,. \end{aligned} \tag{4.5}$$

From this we see that when the tachyon condenses, at large X^+, the path integral becomes suppressed. This results in a truncation of contributions to the integral coming from string oscillations with $X^+ \mapsto \infty$. This is the same mechanism as that of a Liouville wall in Liouville theory: no physical degrees of freedom exist in this region. In fact, one can show that the partition function in (4.4) can be directly related to that of the free theory (with no tachyon deformation) as follows.

After integrating out the zero-mode X_0^+, one can show that:

$$\frac{\partial Z}{\partial \mu_E} = -\frac{1}{\beta \mu_E} \int \mathcal{D}\widehat{X}^+ \mathcal{D}X^- \mathcal{D}X^i \mathcal{D}g \mathcal{D}(\text{ghosts}) e^{-S_E^{\text{free}}} \,, \tag{4.6}$$

where S_E^{free} is the euclidean action of the worldsheet theory without the tachyon potential. Integrating with respect to μ and fixing a cutoff for X^+ such that $\mu_* = e^{\beta X_*^+}$, we obtain:

$$Z_1 = -\frac{\log(\mu_E/\mu_*)}{\beta} \widehat{Z} \,, \tag{4.7}$$

where $\widehat{Z}$ is the partition function for the 2d theory without the tachyon insertion. Hence the partition function Z in the presence of the tachyon background related to that of the theory without the tachyon $\hat{Z}$ via the factor $\frac{\log(\mu_E/\mu_*)}{\beta}$, which thus provides an effective "size" of the direction X^+, which matches that of (4.3).

The interpretation of the exponential tachyon as a wall of nothing receives further support from the dimension quenching mechanism in [36]. In appendix A we review it from the perspective of dynamical cobordisms in the spacetime perspective, to be discussed next.

4.2 Spacetime description and lightlike ETW brane

In this section we study the spacetime description of the wall of nothing corresponding to the lightlike tachyon, and show that it satisfies the properties of (the lightlike version of) and ETW brane. This nicely confirms the worldsheet arguments of the previous section.

4.2.1 Effective action

In order to describe the spacetime dynamics of the lightlike tachyon configuration, we need an effective spacetime action for the relevant fields, in particular for the tachyon. This is already a subtle point, since tachyon condensation processes may in principle backreact on the whole tower of stringy states, hence the validity of the truncation to an effective theory with a finite set of fields is to some extent questionable.

In any event, this approach has been successful enough in open string tachyon effective actions, and we may venture into its use for the closed case, hoping that fortune favors the brave.

The construction of the most general 2-derivative effective action for the metric, dilaton and tachyon in supercritical string theory has been discussed in [35] and [38], whose discussion we follow. In the string frame it has the structure

$$S = \frac{1}{2} \int d^D x \sqrt{-G_{(s)}}\, e^{-2\Phi} \left[f_1 R_{(s)} + 4 f_2 (\nabla \Phi)^2 - f_3 (\nabla T)^2 - 2 f_4 - f_5 \nabla T \cdot \nabla \Phi \right] \,. \tag{4.8}$$

where the $f_i(T)$ are general functions of the tachyon. By demanding that the equations of motion are compatible with the linear dilaton background with an exponential tachyon profile, one can show that the $f_i(T)$ can be expressed in terms of $f_1(T)$:

$$\begin{aligned} f_2(T) &= f_1(T), \qquad f_3(T) = -f_1''(T) - \frac{f_1'(T)}{T}, \qquad f_5(T) = 4\, f_1'(T) \,, \\ f_4 &= \frac{1}{2}\Big[4\, f_1(T) \left(\frac{D-26}{6\alpha'} \right) + T\, f_1'(T) \left(\beta \cdot \beta + \frac{8}{\alpha'} \right) - T^2\, f_1''(T)\, \beta \cdot \beta \Big] \,. \end{aligned} \tag{4.9}$$

For general exponential tachyon profiles, the tachyon background is only a solution at linearized order, hence we expect the above relations to receive corrections. For lightlike tachyons, however, the solution is exact in worldsheet perturbation theory, hence the above relations hold, and the corrections at most modify the behaviour of f_1 at large T. Note also that for lightlike tachyon profiles $\beta\cdot\beta=0$ and the tachyon potential $V_{(s)}=f_4$ becomes β-independent.

Going to the Einstein frame, we redefining the metric to absorb the f_1 prefactor as well as the usual dilaton factor,

$$(G_{(s)})_{MN}=e^{\frac{4}{D-2}\Phi}f_1^{-\frac{2}{D-2}}\,G_{MN}\,, \tag{4.10}$$

the spacetime action is

$$\begin{aligned} S=\frac{1}{2}\int d^Dx\sqrt{-G}\Bigg[&R-\frac{4}{D-2}(\partial_M\Phi\partial^M\Phi-\frac{f_1'}{f_1}\partial_M\Phi\partial^M T) \\ &-\left[\frac{D-1}{D-2}\frac{f_1'^2}{f_1^2}-\frac{f_1''}{f_1}-\frac{f_1'}{f_1T}\right]\partial_M T\partial^M T-\frac{2}{3\alpha'}e^{\frac{4\Phi}{D-2}}f_1^{\frac{-D}{D-2}}((D-26)f_1+12Tf_1')\Bigg]. \end{aligned} \tag{4.11}$$

The complete expression for f_1 is actually not known, beyond its expansion around $T=0$

$$f_1=1-T^2+\dots \tag{4.12}$$

Nevertheless, [38] proposed a set of regularity conditions on the effective action, which to some extent constrain f_1 further, and several explicit solutions were proposed, concretely $f_1=\exp(-T^2)$ and $f_1=1/\cosh(\sqrt{2}T)$. Interestingly, in the large T regime (which is near the wall of nothing, our main focus), both can be parametrized as

$$f_1\sim A\exp\left(-b\,T^k\right). \tag{4.13}$$

Actually, an outcome of [38] is that the behaviour of the system is not particularly sensitive to the precise form f_1. In the following we focus on the dependence (4.13), but later show that the same results hold, even at quantitative level, for very general forms of f_1.

4.2.2 The local scalings

We propose that the lightlike tachyon background, in the weak coupling regime, corresponds to a dynamical cobordism in X^+, and that the tachyon wall corresponds to an ETW brane, namely a singularity in effective theory at finite spacetime distance, and at which some scalar runs of to infinite field theory distance. In the following we show that the scalings derived from the Einstein frame spacetime solution are indeed of the ETW kind.

Note that, because the dynamical cobordism takes place via dependence on the lightlike coordinate X^+, in order to discuss spacetime *distance*, we choose slices of constant X^0, and measure spatial distance along X^1, along which the dilaton remains constant.

Again, recall that we focus on the dependence (4.13), but similar conclusions hold for a very general class of profiles of f_1. The running scalar along X^1 is only the tachyon, hence the distance in field space as we approach the wall is given by:

$$\mathcal{D} = \int^{T_{\rm ETW}} \left(\sqrt{\frac{f_1''}{f_1} + \frac{f_1'}{f_1}\left(\frac{1}{T} - \frac{D-1}{D-2}\frac{f_1'}{f_1}\right)}\right) dT \sim \frac{b}{\sqrt{D-2}}\, T_{\rm ETW}^k\,, \tag{4.14}$$

which diverges (for $k > 1$) since the tachyon goes to infinity at the ETW brane at $X^1 \to \infty$.

Let us now check that the wall is indeed at finite spacetime distance in the Einstein frame. The length along X^1 is

$$\begin{aligned}\Delta =& \int^{\rm ETW} f_1^{\frac{1}{D-2}}\, dx^1 = A^{\frac{1}{D-2}} \int^{\rm ETW} \exp\left(-\frac{bT^k}{D-2}\right) dx^1 \\ =& \frac{\sqrt{2}}{bk} A^{\frac{1}{D-2}} \int^{\rm ETW} \exp\left(-\frac{\mathcal{D}}{\sqrt{D-2}}\right) \frac{d\mathcal{D}}{\mathcal{D}} = \frac{\sqrt{2}}{bk} A^{\frac{1}{D-2}}\, {\rm Ei}\left(-\frac{\mathcal{D}}{\sqrt{D-2}}\right)\,,\end{aligned} \tag{4.15}$$

where this last function is the exponential integral, and is clearly convergent, showing that the tachyon background behaves as dynamical cobordism ending at an ETW brane, where the (tachyon) scalar runs off to infinite field space distance at a finite spacetime distance.

We can check the scaling relations of ETW branes of section 2. We can expand (4.15) for $\mathcal{D} \to \infty$ as

$${\rm Ei}\left(-\frac{\mathcal{D}}{a}\right) \sim e^{-\mathcal{D}/a}\left(\frac{a}{\mathcal{D}} + \ldots\right)\,, \tag{4.16}$$

and get

$$\Delta \sim \exp\left[-\frac{\mathcal{D}}{\sqrt{D-2}} - \log\left(\frac{\mathcal{D}}{\sqrt{D-2}}\right)\right]\,. \tag{4.17}$$

Comparing this with (2.4) gives a value

$$\delta = \frac{2}{\sqrt{D-2}}\,. \tag{4.18}$$

Namely, we recover an exponential relation. It is interesting to point out that the log correction is reminiscent of that encountered in [17, 18] for the EFT strings in [56]. Also note that, restricting to the leading exponential scaling, the critical exponent is independent of k. This is a particular case of the claimed robustness of the results under changes of f_1, and will be explored in general in section 4.2.3.

We can also compute the scaling of the Ricci scalar, which, upon direct computation gives

$$|R| \sim \exp\left[\frac{2\mathcal{D}}{\sqrt{D-2}} + \log\left(\frac{\mathcal{D}^2}{D-2}\right)\right]\,. \tag{4.19}$$

Again, the leading terms gives the scaling corresponding to an ETW brane, with δ given by (4.18), again remarkably independent of k.

The potential, computed in the limit $T \mapsto \infty$ with (4.11), also agrees nicely with the general formula provided by the local analysis in (2.5):

$$V(T) = -\frac{8}{\alpha'} A^{-\frac{2}{D-2}} kb\, T^k e^{\frac{4\phi}{D-2}} e^{\delta \mathcal{D}} = -a\, c(T)\, e^{\delta \mathcal{D}}\,, \tag{4.20}$$

with $a \in [0, 1]$ and the subleading polynomial correction can be absorbed by the function $c(T)$.[9]

Even more remarkably, the value of δ (4.18) for the lightlike tachyon agrees with the critical exponent (3.10) of the ETW brane at the beginning of time of the linear dilaton solution. This shows that both kinds of ETW branes are very similar, and is strongly suggestive that they may admit similar microscopic descriptions. Hence, we claim that the singularity at the beginning of time is a dynamical cobordism to nothing triggered by (the strong coupling version of) the condensation of the closed string tachyon. We look deeper into this argument in section 5 but before then, we show that this surprising matching of the critical exponents holds for general profiles of f_1.

4.2.3 General f_1

Let us now show that the above structure, and in particular the same value for the critical exponent δ, holds for general f_1 under very mild conditions. In particular we demand that f_1 decays at large T faster then $1/T$. This is a very reasonable requirement, in particular notice that this ensures the convergence of the integral for the spacetime distance Δ to the ETW brane. Hence it implements the intuition that the wall of nothing propagating at the spped of light hits in finite time any point at finite spacetime distance.

Consider now the integral for the field space distance

$$\mathcal{D} = \int \left[\frac{f_1''}{f_1} + \frac{f_1'}{f_1} \left(\frac{1}{T} - \frac{D-1}{D-2} \frac{f_1'}{f_1} \right) \right]^{\frac{1}{2}} dT\,. \tag{4.21}$$

We start massaging the integrand of $\mathcal{D}$, by noticing that

$$\frac{f_1''}{f_1} + \frac{f_1'}{f_1} \left(\frac{1}{T} - \frac{D-1}{D-2} \frac{f_1'}{f_1} \right) = \left(\frac{f_1'}{f_1} \right)' + \frac{f_1'}{T\, f_1} + \frac{1}{D-2} \left(\frac{f_1'}{f_1} \right)^2\,, \tag{4.22}$$

it is easy to show that for f_1 decaying faster than $1/T$, the dominant term is the last one. In fact one can see by considering different profiles (e.g. power-law, exponential, exponential of an exponential, etc) that, the faster the decay, the more the last terms dominates. Then

$$\mathcal{D} \sim \int \frac{1}{\sqrt{D-2}} \frac{f_1'}{f_1}\, dT\,. \tag{4.23}$$

We may write this as

$$d\mathcal{D} = \frac{1}{\sqrt{D-2}} \frac{f_1'}{f_1}\, dT = \frac{1}{\sqrt{D-2}} d\log f_1 = \sqrt{D-2}\; d\log f_1^{\frac{1}{D-2}}\,. \tag{4.24}$$

[9] More details about such subleading corrections can be found in the appendix B of [18].

Namely

$$f_1^{\frac{1}{D-2}} = \exp(-\mathcal{D}/\sqrt{D-2})\,, \tag{4.25}$$

where we have chosen the appropriate sign for the distance to be positive (recall that f_1 is a function that decreases to zero). This allow to express the spacetime distance as

$$\Delta \sim \int f_1^{1/(D-2)} dx^1 \sim \int \exp(-\mathcal{D}/\sqrt{D-2}) \frac{dT}{T}\,. \tag{4.26}$$

This has a similar structure to the intermediate expression in (4.15). Similar to the exponential integral there, the above integral behaves just like the exponential in the integrand, leading to the scaling

$$\Delta = \exp(-\mathcal{D}/\sqrt{D-2})\,, \tag{4.27}$$

which reproduces the value of δ in (4.18). Indeed, one can check that the additional terms in the integrand lead to subleading corrections, of the kind in (4.17) (for a proof of this statement under mild assumptions, we refer the reader to appendix C).

5 The strong coupling region and the origin of time

In this section we argue that the microscopic description of the ETW brane at the beginning of time is a region of (the strong coupling version of) closed string tachyon condensation.

The ETW brane recombination. Let us now consider the full lightlike tachyon configuration, including the strongly coupled region, and consider the interplay of the two ETW branes we have encountered.

In the string frame variables there are two asymptotic regions, controlled by seemingly different physics. The first is the region $X^0 \to -\infty$, with X^1 finite (hence $X^+ \to -\infty$), which corresponds to a linear timelike dilaton configuration, with negligible tachyon background. The second is the region $X^+ \to \infty$ at finite X^1 (hence $X^0 \to \infty$), which corresponds to a lightlike tachyon configuration at weak string coupling. Both regions are disjoint, as they only coincide at infinity in $X^0 \to -\infty$, $X^+ \to \infty$ (hence we need $X^1 \to \infty$).

In the Einstein frame, these asymptotic regimes turn into singularities at finite distance in spacetime, triggered by the running off of suitable scalars (the tachyon or the dilaton) to infinite distance in field space. Following the dynamical cobordism interpretation advocated in [16–18], these are ETW branes chopping off the region of spacetime beyond them.

An important observation is that the effective theory in which one describes ETW branes is not valid at arbitrarily short distances to the singularity. The singularity is expected to be smoothed out by new UV physics which implies the existence of a cutoff in the applicability of the effective theory. This translates into cutting of a strip of spacetime around the singularities, hence providing a notion of 'strechted' ETW brane in effective theory. This can be obtained in different ways, for instance by imposing a maximal bound on the scalar curvature. Instead, we use a criterion directly inspired by the swampland distance conjecture [57], as follows.

The distance conjecture states that when an effective theory reaches to a large distance $\mathcal{D}$ in field space, its effective cutoff scales as

$$\Lambda \sim e^{-\alpha \mathcal{D}}\,, \tag{5.1}$$

for some order 1 coefficient α. The actual distance conjecture moreover claims that there is an infinite tower of states becoming light with Λ, but this formulation corresponds to an adiabatic motion in moduli space, and such towers may actually not arise in dynamical situations with spacetime dependence of the scalars [58].[10] Hence we stick to the milder statement that a cutoff is developed, whose origin in our context would stem from the UV completion of the ETW brane.

In our configuration we hence consider the slice of spacetime at which the field space distance (in the combined tachyon-dilaton system) reaches a large but finite value. From the Einstein frame action (4.11), the relevant kinetic terms read

$$\frac{4}{D-2}\partial\Phi\cdot\partial\Phi\,-\,\frac{4}{D-2}\frac{f_1'}{f_1}\partial\Phi\cdot\partial T\,+\,\left(-\frac{f_1''}{f_1}-\frac{f_1'}{Tf_1}+\frac{D-1}{D-2}\frac{f_1'^2}{f_1^2}\right)\partial T\cdot\partial T\,. \tag{5.2}$$

We are interested in the behaviour near the intersection of the two singularities. Since this lies at large T, we can simplify the last term using the argument in section 4.2.3. Using $(f_1'/f_1)\partial T = \partial\log f_1$, the kinetic term may be written

$$\frac{1}{D-2}(\,2\partial\Phi - \partial\log f_1\,)^2\,, \tag{5.3}$$

so that the slices of constant distance are defined by

$$\mathcal{D} \sim \frac{1}{\sqrt{D-2}}(2\Phi - \log f_1) = \text{const}\,. \tag{5.4}$$

Note that interestingly, the swampland distance cutoff (5.1) is

$$\Lambda \sim e^{-\alpha\mathcal{D}} \sim \exp\left(-\frac{\alpha\delta}{2}\,(2\Phi - \log f_1)\right) = \left(e^{-2\Phi}f_1\right)^{\alpha\delta/2}\,, \tag{5.5}$$

where δ is given by (4.18). The factor inside brackets is the prefactor of the Einstein term in the string frame action. The fact that it relates to the cutoff scale shows that one gets the same spacetime slices if one uses a bound in the scalar curvature to limit the applicability of the effective theory, rather than in the field space distance. Indeed, this is expected from the scaling (2.3) of R with $\mathcal{D}$ near ETW branes.

The curve in the (x^0, x^1)-plane defined by (5.4) asymptotes to constant X^0 on one side and to constant X^+ on the other. For illustration we may consider f_1 as in (4.13) and get

$$2qx^0 - b\mu^k e^{\beta k x^+} = \text{const.} \tag{5.6}$$

[10] A simple example is the Taub-NUT geometry, which can be regarded as a spacetime-dependent solution of $\mathbf{S}^1$ compactifications, in which the circle shrinks to zero size at a point in the base. Hence, it attains infinite distance in the naive circle compactification moduli space, but no tower of light particles or other disasters arise.

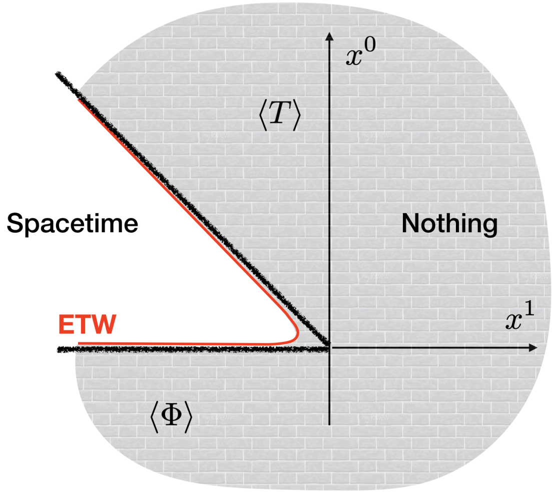

Figure 1. Depiction of a spacetime slice at which the solutions attain a large fixed field space distance in the tachyon-dilaton space. The asymptotes near $x^0 \sim 0$ and $x^+ \sim 0$, should be regarded as corresponding to the physical location of the ETW branes, which are hence recombined in the middle region, and which for a boundary of spacetime, with nothing beyond them. The spray line signal the location of the singularities.

This leads to slices of the form

$$x^1 = \frac{\sqrt{2}}{k\beta} \log(x^0 + cst) - x^0 + \frac{\sqrt{2}}{k\beta} \log \frac{2q}{b\mu^k} \,, \tag{5.7}$$

for some constant cst related to the cutoff. In figure 1 we depict the structure of such curves and of the resulting spacetime picture.

From this it is clear that what seemed to be an intersection between the two ETW walls is in fact a smooth region that interpolates between the two. This hence strongly motivates that this solution describes a one and only recombined ETW wall. Another interesting indication for this is the fact that the dilaton-tachyon mixing in the effective action (scaling like f_1'/f_1 times powers of the string couplings) gets large for large tachyon and strong coupling, namely near the naive intersection of the singularities. Hence, the two walls, which asymptotically correspond to the dilaton or the tachyon running off to infinite distance in their field space, become of a very similar nature in that intersection region.

As a side note, one may wonder if our analysis extends to the case of critical $D = 26$ bosonic string theory. Indeed, this theory is tachyonic and, as mentioned in section 3.1, it admits a lightlike dilaton background. It is then clear from the equation of motion for the tachyon (3.13) that, if we take the dilaton along the direction X^+, the only option is to take the tachyon along X^-. This means that the ETW branes corresponding to large tachyon and strong coupling are of the intersecting kind, exactly like those described in this section. It would be interesting to explore these lightlike linear dilaton backgrounds in other critical theories, and make contact with existing proposals concerning the resulting singularities [36, 51].

6 Conclusions

In this paper we have studied timelike linear dilaton backgrounds of supercritical string theories as time-dependent solutions in string theory, and addressed the question of the resulting spacelike singularity, from the perspective of the cobordism conjecture. We have quantitatively characterized the solution as a Dynamical Cobordism in which the dilaton rolls until it hits infinite field space distance at a singularity at finite time in the past. We have shown that the singularity in effective theory follows the scaling behaviour of ETW branes. In order to clarify its microscopic description, we have considered lightlike tachyon condensation backgrounds, whose microscopic description had been argued to correspond to a stringy version of a bubble of nothing. Using an effective theory approach, we have characterized them as ETW branes and have encountered precisely the same scaling exponent as the beginning of time singularity. Together with the fact that both ETW branes join smoothly from the effective theory perspective, this has motivated our proposal that the spacelike singularity should correspond in string theory to a region of (a strong coupling version of) closed tachyon condensation, giving rise to a cobordism boundary defining the beginning of time.

There are several open questions and new directions:

- We have used an effective theory for the tachyon in terms of the undetermined function f_1. Even though our results are robust under changes of the precise form of this function, it would be interesting to determine it, or at least its asymptotic behavior for large T.

- Conversely, it would be interesting to understand if the criterion that the theory allows for a resolution of spacelike singularities can be used as a constraint on effective theories. For instance, there exist choices of f_1 which lead to lightlike tachyon ETW branes with scalings different from the beginning of time one. Such theories may not be compatible with a microscopic description of the latter, as the ETW brane may not be compatible for recombination. It is thus tantalizing to claim that this can be used as a criterion to exclude such choices of f_1. It would also be interesting to understand this possibility, possibly invoking other swampland constraints or physical considerations.

- Although we have focused on the bosonic theory, there is a rich set of phenomena arising in lightlike tachyon backgrounds in other string theories. We expect our ideas to lead to interesting new insights into this web of transitions.

- Finally, an interesting corner in this circle of ideas is that of lightlike dilaton backgrounds in critical string theories. They are toy models of cosmological singularities, which in certain supersymmetric cases admit interesting proposals for their microscopic description [51]. It would be exciting to use cobordism ideas to make progress on the understanding of such backgrounds.

We hope to report on these and other questions in the near future.

Acknowledgments

We are pleased to thank Luis Ibánez, Fernando Marchesano, Miguel Montero, Lorenz Schlechter and Irene Valenzuela for useful discussions, and Ginevra Buratti, José Calderón, Jesús Huertas for collaboration on related topics. M.D. also wishes to acknowledge the hospitality of the Department of Physics of Harvard University during the development of this work. This work is supported through the grants CEX2020-001007-S and PGC2018-095976-B-C21, funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe. The work by R.A. is supported by the grant BESST-VACUA of CSIC. The work by M.D. is supported by the FPI gran no. FPI SEV-2016-0597-19-3 from Spanish National Research Agency from the Ministry of Science and Innovation.

A Dimension quenching as an interpolating domain wall

In [17] it was argued that, when the scalars remain at finite distance in field space as one hits the wall, the corresponding configuration described an interpolating wall between different QG theories. This scenario is built in contrast with the end-of-the-world walls where the fields reach infinite distance in field-space at the wall. Instead of the solution ending abruptly at the location of a singularity, these interpolating solutions continue across the wall into another theory. On each side of the interpolating wall, the field spaces may have different structures but the location of the wall itself is at finite distance in both of them. As a result, the interpolating wall must have all the right properties for communicating between the two theories, whatever they may be. It is clear from this that the microscopic nature of these walls can be hard to describe; it may in general be non-supersymmetric and may involve strong-coupling physics. The existence of such objects is one of the predictions of the Cobordism Conjecture [3].

The examples of interpolating walls in [17] were simple enough to be described by standard supersymmetric objects. For example, D8 branes in massive type IIA string theory were identified as such walls interpolating between "different" massive IIA theories with different units of 0-form flux. Here, we re-interpret the dimension-quenching bubbles of [36, 37] as interpolating walls between bosonic string theories of different dimensions.

Throughout this paper we have dealt with tachyons with exponential profiles along one light-like direction. These solutions were shown to lead to bubbles of nothing, which fit in the wall-of-nothing description. There are, however, slightly more complicated solutions to the tachyon equation of motion (3.13). Following [36], we can consider a profile with oscillatory dependence on another coordinate, denoted by X^2:

$$T(X) = \mu_0^2 \exp(\beta X^+) - \mu_k^2 \cos(kX^2) \exp(\beta_k X^+) \,. \tag{A.1}$$

This is a solution to the equation of motion with a timelike linear dilaton $\Phi = -qX^0$ background if:

$$q\beta_k = \sqrt{2}\left(\frac{2}{\alpha'} - \frac{1}{2}k^2\right). \tag{A.2}$$

Since the tachyon couples to the worldsheet as a potential, the theory has a vacuum at $X^2 = 0$. One can show that expanding around this vacuum in the limit where the wavelength of oscillations k^{-1} is much larger than the string length l_s yields:

$$T(X^+, X^2) = \frac{\mu^2}{2\alpha'} \exp(\beta X^+) : (X^2)^2 : , \tag{A.3}$$

where $\mu^2 = \alpha' k^2 \mu_k^2$, and dots denote normal ordering. We refer the reader to [36] for additional aspects of the detailed derivation.

The physical interpretation of this is clear. Before the tachyon condenses, at $X^+ \to -\infty$, the string propagates in $D-1$ spatial dimensions. As the string reaches a regime where $T \sim 1$ (namely $X^+ \sim \beta^{-1} \log \mu$), the potential confines the string to the region where it is vanishing, at $X^2 = 0$. Strings that oscillate along the X^2 dimension will be expelled from the region of large tachyon condensate. This bubble thus interpolates between a region of $D-1$ spatial dimensions to one where the string can effectively propagate in $D-2$ dimensions. These types of bubbles were dubbed dimension-quenching or dimension-changing bubbles.

Turning now to the dynamical cobordism perspective, one can see in (A.3) that the tachyon field remains at a finite value (hence at a finite distance in field space) at the location of this bubble at $X^2 = 0$. This fits perfectly with the description of an interpolating wall as described in [17]. We thus interpret these dimension-quenching bubbles as examples of dynamical cobordism interpolating walls between bosonic theories of different dimensions.

As a side note, one can construct similar bubbles that kill more than one dimension by granting oscillatory dependence of the tachyon on extra dimensions. Furthermore, this dimension-quenching mechanism also extends to superstring theories and can be used to draw connections between supercritical Type 0 theories and their 10-dimensional critical counterparts [36].

B The partition function with a lightlike tachyon background

The computation of the partition function in the presence of the lightlike tachyon background is obtained evaluating the path integral without vertex operator insertions:

$$Z(\mu) = \int \mathcal{D}X^+ \mathcal{D}X^- \mathcal{D}X^i \mathcal{D}g \mathcal{D}(\text{ghosts}) e^{iS^{\text{deformed}}} , \tag{B.1}$$

where we have emphasized the fact that the integration along the lightlike directions does not affect the spacelike directions ($i = 2, \ldots, D-1$). Being at weak coupling, we only consider the one-loop contribution and so we have to evaluate the 2d action on a genus one worldsheet:

$$\begin{aligned} S^{\text{deformed}} =& -\frac{1}{2\pi\alpha'} \int_{\mathcal{M}_1} d^2\sigma \sqrt{g} g^{\alpha\beta} \left[-\partial_\alpha X^+ \partial_\beta X^- - \partial_\alpha X^- \partial_\beta X^+ + \partial_\alpha X^i \partial_\beta X_i \right] + \\ &+ \frac{1}{2\pi} \int_{\mathcal{M}_1} d^2\sigma \sqrt{g} R_2 \phi(X) - \frac{1}{2\pi} \mu \int_{\mathcal{M}_1} d^2\sigma \sqrt{g} e^{\beta X^+} . \end{aligned} \tag{B.2}$$

In analogy with the procedure in [45], we decompose the field $X^+(\tau,\sigma)$ into its zero and nonzero modes:

$$X^+(\tau,\sigma) = X_0^+ + \widehat{X}^+(\tau,\sigma)\,, \tag{B.3}$$

we get a standard integration for the zero mode:

$$\mathcal{D}X^+ = dX_0^+\mathcal{D}\widehat{X}^+. \tag{B.4}$$

Choosing the following convention to perform a Wick rotation:

$$\tau \mapsto i\tau_E \qquad X^i \mapsto iX_E^i \qquad \mu \mapsto -i\mu_E\,, \tag{B.5}$$

equation (B.1) becomes:

$$Z(\mu) = \int dX_0^+ \int \mathcal{D}\widehat{X}^+\mathcal{D}X^-\mathcal{D}X^i\mathcal{D}g\mathcal{D}(\text{ghosts})e^{-S_E^{\text{deformed}}}\,. \tag{B.6}$$

where the tachyonic potential gives an oscillating contribution to the integral in the condensate region. Such a behavior produces a truncation of the contributions to the integral coming from configurations with $X^+ \mapsto \infty$. The Euclidean 2d action is:

$$\begin{aligned} S_E^{\text{deformed}} =& \frac{1}{2\pi\alpha'}\int d^2\sigma_E\sqrt{g}\left[\partial_{\sigma^0}\widehat{X}^+\partial_{\sigma^0}X^- + \partial_{\sigma^1}X^-\partial_{\sigma^1}\widehat{X}^+ + \partial_\alpha X^i\partial_\alpha X^i\right] \\ &+ \frac{1}{2\pi}\int d^2\sigma_E\sqrt{g}R_2\phi(X) + \frac{i\mu_E}{2\pi}\int d^2\sigma_E\sqrt{g}e^{\beta X^+}\,. \end{aligned} \tag{B.7}$$

Using the variable change:

$$y = e^{\beta X_0^+} \qquad \longrightarrow \qquad dX_0^+ = \frac{dy}{\beta y}, \tag{B.8}$$

and making the dependence of the integrand on X_0^+ explicit, we obtain:

$$Z(\mu_E) = \int \mathcal{D}\widehat{X}^+\mathcal{D}X^-\mathcal{D}X^i\mathcal{D}g\mathcal{D}(\text{ghosts})\int_0^\infty \frac{dy}{\beta y}e^{-S_E^{\text{kinetic}}-S_E^{\text{dilaton}}-\frac{i\mu_E}{2\pi}\int d^2\sigma_E\sqrt{g}ye^{\beta\widehat{X}^+}}\,, \tag{B.9}$$

where $S_E^{\text{kinetic}} + S_E^{\text{dilaton}} = S_E^{\text{free}}$ are respectively the kinetic and the dilaton contributions in the Euclidean action (B.7).

Now, let us consider the following quantity:

$$\frac{\partial Z}{\partial\mu_E} = \int \mathcal{D}\widehat{X}^+\mathcal{D}(\text{others})\int_0^\infty \frac{dy}{\beta}e^{-S_E^{\text{free}}+\frac{-i\mu_E}{2\pi}\int d^2\sigma_E\sqrt{g}ye^{\beta\widehat{X}^+}}\left(\frac{-i}{2\pi}\int d^2\sigma_E\sqrt{g}e^{\beta\widehat{X}^+}\right)\,. \tag{B.10}$$

Let us finally perform the integration on the zero mode. We obtain:

$$\frac{\partial Z}{\partial\mu_E} = -\frac{1}{\beta\mu_E}\int \mathcal{D}\widehat{X}^+\mathcal{D}X^-\mathcal{D}X^i\mathcal{D}g\mathcal{D}(\text{ghosts})e^{-S_E^{\text{free}}}\,, \tag{B.11}$$

where S_E^{free} is the euclidean action of the world-sheet theory without the tachyon potential. Integrating with respect to μ and fixing a cutoff for X^+ such that $\mu_* = e^{\beta X_*^+}$, we obtain:

$$Z_1 = -\frac{\log(\mu_E/\mu_*)}{\beta}\widehat{Z}\,, \tag{B.12}$$

where $\widehat{Z}$ is the partition function for the free 2d theory, namely without the tachyon insertion and without integrating the zero modes of X^+. Note that the tachyon's contribution to the partition function is entirely encoded in the zero modes.

We can interpret this factor as a "size" of the direction X^+. Indeed, because of the potential barrier created by the condensation of the tachyon, no physical degrees of freedom penetrate inside the bubble wall, beyond $X^+ \sim 1$. As mentioned previously, the path integral is suppressed in this region. The direction X^+ thus has an effectively finite "size" that agrees with the estimate in (4.3).

C The critical exponent for general f_1

In this appendix we provide more details regarding how we obtain the scaling relation (4.27) for a general f_1 decaying faster than T^{-1}. The starting point is (4.26), which we rewrite as follows:

$$\Delta \sim \int \exp\big(- \frac{\mathcal{D}}{\sqrt{D-2}} - \log(T\mathcal{D}')\big) d\mathcal{D}\,, \tag{C.1}$$

where the prime stands for derivation with respect to T. Proving that the first term in the exponential is the dominant one in the limit $T \to \infty$ comes down to comparing the two terms:

$$-\frac{\mathcal{D}}{\sqrt{D-2}} \sim \log f_1 \quad \text{and} \quad -\log T\mathcal{D}' \sim \log\left(\frac{f_1}{|f_1'|T}\right)\,. \tag{C.2}$$

Notice that in the special case where f_1 is power-like $f_1 = T^{-k}$, with $k > 0$, the second term in the exponential is constant so one automatically obtains the scaling relation (4.27). For other choices of f_1, we wish to check that in the limit $T \to \infty$,

$$\frac{|\log(\frac{f_1}{|f_1'|T})|}{|\log f_1|} \to 0\,. \tag{C.3}$$

We consider a positive and monotonically decreasing function f_1, we require:

$$\left|\log\left(\frac{f_1}{|f_1'|T}\right)\right| \ll |\log f_1| \quad \text{as} \quad T \to \infty\,. \tag{C.4}$$

We know for a fact that $\log f_1$ is negative when $T \to \infty$. As we will show shortly, $\log(\frac{f_1}{|f_1'|T})$ is negative, then one can easily show that (C.4) implies:

$$|f_1'| \ll T^{-1}\,, \tag{C.5}$$

which is true for any function f_1 under consideration since $f_1 \ll T^{-1}$.

The question of whether (C.3) is verified is thus recast into the question of the sign of $\log(\frac{f_1}{|f_1'|T})$. In order to determine if this term is negative, we can write it as follows,

$$\log\left(\frac{f_1}{|f_1'|T}\right) = \log\left(\frac{f_1}{T}\right) - \log(|f_1'|)\,. \tag{C.6}$$

One clearly sees that the first term is negative whilst the second is positive. We would therefore like to show that the first one dominates:

$$|\log\left(\frac{f_1}{T}\right)| \gg |\log(|f_1'|) \quad \rightarrow \quad \frac{f_1}{T} \ll |f_1'| \tag{C.7}$$

This is condition is verified for all valid choices of f_1 that are not power-like. Indeed, we can use the trivial fact that $f_1 \ll T$ as $T \to \infty$ to show that $\frac{f_1}{|f_1'|} \ll T$. In the case where f_1 is a power-like function the inequality is not strict as $\frac{f_1}{T} \sim |f_1'|$, but we still recover the right scaling relations as mentioned previously.

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Published for SISSA by Springer

Received: *April 6, 2023*
Accepted: *May 29, 2023*
Published: *June 13, 2023*

Small black hole explosions

Roberta Angius, Jesús Huertas and Angel M. Uranga

Instituto de Física Teórica IFT-UAM/CSIC,
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain

E-mail: roberta.angius@csic.es, j.huertas@csic.es, angel.uranga@csic.es

Abstract: Small black holes are a powerful tool to explore infinite distances in moduli spaces. However, we show that in 4d theories with a scalar potential growing fast enough at infinity, it is energetically too costly for scalars to diverge at the core, and the small black hole puffs up into a regular black hole, or follows a runaway behaviour.

We derive a critical exponent characterizing the occurrence or not of such small black hole explosions, both from a 4d perspective, and in the 2d theory after an $\mathbf{S}^2$ truncation. The latter setup allows a unified discussion of fluxes, domain walls and black holes, solving an apparent puzzle in the expression of their potentials in the 4d $\mathcal{N}=2$ gauged supergravity context.

We discuss the realization of these ideas in 4d $\mathcal{N}=2$ gauged supergravities. Along the way we show that many regular black hole supergravity solutions in the literature in the latter context are incomplete, due to Freed-Witten anomalies (or duals thereof), and require the emission of strings by the black hole.

From the 2d perspective, small black hole solutions correspond to dynamical cobordisms, with the core describing an end of the world brane. Small black hole explosions represent obstructions to completing the dynamical cobordism. We study the implications for the Cobordism Distance Conjecture, which states that in any theory there should exist dynamical cobordisms accessing all possible infinite distance limits in scalar field space. The realization of this principle using small black holes leads to non-trivial constraints on the 4d scalar potential of any consistent theory; in the 4d $\mathcal{N}=2$ context, they allow to recover from a purely bottom-up perspective, several non-trivial properties of vector moduli spaces near infinity familiar from CY_3 compactifications.

Keywords: Black Holes in String Theory, Flux Compactifications, String and Brane Phenomenology

ArXiv ePrint: 2303.15903

https://doi.org/10.1007/JHEP06(2023)070

Contents

1 Introduction

The exploration of regions at infinite distance in moduli spaces (or in general, scalar field spaces) is a powerful source of new physics results. One instance is the celebrated swampland distance conjecture [1],[1] derived from adiabatic motion in exact moduli spaces. However, a physically more realistic avenue for the exploration of infinite distances in field space, pioneered in [5] (see also [6]), is the use of spacetime dependent solutions in effective field theories, in which certain scalars run along some spacetime coordinate.

In several setups, such solutions involve scalars which run off to infinite distance at a finite distance in spacetime. The resulting singularities describe physical objects whose properties relate to the infinite distance in interesting ways. In particular, the end of the world branes in dynamical cobordisms [7–11] (see also [12] for the swampland cobordism conjecture, [13–16] for early work and [17–22] for other related recent developments), display interesting scaling relations among the spacetime curvature and the spacetime and field theory distances. Other setups are 4d EFT strings [23–25], and small black holes (see e.g. [26] for a review, and [27] for discussion in the swampland context). In fact, they can also be regarded as dynamical cobordisms of the theories upon compactification on $\mathbf{S}^1$ for EFT strings and $\mathbf{S}^2$ for 4d small black holes.

An important advantage of dynamical explorations of infinity in field space, as compared with the adiabatic one, is that the latter may be obstructed in the presence of non-trivial scalar potentials growing at infinity,[2] as emphasized in [28]. In the context of ETW branes, the introduction of non-trivial potentials growing at infinity was included in the analysis in [9]. In the present work, we deal with this question for small black holes. For concreteness, we focus on 4d small black holes, but we expect similar conclusions to apply in other dimensions.

Small black holes are very robust solutions in theories with U(1)'s and scalar-dependent gauge kinetic functions. Extremal small black holes may be regarded as the endpoint of the evaporation of large black holes with some charge. In addition, as emphasized in [27], they are interesting probes of infinity in scalar moduli space.[3] As you approach the black

[1] See e.g. [2–4] for recent developments in CY moduli spaces.

[2] In fact, taking fixed vevs in the slope of a non-trivial potential leads to dynamical tadpoles, which besides being unaesthetical, can also lead to specific incompatibilities with swampland constraints [19].

[3] See [29] for a different exploration of infinity, using *large* black holes.

hole core, a black hole potential drives the scalars towards infinite distance in field space, in a most radical realization of the attractor mechanism. Indeed, the near-core solution is independent of the value of moduli in the asymptotic vacuum infinitely away from the black hole.

One could even expect a similar phenomenon, even in the presence of a 4d potential fixing the asymptotic moduli; the 4d scalar potential would be irrelevant near the black hole core, still allowing scalars to attain infinity in field space. However, this may not be so if the potential grows fast enough near infinity in field space, so that it is able to drag the scalars back from infinity. In such case, the small black hole core explodes,[4] possibly puffing up into a black hole solution of finite size horizon, or perhaps following a runaway with no static endpoint configuration.

In this paper we provide explicit examples of the different possibilities, and give precise criteria on the 4d potential for each of them. In particular, we find that for small black holes exploring an infinite distance limit (with gauge coupling scaling as a decreasing exponential along it), the small black hole explosion occurs when the 4d potential grows faster (or equal) than a critical exponent, related to that of the gauge coupling, but independent of the black hole charge.

We study the implications of small black hole explosions for the capability of small black holes to explore infinite distance limits in scalar moduli spaces. Our findings motivate interesting non-trivial constraints on the growth near infinity of 4d scalar potentials in any consistent theory of quantum gravity. In fact, in 4d $\mathcal{N}=2$ theories we obtain constraints on the behaviour of the Kähler potential near infinity in moduli space, matching, from a fully bottom-up approach, non-trivial properties of CY_3 compactifications.

We carry out the analysis both in the 4d picture, as well as in terms of the 2d truncation on the angular $\mathbf{S}^2$. The latter viewpoint is interesting in itself [9], since the spacetime dependent solutions correspond to dynamical cobordisms, i.e. running configurations (for the scalars and the size of $\mathbf{S}^2$), with the small black hole cores corresponding to the ETW branes, i.e. the cobordism defect required to absorb the charges. From this perspective, small black holes puffed up into regular black holes correspond to running solutions, which, instead of hitting an ETW brane which ends spacetime, relax to a minimum of the effective potential describing the near horizon AdS_2 vacuum. They represent solutions which fail to realize dynamical cobordisms ending spacetime. In this respect, they provide a large class of realizations of the idea, introduced in [9] (in the context of the D2/M2-brane solution [31]), that crouching corrections in the near core region of an ETW brane can reveal hidden vacua coming down from infinity in moduli space.

Along the way in this work, we find several other interesting results. We show that this 2d perspective is optimal to discuss the interplay of domain walls, black holes and fluxes recently reviewed in [32] in the context of in 4d $\mathcal{N}=2$ (gauged) supergravities. Moreover, our 2d perspective explains that the seemingly different expression for the black hole and

[4]The name evokes the similar (but converse) effect in [30], where a black hole attractor drags the 4d scalar vevs from an asymptotic vacuum to another. Here we have a 4d potential dragging the scalar from the infinite values at the small black hole core to some finite values at a regular horizon. Both setups deal with the competition of black hole attractor potentials and 4d potentials.

the 4d potentials in terms of the covariantly holomorphic quantities, actually agree upon the inclusion of an extra field associated to the $\mathbf{S}^2$ size (see [32] for an alternative proposed explanation). We also use a 2d version the entropy function formalism [33] (see [34] for a review) to provide a novel derivation of the effective potential in [35], controlling the attractor mechanism in the presence of both black hole and 4d scalar potentials.

Although our discussion is general, a particular setup in which to study the competition of black hole attractor potentials and 4d potentials is 4d $\mathcal{N} = 2$ gauged supergravity. Indeed, regular black holes which feature a balance of both kinds of potentials have appeared in the literature, including BPS solutions [36–38], and non-supersymmetric solutions [39–41] (see [42] and references therein for a recent discussion). However, we argue that, upon a closer look, there is a seemingly unnoticed consistency condition in general not satisfied by these solutions. The problem is related to the non-invariance of the configurations under some symmetries underlying Abelian gaugings, and in certain microscopic setups shows up as a Freed-Witten anomaly. The configurations can thus be amended by having the black hole emit a number of 4d strings (or equivalently having those strings fill up the near horizon AdS_2). For regular black holes with large charges Q, the anomaly is a $1/Q$ effect, meaning that these strings can be described as probes in the above classical geometries. On the other hand, for small black holes, the strings have a dramatic effect, and lead to a second mechanism for small black hole explosion, which is possible even for subcritical 4d potentials.

The paper is organized as follows. In section 2 we discuss small black hole explosions in 4d: in section 2.1 we introduce small black hole solutions with no 4d potential, in section 2.2 we derive the equations of motion in the presence of 4d potentials and the critical exponent, and in section 2.3 we construct the critical small black hole solution. In section 3 we approach the systems from the perspective of its truncation to 2d, performed in section 3.1. In section 3.2 we describe the interplay of domain walls, black holes and fluxes, and explain the relation between the seemingly different expression for the potentials by using the $\mathbf{S}^2$ breathing mode. In section 3.3 we adapt the 4d entropy functional formalism to our 2d setup and derive the effective potential for the scalars. In section 3.4 we use the effective potential to discuss small black hole explosions and recover the critical exponent. In section 4 we discuss running solutions of the 2d theory: in section 4.1 we use this perspective to recover the equations of motion derived from 4d approaches in the literature. In section 4.2 we describe solutions for small black holes in the subcritical and critical cases. In section 4.3 we match this picture with the local description of dynamical cobordisms. In section 5 we describe the consistency conditions requiring certain black holes to emit strings, and their implications for small and large black holes. In section 5.1 we discuss dual descriptions of Abelian gaugings and their corresponding symmetries, and the string emission effect. In section 5.2 we review this in the illustrative example of the Freed-Witten anomaly. In section 5.3 we discuss large black holes in the 4d $\mathcal{N} = 2$ gauged supergravity context, and their emission of strings. In section 5.4 we show that small black holes emitting strings explode, even if the 4d potential is subcritical. In section 6 we discuss the ability of small black holes to continue exploring infinite distance limits in moduli space, even in the presence of 4d potentials. After reviewing the situation in exact moduli spaces in section 6.1, we introduce 4d potentials in section 6.2 in an illustrative example in the context of the *STU* model in 4d $\mathcal{N} = 2$ gauged supergravity. This motivates the proposal of a general criterion restricting the properties of general 4d theories in section 6.3. In section 7 we offer some final thoughts. Appendix A contains some technical aspects complementing the discussion of section 2. Appendix B provides a derivation of the effective potential of [35] from the entropy function formalism in 4d. In appendix C we provide an alternative rederivation of the equations of motion for running solutions of the 2d theory. Appendix D collects some basic ingredients of 4d $\mathcal{N} = 2$ (possibly gauged) supergravity.

Note added. As this work was being completed, we noticed [43] which also explores the use of small black holes to constrain the properties of 4d theories near infinite distance limits. It would be interesting to explore the interplay of both approaches to these systems.

2 Small black hole explosions: the 4d view

In this section we discuss small black holes and their fate under the introduction of a 4d potential, in a fairly direct 4d description. We follow a formulation similar to [44, 45] (see also [27] for a recent discussion).

2.1 Small black holes

We start with the structure of small black holes in the absence of 4d potential. We consider 4d Einstein-Maxwell theory coupled to a real scalar ϕ:

$$S = \int d^4x\sqrt{-g}\left[R - 2|d\phi|^2 + \frac{1}{2g^2}|F|^2\right] \tag{2.1}$$

where the gauge coupling is a function $g(\phi)$.

We are looking for spherically symmetric static solution with electric charge Q. We take the ansatz

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left(\frac{d\tau^2}{\tau^4} + \frac{1}{\tau^2}d\Omega_2^2\right) \tag{2.2}$$

$$F_2 = 2\sqrt{2}g^2Qe^{2U}d\tau\wedge dt \tag{2.3}$$

$$\phi = \phi(\tau) \tag{2.4}$$

Here $d\Omega_2$ is the volume form on a unit $\mathbf{S}^1$, and τ runs from the small black hole core (located at $\tau\to-\infty$) to asymptotic Minkowski space (located at $\tau = 0$). In general, one can introduce an additional function $h(\tau)$, but it is possible to set $h = 1$, as discussed in appendix A.1.

Plugging the ansatz into the action, we get the 1-dimensional action:

$$S_{1d} = \int d\tau\left\{\dot{U}^2 + \dot{\phi}^2 + 2g^2Q^2e^{2U}\right\} \tag{2.5}$$

This describes the dynamics of one particle in two dimensions under a potential given by (minus) the last term. The equations of motion read

$$\ddot{U} = 2Q^2e^{2U}g(\phi)^2\ , \qquad \ddot{\phi} = Q^2e^{2U}\left(g(\phi)^2\right)' \tag{2.6}$$

where prime denotes derivative with respect to ϕ. This must be supplemented with the ('zero energy') constraint

$$\left[\frac{1}{2}(\dot{U}^2 + \dot{\phi}^2)\right] - g^2 Q^2 e^{2U} = 0 \tag{2.7}$$

In this work we are going to focus on the particular class of theories with gauge coupling which for $\phi \to \infty$ behave as

$$g = e^{-\alpha\phi} \tag{2.8}$$

This includes theories arising from 4d $\mathcal{N} = 2$ supergravity. Moreover, the exponential dependence is the behaviour required by the distance conjecture, see [27] for a recent discussion. It would be straightforward to carry out the analysis of our work for other dependencies, which are left as an exercise for the interested reader.

In this case, the equations of motion read

$$\begin{aligned} \ddot{\phi} &= -2\alpha Q^2 e^{2U-2\alpha\phi} \\ \ddot{U} &= 2Q^2 e^{2U-2\alpha\phi} \end{aligned} \tag{2.9}$$

The combination $\alpha U + \phi$ does not appear in the potential, and corresponds to free motion. Using the initial conditions $U(\tau = 0) = 0$ (so that we recover asymptotic flat space) and $\phi(\tau = 0) = \phi_0$ (the asymptotic vev at an arbitrary value in the scalar moduli space), we have

$$\alpha U + \phi = \phi_0 + v\tau \tag{2.10}$$

Let us go for the combination that does appear in the potential. For shorthand notation, we denote

$$\varphi \equiv U - \alpha\phi \ , \qquad q^2 = 2(1 + \alpha^2) Q^2 \tag{2.11}$$

The equation $\ddot{\varphi} = q^2 e^{2\varphi}$ admits the solution

$$\varphi = -\log(-q\tau + c)\,, \tag{2.12}$$

where the initial conditions at $\tau = 0$ implies that the constant $c = e^{\alpha\phi_0}$.

The solution is thus

$$\begin{aligned} U - \alpha\phi &= -\log(-q\tau + e^{\alpha\phi_0}) \\ \alpha U + \phi &= \phi_0 \end{aligned} \tag{2.13}$$

where we have already set $v = 0$ in (2.10), as this turns out to be required by the constraint (2.7). This gives

$$\begin{aligned} e^{-2U} &= e^{-\frac{2\alpha}{1+\alpha^2}\phi_0} (-q\tau + e^{\alpha\phi_0})^{\frac{2}{1+\alpha^2}} \\ \phi &= \frac{1}{1+\alpha^2}\phi_0 + \frac{\alpha}{1+\alpha^2}\log(-q\tau + e^{\alpha\phi_0}) \end{aligned} \tag{2.14}$$

The region $\tau \to -\infty$ (the small black hole core) is a point at finite spacetime distance, at which the scalar goes off to infinite distance with a logarithmic profile, and the size of the $\mathbf{S}^2$, given by e^{-2U}/τ^2 in (2.2), goes to zero. It is important to notice that this local behaviour is independent of the asymptotic value of ϕ. As will be clear in later sections, this is the small black hole version of the attractor mechanism.

For completeness we also see that at $\tau \to -\infty$, g tends to zero as

$$g \sim \tau^{-\frac{\alpha^2}{1+\alpha^2}} \tag{2.15}$$

2.2 Perturbing the small black hole with a 4d potential

We now consider introducing a 4d potential $V(\phi)$ in the theory. The action reads

$$S = \int d^4x\sqrt{-g}\left[R - 2|d\phi|^2 + \frac{1}{2g^2}|F|^2 + 2V\right] \tag{2.16}$$

The discussion of h now is more involved, but, as explained in appendix A.2, one can show that fairly generically $h \simeq 1$ near $\tau \to -\infty$. Since we are interested in the fate of the small black hole core, we simplify the discussion and just proceed with $h = 1$. The more careful analysis will be accounted for in the explicit solutions in the next sections.

The dynamics is described by the following 1-dimensional action:

$$S_{1d} = \int d\tau \left\{\dot{U}^2 + \dot{\phi}^2 + 2g^2Q^2e^{2U} - e^{-2U}\tau^{-4}V\right\} \tag{2.17}$$

The 4d scalar potential can modify the solution substantially. For instance, near $\tau = 0$, the second term dominates, and the scalar will be fixed at the minimum of the 4d potential (if it exists[5]). On the other hand, near $\tau \to -\infty$, the black hole potential term would seem to dominate and maintain the basic features of the small black hole: a scalar going off to infinity at the black hole core. This is indeed the case unless the 4d potential grows too fast near $\phi \to \infty$ (i.e. near infinity in field space), as we explore now in more detail.

In order to proceed, we need to parametrize the asymptotic behaviour of the 4d scalar near $\phi \to \infty$. As discussed in [9], a well-motivated proposal is an exponential[6]

$$V = V_0 e^{\delta\phi}\ , \qquad V_0 > 0 \tag{2.18}$$

We emphasize that we focus on potentials growing at infinity $\phi \to \infty$, namely $V_0 > 0$ and $\delta > 0$, so that they have a chance of competing with the black hole and prevent the scalars from reaching off to infinity. As we will see, (2.18) is also a natural parameterization to make an easy comparison with the black hole pull on scalars, controlled by the exponential gauge coupling (2.8).

[5] We assume that such minimum exists, and consider small black holes whose asymptotic vev ϕ_0 is precisely that minimum. The fate of the small black hole core will nevertheless be independent of this assumption.

[6] We note that this δ differs from that in [9] by a factor of $\sqrt{2}$, due to different normalization of the scalar kinetic term.

We can now check when the last term in (2.17) becomes relevant as compared with that for a small black hole solution, by simply evaluating them on the small black hole solution (2.14). We have that, as $\tau \to -\infty$,

$$\begin{aligned} g^2 Q^2 e^{2U} &= Q^2 e^{2(U-\alpha\phi)} \sim \tau^{-2} \\ \tau^{-4} e^{-2U} V &\sim \tau^{-2+\frac{\alpha(\delta-2\alpha)}{1+\alpha^2}} \end{aligned} \tag{2.19}$$

Hence, for the two terms to be comparable as $\tau \to -\infty$, we get the criticality condition

$$\delta = 2\alpha \tag{2.20}$$

which remarkably relates the behaviour of the 4d potential and the gauge kinetic functions near infinity in field space.

For $\delta < 2\alpha$, the 4d potential becomes subdominant near $\tau \to -\infty$, so the small black hole behaviour prevails, and the small black hole remains small. On the other hand, for $\delta > 2\alpha$, the 4d potential is too strong near the putative small black hole core, so that it is energetically too costly to maintain the small black hole behaviour; the small black hole must explode into a finite size solution, either in a runaway towards larger and larger sized, or to a puffed-up finite size black hole whose horizon scalar value is located in a regime where both terms can attain a balance. Which of these two possibilities is realized depends on details beyond the simple exponential approximation to the 4d potential and gauge kinetic function.

We will see examples in later sections.

2.3 Solution for the critical case

Let us consider the critical case $\delta = 2\alpha$. The equations of motion are

$$\begin{aligned} h_0^2 \ddot{U} &= 2Q^2 e^{2\varphi} + \frac{V_0}{\tau^4} e^{-2\varphi} \\ h_0^2 \ddot{\phi} &= -2\alpha Q^2 e^{2\varphi} - \alpha \frac{V_0}{\tau^4} e^{-2\varphi} \end{aligned} \tag{2.21}$$

where h_0 is a constant discussed in appendix A.2, and we have defined

$$\varphi = U - \alpha\phi \tag{2.22}$$

There is one freely moving combination

$$\alpha \ddot{U} + \ddot{\phi} = 0 \quad \to \quad \alpha U + \phi = 0 \tag{2.23}$$

(with hindsight we have chosen integration constants equal to zero, since ϕ_0 is irrelevant for $\tau \to -\infty$, and the velocity is eventually set to zero by the hamiltonian constraint).

The orthogonal combination φ obeys

$$\ddot{\varphi} = q^2 e^{2\varphi} + v_0 \tau^{-4} e^{-2\varphi} \tag{2.24}$$

with

$$q^2 = \frac{2Q^2(1+\alpha^2)}{h_0^2} \ , \qquad v_0 = \frac{V_0(1+\alpha^2)}{h_0^2} \tag{2.25}$$

We now explore the existence of small black hole solutions, by using the following profile for φ

$$\varphi = -\log(-A\tau) \tag{2.26}$$

Plugging the trial function (2.26) in the eom (2.24), we have

$$\frac{1}{\tau^2} = \frac{q^2}{A^2\tau^2} + \tau^{-4} v_0 A^2 \tau^2 \tag{2.27}$$

So we get a solution if

$$v_0 A^4 - A^2 + q^2 = 0 \tag{2.28}$$

This is bi-quadratic. We solve for A^2 as

$$A^2 = \frac{1 - \sqrt{1 - 4q^2 v_0}}{2v_0} \tag{2.29}$$

where we choose the negative signs because it reproduces the usual small black hole when v_0 is very small. We then have

$$U = -\frac{1}{1+\alpha^2}\log(-A\tau) \ , \qquad \phi = \frac{\alpha}{1+\alpha^2}\log(-A\tau) \tag{2.30}$$

Notice that this is compatible with actually having constant h in (A.12),

$$h_0 = 1 + V_0 A^2 \tag{2.31}$$

for a suitable choice of h_0 (determined by combining the equation above with (2.29) and using the definitions (2.25)). It is also straightforward to compute that the constraint in the last equation in (A.10) is solved.

Since we are interested in $v_0 > 0$, we get the constraint $q^2 v_0 < 1/4$. This means that even black holes of moderate charge admit no solution, hence cannot remain small, but rather explode. In later sections we will describe tools characterizing the ultimate fate of the former small black holes. In particular, in the critical case it is possible that the black holes are stabilized at a finite horizon puffed up black holes, as we will see in later sections.[7]

3 Small black hole explosions: the 2d view

In this section we describe small black holes and their possible explosions from the perspective of the 2d theory resulting after reduction on the angular $\mathbf{S}^2$. Note that, because the compactification scale of the $\mathbf{S}^2$ will ultimately be comparable with the energy scales of the 2d theory this is intended as a consistent truncation rather than a reduction to a 2d effective field theory.

[7] In addition to the 4d $\mathcal{N} = 2$ examples discussed in section 5, see [46, 47] for examples of puffed up black holes in a phenomenological bottom up approach. It is easy to check they correspond to the critical case.

3.1 The 2d reduction

In this section we consider the 2d theory resulting from compactification of the 4d theory on $\mathbf{S}^2$, with gauge fluxes.[8]

Consider the following 4d action for a set of complex scalars z^i coupled to a set of Abelian electric and magnetic gauge bosons[9]

$$S_{4d} = \int d^4x\sqrt{-g_4}\left\{R_4 - 2g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu\bar{z}^{\bar{\jmath}} - \frac{1}{2}\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} - \frac{1}{2}\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma}G_{\Lambda\mu\nu}G^{\mu\nu}_{\Sigma} - 2V\right\}. \tag{3.1}$$

Here $\mathcal{N}_{\Lambda\Sigma}$ and $\mathcal{N}^{\Lambda\Sigma}$ are the (possibly non-diagonal) electric and magnetic couplings, F^Λ and G_Λ are the electric and magnetic field strengths, and V is a general 4d scalar potential.

This action is general, but we maintain a notation adapted to future use in 4d $\mathcal{N}=2$ examples.
We perform a compactification on $\mathbf{S}^2$ with the ansatz:

$$\begin{aligned} ds_4^2 &= ds_2^2 + e^{2\sigma}d\Omega_2^2 \\ F^\Lambda &= \sqrt{2}p^\Lambda \sin\theta d\theta\wedge d\varphi\ , \qquad G_\Lambda = \sqrt{2}q_\Lambda\sin\theta d\theta\wedge d\varphi \end{aligned} \tag{3.2}$$

where p^Λ and q_Λ are respectively the electric and magnetic charges[10] Also, the 2d spacetime dependent warp factor $e^{2\sigma}$ controls the size of the $\mathbf{S}^2$. The reduced 2-dimensional action is (after removing an overall 4π factor):

$$S_{2d} = \int d^2x\sqrt{-g_2}e^{2\sigma}\left[R_2 - 4\Delta\sigma - 6\left(\partial\sigma\right)^2 - 2g_{i\bar{\jmath}}\partial_\mu z^i\partial^\mu\bar{z}^{\bar{\jmath}} + 2e^{-2\sigma} - 2e^{-4\sigma}V_{BH} - 2V\right]$$

Where we have introduced the black hole potential

$$V_{\mathrm{BH}} = -\frac{1}{2}\left(p^\Lambda\mathrm{Im}\,\mathcal{N}_{\Lambda\Sigma}p^\Sigma + q_\Lambda\mathrm{Im}\,\mathcal{N}^{\Lambda\Sigma}q_\Sigma\right) \tag{3.3}$$

Integrating by parts the term $\Delta\sigma(r)$, we get an additional contribution for its kinetic term, giving:

$$S_{2d} = \int d^2x\sqrt{-g_2}e^{2\sigma}\left[R_2 + 2\left(\partial\sigma\right)^2 - 2g_{ij}\partial z^i\partial\bar{z}^{\bar{j}} + 2e^{-2\sigma} - 2e^{-4\sigma}V_{BH} - 2V\right] \tag{3.4}$$

3.2 Interplay of black holes, fluxes and domain walls

Let us make a small aside in this section, which is independent from the general discussion.

A good motivation for the 2d perspective is that it puts both the 4d potential and the black hole potential on a similar footing, namely, they both contribute to the potential of

[8] We focus on solutions with $\mathbf{S}^2$ horizons, although in asymptotically AdS spaces there can be horizons with zero or negative curvature.

[9] As compared with section 2, we have changed the sign of the gauge kinetic terms to adapt it to the compactification below, and the sign of the scalar potential to adapt to complex fields and explore asymptotic regions towards $z^i \mapsto -i\infty$.

[10] Note that this ansatz is implicitly consistent with electric/magnetic duality, with the proviso that one introduces additions 2d spacetime filling components of the fields strengths.

the 2d theory. A particularly nice realization of this is in the context of 4d $\mathcal{N} = 2$ theories, see appendix D for notation and definitions, as we discuss next.

In 4d $\mathcal{N} = 2$ gauged supergravity, the scalar potential can be regarded as arising from (possibly generalized) fluxes in the compactification. These can be regarded as having been produced by domain wall, carrying (3-form) charges g_Λ, g^Λ whose central charge $\mathcal{L}$ gives the superpotential (D.12)

$$\mathcal{L} = e^{\mathcal{K}/2}\langle \mathcal{G}, v\rangle = e^{\mathcal{K}/2}(Z^\Lambda g_\Lambda - \partial_\Lambda F g^\Lambda) \tag{3.5}$$

leading to a 4d scalar potential (D.13)

$$V = g^{i\bar{\jmath}}\mathcal{D}_i\mathcal{L}\,\bar{\mathcal{D}}_{\bar{\jmath}}\bar{\mathcal{L}} - 3|\mathcal{L}|^2 \tag{3.6}$$

where we recall that $\mathcal{D}_i f = \partial_i f + \frac{1}{2}(\partial_i\mathcal{K}\cdot f)$

On the other hand, black holes with charges q_Λ, q^Λ arise from 4d particles characterized by a central charge (D.19)

$$\mathcal{Z} = e^{\mathcal{K}/2}(Z^\Lambda q_\Lambda - \partial_\Lambda F q^\Lambda) \tag{3.7}$$

The black hole potential (D.20) is then given by

$$V_{BH} = g^{i\bar{\jmath}}\mathcal{D}_i\mathcal{Z}\,\bar{\mathcal{D}}_{\bar{\jmath}}\bar{\mathcal{Z}} + |\mathcal{Z}|^2 \tag{3.8}$$

The identical structure of the central charges $\mathcal{L}$ and $\mathcal{Z}$ arises basically from the fact that, in the microscopic description, both the domain walls and the 4d particles are obtained from the same kind of object in what respects to the internal geometry. For instance, in type IIB compactifications, domain walls arise from D5-branes wrapped on 3-cycles of the internal CY, while particles arise from D3-branes wrapped on 3-cycles as well. Hence, the corresponding central charges are controlled by the same geometric quantities. Clearly, we have a similar picture for the (mirror) type IIA compactifications, where domain walls arise from D2-, D4-, D6, D8-branes wrapped on 0-, 2-, 4-, 6-cycles, while 4d particles arise from D0-, D2-, D4-, D6-branes on the same kind of even-dimensional cycles.

On the other hand, there is a seemingly puzzling difference in the expression of the corresponding potentials V and V_{BH}. This was recently considered in [32], but here we propose a more natural explanation. The expression for the potentials differ because they correspond to potentials in theories of different dimensionality. However, in the context of our $\mathbf{S}^2$ compactification, both 4d domain walls and 4d particles compactify onto 2d particles (or 2d domain walls, since in 2d both kind of objects are the same). Hence in this context, the contribution of both potentials must be on equal footing, as we saw in (3.4), and morally, given by the 2d formula $V = |\mathcal{D}\mathcal{Z}|^2 + |\mathcal{Z}|^2$, where here $\mathcal{Z}$ means the central charge of either kind of object.

The point is that, the different microscopic origin of V and V_{BH} manifests in their different coupling with the new field σ arising from the $\mathbf{S}^2$ compactification. The presence of this extra field in the theory accounts for the factor of $+4|\mathcal{L}|^2$ necessary to turn (3.6) into the form (3.8). The argument is as follows[11]

[11]This is analogous to how in 4d $\mathcal{N} = 1$ 'no-scale' models, the scalar potential $|\tilde{D}W|^2 - 3|W|^2$ turns into $|DW|^2$ (where the $\tilde{D}$ in the first expression includes the Kähler moduli and in the second does not).

Consider complexifying the $\mathbf{S}^2$ breathing modulus (e.g. by including the NSNS 2-form background in type II models) into a new complex modulus $z = b + ie^{-\sigma}$, and let us introduce a Kähler potential

$$k(z, \overline{z}) = -4\log(-i(z - \overline{z})) \tag{3.9}$$

Note that this reproduces the kinetic term of σ in (3.4).

There is thus a total Kähler potential given by

$$\tilde{\mathcal{K}} = k(z, \overline{z}) + \mathcal{K} \tag{3.10}$$

where the last piece is the usual 4d Kähler potential for vector multiplet moduli. In what follows, we use a tilde to denote quantities defined with this modified Kähler potential. For instance, this also modifies the Kähler derivatives to

$$\tilde{D}_i f = \partial_i f + \frac{1}{2}\partial_i\tilde{\mathcal{K}} \cdot f = \partial_i f + \frac{1}{2}\partial_i(k + \mathcal{K}) \cdot f \tag{3.11}$$

Let us now consider the modified version of the superpotential (3.5). We have

$$\tilde{\mathcal{L}} = e^{\tilde{\mathcal{K}}/2}\langle \mathcal{G}, v\rangle = e^{k/2}\mathcal{L} \tag{3.12}$$

The 2d potential will be given by the tilded version of (3.6)

$$V = |\tilde{D}\tilde{\mathcal{L}}|^2 - 3|\tilde{\mathcal{L}}|^2 \tag{3.13}$$

Computing the derivatives

$$\tilde{D}_z\tilde{\mathcal{L}} = \frac{4}{(z - \overline{z})^3}\mathcal{L} \ , \qquad \tilde{D}_i\tilde{\mathcal{L}} = e^{k/2}D_i\mathcal{L} \tag{3.14}$$

we easily get

$$V = g^{z\overline{z}}\tilde{D}_z\tilde{\mathcal{L}}\overline{\tilde{D}_{\overline{z}}\tilde{\mathcal{L}}} + e^k g^{i\overline{j}}D_i\mathcal{L}\overline{D_{\overline{j}}\mathcal{L}} - 3e^k|\mathcal{L}|^2 = |D\tilde{\mathcal{L}}|^2 + |\tilde{\mathcal{L}}|^2 \tag{3.15}$$

As promised, we recover an expression identical to the black hole potential (3.8), by simply replacing the corresponding central charges.

Let us finish by mentioning another argument for the fact that the 4d superpotential must be appropriately dressed with the $\mathbf{S}^2$ breathing mode to becomes *on par* with the 2d black hole central charge. From our above discussion, the natural covariantly holomorphic quantity in the 2d theory is a combination of $\mathcal{Z}$ and the dressed superpotential (3.12),

$$e^{k/2}\mathcal{L} = -\frac{1}{(z - \overline{z})^2}\mathcal{L} \sim e^{2\sigma}\mathcal{L} \tag{3.16}$$

This has precisely the required structure of the holomophic superpotential (D.23) introduced in [37].

3.3 The entropy function and the effective potential

In this section we use the 2d action (3.4) to obtain the conditions holding in the near horizon limit of a general regular black hole solution. The computation turns out to be a 2d version of the entropy functional formalism computation, see appendix B for the 4d computation, following [33] (see [34] for a review).

We consider an AdS_2 ansatz[12] for the metric

$$ds_2^2 = v_1 \left(-r^2 dt^2 + \frac{dr^2}{r^2} \right) \tag{3.17}$$

We also take a constant value for the moduli z, and denote the size of the $\mathbf{S}^2$ by v_2

$$e^{2\sigma} = v_2 \tag{3.18}$$

Evaluating (3.4) on this solution, we get the quantity

$$\mathcal{E}(v_1, v_2, p, q) = -4\pi\sqrt{-g_2}\mathcal{L}_2 = 8\pi v_2 - 8\pi v_1 + 8\pi V_{BH}\frac{v_1}{v_2} + 8\pi V v_1 v_2 \tag{3.19}$$

where the factor of 4π is just to meet the conventions of the entropy functional formalism. Note that in this formalism, as compared with appendix B, it is not necessary to perform the Legendre transform, since the magnetic charges were included in the original action.

The condition to get AdS_2 vacua corresponds to the minimization of (3.19) with respect to the variables v_1, v_2 and the moduli z. We have

$$\begin{aligned}
&\text{(i)} && \frac{\partial\mathcal{E}}{\partial v_1} = 0\,, && V v_2^2 - v_2 + V_{BH} = 0 \\
&\text{(ii)} && \frac{\partial\mathcal{E}}{\partial v_2} = 0\,, && 1 - V_{BH}\frac{v_1}{v_2^2} + V v_1 = 0 \\
&\text{(iii)} && \left.\frac{\partial\mathcal{E}}{\partial z^i}\right|_{z^i_H} = 0\,, && \frac{v_1}{v_2}\left.\frac{\partial V_{BH}}{\partial z^i}\right|_{z^i_H} + v_1 v_2 \left.\frac{\partial V}{\partial z^i}\right|_{z^i_H} = 0
\end{aligned} \tag{3.20}$$

The first two give

$$v_2 = \frac{1 - \sqrt{1 - 4VV_{BH}}}{2V}\,, \qquad v_1 = \frac{v_2}{\sqrt{1 - 4V_{BH}V}} \tag{3.21}$$

and replacing into the third, we recover the extremization of the quantity (which in fact corresponds to the $\mathbf{S}^2$ size, hence the horizon area)

$$V_{\text{eff}} = \frac{1 - \sqrt{1 - 4V_{BH}V}}{2V} \tag{3.22}$$

This turns out to be the effective potential derived in [35], in a different approach related to that in section 4.1. This is just a reflection of the equivalence of the entropy functional and attractor mechanism.

[12] We focus on this, rather than flat of dS_2, at least for the time being.

3.4 Small black hole explosions from effective potential

It is now a simple exercise to exploit V_{eff} to determine the fate of small black holes upon the introduction of a 4d potential. Consider a small black hole with purely electric charge Q. We take the U(1) gauge coupling $g(\phi) = e^{-\alpha\phi}$, as in previous discussions (generalization to other possible functional dependencies is straightforward), hence

$$V_{BH} = 2Q^2 e^{-2\alpha\phi} \tag{3.23}$$

Since we are interested in the fate of the core of the small black hole, where $\phi \to \infty$, we parametrize the 4d scalar potential by an exponential as in (2.18)

$$V = V_0 e^{\delta\phi} \quad V_0 > 0 \tag{3.24}$$

The effective potential (3.22) is

$$V_{\text{eff}} = \frac{1 - \sqrt{1 - 8V_0 Q^2 e^{(\delta-2\alpha)\phi}}}{2V_0 e^{\delta\phi}} \tag{3.25}$$

We can use it to easily recover the different cases introduced in section 2:

- If $\delta < 2\alpha$, we may expand the square root and get

$$V_{\text{eff}} \simeq 2Q^2 e^{-2\alpha\phi} = V_{BH} \tag{3.26}$$

  The 4d potential is too weak at the black hole core and the attractor is unchanged. The scalar is dragged to $\phi \to \infty$, and the small black hole remains small. Note that this does not rule out the possible existence of finite size black holes, although this is model dependent as it involves the behaviour of the gauge coupling and 4d potential at finite values of the moduli; hence, the model independent results on which our analysis focuses is that the small black hole persists.

- The critical case $\delta = 2\alpha$ leads to

$$V_{\text{eff}} = \frac{1 - \sqrt{1 - 8V_0 Q^2}}{2V_0}\, e^{-2\alpha\phi} \tag{3.27}$$

  For $V_0 Q^2 \leq 1/8$, the effective potential is exponentially damped as in V_{BH}, but with different prefactor. The scalar is still dragged to $\phi \to \infty$ by the black hole, which remains small. On the other hand, for $V_0 Q^2 > 1/8$, the effective potential is not well defined, meaning that there is no horizon, neither at finite values of the moduli (and black hole size) nor at infinity. Of course this conclusion is reliable only as long as the approximation of the gauge coupling and 4d potential as simple exponentials holds. If their behaviour away from infinity in moduli space is different, the small black hole may stabilize into a puffed-up finite size black hole.

- When $\delta > 2\alpha$, the exponential in the square root blows up and the effective potential is not well defined. As above, the small black hole explodes in a runaway fashion, at least in the region where the exponential approximations to the gauge coupling and 4d potential are reliable.

4 Small black hole explosions: the dynamical cobordism view

As explained in the introduction, the $\mathbf{S}^2$ reduction in the previous section is the natural setup in which small black hole solutions can be regarded as dynamical cobordisms of the resulting 2d theory, given by the running of the 4d scalar and the $\mathbf{S}^2$ size along the radial direction. The small black hole core plays the role of the ETW brane in the sense of [9], namely the cobordism defect that removes the fluxes and allows spacetime to end in the radial direction. From this perspective, small black hole explosions signal a failure of the dynamical cobordism, due to growing corrections as one approaches the putative ETW brane; in case a puffed-up black hole solution exists, the running solution describes the eventual stabilization of the running solution in a (previously hidden) AdS_2 vacuum.

4.1 The 1d running system

In this section we display black holes as running solutions of the 2d theory (3.4) derived in section 3.1. For convenience we perform the following redefinition

$$e^{2\sigma} = r^2 e^{2C} \tag{4.1}$$

The effective action in terms of C reads

$$\begin{aligned} S_{2d} = \int d^2x\sqrt{-g_2}r^2e^{2C}\Big\{ & R_2 - 2g_{i\bar{\jmath}}\partial_\gamma z^i\partial^\gamma\bar{z}^{\bar{\jmath}} + 2\partial_\gamma C\partial^\gamma C + \frac{4}{r}C'e^{-2B} + \\ & + \frac{2}{r^2}(e^{-2B}+e^{-2C}) - 2\frac{V_{BH}}{r^4}e^{-4C} - 2V_{4d}\Big\} \end{aligned} \tag{4.2}$$

where the prime denotes the derivative respect to the radial coordinate.

Now we consider the following ansatz for the 2d metric:

$$ds_2^2 = -e^{2A}dt^2 + e^{2B}dr^2 \tag{4.3}$$

and we assume that the functions appearing in the metric and the 2d scalars z^i and C just depend by the radial coordinate. We also restrict to the simplest setup of one single real scalar ϕ with $g_{\phi\phi} = 1$.

Plugging in the ansatze in the 2d action (4.2), we get the following 1d action

$$\begin{aligned} S = \int dr e^{A-B+2C}r^2\Big\{ & C'(C'+2A') + \frac{2}{r}(C'+A') + \frac{1}{r^2}(1+e^{2(B-C)}) - \phi'^2 + \\ & - \frac{e^{2B-4C}}{r^4}V_{BH} - e^{2B}V\Big\} \end{aligned} \tag{4.4}$$

where we have integrated by parts to remove some total derivatives.

This is an effective action controlling the dynamics of running solutions of the 2d theory. The equations of motion from variations of A, B, C and ϕ, respectively, are

$$
\begin{aligned}
-e^{-2B}\left[2C'' + C'(3C' - 2B') + \frac{2}{r}(3C' - B') + \frac{1}{r^2}\left(1 - e^{2(B-C)}\right) + \phi'^2\right] &= \frac{e^{-4C}}{r^4}V_{BH} + V \\
-e^{-2B}\left[C'(C' + 2A') + \frac{2}{r}(C' + A') + \frac{1}{r^2}\left(1 - e^{2(B-C)}\right) - \phi'^2\right] &= \frac{e^{-4C}}{r^4}V_{BH} + V \\
-e^{-2B}\left[A'' + C'' + A'(A' - B') + C'(C' - B' + A') + \frac{1}{r}(A' - B' + 2C') + \phi'^2\right] &= \\
= -\frac{e^{-4C}}{r^4}V_{BH} + V & \\
-e^{-2B}\left[2\phi'' + 2\phi'\left(A' - B' + 2C' + \frac{2}{r}\right)\right] &= -\frac{e^{-4C}}{r^4}\frac{\partial V_{BH}}{\partial \phi} - \frac{\partial V}{\partial \phi}
\end{aligned} \tag{4.5}
$$

The above ansatz and the resulting equations of motion correspond to those in [35] (for the particular case of a single scalar with canonical kinetic term). We have rederived them as controlling the running solution of the 2d theory, triggered by the presence of the non-trivial potential in (3.4), in the spirit of [7–9]. In appendix C we provide an alternative rederivation of these equations of motion.

It is interesting to mention that, as explained in [35], it is possible to derive the effective potential (3.22) from these equations, by plugging in an AdS_2 ($\times\mathbf{S}^2$) ansatz with

$$
A = -B = \log\frac{r}{r_A}\ , \qquad C = \log\frac{r_H}{r} \tag{4.6}
$$

with r_A, r_H the horizon value of AdS_2 and $\mathbf{S}^2$ radii, respectively. We refer the reader to [35] for details.

4.2 Small black hole solutions redux

In this language, small black hole solutions correspond to dynamical cobordisms, in which the 1d running solution hits a singularity, at which the scalars (including the $\mathbf{S}^2$ size) go off to infinite distance in field space, with the small black hole core playing the role of cobordism defect. On the other hand, finite size black holes correspond to setups in which the running halts and the 2d theory reaches an AdS minimum.

In the previous section we derived the equations of motion (4.5), which control this running. Using reparametrization of r, one can set $A = -B \equiv U$. After some algebra, the equations can be recast as

$$
\begin{aligned}
U'^2 + \phi'^2 + \frac{1}{2}\left(\psi'' + e^{-2\psi}\right) &= \frac{e^{-4C-2U}}{r^4}V_{BH}; \\
C'' + C'^2 + \frac{2}{r}C' + \phi'^2 &= 0; \\
\phi'' + 2\phi'\psi' &= \frac{1}{2}\frac{e^{-4C-2U}}{r^4}\frac{\partial V_{BH}}{\partial \phi} + \frac{1}{2}e^{-2U}\frac{\partial V}{\partial \phi}; \\
\psi'' + 2\psi'^2 - e^{-2\psi} &= -2e^{-2U}V.
\end{aligned} \tag{4.7}
$$

where we have introduced ψ, defined as

$$C + U \equiv \psi - \log r \tag{4.8}$$

It is worthwhile to point out that the functions B and ψ are related to U and h in section 2 by

$$B = -U \ , \quad h(\tau) = \psi'(r)e^{\psi} \ , \quad \tau = e^{-\psi} \tag{4.9}$$

In particular the last equation in (4.7) becomes equivalent to the hamiltonian constraint from the vanishing of the integrand in (A.11).

We now turn to describing diverse small black hole solutions.

4.2.1 Small black holes with zero or subcritical 4d potential

As we anticipated in section 2, the small black hole solution for subcritical 4d potential is just the small black hole solution in the absence of 4d potential, $V = 0$. One can check that this corresponds to $C = -U$, hence, from (4.8), we have $\psi = \log r$. The equations of motion (4.7) read:

$$\begin{aligned} U'^2 + \phi'^2 &= \frac{e^{2U}}{r^4} V_{BH}; \\ U'' + \frac{2}{r}U' - U'^2 - \phi'^2 &= 0; \\ \phi'' + \frac{2}{r}\phi' &= \frac{1}{2}\frac{e^{2U}}{r^4}\frac{\partial V_{BH}}{\partial \phi}, \end{aligned} \tag{4.10}$$

We now focus on the behaviour $V_{BH} = Q^2 e^{-2\alpha\phi}$ near $r \to 0$, and perform a change of coordinates $r \to -1/\tau$ to get

$$\begin{aligned} \dot{U}^2 + \dot{\phi}^2 &= Q^2 e^{2(U-\alpha\phi)}; \\ \ddot{U} &= Q^2 e^{2(U-\alpha\phi)}; \\ \ddot{\phi} &= -\alpha Q^2 e^{2(U-\alpha\phi)}, \end{aligned} \tag{4.11}$$

These are precisely the constraint (2.7) and the equations (2.6). The resulting solution is therefore

$$U = -\frac{1}{1+\alpha^2}\log(-q\tau) \ , \quad \phi = \frac{\alpha}{1+\alpha^2}\log(-q\tau), \tag{4.12}$$

where we have dropped some possible integration constants, irrelevant in the near-core $\tau \to -\infty$ region. These are just the near core behaviour of the solution (2.14).

4.2.2 Small black holes in the critical case

The critical case of the small black hole is given when in the near horizon limit ($r \to 0$), the V term is not negligible with respect to V_{BH} but comparable. Restoring the V term, the equations of motion (4.10) are:

$$\begin{aligned} U'^2 + \phi'^2 - \psi'^2 + e^{-2\psi} &= e^{2U-4\psi}V_{BH} + e^{-2U}V; \\ U'' + 2U'\psi' &= e^{2U-4\psi}V_{BH} - e^{-2U}V; \\ \phi'' + 2\phi'\psi' &= \frac{1}{2}e^{2U-4\psi}\frac{\partial V_{BH}}{\partial\phi} + \frac{1}{2}e^{-2U}\frac{\partial V}{\partial \phi}; \\ \psi'' + 2\psi'^2 - e^{-2\psi} &= -2e^{-2U}V. \end{aligned} \tag{4.13}$$

Already here, we can understand the critical case as having $C + U = \text{const} \neq 0$. Using the last equation of (4.13), this implies that $e^{-2U}V = c_0 e^{-2\psi}$, which implies $\psi = \log r + \frac{1}{2}\log(1-2c_0)$, where c_0 is a constant that satisfies $c_0 < \frac{1}{2}$.

Since in the critical case both potentials must give comparable contributions, we have that $e^{2U-4\psi}V_{BH} \sim e^{-2U}V \propto e^{-2\psi}$, which implies that $V_{BH} \propto e^{-2U+2\psi}$. Hence $V \sim 1/V_{BH}$, in agreement with the analysis in terms of V_{eff} in section 3.4. Using now $V_{BH} = Q^2 e^{-2\alpha\phi}$, hence $V = V_0 e^{2\alpha\phi}$, and introducing the $r = -(\tau\sqrt{1-2v_0})^{-1}$ change of coordinates, we have

$$\begin{aligned} \dot{U}^2 + \dot{\phi}^2 &= \frac{1}{1-2c_0}\left[Q^2 e^{2(U-\alpha\phi)} - \frac{V_0}{\tau^4}e^{-2(U-\alpha\phi)}\right]; \\ \ddot{U} &= \frac{1}{1-2c_0}\left[Q^2 e^{2(U-\alpha\phi)} - \frac{V_0}{\tau^4}e^{-2(U-\alpha\phi)}\right]; \\ \ddot{\phi} &= \frac{-1}{1-2c_0}\left[\alpha Q^2 e^{2(U-\alpha\phi)} - \alpha\frac{V_0}{\tau^4}e^{-2(U-\alpha\phi)}\right]. \end{aligned} \tag{4.14}$$

These are precisely the equations (2.21) and the corresponding constraint cf. (A.10). The resulting solution is

$$U = -\frac{1}{1+\alpha^2}\log(-A\tau) \ , \qquad \phi = \frac{\alpha}{1+\alpha^2}\log(-A\tau), \tag{4.15}$$

with

$$A^2 = \frac{1-\sqrt{1-4q^2v_0}}{2v_0} \tag{4.16}$$

where we have chosen the negative sign, since it reproduces the usual small black hole solution when v_0 is very small. Note that we have dropped some possible integration constants which are irrelevant in the near-core $\tau \to -\infty$ region. These are just (2.30) with A given by (2.29).

4.3 Small black holes and 2d dynamical cobordism

In this section we would like to make more precise the statement that small black hole solutions correspond to a 2d dynamical cobordism. Following [8, 9], dynamical cobordisms are spacetime dependent solutions running along one dimension y which hit a singularity at finite spacetime distance, at which scalars run off to infinite field space distance and at which spacetime ends. In [9] it was shown that, near this end of the world brane, dynamical cobordism solutions are described by a simple local model, which implies specific universal scaling relations among the spacetime distance Δ to the singularity, the field theory distance

D traversed, the scalar curvature R, and the scalar potential. In particular, in our 2d context

$$D \simeq -\frac{2}{\delta_{2d}} \log \Delta \ , \qquad R \sim \Delta^{-2} \ , \qquad V_{2d}(\phi) \simeq e^{\delta_{2d} D} . \tag{4.17}$$

where we have added a $2d$ subscript to the potential and critical exponent, to distinguish them from their 4d counterparts.

Let us consider the small black hole in the critical case (the case with subcritical (aka zero) 4d potential can be worked out similarly, as in fact done in [9]), namely the solution (4.15), and the 2d action (4.2). The metric is given by:

$$ds^2 = -e^{2U} dt^2 + e^{-2U} dr^2 \tag{4.18}$$

Using (4.15), we have that $R \propto r^{\frac{-2\alpha^2}{1+\alpha^2}}$. For the spacetime distance we get

$$\Delta \simeq \int e^{-U} dr \propto r^{\frac{\alpha^2}{1+\alpha^2}} \tag{4.19}$$

Hence we recover the scaling $R \propto \Delta^{-2}$ in (4.17) near $r = 0$.

Consider now the field distance, from the kinetic terms for the original scalar field and the S^2 radion. Using the profiles for the scalars, we obtain $D(r) = \pm \frac{\sqrt{1-\alpha^2}}{(1+\alpha^2)} \log r$, which in terms of the spacetime distance Δ becomes.[13]

$$D \simeq -\frac{\sqrt{1-\alpha^2}}{\alpha^2} \log \Delta \, , \tag{4.20}$$

namely the scaling (4.17) with the critical exponent

$$\delta_{2d} = \frac{2\alpha^2}{\sqrt{1-\alpha^2}} \, . \tag{4.21}$$

Finally, consider the 2d potential in (4.2). It is easy to check that, using the solution, all terms in the potential scale as $V_{2d} \sim e^{2\alpha\phi}$. Hence we have

$$V_{2d} \sim \exp\left(\frac{2\alpha^2}{\sqrt{1-\alpha^2}} D \right) \tag{4.22}$$

Hence reproducing the scaling (4.17) with the critical exponent (4.21).

In the language of dynamical cobordisms, small black hole explosions can be seen as a failure of the dynamical cobordism, which is obstructed by the existence of a too fastly growing 4d potential at infinity in scalar field space. In cases where the small black holes puffs up into a regular black hole, instead of an ETW brane, the 2d running configuration ends up in an AdS_2 vacuum corresponding to the near horizon geometry of the finite size black hole. Hence, the failure of the dynamical cobordism can be blamed on the fact that we do not reach a singularity at a finite distance, but rather is replaced by the infinite AdS_2 throat. We will comment more on the impact of such missing dynamical cobordisms in section 6.

It would be interesting to extend these ideas to higher-dimensional theories to build non-trivial AdS vacua, possibly near infinity (hence in weakly coupled regimes) in moduli space.

[13]The result below corrects a typo in eq. (5.40) in [9].

5 Explosions from topological obstructions

In this section we present a different mechanism which can lead to explosions for small black holes, even for subcritical exponential behaviour of the potential. It is based on a simple and familiar mechanism, which has nevertheless been gone unnoticed in the literature on (finite size) black holes in 4d $\mathcal{N} = 2$ gauged supergravity.

5.1 Topological obstructions from gaugings

We explain the mechanism in a simplified, but illustrative enough, setup. It amounts to the statement that in Abelian gaugings, when the U(1) gauge symmetry is broken, its magnetic charges are confined. We follow the discussion in [48].

Consider a U(1) gauge theory and an axion scalar, which we denote φ (we note that the only relation with ϕ is that they often appear as axion-saxion partners in complex scalars, e.g. in supersymmetric theories). We ignore gravity in what follows, since it can be included straightforwardly. Sketchily, we start from the action

$$S = \int d^4x (|d\varphi|^2 + |F|^2) \tag{5.1}$$

The theory has a U(1) gauge symmetry, and a global U(1) shift symmetry for φ (i.e. a (-1)-form global symmetry). Given that the moduli space has a U(1) isometry, we can now define new theories in which the U(1) gauge transformation acts as a shift along it. The resulting action is

$$\int d^4x \, (\, |d\varphi - kA_1| + |F|^2 \,). \tag{5.2}$$

where k is an integer characterizing the winding number of the gauge U(1) around the U(1) isometry of the scalar. The gauge transformation is

$$A_1 \to A_1 + d\Lambda(x) \ , \qquad \varphi \to \varphi + k\Lambda(x) \tag{5.3}$$

If the axion can be regarded as the phase of a charge k complex field, this just corresponds to a Higgs mechanism, in which the gauge U(1) is broken down to $\mathbf{Z}_k$.

Upon dualizing φ into a 2-form B_2, the action shows a Stückelberg BF coupling

$$\int d^4x \, (\, |d\varphi|^2 + kB_2F_2|H_3|^2 \,) \tag{5.4}$$

where $H_3 = dB_2$. We can regard this action as the gauging of a global 1-form symmetry of electromagnetism, by coupling the current $j = F_2$ to the B_2 gauge field. This can be made more manifest by dualizing the gauge potential into its magnetic potential V_1, the action becomes

$$\int d^4x \, (\, |dV_1 - kB_2|^2 + |H_3|^2 \,) \tag{5.5}$$

So the gauge symmetry is

$$B_2 \to B_2 + d\Lambda_1(x) \ , \qquad V_1 \to V_1 + k\Lambda_1 \tag{5.6}$$

The gauging has important implications for the structure of observable operators. In particular, charged operators under the (-1)-form symmetry of φ (instantons), which have the structure $e^{i\varphi(x)}$ in the absence of gauging, are now no longer gauge-invariant under (5.3). They must be dressed by the emission of k electrically charged particles, i.e. operators charged under the A_1 (equivalently the 1-form global symmetry whose current is $*F_2$) along semi-infinite lines L ending at x, as

$$\exp\left[i\varphi(x) + k\int_L A_1\right] , \quad \partial L = x \tag{5.7}$$

Similarly, the gauge transformation (5.6) implies that magnetic monopoles, charged under V_1 (equivalently line operators along L charged under the 1-form symmetry whose current is F_2), must be dressed by the emission of k strings charged under B_2 (equivalently, the 2-form global symmetry whose current is H_3), along a surface Σ whose boundary is L

$$\exp\left[i\int_L V_1 + k\int_\Sigma B_2\right] , \quad \partial\Sigma = L \tag{5.8}$$

Hence, magnetic monopoles come with strings attached. There is a topological obstruction to the existence of free magnetic charges.

This *gauging* procedure is standard in supergravity, where it takes ungauged to gauged supergravity theories. In this context, the Abelian gauging parameters (k in the above example) are also known as FI terms, since they are related by supersymmetry to BF couplings. Moreover, in 4d $\mathcal{N}=2$ theories, such gaugings are related to the apperance of a scalar potential, see appendix D. We will see more about it later.

5.2 A microscopic realization: the Freed-Witten anomaly

The realization of the above ideas in the context of flux compactifications was discussed in [49], where a prominent role is played by Freed-Witten type consistency conditions for fluxes and branes.[14] In the following we describe one such realization for later use.

Consider type IIB compactified on a CY threefold $\mathbf{X}_6$ and introduce RR 3-form fluxes on a symplectic basis $\{a_\Lambda, b^\Lambda\}$ of 3-cycles

$$\int_{a_\Lambda} F_3 = g^\Lambda \ , \quad \int_{b^\Lambda} F_3 = -g_\Lambda \ , \quad g^\Lambda, g_\Lambda \in \mathbf{Z} \tag{5.9}$$

We ignore other possible fluxes (such as NSNS 3-form fluxes), and focus on these. Consider the electric and magnetic gauge potentials arising from the RR 4-form

$$A^\Lambda = \int_{a_\Lambda} C_4 \ , \quad A_\Lambda = \int_{b^\Lambda} C_4 \tag{5.10}$$

The reduction of the 10d Chern-Simons coupling $B_2F_3F_5$ leads to Stückelberg couplings for their field strengths F^Λ, F_Λ

$$\int_{4d}\int_{\mathbf{X}_6} B_2F_3F_5 ; \to \int_{4d} B_2\,(\,g^\Lambda F_\Lambda + g_\Lambda F^\Lambda\,). \tag{5.11}$$

[14] Actually [50] considered the case D-branes in the presence of torsion H_3. The physical picture for D-branes and general H_3 appeared in [51], and for Dp-branes and RR field strength flux F_p appeared in [52]. Still, we stick to the widely used term FW anomaly / consistency condition.

This implies the gauging of a particular linear combination of U(1)'s. Equivalently, the different (electric and magnetic) U(1)'s act on the same scalar φ, the dual of B_2. We can introduce the vector of electric and magnetic gauging

$$\mathcal{G} = (g^\Lambda; g_\Lambda) \tag{5.12}$$

Consider now a particle carrying charges q_Λ, q^Λ, under A^Λ, A_Λ, respectively. We introduce the charge vector

$$Q = (q^\Lambda; q_\Lambda) \tag{5.13}$$

This corresponds to D3-branes wrapped on the 3-cycle

$$\Pi_{\rm D3} = \sum_\Lambda (q_\Lambda a_\Lambda + q^\Lambda b^\Lambda) \tag{5.14}$$

As is familiar, a Dp-brane wrapped on a p-cycle with k units of RR F_p field strength flux is not consistent by itself, due to a worldvolume tadpole [52], and must emit k fundamental strings. In our example

$$\int_{\Pi_{\rm D3}} F_3 = q_\Lambda g^\Lambda - q^\Lambda g_\Lambda \equiv -\langle \mathcal{G}, Q\rangle \tag{5.15}$$

where $\langle\cdot,\cdot\rangle$ denotes the symplectic pairing. Hence, when $k \equiv -\langle \mathcal{G}, Q\rangle \neq 0$, there must be k fundamental strings emitted by the D3-brane.

There are other interesting related ways to reach this conclusion, as follows. We can regard the F_3 flux as having been created by a domain wall given by a D5-brane on the dual 3-cycle

$$\Pi_{\rm D5} = \sum_i (g^\Lambda b^\Lambda + g_\Lambda a_\Lambda)\,. \tag{5.16}$$

Now, consider the D3-brane wrapped on (5.14) on the flux-less side of the domain wall, hence with no strings attached. When the D3-branes is moved across the D5-brane domain wall to the fluxed side, there is a string creation effect [53–55] (dual to the Hanany-Witten effect [56]) at each intersection of the corresponding 3-cycles. Hence, the D3-brane ends up with k strings attached, with

$$k = \Pi_{\rm D3}\cdot \Pi_{\rm D5} = q_\Lambda g^\Lambda - q^\Lambda g_\Lambda = -\langle \mathcal{G}, Q\rangle \tag{5.17}$$

In this picture, the strings attached to the D3-branes can be derived from the FW anomaly due to the F_3 flux created by the D5-branes (as discussed above), and the strings attached to the D5-brane arise from the FW anomaly created by the F_5 flux created by the D3-brane.

It is straightforward to derive a similar picture for other string setups, for instance type IIA CY3 compactifications with F_0, F_2, F_4 and F_6 flux. In fact, we would like to emphasize that the effect is present independently of the particular microscopic realization, since it follows from the structure of topological operators in the presence of gauging cf. section 5.1.

5.3 4d $\mathcal{N}=2$ gauged supergravity black holes come with strings attached

There is a substantial literature on black holes in 4d $\mathcal{N}=2$ gauge supergravity, including BPS solutions [36–38], and non-supersymmetric solutions [39–41] (see [42] and references therein for a recent discussion).[15]

Interestingly, there are many examples of regular black holes based on (sometimes BPS) solutions whose existence depends on having a non-zero value of the symplectic product

$$\langle \mathcal{G}, Q \rangle = -k \in \mathbf{Z} \tag{5.18}$$

From our above discussion, these black holes solutions are not totally complete, due to a subtle lack of gauge invariance which requires the introduction of k strings sticking out of the black hole (equivalently, filling out the AdS_2 2d geometry in the near horizon region).

These strings are not included in the discussions in the supergravity literature, and the detailed study of their properties and backreaction is beyond the scope of this work. Let us however point out that their effect is very different (and potentially much milder) from the one in small black holes. Indeed, for regular black holes, the value of V_{BH} at the horizon is a finite quantity, which moreover is controlled by Q^2, whereas the number k of attached strings does not scale with Q, and can in fact remain small (typical values being of order 1). Hence for large black holes with large charges, the introduction of the strings corresponds to a $1/Q^2$ modification of V_{BH}, and hence to subleading corrections to the properties of the black hole.

It is interesting that the topological obstruction explained in previous sections provides a complementary explanation for the quantization condition (5.18); it simply relates to the integer number of strings emitted by the black hole. Also, in the next section we will argue that the fact that regularity of the black hole solutions is naturally related to a non-zero value of $\langle \mathcal{G}, Q \rangle$, is related to a novel mechanism for small black hole explosions.

In section 6.3 we will see another interesting avatar of the constraint in $\langle \mathcal{G}, Q \rangle$, related to the capability of small black holes to explore all possible infinite distance limits in moduli space in the presence of 4d $\mathcal{N}=2$ gauging potentials.

5.4 Explosions via string banquets

As we have discussed in section 3.2, in certain theories (such as 4d $\mathcal{N}=2$ supergravity, in previous section) the presence of gauging also implies the appearance of a 4d potential V (3.6). From the above discussion, it is clear that in the presence of a potential arising from a gauging vector $\mathcal{G}$, a black hole with charge vector Q will emit a number of strings (magnetically charged under the axion being gauged) given by the symplectic pairing $\langle \mathcal{G}, Q \rangle$. This leads to an interesting mechanism for the explosion of small black holes.[16]

[15] As in previous sections, we focus on black holes with spherical horizons. In asymptotic AdS vacua, it is possible to also have black holes with zero or negative curvature. Although we do not discuss them, we note that the negative curvature cases also experience the phenomenon explained in this section.

[16] Strictly speaking, such small black holes are not gauge invariant as isolated objects, hence we regard them as defined as endpoint configurations of the corresponding strings. It is in this sense that one may ask whether it is dynamically preferred for these configurations to remain small (and hence, continue exploring infinite distance in field space) or puff up into a finite size configuration (or possibly a runaway version thereof).

We can use the effective potential approach to address this question. Indeed, if one regards a possibly puffed up finite size version of the charged black hole, its near horizon limit corresponds to an $\mathrm{AdS}_2 \times \mathbf{S}^2$ configuration, with the black hole charges realized as fluxes piercing the $\mathbf{S}^2$. Due to the latter, the BF couplings (5.4) implied by the gauging lead to a tadpole for B_2 in the 2d non-compact dimensions. Cancellation of this tadpole is achieved by the introduction of strings (charged under B_2) spanning the AdS_2 factor, and located at a point on $\mathbf{S}^2$. This is just the near horizon version of the emitted strings mentioned above.

It is easy to derive this tadpole in certain microscopic realizations. For instance, in the type IIB setup in section 5.2, it arises from the reduction of the 10d Chern-Simons term

$$\int_{2d} \int_{\mathbf{X}_6 \times \mathbf{S}^2} B_2 F_3 F_5 ; \rightarrow \int_{2d} B_2 \left(g^i q_i - g_i q^i \right) = \int_{2d} B_2 \langle \mathcal{G}, Q \rangle . \tag{5.19}$$

Hence we need to add $-\langle \mathcal{G}, Q \rangle = -k$ strings filling AdS_2.

The presence of these strings modifies the system. A full discussion of this backreaction is beyond the scope of this work, but we can describe the key feature by modifying the structure of the 2d action, by introducing an extra source of tension, localized on the $\mathbf{S}^2$ and filling the AdS_2, namely.

$$2kT_0 \int d^2x \sqrt{-g_2} \tag{5.20}$$

with the tension $T_0 > 0$ and where the factor of 2 is for convenience. As such, they behave as an extra term $2kT_0 e^{-2\sigma}$ inside the bracket of the 2d action (3.4), with T_0 taken constant for simplicity.[17] Note that a positive tension T_0 contributes with same sign as the black hole or 4d potentials, and opposite to the contribution of the $\mathbf{S}^2$ curvature.

Let us now consider the fate of the core of former small black holes in the presence of the gauging potential, and thus, of the extra strings. Consider the simplified case of a single scalar. Following the steps in section 3.3, the structure of the effective potential (3.22) is

$$V_{\text{eff}} = \frac{(1 - kT_0) - \sqrt{(1 - kT_0)^2 - 4V_{BH}V}}{2V} \tag{5.21}$$

Note that the string contributes as the tadpole generated by the $\mathbf{S}^2$ compactification, and they contribute with opposite sign. Moreover, in microscopic realizations, the strings are locally BPS objects (such as the fundamental strings in Freed-Witten anoamlies), and by charge quantization we expect that $T_0 = 1$. Hence, in the presence of such strings, V_{eff} goes negative and according to the solutions in section 4.2, there is no small hole solution. Remarkably, this conclusion is independent of the value of δ and how it compares with α. Namely, in the presence of topological obstructions which force the introduction of strings sticking out of the small black hole, the small black holes explodes even for subcritical values of the exponent in the 4d potential.

[17] Let us note that by constant, we mean that it does not depend on vector multiplet moduli. This statement however may be model-dependent and depend on the particular setup or microscopic embedding in string theory.

An intuitive interpretation is that small black holes with strings attached tend to feast on them and grow. As usual, our exponential approximations near infinity in field space do not suffice to assess or not the existence of an endpoint for this process, a topic to which we turn in the next section.

Incidentally, we note that certain regular black holes, such us the example in the STU model in [37], do not correspond to small black holes even in the absence of 4d potential. In this sense, they cannot be regarded as a puffed up version of small black holes.

6 Interplay with the Cobordism Distance Conjecture

6.1 Small black holes and the Cobordism Distance Conjecture

The Cobordism Distance Conjecture proposed in [8] states that in a consistent theory of quantum gravity, every infinite field distance limit can be realized as a dynamical cobordism running into an end of the world brane (possibly in a suitable compactification of the theory). In fact, for 4d theories and upon compactification on $\mathbf{S}^1$ this encompasses the Distant Axionic String Conjecture in [24]. Similarly, this suggests that small black holes, which are dynamical cobordisms of the 4d theory compactified on $\mathbf{S}^2$, are a powerful tool to explore infinite distance limits of the theory. In this section we explore to what extent the set of small black holes is rich enough to actually explore all those possible limits, and to what extent the obstructions due to small black hole explosions imply some limitations in this exploration.

A clear restriction in this regard is that small black holes allow to explore infinite distance limits for scalars controlling gauge couplings of the theory. This is implicit in the discussion below. In particular, we will frame the discussion in the context of 4d $\mathcal{N}=2$ supergravity, which provides a rich arena to carry out the discussion in explicit examples.

We start with the theory in the absence of 4d potentials (see [57] for a related discussion). In this context, the structure of infinite distance limits in the vector multiplet moduli space has been studied in [2, 3, 58]. In particular, every infinite distance limit corresponds to a regime in which some gauge coupling goes to zero, hence allowing the construction of small black hole solution exploring that infinite distance limit [27]. Note that in general an infinite distance limit can involve combinations of different moduli to go off to infinity (as classified by the corresponding growth sector), hence the corresponding small black hole must carry the appropriate combination of charges to couple to the gauge coupling becoming small. Hence, this implicitly profits from the Completeness Conjecture, which ensures that for any possible charge vector one may need, there are appropriate charged states in the theory (and which for large charges constitute a small black hole, rather than a fundamental particle).

6.2 Adding the potential: the *STU* model

6.2.1 Generalities

Let us now consider the introduction of a 4d potential. In order to be explicit, we focus on those arising from Abelian gaugings of 4d $\mathcal{N}=2$ theories, as in the previous section, see appendix D for background information. In particular, we can focus on the illustrative example of the STU model. We will subsequently study the extension to more general 4d $\mathcal{N} = 2$ models, in particular covering the moduli space of any CY_3 compactification.

We follow closely the conventions in [37], and refer the reader to appendix D for details.

The model (in a specific symplectic frame) is defined by the prepotential

$$F = \frac{X^1 X^2 X^3}{X^0} \,. \tag{6.1}$$

We introduce the coordinates $S = X^1/X^0$, $T = X^2/X^0$ and $U = X^3/X^0$, in terms of which the Kähler potential is

$$K = -\log[-i(S-\bar{S})(T-\bar{T})(U-\bar{U})], \tag{6.2}$$

and $\mathcal{V}$ is given by

$$\mathcal{V} = e^{K/2}\,(1, S, T, U, -STU, TU, SU, ST)^T. \tag{6.3}$$

We consider a general set of charges/gaugings, given by

$$Q = \left(p^0, p^1, p^2, p^3, q_0, q_1, q_2, q_3\right)^T \qquad \mathcal{G} = \left(g^0, g^1, g^2, g^3, g_0, g_1, g_2, g_3\right)^T \tag{6.4}$$

A rich class of models was introduced in [37] by taking gauging charges of the form $\mathcal{G} = (0, g^1, g^2, g^3, g_0, 0, 0, 0)^T$ and black hole charges $Q = (p^0, 0, 0, 0, 0, q_1, q_2, q_3)$. For the following discussion, we prefer to allow for general choices (in principle constrained only by the Freed-Witten like consistency condition discussed in the previous section).

The black hole central charge (D.19) and 4d superpotential (D.12) are thus

$$\begin{aligned}\mathcal{Z} &= e^{K/2}(p^0 STU - p^1 TU - p^2 SU - p^3 ST + q_0 + q_1 S + q_2 T + q_3 U)\\ \mathcal{L} &= e^{K/2}(g_0 + g^0 STU - g^1 TU + g_1 S - g^2 SU + g_2 T - g^3 ST + g_3 U).\end{aligned} \tag{6.5}$$

Since we are interested in exploring infinite distance limits in moduli space, the asymptotic behaviours are controlled by the imaginary parts of the moduli, hence we replace $S \mapsto is$, $T \mapsto it$ and $U \mapsto iu$, with $s, t, u < 0$.

From the expressions for the black hole potential (3.8), namely (D.20), and the 4d scalar potentials (3.6), namely (D.13), we have

$$\begin{aligned}V_{BH} &= \frac{q_0^2 + q_1^2 s^2 + q_2^2 t^2 + q_3^2 u^2 + (p^1)^2 t^2 u^2 + (p^2)^2 s^2 u^2 + (p^3)^2 s^2 t^2 + (p^0)^2 s^2 t^2 u^2}{-2stu}\\ V &= \frac{-g_0(g^1 tu + g^2 su + g^3 st) - g_1 g_2 st - g_2 g_3 tu - g_1 g_3 su}{-stu} +\\ &\quad + \frac{(g^0 g_1 - g^2 g^3) s^2 tu + (g^0 g_2 - g^1 g^3) st^2 u + (g^0 g_3 - g^1 g^2) stu^2}{-stu}\end{aligned} \tag{6.6}$$

It is now a simple exercise to pick sets of gauging/charges which illustrate different behaviours for small black holes. We refrain from the discussion of random examples, but rather focus on a more fundamental question in the next section.

6.2.2 Small black holes: exploration or explosion

As explained, a natural question to what extent small black hole explosions may hamper their ability to explore infinite distance limits in a given theory. In this section, we carry out this discussion in the *STU* model, we may consider the simplified setup in which all moduli are identified i.e. scale to the limit in the same way. More general situations will be discussed later on.

For a single modulus case $s = t = u \equiv z$, we have

$$V_{BH} = \frac{q_0^2 + (q_1^2 + q_2^2 + q_3^2)z^2 + [\,(p^1)^2 + (p^2)^2 + (p^3)^2\,]\, z^4 + (p^0)^2 z^6}{-2z^3} \tag{6.7}$$

$$\begin{aligned} V &= \frac{(-g_0(g^1 + g^2 + g^3) - g_1 g_2 - g_2 g_3 - g_1 g_3)z^2}{-z^3} + \\ &+ \frac{(g^0 g_1 + g^0 g_2 + g^0 g_3 - g^1 g^2 - g^2 g^3 - g^1 g^3)z^4}{-z^3} \end{aligned} \tag{6.8}$$

As explained in section 6.1, we are interested in testing possible limitations in the ability of small black holes to explore infinite distance limits, due to small black hole explosions.

Hence, we focus on 4d potentials with the highest possible value of δ in the corresponding limit.[18] Considering the limit of large moduli,[19] the most dangerous term is the second line in (6.8), which scales as $V \sim z$. There are basically two possibilities (or simple combinations thereof):

- We may choose $g^0 \neq 0$, and get $V \sim z$ by choosing some of the $g_i \neq 0$. In order to explore this infinite distance limit, one may be tempted to just take a black hole with $q_0 \neq 0$ as only charge, so that $V_{BH} \sim z^{-3}$, for which the 4d potential is subcritical. However, this is not possible, since it would lead to Freed-Witten-like anomalies. However, it is possible to choose black holes with some $q_0 = 0$, $q_i \neq 0$ for $i = 1, 2, 3$, which avoid the Freed-Witten-like anomalies, and lead to $V_{BH} \sim z^{-1}$, for which the 4d potential is critical.

- We may get $V \sim z$ by keeping $g^0 = 0$ and turning on two different g^i's, for instance g^1 and g^2. In such case, we can take a black hole with $q_1, q_2 \neq 0$, with lead to $V_{BH} \sim z^{-1}$, hence a critical case, and which easily avoid the Freed-Witten-like anomalies by demanding that $g^1 q_1 + g^2 q_2 = 0$ (e.g. $q_1 = g^2 n$, $q_2 = -g^1 n$, with arbitrary $n \in \mathbf{Z}$).

We have thus checked that, even for the most dangerous 4d potential, there are specific small black holes for which the 4d potential is critical; hence they in principle have a chance to remain small and be able to explore this infinite distance limit. We thus conclude that, also in the presence of 4d potentials, the Completeness Conjecture is enough to guarantee that, for any gauging potential and any infinite distance limit, there is always

[18] Note that we are also not insisting on the 4d theory to admit an asymptotic AdS_4 vacuum, since the discussion of the small black hole cores does not require information about the asymptotics, which may well correspond to a 4d running solution.

[19] We Note that there is a similar limit, corresponding to $z \to 0$, which can be analyzed similarly, with identical results up to reshuffling of terms. This is just a reflection of the duality of the *STU* model.

some small black hole in the theory which remains small and still allows to explore the infinite distance limit.

We finish by noting that we have not discussed other possible sources of 4d potentials, which in principle could lead to a modification of the above discussion. On the other hand, turning things around, the above discussion motivates a proposal restricting the form of the possible 4d potentials, as we discuss in the next section.

6.3 Constraints on potentials from small black holes explosions

In this section we propose the use of our ideas above to constrain the possible 4d scalar potentials, from their properties regarding small black holes and their explosions. We show that this can lead to powerful restrictions, not only on 4d potentials but (in theories with extended supersymmetry) also on the geometry of the moduli space near infinity.

6.3.1 The Small Black Hole Criterion

In the example of the *STU* model in the previous section, we showed the interesting fact that the available 4d potentials from gaugings are such that there still remain enough small black holes (i.e. those possibly not exploding under the effect of the potential) to still allow the exploration of all infinite distance limits for the scalars.

It is natural to propose promoting this idea to a general principle, as follows:

Small Black Hole Criterion. *In a consistent theory of quantum gravity with scalar-dependent Abelian gauge couplings, the allowed 4d potentials are constrained by the requirement that they allow for the existence of a set of small black holes whose scalars are still able to explore any infinite distance limit of the theory (i.e. remain subcritical or critical).*

In a sense, this idea is not a new conjecture, but rather a mere reformulation of the Cobordism Distance Conjecture in [8] for the 2d theories resulting from the 4d theories upon $\mathbf{S}^2$ compactification. On the other hand, its explicit formulation facilitates its use to put constraints on the 4d theories, as we discuss in the next section.

6.3.2 Recovering the CY vector moduli space asymptotics

In this section we put the above proposal to work and show that it leads to remarkably strong results. In particular, we will argue it implies that, for any 4d $\mathcal{N} = 2$ theory, the Kähler potential for the vector multiplets satisfies, in the large moduli limit, the equality

$$g^{i\bar{\jmath}}\partial_i\mathcal{K}\partial_{\bar{\jmath}}\mathcal{K} = 3 \tag{6.9}$$

This is precisely the asymptotic identity obeyed by vector moduli in a CY threefold compactification of type II string theory (and via duality, of K3 compactifications of heterotic, or other dual realizations of 4d $\mathcal{N} = 2$). We are however about to derive it from a purely 4d bottom-up perspective.

The argument is as follows. Consider a general 4d $\mathcal{N} = 2$ with prepotential $F(X)$. Using homogeneity of the prepotential, the covariantly holomorphic vector $\mathcal{V}$ always contains an entry equal to $e^{\mathcal{K}/2}$, its symplectic dual contains and entry $e^{\mathcal{K}/2}F(z)$, where z are the moduli (i.e. affine coordinates). Let us focus on the limit of all moduli becoming large, in which this entry dominates the dynamics. For instance, from (D.6) the Kähler potential scales as

$$\mathcal{K} \sim -\log[i(F-\bar{F})] \tag{6.10}$$

Consider now a possible gauging of the theory, analogous to g^0 in the STU model in section 6.2, namely

$$\mathcal{L} \sim g^0 F(z) + \ldots \tag{6.11}$$

The corresponding scalar potential (3.6), namely (D.13), is

$$V = \frac{1}{i(F-\bar{F})} \left[\left(g^{i\bar{\jmath}} \partial_i \mathcal{K} \partial_{\bar{\jmath}} \mathcal{K} - 3 \right) (g^0)^2 \, F\bar{F} \right] \tag{6.12}$$

which scales as $V \sim F$ in the large moduli limit unless (6.9). We now show that this would be in contradiction with our Small Black Hole Criterion, hence (6.9) must be satisfied.

Consider the above potential and demand that there must exist some black holes which is critical with respect to that term in the potential, namely $V_{BH} \sim F^{-1}$. Clearly, this must correspond to a black hole whose only charge, q_0, is associated to the entry $e^{\mathcal{K}/2}$ in $\mathcal{V}$, namely the symplectic dual to g^0. This would superficially allow to satisfy the Small Black Hole Criterion. However, we should realize that precisely this required charge vector suffers the problem pointed out in section 5, namely

$$\langle \mathcal{G}, Q \rangle = g^0 q_0 \neq 0 \tag{6.13}$$

Namely, it is not an allowed state in the theory, as it is forbidden by gauge invariance. Alternatively, even if the black hole is made consistent by allowing it to emit the required number of strings, it does not remain small, but rather explodes by the string banquet mechanism in section 5.4, and does not explore the infinite distance limit. Hence, as announced, the Small Black Hole Criterion implies the property (6.9) in the infinite moduli limit of the $\mathcal{N}=2$ vector moduli space of any consistent theory.

The above argument is a general description of how the $(g^0)^2$ term does not arise in the potential (6.6) of the STU model in section 6.2. In fact this model is one of the simplest toy models to characterize the large moduli behaviour of any CY compactification

$$F = -D_{IJK} X^I X^J X^k / X^0 \tag{6.14}$$

with $6D_{IJK}$ are the familiar (integer) triple intersection numbers of CY compactifications.

It is an amusing fact that we recover the very familiar property (6.9) of CY_3 compactifications in a 4d theory which has a priori no information about CY's or compactifications. In this sense, the result resonates with the reconstruction of compactification geometries from cobordism considerations in [59].

7 Conclusions

In this work we have explored the ability of 4d small black holes to explore infinite distance limits in scalar field space in the presence of scalar potentials growing near infinity. We have derived the critical relation distinguishing when the small black hole remains small from when it explodes by puffing up into a regular black holes or following a runaway behaviour. Although the discussion is general we have studied the particular case of 4d $\mathcal{N} = 2$ gauged supergravity, which allows a rich arena for this exploration. In fact, in that context we have uncovered a constraint, previously unnoticed in the literature on regular black holes in this setup, which requires the black holes to emit strings.

We have used these findings to motivate a general constraint on scalar potentials in 4d theories, based on the principle that the set of small black holes should still be rich enough to allow for exploration of all possible infinite distance limits. We have shown that this leads to non-trivial constraints, and that in 4d $\mathcal{N} = 2$ theories it allows to reproduce some of the properties of vector moduli spaces of CY_3 compactifications.

Our discussion has emphasized the use of a 2d description, after truncation on $\mathbf{S}^2$. This allowed us to discuss 4d potentials and black hole potentials on an equal footing, and, in the 4d $\mathcal{N} = 2$ context, to satisfactorily explain the difference between their expressions in terms of the underlying covariantly holomorphic quantities. More importantly, the 2d perspective permits the translation of the result to the language of dynamical cobordisms, where the small black hole core corresponds to the end of the world brane, and small black hole explosions signal the failure of completing a cobordism ending spacetime. In this picture, puffed up black holes correspond to running solutions of the 2d theory which end up relaxing onto a newly appeared AdS_2 vacua.

There are many possible generalizations of our work and other related ideas. For instance:

- We expect that further results about the structure of infinite distance regimes in moduli space can be derived for further exploration of large and small black holes, in particular considerations about their entropy and its microscopic explanation, possibly along the lines of [43].
- Generalization to black strings and other extended objects, in general dimensions, and their possible explosions upon the introduction of scalar potentials. A particular instance of this is the fate of the 4d EFT strings in [23–25] in the presence of potentials growing fast at the string core.
- The failure of 2d running solutions to complete dynamical cobordisms and rather end up relaxing onto AdS_2 vacua is a particularly rich instance of a phenomenon which also exists in higher dimensions [9]. It would be interesting to explore its possible use as a technique to build or detect AdS vacua hidden near infinity in moduli space.
- In the supercritical case, the small black hole explosion is expected to lead to time dependent runaway configurations. The construction of such full solution is presumably difficult, due to non-trivial dependence in two coordinates. However, its asymptotic

behaviour at late times/distances may admit a simpler description, depending only on a single light-cone coordinate, in analogy with expanding domain walls in e.g. [60–65].

- The Freed-Witten-like constraint in 4d $\mathcal{N} = 2$ gauged supergravity models forces the introduction of strings, which in the near horizon regime of regular black holes are spacetime filling in the 2d AdS_2. In cases where this extra ingredient breaks supersymmetry i.e. is an anti-string (see [66–68], also [69–71] for similar phenomena in higher dimensions), this is reminiscent of the antibrane uplift in [72]. It would be interesting to explore actual quantitative connections between the two phenomena.

We hope to come back to these and other interesting topics in future work.

Acknowledgments

We are pleased to thank Gianguido Dall'Agata, Iñaki Garcia-Etxebarria, Luis Ibáñez, Fernando Marchesano, Miguel Montero, Tomás Ortín, Irene Valenzuela for useful discussions, and José Calderón-Infante and Matilda Delgado for collaboration on related topics. This work is supported through the grants CEX2020-001007-S and PID2021-123017NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe. The work by R.A. is supported by the grant BESST-VACUA of CSIC. The work by J. H. is supported by the FPU grant FPU20/01495 from the Spanish Ministry of Education.

A All about h

A.1 No 4d potential

The most general ansatz for an spherically symmetric static solution with electric charge Q in the theory described by (2.1) is actually

$$ds^2 = -e^{2U(\tau)}dt^2 + e^{-2U(\tau)}\left(\frac{d\tau^2}{\tau^4 h(\tau)^2} + \frac{1}{\tau^2}d\Omega_2^2\right), \qquad F_2 = 2\sqrt{2}Qg^2e^{2U}d\tau\wedge dt \tag{A.1}$$

Plugging the ansatz in the 4d action (2.1) leads to the 1d action

$$S_{1d} = \int d\tau \left\{\frac{h}{2}\left(\dot{U}^2+\dot{\phi}^2\right) + \frac{1}{h}g^2Q^2e^{2U} - \frac{1}{\tau^2}\left(\frac{1}{h} - h + 2\tau\dot{h}\right)\right\} \tag{A.2}$$

The equations of motion read

$$\frac{d}{d\tau}\left(h\dot{U}\right) = \frac{2}{h}Q^2e^{2U}g(\phi)^2 \tag{A.3}$$

$$\frac{d}{d\tau}\left(h\dot{\phi}\right) = \frac{1}{h}Q^2e^{2U}\left(g(\phi)^2\right)' \tag{A.4}$$

$$\left[\frac{1}{2}(\dot{U}^2+\dot{\phi}^2) - \frac{1}{\tau^2}\right]h^2 - g^2Q^2e^{2U} + \frac{1}{\tau^2} = 0\,. \tag{A.5}$$

As mentioned in [44], the action (A.2) implies that h is not an actual dynamical variable, but rather imposes a constraint, associated to reparametrizations of τ. To make this explicit, consider the variation of the action under changes of U, ϕ of the form

$$U(\tau+\delta\tau) = U(\tau) + \dot{U}(\tau)\delta\tau\ , \qquad \phi(\tau+\delta\tau) = \phi(\tau) + \dot{\phi}(\tau)\delta\tau\,. \tag{A.6}$$

Using (A.3), (A.4) we get:

$$\delta S_{1d} = \int d\tau \left\{ \frac{d}{d\tau}\left(h\dot{U}^2\delta\tau\right) + \frac{d}{d\tau}\left(h\dot{\phi}\delta\tau\right) + \frac{2}{\tau^3}\left(\frac{1}{h} - h\right)\delta\tau \right\}, \tag{A.7}$$

namely, up to total derivatives, demanding $\delta S_{1d} = 0$ requires

$$\int d\tau \frac{2}{\tau^3}\left(\frac{1}{h} - h\right)\delta\tau = \text{const} \tag{A.8}$$

which can be solved by setting $h = 1$. Then (A.3), (A.4) correspond to the equations of motion (2.6), and (A.5) leads to the constraint (2.7).

A.2 Including 4d potential

Let us now introduce a 4d potential term in the action cf. (2.16)

$$S = \int d^4x\sqrt{-g}\left[R - 2(d\phi)^2 + \frac{1}{2g^2}F^2 + 2V\right]. \tag{A.9}$$

Using the ansatz (A.1), we get the 1d action

$$S_{1d} = \int d\tau \left\{ \frac{h}{2}\left(\dot{U}^2 + \dot{\phi}^2\right) + \frac{1}{h}Q^2 g(\phi)^2 e^{2U} - \frac{1}{h}\frac{V(\phi)}{2\tau^4}e^{-2U} - \frac{1}{2\tau^2}\left(\frac{1}{h} - h + 2\tau\dot{h}\right)\right\}.$$

The equations of motion are

$$\begin{aligned}
&\frac{d}{d\tau}\left(h\dot{U}\right) = \frac{2}{h}Q^2 g(\phi)^2 e^{2U} + \frac{1}{h}\frac{V(\phi)}{\tau^4}e^{-2U} \\
&\frac{d}{d\tau}\left(h\dot{\phi}\right) = \frac{1}{h}Q^2\left(g(\phi)^2\right)' e^{2U} - \frac{1}{h}\frac{V'(\phi)}{2\tau^4}e^{-2U}, \\
&\left[\frac{1}{2}\left(\dot{U}^2 + \dot{\phi}^2\right) - \frac{1}{2\tau^2}\right]h^2 - g(\phi)^2 Q^2 e^{2U} + \frac{V}{2\tau^4}e^{-2U} + \frac{1}{2\tau^2} = 0
\end{aligned} \tag{A.10}$$

The change of the action under variations (A.6) requires

$$\delta S_{1d} = \int d\tau \frac{1}{\tau^3}\left(\frac{1}{h} - h + 2\tau\dot{h} + \frac{1}{h}\frac{2V(\phi)}{\tau^2}e^{-2U}\right)\delta\tau = 0. \tag{A.11}$$

which can be solve by choosing

$$\dot{h} \simeq 0 \qquad h^2 = 1 + \frac{2V(\phi)}{\tau^2}e^{-2U} \tag{A.12}$$

Although this is more involved, it simplifies in particular cases. For instance, if Ve^{-2U} grows slower that τ^2 we recover $h^2 = 1$ near $\tau \to -\infty$, just like in the case with no 4d potential; this corresponds to the subcritical case in the main text. On the other hand, the critical case solution in section 2.3 corresponds to a constant $h \equiv h_0 \neq 1$.

B The 4d entropy functional computation of $V_{\rm eff}$

For completeness we include a quick derivation of $V_{\rm eff}$ from the entropy functional from the perspective of the 4d solutions. The 2d computation of $V_{\rm eff}$ in section 3.3 is a version of this.

We start with the following 4d action[20]

$$S_{4d} = \int d^4x\sqrt{-g_4}\left\{R_4 - 2g_{i\bar{j}}\partial_\mu z^i\partial^\mu\bar{z}^{\bar{j}} + Im\mathcal{N}_{\Lambda\Sigma}F^\Lambda_{\mu\nu}F^{\Sigma\mu\nu} - 2V\right\} \tag{B.1}$$

Recall that, although it has the structure of the bosonic sector of $\mathcal{N}=2$ (possibly gauged) supergravity action, but is intended as a general theory.

We are interested in the near horizon limit of extremal black hole solutions with spherical horizon, which corresponds to $\mathrm{AdS}_2\times\mathbf{S}^2$ with a 2-form field strength background, of the form:

$$ds_4^2 = v_1\left(-r^2dt^2+\frac{dr^2}{r^2}\right)+v_2\left(d\theta^2+\sin^2\theta d\varphi^2\right) \tag{B.2}$$

$$F^\Lambda_{rt} = \frac{e^\Lambda}{8\pi} \qquad\qquad F^\Lambda_{\theta\phi} = p^\Lambda\sin\theta \tag{B.3}$$

where v_1, v_2, e and p are constants. In addition we require that the scalars be regular when we approach the horizon: $\lim_{r\mapsto r_H} z^i = z^i_H$.

Following [33] (see [34] for a review), we introduce the *entropy function*, defined by

$$\mathcal{E}(v_1,v_2,z^i,p,q) = 2\pi\left(q_\Lambda e^\Lambda - f(v_1,v_2,z^i,p,e)\right) \tag{B.4}$$

where q is the conjugate quantity to e:

$$\frac{\partial\mathcal{E}}{\partial e} = 0 \tag{B.5}$$

and f is the action evaluation on the near horizon ansatz solution

$$f(v_1,v_2,z^i,p,e) = \int d\theta d\varphi\sqrt{-g_4}\mathcal{L}_4. \tag{B.6}$$

The parameters for the actual solution, including the attractor values for the scalars, are obtained by extremizing the entropy function

$$\text{(i)}\quad \frac{\partial\mathcal{E}}{\partial v_1}=0\ ,\qquad \text{(ii)}\quad \frac{\partial\mathcal{E}}{\partial v_2}=0\ ,\qquad \text{(iii)}\quad \left.\frac{\partial\mathcal{E}}{\partial z^i}\right|_{z^i_H}=0 \tag{B.7}$$

Moreover, the value of $\mathcal{E}$ at this extremum gives the black hole entropy.

Using the ansatzs (B.2) and (B.3) the entropy functional (B.4) reads:

$$\begin{aligned}\mathcal{E}(v_1,v_2,z^i_H,p,q) = 2\pi\Big\{&q_\Lambda e^\Lambda + 8\pi v_2 - 8\pi v_1 + \\ &- 4\pi Im\mathcal{N}_{\Lambda\Sigma}p^\Lambda p^\Sigma\frac{v_1}{v_2} + \frac{1}{16\pi}Im\mathcal{N}_{\Lambda\Sigma}e^\Lambda e^\Sigma\frac{v_2}{v_1} + 8\pi V v_1 v_2\Big\}\end{aligned} \tag{B.8}$$

[20] This action is equivalent to the 4d action (3.1) used at the beginning of section 3.1, where we made explicit the democratic treatment of the electric and magnetic gauge fields. Here it is the Legendre transformation (B.4) with the condition (B.5) that will restore the not-manifest electric/magnetic duality.

Here the conjugate variable q is defined in terms of the other parameters by the (B.5):

$$e^{\Lambda} = -8\pi \frac{v_1}{v_2} Im\mathcal{N}^{\Lambda\Sigma} q_{\Sigma}. \tag{B.9}$$

where $Im\mathcal{N}^{\Lambda\Sigma}$ denotes the inverse matrix of $Im\mathcal{N}_{\Lambda\Sigma}$.

Using this last equation, (B.8) becomes:

$$\mathcal{E} = 2\pi \left\{ 8\pi v_2 - 8\pi v_1 - 4\pi Im\mathcal{N}_{\Lambda\Sigma} p^{\Lambda} p^{\Sigma} \frac{v_1}{v_2} - 4\pi Im\mathcal{N}^{\Lambda\Sigma} q_{\Lambda} q_{\Sigma} \frac{v_1}{v_2} + 8\pi V v_1 v_2 \right\} \tag{B.10}$$

Note that the electric and magnetic charge contributions couple to v_1 and v_2 in the same way. This motivates introducing the black hole potential:

$$V_{BH} = -\frac{1}{2} \left(p^{\Lambda} Im\mathcal{N}_{\Lambda\Sigma} p^{\Sigma} + q_{\Lambda} Im\mathcal{N}^{\Lambda\Sigma} q_{\Sigma} \right) \tag{B.11}$$

Now (B.10) takes the form:

$$\mathcal{E}(v_1, v_2, p, q) = 2\pi \left[8\pi v_2 - 8\pi v_1 + 8\pi V_{BH} \frac{v_1}{v_2} + 8\pi V v_1 v_2 \right] \tag{B.12}$$

The extremization conditions (B.7) give:

$$\begin{aligned}
&\text{(i)} \qquad V v_2^2 - v_2 + V_{BH} = 0 \quad \longmapsto \quad v_2 = \frac{1 \pm \sqrt{1 - 4 V V_{BH}}}{2V} \\
&\text{(ii)} \qquad 1 - V_{BH} \frac{v_1}{v_2^2} + V v_1 = 0 \quad \longmapsto \quad v_1 = \frac{v_2}{\sqrt{1 - 4 V_{BH} V}} \\
&\text{(iii)} \quad \frac{v_1}{v_2} \frac{\partial V_{BH}}{\partial z^i}\bigg|_{z^i_H} + v_1 v_2 \frac{\partial V}{\partial z^i}\bigg|_{z^i_H} = 0
\end{aligned} \tag{B.13}$$

These conditions exactly reproduce those in [35], derived from the attractor mechanism. Replacing the values of v_1 and v_2, one can define the effective potential

$$V_{\text{eff}} = \frac{1 - \sqrt{1 - 4 V_{BH} V}}{2V} \tag{B.14}$$

introduced in (3.22) in the main text.

C Alternative derivation of the 1d equations of motion

In this appendix we provide an alternative derivation of the equations of motion in section 4.1.

We slightly generalize the ansatz (3.2) for the reduction of the 4d metric to 2d:

$$ds_4^2 = e^{\alpha\sigma} ds_2^2 + e^{\beta\sigma} d\Omega_2^2 \tag{C.1}$$

The resulting 2d action, after integrating by parts some terms, is:

$$\begin{aligned}
S_{2d} = \int d^2x \sqrt{-g_2} e^{\beta\sigma(r)} \Big\{ & R_2 - \beta \left(\alpha + \frac{\beta}{2} \right) (\partial\sigma)^2 - 2 g_{i\bar{j}} \partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} \\
& + 2 e^{(\alpha-\beta)\sigma(r)} - 2 V_{BH} e^{(\alpha - 2\beta)\sigma(r)} - 2 V_{4d} e^{\alpha\sigma(r)} \Big\}
\end{aligned} \tag{C.2}$$

One can then obtain the equation of motion for this action upon variations of the 2d metric, α, β, σ and the moduli z^i.

One can then use the following 2d ansatz

$$ds_2^2 = -dt^2 + e^{2b(r)}dr^2. \tag{C.3}$$

The relation with the quantities A, B, C in the main text is

$$e^{\alpha\sigma(r)} = e^{2A(r)} \qquad e^{\alpha\sigma(r)+2b(r)} = e^{2B(r)} \qquad e^{\beta\sigma(r)} = r^2 e^{2C(r)}. \tag{C.4}$$

Upon setting, without loss of generality, $\beta = 2$, we recover the equation of motion (4.5).

D 4d $\mathcal{N} = 2$ supergravity

A general setup in which 4d potentials are naturally included is gauged 4d $\mathcal{N} = 2$ supergravity. Moreover this provides a template for flux compactifications. Here we review its basic ingredients, see [73] for a thorough discussion.

D.1 Ungauged 4d $\mathcal{N} = 2$ supergravity

We start with the ungauged 4d $\mathcal{N} = 2$ coupled to n_V abelian vector multiplets, and ignore any hypermultiplets in the discussion. There are n_V complex moduli z^i labelled by $i = 1, \ldots, n_V$. Including the graviphoton, there are $n_V + 1$ gauge bosons, labelled by $\Lambda = 0, \ldots, n_V$. The structure of the bosonic lagrangian is

$$\mathscr{L} = R - 2g_{i\bar{j}}\,\partial_\mu z^i \partial^\mu \bar{z}^{\bar{j}} + \mathrm{Im}\mathcal{N}_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, F^{\Sigma\,\mu\nu} + \mathrm{Re}\mathcal{N}_{\Lambda\Sigma}\, F^\Lambda_{\mu\nu}\, \frac{\epsilon^{\mu\nu\rho\sigma}}{2\sqrt{-g}} F^\Sigma_{\rho\sigma} \tag{D.1}$$

where the different quantities are defined using special geometry. The scalars parametrize a special Kähler moduli space, i.e. is the base of a symplectic bundle, with covariantly holomorphic sections

$$\mathcal{V} = \begin{pmatrix} X^\Lambda \\ F_\Lambda \end{pmatrix}, \qquad \mathcal{D}_{\bar{i}}\mathcal{V} = \partial_{\bar{i}}\mathcal{V} + \frac{1}{2}\partial_{\bar{i}}\mathcal{K}\,\mathcal{V} = 0, \tag{D.2}$$

(where $\mathcal{K}$ is the Kähler potential) and obeying the symplectic constraint

$$i\langle \mathcal{V}, \bar{\mathcal{V}}\rangle = F_\Lambda \bar{X}^\Lambda - X^\Lambda \bar{F}_\Lambda = 1 \tag{D.3}$$

It is useful to write

$$\mathcal{V} = e^{\mathcal{K}(z,\bar{z})/2} v \tag{D.4}$$

The holomorphic symplectic vector

$$v(z) = \begin{pmatrix} Z^\Lambda \\ \partial_\Lambda F \end{pmatrix}, \qquad \partial_{\bar{i}} v = 0 \tag{D.5}$$

is defined using the prepotential[21] $F(X)$, a holomorphic function of degree 2. Then, (D.3) becomes

$$e^{-\mathcal{K}} = i\langle v, \bar{v}\rangle \tag{D.6}$$

The matrix $\mathcal{N}_{\Lambda\Sigma}$ determining the coupling between the scalars and the vectors is defined by the relations

$$F_\Lambda = \mathcal{N}_{\Lambda\Sigma} X^\Sigma \ , \qquad \mathcal{D}_{\bar{i}} \bar{F}_\Lambda = \mathcal{N}_{\Lambda\Sigma} \mathcal{D}_{\bar{i}} \bar{X}^\Sigma \tag{D.7}$$

For completeness, we present the microscopic realization of this in type II compactification on a Calabi-Yau threefold $\mathbf{X}_6$. Start with type IIB, where the vector moduli space corresponds to the complex structure moduli. Introduce a symplectic basis of 3-cycles A_Λ, B^Σ (namely we have intersection numbers $A_\Lambda \cdot A_\Sigma = B^\Lambda \cdot B^\Sigma = 0, A_\Lambda \cdot B^\Sigma = \delta_\Lambda^\Sigma$). The holomorphic vector v is given by

$$Z^\Lambda = \int_{A_\Lambda} \Omega \ , \qquad \partial_\Lambda F = \int_{B^\Lambda} \Omega \tag{D.8}$$

and

$$K = -\log \big(i \int_{\mathbf{X}_6} \Omega \wedge \bar{\Omega} \big) \tag{D.9}$$

The electric and magnetic gauge potentials arise from the redution of the 10d RR 4-form

$$A_1^\Lambda = \int_{A_\Lambda} C_4 \ , \qquad A_{1\,\Lambda} = \int_{B^\Lambda} C_4 \tag{D.10}$$

For type IIA one has a similar story with even-dimensional cycles replacing the 3-cycles and e^J playing the role of Ω.

D.2 Gauged 4d $\mathcal{N} = 2$ supergravity

In this section we review some useful formulas for Abelian gaugings in supergravity. Gaugings are defined by introducing a vector of parameters (FI terms)

$$\mathcal{G} = \begin{pmatrix} g^\Lambda \\ g_\Lambda \end{pmatrix} , \tag{D.11}$$

corresponding to electric or magnetic gaugings, respectively. The physical interpretation of the gauging is reviewed in section 5.1 in the main text.

The gauging introduces a 4d superpotential

$$\mathcal{L} = \langle \mathcal{G}, \mathcal{V} \rangle = e^{\mathcal{K}/2} (Z^\Lambda g_\Lambda - \partial_\Lambda F g^\Lambda) \tag{D.12}$$

This leads to a 4d scalar potential[22]

$$V = g^{i\bar{j}} \mathcal{D}_i \mathcal{L} \bar{\mathcal{D}}_{\bar{j}} \bar{\mathcal{L}} - 3|\mathcal{L}|^2 \tag{D.13}$$

[21] Although there are 4d $\mathcal{N} = 2$ theories with no prepotential, we focus on the usual case in which it exists.

[22] Notice that this reproduces the usual formula of 4d $\mathcal{N} = 1$ with the $e^{\mathcal{K}}$ prefactor arising from squaring the $e^{\mathcal{K}/2}$ in the defining of $\mathcal{V}$, manifest in the last expression in (D.12).

The intuition is very simple. The gaugings correspond to the introduction of fluxes in the compactification (including field strength fluxes, but also geometric or generalized fluxes). This can be described in terms of a domain wall connecting the vacuum of the ungauged theory (i.e. no fluxes) and the vacuum of the theory with gauging $\mathcal{G}$. The domain wall is a codimension 1 object in 4d, and its internal structure is such that its tension is given by (D.12). This is the standard argument for the superpotential from p-form field strength fluxes in M-theory or type II compactifications [74], which for completeness we repeat for type IIB 3-form fluxes [75].

Consider type IIB on a CY3 $\mathbf{X}_6$ as described around (D.8) and introduce RR 3-form fluxes

$$\int_{A_\Lambda} F_3 = g^\Lambda \ , \qquad \int_{B^\Lambda} F_3 = g_\Lambda \tag{D.14}$$

For clarity here and in what follows we ignore constant factors.

Denoting the basis of 3-forms α_Λ, β^Σ Poincaré dual to A_Λ, B^Σ, the flux quanta (D.14) give the 3-form flux cohomology class

$$[F_3] = g^\Lambda \alpha_\Lambda + g_\Lambda \beta^\Lambda \tag{D.15}$$

This flux can be regarded as being created by a domain wall, given by a D5 wrapped on the 3-cycle Π_{D5} whose Poincaré dual is

$$\delta(\Pi_{\mathrm{D5}}) = F_3 = g^\Lambda \alpha_\Lambda + g_\Lambda \beta^\Lambda \tag{D.16}$$

The tension of this domain wall can be obtained as

$$e^{\mathcal{K}} \int_{\Pi_{\mathrm{D5}}} \Omega = e^{\mathcal{K}} \int_{\mathbf{X}_6} \Omega \wedge \delta(\Pi_{\mathrm{D5}}) = e^{\mathcal{K}} (Z^\Lambda g_\Lambda - \partial_\Lambda F g^\Lambda) = e^{\mathcal{K}/2} \langle \mathcal{G}, v \rangle = \mathcal{L} \tag{D.17}$$

D.3 Black hole central charge

In 4d $\mathcal{N} = 2$ theories there are BPS particle states, which can correspond to charged black holes. We introduce a vector of gauge charges

$$Q = \begin{pmatrix} p^\Lambda \\ q_\Lambda \end{pmatrix} , \tag{D.18}$$

They have an associated central charge

$$\mathcal{Z} = \langle Q, \mathcal{V} \rangle = e^{\mathcal{K}/2} (Z^\Lambda q_\Lambda - \partial_\Lambda F p^\Lambda) \tag{D.19}$$

The black hole potential which controls the radial flow of scalars in the attractor mechanism is given by

$$V_{BH} = g^{i\bar{j}} \mathcal{D}_i \mathcal{Z} \, \bar{\mathcal{D}}_{\bar{j}} \bar{\mathcal{Z}} + |\mathcal{Z}|^2 \tag{D.20}$$

The intuition is very simple. For instance in the type IIB setup, BPS particles arise from D3-branes wrapped on a (special lagrangian) 3-cycle Π_{D3} whose Poincaré dual is

$$\delta(\Pi_{\mathrm{D3}}) = F_3 = p^\Lambda \alpha_\Lambda + q_\Lambda \beta^\Lambda \tag{D.21}$$

in the symplectic basis introduced in the previous section. The BPS mass can be obtained as

$$e^{\mathcal{K}}\int_{\Pi_{\rm D3}}\Omega = e^{\mathcal{K}}\int_{\mathbf{X}_6}\Omega\wedge\delta(\Pi_{\rm D3}) = e^{\mathcal{K}}(Z^\Lambda q_\Lambda - \partial_\Lambda F p^\Lambda) = e^{\mathcal{K}/2}\langle Q, v\rangle = \mathcal{Z} \tag{D.22}$$

In the presence of a gauging introducing a potential, the complete attractor flow is determined by a combination of $\mathcal{Z}$ and $\mathcal{L}$. In particular, in [37] it was shown that for BPS solutions the attractor flow is controlled by the superpotential

$$\mathcal{W} = e^U|\mathcal{Z} + ie^{2\sigma}\mathcal{L}| \tag{D.23}$$

where U and σ are functions of the radial coordinate controlling the non-compact 2d geometry and the $\mathbf{S}^2$ size, respectively.

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Received: *January 10, 2024*
Accepted: *February 25, 2024*
Published: *March 19, 2024*

Intersecting end of the world branes

Roberta Angius, Andriana Makridou and Angel M. Uranga

Instituto de Física Teórica IFT-UAM/CSIC,
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain

E-mail: roberta.angius@csic.es, andriana.makridou@ift.csic.es, angel.uranga@csic.es

Abstract: Dynamical cobordisms implement the swampland cobordism conjecture in the framework of effective field theory, realizing codimension-1 end of the world (ETW) branes as singularities at finite spacetime distance at which scalars diverge to infinite field space distance. ETW brane solutions provide a useful probe of infinity in moduli/field spaces and the associated swampland constraints, such as the distance conjecture.

We construct explicit solutions describing intersecting ETW branes in theories with multiple scalars and general potentials, so that different infinite field space limits coexist in the same spacetime, and can be simultaneously probed by paths approaching the ETW brane intersection. Our class of solutions includes physically interesting examples, such as intersections of Witten's bubbles of nothing in toroidal compactifications, generalizations in compactifications on products of spheres, and possible flux dressings thereof (hence including charged objects at the ETW branes). From the cobordism perspective, the intersections can be regarded as describing the end of the world for end of the world branes, or as boundary domain walls interpolating between different ETW brane boundary conditions for the same bulk theory.

Keywords: String and Brane Phenomenology, D-Branes

ArXiv ePrint: 2312.16286

https://doi.org/10.1007/JHEP03(2024)110

Contents

1 Introduction

One of the main outcomes of the swampland program (see [1–4] for reviews) is a renewed interest in the exploration of regions at infinite distance in moduli space. A prominent tool and motivation is the Distance Conjecture [5], which posits the existence of towers of particles becoming exponentially light along trajectories reaching such infinite distance regions.

In theories with exact moduli spaces, such as the much studied case of 4d $\mathcal{N} = 2$ supersymmetry, the exploration of infinite distances is possible using spacetime independent scalar vevs. In this context there is a rich industry of various approaches and results, including [6–19]. An interesting feature in theories with several scalars, in particular in CY moduli spaces [6, 8, 9], is the existence of a rich network of infinite distance loci with different components which in general intersect in non-trivial ways, and for which the Distance Conjecture requires formulations including the interplay of multiple towers [15–19].

In the presence of general scalar potentials, however, the above adiabatic exploration of infinite distances by constant vevs may result in inconsistencies [20] and may even be forbidden [21] (see [22] for a recent discussion). One is thus bound to the study of spacetime-dependent solutions, as pioneered in [23] (see [17] for recent discussions, and [24, 25] for recent time-dependent running solutions and the Distance Conjecture). In this context, there are several classes of solutions describing scalars running to infinite field space distance at finite spacetime distance. These include dynamical cobordisms [26–30] (see also [31–37] for related early work and [20, 38–48] for other related recent developments), 4d EFT strings [49–51], and small black holes [28, 47, 52, 53] (see [54] for the exploration of infinity in moduli space using *large* black holes).

Dynamical cobordisms describe configurations of scalars running in one spacetime dimension, along which spacetime ends at finite distance when the theory hits a spacetime singularity at which scalars run off to infinite field space distance [26, 27]. They can be regarded as describing boundaries of spacetime at a codimension-1 end of the world (ETW) brane, which provide a dynamical realization of the cobordism defects predicted by the Cobordism Conjecture of [55] (see [56–67] for other applications). Interestingly, they admit a universal local description introduced in [28] in terms of single parameter (dubbed critical exponent), which moreover controls interesting scaling relations between the spacetime and field theory distances in the solution.

A natural question is how to use running solutions to explore the network of infinite distance limits in theories with multiple scalars. To achieve this goal, we consider the generalization of the above configurations, by considering solutions which include different spacetime regions at which different infinite distance limits are attained, and which intersect in spacetime so as to allow the exploration of the intersection of components of the infinite distance loci in field space. In particular we focus on the realization of this idea using dynamical cobordisms as building blocks, and build a large class of explicit solutions describing intersecting ETW brane configurations.

Intersecting ETW branes have further interesting interpretations in the light of other swampland conjectures besides the Distance Conjecture. Being a key ingredient in dynamical cobordism, they have a natural home in the Cobordism Conjecture [55]. Indeed, a configuration of two intersecting ETW branes (ETW_1 and ETW_2) can be regarded as a dynamical cobordism where the ETW_2 brane defines a boundary for the configuration of the bulk theory ending on the ETW_1 (and viceversa), see figure 1a. In short, the intersection provides the end of the world for end of the world branes.

A second cobordism interpretation for the intersecting ETW brane configurations, illustrated in figure 1b, is as providing a domain wall between different boundaries, defined by the ETW_1 and ETW_2 branes, for the same bulk theory. This is again in the spirit of the Cobordism Conjecture, which implies that in quantum gravity theories there must exist domain walls interpolating between any two configurations.

The full expicit solutions we construct have a remarkably simple structure, realizing a superposition of the individual ETW branes, and are fully characterized by the individual critical exponents. In fact we compute the source terms and find they correspond to localized terms associated to the ETW brane tension and scalar couplings, with no additional source terms localized at their intersection.

Our approach should be regarded as local near the intersection, in the spirit of [28], and similarly leads to a universal scaling properties. We explore them in the context of

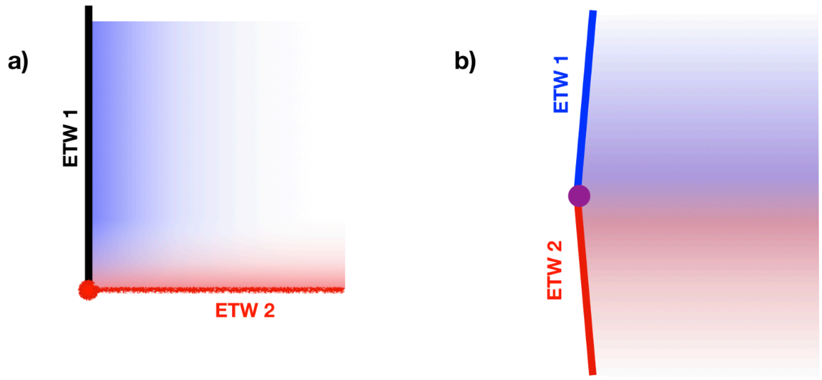

Figure 1. Two possible cobordism interpretations of intersecting ETW branes: a) The ETW_2 brane defines a cobordism to nothing for the configuration of the bulk theory ending at the ETW_1 brane boundary. b)The bulk theory ends on a cobordism to nothing boundary which switches from and ETW_1 brane to and ETW_2 brane.

versions of the Distance Conjecture in theories with multiple scalars, including the Convex Hull formulation [15].

Although we do not expect our solutions to provide the most general intersecting ETW brane solutions, we show that they include a rich set of physically relevant systems, such as intersecting Witten's bubbles of nothing [68] (see [69–72] for recent related systems) in toroidal compactifications, or generalizations in compactifications in products of spheres, possibly dressed with fluxes (and hence including D-brane sources at the ETW brane). The setup also provides an arena for the actual exploration of the network of infinite distance loci in CY moduli spaces, which will be discussed in [73].

For simplicity we mostly focus on configurations of two ETW branes intersecting orthogonally (see [29, 30, 32, 33] for other discussions of codimension-2 solutions), but also present generalizations for more than two ETW branes, and for general angles. We also compare our solutions with intersecting ETW branes associated to the *same* scalar, and show the latter actually are better regarded as singular limits of a single recombined ETW brane. This fits with the interpretation in [45] for a particular example with tachyon condensation in supercritical strings.

A cosmological application of our solutions would be the description of collisions of cosmological bubbles (see [74] for a review). It would be interesting to exploit our local models to extract universal signatures of these phenomena. We leave the exploration of phenomenological applications of our solutions to future work.

The paper is organized as follows. In section 2 we review the codimension-1 ETW brane solution, following [28]. In section 3 we construct the intersecting ETW brane solutions and discuss their properties. The ansatz and explicit solutions are constructed in section 3.1, section 3.2 discusses the associated ETW brane worldvolume source terms, and in section 3.3 we describe the scaling properties of the class of solutions. Section 4 contains explicit examples, including intersections of Witten's bubbles of nothing in $\mathbf{S}^1 \times \mathbf{S}^1$ compactifications (section 4.1), and in $\mathbf{S}^{p_1} \times \mathbf{S}^{p_2}$ compactifications (section 4.2), ETW branes with charged

D-brane defects (section 4.3), and the intersection of a bubble of nothing with a general ETW brane (section 4.4). In section 5 we discuss the interplay with swampland constraints, including the cobordism conjecture (section 5.1), the distance conjecture (section 5.2), including the convex hull formulation in [15] and the infinite distance pattern in [18, 19]. Finally, we offer some concluding remarks in section 6. Appendix A discusses several generalizations and related systems, including intersections at general angles (section A.1), intersections of more than two ETW branes (section A.2), and intersecting ETW branes with a single scalar (section A.3)

2 Overview of codimension-1 ETW branes

In this section we overview the local description for codimension-1 ETW branes in [28]. Consider d-dimensional gravity coupled to a real scalar ϕ with general potential

$$S = \int d^d x \sqrt{-g} \left(\frac{1}{2} R - \frac{1}{2} (\partial\phi)^2 - V(\phi) \right) , \tag{2.1}$$

Here and in the rest of this work we set $M_{Pl} = 1$ units and consider $d > 2$. The scalar may correspond to a combination of underlying moduli/scalar fields.

The codimension-1 ETW brane solution has the structure

$$\begin{aligned} ds_d^2 &= e^{-2\sigma(y)} ds_{d-1}^2 + dy^2 , \\ \phi &= \phi(y) , \end{aligned} \tag{2.2}$$

As pioneered in [27], the dynamical cobordism is characterized by the scalar ϕ going off to infinite distance in field space $\phi \to \infty$ at finite distance in spacetime $y \to 0$. Imposing the equations of motion, the local description near the ETW brane is

$$\begin{aligned} \phi(y) &\simeq -\frac{2}{\delta} \log y \\ \sigma(y) &\simeq -\frac{4}{(d-2)\delta^2} \log y + \frac{1}{2} \log c , \end{aligned} \tag{2.3}$$

where the parameter δ describes the leading exponential behaviour of the potential

$$V(\varphi) = -ace^{\delta\varphi} \tag{2.4}$$

with c a free parameter and a is related to δ by

$$\delta = \sqrt{\frac{d-1}{d-2}(1-a)} \tag{2.5}$$

As explained in [28] the above solution leads to universal scaling relations among the spacetime distance to the singularity , the traverse scalar field space distance $\mathcal{D}$ and the spacetime curvature scalar, in terms of the parameter δ:

$$\sim e^{-\frac{1}{2}\delta\mathcal{D}}, \quad |R| \sim e^{\delta\mathcal{D}}. \tag{2.6}$$

A general warning, for these solutions and those in coming sections, is that at infinity in field space one expects the theory to have a lowered cutoff, which limits the validity of the effective field theory. Indeed, the appearance of corrections at the species scale has been discussed in small black holes in [53], and we may expect similar phenomena in ETW brane solutions. Still, effective field theory remains a useful tool to describe the systems, and even to quantify these corrections. Our solutions in this work should be understood in this spirit.

We now discuss a simple example of ETW brane, given by an analogue of the bubble of nothing of $\mathbf{S}^1$ compactifications [68], with the role of the expanding bubble replaced by a flat static wall of nothing (see [28] for the spherical bubble case).

We start with $(d+1)$-dimensional gravity with an action

$$S_{d+1} = \frac{1}{2}\int d^{d+1}x\sqrt{-g}R. \tag{2.7}$$

We consider compactifying on $\mathbf{S}^1$ parametrized by θ, with the compactification ansatz

$$ds^2_{d+1} = e^{\alpha\rho}ds^2_d + e^{-\beta\rho}d\theta^2. \tag{2.8}$$

The parameters α, β are fixed by requiring the d-dimensional metric is in the Einstein frame, and by fixing the normalization of the radion kinetic term. We have

$$\beta = (d-2)\alpha, \quad \alpha^2 = \frac{4}{(d-1)(d-2)}\,. \tag{2.9}$$

The resulting d-dimensional action is

$$S = \frac{1}{2}\int d^dx\,\sqrt{-g}\left[R - (\partial\rho)^2\right]. \tag{2.10}$$

The theory is like (2.1) with the scalar $\phi = \rho$ and zero potential, since $\mathbf{S}^1$ has no curvature. It thus admits a solution of the kind (2.2), (2.3) with

$$\delta = 2\sqrt{\frac{d-1}{d-2}}. \tag{2.11}$$

It is easy to uplift this solution and check that it corresponds to taking an $\mathbf{S}^1$ slicing of $(d+1)$ dimensional flat space

$$ds^2_{d+1} = ds^2_{d-1} + dr^2 + r^2d\theta^2, \tag{2.12}$$

with ds^2_{d-1} describing a flat metric along the ETW brane worldvolume, and r and y related by

$$y = \left(\frac{d-2}{d-1}\right) r^{\frac{d-1}{d-2}}. \tag{2.13}$$

We refer to [28] for other examples, some of which will be arise as building blocks in our examples of intersecting ETW branes in section 4. We now turn to the general description of codimension-2 intersections of ETW branes.

3 Intersecting ETW branes

In this section we consider configurations describing the intersection of two ETW branes of the kind considered in the previous section. As we will see, they remarkably satisfy a simple superposition ansatz. This is reminiscent of the superposition of harmonic functions for supergravity solutions of (suitably smeared) intersecting BPS branes, with the differences that we do not require supersymmetry of the solutions, or even of the underlying theory, and that our solutions are fully localized and require no smearing.

We note that we focus on solutions describing the local behaviour near the intersection. The global structure in a general setup may differ in a model-dependent way. Hence, we focus on the universal behaviour of the configurations, much in the spirit of [28] for the codimension-1 case.

3.1 Codimension-2 ansatz and solutions

We consider the following $(n+2)$-dimensional action for gravity coupled to two real scalars with a general potential $V(\phi_1, \phi_2)$:

$$S = \int d^{n+2}x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}\left(\partial\phi_1\right)^2 - \frac{1}{2}\left(\partial\phi_2\right)^2 - \frac{\alpha}{2}\partial_\rho\phi_1\partial^\rho\phi_2 - V(\phi_1,\phi_2)\right\}. \tag{3.1}$$

Note that we have introduced a mixed kinetic term, which could be removed by diagonalization. However, maintaining it will allow for a simpler solution for the scalar profiles. As in the codimension-1 case in section 2, the scalars can be combinations of several moduli/scalar fields.

The above action is regarded as describing the theory around the infinite distance locus. An important observation in this respect is that we are using a locally flat metric around that point. Points at infinity are actually singular in general, but can admit such description if one restricts to specific directions in field space. An illustrative example is provided by considering two complex scalars   with Kähler potential

$$K = \log(\ _1 + \bar{\ }_1) + \log(\ _1 + \bar{\ }_2). \tag{3.2}$$

Introducing the axion and saxion components $\ _i = \varphi_i + it_i$, the metric is given by two decoupled hyperbolic spaces

$$\frac{1}{t_1^2}\left(dt_1^2 + d\varphi_1^2\right) + \frac{1}{t_2^2}\left(dt_2^2 + d\varphi_2^2\right) = d\phi_1^2 + d\phi_2^2 + e^{-2\phi_1}d\varphi_1^2 + e^{-2\phi_2}d\varphi_2^2, \tag{3.3}$$

where we have introduced the canonically normalized saxion fields $\phi_i = \log t_i$. Clearly, the metric for the axions φ_i is singular at the infinite distance locus for $\phi_i \to \infty$. On the other hand, restricting to solutions where the axions are inactive, the dynamics for the saxions is controlled by flat metric kinetic terms. The action (3.1) should be understood as describing such smooth slices around the infinite distance point.[1]

In order to solve the equations of motion for the above action, we consider the ansatz for the metric:

$$ds_{n+2}^2 = e^{2A(y_1,y_2)}ds_n^2 + e^{2B(y_1,y_2)}dy_1^2 + e^{2C(y_1,y_2)}dy_2^2\,, \tag{3.4}$$

[1] More formally, in the formalism of [15], the smooth slice belongs to the subspace $\mathbb{G}$ spanned by asymptotic tangent vectors of asymptotically geodesic trajectories.

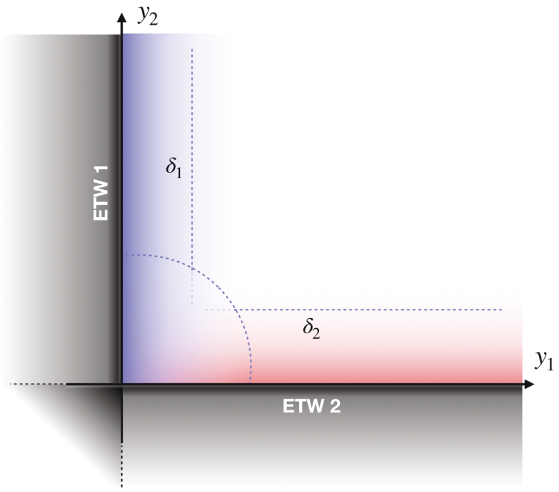

Figure 2. Intersection of two orthogonal ETW branes of type δ_1 and δ_2. Our solutions zoom into the region near the intersection, denoted with a dashed curve.

and for the profiles of the two scalars:

$$\phi_1 = \phi_1(y_1), \qquad \phi_2 = \phi_2(y_2). \tag{3.5}$$

A linear independent set of combinations of the equations of motion is

$$\begin{aligned}
&(1) \quad e^{-2C} n \left[n\left(A'\right)^2 + A'\left(B'-C'\right) + A''\right] + e^{-2B} n \left[n\left(\dot{A}\right)^2 + \dot{A}\left(\dot{C}-\dot{B}\right) + \ddot{A}\right] = -2V,\\
&(2) \quad e^{-2C}\left[-n\left(A'\right)^2 + (n-2)A'\left(B'-C'\right) + (n-2)A'' + 2B'\left(B'-C'\right) + 2B'' + \left(\phi_2'\right)^2\right] +\\
&\qquad + e^{-2B}\left[-n\left(\dot{A}\right)^2 + (n-2)\dot{A}\left(\dot{C}-\dot{B}\right) + (n-2)\ddot{A} + 2\dot{C}\left(\dot{C}-\dot{B}\right) + 2\ddot{C} + \left(\dot{\phi}_1\right)^2\right] = 0,\\
&(3) \quad n\left[\dot{A}B' - \dot{A}A' + \dot{C}B' - \dot{A}'\right] = \frac{\alpha}{2}\dot{\phi}_1\phi_2',\\
&(4) \quad e^{-2B}\left[\left(n\dot{A} - \dot{B} + \dot{C}\right)\dot{\phi}_1 + \ddot{\phi}_1\right] + \frac{\alpha}{2}e^{-2C}\left[\left(nA' + B' - C'\right)\phi_2' + \phi_2''\right] - \frac{\partial V}{\partial \phi_1} = 0,\\
&(5) \quad e^{-2C}\left[\left(nA' + B' - C'\right)\phi_2' + \phi_2''\right] + \frac{\alpha}{2}e^{-2B}\left[\left(n\dot{A} - \dot{B} + \dot{C}\right)\dot{\phi}_1 + \ddot{\phi}_1\right] - \frac{\partial V}{\partial \phi_2} = 0,
\end{aligned} \tag{3.6}$$

where we have introduced the notation $\dot{f} \equiv \partial_{y_1} f$, $f' \equiv \partial_{y_2} f$.
In particular, eq.(1) is proportional to the sum of the $\{y_1 y_1\}$ and $\{y_2 y_2\}$ components of the Einstein equations. Eq.(2) is the $\{ij\}$ component, using eq.(1) to eliminate the potential. Eq.(3) is the mixed $\{y_1 y_2\}$component, and eqs.(4) and (5) are the equations of motion for the scalars.

We are interested in solutions describing intersecting ETW branes, with the requirement that each scalar ϕ_i runs off to infinite distance in field space as it approaches the origin in the coordinate y_i it depends on; consequently both scalars diverge at the codimension-2 locus $y_1 = y_2 = 0$. This is depicted in figure 2.

Let us emphasize again that the solution is intended to be a local description near the intersection point. Hence the center piece of figure 2 is meant to describe a local patch near the intersection where this local description holds.

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A simple proposal for the solution to describe two intersecting ETW branes is that for e.g. constant non-zero y_2, one recovers a codimension-1 ETW brane solution (and similarly for y_1). Hence, the metric should be of the form (2.2). This is achieved if we impose an additive structure in our warp factors

$$A(y_1, y_2) = -\sigma_1(y_1) - \sigma_2(y_2), \quad B(y_1, y_2) = -\sigma_2(y_2), \quad C(y_1, y_2) = -\sigma_1(y_1). \tag{3.7}$$

Note that we have not included additive factors depending on y_1 on B, or on y_2 on C, since they can be reabsorbed by redefining those coordinates. In fact, the above parametrization makes the connection with (2.2) manifest. The metric (3.4) becomes

$$ds^2_{n+2} = e^{-2\sigma_1 - 2\sigma_2} ds^2_n + e^{-2\sigma_2} dy_1^2 + e^{-2\sigma_1} dy_2^2 \,, \tag{3.8}$$

so that for e.g. in a constant non-zero y_2 slice, the resulting $(n+1)$-dimensional metric is (up to a constant)

$$ds^2_{n+1} = e^{-2\sigma_1} ds^2_n + dy_1^2 \tag{3.9}$$

with a running scalar $\phi_1(y_1)$, and ϕ_2 remains constant along the slice. This is precisely the structure of the local description of a codimension-1 ETW brane. Obviously, a similar pattern holds for constant y_1 slices. Motivated by this, we can propose logarithmic profiles for the functions σ_i, ϕ_i:

$$\begin{aligned} \sigma_1 &= -a_1 \log y_1 + \tfrac{1}{2} \log c_1, \qquad & \sigma_2 &= -a_2 \log y_2 + \frac{1}{2} \log c_2, \\ \phi_1 &= b_1 \log y_1, & \phi_2 &= b_2 \log y_2, \end{aligned} \tag{3.10}$$

where c_1 and c_2 correspond to (subleading) constant terms related to the two independent integration constants of the equations of motion.

Replacing these profiles in (3.6), we get the following constraints:

$$\begin{aligned} V &= -\frac{1}{2} c_1 n a_1 [\,(n+1) a_1 - 1\,] y_1^{-2} y_2^{-2a_2} - \frac{1}{2} c_2 n a_2 [\,(n+1) a_2 - 1\,] y_1^{-2a_1} y_2^{-2} \\ &\equiv -c_1 v_1 y_1^{-2} y_2^{-2a_2} - c_2 v_2 y_1^{-2a_1} y_2^{-2} \\ \alpha &= 2\sqrt{a_1 a_2} \\ b_i^2 &= n a_i \end{aligned} \tag{3.11}$$

Note the prefactors in the scalar potential, which are controlled by the constants c_i in the logarithmic ansatz (3.10). In the first equation, assuming $a_1, a_2 \neq 1$, the potential splits in two pieces with different dependence on y_1, y_2. We thus split

$$V = V_1 + V_2, \quad V_1 \sim -c_1 v_1 \, y_1^{-2} y_2^{-2a_2}, \quad V_2 \sim -c_2 v_2 \, y_1^{-2a_1} y_2^{-2} \tag{3.12}$$

Note that the asymptotic behaviour of V fixes the values of a_1, a_2 and then there is no freedom to change α. Hence this class of solutions requires a tuning of the mixed kinetic term for the scalars. This may seem a strong restriction on the theory; however, starting from a theory with e.g. no mixed terms, one can always redefine the scalars such that the appropriate mixed term arises. Hence the above condition can be regarded as specifying in which basis of the scalar fields the solution takes the above simple form. Interestingly we will show that the above set of solutions includes large classes of interesting examples, as we describe in section 4.

Using the above, we get

$$a_1 = \frac{1 \pm \sqrt{1 + 8v_1\left(1 + \frac{1}{n}\right)}}{2(n+1)}, \quad a_2 = \frac{1 \pm \sqrt{1 + 8v_2\left(1 + \frac{1}{n}\right)}}{2(n+1)} \tag{3.13}$$

In analogy with the codimension-1 ETW branes, we introduce the quantities δ_1, δ_2, such that the local ansatz reads

$$\begin{aligned} &V = V_1 + V_2 = -c_1 v_1\, e^{\delta_1 \phi_1} e^{a_2 \delta_2 \phi_2} - c_2 v_2\, e^{a_1 \delta_1 \phi_1} e^{\delta_2 \phi_2} \\ &\phi_1 = -\frac{2}{\delta_1} \log y_1, \qquad\qquad \phi_2 = -\frac{2}{\delta_2} \log y_2 \\ &\sigma_1 = -\frac{4}{n\delta_1^2} \log y_1, \qquad\qquad \sigma_2 = -\frac{4}{n\delta_2^2} \log y_2 \end{aligned} \tag{3.14}$$

This is the codimension-2 local description of intersecting ETW branes, which generalizes the structure of the local description for codimension-1 ETW branes in [28]. The full solution is determined by the critical exponents δ_i associated with the two individual ETW branes. From the above equations, they are given by

$$\delta_1^2 = \frac{8(n+1)}{n \pm \sqrt{n\left[n + 8v_1(n+1)\right]}}, \quad \delta_2^2 = \frac{8(n+1)}{n \pm \sqrt{n\left[n + 8v_2(n+1)\right]}}. \tag{3.15}$$

Let us mention that our solution is even more general than what the derivation above suggests. Indeed, if one starts from (3.4) and requires a general additive structure with independent functions (i.e. beyond (3.7)) with general logarithmic profiles, the equations of motion end up leading to the above solution. Hence, (3.7) can be regarded as a derived structure, once the logarithmic profiles are imposed.

We incidentally note that the ansatz (3.8) is conformally flat, generalizing the situation encountered in the codimension-1 case in section 2. This is easily shown by changing to new coordinates x^1, x^2 such that

$$dy_1 = e^{-\sigma_1} dx_1, \quad dy_2 = e^{-\sigma_2} dx_2 \tag{3.16}$$

so that (3.8) becomes

$$ds_{n+2}^2 = e^{-2\sigma_1 - 2\sigma_2} \left[ds_n^2 + dx_1^2 + dx_2^2 \right]. \tag{3.17}$$

where σ_i are obviously regarded as functions of x^i. Conformally flat codimension-2 solutions have been discussed in setups with a single scalar e.g. in [33]. It would be interesting to explore possible relations between the two setups.

For our purposes, the expression (3.17) will allow to generalize our solutions to ETW branes intersecting at general angles in appendix A.1. In coming sections we restrict to orthogonal intersections (3.8) for simplicity, the generalization being trivial.

3.2 The sources

In this section we discuss the source terms that correspond to our above intersecting ETW brane solutions, and check that they are just a superposition of source terms at $y_1 = 0$ and at $y_2 = 0$ closely related to the source terms of codimension-1 ETW branes. In particular, there is no further source term localized purely at the intersection $y_1 = y_2 = 0$. For simplicity of the discussion, we focus on the case of zero scalar potential; the general case can be worked out similarly.

We start with recovering the source term for the codimension-1 ETW brane solutions following [30] (see also [29, 32, 71, 72]). We start with the d-dimensional action with a source term describing the tension and coupling to the scalar of the ETW brane:

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} (\partial\phi)^2 \right] - \lambda \int d^{d-1} x \sqrt{-g}\, e^{\tilde{\alpha}\phi} \delta(y), \tag{3.18}$$

where now g denotes the pullback of the d-dimensional metric g to the worldvolume of the codimension-1 defect. Using the ansatz (2.2), the two equations of motion coming from the variation with respect to the metric become:

$$\begin{aligned} \{i,j\}: \ & \frac{1}{2}\phi'^2 + \frac{(d-1)(d-2)}{2}\sigma'^2 - (d-2)\sigma'' + \lambda e^{\tilde{\alpha}\phi}\delta(y) = 0, \\ \{y,y\}: \ & \phi'^2 - (d-1)(d-2)\sigma'^2 = 0, \end{aligned} \tag{3.19}$$

where the prime denotes derivation with respect to y. The variation of the action with respect to the field ϕ gives:

$$\phi'' - (d-1)\phi'\sigma' = \alpha\lambda e^{\tilde{\alpha}\phi}\delta(y). \tag{3.20}$$

Making the ansatz $\phi = -\sqrt{\frac{d-2}{d-1}} f(y)$ and $\sigma = -\frac{1}{d-1} f(y)$ automatically satisfies the $\{y, y\}$ equation of motion, and the remaining two equations read:

$$\begin{aligned} \frac{d-2}{d-1}(f'^2 + f'') &= -\lambda e^{\tilde{\alpha}\phi}\delta(y), \\ \sqrt{\frac{d-2}{d-1}}(f'^2 + f'') &= -\alpha\lambda e^{\tilde{\alpha}\phi}\delta(y). \end{aligned} \tag{3.21}$$

By direct comparison one can read off the value of α

$$\alpha = \sqrt{\frac{d-1}{d-2}}. \tag{3.22}$$

Setting now $f(y) = \log(h(y))$ (with $h(y) \geq 0$) this becomes

$$\frac{h''}{h} = -\lambda \frac{d-1}{d-2} e^{-\log h}\delta(y) \quad \Rightarrow \quad h'' = -\lambda \frac{d-1}{d-2}\,\delta(y)\,. \tag{3.23}$$

Now we can integrate this over $[0, x_0)$ and take the limit $x_0 \to 0$. The left hand side is just the discontinuity of h'. Using the solution to the equations of motion, for $y < 0$ we take $h = 1$, so that $f(y) = 0$ and all fields vanish beyond the ETW brane, while for $y > 0$,

$h = y - y_0$, with y_0 an integration constant (with $y_0 < 0$ to have $h(y) \geq 0$ near $y = 0$), which we will eventually take to $y_0 \to 0^-$ to match our solution.

Then after the integration/limit we have:

$$1 = -\frac{d-1}{d-2}\lambda \quad \Rightarrow \quad \lambda = -\frac{d-2}{d-1}. \tag{3.24}$$

The negative tension of this ETW brane was already explicitly noticed in [30] (see also [72]), for the physical choice of signs we have implicitly assumed in our solution.

We now turn to the intersecting ETW-brane solution, and show that codimension-1 sources of the kind studied above suffice to support the solution. We thus use a d-dimensional action (and set $d = n + 2$) including sources of the following form:

$$\begin{aligned} S = &\int d^{n+2}x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}\left(\partial\phi_1\right)^2 - \frac{1}{2}\left(\partial\phi_2\right)^2 - \frac{\alpha}{2}\partial_\rho\phi_1\partial^\rho\phi_2\right\} \\ &- \lambda_1\int d^{n+1}x\sqrt{-g_1}e^{\alpha_{11}\phi_1+\alpha_{12}\phi_2}\delta(y_1) - \lambda_2\int d^{n+1}x\sqrt{-g_2}e^{\alpha_{22}\phi_2+\alpha_{21}\phi_1}\delta(y_2)\,. \end{aligned} \tag{3.25}$$

where g_i, $i = 1, 2$ is the pullback of the metric on the worldvolume of the $(n+1)$-dimensional ETW branes. Note that we have not included a term $\delta(y_1)\delta(y_2)$, as it is not necessary (in fact, it is forced to be absent) in our solution.

The equations of motion arising from the variation of the above action are very similar to those of (3.6), when appropriately replacing the potential with the source terms. Using the ansatz (3.4) for the metric and the additive structure (3.7) for the warp factors, the equations of motion coming from the variations of (3.25) with respect to the metric components are:

$$\begin{aligned} \{i,j\}: \quad &\frac{1}{2}\Big[e^{-2\sigma_1}\Big(\phi_1^2 + n(n+1)\sigma_1^2 - 2n\sigma_1\Big) + e^{-2\sigma_2}\Big(\phi_2'^2 + n(n+1)\sigma_2'^2 - 2n\sigma_2''\Big)\Big] \\ &+ \lambda_1 e^{\alpha_{11}\phi_1+\alpha_{12}\phi_2}e^{-2\sigma_1-\sigma_2}\delta(y_1) + \lambda_2 e^{\alpha_{22}\phi_2+\alpha_{21}\phi_1}e^{-\sigma_1-2\sigma_2}\delta(y_2) = 0, \\ \{y_1,y_1\}: \quad &\frac{1}{2}\Big[-\phi_1^2 + n(n+1)\sigma_1^2 + e^{2\sigma_1-2\sigma_2}(\phi_2'^2 + n(n+1)\sigma_2'^2 - 2n\sigma_2'')\Big] \\ &+ \lambda_2 e^{\alpha_{22}\phi_2+\alpha_{21}\phi_1}e^{-2\sigma_2+\sigma_1}\delta(y_2) = 0, \\ \{y_2,y_2\}: \quad &\frac{1}{2}\Big[-\phi_2'^2 + n(n+1)\sigma_2'^2 + e^{-2\sigma_1+2\sigma_2}(\phi_1^2 + n(n+1)\sigma_1^2 - 2n\sigma_1)\Big] \\ &+ \lambda_1 e^{\alpha_{11}\phi_1+\alpha_{12}\phi_2}e^{-2\sigma_1+\sigma_2}\delta(y_1) = 0, \\ \{y_1,y_2\}: \quad &-n\sigma_1\sigma_2' + \frac{1}{2}\alpha\phi_1\phi_2' = 0. \end{aligned} \tag{3.26}$$

Additionally, one has two equations of motion coming from the variations with respect to the fields ϕ_i:

$$\begin{aligned} \phi_1: \quad &e^{-\sigma_1+\sigma_2}\big(-(n+1)\phi_1\sigma_1 + \phi_1\big) + \frac{\alpha}{2}e^{\sigma_1-\sigma_2}\big(-(n+1)\phi_2'\sigma_2' + \phi_2''\big) \\ &- \alpha_{11}\lambda_1 e^{\alpha_{11}\phi_1+\alpha_{12}\phi_2}e^{-\sigma_1}\delta(y_1) - \alpha_{21}\lambda_2 e^{\alpha_{22}\phi_2+\alpha_{21}\phi_1}e^{-\sigma_2}\delta(y_2) = 0, \\ \phi_2: \quad &\frac{\alpha}{2}e^{-\sigma_1+\sigma_2}\big(-(n+1)\phi_1\sigma_1 + \phi_1\big) + e^{\sigma_1-\sigma_2}\big(-(n+1)\phi_2'\sigma_2' + \phi_2''\big) \\ &- \alpha_{12}\lambda_1 e^{\alpha_{11}\phi_1+\alpha_{12}\phi_2}e^{-\sigma_1}\delta(y_1) - \alpha_{22}\lambda_2 e^{\alpha_{22}\phi_2+\alpha_{21}\phi_1}e^{-\sigma_2}\delta(y_2) = 0. \end{aligned} \tag{3.27}$$

In analogy with the codimension-1 case, let us make the change $\phi_i = -\sqrt{\frac{n}{n+1}}f_i(y_i)$, $\sigma_i = -\frac{1}{n+1}f_i$ to simplify the equations, in particular the $\{y_1, y_2\}$ equation in (3.26) is satisfied identically. Each of the remaining equations splits into two contributions depending on each the two coordinates, provided that $\alpha_{21}\phi_1 = \sigma_1$ and $\alpha_{12}\phi_2 = \sigma_2$, hence

$$\alpha_{12} = \alpha_{21} = \frac{1}{\sqrt{n(n+1)}}. \tag{3.28}$$

For the scalar equations (3.27) to be compatible one needs

$$\alpha = \frac{2}{n+1}. \tag{3.29}$$

which corresponds to the appropriate value for the case of vanishing potential. The set of equations decouples into two sets basically identical to the codimension-1 equations (3.21) for the f_i, namely

$$\begin{aligned} \frac{n}{n+1}\left(f_i'^2 + f_i''\right) &= -\lambda_i e^{\alpha_{ii}\phi_i}\delta(y_i), \\ \sqrt{\frac{n}{n+1}}\left(f_i'^2 + f_i''\right) &= -\alpha_{ii}\lambda_i e^{\alpha_{ii}\phi_i}\delta(y_i), \end{aligned} \tag{3.30}$$

with no sum over i, and where prime denotes derivative with respect to their argument, now also for $f_1(y_1)$. The equations can be analyzed as in the codimension-1 case, so their compatibility requires

$$\alpha_{11} = \alpha_{22} = \sqrt{\frac{n+1}{n}} = \sqrt{\frac{d-1}{d-2}} \tag{3.31}$$

and the computation of the discontinuities imply

$$\lambda_1 = \lambda_2 = -\frac{n-1}{n} = -\frac{d-2}{d-1}, \tag{3.32}$$

just like in the codimension-1 case, cf. (3.24).

Hence the sources are a simple superposition of two terms along $y_i = 0$, with tensions λ_i and couplings α_{ii} to the scalar ϕ_i given by those of the corresponding codimension-1 ETW brane solution. The extra coupling α_{12} of the ETW brane along $y_1 = 0$ to ϕ_2, and α_{21} of the ETW along $y_2 = 0$ to ϕ_1 imply an interesting variation of the effective tension of the ETW branes as one moves further away from the intersection. Morally, it accounts for the extra factor involved in expressing the codimension-2 solution as a codimension-1 solution, mentioned just above (3.9).

Let us finally explain the solution leaves no room for a codimension-2 $\delta(y_1)\delta(y_2)$ source term. Such term would lead to discontinuities in mixed derivatives of the fields, which are absent in the equations of motion (3.26), (3.27). This is already built in from the use of the additive structure (3.7). It would be interesting to explore more general solutions involving this extra sources, but this lies beyond the scope of this work.

3.3 Scaling relations

In this section we discuss the analogue of the scaling relations (2.6), in particular the relation between the spacetime distance to the singularity along some path and the traversed scalar field space distance. Clearly, the relation will be path-dependent, albeit in a simple way.

Consider a general path $y_i(t)$ in spacetime, parametrized by t, with $y^i \to 0$ as $t \to 0$. For instance, we can choose

$$y_1 = t^{\gamma_1}, \quad y_2 = t^{\gamma_2} \tag{3.33}$$

in terms of two positive real numbers $\gamma_i \geq 0$. Clearly, a change $t \to t^\lambda$ is just a reparametrization of the same path, so γ_i are defined up to an overall rescaling, so only its ratio is meaningful. The tangent vector is

$$\partial_t y_i = \gamma_i \, t^{\gamma_i - 1}. \tag{3.34}$$

The spacetime distance to the origin along this path is

$$= \int [\, e^{-2\sigma_2}(\partial_t y_1)^2 + e^{-2\sigma_1}(\partial_t y_2)^2 \,]^{\frac{1}{2}} \, dt \; = \; \int [\, \gamma_1^2 t^{2r_1} + \gamma_2^2 t^{2r_2} \,]^{\frac{1}{2}} \, dt \tag{3.35}$$

with

$$r_1 = \frac{4\gamma_2}{n\delta_2^2} + \gamma_1 - 1, \quad r_2 = \frac{4\gamma_1}{n\delta_1^2} + \gamma_2 - 1 \tag{3.36}$$

We can consider two regimes, depending on which contribution dominates in the $t \to 0$ limit. The two regions are separated by the line $r_1 = r_2$, equivalently

$$\frac{\gamma_1}{\gamma_2} = \frac{\frac{4}{n\delta_2^2} - 1}{\frac{4}{n\delta_1^2} - 1}. \tag{3.37}$$

The two regimes correspond to the path being closer to each of the two ETW branes. Assuming $r_1 \neq r_2$, the spacetime distance in the two regions is given by

$$= \int \gamma_i \, t^{r_i} \, dt = \frac{\gamma_i}{r_i + 1} t^{r_i + 1}. \tag{3.38}$$

We incidentally note that the r_i have a natural interpretation by writing the tangent vector (3.34) in the tangent space frame[2] $\tau^a = e_i^a \partial_t y_i$ with e_i^a defined by $G_{ij} = e_i^a e_j^b \delta_{ab}$

$$\vec{\tau} = (e^{-\sigma_2}\partial_t y_1, e^{-\sigma_1}\partial_t y_2) = (\gamma_1 t^{r_1}, \gamma_2 t^{r_2}). \tag{3.39}$$

Products of vectors in the tangent space are with the flat metric δ_{ab}, so this reproduces the distance element (3.35). The tangent vector $\vec{\tau}$ will play an interesting role in the discussion of the Distance Conjecture in section 5.2.

The profiles for the scalars allow to translate the path $y_i(t)$ in spacetime into a path in field space $\phi_i(y(t))$. Let us now compute the field theory distance traversed along the path from a point located at some small non-zero value t to the origin $t = 0$. From the action (3.1), the line element in field space is given by

$$d\mathcal{D}^2 \; = d\phi_1^2 + d\phi_2^2 + \alpha d\phi_1 d\phi_2 = 4 \left(\frac{\gamma_1^2}{\delta_1^2} + \frac{\gamma_2^2}{\delta_2^2} + \frac{\alpha\gamma_1\gamma_2}{\delta_1\delta_2} \right) \frac{dt^2}{t^2}. \tag{3.40}$$

[2] We warn the reader that we are using lowercase indices for our spacetime coordinates y_i, which leads to some funny contraction of indices, like in this formula.

This can be recast in terms of the spacetime distance, for each of the two regions in (3.38), as

$$\mathcal{D} = -2\left(\frac{\gamma_1^2}{\delta_1^2}+\frac{\gamma_2^2}{\delta_2^2}+\frac{\alpha\gamma_1\gamma_2}{\delta_1\delta_2}\right)^{\frac{1}{2}}\log t \sim -\frac{2}{|r_i+1|}\left(\frac{\gamma_1^2}{\delta_1^2}+\frac{\gamma_2^2}{\delta_2^2}+\frac{\alpha\gamma_1\gamma_2}{\delta_1\delta_2}\right)^{\frac{1}{2}}\log \quad . \tag{3.41}$$

The explicit dependence on the parametrization i.e. on the γ_i, is simply due to the fact that the choice of initial point for the computation of the distance in terms of a value t does depend on the parametrization (3.33).

We thus obtain a scaling relation near the intersection, of the kind (2.6), namely

$$\sim e^{-\frac{1}{2}\delta_{\rm int}\,\mathcal{D}}\,, \tag{3.42}$$

with a path-dependent coefficient, which in each of the two regions reads

$$\delta_{\rm int} = \left(\frac{\gamma_1^2}{\delta_1^2}+\frac{\gamma_2^2}{\delta_2^2}+\frac{\alpha\gamma_1\gamma_2}{\delta_1\delta_2}\right)^{-\frac{1}{2}}(r_i+1). \tag{3.43}$$

This interpolates between the values of the critical exponents of the two individual ETW branes, which are attained for paths orthogonal to the individual ETW branes:

$$\begin{array}{lcl} \gamma_1=1\,,\,\gamma_2=0 & \Rightarrow & \delta_{\rm int}\to\delta_1 \\ \gamma_1=0\,,\,\gamma_2=1 & \Rightarrow & \delta_{\rm int}\to\delta_2 \end{array} \tag{3.44}$$

In order to visualize the interpolation, let us parametrize $\gamma_1=\sqrt{\gamma}$ and $\gamma_2=1/\sqrt{\gamma}$ such that we have the ratio $\gamma_1/\gamma_2=\gamma$. The value of γ separating the two regimes is (3.37). In terms of this parametrization, we have

$$\delta_{\rm int} = \left\{ \begin{array}{lr} \left(\frac{\gamma}{\delta_1^2}+\frac{1}{\gamma\delta_2^2}+\frac{\alpha}{\delta_1\delta_2}\right)^{-1/2}\left(\sqrt{\gamma}+\frac{4}{n\delta_2^2\sqrt{\gamma}}\right) & \text{for } \gamma>\gamma^* \\ \left(\frac{\gamma}{\delta_1^2}+\frac{1}{\gamma\delta_2^2}+\frac{\alpha}{\delta_1\delta_2}\right)^{-1/2}\left(\frac{1}{\sqrt{\gamma}}+\frac{4\sqrt{\gamma}}{n\delta_1^2}\right) & \text{for } \gamma<\gamma^*. \end{array}\right. \tag{3.45}$$

Note that we recover the limits δ_1, δ_2 for $\gamma\mapsto\infty, 0$, respectively.

Consider for instance the case $\delta_1=\delta_2\equiv\delta$. The two regimes are separated by $\gamma^*=1$ and the scaling parameter simplifies:

$$\delta_{\rm int} = \left\{ \begin{array}{lr} \left[1+\gamma(\gamma+\alpha)\right]^{-1/2}\left(\frac{4+n\gamma\delta^2}{n\delta}\right) & \text{for } \gamma>1 \\ \left[1+\gamma(\gamma+\alpha)\right]^{-1/2}\left(\frac{4\gamma+n\delta^2}{n\delta}\right) & \text{for } \gamma<1. \end{array}\right. \tag{3.46}$$

For $\delta_1\neq\delta_2$ the separation between the two regimes lies at $\gamma^*\neq 1$. In figures 3 we display $\delta_{\rm int}$ as a function of $\log\gamma$, for some illustrative examples with equal (figure a) or different (figure b) values of δ_1,δ_2.

The scaling properties between the spacetime and field theory distance will play an important role in the discussion of swampland conjectures in section 5. In the following section we turn to show several explicit examples of systems described by the solution we have discussed.

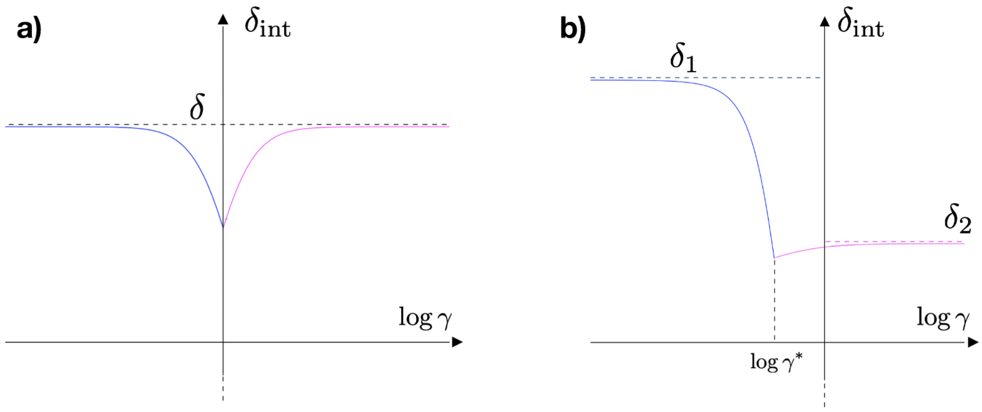

Figure 3. Plot of δ_{int} as a function of the path for two illustrative examples with $\delta_1 = \delta_2 \equiv \delta$ (figure a) and $\delta_1 \neq \delta_2$ (figure b).

4 Explicit examples

In this section we consider explicit examples of intersecting ETW brane solutions. Many are simply obtained from simple configurations, like flat space, by considering reduction along some isometry orbits, which potentially diverge at large distances from the intersection. The solutions we describe should be regarded as local descriptions near the intersection of more involved solutions where such orbits have finite size at infinity. We will mention explicit realizations for several of our examples.

4.1 $\mathbf{S}^1 \times \mathbf{S}^1$ compactifications

A simple example of a codimension-1 ETW brane is the wall of nothing in $\mathbf{S}^1$ compactifications (analogue of the bubble of nothing in [68]) described in section 2. From the higher-dimensional perspective, the local description corresponds to taking flat space, splitting of an $\mathbf{R}^2$, and slicing it along the angular $\mathbf{S}^1$, regarding it as a compactification circle, with radius varying along the radial coordinate, cf. (2.12).

In the same spirit, we can consider the intersection of two wall of nothing ETW branes, which provides a local model for two[3] intersecting bubbles of nothing for different $\mathbf{S}^1$'s. The idea is simply to consider the flat $(n+4)$-dimensional space, written as

$$ds^2_{n+4} = ds^2_n + dr_1^2 + dr_2^2 + r_1^2 d\theta_1^2 + r_2^2 d\theta_2^2 \tag{4.1}$$

where ds^2_n is just flat n-dimensional space.

The above will soon be rewritten in the Einstein frame of the $(n+2)$-dimensional theory obtained upon reduction on the $\mathbf{S}^1 \times \mathbf{S}^1$ parametrized by θ_i; but the intuition is already clear. We have an $(n+2)$-dimensional theory with two scalars, the $\mathbf{S}^1$ sizes, depending of two coordinates r_i. Each scalar shrinks to zero size at the codimension-1 locus $r_1 = 0$ or $r_2 = 0$, respectively, and both shrink simultaneously at the codimension-2 locus $r_1 = r_2 = 0$.

[3]The case of two intersecting bubbles of nothing for the same $\mathbf{S}^1$ belongs to the class of solutions considered in section A.3.

We now turn to carrying this out explicitly. We start with $(n+4)$-dimensional gravity with action

$$S_{n+4} = \frac{1}{2}\int d^{n+4}x\sqrt{-g_{n+4}}\,R_{n+4} \tag{4.2}$$

and compactify on $\mathbf{S}^1 \times \mathbf{S}^1$, parametrized by coordinates θ_1, θ_2. We consider the following ansatz for the $(n+2)$-dimensional theory:

$$ds^2_{n+4} = e^{\alpha_1\rho_2+\alpha_2\rho_2}ds^2_{n+2} + e^{-\beta_1\rho_1}d\theta_1^2 + e^{-\beta_2\rho_2}d\theta_2^2, \tag{4.3}$$

where the breathing modes ρ_i are functions of the non-compact $n+2$ dimensions.[4] The parameters are fixed by the $(n+2)$-dimensional Einstein frame condition, and normalization of the scalar kinetic terms, giving

$$n\alpha_i = \beta_i, \quad \beta_i^2 = \frac{4n}{n+1}. \tag{4.4}$$

Upon compactification, the $(n+2)$-dimensional action for these fields is

$$S_{n+2} \propto \frac{1}{2}\int d^{n+2}x\sqrt{-g_{n+2}}\left[\,R_{n+2} - |d\rho_1|^2 - |d\rho_2|^2 - \alpha\partial_\mu\rho_1\partial^\mu\rho_2\,\right], \tag{4.5}$$

with

$$\alpha = \frac{2}{n+1}. \tag{4.6}$$

This corresponds to an action of the kind (3.1), with the two scalars corresponding to the $\mathbf{S}^1$ sizes, namely $\phi_i \equiv \rho_i$. The scalar potential is zero because the $\mathbf{S}^1$'s have no curvature.

The flat space slicing (4.1) corresponds to an intersecting ETW brane solution of the kind in section 3 with

$$\begin{aligned} ds^2_{n+2} &= r_1^{\frac{2}{n}} r_2^{\frac{2}{n}}\left(\eta_{\mu\nu}dx^\mu dx^\nu + dr_1^2 + dr_2^2\right), \\ \phi_1 &= -\sqrt{\frac{n+1}{n}}\log r_1, \qquad \phi_2 = -\sqrt{\frac{n+1}{n}}\log r_2. \end{aligned} \tag{4.7}$$

With a change of variables

$$y_i = \frac{n}{n+1}\,r_i^{\frac{n+1}{n}}, \tag{4.8}$$

we can go from the above conformally flat solution to one of the form (3.4) with

$$\begin{aligned} ds^2_{n+2} &= y_1^{\frac{2}{n+1}}y_2^{\frac{2}{n+1}}\,ds^2_n + y_2^{\frac{2}{n+1}}dy_1^2 + y_1^{\frac{2}{n+1}}dy_2^2, \\ \phi_i &= -\sqrt{\frac{n}{n+1}}\log y_i. \end{aligned} \tag{4.9}$$

This corresponds to the critical exponents

$$\delta_i = 2\sqrt{\frac{n+1}{n}}. \tag{4.10}$$

[4] Although in this particular example they will turn out to be the relevant lower dimensional scalars ϕ_i, we choose to maintain a specific notation for breathing modes, as in general there may be additional components entering the scalars ϕ_i (see sections 4.3, 4.4 for examples).

4.2 $\mathbf{S}^{p_1} \times \mathbf{S}^{p_2}$ compactifications

We now consider a generalization of the above example, by starting with an D-dimensional space, with $D = n + 2 + p_1 + p_2$, and compactifying on $\mathbf{S}^{p_1} \times \mathbf{S}^{p_2}$. The intersecting ETW brane solution, in which the $\mathbf{S}^{p_1}$ and $\mathbf{S}^{p_2}$ shrink to zero size at two interescting codimension-1 loci, is locally given by slicing D-dimensional flat space as

$$ds_D^2 = \eta_{\mu\nu} dx^\mu dx^\nu + dr_1^2 + dr_2^2 + r_1^2 d\ {}^{2}_{p_1} + r_2^2 d\ {}^{2}_{p_2}, \tag{4.11}$$

where x^μ are coordinates along the Poincaré invariant directions along the intersection and $d\ {}^{2}_{p_1}$ and $d\ {}^{2}_{p_2}$ are the line elements in $\mathbf{S}^{p_1}$ and $\mathbf{S}^{p_2}$, respectively.

Let us quickly describe this construction. We start with D-dimensional gravity with action

$$S_D = \frac{1}{2} \int d^D x \sqrt{-g_D} R_D. \tag{4.12}$$

The compactification ansatz is

$$ds_D^2 = e^{\alpha_1 \rho_2 + \alpha_2 \rho_2} ds_{n+2}^2 + e^{-\beta_1 \rho_1} d\ {}^{2}_{p_1} + e^{-\beta_2 \rho_2} d\ {}^{2}_{p_2}, \tag{4.13}$$

with the requirement to land on the $(n+2)$-dimensional Einstein frame leading to

$$n\alpha_i = p_i \beta_i. \tag{4.14}$$

Also, we normalize the kinetic terms of the two radions ρ_i via the following relations:

$$\beta_i^2 = \frac{4n}{p_i(n+p_i)} \tag{4.15}$$

The $(n+2)$-dimensional Einstein frame action for gravity and the two radions is

$$\begin{aligned} S_{n+2} \propto \frac{1}{2} \int d^{n+2}x \sqrt{-g_{n+2}} \Bigg[& R_{n+2} - |d\rho_1|^2 - |d\rho_2|^2 - \alpha \partial_\mu \rho_1 \partial^\mu \rho_2 \\ & + \frac{p_1(p_1-1)}{R_{p_1}^2} e^{(\alpha_1+\beta_1)\rho_1 + \alpha_2 \rho_2} + \frac{p_2(p_2-1)}{R_{p_2}^2} e^{\alpha_1 \rho_1 + (\alpha_2+\beta_2)\rho_2} \Bigg], \end{aligned} \tag{4.16}$$

with

$$\alpha = \frac{2\sqrt{p_1 p_2}}{\sqrt{(n+p_1)(n+p_2)}}. \tag{4.17}$$

The action (4.16) has precisely the structure (3.1) with the two radions corresponding to the two scalars i.e. $\phi_i \equiv \rho_i$, which have an exponential potential due to the curvature of the internal spheres.

The flat space slicing (4.11) provides a solution to this theory with the structure (3.8), (3.10) with

$$\begin{aligned} ds_{n+2}^2 &= y_1^{\frac{2p_1}{p_1+n}} y_2^{\frac{2p_2}{p_2+n}} ds_n^2 + y_2^{\frac{2p_2}{p_2+n}} dy_1^2 + y_1^{\frac{2p_1}{p_1+n}} dy_2^2, \\ \phi_i &= -\sqrt{\frac{np_i}{n+p_i}} \log y_i. \end{aligned} \tag{4.18}$$

This corresponds to the critical exponents

$$\delta_i = 2\sqrt{\frac{n+p_i}{np_i}}. \tag{4.19}$$

The $\mathbf{S}^1 \times \mathbf{S}^1$ example in section 4.1 is clearly recovered for $p_1 = p_2 = 1$.

One particular example realizing this local behaviour (albeit with AdS_4 rather than 4d Poincaré invariance along the ETW brane) is the gravity dual of 4d $\mathcal{N} = 4$ SU(N) SYM with a boundary coupled to a 3d $\mathcal{N} = 4$ BCFT [75, 76]. It is given by the supergravity solution of D3-branes ending on NS5- and D5-branes), see [48, 77–85], which as emphasized in [48, 84, 85] corresponds to an ETW brane in $AdS_5 \times \mathbf{S}^5$. It is described as an AdS_4 times $(\mathbf{S}^2)^2$ fibered over a Riemann surface given by the first quadrant (y_1, y_2) with $y_i \geq 0$. The two $\mathbf{S}^2$'s shrink to zero size at $y_1 = 0$ and $y_2 = 0$ respectively, while both shrink simultaneously at $y_1 = y_2 = 0$ (so, together with the polar angle in the (y_1, y_2)-plane, there is a shrinking $\mathbf{S}^5$). Hence, this setup reproduces locally the structure of our solution, while globally provides an explicit example in which the $\mathbf{S}^2$'s have constant radius at infinity (being part of the constant radius $\mathbf{S}^5$ in the asymptotic $AdS_5 \times \mathbf{S}^5$).

4.3 Adding D-brane defects

The above examples correspond to variants of bubbles of nothing, in which the ETW brane is realized geometrically, by shrinking (parts of) the compact space. In general, one expects ETW branes to be dressed with topological defects necessary to remove non-trivial cobordism charges of the compactification. A prototypical example is compactifications with field strength fluxes, which require the presence of charged branes at the ETW brane to remove the flux. In this section we consider a simple example describing the local behaviour of an ETW brane for a compactification on $\mathbf{S}^{8-p}$ with N units of RR flux (hence the ETW brane is dressed with N Dp-branes) intersecting an ETW brane of a fluxless $\mathbf{S}^q$ (i.e. a bubble of nothing).

Actually, the solution is described by simply taking the solution for a stack of N Dp-branes in flat space. For simplicity we focus on $p < 7$, for which the metric and dilaton read

$$\begin{aligned} ds_{10}^2 &= Z(r_1)^{\frac{p-7}{8}}(\eta_{\mu\nu}dx^\mu dx^\nu \ + \ dr_2^2 + r_2^2 d\ \ {}_q^2) + Z(r_1)^{\frac{p+1}{8}}\left(dr_1^2 + r_1^2 d\ \ {}_{8-p}^2\right), \\ &= \frac{(3-p)}{4\sqrt{2}}\log Z(r_1), \\ Z(r_1) &= 1 + \left(\frac{\rho}{r_1}\right)^{7-p}, \qquad \rho^{7-p} = g_s N \alpha'^{(7-p)/2}(4\pi)^{(5-p)/2} \quad \left(\frac{7-p}{2}\right). \end{aligned} \tag{4.20}$$

Here x^μ parametrize $p-q$ of the Dp-brane worlvolume directions, and r_2 is the radial coordinate in the remaining $\mathbf{R}^{p+1}$ part of the worldvolume, and $d\ \ {}_q^2$ is the line element in the angular $\mathbf{S}^q$. Namely, we slice the Dp-brane solution along the transverse $\mathbf{S}^{8-p}$ and a worldvolume $\mathbf{S}^q$, and regard it as a solution of an $\mathbf{S}^{8-p} \times \mathbf{S}^q$ compactification.

We thus start with the 10d action for gravity coupled to the dilaton and the RR $(p+1)$-form

$$S_{10} \sim \frac{1}{2}\int d^{10}x\sqrt{-g_{10}}\left\{R_{10} - |d\ \ |^2 - \frac{1}{2(8-p)!}e^{a\Phi}|F_{8-p}|^2\right\} \tag{4.21}$$

with

$$a = \frac{p-3}{2}. \tag{4.22}$$

We use the compactification ansatz

$$
\begin{aligned}
ds^2_{10} &= e^{\alpha_1\rho_1+\alpha_2\rho_2} ds^2_{n+2} + e^{-\beta_1\rho_1} d\ \ {}^2_{8-p} + e^{\gamma_1\rho_1-\gamma_2\rho_2} d\ \ {}^2_{q} \\
F_{8-p} &= N dVol\left(S^{8-p}\right).
\end{aligned}
\tag{4.23}
$$

We now impose the Einstein frame condition, and fix the normalization of the scalars ρ_i

$$
\begin{aligned}
\beta_1 &= \frac{n\alpha_1+q\gamma_1}{8-p} = \frac{1}{8+n-p}\left[(p-n)\gamma_1 \pm 2\frac{\sqrt{n\left[8+n-p+2(p-n)\gamma_1^2\right]}}{\sqrt{8-p}}\right] \\
\gamma_2 &= \frac{n}{q}\alpha_2 = \pm 2\sqrt{\frac{n}{q(n+q)}}
\end{aligned}
\tag{4.24}
$$

The resulting $(n+2)$-dimensional action (with $n = p - q$) is:

$$
\begin{aligned}
S_{n+2} = \frac{1}{2}\mathcal{C}\int d^{n+2}x\sqrt{-g_{n+2}}\Bigg\{ & R_{n+2} - |d\ \ |^2 - |d\rho_1|^2 - |d\rho_2|^2 + \frac{\gamma_2 q\left[\beta_1(p-8)+\gamma_1 p\right]}{2n}\partial_\mu\rho_1\partial^\mu\rho_2 + \\
& + (8-p)(7-p)e^{(\alpha_1+\beta_1)\rho_1+\alpha_2\rho_2} + q(q-1)e^{(\alpha_1-\gamma_1)\rho_1+(\alpha_2+\gamma_2)\rho_2} \\
& - \frac{N^2}{2(8-p)!}e^{a\Phi}e^{[\alpha_1+(8-p)\beta_1]\rho_1+\alpha_2\rho_2}\Bigg\},
\end{aligned}
\tag{4.25}
$$

with the coefficient

$$
\mathcal{C} = \left(\frac{2\pi^{(8-p)/2}}{\left(\frac{(8-p)}{2}\right)}\right)\left(\frac{2\pi^{q/2}}{\left(\frac{q}{2}\right)}\right).
\tag{4.26}
$$

We can now write this in the form (3.1) using the redefinitions

$$
\begin{aligned}
\phi_1 &= \int\left[|d\ \ |^2 + |d\rho_1|^2\right]^{1/2} \\
\phi_2 &= \rho_2
\end{aligned}
\tag{4.27}
$$

and the following relation between the 10 dimensional dilaton and the radion:

$$
= \sqrt{2}\frac{(p-7)}{(p-3)}\rho_1,
\tag{4.28}
$$

from which we have:

$$
\phi_1 = \left[1 - 2\frac{\gamma_1}{\beta_1}\right]^{1/2}\rho_1.
\tag{4.29}
$$

The action becomes:

$$S_{n+2} = \frac{\mathcal{C}}{2} \int d^{n+2}x \sqrt{-g_{n+2}} \Bigg\{ R_{n+2} - |d\phi_1|^2 - |d\phi_2|^2 - \alpha \partial_\rho \phi_1 \partial^\rho \phi_2 + \\ + \frac{(8-p)(7-p)}{R_{8-p}^2} e^{\frac{1}{n}\left(1-2\frac{\gamma_1}{\beta_1}\right)^{-1/2}[(8+n-p)\beta_1+(n-p)\gamma_1]\phi_1} e^{2\sqrt{\frac{q}{n(n+q)}}\phi_2} + \\ + \frac{q(q-1)}{R_q^2} e^{\frac{1}{n}\left(1-2\frac{\gamma_1}{\beta_1}\right)^{-1/2}[(8-p)\beta_1-p\gamma_1]\phi_1} e^{2\sqrt{\frac{n+q}{nq}}\phi_2} + \\ - \frac{N^2}{2(8-p)!} e^{\left\{\frac{(p-3)}{2}\left(1-\frac{1}{2}\frac{\beta_1}{\gamma_1}\right)^{-1/2} - \frac{1}{n}\left(1-2\frac{\gamma_1}{\beta_1}\right)^{-1/2}[(n+1)(p-8)\beta_1+q\gamma_1]\right\}\phi_1} e^{2\sqrt{\frac{q}{n(n+q)}}\phi_2} \Bigg\} \tag{4.30}$$

with

$$\alpha = \frac{\gamma_2 q \left[\beta_1(8-p) - \gamma_1 p\right]}{2n} \left(1 - 2\frac{\gamma_1}{\beta_1}\right)^{-1/2}. \tag{4.31}$$

The Dp-brane solution descends to a solution of the kind (3.4) with the redefinitions

$$\begin{aligned} y_1 &= \left(\frac{2n}{9+n(p-5)-p}\right) r_1^{\frac{9+n(p-5)-p}{2n}}, \\ y_2 &= \left(\frac{n}{q+n}\right) r_2^{\frac{q}{n}+1}. \end{aligned} \tag{4.32}$$

The solution corresponds to

$$ds_{n+2}^2 = e^{-2\sigma_1 - 2\sigma_2} ds_n^2 + e^{-2\sigma_2} dy_1^2 + e^{-2\sigma_1} dy_2^2, \tag{4.33}$$

$$\phi_1 = -\sqrt{\frac{n(9-p)}{9+n(p-5)-p}} \log y_1, \tag{4.34}$$

$$\phi_2 = -\sqrt{\frac{qn}{q+n}} \log y_2, \tag{4.35}$$

$$\sigma_1 = -\frac{(9-p)}{9+n(p-5)-p} \log y_1, \tag{4.36}$$

$$\sigma_2 = -\frac{q}{q+n} \log y_2. \tag{4.37}$$

Hence the resulting critical exponents are

$$\delta_1 = 2\sqrt{\frac{9+n(p-5)-p}{n(9-p)}}, \quad \delta_2 = 2\sqrt{\frac{q+n}{qn}}. \tag{4.38}$$

4.4 One general ETW brane

In this section we present a generalization of the previous sections, and consider the intersection of a bubble of nothing ETW brane, corresponding to a shrinking $\mathbf{S}^p$, with a completely general ETW brane characterized by a critical exponent δ. The solution is actually very simple. We start with a theory in $d = n + p + 2$ dimensions and action

$$S_d = \int d^d x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} (\partial\varphi)^2 - V(\varphi)\right] \tag{4.39}$$

We consider a general codimension-1 local ETW brane solution as in (2.2), (2.3), sliced along an angular $\mathbf{S}^p$ on its worldvolume dimensions, namely

$$\begin{aligned} ds_d^2 &= e^{-2\sigma(y)}\left[ds_n^2 + dr^2 + r^2 d\Omega_p^2\right] + dy^2 \\ \varphi(y) &\simeq -\frac{2}{\delta}\log y, \qquad \sigma(y) \simeq -\frac{4}{(d-2)\delta^2}\log y, \end{aligned} \tag{4.40}$$

The potential in the regime near the ETW brane is of the exponential form (2.4):

$$V(\varphi) = -ace^{\delta\varphi}. \tag{4.41}$$

Taking the coordinate r to parametrize a coordinate along with the $\mathbf{S}^p$ varies, this realizes the intersection of ETW branes of interest.

In order to describe it from the perspective of the $(n+2)$-dimensional action after reduction along the $\mathbf{S}^p$, we take the compactification ansatz

$$ds_d^2 = e^{\alpha\rho} ds_{n+2}^2 + e^{-\beta\rho} d\Omega_p^2, \tag{4.42}$$

The coefficients α, β are fixed by the Einstein frame condition in $(n+2)$-dimensional action and the normalization of the scalar ρ. We require

$$n\alpha = p\beta, \quad \beta^2 = \frac{4n}{p(p+n)}. \tag{4.43}$$

The resulting $(n+2)$-dimensional action is

$$S_{n+2} = \frac{1}{2}\int d^{n+2}x\sqrt{-g_{n+2}}\left\{R_{n+2} - (\partial\varphi)^2 - (\partial\rho)^2 + \frac{p(p-1)}{R_p^2}e^{2\sqrt{\frac{n+p}{np}}\rho} - 2V(\varphi)e^{2\sqrt{\frac{p}{n(n+p)}}\rho}\right\}, \tag{4.44}$$

where φ is the scalar associated to the generic ETW brane in $d-$dimensions and ρ is the breathing mode of the $\mathbf{S}^p$ compactification. In order to get an action in the form (3.1), we redefine the fields via:

$$\varphi = \delta\sqrt{\frac{n(n+p)}{4p+n(n+p)\delta^2}}\,\phi_1, \qquad \rho = \phi_2 + 2\sqrt{\frac{p}{4p+n(n+p)\delta^2}}\,\phi_1\,, \tag{4.45}$$

and the (4.44) becomes:

$$\begin{aligned} S_{n+2} = \frac{1}{2}\int d^{n+2}x\sqrt{-g_{n+2}}\Bigg\{& R_{n+2} - (\partial\phi_1)^2 - (\partial\phi_2)^2 - \alpha\partial_\rho\phi_1\partial^\rho\phi_2 + \\ &+ \frac{p(p-1)}{R_p^2}e^{2\sqrt{\frac{n+p}{np}}\left(\phi_2 - 2\sqrt{\frac{p}{4p+n(n+p)\delta^2}}\phi_1\right)} + 2ace^{2\sqrt{\frac{p}{n(n+p)}}\phi_2 + \sqrt{\frac{4p+n(n+p)\delta^2}{n(n+p)}}\phi_1}\Bigg\}, \end{aligned} \tag{4.46}$$

with

$$\alpha = 2\sqrt{\frac{p}{4p+n(n+p)\delta^2}}. \tag{4.47}$$

It is easy to check that the configuration (4.40) is a local intersecting ETW brane solution (3.14). Performing the change of variables

$$y_1 = \left(\frac{n(n+p)\delta^2}{4p+n(n+p)\delta^2}\right) y^{\frac{4p}{n(n+p)\delta^2}+1}, \quad y_2 = \left(\frac{n}{p+n}\right) r^{\frac{p}{n}+1} \tag{4.48}$$

we obtain

$$\begin{aligned}
&ds^2_{n+2} = e^{-2\sigma_1-2\sigma_2} ds_n^2 + e^{-2\sigma_2} dy_1^2 + e^{-2\sigma_1} dy_2^2 \\
&\quad \sigma_1 = -\frac{4(n+p)}{\delta^2 n(n+p)+4p} \log y_1, \qquad\qquad \sigma_2 = -\frac{p}{n+p} \log y_2 \\
&\quad \phi_1 = -2\sqrt{\frac{n(n+p)}{4p+n(n+p)\delta^2}} \log y_1, \qquad\qquad \phi_2 = -\sqrt{\frac{np}{n+p}} \log y_2
\end{aligned} \tag{4.49}$$

This corresponds to the critical exponents

$$\delta_1 = \sqrt{\delta^2 + \frac{4p}{n(n+p)}}, \qquad \delta_2 = 2\sqrt{\frac{n+p}{np}}. \tag{4.50}$$

Note that δ_2 nicely agrees with the value obtained in section 4.2 cf. (4.19). We can also recover examples of previous sections for different choices of δ. For instance, for $\delta^2 = 4\frac{n+p+q}{q(n+p)}$ we recover the case of $\mathbf{S}^p \times \mathbf{S}^q$ compactification studied in section 4.2. Also, for $\delta^2 = \frac{4(q-3)^2}{q(9-q)}$, with $q < 7$, we recover the case of a Dq-brane solution reduced along the transverse $\mathbf{S}^{8-q}$ times an $\mathbf{S}^p$ along the brane worldvolume, as studied in section 4.3 (with reversed labels p, q).

We hope that these examples suffice to illustrate that our class of solutions includes many physically relevant cases of ETW branes.

5 Swampland applications

In this section we discuss the interplay of our solutions with various swampland constraints. We mainly focus on the cobordism conjecture and the distance conjecture, but mention others along the way.

5.1 Cobordism conjecture

Since ETW branes are motivated by the Cobordism Conjecture, there are several interesting interpretations of our intersecting ETW brane solutions from this perspective.

5.1.1 The end of the world for end of the world branes

Dynamical cobordisms arise in the exploration of the Cobordism Conjecture [55] beyond its merely topological avatar. They provide explicit effective field theory descriptions of cobordisms to nothing, including the possibly necessary defects to remove any non-trivial cobordism classes or other charges in the configuration.

From the perspective of the effective theory, the microscopic structure of the cobordism defect remains mostly shrouded in the mist of the UV completion, but there may be features of the ETW brane worldvolume dynamics amenable to the effective theory approach. One way to probe them is to let the ETW brane interact with other defects of the theory. In the familiar context of string theory branes, the worldvolume gauge field content in a brane can be read from the objects able to end on it [86], and some such BIon configurations are accessible in supergravity [87]. In the context of cobordism defects, for instance [65] obtained the type IIB R7-brane worldvolume theory by the characterization of branes allowed to end on its worldvolume. Namely, cobordism defects must be able to explain not just the end of bulk spacetime, but also the disappearance of the possible defects present in the bulk theory.

Intersecting ETW brane configurations can be regarded as a radical case of this last idea: cobordism defects must be able to explain not just the end of bulk spacetime, but also of bulk configurations bounded by other pre-existing cobordism defects. From the effective theory perspective, when the bulk theory is bounded by a first ETW brane (ETW_1), the resulting configuration must admit being bounded by a second ETW brane (ETW_2), with the bulk ending on the ETW_2 brane and its boundary ETW_1 brane ending on its intersection with the ETW_2 brane, see figure 1a. Obviously there is the converse picture, in which the ETW_1 brane provides a boundary of the configuration given by the bulk theory ending on the ETW_2 image. Hence our intersecting ETW brane solutions explain the end of the world for end of the world branes.

5.1.2 Interpolating domain walls between ETW branes

There is another related but complementary interpretation of intersecting brane configurations from the Cobordism Conjecture perspective, focusing on its implication that in theories of quantum gravity any two configurations can be connected by some domain wall. Hence we may consider two configurations given by the bulk theory bounded by the ETW_1 and the *same* bulk theory bounded by a different ETW_2 brane. We can now consider the interpolating configuration across which the bulk theory is unchanged but the ETW_1-brane turns into the ETW_2 brane. This can be regarded as a configuration of two intersecting ETW branes in the limit where their angle is close to π, with the intersection playing the role of interpolating domain wall between the ETW brane boundaries, see figure 1b.

ETW branes at general angle θ are explicitly constructed in appendix A.1. The limit $\theta \to \pi$ is singular in that description, but only because the original coordinate system becomes degenerate. It would be interesting to extract the resulting limit configuration, or build its solution from scratch, but we refrain from doing so in the hope that the conceptual picture is clear enough.

The above links with the cobordism conjecture relate to the structure of spacetime and its boundaries (and boundaries of their boundaries). The realization of these configurations via dynamical cobordisms with scalars blowing up at the ETW branes furthermore leads to an in interesting map between spacetime physics and field space, to which we turn next.

5.2 Distance conjecture

Our solutions are spacetime-dependent configurations probing infinite distance limits in field space at finite spacetime distance, and therefore have a direct interplay with the Distance Conjecture [5]. We explore various such connections in this section.

5.2.1 General distance conjecture

The Distance Conjecture [5] states that effective field theories break down along geodesic paths extending to infinite distance limits in moduli space due to the appearance of an infinite tower of states at a scale m falling exponentially with the field theory distance

$$m \sim e^{-\lambda \mathcal{D}} \tag{5.1}$$

with some $\mathcal{O}(1)$ coefficient λ (with $\lambda \geq \frac{1}{\sqrt{d-2}}$ according to the sharpened distance conjecture [16]).

In spacetime-dependent configurations probing infinite field space distances at finite spacetime distance, such as ETW branes [28–30], there are interesting relations between the field space distance and the spacetime distance, cf. (2.6). This allows to provide a spacetime version of the Distance Conjecture which expresses the falloff of the cutoff scale along some path in spacetime.

In our intersecting ETW brane setup, we have found similar scaling relations between the field space distance and the spacetime distance (3.42), controlled by the path-dependent coefficient δ_{int}. Combining this expression with (5.1), we obtain

$$m \sim \quad {}^{\frac{2\lambda}{\delta_{\text{int}}}} \tag{5.2}$$

In spacetime-dependent solutions, i.e. beyond the adiabatic approximation, the interpretation of m is not necessarily the appearance of an infinite tower [23]; it instead indicates the scale at which new UV physics kicks in. This played an important role in the context of small black holes, where it fixes the size of the smallest possible black hole in the effective theory [53, 60]. It would be interesting to explore similar mechanisms for ETW branes.

5.2.2 The convex Hull distance conjecture

In theories with several scalar fields, there are different infinite distance limits, which probe the existence of different UV towers. Hence, the cutoff along general infinite distance path in field space can be sensitive to multiple individual towers. A general recipe to obtain the exponential falloff along such a general path is provided by the Convex Hull Distance Conjecture [15] (see also [17]), as follows.

Consider all the towers of the theory corresponding to all possible infinite distance limits, and denote by $m(\phi^i)$ the moduli-dependent mass scale of each of these towers. Let us introduce the scalar charge to mass ratios

$$\zeta_i \equiv \partial_i \log m. \tag{5.3}$$

Consider a trajectory $\phi^i(t)$ going to some infinite distance limit, and define the (normalized) tangent vector

$$\tau^i \equiv \frac{\phi'^i}{||\phi'||}, \quad \text{with } \phi'^i = \frac{d\phi^i}{dt} \tag{5.4}$$

Along this trajectory the tower scale falls off as in (5.1) with

$$\lambda = -\zeta_i \, \tau^i = -\frac{\zeta_i \phi'^i}{||\phi'||}. \tag{5.5}$$

Notice that there is a dependence on the moduli space metric in the normalization of the tangent vector.

Introducing the tangent frame e_i^a in field space $G_{ij} = \delta_{ab} e_i^a e_j^b$, and its inverse e_a^i, this can be recast in terms of vector scalar products

$$\lambda = -\zeta^a \tau_a, \quad \text{with } \zeta^a = e_a^i \zeta_i \ , \ \tau_a = e_a^i \tau_i. \tag{5.6}$$

Combining all towers, the falloff rate is controlled by the convex hull defined by the scalar charge to mass ratios for all towers in the theory [15].

We have shown that theories with several scalars admit intersecting ETW brane solutions, and that different spacetime paths approaching the intersection define different field space paths traversing infinite distance. Formally, one can regard the scalar profiles $\phi^i(y^\mu)$ in our solution[5] as defining an embedding of two spacetime dimensions into the scalar field space. This allows to define pullbacks of moduli space quantities onto the spacetime dimensions, and formulate a spacetime avatar of the Convex Hull Distance Conjecture.

Indeed, the pullback onto spacetime of the scalar charge to mass ratio is

$$\zeta_\mu = \partial_\mu \log m(\phi^i(x^\mu)) = \partial_\mu \phi^i \, \zeta_i \tag{5.7}$$

Also, for a path in spacetime $x^\mu(\lambda)$, with (unnormalized) tangent vector

$$v^\mu = x'^\mu(\lambda), \quad \text{with } x'^\mu = \frac{dx^\mu}{d\lambda}\,, \tag{5.8}$$

we get a path $\phi^i(x^\mu(\lambda))$ in field space, with (unnormalized) tangent vector

$$\frac{d}{d\lambda}\phi^i(x^\mu(\lambda)) = \partial_\mu \phi^i \, x'^\mu = \partial_\mu \phi^i \, v^\mu \tag{5.9}$$

We then have the spacetime version of the numerator of (5.5)

$$\zeta_i \, \phi'^i \ = \ \zeta_i \partial_\mu \phi^i \, v^\mu = \zeta_\mu v^\mu \tag{5.10}$$

The Convex Hull criterion requires using normalized tangent vectors in field space. Using the scalar field metric G_{ij}, we have

$$||\phi'|| = (G_{ij}\phi'^i \phi'^j)^{\frac{1}{2}} = (G_{ij}\partial_\mu \phi^i \partial_\nu \phi^j x'^\mu x'^\nu)^{\frac{1}{2}} = (h_{\mu\nu} v^\mu v^\nu)^{\frac{1}{2}}\,, \tag{5.11}$$

namely, the norm of v^μ but computed with the induced metric

$$h_{\mu\nu} = G_{ij} \partial_\mu \phi^i \partial_\nu \phi^j \tag{5.12}$$

Hence one can formulate the Distance Conjecture as a statement in spacetime in terms of the metric $h_{\mu\nu}$. Note that this is actually different from the spacetime metric $g_{\mu\nu}$. In particular, the field space metric contains mixed terms, whereas the actual spacetime metric is diagonal. It would be interesting to discuss dynamical properties in spacetime of this induced metric from the field space.

[5] We momentarily change to upper indices for fields and spacetime coordinates in order to match usual mathematical conventions in the following argument.

5.2.3 The infinite distance pattern

The above ideas can be easily extended to other swampland criteria. For instance, in [18, 19], an interesting pattern was proposed to hold at infinite distance limits (see also [88] for proposals in the interior) between the tower scale $m(\phi^i)$ and the species scale (the effective cutoff scale of quantum gravity [89–93]) $\;_s(\phi^i)$, which in general is moduli-dependent. Specifically, they are claimed to satisfy

$$G^{ij}\,\partial_i \log m\,\partial_j \log \;_s = \frac{1}{d-2} \tag{5.13}$$

In the context of our solutions, there is a spacetime version of this condition using the (inverse) induced metric

$$h^{\mu\nu}\,\partial_\mu \log m(\phi^i(y^\mu))\,\partial_\nu \log \;_s(\phi^i(y^\mu)) = \frac{1}{d-2} \tag{5.14}$$

Note that the species scale and its relation to the distance conjecture has already been studied from the perspective of the link between spacetime and field space structures for codimension-1 ETW brane solutions in [94]. We expect that similarly exciting ideas may arise in the codimension-2 case of intersecting ETW branes. We leave these explorations as well as links to other swampland conjectures for future work.

6 Conclusions

The exploration of infinite distance limits has led to the construction of diverse defects, such as ETW branes, small black holes or 4d EFT strings, defined by the fact that scalars reach infinite field theory distance at their cores. It is natural to ask about the interplay of such objects, and their use to explore the network of infinite field theory distance limits, i.e. their different components and their intersections.

In this paper we have initiated this exploration by constructing explicit solutions describing intersecting ETW branes in theories with multiple scalars. The configurations behave as the superposition of two codimension-1 ETW branes, and display interesting path-dependent scaling properties along trajectories approaching the codimension-2 intersection. We have explored the interplay of these solutions with swampland conjectures, and in particular with the convex hull description of the Distance Conjecture in theories with several scalars. Finally, we have explicitly shown that many interesting systems correspond to solutions within our class, including intersections of bubbles of nothing and several generalizations thereof.

Some of the interesting questions opened up by our work are:

- Our solutions can be regarded as a mere superposition of ETW branes, in the sense that their source terms are localized on the individual codimension-1 ETW branes. It would be interesting to describe more general intersections supported also by codimension-2 source terms. In this respect, it would be interesting to connect with the codimension-2 objects in [29, 30].

- There are localized D-brane solutions in the literature (see [95] for review and references), which upon suitable reductions along transverse space may be rephrased as intersections of codimension-1 ETW branes. However, the reduction involves directions along which there are no isometries and it is likely it leads to solutions not describable within our ansatz. It would be interesting to study these examples as a tool to generate more general solutions, in particular including examples with non-trivial degrees of freedom at the intersection of ETW branes.

- In our solutions the scalars diverging at the ETW branes had a non-trivial mixed kinetic term. It would be interesting to describe intersecting configurations for decoupled scalar fields, in particular to better connect with CY moduli spaces near infinite distance limits [6, 8, 9]. This will be addressed in [73].
- We have focused on ETW branes and their interplay via intersections. More generally, it would be interesting to understand the interplay of other defects defined by scalars running off to infinity in field space. For instance, the crossing of two EFT strings may lead to the creation of new strings, unveiling non-abelian structures at infinity in moduli space, in the spirit of [96].

We hope to come back to these and other interesting questions in future work.

Acknowledgments

We are pleased to thank Bruno Bento, José Calderón-Infante, Matilda Delgado, Jesús Huertas, Luis Ibáñez, Christian Kneissl, Fernando Marchesano, Miguel Montero, Irene Valenzuela for useful discussions. This work is supported through the grants CEX2020-001007-S and PID2021-123017NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe. The work by R.A. is supported by the grant BESST-VACUA of CSIC.

A Some generalizations

In this appendix we consider several generalizations and variants of the solutions discussed in the main text.

A.1 Intersection at angles

In the main text we have restricted the intersection of ETW branes to be orthogonal in the conformally flat coordinates (3.17). It is a natural generalization to consider a general off-diagonal term

$$\begin{aligned} ds_{n+2}^2 &= e^{-2\sigma_1-2\sigma_2}\left[\, ds_n^2 + dx_1^2 + dx_2^2 + f\, dx_1 dx_2 \,\right] \\ &= e^{-2\sigma_1-2\sigma_2}\, ds_n^2 + e^{-2\sigma_2}\, dy_1^2 + e^{-2\sigma_1}\, dy_2^2 + f\, e^{-\sigma_1-\sigma_2}\, dy_1 dy_2 \,, \end{aligned} \tag{A.1}$$

where σ_i are regarded as functions of x_i in the top line and of y_i in the bottom one. We will look for solutions in which each scalar still runs along one coordinate $\phi_1(y_1)$, $\phi_2(y_2)$. In fact, we try and solve the equations of motion by using logarithmic profiles for σ_i and ϕ_i as in (3.10), which we repeat for convenience

$$\begin{aligned} \sigma_1 &= -a_1 \log y_1 + \frac{1}{2}\log c_1, & \sigma_2 &= -a_2 \log y_2 + \frac{1}{2}\log c_2, \\ \phi_1 &= b_1 \log y_1, & \phi_2 &= b_2 \log y_2, \end{aligned} \tag{A.2}$$

The equations of motion are satisfied if the coefficients satisfy:

$$b_i^2 = na_i^2, \quad \alpha = 2\sqrt{a_1 a_2} \tag{A.3}$$

and the potential behaves as

$$\begin{aligned} V &= V_1 + V_2 + \mathcal{O}\left(y_1^{-1-a_1} y_2^{-1-a_2}\right) \\ V_1 &= c_1 v_1 y_1^{-2} y_2^{-2a_2} = \frac{2c_1 n a_1}{(f^2 c_1 c_2 - 4)} \left[(na_1 + a_1 - 1)\right] y_1^{-2} y_2^{-2a_2} \quad &(A.4) \\ V_2 &= c_2 v_2 y_1^{-2a_1} y_2^{-2a} = \frac{2c_2 n a_2}{(f^2 c_1 c_2 - 4)} \left[(na_2 + a_2 - 1)\right] y_1^{-2a_1} y_2^{-2} \quad &(A.5) \\ \mathcal{O}\left(y_1^{-1-a_1} y_2^{-1-a_2}\right) &= -c_1 c_2 \frac{2n^2 f a_1 a_2}{(f^2 c_1 c_2 - 4)} y_1^{-1-a_1} y_2^{-1-a_2} \quad &(A.6) \end{aligned}$$

Note that the equations of motions provide us more than two terms for the potential. Assuming that $a_1 < 1$ and $a_2 < 2$, the last term is subleading respect to the previous ones which contain the information to specify the kind of intersecting ETW branes.

Moreover the coefficients are given by a generalization of (3.13), namely

$$a_i = \frac{n \pm \sqrt{n + 2(n+1)v_i \left(f^2 c_1 c_2 - 4\right)}}{2n(n+1)} \tag{A.7}$$

Finally, the critical exponents are given by

$$\delta_i^2 = \frac{4}{a_1 n} = \frac{8(n+1)}{n \pm \sqrt{n + 2(n+1)v_i(f^2 c_1 c_2 - 4)}} \tag{A.8}$$

Hence, even though (A.3) has the same structure as (3.11), there is a non-trivial dependence on f in the potential and the coefficients of the logarithms. This implies that, given the potential of the theory, one can read off the values of a_i and v_i from its leading exponential behaviour, and determine the value (or values) of f that provides a solution, which in general corresponds to non-orthogonal intersections.

Note that when $f^2 c_1 c_2 - 4 = 0$ the values above become singular. This corresponds to the limit where the angle between the two ETW branes is 0 or π and the two ETW branes overlap. It would be interesting to explore the limiting behaviour near this regime.

A.2 Triple intersections and beyond

In this section we discuss a natural generalization of the intersecting ETW branes of section 3, by considering triple intersections of three independent ETW branes.

We consider the following action for $(n+3)-$dimensional gravity coupled to three real scalar fields with general potential $V(\phi_1, \phi_2, \phi_3)$:

$$S_{n+3} = \int d^{n+3}\sqrt{-g} \left[\frac{1}{2}R - \frac{1}{2}\sum_{i=1}^{3} (\partial\phi_i)^2 - \frac{1}{2}\sum_{i \neq j} \alpha_{ij} \partial_\rho \phi_i \partial^\rho \phi_j - V(\phi_1, \phi_2, \phi_3)\right]. \tag{A.9}$$

We consider the following ansatz for the metric:

$$ds_{n+3}^2 = e^{2A(y_1,y_2,y_3)} ds_n^2 + e^{2B(y_1,y_2,y_3)} dy_1^2 + e^{2C(y_1,y_2,y_3)} dy_2^2 + e^{2D(y_1,y_2,y_3)} dy_3^2. \tag{A.10}$$

The equations of motion admit solutions with each scalar ϕ_i depends only on the coordinate y_i, given by a simple generalization of (3.4), namely

$$ds_{n+3}^2 = e^{-2\sigma_1-2\sigma_2-2\sigma_3}ds_n^2 + e^{-2\sigma_2-2\sigma_3}dy_1^2 + e^{-2\sigma_1-2\sigma_3}dy_2^2 + e^{-2\sigma_1-2\sigma_2}dy_3^2, \qquad (A.11)$$

while the different functions are locally of the form

$$\phi_i = b_i \log y_i, \quad \sigma_i = -a_i \log y_i + \frac{1}{2}\log c_i \qquad (A.12)$$

and the scalar potential splits into three terms $V = V_1 + V_2 + V_3$ encoding the dominant terms as the different scalars go off to infinity, with

$$V_i = -c_i v_i y_i^{-2} y_j^{-2a_j} y_k^{-2a_k},\ i \neq j \neq k \qquad (A.13)$$

The parameters of the solution are related by

$$b_i^2 = (n+1)a_i, \quad a_i = \frac{1 \pm \sqrt{1 + 8v_i\frac{n+2}{n+1}}}{2(n+2)}, \quad \alpha_{ij} = (a_i a_j)^{-\frac{1}{2}}. \qquad (A.14)$$

Hence the critical exponents are

$$\delta_i^2 = \frac{8(n+2)}{(n+1) \pm (n+1)\sqrt{1 + 8v_i\frac{n+2}{n+1}}}. \qquad (A.15)$$

in terms of which the analogue of (3.14) is

$$\begin{aligned} V_i &= -c_i v_i e^{\delta_i \phi_i} e^{a_j \delta_j \phi_j} e^{a_k \delta_k \phi_k}, & i \neq j \neq k \\ \phi_i &= -\frac{2}{\delta_i}\log y_i, & \sigma_i = -\frac{4}{(n+1)\delta_i^2}\log y_i\,. \end{aligned} \qquad (A.16)$$

It is clear that one can generalize to even higher-codimensional intersections. Triple or higher intersections can be useful to further understand intersection of loci corresponding to multiple infinite distance limits.

A.3 ETW brane configurations with a single scalar field

One may wonder to what extent we need two scalars to achieve intersecting ETW brane configurations. In this appendix we consider candidates for codimension-2 intersections of ETW branes in a theory with a single scalar. We will show that the configuration is actually better described as a single recombined codimension-1 ETW brane.

We start with $(n+2)$-dimensional gravity coupled to one real scalar with general potential

$$S = \int d^{n+2}x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}(\partial\phi)^2 - V(\phi)\right]. \qquad (A.17)$$

We consider a codimension-2 ansatz Considering the following ansatz for the solution:

$$\begin{aligned} ds_{n+2}^2 &= e^{2A(y_1,y_2)}ds_n^2 + e^{2B(y_1,y_2)}dy_1^2 + e^{2C(y_1,y_2)}dy_2^2 \\ \phi &= \phi(y_1, y_2), \end{aligned} \qquad (A.18)$$

and we solve the equations of motion with a set of logarithmic profiles for the fields

$$\begin{aligned}&A = a_1 \log y_1 + a_2 \log y_2, \quad B = b_2 \log y_2, \quad C = c_1 \log y_1 \\ &\phi = d_1 \log y_1 + d_2 \log y_2\end{aligned} \tag{A.19}$$

We get the conditions

$$\begin{aligned}
&(a_2 - b_2)(na_2 + b_2 - c_2) - a_2 + b_2 = 0, && (a_1 - c_1)(na_1 + c_1 - b_2) - a_1 + c_1 = 0 \\
&d_1^2 = c_1(a_1 - c_1) + (n-1)a_1 + c_1, && d_2^2 = b_2(a_2 - b_2) + (n-1)a_2 + b_2 \\
&d_1^2 = \frac{1}{2}na_1\left(1 + \frac{d_1}{d_2}b_2\right), && \delta_2^2 = \frac{1}{2}na_2\left(1 + \frac{d_2}{d_1}c_1\right) \\
&V_1 = -\frac{n}{2}a_1\left(na_1 + c_1 - 1\right) y_1^{-2} y_2^{-2b_2}, && V_2 = -\frac{n}{2}a_2\left(na_2 + b_2 - 1\right) y_1^{-2c_1} y_2^{-2}
\end{aligned}$$

This system is solved by the following choice of the parameters:

$$d_1 = d_2 = -\sqrt{n}, \quad a_1 = c_1 = a_2 = b_2 = 1 \tag{A.20}$$

The scaling relations are now satisfied with the critical exponent:

$$\delta = \frac{2}{\sqrt{n}}\,, \tag{A.21}$$

independently of the path. Indeed, the computation is basically identical to that of section 3.3 with a single field, so that $\delta_1 = \delta_2 \equiv \delta$. The analogue of (**??**) is

$$\delta_{\text{int}} = (r_i + 1)\,\delta \tag{A.22}$$

with r_i as in (3.36). Actually, using a parametrization satisfyin $\gamma_1 + \gamma_2 = 1$, the latter are $r_i = 0$, and hence $\delta_{\text{int}} = \delta$ independently of the path.

The interpretation is that the configuration, rather than the intersection of two individual ETW branes, describes a singular limit of a single recombined ETW brane. This is also motivated by the fact that the solution (A.18), (A.19), with parameters (A.20), is

$$ds_{n+1}^2 = y_1^2 y_2^2 ds_n^2 + y_2^2 dy_1^2 + y_1^2 dy_2^2, \quad \phi = -\sqrt{n}\log(y_1 y_2) \tag{A.23}$$

Performing a change of coordinates

$$y = y_1 y_2, \quad x = \log(y_1/y_2) \tag{A.24}$$

we see that the scalar profile varies non-trivially only along y. In fact, the full solution reads

$$ds_{n+1}^2 = y^2(\, ds_n^2 + 2dx^2\,) + 2dy^2, \quad \phi = -\sqrt{n}\log(y) \tag{A.25}$$

Reabsorbing the factors of 2 via simple redefinitions, this corresponds to a codimension-1 ETW brane solution of type (2.2), (2.3)), for δ given in (A.21). Note the interesting way in which the coordinate x, along which the scalar is constant, becomes a coordinate along the codimension-1 ETW brane, by combining with the n original ones.

Actually, this phenomenon of recombination of ETW branes was already proposed in [45] in the particular context of ETW branes in supercritical bosonic strings with light-like tachyon condensation. The system contains two ETW branes with precisely the critical exponent (A.21). One is spacelike and corresponds to the dilaton growing towards infinitely strong coupling at a point in a finite past time in the Einstein frame; the second is lightlike and corresponds to closed string tachyon condensation. Both ETW branes meet in a codimension-2 $(D-2)$-dimensional locus. Although the ETW branes seem to involve two different scalars, the dilaton and the tachyon, it was shown that they mix together into a single combination. This motivated the proposal that the two ETW branes should be regarded as a single recombined one, so that the beginning of time can be thought of as a strong coupling avatar of closed string tachyon condensation. It is very satisfactory that our general analysis here provides extra support for this picture.

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Received: *June 12, 2024*
Accepted: *August 22, 2024*
Published: *September 25, 2024*

End of the world brane networks for infinite distance limits in CY moduli space

Roberta Angius

Instituto de Física Teórica IFT-UAM/CSIC,
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain

E-mail: roberta.angius@csic.es

Abstract: Dynamical Cobordism provides a powerful method to probe infinite distance limits in moduli/field spaces parameterized by scalars constrained by generic potentials, employing configurations of codimension-1 end of the world (ETW) branes. These branes, characterized in terms of critical exponents, mark codimension-1 boundaries in the spacetime in correspondence of finite spacetime distance singularities at which the scalars diverge. Using these tools, we explore the network of infinite distance singularities in the complex structure moduli space of Calabi-Yau fourfolds compactifications in M-theory with a four-form flux turned on, which is described in terms of normal intersecting divisors classified by asymptotic Hodge theory. We provide spacetime realizations for these loci in terms of networks of intersecting codimension-1 ETW branes classified by specific critical exponents which encapsulate the relevant information of the asymptotic Hodge structure characterizing the corresponding divisors.

Keywords: Flux Compactifications, String and Brane Phenomenology

ArXiv ePrint: 2404.14486

https://doi.org/10.1007/JHEP09(2024)178

Contents

1 Introduction

The study of compactifications in String Theory represents a powerful method for understanding the rich landscape of possible derived vacua and their corresponding phenomenology (see [1–3]). In this realm, the most interesting classes of compactifications are those that preserve only a small number of supersymmetries in the non-compact dimensions. Among these classes, a prominent one arises from compactifying M-theory on a Calabi-Yau fourfold manifold Y_4 [4–6], which leads to three-dimensional effective supergravity theory with $\mathcal{N} = 2$ supersymmetry. Some of these vacua can be lifted to four-dimensional $\mathcal{N} = 1$ F-vacua up to T-duality (see [7, 8] for applications to phenomenology). Considering F-theory on Y_4, for specific choices of the compactification background, in particular when the manifold Y_4 admits an elliptic fibration with base $\mathcal{B}$, the limit in which the elliptic fiber shrinks to zero can be read as a four-dimensional vacuum obtained compactifying type IIB on the base $\mathcal{B}$.

A central aspect in the study of Calabi-Yau compactifications controlled by several moduli is the intricate structure of their moduli space. These moduli parameterize geometric deformations of the compact manifold Y_4 controlling its total volume or the size of the internal cycles. It has been proven in [9] that infinite distance limits of this moduli space are singular loci corresponding to (de-)compactification regimes in the Kähler sector or points in the complex structure sector where some internal cycle shrinks to zero size. For effective field theories with exact moduli space it is possible to explore these singular loci using spacetime independent scalar vevs.

In the context of the swampland program [10] (see also [11, 12] for more recent reviews), many of the conjectures ruling out effective field theories that cannot be lifted into a consistent quantum gravity theory put constraints in the asymptotic behavior of these theories near the boundaries of their corresponding moduli space. This is the case of the Distance conjecture [13], which predicts a tower of exponentially light particles emerging when we approach one of these limits, and its sharpened versions [14, 15], which provide information about the nature of these towers and set a lower bound on the decay rate of their masses. Similar constrains are imposed in terms of a Convex Hull condition [16] for the geodesic trajectories approaching the asymptotic regions of the field spaces of scalars constrained by non-trivial potentials.

In the presence of effective scalar potentials the adiabatic approach to investigate infinite distance limits with constant vevs is in general inconsistent [17] or forbidden [18] and the way to probe these limits is to make use of spacetime dependent solutions describing scalars that go to infinity in a finite distance in spacetime. These tools, first proposed in [19, 20] (see also [15, 21, 22] for recent discussions on spacetime dependent solutions), have been subsequently defined within the framework of Dynamical Cobordisms, [23–29] (see also [30–33] for early related works and [34–41] for more recent developments and [42, 43] for holographic applications), describing configurations where the scalars run to infinity along a spacetime direction that ends at a finite distance in spacetime at which the spacetime metric features a Ricci singularity. These solutions can be regarded as networks of codimension-1 boundaries dressed by extended objects, i.e. End of The World (ETW) branes, sourcing the singularities and allowing spacetime to end. In this sense, Dynamical Cobordism solutions are especially interesting for the bottom-up exploration of these infinite distance limits. Some of these ETW branes do not have a UV resolution, or sometimes their corresponding microscopic description is unknown, see [44] for discussion in purely EFTs with a cutoff.

Beyond these motivations, Dynamical Cobordism solutions represent the natural way to implement the Cobordism Conjecture [45] in the framework of the effective field theories. In particular, they provide effective realizations of the End of The World configurations predicted by that conjecture at the topological level and hint at the presence of defects in the complete theory, realized as ETW branes in this effective approach, which are able to trivialize the corresponding cobordism group and make the compactification background bordant to nothing.

In the context of three dimensional supergravity theories, obtained by M-theory compactification on Calabi-Yau fourfolds, the effective potential admits a purely geometric description in terms of a G_4-flux turned on in some internal cycles of the compact manifold Y_4. The complex structure sector $\mathcal{M}_{cs}$ of the field space of these theories still holds a very intricate net of infinite distance singularities described in terms of normal crossing divisors [46] at which some internal four-cycle, maybe dressed with a G_4 flux, shrinks to zero size and produces a singular geometry for the corresponding Y_4. The mathematical formalism best suited to explore the structure of this network is encoded in the asymptotic Hodge theory [47, 48]. While in the bulk of the moduli space the middle cohomology $H^4(Y_4, \mathbb{C})$ admits a pure Hodge decomposition, as we approach a point in the boundary this structure is no longer valid and we need to define a finer structure, known as Deligne splitting, which includes the possibility to enhance four-forms to higher forms. The new splitting is completely determined by the nilpotent orbit approximation of the period vector and by the properties of local monodromy around the putative singular locus. This structure allows to provide a classification of the types of possible singular divisors forming the network and the allowed enhanced singularities occurring at their intersections [48, 49]. Remarkably, the asymptotic structure of the middle cohomology also furnishes a suitable framework to formulate a growth theorem that is able to capture the leading growth of the Hodge norm of the four-forms near the boundary. The theorem allows to compute all the terms to construct the three-dimensional effective action controlling the dynamics of the scalars parameterizing the divisors involved in the putative local patch of the network. Applications of this mathematical machinery to the study of Calabi-Yau compactifications have been developed in several works [9, 50–53], as well as applications to the computations of scattering amplitudes are starting to generate some interest [54, 55].

In this paper we will start to probe the network of infinite distance singularities of the complex structure sector of the moduli space associated with Calabi-Yau four-folds flux compactifications of M-theory using Dynamical Cobordism solutions of the corresponding three-dimensional effective action. We will present a dictionary associating to each singular divisor in $\mathcal{M}_{cs}$, classified in terms of its asymptotic Hodge-Deligne structure and equipped with the flux information, a specific codimension-1 ETW brane in spacetime, classified in terms of its critical exponent. Highly non-trivially, we will show that the consistency of the construction requires that the spacetime solution be time-dependent, and the codimension-1 ETW brane mark a boundary for a timelike coordinate.

In order to explore the intersections between distinct singular divisors in the moduli space, we will construct a new class of Dynamical Cobordism solutions involving intersecting ETW branes associated to two scalars attaining infinite field space distance in finite spacetime distance. The solution explores the infinite distance network of intersecting divisors, albeit in a subtle way, different from the naive expectation to associate to each intersecting ETW brane an intersecting divisor in the moduli space. The resulting spacetime picture nicely matches the moduli space view in [9, 50] that the intersection of divisors can be described as the enhancement of singularities in specific growth sectors of the moduli along infinite distance paths. We will show that the structure of the scalar flux potentials has exactly the structure required to support the new spacetime dependent intersecting ETW brane Dynamical Cobordism solution.

Although we are dealing with singular solutions in a regime where the effective field theory exhibits a lowered cutoff that limits its validity, these solutions should describe the EFT version of objects that are well defined in the UV. This has been checked on large classes of string theory examples in [19, 20, 23] (see also [56, 57] for other setups in which singularities in supergravity are resolved by sources in the complete theory), but a general proof still remains an open question. For the class of solutions treated in the present work, whose corresponding EFTs come from specific top-down constructions, we assume that this UV completion exists.

The paper is organized as follows. In section 2 we review the main mathematical tools of the asymptotic Hodge theory necessary to characterize the network of infinite distance singularities in the complex structure sector of the Calabi-Yau moduli space. In section 3 we will start reviewing the codimension-1 ETW brane solutions, following [23], and their intersecting configurations introduced in [26]. In section 3.3 we will show how these intersecting configurations can be read in terms of decoupled scalar fields up to an appropriate redefinition of the fields. In section 3.4 we will construct a new class of Dynamical Cobordism solutions involving two distinct divergent scalar fields and a non-conformally flat ansatz for the spacetime metric. These solutions can still be interpreted as intersecting configurations of two codimension-1 ETW branes up to an appropriate redefinition of the fields. In section 4 we will exploit these Dynamical Cobordism solutions to probe the network of infinite distance singularities of $\mathcal{M}_{cs}$. In section 4.1 we will present the dictionary between singular divisors and ETW branes; in section 4.2 we will apply the results of section 3.4 to the enhanced singularities occurring at the loci of intersections of singular divisors in $\mathcal{M}_{cs}$. Finally, we will provide some final considerations in section 5.

2 Generalities on Calabi Yau moduli space and flux potentials

In this introductory section we will review the mathematical tools necessary to provide a powerful local description of the complex structure moduli space of Calabi-Yau manifolds around its singular loci. Although the results are already in the literature, we review them to make our discussion self-contained, and to emphasize the key ideas relevant for the construction of our solutions in later sections. We keep the discussion brief, and refer the reader to [47–49, 58] for more detailed explanations in the mathematical part and to [9, 50–52, 59, 60] for the physical interpretation. The reader not interested in the mathematical details may take the results summarized in figure 4 and table 1 and safely jump to section 3.

A Calabi-Yau *D-folds* manifold is a Kähler manifold of complex dimension D admitting everywhere a non-vanishing $(D,0)$ form . One way to fully determine a particular Calabi-Yau manifold Y_D in a family of CY_D manifolds is to specify its holomorphic form   and its Kähler $(1,1)-$form J. Deformations of these choices that keep the manifold in the same family can be of two different types: we can have *complex structure* deformations parameterized by the choice of a $(D-1,1)$ form, or *Kähler structure* deformations encoding all the possibilities to fix the Kähler form J. Geometrically, deformations of the first type control the size of the internal $D-$cycles, while deformations of the second type control the sizes of the even internal cycles of the Calabi-Yau.

From now on we will consider the Kähler moduli to be fixed and we will focus our attention on the complex structure sector of the moduli space parameterized by deformations of $D-$cycles. For Calabi-Yau D-folds the complex structure manifold $\mathcal{M}_{cs}$ is still a Kähler manifold of complex dimension $h^{D-1,1}$, and it has the property of being neither smooth nor compact, [61, 62]. This implies the existence of singular loci on it, whose corresponding Calabi-Yau manifolds are singular. Let us call this set of points   the *discriminant locus*. Since $\mathcal{M}_{cs}$ is a quasi-projective manifold [63], the Hironaka's resolution theorem [46] assures us that we can always resolve   in terms of the union of normally intersecting divisors:

$$= \bigcup_k \;\; _k . \tag{2.1}$$

Far away from these critical divisors, in the bulk of the moduli space, the Kähler potential $\mathcal{K}^{cs}(z, z)$ is always a well-defined function:

$$\mathcal{K}^{cs}(z, z) = -\log \int_{Y_D} \quad (z) \wedge * \overline{}(z) \tag{2.2}$$

where the dependence on the moduli $\left\{z^I\right\}$, with $I = 1, \ldots, h^{D-1,1}$, is made explicit. This function induces a natural metric in $\mathcal{M}_{cs}$:

$$G_{I\overline{J}} = \partial_{z^I} \partial_{\overline{z}^J} \mathcal{K}^{cs} \tag{2.3}$$

known as Weil-Petersson metric.

In the rest of the paper we will concentrate our attention on the study of $\mathcal{M}_{cs}$ associated with Calabi-Yau 4-folds. M-theory compactifications on this class of manifolds leads to three-dimensional supergravity with $\mathcal{N} = 2$ supersymmetry where the complex structure moduli become the scalar fields for the effective action with kinetic terms induced by the metric (2.3). An effective three-dimensional potential for these fields can be generated turning-on G_4 fluxes in the four-cycles of the internal CY_4 [4, 5, 64, 65]. Its definition in the three-dimensional Einstein frame is [6]:

$$V_M = \frac{1}{\mathcal{V}_4^3} \left(\int_{Y_4} G_4 \wedge \star \overline{G}_4 - \int_{Y_4} G_4 \wedge G_4 \right). \tag{2.4}$$

The second term of this expression is topological and it is constrained by a *tadpole cancellation* condition:

$$\frac{1}{2} \int_{Y_4} G_4 \wedge G_4 = \frac{\chi(Y_4)}{24}, \tag{2.5}$$

where $\chi(Y_4)$ is the Euler characteristic of Y_4. Assuming that this condition is satisfied, for the rest of the discussion we will limit our attention on the first contribution.

The expression (2.4) depends on both the complex structure and the Kähler moduli through the Hodge star operator and the volume $\mathcal{V}_4$ of Y_4. Since we are focusing our analysis in the complex structure sector, we impose the following condition on the flux G_4:

$$G_4 \wedge J = 0, \tag{2.6}$$

which limits our possibilities on the *primitive cohomology* $H^4_p(Y_4, \mathbb{R})$,[1] and isolates all dependence on the Kähler moduli in the volume prefactor.

2.1 Variation of Hodge structure

All the relevant quantities we introduced so far to construct the three-dimensional physical action depend on the complex structure through the holomorphic form or the Hodge star operator in (2.4). In this section we will review some essential mathematical tools that allow us to encode all these dependences in the language of variation of the Hodge structure.

[1]For the rest of the paper we restrict our attention on this primitive cohomology but we will omit the subindex p.

The nice property that the $k-th$ cohomology groups $H^k(Y_D,\mathbb{C})$ of a smooth Calabi-Yau manifold admit a pure Hodge structure of weight k means that for each level k we can define a vector space $V_{\mathbb{C}} = H^k(Y_D,\mathbb{C})$, which always admits an Hodge decomposition:

$$V_{\mathbb{C}} = H^{k,0} \oplus H^{k-1,1} \oplus \ldots \oplus H^{1,k-1} \oplus H^{0,k} = \bigoplus_{k=p+q} H^{p,q} \tag{2.7}$$

where the building subspaces satisfy the following complex-conjugation property:

$$H^{p,q} = \overline{H}^{q,p}, \tag{2.8}$$

and the weight k is the constant sum of the indices $p+q=k$ of all the blocks in (2.7).

An equivalent way to rephrase the same property is saying that each cohomology group $H^k(Y_D,\mathbb{C})$ defines a decreasing Hodge filtration:

$$0 \subset H^{k,0} = F^k \subset F^{k-1} \subset \ldots \subset F^1 \subset F^0 = V_{\mathbb{C}} \tag{2.9}$$

where $H^{p,q} = F^p \cap \overline{F}^q$ and such that $F^p \oplus \overline{F}^{k+1-p} \cong H^k$, for any $p \leq k$.

If we can further equip our Hodge structure of a bilinear form $S(\cdot,\cdot)$ on $V_{\mathbb{C}}$ satisfying the following two properties:

$$\begin{aligned} &(i) \quad \text{ORTHOGONALITY} \quad S\left(H^{p,q}, H^{r,s}\right) = 0 \quad \text{for} \quad p \neq s, q \neq r; \\ &(ii) \quad \text{NON-DEGENERACY} \quad i^{p-q} S\left(v, \overline{v}\right) > 0 \quad \text{for} \quad v \in H^{p,q}, \quad v \neq 0, \end{aligned} \tag{2.10}$$

the structure is called *polarized*.

Since we are dealing with four-form in Calabi-Yau fourfolds we are interested in a deeper exploration of the middle cohomology $H^4(Y_4,\mathbb{C})$. In such a case the bilinear form is the cup product:

$$S(v,w) = \int_{Y_4} v \wedge w \qquad v, w \in H^4(Y_4,\mathbb{C}), \tag{2.11}$$

and it induces a norm for the vectors of the whole filtration:

$$||v||^2 = S(v, *\overline{v}) \tag{2.12}$$

called Hodge norm.

When we move on the moduli space the Hodge decomposition (2.7) changes. Roughly speaking due to change of what we call holomorphic and anti-holomorphic. One way to express this variation is in terms of the variation of the holomorphic four form with respect to a fixed basis $\left\{\gamma^{\mathcal{I}}\right\}$ of F^0, with $\mathcal{I} = 1, \ldots, dim H^4(Y_4,\mathbb{C}) = 2 + 2h^{3,1} + h^{2,2}$, such that the pairing:

$$\eta_{\mathcal{I}\mathcal{J}} = -\int_{Y_4} \gamma^{\mathcal{I}} \wedge \gamma^{\mathcal{J}} \tag{2.13}$$

has signature $(2h^{3,1}, 2+h^{2,2})$.

In particular we can expand the form along this basis writing:

$$= {}^{\mathcal{I}}(z)\gamma_{\mathcal{I}}. \tag{2.14}$$

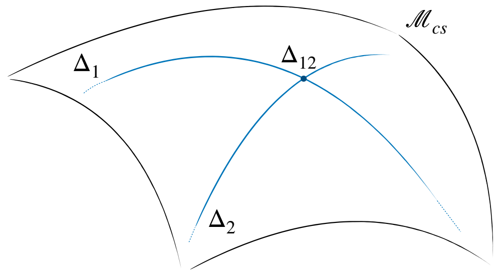

Figure 1. Intersection of two codimension-1 singular divisors $_1$ and $_2$ in the codimension-2 locus $_{12}$ of the two-dimensional complex moduli space $\mathcal{M}_{cs}$.

The coefficients of such expansion are called *periods* and they are in general complicated holomorphic trascendental functions of the moduli. Equivalently, they are defined by:

$$ {}^{\mathcal{I}}(z) = \int_{\Gamma^{\mathcal{I}}} \quad , \tag{2.15} $$

where ${}^{\mathcal{I}}$ are the 4-cycles Poincaré-dual of the forms $\gamma^{\mathcal{I}}$.

At this point we can understand the variations of the spaces F^p on $V_{\mathbb{C}}$ over the space $\mathcal{M}_{cs}$ in terms of the variations of the periods (2.15) with respect to the coordinates z.

This whole structure is completely well defined in the bulk of the moduli space, as long as the manifold Y_4 is smooth. However, when we approach a singular locus in $\mathcal{M}_{cs}$ the periods (2.15) diverge, the Hodge filtration F^p and the corresponding Hodge decomposition $H^{p,q}$ are not well defined anymore and we lose all information about what happens at these points. This pushed the mathematical research to define analogous quantities at these limits, truncating the divergences of F^p and $H^{p,q}$.

2.2 Asymptotic regime and N.O.T.

In this section we start exploring the boundaries of the complex structure moduli space $\mathcal{M}_{cs}$, in particular studying the asymptotic behavior of the Hodge structure of the middle cohomology within the nilpotent orbit approximation.

As we pointed out above these asymptotic regions are resolved in terms of normal crossing divisors (2.1), where each individual critical divisor $_k$ identifies a codimension-1[2] locus in $\mathcal{M}_{cs}$ and each normal intersection $_{k_1 \dots k_n} = {}_{k_1} \cap \dots \cap {}_{k_n}$ identifies a codimension-n locus. See figure 1 for an illustrative representation in a complex two-dimensional moduli space.

In order to understand what happens around these loci it is useful to introduce an adapted set of local coordinates $\left\{z^i, \xi^{\bar{j}}\right\}$ describing the patch $\mathcal{L} \subset \mathcal{M}_{cs}$ that contain the sub-discriminant locus $_{k_1 \dots k_n}$ identified by the conditions $z^i = 0$, with $i = 1, \dots, n$ and $j = n+1, \dots, h^{3,1}$. The structure of $\mathcal{L}$ is given by the product of $(h^{3,1} - n)$ disks $\mathcal{D} = \{\xi \in \mathbb{C} | |\xi| < 1\}$, spanning the transverse directions to the singularity, and n punctured disks

[2]Note that when we talk about *codimension* in the moduli space we mean a complex codimension. In later sections, when we will refer to spacetime codimension, it will mean a real codimension.

$\mathcal{D}^* = \{z \in \mathbb{C} | 0 < |z| < 1\}$, which span the longitudinal directions:

$$\mathcal{L} = (\mathcal{D}^*)^n \times (\mathcal{D})^{h^{3,1}-n} \,. \tag{2.16}$$

Using a conformal transformation we can map the punctured disks $\mathcal{D}^*$ in the upper half complex plane via:

$$z^i \quad \mapsto \quad t^i = \frac{1}{2\pi i} \log z^i = a^i + i s^i. \tag{2.17}$$

With respect to these new suggestive coordinates the boundary $z^i = 0$ is reached sending to infinity the imaginary part s^i of t^i and leaving the real part a^i constant. Moreover, the periods are multivalued functions of the moduli, due to monodromies in the phase of the coordinates z^i, which in the new coordinates are thus associated to shifts of a^i. Let us encode the transformation properties of the periods around each singular divisor $\;_k$ in the *monodromy matrices* T_k acting conventionally as:

$$\mathbf{\Pi}\left(\ldots, e^{2\pi i} z^k, \ldots\right) = T_k^{-1} \quad \left(\ldots, z^k, \ldots\right). \tag{2.18}$$

These local monodromies will be sufficient to classify all the types of singularities in $\mathcal{M}_{cs}$ and to extract information about the behavior of the Hodge structure of the middle cohomology around these limits. In particular, the crucial information for this analysis is contained in the infinite order part of these matrices, which can be extracted through the factorization:

$$T_k = T_k^{(s)} \cdot T_k^{(u)}, \tag{2.19}$$

where $T_k^{(s)}$ is the finite order part and $T_k^{(u)}$ is the *unipotent* part. For the rest of the discussion we will only care about this last factor defining the following related matrix:

$$N_k = \log\left(T_k^{(u)}\right), \tag{2.20}$$

which is *Nilpotent*, namely there exists a positive integer n_k such that $N_k^{n_k+1} = 0$. Matrices N_k associated with different divisors $\;_k$ are commutative.

In the asymptotic regime, reached in the limit $t_1, \ldots, t_n \mapsto +i\infty$, the *Nilpotent Orbit Theorem* (N.O.T.) [47] states that the period vector is represented by the following expansion:

$$\mathbf{\Pi}(t^i, \xi^{\bar{j}}) = e^{\sum_{i=1}^n t^i N_i} \left(\mathbf{a}_0(\xi) + \mathbf{a}_j(\xi) e^{2\pi i t^j} + \mathbf{a}_{ij}(\xi) e^{2\pi i (t^i + t^j)} + \ldots\right), \tag{2.21}$$

where the entries of the vectors $\mathbf{a}_\bullet(\xi)$ are holomorphic functions, in general non-polynomial, of the non-singular coordinates $\xi^{\bar{j}}$. This implies that, up to exponential corrections, the nilpotent orbit:

$$\mathbf{\Pi}_{nil}(t^i, \xi^{\bar{j}}) = e^{\sum_{i=1}^n t^i N_i} \mathbf{a}_0(\xi) \tag{2.22}$$

is a good approximation of the period vector near the locus $\;_{k_1 \ldots k_n}$.

Following this expansion, we can define near each asymptotic region the *limiting filtration*:

$$F^p_{lim}(\;_{k_1 \ldots k_n}) = \lim_{t_{k_1}, \ldots, t_{k_n} \mapsto +i\infty} e^{-\sum_{i=1}^n t^i N_i} F^p(t), \tag{2.23}$$

which is related with the pure Hodge structure F^p by the N.O.T. and it has the property that is stays finite.

Just like the Hodge filtration F^p produces the decomposition (2.7) of the middle cohomology, the limiting filtration F_0^p, together with the information about the monodromies $N_1, \ldots, N_n$, packaged in any element N of the cone $\sigma(N_1, N_2, \ldots, N_n) = \{\sum_{i=1}^n a^i N_i | a^i > 0\}$, contains all the ingredients to construct a *mixed* Hodge decomposition where the middle cohomology lifts into a finer splitting $\{I^{p,q}\}$, with $0 \leq p, q \leq 4$, known as *Deligne splitting*:

$$H_p^4\left(Y_4, \mathbb{C}\right) \quad \longrightarrow \quad \bigoplus_{0 \leq p,q \leq 4} I^{p,q} \tag{2.24}$$

The significant ingredient that allows us to give a formal definition for this splitting is encoded in the vector spaces:

$$W_l(N) = \sum_{j \geq max(-1, l-4)} ker N^{j+1} \cap Im N^{j-l+4}, \tag{2.25}$$

producing a monodromy weight filtration under the action of N such that [66]:

$$N W_l \subseteq W_{l-2} \tag{2.26}$$

Using the vector spaces W_i and F_{lim}^p we can define the Deligne splitting through the formula:

$$I^{p,q} = F_{lim}^p \cap W_{p+q} \cap \left(\overline{F}_{lim}^q \cap W_{p+q} + \sum_{j \geq 1} \overline{F}_{lim}^{q-j} \cap W_{p+q-j-1} \right). \tag{2.27}$$

This is the unique definition such that the following three properties are satisfied:

(*i*) $F_{lim}^p = \bigoplus_{r \geq p} \bigoplus_s I^{r,s}$;

(*ii*) $W_l = \bigoplus_{p+q \leq l} I^{p,q}$;

(*iii*) $I^{p,q} = \overline{I^{q,p}} \mod \bigoplus_{r<p, s<q} I^{r,s}$.

Acting on the space $I^{p,q}$ with the Nilpotent matrix N, we have:

$$N I^{p,q} \subset I^{p-1,q-1}. \tag{2.28}$$

However, in general not the whole lower $(p,q)-$ spaces can be obtained through this action. The subspaces $P^{p,q} \in I^{p,q}$ that cannot be obtained acting with N^k on $I^{p+k,q+k}$ form the *primitive part* of the splitting, and we have:

$$I^{p,q} = \oplus_{i \geq 0} N^i \left(P^{p+i,q+i}\right), \tag{2.29}$$

The elements of $P^{p,q}$ satisfy the following *polarization conditions*:

$$\begin{aligned} S\left(P^{p,q}, N^l P^{r,s}\right) &= 0 \qquad \text{for} \quad p+q = r+s = l+4 \quad \text{and} \quad (p,q) \neq (s,r) \\ i^{p-q} S\left(v, N^{p+q-4} \overline{v}\right) &> 0 \qquad \text{for} \quad v \in P^{p,q}, \quad v \neq 0, \end{aligned} \tag{2.30}$$

which guarantees us that the elements belonging to the primitive part have strictly-positive norm. These conditions will become important to study the allowed Deligne splittings occurring at the enhanced singularities. It will be crucial that they are correctly transmitted, and this will put severe constraints on the form of the enhancement.

2.3 Strict asymptotic regime and SL(2)-orbit theorem

In this work we are especially interested to construct the effective three-dimensional actions for the complex structure moduli near any asymptotic region of . This requires to have a prescription to approximate the Hodge norm of four-forms in order to explicitly compute the kinetic metric (2.3) and the leading behavior of the scalar potential (2.4). The prescription is given by the *Growth Hodge-norm theorem*, proven in [48], using the information of the monodromy matrices N_i and the vector $\mathbf{a}_0$.

Another fundamental data we need when we treat with singular loci of codimension higher than 1 is the specification of the growth sector we use to reach the final singularity. This means to choose an order to send the moduli to infinity in the asymptotic region around $_{k_1\dots k_n}$. Each of these choice defines a growth sector:

$$\mathcal{R}_{12\dots n} = \left\{ t^i = a^i + is^i \middle| \frac{s^1}{s^2} > \gamma, \frac{s^2}{s^3} > \gamma, \dots \frac{s^{n-1}}{s^n} > \gamma, s^n > \gamma, a^i < \delta \right\}. \tag{2.31}$$

For $\gamma \gg 1$ this definition specifies what we call a *strict asymptotic regime* (SAR) around $_{k_1\dots k_n}$. For application to the growth theorem, the objects that better encode the previous data are a set of n $sl(2,\mathbb{C})$ commuting algebras:

$$sl\,(2,\mathbb{C})_i = \langle N_i^-, N_i^+, Y_i \rangle, \tag{2.32}$$

associated to the $i-$th divisor $_{k_i}$ involved in the intersection $_{k_1\dots k_n}$. The triplets $\left(N_i^-, N_i^+, Y_i\right)$ can be computed following a recursive method applied to the splittings $\left\{I_{(i)}^{p,q}\right\}$ which are associated with all the loci $_{k_1\dots k_i}$, with $i < n$, traversed to reach the highest intersection $_{k_1\dots k_n}$ following the order of (2.31). These refined splittings are $\mathbb{R}-$split[3] Deligne splitting computed using the filtration F^p related to F^p up to the action of two operators ξ and δ:

$$F^p = e^{\hat{\xi}} e^{-i\hat{\delta}} F^p. \tag{2.33}$$

The $Sl(2)-$*orbit theorem* in [48] prove this statement and provide an explicit construction of the matrices ξ and δ.

Now, starting from the higher Deligne splitting $\left\{I_{(n)}^{p,q}\right\}$ we can compute the operator $Y_{(n)} = Y_1 + \dots + Y_n$ such that:

$$Y_{(n)} I_{(n)}^{p,q}(\;_{k_1\dots k_n}) = (\underbrace{p+q}_{l_n} - 4) I_{(n)}^{p,q}. \tag{2.34}$$

Then, repeating the construction for all the subsequent splittings $\left\{I_{(i)}^{p,q}\right\}$ until the operator Y_1 associated to the divisor $_1$, we can reconstruct all the operators $Y_i = Y_{(i)} - Y_{(i-1)}$. At this point, decomposing N_i into the basis of the eigenvectors of $Y_{(i-1)}$, namely $N_i = \sum_\alpha N_i^\alpha$, we can define $N_i^- := N_i^0$, which is the only element in the decomposition that commutes with $Y_{(i-1)}$. Finally, we can complete the triples looking for the elements N_i^+ satisfying the correct commutation relations and preserving the polarization.

[3] They satisfy the property *(iii)* below (2.24) with zero modulo.

The first important consequence of the $Sl(2)$-orbit theorem is the definition of a further approximation of the nilpotent orbits in the strict asymptotic regime, specified by the sector $R_{12\ldots n}$, with the $Sl(2)-$orbits:

$$\mathbf{\Pi}_{Sl(2)} = e^{i\sum_{i=1}^{n} s^i N_i^-} \mathbf{a}_0^{(n)}(\xi) \tag{2.35}$$

related with $\mathbf{\Pi}_{nil}$ via:

$$\mathbf{\Pi}_{nil} = e^{\sum_{i=1}^{n} t^i N_i} \mathbf{a}_0(\xi) = e^{\sum_{i=1}^{n} a^i N_i} \cdot M(s) \cdot \mathbf{\Pi}_{Sl(2)}, \tag{2.36}$$

where $M(s)$ is a s^i-dependent matrix introduced in section 5 of [47].

This new approximation drops the subleading polynomial corrections in $\mathbf{\Pi}_{nil}$.

The second relevant consequence for our aim is the fact that the $sl\,(2,\mathbb{C})$ triplets allow to decompose the cohomology group as:

$$H_p^4\,(Y_4,\mathbb{C}) = \bigoplus_{\mathbf{l}\in\mathcal{E}} V_{\mathbf{l}} \tag{2.37}$$

where $\mathcal{E}$ is the set of vectors $\mathbf{l} = (l_1, l_2, \ldots, l_n)$ which entries $0 \leq l_i \leq 8$ are related to the eigenvalues of $\mathbf{v_l}$ with respect the operators $Y_{(i)}$ as:

$$Y_{(i)} v_{\mathbf{l}} = (l_i - 4) v_{\mathbf{l}}. \tag{2.38}$$

Note that this decomposition holds near the locus $\;_{k_1\ldots k_n}$ and strictly depends on the growth sector we use to reach that locus. Now, any vector $w \in H_p^4\,(Y_4,\mathbb{C})$ can be written in an unique way as:

$$w = \sum_{\mathbf{l}\in\mathcal{E}} w_{\mathbf{l}}, \tag{2.39}$$

where $w_{\mathbf{l}} \in V_{\mathbf{l}}$. For each of these vectors the growth Hodge-norm theorem ensures that the leading growing of its norm in the $Sl(2)-$approximation is captured by a term of the form:

$$||w||^2 \sim \text{const} \cdot \left(\frac{s^1}{s^2}\right)^{l_1-4} \ldots \left(\frac{s^{n-1}}{s^n}\right)^{l_{n-1}-4} (s^n)^{l_n-4}, \qquad \text{const} > 0, \tag{2.40}$$

where the list of numbers $(l_1, \ldots, l_n)$ identifies the location of w in the intersection of monodromy filtrations:

$$w \in W_{l_1}(N_1) \cap W_{l_2}(N_1 + N_2) \cap \ldots \cap W_{l_n}(N_1 + N_2 + \ldots + N_n). \tag{2.41}$$

Note that the axions contribution to the vectors norm is subdominant with respect to the saxion contribution and it is included in the constant term of (2.40).

We will exploit this theorem to compute the kinetic metrics and the flux potentials for the three-dimensional spacetime action in section 4.

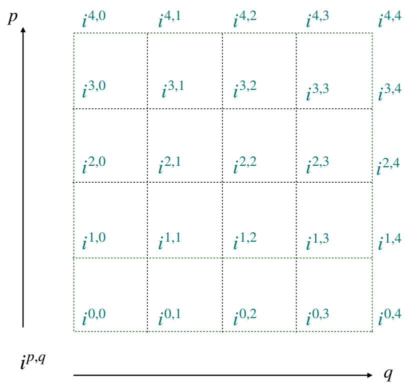

Figure 2. Hodge-Deligne diamond encoding the complex dimensions of the Deligne spaces $\dim_{\mathbb{C}} I^{p,q} = i^{p,q}$.

2.4 Classification of singularities

In this section we make use of the mathematical tools explained above in order to classify the singularities that can occur in the complex structure moduli space $\mathcal{M}_{cs}$ of Calabi-Yau 4-folds compactifications. This implies a classification of all the possible Deligne splittings (2.24) associated with the points of . For the moment we consider to approach the discriminant locus from the bulk of the moduli space towards one of its singular divisors $\;_k$ in a region very far away from any higher intersection.

In order to better visualize the discussion we associate to each splitting $\{I^{p,q}\}$ a lattice representing the associated Hodge-Deligne diamond, as in figure 2, where each lattice point (p, q) is labeled with the dimension $i^{p,q}$ of the corresponding space $I^{p,q}$.

These complex dimensions satisfy the following properties:

$$(i) \quad i^{p,q} = i^{q,p}; \qquad\qquad (iii) \quad i^{p,q} \leq i^{p+1,q+1}, \quad \text{for } p+q \leq 2$$

$$(ii) \quad i^{p,q} = i^{4-q,4-p}; \qquad\qquad (iv) \quad \sum_{q=0}^{4} i^{p,q} = h^{p,4-p}.$$

For Calabi-Yau 4-folds we have a unique holomorphic four form , then $h^{4,0} = h^{0,4} = 1$. Moreover the property *(iv)* tells us that we have only five possibilities to spread this value in the new Hodge-Deligne decomposition:

$$1 = h^{4,0} = i^{4,d} \qquad \text{with} \qquad d = 0, 1, 2, 3, 4. \tag{2.42}$$

According to [49], these possibilities fix the Roman label of five large classes of singularities:

$$\begin{array}{cccccccc} d & = & 0, & 1, & 2, & 3, & 4 \\ & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow \\ & & I & II & III & IV & V \end{array}$$

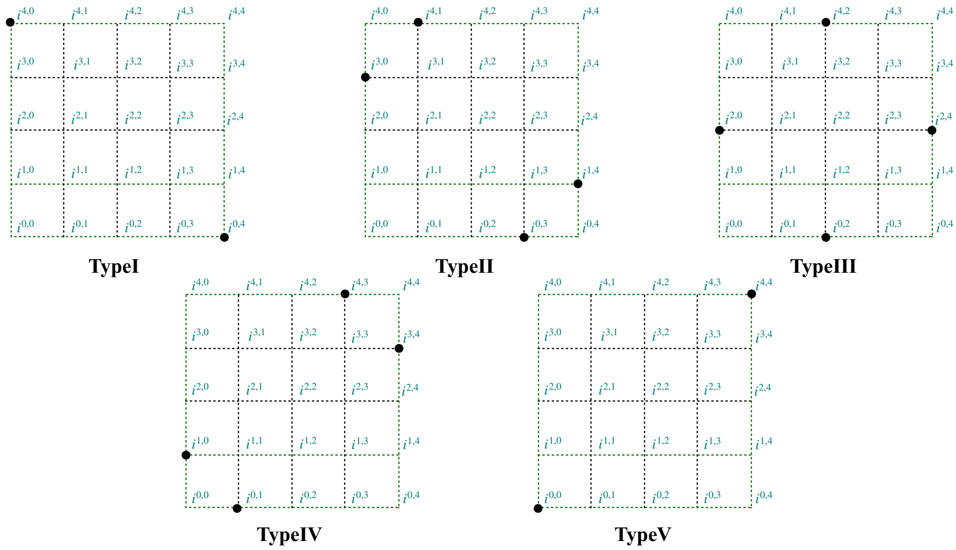

Figure 3. Basic lattices classifying the five large classes of singularities for Calabi-Yau 4-folds.

Let us now introduce the notation to indicate the dimensions of the Deligne spaces with the number of dots on the corresponding lattice points. Using the properties *(i)-(ii)* we can associate to each Roman class a basic lattice over which we can build the remaining unspecified components of the splitting. These basic lattices are summarized in figure 3. Note that the Roman label of the class uniquely determines the dots on the external perimeter of the diamond. In order to fully specify the type of singularity, we still need to fix the number of dots on the internal square.

Before proceeding, we need to specify the class of Calabi-Yau 4-folds manifold we are dealing with. In particular we have to fix the dimension $h^{1,3} = h^{3,1}$ of the groups $H^{1,3}$ and $H^{3,1}$. Such information specifies the dimension of $\mathcal{M}_{cs}$ and, with the property *(iv)*, it tells us that the total number of dots along the columns $(1,q),(3,q)$ and the rows $(p,1),(p,3)$ has to be $h^{1,3}$.

To make the description more explicit, we will focus on the case $h^{1,3} = 2$, but the same procedure can be repeated analogously for other classes of Calabi-Yau manifolds.

Let us also fix $h^{2,2} = \widehat{m}$, so that the sum of dots along the central column and row in the diamonds must be $\widehat{m}$.

Starting with the five models depicted in figure 3, and using the properties *(i)–(iv)*, we can construct all the Hodge-Deligne diamonds representing all the possible singularities [50]. To distinguish between all of these possibilities we add two subindices (a, a') to each Roman number: the first index indicates the dimension $i^{3,3}$ and the second one the sum $i^{3,3} + i^{3,2}$. All allowed diamonds are summarized in the figure 4. In the next sections we will not consider type $I_{a,a'}$ singularities because they are located at finite distance in the moduli space as explained in section 4.1.

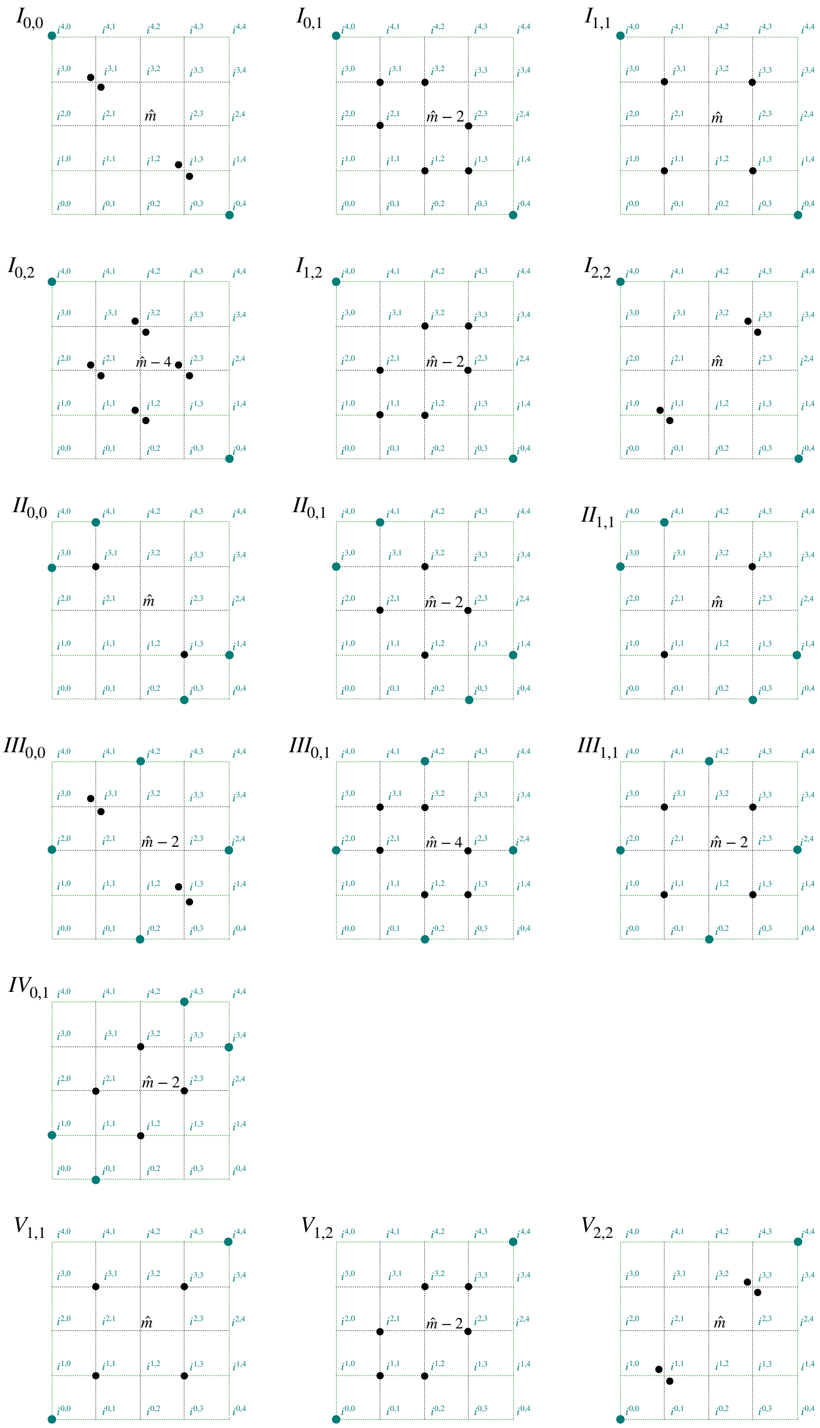

Figure 4. Hodge-Deligne diamonds associated with all the possible singularities of the $h^{3,1} = 2$ dimensional complex structure moduli space of Calabi-Yau 4-folds with $h^{2,2} = m$.

2.5 Allowed enhancements

In the previous section we classified all the allowed simple singularities that occur in $\mathcal{M}_{cs}$ approaching a singular divisor $\;_k$. Now we want to consider enhanced singularities which occur when two singular divisors intersect. In the case $h^{1,3} = 2$ the intersection loci $\ _{k_1 k_2} = \ _{k_1} \cap \ _{k_2}$ are 0-dimensional loci (points).

As explained in section 2.3 we can reach the locus $\ _{k_1 k_2}$ following paths in $\mathcal{M}_{cs}$ belonging to different growth sectors. From now on we will focus our attention on the sectors R_{12}, but the same procedure can be implemented also for the sectors R_{21}. Starting from a generic point in the bulk of the moduli space, we send the first modulus $s^1 \mapsto \infty$ to get the singularity $\mathit{Type}_A(\ _1)$. Then we send also the second modulus $s^2 \mapsto \infty$ to reach the intersection $\ _{12}$ with the divisor $\ _2$. The singularity occurring at this point must be one of those classified in the previous section. Let us call such singularity $\mathit{Type}_B(\ _{12})$ and indicate the enhancement through the sector R_{12} with the following notation:

$$\mathit{Type}_A(\ _1) \quad \longrightarrow \quad \mathit{Type}_B(\ _{12})\,. \tag{2.43}$$

Roughly speaking we have to understand which possible Deligne-splitting $\{I^{p,q}(\ _{12})\}$ can enhance from the Deligne splitting $\{I^{p,q}(\ _1)\}$. The fundamental mathematical object containing this information is the primitive part of $\{I^{p,q}(\ _{k_1})\}$, which can be written as:

$$P^{p,q} = I^{p,q} \cap \ker N_1^{p+q-3}. \tag{2.44}$$

Using this definition and the equation (2.29), we make explicit in figure 5 which spaces $I^{p,q}$ could contain a non-trivial primitive part. These primitive parts can be grouped in the following vector spaces:

$$\begin{aligned}
P^8 &= P^{4,4} \\
P^7 &= P^{3,4} \oplus P^{4,3} \\
P^6 &= P^{2,4} \oplus P^{3,3} \oplus P^{4,2} \\
P^5 &= P^{1,4} \oplus P^{2,3} \oplus P^{3,2} \oplus P^{4,1} \\
P^4 &= P^{0,4} \oplus P^{1,3} \oplus P^{2,2} \oplus P^{3,1} \oplus P^{4,0}
\end{aligned} \tag{2.45}$$

each of which can be used to construct a *pure Hodge structure* of respective weight $8, 7, 6, 5, 4$, exactly as H^4_p admits a pure Hodge structure of weight 4. These new Hodge structures will be the basis to construct the next Deligne splitting $\{I^{p,q}(\ _{12})\}$ occurring at the intersection. They will play the role of $H^4_p(Y_4, \mathbb{C})$ for simple singularities.

For each P^j we can construct a Deligne-splitting $\{I^{p,q}(\ _{12})\}^j$, with $0 \leq p+q \leq 2j$, following the standard procedure described in section 2.2. Then, there is a systematic way to rearrange these set of splittings to construct the Deligne-splitting $\{I^{p,q}(\ _{12})\}$ occuring at the enhancement. This systematic way contains in itself the rules to determine whether a particular enhancement is admissible with respect to its capability to correctly transmit the polarization conditions. We refer to [49] for a complete description of the procedure, while in the following we just summarize the result.

Given an Hodge-Deligne diamond associated to a $j-$weight Hodge structure polarized by $S(\cdot, N\cdot)$, it is possible to define an integer-valued function on its corresponding lattice:

$$\diamondsuit \quad : \quad (p,q) \quad \longrightarrow \quad \mathbb{Z} \tag{2.46}$$

such that the following properties are satisfied:

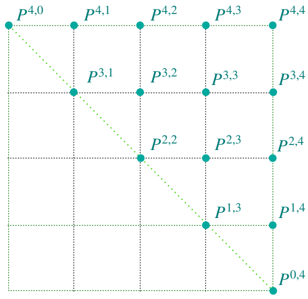

Figure 5. Principal parts of the Deligne splitting for the middle cohomology of CY 4−folds.

(i) $\sum_{q=0}^{j} \diamondsuit(p,q) = h^{p,j-p}$ for all p;

(ii) $\diamondsuit(p,q) = \diamondsuit(q,p) = \diamondsuit(j-q, j-p)$ for all p, q;

(iii) $\diamondsuit(p-1, q-1) \leq \diamondsuit(p,q)$ for $p+q \leq j$.

A simple choice for such a function is $\diamondsuit(p,q) = i^{p,q}$. Using this definition we can easily define new Hodge-Deligne diamonds taking the sum of two diamonds:

$$\diamondsuit(p,q) = \diamondsuit_1(p,q) + \diamondsuit_2(p,q) \tag{2.47}$$

or shifting the entries of one initial diamond:

$$\diamondsuit\,[a]\,(p,q) = \diamondsuit(p+a, q+a). \tag{2.48}$$

Now let $\diamondsuit\,(F_1, N_1)$ be the Hodge-Deligne diamond associated with the singularity $\mathit{Type}_A\left(\ \ _1\right)$ and $\diamondsuit\,(F_2, N_2)$ the one associated with the singularity $\mathit{Type}_B\left(\ \ _{12}\right)$. From $(F_1, W(N_1))$ we can construct the primitive vector spaces $P^j(N_1)$, and we can associate to each of these a $j-$weight pure Hodge structure polarized by $S(\cdot, N_1^j \cdot)$ with Hodge-Deligne diamond $\diamondsuit\left(F_j', N_1^j\right)$.

If the following decomposition is possible:

$$\diamondsuit\,(F_2, N_2) = \sum_{4 \leq j \leq 8} \sum_{0 \leq a \leq j-4} \diamondsuit\left(F_j', N_1^j\right)[a] \tag{2.49}$$

then the enhancement (2.43) is allowed.

In order to classify all the allowed enhancements in the space $\mathcal{M}_{cs}$ we have to check the condition (2.49) for all the enhancements we can construct using the singularities in figure 4. The results are summarized in table 1, in agreement with table (5.2) of [50].

Since we are considering flux compactifications and we need to compute the leading behavior of the potential (2.4) near each possible enhancement using the theorem (2.40), we have to know all the possible location of G_4 in the intersection of the monodromy filtrations:

$$G_4 \in W_{l_1}(N_1) \cap W_{l_2}(N_1 + N_2)\,. \tag{2.50}$$

This is equivalent to classify all the possible doublets (l_1, l_2) that can contain some component of G_4. The table 1 summarizes the results for all available enhancements which corresponding expression of the flux potential will be explicitly studied in section 4.2. As mentioned before, we do not consider enhancements from type $I_{a,a'}$ singularities because they are located at finite distance in the moduli space.

Enhancement			$(l_1, l_2) \in \mathcal{E}$
$II_{0,0}$	$\longrightarrow$	$II_{0,1}$	$(3,3), (4,3), (4,4), (4,5), (5,5)$
		$II_{1,1}$	$(3,3), (4,2), (4,4), (4,6), (5,5)$
$II_{0,1}$	$\longrightarrow$	$II_{1,1}$	$(3,2), (3,3), (3,4), (4,4), (5,4), (5,5), (5,6)$
		$III_{0,0}$	$(3,2), (3,4), (4,4), (5,4), (5,6)$
		$V_{2,2}$	$(3,0), (3,2), (3,4), (3,6), (4,4), (5,2), (5,4), (5,6), (5,8)$
$III_{0,0}$	$\longrightarrow$	$III_{0,1}$	$(2,2), (4,3), (4,4), (4,5), (6,6)$
		$III_{1,1}$	$(2,2), (4,2), (4,4), (4,6), (6,6)$
$III_{0,1}$	$\longrightarrow$	$III_{1,1}$	$(2,2), (3,2), (3,4), (4,4), (5,4), (5,6), (6,6)$
$III_{1,1}$	$\longrightarrow$	$V_{2,2}$	$(2,0), (2,2), (4,2), (2,4), (4,4), (6,4), (4,6), (6,6), (6,8)$
$IV_{0,1}$	$\longrightarrow$	$V_{2,2}$	$(1,0), (1,2), (3,2), (3,4), (4,4), (5,4), (5,6), (7,6), (7,8)$
$V_{1,1}$	$\longrightarrow$	$V_{1,2}$	$(0,0), (2,2), (4,3), (4,4), (4,5), (6,6), (8,8)$
		$V_{2,2}$	$(0,0), (2,2), (4,2), (4,4), (4,6), (6,6), (8,8)$
$V_{1,2}$	$\longrightarrow$	$V_{2,2}$	$(0,0), (2,2), (3,2), (3,4), (4,4), (5,4), (5,6), (6,6), (8,8)$

Table 1. Allowed Enhancements for Calabi-Yau fourfolds with $h^{1,3} = 2$.

3 Dynamical Cobordisms

Dynamical Cobordism techniques [19, 20, 23–29] (see [30–33]for early related works and [34–41] for more recent developments, and [42, 43] for holographic applications) represent a powerful method to explore large field regimes through dynamical solutions of generic spacetime actions. Moreover, in the context of the Swampland program, they are able to provide effective realizations of the configurations cobordant to nothing predicted at the topological level by the Cobordism Conjecture [45].

3.1 Codimension-1 ETW branes

In this first section we provide a quick review of the method, that generalizes the construction in [23] including also time-dependent solutions as discussed in [24] (see also [32, 34, 35, 38] for early works on time-dependent solutions) in the simplest case of codimension-1 ETW branes.

Consider the $d-$dimensional action in $M_P = 1$ units:

$$S = \int d^d x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} (\partial \phi)^2 - V(\phi) \right] \tag{3.1}$$

containing Einstein-gravity coupled to a real scalar with arbitrary potential.

We consider solutions for the equations of motion associated to this action that run along one spacetime coordinate y according with the ansatz:

$$\begin{aligned} ds_d^2 &= e^{-2\sigma(y)} ds_{d-1}^2 \pm dy^2 \\ \phi &= \phi(y), \end{aligned} \tag{3.2}$$

where the sign $\pm$ encompasses the possibilities to consider y as a spacelike or a timelike coordinate. The condition for such a solution to realize a dynamical cobordism to nothing is the presence of a metric singularity at finite distance in spacetime corresponding to a divergent regime for the scalar. Imposing these requirements within the equations of motion, we obtain a very simple class of solutions with the following behavior for the fields:

$$\begin{aligned} \phi(y) &\simeq -\frac{2}{\delta}\log y \\ \sigma(y) &\simeq -\frac{4}{\delta^2(d-2)}\log y. \end{aligned} \tag{3.3}$$

The real number δ parameterizes the class of solutions and controls the growing of the scalar potential:

$$V(\phi) = \mp a c e^{\delta\phi}, \tag{3.4}$$

where c is a free parameter, a is related to δ by:

$$\delta = 2\sqrt{\frac{d-1}{d-2}(1-a)} \tag{3.5}$$

and the overall sign depends on whether y is a spacelike or timelike coordinate. The equations of motion also impose the condition $a < 1$ for this family, which imposes some bounds on δ when we choose a spacelike or a timelike running coordinate. In particular, if $V > 0$ we have that, for $a < 0$ (hence $\delta > 2\sqrt{\frac{d-1}{d-2}}$) the coordinate y is spacelike, and for $0 < a < 1$ (hence $\delta < 2\sqrt{\frac{d-1}{d-2}}$) the coordinate y is timelike; while if $V < 0$, for $a < 0$ (hence $\delta > 2\sqrt{\frac{d-1}{d-2}}$) the coordinate y is timelike, and for $0 < a < 1$ (hence $\delta < 2\sqrt{\frac{d-1}{d-2}}$) the coordinate y is spacelike.

Another interesting feature of these solutions is the existence of universal scaling relations linking the spacetime and the field space distances and $\mathcal{D}$ with the spacetime scalar curvature $|R|$ in the following way:

$$\sim e^{-\frac{\delta}{2}\mathcal{D}}, \qquad |R| \sim e^{\delta\mathcal{D}}. \tag{3.6}$$

Such relations, controlled by the *critical exponent* δ, encode the defining properties for the realization of a dynamical cobordism in the spacetime solutions.

3.2 Intersecting ETW branes

In this section we overview the local description of intersecting ETW branes configuration done in [29]. These solutions, given by a mere superposition of branes, represent the simplest building block to probe multiple infinite distance limits in the field space. However, as we will discuss in section 4.2, these solutions are not adequate to probe infinite distance limits involving multiple scalars in the Calabi-Yau moduli space, but it will be necessary a generalization of the ansatz as described in section 3.4.

We consider the following $(n+2)-$dimensional action containing Einstein gravity coupled to two real scalar fields constrained by the general potential $V(\phi_1, \phi_2)$:

$$S = \int d^{n+2}x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}(\partial\phi_1)^2 - \frac{1}{2}(\partial\phi_2)^2 - \frac{\alpha}{2}\partial_\rho\phi_1\partial^\rho\phi_2 - V(\phi_1,\phi_2)\right\}. \tag{3.7}$$

Note that the presence of a mixed term in the kinetic sector of the action is essential to solve the equations of motion for the class of solutions considered in [29].

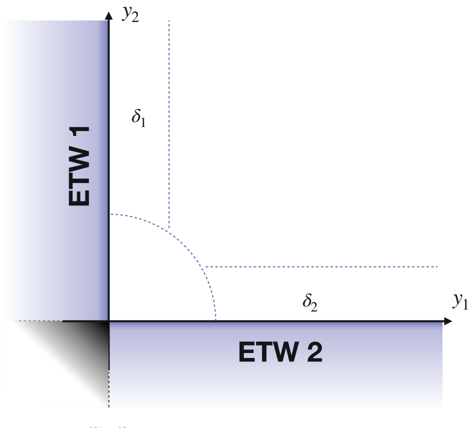

Figure 6. Spacetime configuration representing two intersecting ETW branes of type δ_1 and δ_2, as figure 2 of [29].

We consider the following structure for the spacetime metric:

$$ds^2_{n+2} = e^{-2\sigma_1 - 2\sigma_2} ds^2_n + e^{-2\sigma_2} dy_1^2 \pm e^{-2\sigma_1} dy_2^2 \,, \tag{3.8}$$

which, by setting only one of the two running coordinates y_i to a constant, nicely reduces to a local codimension-1 ETW metric up to a constant. This is equivalent to restricting our investigation to an $(n+1)$ dimensional y_i-constant slice in the $(n+2)$dimensional spacetime. The sign $\pm$ in the metric indicates the possibilities to consider the coordinate y_2 as a spacelike or a timelike direction.

Moreover, we require the following coordinate dependence for the scalars:

$$\phi_1 = \phi_1(y_1) \ , \qquad \phi_2 = \phi_2(y_2). \tag{3.9}$$

A picture of the spacetime configuration described in this structure is shown in figure 6, with two codimension-1 ETW branes intersecting at the codimension-2 locus $y_i = 0$.

Imposing the equations of motion, the local description near the codimension-2 locus is given by logarithmic functions:

$$\begin{aligned} \phi_1 &= -\frac{2}{\delta_1} \log y_1 \ , \qquad \phi_2 = -\frac{2}{\delta_2} \log y_2 \\ \sigma_1 &= -\frac{4}{n\delta_1^2} \log y_1 \ , \quad \sigma_2 = -\frac{4}{n\delta_2^2} \log y_2 \end{aligned} \tag{3.10}$$

where the parameters δ_i specify the type of intersecting branes and control the growing of the potential in the quarter circle area of figure 6:

$$V = V_1 + V_2 = -c_1 v_1 \, e^{\delta_1 \phi_1} e^{\frac{4}{n\delta_2}\phi_2} \mp c_2 v_2 \, e^{\frac{4}{n\delta_1}\phi_1} e^{\delta_2 \phi_2} \tag{3.11}$$

with c_i positive integration constants. The sign $\mp$ in the second term of V refers to the cases where y_2 is a spacelike or a timelike coordinate, respectively. The coefficients v_i are related to the parameters δ_i as:

$$\delta_i^2 = \frac{8(n+1)}{n + \sqrt{n\,[n + 8v_i(n+1)]}}. \tag{3.12}$$

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The equations of motion also constrain the coefficient α, controlling the mixed kinetic term in the action, to a specific value:

$$\alpha = \frac{8}{n\delta_1\delta_2}. \tag{3.13}$$

This highlights the fact that the solutions apply to a preferential basis of the field space which metric contain a non-trivial mixed term weighed by the parameter (3.13). As we will explain in section 4.2, this invalidates their naive application in the Calabi-Yau moduli space, where each Calabi-Yau modulus is a scalar field, because in the corresponding infinite distance loci the asymptotic metrics are diagonal.

3.3 Intersecting configurations for decoupled scalar fields

Since the Calabi-Yau moduli space metrics are diagonal in the asymptotic limit, as we will discuss in detail in section 4.2, we consider the following $(n+2)-$dimensional action for gravity coupled with two decoupled real scalar fields constraint by the general potential $V(\psi_1, \psi_2)$:

$$S = \int d^{n+2}x\sqrt{-g}\left\{\frac{1}{2}R - \frac{1}{2}\left(\partial\psi_1\right)^2 - \frac{1}{2}\left(\partial\psi_2\right)^2 - V(\psi_1,\psi_2)\right\} \tag{3.14}$$

Note that in this new action the dynamics of the scalars is controlled by a diagonal metric for the kinetic term.

To solve the equations of motion, we consider the following ansatz for the fields:

$$ds^2_{n+2} = e^{2A(y_1,y_2)}ds^2_n + e^{2B(y_1,y_2)}dy_1^2 \pm e^{2C(y_1,y_2)}dy_2^2 \tag{3.15}$$

$$\psi_1 = \psi_1(y_1,y_2)\ , \qquad \psi_2 = \psi_2(y_2). \tag{3.16}$$

Note that, in order to find intersecting ETW brane solutions for scalars with diagonal kinetic terms, we now allow one of the scalars to depend on two coordinates (although the equations of motion make this dependence relatively simple). Despite this could seem like a complication, we will actually show that it leads to an interpretation very much in the spirit of enhancements of singularities in CY moduli space in section 4.2. A linearly independent set of combination of equations of motion for these functions is:

$$\begin{aligned}
&(1)\quad \mp e^{-2C}n\left[A'\left(nA'+B'-C'\right)+A''\right] - e^{-2B}\left[A\left(nA-B+C\right)+A\right] = 2V\\
&(2)\quad e^{-2B}\left[-nA^2+(n-2)A\left(C-B\right)+(n-2)A+2C\left(C-B\right)+2C+\left(\psi_1\right)^2\right]+\\
&\qquad \pm e^{-2C}\left[-nA''+(n-2)A'\left(B'-C'\right)+(n-2)A''+2B'\left(B'-C'\right)+2B''+(\psi_1')^2+(\psi_2')^2\right]=0\\
&(3)\quad n\left[AB'-AA'+CB'-A'\right] = \psi_1\psi_1'\\
&(4)\quad e^{-2B}\left[\left(nA-B+C\right)\psi_1+\psi_1\right] \pm e^{-2C}\left[\left(nA'+B'-C'\right)\psi_1'+\psi_1''\right] - \frac{\partial V}{\partial\psi_1} = 0\\
&(5)\quad \pm e^{-2C}\left[\left(nA'+B'-C'\right)\psi_2'+\psi_2''\right] - \frac{\partial V}{\partial\psi_2} = 0,
\end{aligned} \tag{3.17}$$

where we have used the notation $f := \partial_{y_1}f$ and $f' := \partial_{y_2}f$. The upper signs refer to the case where y_2 is a spacelike coordinate and the bottom signs to the case where y_2 is a timelike coordinate.

As in the previous section, we are interested in studying classes of solutions for these equations realizing a dynamical cobordism, with the scalars running to infinity in a finite distance in spacetime. We can conventionally choose that this happens as we approach the origin of the coordinates y_i.

Let us focus our attention in the simple class of solutions featuring an additive structure for the warp factors of the metric:

$$A(y_1, y_2) = -\sigma_1(y_1) - \sigma_2(y_2) \ , \quad B(y_1, y_2) = -\sigma_2(y_2) \ , \quad C(y_1, y_2) = -\sigma_1(y_1) \tag{3.18}$$

and logarithmic profiles for the functions σ_i and ψ_i:

$$\begin{aligned} \sigma_1 &= -a_1 \log y_1 + \frac{1}{2}\log c_1 \ , \quad \sigma_2 = -a_2 \log y_2 + \frac{1}{2}\log c_2 \\ \psi_1 &= b_{11}\log y_1 + b_{12}\log y_2 \ , \quad \psi_2 = b_2 \log y_2 \end{aligned} \tag{3.19}$$

Note that this means that at $y_1 = 0$ the scalar ψ_1 diverges, whereas at $y_2 = 0$ both scalars diverge. This will provide a nice match with the picture of singularity enhancement in CY moduli space of section 2.5, as we explain later on. The functions B and C do not have a respectively y_1 and y_2 dependence because such a dependence can be easily reabsorbed by a change of variables in the metric. The real numbers c_1 and c_2 are integration constants, giving only subleading contributions in the $y_i \to 0$ limits.

Replacing these profiles in (3.17) we get:

$$\begin{aligned} V =& -\frac{1}{2}c_1 n a_1 \left(n a_1 + a_1 - 1\right) y_1^{-2} y_2^{-2a_2} \mp \frac{1}{2} c_2 n a_2 \left(n a_2 + a_2 - 1\right) y_1^{-2a_1} y_2^{-2} = \\ :=& -c_1 v_1 y_1^{-2} y_2^{-2a_2} \mp c_2 v_2 y_1^{-2a_1} y_2^{-2} \\ b_{11}^2 &= n a_1 \\ b_{12}^2 &= n a_1 a_2^2 \\ b_2^2 &= n a_2 \left(1 - a_1 a_2\right) \end{aligned} \tag{3.20}$$

For $a_1, a_2 \neq 1$ the scalar potential splits in two pieces with a different dependence on y_1 and y_2, which means a different dependence on ψ_1 and ψ_2. These terms encode the leading contribution of the function V in the asymptotic regime near the locus $y_i = 0$. The constant coefficients v_i are related to the parameters a_i by:

$$a_i = \frac{1 \pm \sqrt{1 + 8 v_i \left(1 + \frac{1}{n}\right)}}{2(n+1)} \tag{3.21}$$

In order to give to the reader a clear overview on the results, we summarize below the spacetime solutions written in terms of the parameters a_i:

$$\begin{aligned} ds_{n+2}^2 &= y_1^{2a_1} y_2^{2a_2} ds_n^2 + y_2^{2a_2} dy_1^2 \pm y_1^{2a_1} dy_2^2 \\ \psi_1 &= -\sqrt{n a_1}\log y_1 - a_2\sqrt{n a_1}\log y_2 \\ \psi_2 &= -\sqrt{n a_2 (1 - a_1 a_2)}\log y_2 . \end{aligned} \tag{3.22}$$

The potential takes the form:

$$V = -c_1 v_1 e^{\frac{2}{\sqrt{n a_1}}\psi_1} \mp c_2 v_2 e^{2\sqrt{\frac{a_1}{n}}\psi_1} e^{2\sqrt{\frac{1 - a_1 a_2}{n a_2}}\psi_2} . \tag{3.23}$$

Note that the spacetime metric in (3.22) is conformally flat: this is a straightforward generalization of the metric (3.2) of the codimension-1 case.

Actually, although we have constructed the above solutions from scratch from the action (3.14), they can be regarded as a reinterpretation of the solutions in section 3.2. This is because the two solutions are related by a simple redefinition of the fields:

$$\begin{aligned}\phi_1 &= \psi_1 - \left(\frac{a_1 a_2}{1 - a_1 a_2}\right)^{1/2} \psi_2 \\ \phi_2 &= \left(\frac{1}{1 - a_1 a_2}\right)^{1/2} \psi_2\end{aligned} \tag{3.24}$$

for which the spacetime action (3.14) takes the form (3.7) with

$$\alpha = 2\sqrt{a_1 a_2}. \tag{3.25}$$

Respect to these new fields we can read the solutions as the superposition of two codimension 1 ETW branes located at the respective positions $y_1 = 0$ and $y_2 = 0$, as depicted in figure 6, and characterized by the critical exponents δ_i given in (3.12), which relation with the parameters a_i is:

$$\delta_i^2 = \frac{4}{n a_i}. \tag{3.26}$$

Note that when we approach the locus $y_1 = 0$ the new redefined scalar ϕ_1 runs to infinity such as the scalar ψ_1 of the decoupled setup, while when we approach the locus $y_2 = 0$ only the scalar ϕ_2 goes to infinity in the new fields redefinition but both the scalars ψ_1 and ψ_2 diverge at the same time in the decoupled picture. In section 4.2 we will take advantage of this feature to associate to the ETW brane located at the position $y_1 = 0$, where only the scalar ψ_1 diverges, a codimension-1 divisor in the Calabi-Yau moduli space, and to the other ETW brane, the one located at the position $y_2 = 0$ where both the scalars ψ_1 and ψ_2 diverge, not a second divisor intersecting with the first one but the intersection of the two divisors itself, i.e. the enhancement.

Although this is a nice interpretation to solve the problem to have diagonal asymptotic metrics, the structure of the potential that support this class of solutions is not of the same form of the potentials that we get in the Calabi-Yau moduli space near the enhanced singularities, as we will see in section 4.2. In the next section we will introduce a generalization of the above solution, which is similar in spirit in the interpretation of the ETW branes, but it is supported by scalar potentials having the same form of the flux potentials computed using the growth theorem for the Hodge norm applied to the G_4 fluxes supported by the asymptotic Hodge structures allowed at the enhancements and summarized in table 1.

3.4 A new solution beyond the conformal flatness

In this section we present a generalization of the solutions described in the previous section and associated with the same action (3.14). These new solutions are governed by a scalar potential of the right form to match the flux potential supported in the asymptotic regions of the Calabi-Yau moduli space around the enhancements. The simple modification with respect to the ansatz (3.18) that we will consider in this section consists to go beyond the conformal flatness of the metric.

We consider a non-conformally flat ansatz for the spacetime metric (3.15):

$$\begin{aligned} A(y_1,y_2) &= a_1\log y_1 + a_2\log y_2 \\ B(y_1,y_2) &= a_2\log y_2 - \frac{1}{2}\log c_2 \\ C(y_1,y_2) &= (1-a_1 n)\log y_1 - \frac{1}{2}\log c_1 \end{aligned} \tag{3.27}$$

and we keep the same logarithmic profiles for the scalar fields:

$$\psi_1 = b_{11}\log y_1 + b_{12}\log y_2\ , \quad \psi_2 = b_2\log y_2. \tag{3.28}$$

The profile of the scalars is the same as in the previous section, so these solutions also follow the interpretation that one ETW brane corresponds to a singular divisor and the other ETW brane corresponds to the intersection of divisors, i.e. to the enhancement of the singularity.

Replacing these functions in the equations of motion (3.17) we obtain the following constraints for the parameters:

$$\begin{aligned} b_{11} &= -\sqrt{a_1 n(2-a_1-a_1 n)} \\ b_{12} &= -a_2(1-a_1 n)\sqrt{\frac{n}{a_1(2-a_1-a_1 n)}} \\ b_2 &= -\sqrt{a_2 n\left[1+\frac{a_2(a_1 n-1)^2}{a_1(-2+a_1+a_1 n)}\right]} \end{aligned} \tag{3.29}$$

and the following shape for the potential:

$$V = \mp\frac{c_2}{2}a_2 n\,(a_2 n + a_2 - 1)\,y_1^{-2+2a_1 n}y_2^{-2} = \mp c_2 v_2 y_1^{-2+2a_1 n}y_2^{-2}. \tag{3.30}$$

As in the previous section, the upper signs refer to the case where y_2 is a spacelike coordinate and the bottom ones to the case where y_2 represents a timelike direction. Note that if y_2 is timelike, and $a_2 > \frac{1}{1+n}$, the potential is positive. For positive a_i the relations (3.29) are well defined only if $a_1 < \frac{2}{1+n}$.

For a clearer overall picture of the result, let us summarize the solutions by making the dependence on the spacetime coordinates explicit:

$$\begin{aligned} ds^2_{n+2} &= y_1^{2a_1}y_2^{2a_2}ds_n^2 + y_2^{2a_2}dy_1^2 - y_1^{2(1-a_1 n)}dy_2^2 \\ \psi_1 &= -\sqrt{a_1 n(2-a_1-a_1 n)}\log y_1 - a_2(1-a_1 n)\sqrt{\frac{n}{a_1(2-a_1-a_1 n)}}\log y_2 \\ \psi_2 &= -\sqrt{a_2 n\left[1+\frac{a_2(a_1 n-1)^2}{a_1(-2+a_1+a_1 n)}\right]}\log y_2. \end{aligned} \tag{3.31}$$

The potential written in terms of these fields takes the form:

$$V = \mp c_2 v_2 e^{\frac{2-2a_1 n}{\sqrt{a_1 n(2-a_1-a_1 n)}}\psi_1}e^{2\sqrt{\frac{a_2+a_1(-2+a_1+a_1 n-2a_2 n+a_1 a_2 n^2)}{a_1 a_2 n(-2+a_1+a_1 n)}}\psi_2}. \tag{3.32}$$

Note that with respect to the class of solutions treated in section 3.3, here we have only a single term in the potential driving the dynamics of the fields near the locus $y_i = 0$. This feature makes these solutions appropriate for the application to the infinite distance network of the Calabi-Yau moduli space. In fact, the structure (3.32) of the potential is precisely the one that matches the flux potentials obtained from the Hodge norm of G_4 for all the admissible enhancements in the space $\mathcal{M}_{cs}$ with $h^{3,1} = 2$.

In order to make more manifest the interpretation of this solution to describe the intersection of two codimension-1 ETW branes of the kind in section 3.1, we make use of the following redefinition of the fields:

$$\begin{aligned} \phi_1 &= \psi_1 - (1 - a_1 n)\sqrt{\frac{a_2}{a_1(2 - a_1 - a_1 n) - a_2(a_1 n - 1)^2}}\psi_2 \\ \phi_2 &= \left(1 + \frac{a_2(a_1 n - 1)^2}{a_1(-2 + a_1 + a_1 n)}\right)^{-1/2} \psi_2 \end{aligned} \tag{3.33}$$

to write the spacetime action (3.14) in the form (3.7), with

$$\alpha = 2(1 - a_1 n)\sqrt{\frac{a_2}{a_1(2 - a_1 - a_1 n)}} \tag{3.34}$$

and

$$V = \mp c_2 v_2 e^{\frac{2-2a_1 n}{\sqrt{a_1 n(2-a_1-a_1 n)}}\phi_1} e^{\frac{2}{\sqrt{a_2 n}}\phi_2}, \tag{3.35}$$

Importantly, one has to remember that the fields ϕ_1 and ϕ_2 should not be regarded as independent Calabi-Yau moduli, but rather that ϕ_1 is a Calabi-Yau modulus and ϕ_2 is a combination of ϕ_1 and another orthogonal modulus. In this interpretation, the non-trivial mixed term (3.34) emphasizes the fact that ϕ_2 is not orthogonal to ϕ_1.

With respect to these new fields we can read the solutions as the intersection of two codimension 1 ETW branes located at the respective positions $y_1 = 0$ and $y_2 = 0$. And we characterize these branes using the critical exponents:

$$\begin{aligned} \delta_1 &= \frac{2 - 2a_1 n}{\sqrt{a_1 n(2 - a_1 - a_1 n)}} \\ \delta_2 &= \frac{2}{\sqrt{a_2 n}} \end{aligned} \tag{3.36}$$

controlling the growing of the potential as we approach the codimension-2 intersection between the two codimension-1 branes.

As anticipated, the new key feature of these solutions is that they go beyond the conformally flat ansatz of sections 3.2 and 3.3. Indeed, using a convenient redefinition of the spacetime coordinates we can write the metric (3.31) in the following form:

$$\begin{aligned} ds^2_{n+2} &= y_1^{2(1-a_1 n)} y_2^{2a_2}\left[y_1^{4a_1 n-2} ds_n^2 + y_1^{-2(1-a_1 n)} dy_1^2 - y_2^{-2a_2} dy_2^2\right] = \\ &\propto y_1^{2\frac{(1-a_1 n)}{a_1 n}} y_2^{\frac{2a_2}{1-a_2 n}}\left[y_1^{4-\frac{2}{a_1 n}} ds_n^2 + dy_1^2 - dy_2^2\right], \end{aligned} \tag{3.37}$$

which is not conformally flat due to the presence of the extra factor in front of the n dimensional metric ds_n^2.

The conformally flat solutions in section 3.2 were shown in [29] to correspond to the backreaction of the superposition of two codimension-1 source terms, with no codimension-2 sources. In our solutions the extra factor beyond the conformally flat ansatz suggests the presence of an additional codimension-2 source localized at the intersection between the two ETW branes.. It would be interesting to explore this further.

3.4.1 Scaling relations

As we point out above, one interesting feature of the Dynamical Cobordism solutions is the existence of specific scaling relations linking the spacetime and the field space distances and $\mathcal{D}$ with the spacetime scalar curvature $|R|$. Such relations are summarized in (3.6) for the case of codimension-1 singularities and they are controlled by the critical exponent δ characterizing the type of ETW brane dressing the singularity. In [26] these scaling relations were extended to the case when we approach the codimension-2 singularity located at the intersection between two distinct codimension-1 ETW branes. In these setups, the parameter controlling the scaling is a path-dependent combination of the critical exponents associated with the two individual intersecting branes. In this section we discuss the same relations for the new class of non-conformally flat solutions.

Consider the following parameterization for the spacetime paths in the region $y_i < 1$ that approach the intersecting locus $y_i = 0$ as $t \mapsto 0$:

$$(\gamma_1, \gamma_2): \quad \begin{cases} y_1 = t^{\gamma_1} \\ y_2 = t^{\gamma_2} \end{cases}, \tag{3.38}$$

where γ_i are two positive real numbers.

The spacetime distance to the origin for the path identified by the pair of numbers (γ_1, γ_2), is:

$$= \int_\Gamma \left| y_2^{2a_2} \left(\partial_t y_1\right)^2 - y_1^{2(1-a_1 n)} \left(\partial_t y_2\right)^2 \right|^{1/2} dt = \int_\Gamma \left| \gamma_1^2 t^{2r_1} - \gamma_2^2 t^{2r_2} \right|^{1/2}, \tag{3.39}$$

with:

$$\begin{aligned} r_1 &= \gamma_1 - 1 + a_2 \gamma_2 \\ r_2 &= \gamma_2 - 1 + (1 - a_1 n)\gamma_1. \end{aligned} \tag{3.40}$$

The two contributions to the integral are comparable in the case $r_1 = r_2$:

$$\frac{\gamma_1}{\gamma_2} = \frac{1 - a_2}{a_1 n}, \tag{3.41}$$

at which the distance is minimized. In all the other cases one of the two terms dominates over the other and the spacetime distance behaves as:

$$= \int_\Gamma \gamma_i^2 t^{r_i} dt = \frac{\gamma_i}{r_i + 1} t^{r_i + 1}, \tag{3.42}$$

with $i = 1$ for the paths above the tangent line to the path (3.38) at $r_1 = r_2$, and $i = 2$ for the paths below that line.

Since we know the profiles of the scalars $\phi_i(y_i)$ in terms of the spacetime coordinates, we can translate each spacetime path in (3.38) into a path in the field space $\phi_i\left(y(t)\right)$ and compute the corresponding distance using the kinetic metric appearing in the spacetime action:

$$\mathcal{D} = \int \left|d\phi_1^2 + d\phi_2^2 + \alpha d\phi_1 d\phi_2\right|^{1/2} \tag{3.43}$$

with α given in (3.34). For each of the two regimes in (3.42) we can write the distance $\mathcal{D}$ in the field space in terms of the distance in the spacetime:

$$\mathcal{D} \sim \frac{\sqrt{n\left[a_1\left(2-a_1-a_1n\right)\gamma_1^2 + 2a_2(1-a_1n)\gamma_1\gamma_2 + a_2\gamma_2^2\right]}}{r_i+1} \log \quad . \tag{3.44}$$

Using the parameterization $\gamma_1 = \sqrt{\gamma}$ and $\gamma_2 = 1/\sqrt{\gamma}$ such that the ratio $\gamma_1/\gamma_2 = \gamma$ and the value of γ separating the two regimes is $\gamma^* = (1-a_2)/a_1 n$ we obtain the scaling relation:

$$\sim e^{-\frac{1}{2}\delta_{\text{int}}\mathcal{D}} \tag{3.45}$$

with

$$\delta_{\text{int}} = \begin{cases} \dfrac{2\left(\frac{a_2}{\sqrt{\gamma}}+\sqrt{\gamma}\right)}{\sqrt{n\left[\frac{a_2}{\gamma}+a_1(2-a_1-a_1n)\gamma+2a_2(1-a_1n)\right]}} & \text{for} \quad \gamma > \gamma^*, \\ \dfrac{2\left(\frac{1}{\sqrt{\gamma}}+\sqrt{\gamma}(1-a_1n)\right)}{\sqrt{n\left[\frac{a_2}{\gamma}+a_1(2-a_1-a_1n)\gamma+2a_2(1-a_1n)\right]}} & \text{for} \quad \gamma < \gamma^*. \end{cases} \tag{3.46}$$

This result is different with respect to the one discussed in section 3.3 of [29] due to the modified relation between the parameter a_1 and the critical exponent δ_1 in (3.36). Nevertheless, the generalization to non-conformally flat metrics preserves the fact that one still gets nice scaling relations encoding the fundamental properties defining the realization of a dynamical cobordism in the spacetime.

Notice that in the limits $\gamma \mapsto 0$ and $\gamma \mapsto \infty$ we recover respectively the critical exponent δ_2 and the combination $\delta_1/(1-a_1n)$ controlling the asymptotic behavior of the fields (3.33).

4 ETW networks for infinite distance limits in CY moduli space

In this section we are finally ready to explore the network of singular divisors located at infinite distance in the complex structure sector of the Calabi-Yau moduli space using ETW brane solutions in spacetime.

The key tool in this dictionary between infinite distance limits in moduli space and cobordisms to nothing in spacetime is the translation, for each specific infinite distance limit, of the information encoded in the Hodge-Deligne structure and in the flux with that encoded in the critical exponents for ETW branes. As already anticipated, each ETW brane explores a singular divisor, while intersecting configurations explore intersections of divisors in a different way with respect to the naive expectation.

4.1 ETW branes for simple singularities

In this section we exploit the classification of singularities that can occur in a two-moduli family of Calabi-Yau fourfolds with primitive Hodge number $m \geq 4$, done in [50] and reviewed in section 2.4, to write for each of these boundaries an effective spacetime action and to build codimension-1 Dynamical Cobordism solutions, characterized by their appropriate critical exponent, which explore the infinite distance limit for the corresponding divergent scalar.

4.1.1 Generalities

We consider the three-dimensional effective action obtained compactifying M-theory on the above mentioned Calabi-Yau fourfold. The resulting theory is completely specified by the Kähler potential, which determines the Weil-Petersson metric G_{WP}, and the three-dimensional scalar potential induced by turning on four-form fluxes G_4 on certain four-cycles of the internal space.

In the regime we are interested, namely the asymptotic regime near the singular divisor $_1$ and very far away from any intersection, we can choose the local set of coordinate in $\mathcal{M}_{cs}$ introduced above (2.16), such that the putative divisor is parameterized by the condition $z_1 = 0$. The leading term of the Kähler potential in the strict asymptotic regime near the divisor $_1$ is computed applying the growth theorem (2.40) to the holomorphic form :

$$\mathcal{K}^{cs}_{Sl(2)} \simeq -\log\left[(s_1)^{d_1} f(\xi)\right], \tag{4.1}$$

where d_1 identifies the location of the form in the monodromy filtration $W(N_1)$. Note that d_1 is exactly the number used in section 2.4 to classify the five classes of singularities.

Using this result we can compute the following expansion for the Weil-Petersson metric:

$$G_{s_1 s_1} = \partial_{t_1}\partial_{\bar{t}_1}\mathcal{K}^{cs} = \frac{1}{4}\frac{d_1}{(s_1)^2} + \frac{\sharp}{(s_1)^3} + \ldots + \mathcal{O}\left(e^{2\pi i t_1}\right) \tag{4.2}$$

and keep only the leading contribution. Note that the constant coefficient of the leading term is completely determined by the integer d_1 which contains the information about the specific type of singularity. Applying this computation to each infinite distance singularity in $\mathcal{M}_{cs}$, we can obtain, as done in [50], the following few possibilities to construct the kinetic sector of the spacetime action:

Singularity	d_1	Kähler potential	$G_{s_1 s_1}$
II	1	$K^{cs} \sim -\log\left[(s_1)^1 f(\xi)\right]$	$\sim \frac{1}{4}\frac{1}{(s_1)^2}$
III	2	$K^{cs} \sim -\log\left[(s_1)^2 f(\xi)\right]$	$\sim \frac{1}{4}\frac{2}{(s_1)^2}$
IV	3	$K^{cs} \sim -\log\left[(s_1)^3 f(\xi)\right]$	$\sim \frac{1}{4}\frac{3}{(s_1)^2}$
V	4	$K^{cs} \sim -\log\left[(s_1)^4 f(\xi)\right]$	$\sim \frac{1}{4}\frac{4}{(s_1)^2}$

The introduction of a primitive G_4 four-form flux in some internal cycle of the compactification generates an effective potential (2.4) for the three dimensional action in the Einstein frame. According to section 2.3, we have that in the strict asymptotic regimes of the moduli space the middle cohomology split in eigenspaces of the $sl(2)$ operators Y_i, (2.37), and the G_4 flux decomposes in this splitting as:

$$G_4 = \sum_{\mathbf{l}\in\mathcal{E}} G_4^{\mathbf{l}} \tag{4.3}$$

This sum tells us that approaching the putative singular locus, G_4 spreads in the corresponding Hodge-Deligne diamond admitting components only in the admissible sectors $V_{\mathbf{l}}$ of $H_p^4(Y_4, \mathbb{C})$.

All this information allows to use the growth theorem for the Hodge norm (2.40) to extract the leading complex structure dependence of the potential V_M via:

$$V_M = \frac{1}{\mathcal{V}_4^3}||G_4||^2 \sim \frac{1}{\mathcal{V}_4^3}||G_4||^2_{sl(2)} = \sum_{l_1\in\mathcal{E}} (s_1)^{l_1-4}\, ||\rho_{l_1}(G_4, a)||^2 \tag{4.4}$$

in the asymptotic regime near the singular divisor $_1$ parameterized by the divergent coordinate $s_1 \mapsto \infty$. The positive coefficient $||\rho_{l_1}(G_4, a)||^2$ is constraint by the flux quantization conditions and it carries within it the dependence on the real part of the coordinate t_1 as in (2.17).

Since in this section we are considering asymptotic regimes near to a specific singular divisor $_1$ but very far away from any higher intersection, in the previous formula we have a single divergent scalar s_1 and one single component l_1 for the vector $\mathbf{l}$. Depending on the value of l_1 we can have power divergent potentials, whenever $4 < l_1 \leq 8$, or power vanishing potentials, when $0 \leq l_1 < 4$. The leading contribution of the three-dimensional spacetime action for the scalar s_1 is:

$$S = \int d^3x\sqrt{-g}\left[\frac{1}{2}R - \frac{d_1}{4(s_1)^2}\,(\partial s_1)^2 - ||\rho_{l_1}||^2 s_1^{l_1-4}\right]. \tag{4.5}$$

In order to put it in the form (3.1) we redefine the field as:

$$\phi = \sqrt{\frac{d_1}{2}}\log s_1, \tag{4.6}$$

and the action (4.5) becomes:

$$S = \int d^3x\sqrt{-g}\left[\frac{1}{2}R - \frac{1}{2}\,(\partial\phi)^2 - ||\rho_{l_1}||^2 e^{\sqrt{\frac{2}{d_1}}(l_1-4)\phi}\right] \tag{4.7}$$

For our analysis, we consider in the above action only power divergent potentials, with $4 < l_1 \leq 8$. This is because for decreasing potentials the Dynamical Cobordism solutions are all equivalent to the case with no potential, and the value of δ is fixed to a specific number for all the different types of singularities. This excludes the possibility to distinguish between the different cases.

As shown in [23] and revisited in section 3.1, the equations of motion associated with this action admit a special class of solutions (3.3) realizing a codimension-1 Dynamical Cobordism to nothing. It is remarkable that the well-established setup of M-theory flux compactification leads to solutions of this type requiring a non-trivial time dependence. It would be interesting to explore the possible cosmological implications of this result and potential relations with phenomena of nucleation of bubbles of something [44]. The important point is that these solutions provide a way to explore the strict asymptotic regime of the corresponding boundary $\,_1$ in $\mathcal{M}_{cs}$ through its spacetime realizations as an ETW brane configuration. From equation (4.3) we have that the G_4 flux of our compactification can have several components turned on that live in different vector spaces $W_{l_1}(N_1)$. The relevant term in this expansion for our analysis is the one that grows faster than the others in the $s_1 \mapsto \infty$ limit. Considering the label l_1 associates with this leading term we compute the critical exponent identifying the specific solution in the family through the formula:

$$\delta = \sqrt{\frac{2}{d_1}}\,(l_1 - 4)\,. \tag{4.8}$$

Let us emphasize that δ completely determines the behavior of the spacetime fields because it encodes the information about the growth of the potential as we approach to the ETW boundary. From a geometric point of view, such information is completely encoded in the Hodge-Deligne splitting of the middle cohomology near the corresponding singular divisor plus the information about the flux.

4.1.2 Examples

Let us consider the example of the type $II_{0,0}$ singularity in the complex structure moduli space of Calabi-Yau fourfold with $h^{3,1} = 2$. Its Hodge-Deligne diamond is depicted in figure 4 and it encodes the information about how the Hodge structure of $H^4_p(Y_4, \mathbb{C})$ is spread in the Hodge-Deligne splitting as we approach the singularity. We have the following non trivial groups:

$$\begin{aligned}
(i^{0,3}, i^{1,2}, i^{2,1}, i^{3,0}) &= (1, 0, 0, 1)\\
(i^{0,4}, i^{1,3}, i^{2,2}, i^{3,1}, i^{4,0}) &= (0, 1, \widehat{m}, 1, 0)\\
(i^{1,4}, i^{2,3}, i^{3,2}, i^{4,1}) &= (1, 0, 0, 1)
\end{aligned} \tag{4.9}$$

which implies three possibilities for the location of G_4 in the splitting: $l_1 = 3, 4, 5$. We neglect the cases $l_1 = 4, 3$ because, according to (4.7), they produce vanishing potential, while for the case $l_k = 5$ the equation (4.8) gives:

$$\text{Type } II_{0,0} \qquad \text{with } l_k = 5 \qquad \longrightarrow \qquad \delta = \sqrt{2}. \tag{4.10}$$

Following the same procedure for all the singularities classified in section 2.4, with a G_4 flux specified by its location l_1 in the splitting, we are able to associate to each of these possibilities a specific spacetime solution characterized by the corresponding critical exponent. The results are summarized in table 2. Notice that all the critical exponents classified in the table stay inside the upper bound $\delta \leq 2\sqrt{\frac{d-1}{d-2}}$, that for three spacetime dimensions means $\delta \leq 2\sqrt{2}$. Since we are dealing with positive potentials, this implies that the spacetime solutions of the form (3.3) realizing these infinite distance limits involve a timelike running coordinate. The cases $II_{1,1}$ with a non-trivial flux component in $l_1 = 6$ and $V_{1,1}, V_{1,2}$ and $V_{2,2}$ with $l_1 = 8$ are special because they saturate the bound for the critical exponent: they correspond in regimes where the scalar potential is subleading with respect to the kinetic term, so they behave as in the case of zero potential.

Type	d_k	l_k	δ	Type	d_k	l_k	δ
$II_{0,0}$	1	5	$+\sqrt{2}$	$II_{0,1}$	1	5	$+\sqrt{2}$
$II_{1,1}$	1	5	$+\sqrt{2}$	$II_{1,1}$	1	6	$+2\sqrt{2}$
$III_{0,0}$	2	6	$+2$	$III_{1,1}$	2	6	$+2$
$III_{0,1}$	2	5	$+1$	$III_{0,1}$	2	6	$+2$
$IV_{0,1}$	3	5	$+\sqrt{6}/3$	$IV_{0,1}$	3	7	$+\sqrt{6}$
$V_{1,1}$	4	6	$+\sqrt{2}$	$V_{1,1}$	4	8	$+2\sqrt{2}$
$V_{1,2}$	4	5	$+\sqrt{2}/2$	$V_{1,2}$	4	6	$+\sqrt{2}$
$V_{1,2}$	4	8	$+2\sqrt{2}$	$V_{2,2}$	4	6	$+\sqrt{2}$
$V_{2,2}$	4	8	$+2\sqrt{2}$				

Table 2. Critical exponents characterizing the ETW brane solutions in spacetime for each type of singularity that can occur in the complex structure moduli space of Calabi-Yau fourfold with $h^{1,3} = 2$.

4.2 ETW networks for the enhancements

The main claim of the previous section is the capability to explore any singular divisor of the Calabi-Yau moduli space with an ETW brane solution in spacetime. In this section we propose the use of real codimension-2 intersecting ETW brane solutions to explore the network of intersecting divisors in CY moduli space with flux potential.

As anticipated in section 3.3, the naive expectation is that individual ETW branes explore singular divisors in the moduli space, according to the dictionary of the previous section, and their intersection would explore the intersection between the corresponding divisors. However the actual interpretation of the spacetime solutions turns out to be more subtle, and much closer to the mathematical description of the network of divisors in terms of a structure of singularity enhancements in specific growth sectors, as we now explain.

4.2.1 Generalities

The regime we are interested is the strict asymptotic regime that approaches the codimension-2 singularity $_{12} = {}_1 \cap {}_2$ through the growth sector R_{12}, staying very far away from any higher intersection. As explained in section 2.2, in this region we can fix a local set of coordinates $\left\{t_1, t_2, \xi^{\bar{j}}\right\}$, such that the putative locus $_{12}$ is parameterized by the conditions $s_1 \succ s_2 \mapsto \infty$. Applying the growth theorem (2.40) to the Hodge norm of the holomorphic form , we can extract the leading term of the Kähler potential in the sector R_{12}:

$$\mathcal{K}^{cs}_{Sl(2)} \simeq -\log\left[(s_1)^{d_1}(s_2)^{(d_e-d_1)}f(\xi)\right], \tag{4.11}$$

where d_1 labels the type of singularity associated with the divisor $_1$, according with the classification of section 2.4, and d_e labels the type of singularity occurring at the intersection $_{12}$, according with the allowed enhancements listed in the table 1. Using the equation (4.11), we compute for all the possible enhancements of $\mathcal{M}_{cs}$, listed in table 1, the corresponding asymptotic Weil-Petersson metric via (2.3) and we summarize the results in table 3.

Enhancement	d_1	d_e	*Kähler Potential*	*WP Metric*
$II_{0,0} \mapsto II_{0,1}$	1	1	$\mathcal{K}^{cs} \sim -\log[s_1 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$II_{0,0} \mapsto II_{1,1}$	1	1	$\mathcal{K}^{cs} \sim -\log[s_1 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$II_{0,1} \mapsto II_{1,1}$	1	1	$\mathcal{K}^{cs} \sim -\log[s_1 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$II_{0,1} \mapsto III_{0,0}$	1	2	$\mathcal{K}^{cs} \sim -\log[(s_1)(s_2)f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & \frac{1}{4(s_2)^2} \end{pmatrix}$
$II_{0,1} \mapsto V_{2,2}$	1	4	$\mathcal{K}^{cs} \sim -\log[(s_1)(s_2)^3 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & \frac{3}{4(s_2)^2} \end{pmatrix}$
$III_{1,1} \mapsto V_{2,2}$	2	4	$\mathcal{K}^{cs} \sim -\log[(s_1)^2(s_2)^2 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{2(s_1)^2} & 0 \\ 0 & \frac{1}{2(s_2)^2} \end{pmatrix}$
$III_{0,0} \mapsto III_{0,1}$	2	2	$\mathcal{K}^{cs} \sim -\log[(s_1)^2 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$III_{0,0} \mapsto III_{1,1}$	2	2	$\mathcal{K}^{cs} \sim -\log[(s_1)^2 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$III_{0,1} \mapsto III_{1,1}$	2	2	$\mathcal{K}^{cs} \sim -\log[(s_1)^2 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{4(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$IV_{0,1} \mapsto V_{2,2}$	3	4	$\mathcal{K}^{cs} \sim -\log[(s_1)^3(s_2) f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{3}{4(s_1)^2} & 0 \\ 0 & \frac{1}{4(s_2)^2} \end{pmatrix}$
$V_{1,1} \mapsto V_{2,2}$	4	4	$\mathcal{K}^{cs} \sim -\log[(s_1)^4 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$V_{1,2} \mapsto V_{2,2}$	4	4	$\mathcal{K}^{cs} \sim -\log[(s_1)^4 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$
$V_{1,1} \mapsto V_{1,2}$	4	4	$\mathcal{K}^{cs} \sim -\log[(s_1)^4 f(\xi)]$	$G_{s_is_j} \sim \begin{pmatrix} \frac{1}{(s_1)^2} & 0 \\ 0 & 0 \end{pmatrix}$

Table 3. Leading behavior of the infinite distances in the moduli space from a double intersection singularity.

Note that, except for the enhancements:

$$(i) \quad II_{0,1} \mapsto III_{0,0} \qquad\qquad (ii) \quad II_{0,1} \mapsto V_{2,2}$$
$$(iii) \quad III_{1,1} \mapsto V_{2,2} \qquad\qquad (iv) \quad IV_{0,1} \mapsto V_{2,2}$$

whose corresponding Weil-Petersson metrics show a diagonal form, for the rest of the

Enhancement	*Doublets*	*Potential*
$II_{0,1} \mapsto III_{0,0}$	$(5,6)$, $(5,4)$	$\sim c_1(s_1)(s_2) + c_2\frac{s_1}{s_2}$
$II_{0,1} \mapsto V_{2,2}$	$(5,8)$, $(5,6)$, $(5,4)$, $(5,2)$	$\sim c_1(s_1)(s_2)^3 + c_2(s_1)(s_2) + c_3\frac{s_1}{s_2} + c_4\frac{s_1}{(s_2)^3}$
$III_{1,1} \mapsto V_{2,2}$	$(6,8)$, $(6,6)$, $(6,4)$	$\sim c_1(s_1)^2(s_2)^2 + c_2(s_1)^2 + c_3(s_2)^2$
$IV_{0,1} \mapsto V_{2,2}$	$(7,8)$, $(7,6)$, $(5,6)$, $(5,4)$	$\sim c_1(s_1)^3(s_2) + c_2\frac{(s_1)^3}{(s_2)} + c_3(s_1)(s_2) + c_4\frac{s_1}{s_2}$

Table 4. Divergent potentials associated to the enhancements in $\mathcal{M}_{cs}$ with $h^{3,1} = 2$ equipped with a non-degenerate metric. The parameters c_i are positive constant coefficients constrained by the flux quantization conditions.

enhancements we obtain degenerate metrics. This implies that these cases cannot be studied with the methods we developed in section 3. Computations of the periods beyond the polynomial approximation (see [67] and [68]) that include exponential corrections lead to non-degenerate metrics for all the enhancements of table 3. In [69] it was argued that loci associated with leading order degenerate metrics in the vector multiplet moduli space of type IIA string theory compactified on a Calabi-Yau threefold correspond to EFTs containing a subsector that decouples to gravity at infinite distance. It would be interesting to investigate whether such an upcoming gauge theory allows a deep exploration of the ETW branes associated with these regimes.

As explained in the previous section, turning on a primitive G_4-flux in some internal cycles of the compactification we generate an effective three dimensional potential for the Calabi-Yau moduli according with the definition (2.4). The $sl(2)$ decomposition (2.37) of the middle cohomology near the asymptotic locus $\ _{12}$ allows to write the G_4-flux as in (4.3), where the allowed doublets $\mathbf{l} = (l_1, l_e)$ appearing in the expansion are the one classified in table 1. Using these allowed doublets in the growth theorem for the Hodge norm, the authors of [50] compute the complex structure dependence of the flux potential for all the allowed enhancements in $\mathcal{M}_{cs}$ with $h^{3,1} = 2$ through the formula:

$$V_M = \frac{1}{\mathcal{V}_4^3}||G_4||^2 \sim \frac{1}{\mathcal{V}_4^3}||G_4||^2_{sl(2)} = \sum_{(l_1,l_e)\in\mathcal{E}} (s_1)^{l_1-4}\,(s_2)^{l_e-l_1}\,||\rho_{(l_1,l_e)}(G_4,a)||^2 \tag{4.12}$$

Since Dynamical Cobordism solutions are not able to distinguish decreasing potentials from the case with zero potential, we consider only the doublets (l_1, l_e) leading to divergent flux-potentials controlled by positive powers. In the table 4 we summarize the results only for the four enhancements equipped with a non-degenerate metric.

4.2.2 Spacetime solutions and interpretation

Using the Weil-Petersson metric and the dominant contribution of the scalar potential for all the possible enhancements reviewed in the previous section and all the possible components for the G_4-flux, we can write the leading contribution of the spacetime action for the involved moduli, valid in the growth sector R_{12}, near the intersection locus $\ _{12} = \ _1 \cap \ _2$:

$$S = \int d^3x\sqrt{-g}\left[\frac{1}{2}R - \frac{d_1}{4(s_1)^2}\,(\partial s_1)^2 - \frac{d_e - d_1}{4(s_2)^2}\,(\partial s_2)^2 - ||\rho_{\mathbf{l}}||^2 s_1^{l_1-4} s_2^{l_e-l_1}\right]. \tag{4.13}$$

In order to put the action in the form (3.14), we fix the canonical normalization for the scalars through the following redefinition of the fields:

$$\psi_1 = \sqrt{\frac{d_1}{2}} \log s_1 \ , \qquad \psi_2 = \sqrt{\frac{d_e - d_1}{2}} \log s_2, \tag{4.14}$$

and the action (4.13) takes the form:

$$S = \int d^3x \sqrt{-g} \left[\frac{1}{2} R - \frac{1}{2} (\partial \psi_1)^2 - \frac{1}{2} (\partial \psi_2)^2 - ||\rho_{\mathbf{l}}||^2 e^{(l_1 - 4)\sqrt{\frac{2}{d_1}}\psi_1} e^{(l_e - l_1)\sqrt{\frac{2}{d_e - d_1}}\psi_2} \right] . \tag{4.15}$$

As pointed out above, the diagonal kinetic metric appearing in this spacetime action prevents the naive interpretation of the two scalars as two independent moduli. This is due to the fact that intersecting ETW brane configurations contain a mixed kinetic term in their spacetime action (3.7) which is always non-zero, (3.13), as imposed by the equations of motion. The key idea to solve this drawback is encoded in the redefinition of the fields (3.24) and (3.33) relating the decoupled scalars ψ_i with the scalars ϕ_i satisfying the equations of motion associated to the action (3.7) with a specific non-trivial mixing term as in section 3.2. The solutions presented in section 3.3 for decoupled scalars have the nice property that when we approach the locus $y_1 = 0$ only the scalar ψ_1 diverges, while when we approach the locus $y_2 = 0$ both the scalars ψ_1 and ψ_2 diverge at the same time. On the other hand, the solutions with respect to the fields ϕ_i have a clear interpretation for these loci as two intersecting ETW branes. In terms of Calabi-Yau moduli, the ETW brane located at the position $y_1 = 0$, at which the modulus ψ_1 diverge, is interpreted as the spacetime realization of the divisor $_1$ involved in the intersection $_{12}$. The brane located at position $y_2 = 0$, at which both the Calabi-Yau moduli diverge, is then naturally interpreted as the spacetime realization of the intersection $_{12}$ of the two divisors, i.e. the enhancement. This new interpretation matches very well with the description of the network of divisors provided in section 2 where the asymptotic Hodge theory strongly constrains how to jump from a singular divisor $_1$ to the enhanced singularity at the intersection $_{12}$ through the growth sector R_{12}. Here we have something very similar in the spacetime: the solution describes in the same configuration what happens near the singular divisor $_1$, corresponding to the ETW brane located at $y_1 = 0$, and near its enhanced singularity in $_{12}$, corresponding to the ETW brane located at $y_2 = 0$.

Although the interpretation of the intersecting ETW configurations as spacetime realizations of the enhancements in the moduli space along specific growth sectors nicely solve the problem to have asymptotic diagonal metrics, the structure of the potential that supports the class of solutions in section 3.3 is not of the same form of the potential that we have in the Calabi-Yau moduli space near the enhanced singularities. The potential (3.23) contains two different contributions: the first one dominating in a cone region near the ETW brane located at the position $y_1 = 0$, and the second one dominating in the complementary region near the ETW brane located at $y_2 = 0$. On the other hand, in the spacetime action for the Calabi-Yau moduli the potential is given by a single term: the leading contribution in the sum (4.12) when we approach the intersection between the two different divisors. Then, there is no straight direct way to match the two expressions.

One possibility is to focus on trajectories in the moduli space where one term of the potential (3.23) is subleading with respect to the other and match with (4.12) only the dominant term of (3.23). However, this approach does not provide a solution in the full spacetime region between the two intersecting ETW branes, but only in the sector in which the trajectories selecting the dominant term in the potential are supported.

Happily, a fully satisfactory realization can be provided using the new class of solutions described in section 3.4, driven by a single potential term (3.32), and performing a special redefinition of the fields (3.33) which leads to the interpretation of these multiple infinite distance limits as configurations of intersecting ETW branes.

Note that for each particular enhancement $_1 \to\ _{12}$, characterized by its corresponding variation of the mixed Hodge-Deligne structure and equipped with the flux information, we have a specific spacetime action (4.15) defined in terms of the pairs of parameters (d_1, d_e) and (l_1, l_e). The main advantage of the solutions computed in section 3.4, whose potential (3.32) matches perfectly the structure of the Calabi-Yau flux potential given by the $sl(2)$-approximation of the Hodge norm of G_4, is that we can associate to each of these actions a specific solution of the form (3.31) and interpret each growth sector R_{12} as an intersecting configuration of branes in spacetime.

4.2.3 Examples

In this section we will perform the explicit matching between the potential (4.12), appearing in the action (4.15), and the Dynamical Cobordism potential (3.32), with $n = 1$, supporting the class of solutions (3.31), for each enhancement of table 4 and for each possible leading term of the corresponding potential. The matching produces for each of these enhancements the exact values of the parameters a_i, which uniquely determine the spacetime solutions via (3.31). The list of values of a_i obtained for all the allowed enhancements of table 4 which are equipped with a non-trivial diagonal kinetic metric, computed in table 3, and with a divergent flux potential showing an explicit dependence from both the moduli is summarized in table 5.

In order to obtain the correct sign in the potential we are requiring the direction y_2 to be timelike. The corresponding solutions physically describe a configuration of intersecting branes with a boundary in the spacelike direction y_1 and an origin in time at the location $y_2 = 0$. It would be interesting to explore the role of such solutions in cosmological applications. A special configuration with a boundary defining the beginning of time and a boundary with respect to a spacelike coordinate has been explored in [24] in a supercritical bosonic string theory setup with tachyon condensation along a lightlike direction. However, in that case there is an interpretation of the two boundaries as two different phases of a same recombinating ETW brane. In the present case of Calabi-Yau moduli we have an interpretation of the two boundaries in terms of an intersecting configuration of two distinct ETW branes.

Using the definition of critical exponents given in (3.36) we can characterize the two intersecting branes appearing in each of these spacetime realizations in the language of the Dynamical Cobordism. The values of the critical exponents for all the accounted enhancements are summarized in the last two columns of table 5. Notice that the list of values obtained for δ_1 nicely agree with the dictionary between singular divisors and codimension 1 ETW branes provided in the table 2 of the previous subsection. As explained above, the ETW brane at

Enhancement	(l_2, l_e)	a_1	a_2	δ_1	δ_2
$II_{0,1} \mapsto III_{0,0}$	$(5,6)$	$1/2$	1	$\sqrt{2}$	2
$II_{0,1} \mapsto III_{0,0}$	$(5,4)$	$1/2$	1	$\sqrt{2}$	2
$II_{0,1} \mapsto V_{2,2}$	$(5,8)$	$1/2$	$1/2$	$\sqrt{2}$	$2\sqrt{2}$
$II_{0,1} \mapsto V_{2,2}$	$(5,6)$	$1/2$	$3/2$	$\sqrt{2}$	$2\sqrt{6}/3$
$II_{0,1} \mapsto V_{2,2}$	$(5,4)$	$1/2$	$3/2$	$\sqrt{2}$	$2\sqrt{6}/3$
$II_{0,1} \mapsto V_{2,2}$	$(5,2)$	$1/2$	$1/2$	$\sqrt{2}$	$2\sqrt{2}$
$III_{1,1} \mapsto V_{2,2}$	$(6,8)$	$1/3$	$1/4$	2	4
$III_{1,1} \mapsto V_{2,2}$	$(6,6)$	$1/3$	1	2	2
$III_{1,1} \mapsto V_{2,2}$	$(6,4)$	$1/2$	$1/4$	2	4
$IV_{0,1} \mapsto V_{2,2}$	$(7,8)$	$1/4$	$1/2$	$\sqrt{6}$	$2\sqrt{2}$
$IV_{0,1} \mapsto V_{2,2}$	$(7,6)$	$1/4$	$1/2$	$\sqrt{6}$	$2\sqrt{2}$
$IV_{0,1} \mapsto V_{2,2}$	$(5,6)$	$3/4$	$3/2$	$\sqrt{6}/3$	$2\sqrt{6}/3$
$IV_{0,1} \mapsto V_{2,2}$	$(5,4)$	$3/4$	$3/2$	$\sqrt{6}$	$2\sqrt{6}/3$

Table 5. Table showing the parameters a_i appearing in the spacetime solutions (3.31) for each possible enhancement and the corresponding values of the critical exponents characterizing the nature of the intersecting ETW branes.

$y_2 = 0$ is interpreted as the spacetime representation of the enhanced singularity occurring at the intersection locus $_{12}$ in $\mathcal{M}_{cs}$. For that reason the corresponding critical exponent δ_2 does not have to match with the values appearing in table 2: the corresponding ETW brane does not represent a singular divisor in the moduli space but an enhanced singularity.

4.3 Application to Swampland Distance Conjecture

One of the most explored swampland conjecture is the Distance Conjecture [13], which predicts for any infinite distance limit in moduli space the emergence of an infinite tower of states becoming exponentially massless with the field space distance, therefore the cutoff of the corresponding effective theory is lowered as:

$$\sim e^{-\lambda \mathcal{D}} \tag{4.16}$$

where λ is an $\mathcal{O}(1)$ parameter.

On the other hand, Dynamical Cobordism solutions probes infinite distance limits in field spaces, then it is natural to ask for their interplay. Since the Dynamical Cobordism analysis provides scaling relations linking the spacetime and the field space distances, we can offer a spacetime version of (4.16) expressing the lowering of the cutoff in terms of paths in the spacetime:

$$\sim \quad ^{\frac{2\lambda}{\delta_{\text{int}}}} \tag{4.17}$$

This spacetime rephrasing of the Distance Conjecture was proposed in [23] for codimension 1 infinite distance limits explored with codimension-1 ETW brane. In [29], it was shown that the same relation is still valid for codimension-2 infinite distance limits in the field space probed in spacetime through configurations of intersecting ETW branes. The only difference in this second case, with respect to the case of single ETW brane, is that the relation (4.17) is controlled by a path-dependent parameter δ_{int} which takes different values for each specific spacetime direction traveled to reach the intersection between the two ETW branes. Due to the knowledge of the profiles of the divergent scalars that parameterize the codimension-2 infinite distance locus in the field space in terms of the spacetime coordinates, it is possible to map each of these spacetime paths to a specific path in field space that reaches the multiple infinite distance limit, and to obtain the decay rate of for each of these paths.

For the class of solutions reviewed in section 3.2, considering all the spacetime paths reaching the brane intersection along all the possible different directions, we are able to reproduce all the possible paths in the field space reaching the corresponding codimension-2 infinite distance limit. However, this is not true for the new class of solutions studied in section 3.4. The new insight lies in the interpretation of these solutions as the intersection of two ETW branes, where one brane represents a codimension-1 infinite distance locus in field space, corresponding to the singular divisor $_1$, and the other intersecting brane does not represent one of its intersecting divisors $_2$, but the enhanced singularity arising at the codimension-2 locus $_{12} = {}_1 \cap {}_2$. This new interpretation implies that none of the paths parameterized in (3.38) corresponds to a path in field space able to reach the divisor $_2$. Roughly speaking, using the profiles of the fields given in (3.31) to translate the spacetime paths parameterized in (3.38) to the corresponding ones in the field space, we are able to reproduce only the field space sector containing paths that achieve the locus $_{12}$ passing close to the divisor $_1$. In particular, the field space path (γ_1, γ_2), corresponding to the path (γ_1, γ_2) in spacetime, is:

$$(\gamma_1, \gamma_2): \quad \begin{cases} \psi_1 = -\left[\gamma_1 \sqrt{a_1 n(2 - a_1 - a_1 n)} + \gamma_2 a_2 (1 - a_1 n) \sqrt{\frac{n}{a_1(2 - a_1 - a_1 n)}}\right] \log t \\ \psi_2 = -\gamma_2 \sqrt{a_2 n \left[1 + \frac{a_2 (a_1 n - 1)^2}{a_1(-2 + a_1 - a_1 n)}\right]} \log t \end{cases} \tag{4.18}$$

Since the choice $\gamma_1 = 0$ in (3.38) corresponds to approach the ETW brane located at $y_2 = 0$ following a transversal path to the brane, this direction corresponds to the furthest path from $_1$ that we can have in the field space. Therefore the parameterization (4.18) includes all the field space paths that satisfy the condition:

$$\frac{\psi_1}{\psi_2} > \frac{\sqrt{a_2}(1 - a_1 n)}{\sqrt{a_1(-2 + a_1 + a_1 n) + a_2 (a_1 n - 1)^2}}. \tag{4.19}$$

This interpretation nicely reproduces the growth sectors R_{12} of the Calabi-Yau moduli space.

5 Conclusions

M-theory compactifications on Calabi-Yau fourfolds with four-form flux produce a rich landscape of vacua reproducing interesting phenomenology. The infinite distance limits of the corresponding moduli/field spaces offer an intriguing arena to test many of the Swampland conjectures and get information about the mechanism breaking the effective field theory description by quantum gravity effect. Interestingly, the structure of the infinite distance limits in the moduli space displays a rich network of intersecting divisors, characterized by the rich mathematical tools of asymptotic Hodge theory [9, 47, 49–53, 66]. In this paper we have initiated the study of this network of normal crossing divisors using Dynamical Cobordism solutions in the resulting three dimensional effective field theories. Some of our results in this context are:

- We have provided a dictionary associating to each singular divisor in the network its spacetime realization as a codimension-1 ETW brane characterized by a specific critical exponent;
- We have shown that, due to the properties of the flux potential, this match requires the corresponding Dynamical Cobordism to describe a time-dependent solution;
- We have extended this dictionary to codimension-2 loci in the network of divisors at infinity in the Calabi-Yau moduli space by using codimension-2 Dynamical Cobordisms describing intersecting ETW branes;
- We have shown that the two intersecting ETW branes in the spacetime picture, rather than describe two intersecting divisors in the moduli space, describe the singularity enhancement of a divisor as it approaches a singularity in a specific growth sector. It is remarkable that the spacetime picture reproduces the spirit of the mathematical approach to the study of intersections of divisors;
- We have shown that, in order to match the leading behaviour of the flux potential given by the asymptotic growth of the Hodge norm, the required spacetime solutions for intersecting ETW branes are more general than those considered hitherto and we have provided the explicit construction of such generalization, by relaxing the constraint of conformally flat ansatz in the solutions in [29];
- We studied the scaling relations between the spacetime distance and the field space distance along general paths in the new intersecting ETW brane solution, generalizing those in the literature, and we explored its relation to the cutoff implied by the Swampland Distance Conjecture, thus providing a generalization valid for non-trivial scalar potentials in Calabi-Yau flux compactifications.

Some related interesting open questions are the following:

- It would be interesting to further study the class of non-conformally flat solutions involving two divergent scalar fields we constructed in this paper. For instance to display the source terms supporting it and to characterize the properties of the possible codimension-2 source term localized at the intersection of the ETW branes, and study possible connections with other setups including such codimension-2 sources [25].
- We performed explicit computations of the critical exponents characterizing the ETW branes associated to each possible singular divisor and some of the enhanced singularities in the complex structure moduli space of Calabi-Yau fourfolds with $h^{1,3} = 2$. It would

be interesting to reproduce the same analysis in higher dimensional moduli space, thus allowing for the exploration of higher codimension intersections in the Calabi-Yau moduli space.

- In our dictionary between singular divisors in the moduli space and ETW branes in spacetime, we translate the information about the Hodge-Deligne splittings characterizing the moduli space singularities in terms of the critical exponents characterizing the branes. However the inverse is not true: the known class of Dynamical Cobordism solutions does not reproduce all the singularity enhancements in the Calabi-Yau moduli space. In particular, those involving degenerate metrics at infinity in field space may require, from the Calabi-Yau side, a better characterization of the subleading contributions to the Calabi-Yau moduli metric, and from the Dynamical Cobordism side, a generalization of the solutions for scalars with non-trivial metrics for the running scalars. It would be interesting to fully understand the spacetime realizations of the enhancements in order to investigate if the complete picture contains more details allowing a 1-1 correspondence between the two approaches.

Our work has provided an important step in the exploration of the intricate structure of the asymptotic Calabi-Yau moduli space in the presence of general flux potential. We hope to come back to these exciting open questions in the future.

Acknowledgments

We would like to acknowledge Angel Uranga for his fundamental contribution to this manuscript through meticulous comments and extensive discussions. We are also pleased to thank José Calderon-Infante, Matilda Delgado, Jesús Huertas and Andriana Makridou for useful discussions and for collaborations on related topics. We also thank Ivano Basile, Stefano Giaccari, Alvaro Herráez, Elias Kiritsis, Fernando Marchesano, Anthony Massidda, Irene Valenzuela and Roberto Volpato for helpful discussions. R.A. wishes to acknowledge the hospitality of the Max Planck Institute for Physics in Munich during the last stages of this work. The research of R.A. has been supported by the grants CEX2020001007-S and PID2021-123017NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.

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Received: *November 19, 2024*
Accepted: *February 10, 2025*
Published: *March 11, 2025*

End of the world boundaries for chiral quantum gravity theories

Roberta Angius, Angel M. Uranga and Chuying Wang

Instituto de Física Teórica IFT-UAM/CSIC,
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain

E-mail: roberta.angius@csic.es, angel.uranga@csic.es, chuying.wang@ift.csic.es

Abstract: We describe the construction of large classes of explicit string theory backgrounds corresponding to 6d and 4d chiral theories with end of the world boundaries, and describe the strong coupling phenomena involved in gapping the chiral (but non-anomalous) sets of fields, such as strongly coupled phase transitions or symmetric mass generation. One class of 6d constructions is closely related to chirality changing phase transitions, such as those turning heterotic NS5-branes into gauge instantons, in flat space or orbifold singularities. A class of 4d models exploits systems of IIB D3-branes at toric CY3 singularities with an extra $\mathbf{Z}_2$ involution related to G_2 holonomy manifolds in the type IIB picture and its IIA mirror, which we explicitly describe in terms of dimer diagrams.

Keywords: Brane Dynamics in Gauge Theories, String and Brane Phenomenology

ArXiv ePrint: 2410.07322

https://doi.org/10.1007/JHEP03(2025)064

Contents

1 Introduction

The swampland cobordism conjecture [1] implies that any theory of quantum gravity must admit configurations including boundaries ending spacetime (end of the world or ETW configurations). These has been discussed in various contexts (see e.g. [2–10]), but are particularly challenging for chiral theories. Indeed, even for string theory or M-theory in their maximal dimensions, such boundary configurations are essentially known only[1] for 11d M-theory (in the form of Hořava-Witten boundaries) and 10d type IIA (a negatively charged O8-plane with 16 D8-branes as counted in the double cover). This is intimately related to the fact that these theories are non-chiral at the level of their spectrum, and only break parity via topological Chern-Simons terms. In fact, for 10d type IIB, type I or heterotic theories, as well as their non-supersymmetric cousins, which are chiral yet anomaly free, there is no microscopic understanding of such boundary ETW configurations. Similar statements can be made in compactifications to lower dimensions.

Morally, the rationale for this relation is that in theories with vector-like spectrum the boundary conditions pair up opposite-chirality degrees of freedom. This is the equivalent of a gapping vector-like fermions with a Dirac mass. Thus, from this perspective, chirality prevents the existence of weakly coupled mechanisms to gap the set of chiral fermions, hence

[1] One may also wish to include bosonic string theory and some supercritical string theories, for which analogues of bubbles of nothing have been built using light-like tachyon condensation [11–13].

boundary conditions for chiral theories must involve strong coupling. This makes it difficult to formulate such boundary conditions, even in situations with high supersymmetry.

Although examples of mechanisms gapping chiral non-anomalous sets of fermions have been studied in the context of quantum field theory (see e.g. [14, 15], also [16] for a review), examples of boundary configurations for chiral theories in the context of quantum gravity or string theory are very scarce (one example is given by the bubble of nothing in [17], when regarded from the 10d perspective; see also [18] for a proposed construction in 4d compactifications). In this paper we take important steps in improving this situation.

We build explicit boundary ETW configurations for large classes of examples of 6d and 4d chiral theories from string theory compactifications, hence coupled to quantum gravity. The examples are constructed by considering a $(d-1)$-dimensional locus of a d-dimensional localized chiral field theory in D-dimensional spacetime, and regarding the local configuration as a cone over the angular manifold in the $(D-d+1)$-dimensional transverse space around the $(d-1)$-dimensional slice. We are thus left with a compactification on the $(D-d)$-dimensional base of the cone, with a potentially chiral spectrum including the d-dimensional field theory. The cone defines a boundary configuration for the system, with an ETW boundary specified by the $(d-1)$-dimensional slice, which sits at the tip of the cone. The actual appearance of chirality in the d-dimensional theory is highly non-trivial and requires special physics happening at the $(d-1)$-dimensional locus, the tip of the cone. We dub this the Cone Construction or, when it leads to boundary configurations for actual chiral theories, the Chiral Cone Construction.

The Cone Construction provides an explicit link with the Dynamical Cobordisms of the compactified theory, in the sense of [6, 7, 19, 20].[2] In the Cone Construction, the lower-dimensional theory is obtained by compactification on the base of the cone. The evolution in the radial direction, along which the size of the compactification space varies, defines a solution with a running scalar for this lower dimensional theory. At the tip of the cone the corresponding scalar blows up to infinite field theory distance at a finite spacetime distance producing a singularity at which spacetime ends. This precisely agrees with the behaviour near an ETW configurations in Dynamical Cobordisms, and in particular there is a precise match with the local dynamical cobordism solutions in [7] at the quantitative level.

Regarding the special physics at the $(d-1)$ slice, we specifically consider two main classes of models:

- The first involves chirality changing phase transitions: we focus on explicit examples of 6d $\mathcal{N}=1$ theories with heterotic NS5-branes reaching the origin of the Coulomb branch of their tensor multiplets and turning into gauge instantons, effectively trading each tensor multiplet for 29 hypermultiplets [45, 46]. We consider several examples based on 5-branes in flat space or on orbifold singularities [45–54], and apply the Cone Construction to obtain boundary configurations for large classes of 6d chiral theories.

- The second involves fixed planes under $\mathbf{Z}_2$ involutions, closely related to those turning a CY3 conical singularity times $\mathbf{R}$ into a (barely) G_2 holonomy variety [55–57]. We consider large classes of chiral 4d theories arising from IIB D3-branes at toric CY3

[2]For related ideas, see [11–13, 21–24] for early references, and [2, 18, 25–44] for recent works.

singularities, and use $\mathbf{Z}_2$ quotients related to G_2 varieties in the IIA mirror, to define boundary conditions from Chiral Cone constructions. We exploit the powerful language of dimer diagrams as an efficient tool to describe the theories and the quotients leading to boundary configurations.

The paper is organized as follows. In section 2, we consider explicit examples based on chirality changing phase transitions. After a warm-up in section 2.1 revisiting open heterotic strings (section 2.1.1) and building cone construction over their boundaries (section 2.1.2), we move into the non-trivial case of Chiral Cone Constructions for 6d theories in section 2.2. We revisit the chirality changing phase transition for the $E_8 \times E_8$ heterotic NS5-brane in flat space in section 2.2.1, and in section 2.2.2 we use the Cone Construction to define boundary configurations for chiral 6d theories. In section 2.2.3 we relate our discussion to the supergravity solution [58] and its recent worldsheet description in [59]. In section 2.3 we extend our construction to 5-branes at singularities, and in section 2.4 we discuss relations with the cone constructions used in the string theory derivation of SymTFTs [60] in the study of generalized symmetries (see [61–67] for reviews).

In section 3 we focus on boundary configurations for 4d chiral theories. In section 3.1 we emphasize how non-trivial the task is. We review intersecting D6-branes in section 3.1.1 and open D6-branes ending on NS5 branes in section 3.1.2, using them to construct localized 4d fermions on a space with boundary in section 3.1.3. However, in section 3.1.4 we show that the corresponding Cone Construction fails (in an interesting way) to provide boundary conditions for chiral fermions, due to the presence of additional D4-branes. Overcoming this failure motivates the construction in section 3.2 of chiral gauge sectors localized on D3-branes at singularities, whose Cone Construction produces boundary configurations via a mechanism resembling that in [17]. In section 3.2.1 we present one example leading to boundary conditions for the chiral 4d theory of D3-branes at $\mathbf{C}^3/\mathbf{Z}_3$ (the dP_0 theory), which in section 3.2.2 we extend to D3-branes at general CY3 toric singularities. In these models the special physics at the tip of the cone can be associated to brane-antibrane annihilation. In section 3.3 we improve over this class of models, by including a $\mathbf{Z}_2$ quotient ultimately lying at the tip of the cone. In section 3.3.1 we motivate the construction by considering the $\mathbf{Z}_2$ quotients turning CY3$\times\mathbf{R}$ into a barely G_2 holonomy variety. The mirror of such $\mathbf{Z}_2$ actions is applied in section 3.3.2 to construct boundary configurations for theories arising from D3-branes at CY3 cone singularities, with several explicit examples described in section 3.3.3, and 3.3.4. In section 3.4 we describe the relation of the cone constructions with Dynamical Cobordisms. We study the general dimensional reduction in section 3.4.1, particularize to compactification on the base of cones in section 3.4.2, and show our cone constructions agree with the local dynamical cobordisms solutions in [7] in section 3.4.3.

In section 4 we offer some final remarks. In appendix A we extend the analysis of section 3.1 to even more intricate configurations of intersecting D6-branes with boundaries, and show that their cone constructions do not lead to boundary conditions for 4d chiral theories. In appendix B we revisit a system studied in [68] and show it can be regarded as an explicit example of a G_2 cone construction providing boundary configuration for a 4d chiral gauge theory from intersecting D6-branes.

2 Boundaries from chirality changing phase transitions

The problem of gapping a set of chiral non-anomalous fields has appeared in string theory context in a slightly different avatar: the study of chirality changing phase transitions. In this section we argue that this question is closely related to the construction of boundary configurations for chiral theories via the Cone Construction, and present several classes of examples.

2.1 The cone construction: warm-up with the open heterotic string

In this section we introduce the key ideas of building boundary configurations for potentially chiral theories (the Chiral Cone construction), in terms of the example of the open heterotic string. The construction is easily generalized to other setups, as we study in later sections.

2.1.1 Open heterotic string

A prominent manifestation of the difficulty to introduce boundary conditions for chiral theories arises in the context of D-branes as defining boundary conditions for 2d worldsheet CFTs. Indeed, there are no D-branes in heterotic string theory because one cannot introduce suitable boundary conditions on its chiral worldsheet theory.[3] However, there is a remarkable construction of open heterotic strings in [69] in 10d flat space SO(32) heterotic[4] theory (see [70] for a recent discussion), as we now review.

The point is that a heterotic SO(32) string worldsheet can end on configurations of the SO(32) gauge fields with non-trivial value for $\mathrm{tr}\, F^4$ on the $\mathbf{S}^8$ surrounding the worldsheet boundary (i.e. the $\mathbf{S}^8$ around the origin in the $\mathbf{R}^9$ transverse to the string endpoint worldline). This can be shown to be consistent with flux conservation by checking the invariance of the action under gauge transformations of the 10d 2-form $B_2 \to B_2 + d\Lambda_1$. Indeed, the action contains the terms

$$S_{B_2} = \int_{\Sigma_2} B_2 + \int_{10d} B_2 \mathrm{tr}\, F^4, \tag{2.1}$$

where the first term is the coupling of the string worldsheet Σ_2 and the second is the 10d 1-loop term required in the Green-Schwarz mechanism. Under gauge transformations,

$$\delta_{\Lambda_1} S_{B_2} = \int_{\partial\Sigma_2} \Lambda_1 - \int_{10d} \Lambda_1 d\mathrm{tr}\, F^4 = \int_{\partial\Sigma_2} \Lambda_1 - \int_{10d} \Lambda_1 \delta_9(\partial\Sigma_2) = 0\,. \tag{2.2}$$

Here, in the first equality we have used integration by parts, and in the next-to-last equality we have used $d\mathrm{tr}\, F^4 = \delta_9(\partial\Sigma_2)$, where $\delta_9(\partial\Sigma_2)$ is a bump 9-form supported at the boundary $\partial\Sigma_2$ of the worldsheet (namely, by Gauss' law, $\mathrm{tr}\, F^4$ integrates to 1 over the $\mathbf{S}^8$ around $\partial\Sigma_2$).

A second important ingredient in the discussion in [69] is that the gauge configuration carries away the excess of left- over right-moving fermions on the heterotic worldsheet. At the boundary of the open heterotic string, the left-moving fermions transition into fermions of the bulk theory which are carried in the radial direction away from the worldsheet boundary.

[3] Note that although the type IIA string worldsheet is chiral in 2d, due to the opposite GSO projections, it is possible to introduce boundary conditions breaking part of the global symmetry (i.e. 10d Poincaré invariance).

[4] In the $E_8 \times E_8$ theory, the analogous constructions is possible, but it requires the presence of certain singularities in the geometry [69], hence we skip it.

2.1.2 The cone construction

The above configuration represents a non-trivial boundary for a 2d chiral theory, albeit in a theory embedded in a higher-dimensional theory. However, there is a simple way in which we can turn the system into a 2d configuration, which amounts to regarding a local flat space as a cone. This has been exploited in the context of building Local Dynamical Cobordism solutions in [7][5] and in fact it will produce dynamical cobordisms in our setup as well, cf. section 3.4. We advance that, although the Cone Construction does not yield a boundary configuration for a genuine chiral 2d theory in this particular example of open heterotic strings, this construction will do the job in other examples in coming sections.

We hence regard the flat space local geometry around the open heterotic string worldsheet boundary in the previous section as a cone over $\mathbf{S}^8$ (times the time direction along the boundary of the string worldsheet), and consider it from the perspective of the effective 2d theory obtained after a compactification on $\mathbf{S}^8$. The cone configuration, in which the $\mathbf{S}^8$ varies in the radial direction and shrinks at the origin, can thus be regarded as a dynamical cobordism solution of the 2d theory obtained after compactification on $\mathbf{S}^8$, in analogy with [7, 73], thus defining an ETW configuration ending spacetime.

As in [7], the above configurations should be regarded merely as local descriptions near the ETW boundary, which can be part of a more involved global configuration, in which in particular the $\mathbf{S}^8$ may have a finite size further away from the ETW boundary. A template for this behaviour is Witten's bubble of nothing [74], in which the compactification $\mathbf{S}^1$ has a constant asymptotic radius for, but locally near the bubble of nothing it is a polar angle which combines with the radial coordinate to parametrize a local $\mathbf{R}^2$. We thus conceive our cone constructions in a similar spirit.

Let us thus consider the compactification of the 10d theory on $\mathbf{S}^8$ (similar considerations can be made for more general spaces $\mathbf{X}_8$). Since we want to match the cone construction of the previous section, we need to turn on a non-trivial $\mathrm{tr}\, F^4$ background on it. Note that from the 10d 1-loop coupling (2.1), the resulting 2d theory has a non-trivial tadpole for B_2 (the heterotic analogue of the tadpole in [75]), which has to be explicitly cancelled by the introduction of a fundamental string worldsheet, namely the first term in (2.1). This is just a rederivation of the flux conservation argument at the beginning of this section.

Hence the worldsheet fields on this string worldsheet are now degrees of freedom of our 2d spacetime theory. Because they are chiral, one may, as mentioned above, have the expectation that we have a 2d chiral theory, which ends on a codimension 1 boundary where the $\mathbf{S}^8$ shrinks. If true, this would actually be very striking, because the 2d theory on the worldsheet is anomalous and does not make sense by itself in the quantum theory. However we know that there are actually extra ingredients which come to the rescue, in the form of the fermion zero modes of the 10d gauginos in the presence of the gauge background. Indeed, the 10d chiral fermions in the adjoint of SO(32) lead, upon compactification on $\mathbf{S}^8$ with a non-trivial $\mathrm{tr}\, F^4$, to non-trivial 2d chiral fermions due to the index of the Dirac operator. The computation is essentially a reinterpretation of that in [69], with the result that the chiral fermions coming from these zero modes cancel the chirality of the 2d fermions from the

[5]Cone constructions of this kind have also been played a prominent role in the construction of SymTFTs (see [60], also [65] for a review), as well as in holography, starting from [71, 72].

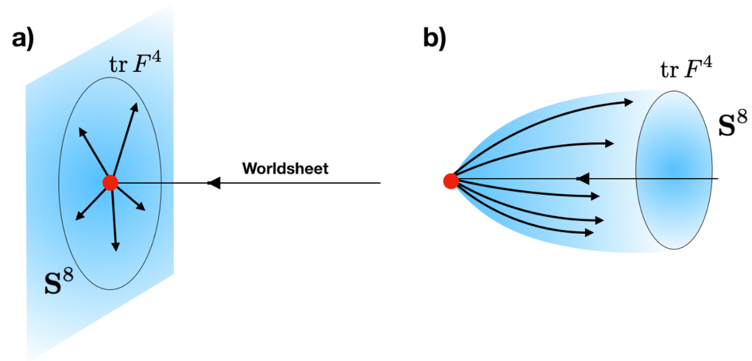

Figure 1. a) Open heterotic string in flat space, with chiral fermions (denoted with arrows) ending on the boundary outflow as bulk zero modes as. b) The configuration in the Cone Construction: the theory upon compactification on $\mathbf{S}^8$ describes a boundary configuration for a running 2d solution. However, the content of chiral fermions is non-chiral.

worldsheet, in fact in a trivial vector-like way. We thus end up with a non-trivial boundary configuration described as a dynamical cobordism ending spacetime, but for a 2d theory with a *vector-like* set of fermions. The situation is depicted in figure 1.

Despite the apparent failure to obtain a Chiral Cone Construction in this concrete example, we will continue exploiting the general strategy in coming examples. Namely, we consider branes or defects supporting chiral theories, introduce boundaries for them ensuring flux conservation, and regard the geometry around the boundary as a cone, and the configuration as a dynamical cobordism solution for the theory compactified on the base of the cone. Following these setups, we will eventually obtain boundary configurations in several large classes of models, discussed in later sections.

Let us finally mention that there is interestingly a very explicit quantitative description of the above cone construction solution, in terms of a precise worldsheet theory for a non-critical heterotic string. In fact [59] recently identified the boundary of the open heterotic string as a non-supersymmetric 0-brane of heterotic theory, and provided the worldsheet description of the near horizon geometry, in terms of a (gapped) 2d $\mathcal{N} = (1,1)$ sigma model describing an $\mathbf{S}^8$ compactification with a non-trivial $\int_{\mathbf{S}^8} \mathrm{tr}\, F^4$ background, and a radial direction with a linear dilaton background. As shown in [33] such linear dilaton backgrounds turn into a dynamical cobordism in the Einstein frame. Hence our picture has a nice agreement with the setup in [59].

2.2 Boundaries for chiral 6d theories from open heterotic NS5-branes

The example in the previous section (see section 3.1 and appendix A for other examples) illustrates an important point. In a defect supporting a chiral theory with an open worldvolume manifold, the boundary defines a transition in which the worldvolume ends and its localized degrees of freedom outflow as bulk modes. In this context, the cone construction allows to turn the system into a boundary configuration for the theory obtained as dimensional reduction on the base of the cone. However, this theory is non-chiral if the worldvolume degrees of freedom and the bulk degrees of freedom after the transition are of the same kind; in the previous examples, they both corresponded to chiral fermions transforming in the same representation of the gauge group, so they end up forming non-chiral pairs. Hence, the strategy to achieve a boundary for a genuinely chiral theory is to consider as starting point a chirality changing phase transition given by a process in which some brane ends on a boundary and the bulk modes outflowing from the boundary are of a totally different kind from the original worldvolume modes.

Chirality changing phase transitions have been a subject of active research in string theory and there is a good number of examples in the literature both in 6d [45–54] as well as in 4d [76–78]. In the following we illustrate the above picture with the paradigmatic case of the small instanton phase transitions in the $E_8 \times E_8$ heterotic theory [45, 46].

2.2.1 The $E_8 \times E_8$ heterotic NS5-brane chirality changing phase transition

The $E_8 \times E_8$ heterotic NS5-brane chirality changing phase transition (which is often discussed in terms of the Hořava-Witten M-theory uplift) is as follows [45, 46]. Consider the 10d heterotic theory in flat spacetime, in the presence of one NS5-brane along the directions 012345. The worldvolume theory has 6d $\mathcal{N} = (1,0)$ supersymmetry and contains a tensor multiplet, whose single real scalar parametrizes a Coulomb branch (corresponding to the position of the M5-brane in the Hořava-Witten interval $\mathbf{S}^1/\mathbf{Z}_2$ in the M-theory uplift), and one hypermultiplet, whose four real scalars parametrize a Higgs branch, the position of the NS5-brane (equivalently the M5-brane in the 11d lift) in the transverse dimensions 6789. By changing the vev of the scalar in the tensor multiplet one can reach the origin in the Coulomb branch (which corresponds to the 11d M5-brane reaching one of the Hořava-Witten boundaries) at which new massless degrees of freedom become light (M2-branes stretched between the M5 and the Hořava-Witten boundary) and the theory becomes strongly interacting. At this point the NS5-brane can be equivalently regarded as a zero-size small instanton (see [79] for the similar process for the SO(32) heterotic), so it is possible to move into a Higgs branch, turning it into a finite size E_8 gauge instanton. The resulting spectrum of zero modes on the instanton can be obtained using the index theorem, which provide a spectrum 6d $\mathcal{N} = (1,0)$ hypermultiplets. Consider the transition of k 5-branes into an instanton background into an $\mathrm{SU}(2) \subset E_8$, for simplicity. From the group theory decomposition

$$\begin{array}{ccl} E_8 & \to & E_7 \times \mathrm{SU}(2) \\ \mathbf{248} & \to & (\mathbf{133},\mathbf{1}) + (\mathbf{56},\mathbf{2}) + (\mathbf{1},\mathbf{3})\,, \end{array} \tag{2.3}$$

the number of instanton fermion zero modes in the different E_7 representations given by the index theorem are

$$\#\mathbf{56} = (k-4)/2, \quad \#\mathbf{1} = 2k-3\,. \tag{2.4}$$

Hence, each single unit of instanton charge contributes a 6d $\mathcal{N} = (1,0)$ half-hypermultiplet in the $\mathbf{56}$ of E_7 and 2 singlets.

Overall, the transition from the heterotic NS5-brane to the finite size gauge instanton has turned a spectrum with 1 tensor multiplet and 1 hypermultiplet into a total of 30 hypermultiplets. These 6d $\mathcal{N} = (1,0)$ spectra are chiral, hence the transition is a chirality changing phase transition, albeit (and very remarkably) in a way compatible with the (highly restrictive) 6d anomaly cancellation conditions.[6] In particular, focusing on purely gravitational anomalies, the contribution from an overall number V of vector multiplets, H hypermultiplets and T tensor multiplest is $H - V - 29T$. This leads to the celebrated fact that a tensor multiplet can be traded for 29 hypermultiplets, as realized in the above phase transition. Let us also mention that, in addition, there are several other gauge anomalies that match in this transition, see [45, 46] for details.

Note that for the SO(32) heterotic there is a similar small instanton transition, but if the NS5-brane is at a smooth point in the transverse space, it does not involve tensor multiplets, and does not lead to chirality change. Hence, in the cone construction in the next section will lead to boundaries for non-chiral theories. The situation changes for 5-branes at singularities, as we explore in section 3.2.

2.2.2 The open heterotic NS5-brane and the chiral cone construction

Let us now exploit the above information to build an open NS5-brane configuration, in analogy with the open heterotic string in section 2.1.

The key point is that in the presence of NS5-branes with worldvolume Σ_6, the couplings for the 6-form B_6 dual of the 2-form field are

$$\int_{\Sigma_6} B_6 + \int_{10d} B_6 \, (\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2) \,. \tag{2.5}$$

Equivalently, the modified Bianchi identity for the 3-form field strength is

$$dH_3 = \mathrm{tr}\, F^2 - \mathrm{tr}\, R^2 + \delta_4(\Sigma_6) \,, \tag{2.6}$$

where $\delta_4(\Sigma_6)$ is a bump 4-form Poincaré dual to Σ_6. The above means that a 5-brane can have a 5d boundary if the latter acts as a source as $d(\mathrm{tr}\, F^2 - \mathrm{tr}\, R^2) = \delta_5(\partial\Sigma_6)$, with δ_5 a bump form for Poincaré dual to the boundary $\partial\Sigma_6$. In other words, if we denote by $\mathbf{X}_4$ the geometry around the 5d boundary $\partial\Sigma_6$, we need

$$\int_{\mathbf{X}_4} \mathrm{tr}\, F^2 - \mathrm{tr}\, R^2 = 1 \,. \tag{2.7}$$

There are several possibilities for this. The most direct is that the space transverse to $\partial\Sigma_6$ is smooth, hence locally $\mathbf{R}^5$, and then $\mathbf{X}_4 = \mathbf{S}^4$ at the topological level. Since $\int_{\mathbf{S}^4} \mathrm{tr}\, R^2 = 0$, then we need a non-trivial gauge instanton bundle

$$\int_{\mathbf{S}^4} \mathrm{tr}\, F^2 = 1 \,. \tag{2.8}$$

Another possibility is that the 5-brane boundary is located at the tip of a singular 5d transverse space, so that we can have some $\mathbf{X}_4$ with non-trivial second Pontryagin class. This will be explored in section 2.3, and here we consider just the case of $\mathbf{X}_4 = \mathbf{S}^4$ with a non-trivial gauge bundle.

[6] Although we are focusing on configurations with non-compact 10 dimensions, it makes sense to consider the 6d anomalies, which in this context are considered as localized anomalies on the volume of the defects. They will become genuine 6d anomalies upon compactification e.g. on K3, or as in the cone construction in the next section.

It is easy to construct a gauge background with instanton number 1 on $\mathbf{S}^4$. One just picks an SU(2) subgroup of the gauge group $E_8 \times E_8$ or SO(32), and regards SU(2) as $\mathbf{S}^3$, and builds the instanton background using the Hopf fibration of $\mathbf{S}^7$ over $\mathbf{S}^4$ with fiber $\mathbf{S}^3$. We will not need this explicit construction and simply proceed at an essentially topological level.

It is easy to describe what is happening at the boundary of the NS5-brane. When the chiral content of the 6d NS5-brane theory reaches the boundary, it encounters a non-trivial gauge background, which produces a set of bulk fermions (from the 10d gaugino zero modes) outgoing radially as 30 hypermultiplets[7] (as $1/2 \cdot \mathbf{56} + 2 \cdot \mathbf{1}$ of E_7) and carrying away the anomaly. The total charge under B_6 (i.e. the H_3 flux) is conserved, and so is anomaly, albeit in a very non-trivial way, because of the trading of 1 tensor for 29 hypers. This is a key different with respect to the open heterotic string in section 2.1, and impacts crucially in the cone construction, to which we now turn.

Let us now regard the $\mathbf{R}^5$ transverse to $\partial\Sigma_6$, as a cone over $\mathbf{S}^4$. We can regard this as a dynamical cobordism of the 6d theory obtained upon compactification of the 10d theory on $\mathbf{S}^4$. On this $\mathbf{S}^4$, the NS5-brane is sitting at a point, and leads to a 6d $\mathcal{N} = (1, 0)$ tensor and a hyper. The 5-brane charge is cancelled by a gauge background

$$\int_{\mathbf{S}^4} \operatorname{tr} F^2 = -1\,. \tag{2.9}$$

The change of sign is due to a change in the orientation of the $\mathbf{S}^4$ when regarded in flat space or in the cone. It implies we get fermions of chirality opposite to that of hypers (reflecting the compactification is non-susy), Namely, we get 'opposite chirality' hypers in the $1/2 \cdot \mathbf{56} + 2 \cdot \mathbf{1}$.

Overall, the total content is (very!) chiral, but non-anomalous, with the anomaly from the tensor cancelling against the 29 'opposite chirality' hypers. The configuration describes a dynamical cobordism in a 6d chiral non-anomalous 6d theory, in which the scalar parametrizing the $\mathbf{S}^4$ size runs until it shrinks to zero size, ending spacetime. We moreover have a fairly good microscopic understanding of the ETW configuration, in terms of the NS5-brane boundary. The theory admits a boundary, with effective boundary conditions relating wildly different fermion fields, thanks to a non-trivial mechanism gapping the chiral non-anomalous content, necessarily at strong coupling.

One interesting perspective of the cone construction is that, in the same way its use to construct SymTFTs allows an efficient way to study singular configurations by means of a smooth compactification, in our present setup it may serve to get further information about the strongly coupled regime of the transition between the NS5-brane and the instanton. We will say a bit more on this in section 2.4.

2.2.3 A related open M5-brane chiral cone construction

The above cone construction describing open heterotic NS5-branes is closely related to the system studied in [58] in supergravity, and recently revisited in [59] from the viewpoint of a worldsheet description. In this section we show that this system, when regarded as a cone construction, also corresponds to a boundary configuration for a chiral compactification of the $E_8 \times E_8$ heterotic theory.

[7]Actually, the configuration is in general not supersymmetric, but the counting of fermions is topological, so it works similarly and we abuse language and still use the susy jargon.

Let us revisit the setup in [58, 59]. Consider an M5-brane extending in the $\mathbf{S}^1/\mathbf{Z}_2$ interval between the two boundaries of the Hořava-Witten theory, and turning into an instanton of the E_8 at each of the boundaries. Equivalently, an E_8 instanton on a first boundary turns into an M5-brane, which travels along the interval, and turns into an E_8 instanton of the second boundary. The system is morally a double copy of the chirality changing phase transition in the previous sections. In the heterotic string limit of small M-theory interval size, we just have one E_8 instanton turning into an instanton of the second E_8. This configuration was discussed in the supergravity approximation [58], while [59] identified the transition region as a non-supersymmetric heterotic 4-brane and provided an explicit worldsheet description of its near horizon regime.

We would like to pursue the latter 4-brane perspective with emphasis in regarding it as a cone construction. The geometry around the 4-brane can be regarded as a cone over $\mathbf{S}^4$, on which there is a non-trivial instanton background with instanton number $(1, -1)$ embedded in $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset E_8 \times E_8$. The near horizon regime was shown in [59] to be given by 2d (gapped) $\mathbf{S}^4$ sigma model, times a quotient of a 4 Majorana-Weyl fermion SO(4) free theory and an $E_7 \times E_7$ current algebra CFT (describing the unbroken gauge symmetry), times a linear dilaton theory describing the radial coordinate.

From the cone construction perspective, the system is providing a dynamical cobordism for the 6d theory obtained by compactifying the 10d heterotic theory on $\mathbf{S}^4$ with a non-trivial instanton background in $\mathrm{SU}(2) \times \mathrm{SU}(2) \subset E_8 \times E_8$. Namely, a boundary configuration for the 6d chiral theory with gauge symmetry $E_7 \times E_7$ and chiral matter content given by a set of fermions charged under the first E_7, with multiplicities dictated by the index theorem, and the corresponding opposite chirality fermions charged under the second E_7. Hence, this simple cone construction provides a boundary configuration for a chiral 6d theory.

Let us remark that, even though our derivation involved a double use of the chirality changing phase transition of the previous sections, in the final chiral cone construction that complicated physics is all hidden at the tip of the cone. In fact, it is possible to propose a simpler description of the boundary conditions at the tip of the cone in terms of exchange of left- and right chiralities, with a simultaneous exchange of the two E_8's, which is a gauge symmetry of the 10d theory. We will encounter similar examples in the 4d context in section 3.2.

We finally note that, although the resulting full configuration is non-supersymmetric and would seem complicated, its behaviour near the tip is explicitly described by the near horizon worldsheet theory in [59]. Moreover, its description of the radial direction as a linear dilaton theory, implies as in [33] that in the Einstein frame it corresponds to a dynamical cobordism in which the dilaton runs and blows up at a finite spacetime distance. This nicely reproduces our intuition that the cone construction correspond to dynamical cobordisms of the theory after compactification on the base of the cone. The relation with dynamical cobordisms will be explicitly recovered, in an analogous class of cone constructions, in section 3.4.

2.3 Chirality changing phase transitions from 5-branes at singularities

In this section we briefly point out that the 5-brane chirality changing phase transition in section 2.2.1 has several generalizations, obtained by locating the 5-brane at the tip of an orbifold singularity. This has been efficiently studied for D5-branes at $\mathbf{C}^2/\mathbf{Z}_N$ singularities in [50–52] (see also similar results and generalization from Hanany-Witten brane constructions [53, 54] and from the perspective of F-theory on CY3 in [48, 49] and in the recent [80, 81]).

For concreteness, we will focus on a particular illustrative example, based on the chirality changing phase transition for type I D5-branes at the $\mathbf{C}^2/\mathbf{Z}_2$ singularity, studied in [50] (see [48] for an earlier derivation in F-theory on CY3, and [53, 54] for a derivation in a T-dual type I' theory with D6-branes suspended among NS5-branes). The discussion generalizes straightforwardly to more general cases, which we leave as an exercise for the interested reader.

Consider type I theory on $M_6 \times \mathbf{C}^2/\mathbf{Z}_N$, with the generator θ of $\mathbf{Z}_N$ acting as $\theta : (z_1, z_2) \to (e^{2\pi i/N} z_1, e^{-2\pi i/N} z_2)$, which preserves 8 supersymmetries, i.e. 6d $\mathcal{N} = 1$ at the tip of the singularity. As explained in [82] there are two choices of the orientifold action on the orbifold twisted sector: the choice without vector structure, which gives a 6d $\mathcal{N} = 1$ hypermultiplet in the twisted sector, and breaks the D9-brane symmetry down to U(16), and the choice with vector structure, which produces a tensor multiplet, and breaks the D9-brane symmetry down to $\mathrm{SO}(w_0) \times \mathrm{SO}(w_1)$, where these integers satisfy $w_0 + w_1 = 32$, and define asymptotic holonomy of the D9-brane gauge bundle. We focus on the latter case, i.e. with vector structure, and for simplicity we choose $w_0 = 32$, $w_1 = 0$, so the unbroken symmetry is SO(32).

We can now locate a number of D5-branes at the tip of the singularity, without further breaking of supersymmetry, so we get a 6d $\mathcal{N} = 1$ gauge theory on their worldvolume. The spectrum is

$$\begin{array}{c} \mathrm{USp}(2k) \times \mathrm{USp}(2k-8) \\ (\square, \square) + 16(\square, \mathbf{1}) + 1 \cdot \mathbf{T}\,, \end{array} \tag{2.10}$$

where $\mathbf{T}$ is the tensor multiplet, and the 16 fundamentals arise from the D5-D9 open string sector and actually correspond to one half-hypermultiplet in the $(\square; \mathbf{32})$ of $\mathrm{USp}(2k-8) \times \mathrm{SO}(32)$, with the latter regarded as a global symmetry from the 6d perspective.

In the limit of strong coupling of the $\mathrm{USp}(2k-8)$ theory, which is the origin of the Coulomb branch for the tensor multiplet, there exists a chirality changing phase transition, in which this gauge factor disappears and so do the bifundamental hypermultiplet and the tensor multiplet, while there appears hypermultiplets in the $\begin{smallmatrix}\square\\\square\end{smallmatrix} + 2 \cdot \mathbf{1}$ of the $\mathrm{USp}(2k)$ factor, which parametrize a Higgs branch. The theory is thus

$$\begin{array}{c} \mathrm{USp}(2k) \\ \begin{smallmatrix}\square\\\square\end{smallmatrix} + 16\square + 2 \cdot \mathbf{1}\,. \end{array} \tag{2.11}$$

The anomalies of the theories before and after the transition fully agree, in particular again effectively trade 1 tensor multiplet for 29 hypermultiplets. For instance, for $k = 4$, the initial theory is simply USp(8) with 16 hypers in the fundamental and a tensor multiplet, and the whole transition amounts to removing the tensor multiplet and replacing it by a hypermultiplet in the **27** of USp(8) plus two singlets.

We can now move into the Higgs branch in particular giving vevs to the 16 fundamentals, which corresponds to dissolving the D5-branes as gauge instantons of the D9-brane theory. This breaks the $\mathrm{USp}(2k)$ group completely, and the SO(32) down to some subgroups depending on the gauge embedding of the instantons. Embedding them as an instanton number k

background in an SU(2) ⊂ SO(32) for simplicity, we can use the decomposition

$$\begin{aligned} \mathrm{SO}(32) &\rightarrow \mathrm{SU}(2)\times\mathrm{SU}(2)\times\mathrm{SO}(28) \\ \mathbf{496} &\rightarrow (\mathbf{3},\mathbf{1},\mathbf{1})+(\mathbf{1},\mathbf{3},\mathbf{1})+(\mathbf{1},\mathbf{1},\mathbf{378})+(\mathbf{2},\mathbf{2},\mathbf{28})\,, \end{aligned} \tag{2.12}$$

and get the hypermultiplet spectrum from the index theorem, which gives

$$\#_{(\mathbf{2},\mathbf{28})}=(k-4)/2, \quad \#_{(\mathbf{1},\mathbf{1})}=2k-3\,. \tag{2.13}$$

The transition is again compatible with the structure of anomalies, once the change in vector multiplets from the breaking SO(32) → SO(28) × SU(2) is taken into account.

For completeness, let us describe this transition from the perspective of Hanany-Witten brane configurations in type I' theory, which is obtained upon T-dualizing the system above along the $\mathbf{S}^1$ corresponding to the U(1) orbit $(z_1,z_2)\to(e^{i\varphi}z_1,e^{-i\varphi}z_2)$ in $\mathbf{C}^2/\mathbf{Z}_N$ (see [53, 54], also [83] for a 4d $\mathcal{N}=2$ version). We have type I' theory, i.e. IIA on $\mathbf{S}^1/\mathbf{Z}_2$, with a $\mathbf{Z}_2$ orientifold quotient introducing O8$^-$ planes at the two fixed loci. The $\mathbf{C}^2/\mathbf{Z}_2$ orbifold is mapped to two NS5-branes in the covering $\mathbf{S}^1$, and the choice with vector structure corresponds to having them at orientifold image points away from the O8$^-$-planes. The choices of w_0, w_1 describe the distribution of the 32 D8-branes in the two intervals separated by the NS5-branes, i.e. on top of each of the O8$^-$-planes, so the choice $w_0=32$, $w_1=0$, leads to the 32 D8-branes on top of one O8$^-$-plane, leaving the other empty. We now stretch $2k$ D6-branes suspended between the NS5-branes in the interval passing through the occupied O8$^-$-plane, and $2k-8$ D6-branes between the NS5-branes but on the interval passing through the empty O8$^-$-plane. The spectrum of the 6d $\mathcal{N}=1$ theory is (2.10), with the tensor multiplet corresponding to the position of the NS5-branes on the $\mathbf{S}^1$. We can now move the NS5-brane and its image on top of the empty O8$^-$-plane, by tuning the scalar in the tensor multiplet. Then there exists a phase transition, corresponding to moving the NS5-branes, as two independent objects, along the O8$^-$-plane and off the D6-branes. The set of left-over D6-branes leads to the USp($2k$) theory with the 2-index antisymmetric hypermultiplet, while the tensor multiplet has disappeared because the NS5-brane position in $\mathbf{S}^1$ is fixed. The positions of the NS5-branes away from the D6-branes parametrize 2 hypermultiplet singlets, and the rest of the Higgs branch is parametrized by the antisymmetric matter and the bifundamentals, whose effect was discussed in the previous paragraph. This picture of the transition in terms of brane motions is completely general and applies to the infinite classes of 6d $\mathcal{N}=1$ theories from type I D5-branes at $\mathbf{C}^2/\mathbf{Z}_N$ singularities with and without vector structures.

Let us now go back to our particular $\mathbf{C}^2/\mathbf{Z}_2$ example and carry out a Chiral Cone construction based on the above chirality changing phase transition. Consider type I theory on $\mathbf{C}^2/\mathbf{Z}_2$ and take the 5d space $\mathbf{R}\times\mathbf{C}^2/\mathbf{Z}_2$, with $\mathbf{R}$ parametrized by one of the 6d Poincaré invariant coordinates, say x^5, and regard it as a cone over a 4d base $\mathbf{S}^4/\mathbf{Z}_2$. The $\mathbf{Z}_2$ action has two fixed points on $\mathbf{S}^4$, locally of the form $\mathbf{C}^2/\mathbf{Z}_2$. We now turn on an instanton number $-k$ gauge background in SU(2) ⊂ SO(32), and locate k D5-branes at one of the $\mathbf{C}^2/\mathbf{Z}_2$ singularities, so as to be compatible with untwisted RR tadpole cancellation for this compactification. The flip of the gauge bundle instanton background is due to the orientation flip between the coordinate x^5 and the radial coordinate of the cone for the two singularities.

The spectrum is given by

$$\begin{aligned}
\text{Vectors}: &\quad \mathrm{USp}(2k)\times\mathrm{USp}(2k-8)\times\mathrm{SU}(2)\times\mathrm{SO}(28)\\
\text{Hypers}: &\quad (\Box,\Box;\mathbf{1},\mathbf{1})+(\Box,\mathbf{1};\mathbf{2},\mathbf{1})+\frac{1}{2}(\Box,\mathbf{1};\mathbf{1},\mathbf{28})\\
\text{Hypers}': &\quad \frac{(k-4)}{2}\,(\mathbf{1},\mathbf{1};\mathbf{2},\mathbf{28})+(2k-3)\,(\mathbf{1},\mathbf{1};\mathbf{1},\mathbf{1})\\
\text{Tensors}: &\quad 1\cdot\mathbf{T}\,.
\end{aligned}\tag{2.14}$$

The second line is the group theory decomposition of the hypermultiplet content in (2.10), while the hypers' in the second indicate 'opposite chirality' hypermultiplets. We recall that our use of susy jargon is merely for convenience, cf. footnote 7.

In analogy with section 2.2.2, the above compactification admits a running dynamical cobordism solution microscopically given by the flat space solution regarded as a cone. The physics at the origin is the chirality changing phase transition described above, namely the transformation of the dynamical tensor multiplet of a pointlike instanton at $\mathbf{C}^2/\mathbf{Z}_2$ into a set of hypermultiplets associated to their fattening into a gauge instanton. Hence the dynamical cobordism provides a boundary configuration for the 6d chiral theory (2.14).

We note that, even though the 6d theory has a highly non-supersymmetric, the final running solution describing the dynamical cobordism is supersymmetric, as it secretly corresponds to the system of D5-branes at an orbifold of flat space. Essentially, the non-trivial spacetime gradients in the non-compact dimensions play a crucial role in making the running solution supersymmetric, but neither the starting sphere compactification, nor the transition point at the tip of the cone, are supersymmetric theories when regarded as static ones. The fact that the solutions display a symmetry which is not manifest in the lower-dimensional theory is related to the absence of a separation of scales, in the effective theory, between the kinetic energy in the gradients and the energy scale associated to the size of the internal dimension. Hence it requires the description of the configuration in the full 10d theory. The fact that dynamical cobordism solutions may enjoy more supersymmetry than the effective theory is familiar from several other examples, see e.g. [84].

We again emphasize that this construction technique generalizes straightforwardly to other chirality changing phase transitions of 6d $\mathcal{N}=1$ theories, and leave further examples for the interested reader.

2.4 Relation to SymTFTs

In this section we would like to highlight an interesting connection. We have exploited the cone construction to regard interesting phenomena occurring in a region localized in a $(d-1)$-dimensional subspace of spacetime in terms of the evolution in the d-dimensional theory obtained by compactification on the angular manifold around it, namely on the base of the cone describing its transverse space. This technique has been applied, at the topological level, in a different context related to generalized symmetries in quantum field theory and string theory (see [61–67] for reviews), as follows.

For a $(d-1)$-dimensional field theory (possibly coupled to gravity), the set of generalized symmetry generators and of generalized charged operators can be encoded as the set of topological operators in a d-dimensional gapped topological field theory, known as the SymTFT (or, more generally, if some degree of non-topological sectors is allowed, Symmetry Theory). The SymTFT is given by a d-dimensional sandwich with two boundaries separated by an interval, one describing the local degrees of freedom of the original $(d-1)$-dimensional theory (referred to as *relative* theory, in the sense of [85]), and a second one providing the gapped topological boundary conditions for the SymTFT fields. The actual (or *absolute*) theory, including the global topological information, is recovered by collapsing the SymTFT interval.

For $(d-1)$-dimensional theories which can be constructed as localized sectors in string theory or M-theory, a useful tool to derive the corresponding d-dimensional SymTFT [60] is to regard the transverse space as a cone, and to perform the dimensional reduction of the topological sector of the 10d string theory or 11d M-theory over the base of the cone.[8] The resulting d-dimensional topological field theory is the SymTFT, with the physical theory realized at the tip of the cone, and the topological boundary given by the asymptotic boundary conditions at infinity in the cone. Moreover, the different topological operators are realized as (the topological sector of) different branes of the compactification; specifically, generalized symmetry operators correspond to branes at infinity, parallel to the boundaries, while charged topological defects arise from branes stretching in the radial direction of the cone.

It is clear that our Cone Constructions are based on a similar viewpoint, and in particular they should be closely related if our Cone Construction is truncated to its topological sector. In this perspective, in our above examples the $(d-1)$-dimensional physical theory at the tip of the cone corresponds to the boundary of the relevant brane (such as the open heterotic string or the 5-branes), while the SymTFT is the topological sector of the 10d string theory compactified on the corresponding sphere, with the corresponding fluxes, branes and any other ingredients.

Specifically, our construction shows that the SymTFT of the open heterotic string boundary in section 2.1 is the topological sector of the compactification of 10d heterotic string on $\mathbf{S}^8$ with one explicit fundamental string at a point and -1 units of gauge 'flux' $\int_{\mathbf{S}^8} \text{tr}\, F^4 = -1$. This is actually related to the comment in section 2.1.2 about [59], where an explicit worldsheet description of this configuration around the 0-brane, in the near horizon limit was provided. It would be interesting to explore the topological structures of this cone construction and possibly uncover novel features about the boundary of the open heterotic string.

Similarly, for the open heterotic NS5-brane in flat space, in section 2.2.2, the boundary of the NS5-brane is a 4-brane, whose SymTFT is the topological sector of the compactification of 10d heterotic string on $\mathbf{S}^4$ with one explicit NS5-brane and -1 units of instanton charge $\int_{\mathbf{S}^4} \text{tr}\, F^2 = -1$. In this case, the 4-brane solution presented in [59] actually corresponds to a system where two such chirality changing phase transitions are combined, as emphasized in section 2.2.3, and the asymptotic cone contains no explicit NS5-branes, but a pair of opposite charge instantons under the two E_8 gauge factors (or rather, SU(2) subgroups thereof). In any event, we expect that the topological structure of the chirality changing phase transition of these NS5-branes (and possibly those from singular geometries) can be unravelled using the SymTFT constructions we have described.

[8]The approach is clearly inspired in the similar role played by cones in holography [71, 72], as pioneered in the generalized symmetries of 4d $\mathcal{N}=4$ SU(N) theory using holography in [86].

One general observation about the d-dimensional theories arising from compactification on the base, is that, when the system describes a boundary configuration for a genuine chiral theory, namely when we have a genuine Chiral Cone Construction, the d-dimensional theory is not trivially gappable. This is simply because the d-dimensional chiral theory is part of the massless spectrum of the theory after compactification, and being chiral but non-anomalous, cannot be trivially gapped.

Hence the use of the familiar term SymTFT, which assumes a gapped topological field theory, involves a slight abuse of language. Indeed, we should rather speak about a Symmetry Theory, which contains some non-topological degrees of freedom, yet whose topological sector is relevant to the generalized symmetries and its operators. The need to generalized beyond the naive concept of SymTFT has occurred in various contexts, leading to nover setups such as SymTrees [87], Nested SymTFTs [88] or SymTFT Fans [43]. In particular, the presence of branes stretching in the radial direction in the cone and carrying the non-topological degrees of freedom associated to a chiral sector, suggests an interesting connection with the flavour branes and their realization in Symmetry Theories in [88]. Hence, it is an interesting question how to deal with the Symmetry Theory associated to these systems. We leave this interesting question for the future, and now turn to the study of 4d theories.

3 Boundaries for 4d chiral theories

In order to construct boundary configurations for 4d chiral theories, one may proceed by considering the 4d version of chirality changing phases transitions, which has been considered for heterotic compactifications on CY3 [76] (see also [77]). We will however focus on alternative approaches, realized in terms of D-branes.

In this section we develop several strategies to use the cone construction over chiral D-brane models to build boundary configurations for 4d chiral theories. After an initial discussion of cone constructions over intersecting D6-branes, we focus on systems of D3-branes at singularities, and obtain large classes of working models in this last setup.

3.1 Cones over D6-brane intersections

In this section we study configurations of intersecting D6-branes, such that the 4d chiral fermions at their intersection are defined on a half-space, and carry out the cone construction around their 3d boundary. The specific example will eventually lead to a non-chiral theory upon this cone construction, albeit in a non-trivial and interesting way. It will thus serve as stepping stone in the construction of successful classes of models in coming sections.

There are two key ingredients in the construction of the 4d chiral fermion defined on a defect with boundary, which we study in turn.

3.1.1 4d chiral fermions from intersecting D6-branes

In flat 10d type IIA theory a configuration of two stacks of N_1 and N_2 D6-branes intersecting over a 4d subspace of their worldvolumes, leads to a 4d chiral fermion transforming in the bifundamental[9] $(\Box_1, \overline{\Box}_2)$ [89]. This is the setup which underlies model building via intersecting D6-brane worlds [90–92] (see [93] for review and references).

More explicitly, let the N_1 D6$_1$-branes span the directions 0123 and a 3-plane Π_1 in the remaining $\mathbf{R}^6$, and let the N_2 D6$_2$-branes span 0123 and a 3-plane Π_2 in the remaining $\mathbf{R}^6$. Even more explicitly, consider the SO(6) rotation in $\mathbf{R}^6$ that takes Π_1 to Π_2, and changes the basis of coordinates in spacetime so that the rotation is block diagonal. In this basis, the $\mathbf{R}^6$ splits into $\mathbf{R}^2 \times \mathbf{R}^2 \times \mathbf{R}^2$, and the 3-planes spanned by the D6-branes look like the product of three real lines in the three 2-planes. Let us denote θ_i the rotation angle that takes the line of the D6$_1$-branes to that of the D6$_2$-branes in the i^{th} 2-plane. The configuration preserves 4 susys ($\mathcal{N} = 1$ in the 4d intersection) if the SO(6) rotation is in SU(3), in other words

$$\theta_1 \pm \theta_2 \pm \theta_3 = 0 \text{ mod } 2\pi\,. \tag{3.1}$$

The open string spectrum at the intersection is a 4d chiral fermion in the bifundamental representation $(\Box_1, \overline{\Box}_2)$ of the $\mathrm{U}(N_1) \times \mathrm{U}(N_2)$ on the D6-branes. In the susy SU(3) case, there are also massless complex scalars that complete the spectrum to a 4d $\mathcal{N} = 1$ chiral multiplet.

In cases where the amount of supersymmetry is not important (e.g. topological aspects), we will use a simple example of 3-planes, and take e.g. the D6$_1$-branes to span the directions 0123456, and the D6$_2$-branes to span the directions 0123789. In this case, in the $\mathbf{R}^2$'s parametrized by 47, 58, 69, respectively, the D6-branes are at angles $\theta_i = \pi/2$, which does not preserve susy. But the key topological ingredients, e.g. the presence of the localized 4d chiral fermion in the $(\Box_1, \overline{\Box}_2)$ are still present.

Notice that the localized anomaly of the above 4d fermion in the $(\Box_1, \overline{\Box}_2)$ is cancelled by an anomaly inflow mechanism [94]. The consistency of inflows is the analogue in this setup of the conservation of fluxes for open branes in previous sections.

3.1.2 Open D6-branes

In order to define boundaries for the above defect supporting the 4d bifundamental fermion, we intend to put boundaries in the above configurations of intersecting D6-branes. This first requires the discussion of how to define boundaries for a single isolated stack of D6-branes. In particular we explore D6-branes ending on NS5-branes (for D6-branes ending on D8-branes, see footnote 10).

As discussed in [95], in type IIA in the presence of a Romans mass m, an NS5-brane must emit m semi-infinite D6-branes. Alternatively, in the presence of a Romans mass m, a set of m D6-brane can end on one NS5-brane. A simple way to derive this, in analogy with the argument in sections 2.1.1, 2.2.2, is the following. We demand invariance of the action of the configuration under a gauge transformation of the RR 7-form $C_7 \to C_7 + d\Lambda_6$. The relevant pieces in the action are

$$S_{C_7} = \int_{\Sigma_7} C_7 + m \int_{10d} C_7 H_3\,. \tag{3.2}$$

[9]Recall that the chirality of the fermion (or equivalently, the fact of getting this bifundamental vs its conjugate) is determined by the relative orientation defined by the two intersecting 3-planes spanned by the D6-brane stacks (besides the Poincaré invariant 4d).

The first term is the coupling of a D6-brane spanning a submanifold Σ_7, and the second is a topological coupling of Romans massive IIA theory. Its gauge variation is

$$\delta_{\Lambda_6} S_{C_7} = \int_{\Sigma_7} d\Lambda_6 + m \int_{10d} d\Lambda_6 H_3 = \int_{\partial\Sigma_7} \Lambda_6 - m \int_{10d} \Lambda_6 dH_3 \,. \tag{3.3}$$

So, if the D6-branes end on an NS5-brane, we have $dH_3 = \delta_4(\partial\Sigma_7)$, with $\delta_4(\partial\Sigma_7)$ a bump form Poincaré dual to $\partial\Sigma_7$, and hence

$$\delta_{\Lambda_6} S_{C_7} = \int_{\partial\Sigma_7} \Lambda_6 - m \int_{10d} \Lambda_6 \delta_4(\partial\Sigma_7) = 0 \,. \tag{3.4}$$

An equivalent derivation is that the 10d coupling mH_3C_7 turns the U(1) gauge symmetry of C_7 into a discrete $\mathbf{Z}_m$ symmetry, so that the electrically charged objects (D6-branes) are conserved only modulo m [96]. The NS5-brane is the operator which must be dressed with electric D6-brane operators to be gauge invariant. Similarly, the emission effect can be regarded as a Freed-Witten anomaly on the NS5-brane [97, 98] (see also [96]), or equivalently from a D6-brane creation effect upon bringing m D8-branes from infinity and crossing them over the NS5-branes as domain walls to introduce the Romans mass.

3.1.3 4d chiral fermion on an intersection with boundary

Consider an intersecting brane configuration, with a stack of N_1 D6$_1$-branes along 0123 456, and a second one of N_2 D6$_2$-branes along 012 789, the latter of semi-infinite extent in the direction 3 with the D6-branes ending on one NS5-brane located at $x^3 = 0$ and spanning 012 789. For this to be consistent we turn on a Romans mass $m = N_2$, as discussed in the previous section. The D6$_1$-branes are instead taken infinite (see appendix A for the case of both kinds of D6-branes being semi-infinite).

The spectrum gives 7d gauge fields on the D6$_1$-branes, 7d gauge fields on the half-space on the D6$_2$-branes, and a 4d chiral fermion on a half-space corresponding to the intersection, see figure 2a.

Although it seems that we are harmlessly combining the two ingredients introduced in the previous section, it is clear that the above configuration cannot be complete, as can be argued in several ways. For instance, there is no consistent inflow mechanism, since the inflow from the D6$_1$-branes to the 4d intersection must suddenly stop when the intersection ceases to exist. Related to this, in the open heterotic string example we saw that chiral fermions reaching a boundary must outflow in some way, which is not obvious in the above description. Finally, if we turn the geometry into a cone, the missing fermions degrees of freedom imply we get an effective anomalous theory.

For illustration, let us be more explicit about this last argument, by performing the cone construction, depicted in figure 2b. We regard the $\mathbf{R}^7$ spanned by 3456789 as a cone over $\mathbf{S}^6$. The D6$_1$-branes span the directions 3456, so they span a cone over an $\mathbf{S}^3 \subset \mathbf{S}^6$ defined by $(x^3)^2 + (x^4)^2 + (x^5)^2 + (x^6)^2 = R^2$. The NS5-brane spans the direction 789, namely a cone over an $\mathbf{S}^2 \subset \mathbf{S}^6$ defined by $(x^7)^2 + (x^8)^2 + (x^9)^2 = R^2$. The $\mathbf{S}^2$ and $\mathbf{S}^3$ do not intersect but are linked on $\mathbf{S}^6$. The D6$_2$-branes span the direction 789 and are semi-infinite in 3 (because they end on the NS5-brane), so the span a cone over $(x^3)^2 + (x^7)^2 + (x^8)^2 + (x^9)^2 = R^2$ with $x^3 > 0$, namely a half-$\mathbf{S}^3$ bounded by the $\mathbf{S}^2$ wrapped by the NS5-brane. The half-$\mathbf{S}^3$ of the D6$_2$-branes

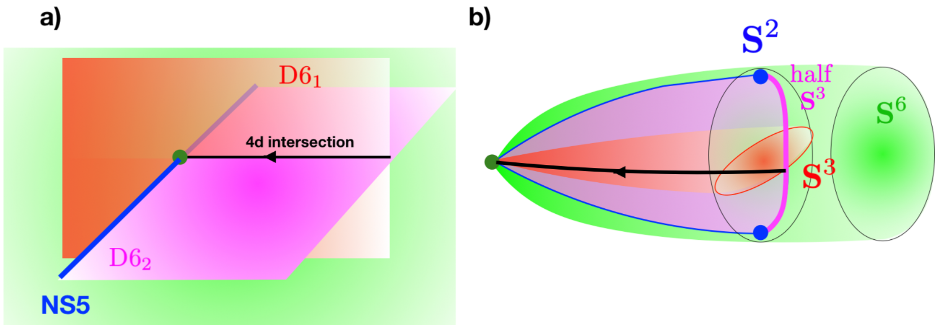

Figure 2. Stack of infinite D6-branes intersecting a stack of semi-infinite D6-branes (ending on an NS5-brane) over 4d half-space in flat 10d space. a) The view in flat space. b) The cone perspective: the green cone represents the flat 7d space spanned by 3456789, regarded as a cone over $\mathbf{S}^6$. The red area is the D6$_1$-brane worldvolume along the 4d space spanning 3456, regarded as a cone over $\mathbf{S}^3_1$; the magenta area is the semi-infinite D6$_2$-brane, along 3789, which is a cone over a half $\mathbf{S}^3_2$; it ends on the NS5-brane, in blue, which spans 789, regarded as a cone over $\mathbf{S}^2$. The $\mathbf{S}^3_1$ and the half $\mathbf{S}^3_2$ intersect at one point on the $\mathbf{S}^6$, so the intersection spans the black line along the radial direction. For clarity, the cone over $\mathbf{S}^6$ has been extended slightly longer than the other cones.

intersects the $\mathbf{S}^3$ of the D6$_1$-branes at one point, $x^4 = x^5 = x^6 = x^7 = x^8 = x^9 = 0$, $x^3 = R$; the cone over this point is the direction supporting the 4d fermion over the semi-infinite radial direction.

So in the compactification of the 10d theory on $\mathbf{S}^6$ we have D6$_1$-branes wrapped on an $\mathbf{S}^3_1$ and D6$_2$-branes wrapped on a half-$\mathbf{S}^3_2$ ending on an NS5-brane wrapped on the $\mathbf{S}^2$ at the equator of $\mathbf{S}^3_2$. The two sets of D6-branes intersect at one point in $\mathbf{S}^6$ leading to one 4d chiral fermion in the $(\square_1, \overline{\square}_2)$. Hence, the resulting 4d theory is anomalous, making it manifest that we are missing some degrees of freedom.

3.1.4 The missing D4-branes

The appearance of anomalies suggests that the configuration in the previous section must be inconsistent as it stands. In fact, it is easy to see why, and to solve the problem.

Consider the intersection of the NS5-brane and the D6$_1$-branes, namely the locus parametrized by 012 and located at the origin in 3456789. This locus is real codimension 4 in the D6$_1$-brane worldvolume. Then, in the D6$_1$-brane worldvolume, we can take an $\mathbf{S}^3$ which surrounds the NS5-brane , namely the angular part of the $\mathbf{R}^4$ spanning 3456. Since the NS5-brane is magnetically charged under the NSNS 2-form

$$\int_{\mathbf{S}^3} H_3 = 1\,. \tag{3.5}$$

So, if we excise the location of the NS5-brane intersection from the D6$_1$-brane worldvolume, we have a non-trivial 3-cycle on which there is one unit of H_3 flux, leading to a Freed-Witten inconsistency [97, 98]. This forces each of the D6$_1$-branes to emit one D4-brane, spanning 012 times the radial direction in 3456 times one direction away from the D6$_1$-brane worldvolume.

Conversely, the flux created by the D6$_1$-branes implies a Freed-Witten inconsistency on the NS5-brane, as follows. The intersection of the D6$_1$-brane with the NS5-brane is codimension 3 in the NS5-brane worldvolume, hence an $\mathbf{S}^2$ surrounding the D6$_1$-brane in

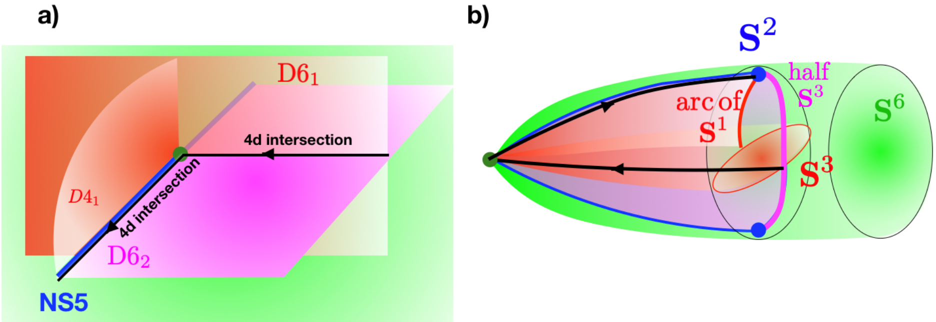

Figure 3. Stack of infinite D6$_1$-branes intersecting a stack of semi-infinite D6$_2$-branes (ending on an NS5-brane) over 4d half-space in flat 10d space, with the D4-branes suspended between the infinite D6$_1$-branes and the NS5-brane. a) The view in flat space. b) The cone perspective. Comparing with figure 2, the addition of the D4-branes lead to additional intersections supporting 4d chiral fermions in the bifundamental representation, so that this spectrum is now defined on an infinite line.

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the NS5-brane worldvolume (namely the angular part in the $\mathbf{R}^3$ spanned by 789) supports a RR 2-form field strength flux

$$\int_{\mathbf{S}^2} F_2 = N_1 \,. \tag{3.6}$$

This forces the NS5-brane to emit N_1 D4-branes, spanning 012 times the radial direction in 789 times a direction transverse to the NS5-brane worldvolume.

Overall, and keeping track of the orientations, we end up with N_1 D4-branes stretching between the D6$_1$-branes and the NS5-brane, see figure 3a. Note that the D4-branes indeed span a radial direction away from the intersection of the D6$_1$-branes and the NS5-branes, both on the worldvolume of the D6$_1$-branes and of the NS5-brane, and one direction transverse to the D6$_1$-brane worldvolume and one direction transverse to the NS5-brane worldvolume. In the cone construction, the wedge spanned by the D4-branes is a cone over an arc of $\mathbf{S}^1$stretching between (a point in) the $\mathbf{S}^3$ of the D6$_1$ and (a point in) the $\mathbf{S}^2$ of the NS5-brane, see figure 3b.

Consider now the implications of the D4-branes for the spectrum of the theory, starting in the flat space configuration, see figure 3a. A crucial observation is that when the D4-brane ends on the D6-brane, their gauge groups are identified. This is analogous to the familiar statement that D3-branes ending on D5-branes have Dirichlet boundary conditions for the vector multiplets [99]. To emphasize this, in figure 3 we have labeled the D4-branes with a subindex 1, and we have colored them in red, just like the D6$_1$-branes. This also agrees well with the fact that there is one D4-brane per D6$_1$-brane, so the two stacks collectively carry both a single U(N_1).

Hence, in the sector of open strings stretching between the D4-branes and the D6$_2$-branes, we obtain matter in the bifundamental of the SU(N_2) on the D6$_2$-branes and the SU(N_1) on the D6$_1$/D4$_1$-branes. We note that the spectrum at the intersection of D4-branes and D6-branes ending on the same NS5-brane was shown in [95] to indeed correspond to this kind of 4d chiral bifundamental $(\square_1, \overline{\square}_2)$. This is precisely the content we had on the 4d intersection of the D6$_1$- and D6$_2$-branes, so we have indicated it figure 3 with the same black arrow line. Note that the new black line from the D6$_2$-D4$_1$ intersection continues the formerly semi-infinite black line from the $D6_1$-$D6_2$ intersection. This implements the analogue of the fate of worldsheet fermions in the open heterotic string in section 2.1.1: the 4d chiral fermion at the $D6_1$-$D6_2$ intersection reaches the boundary of its support, but it is carried away by some additional degrees of freedom, in this case the $D6_2$-$D4_1$ intersection.

This picture makes the anomaly inflow consistent. The inflow from the $D6_2$-brane bulk into the intersection with the $D6_1$-brane continues as an inflow towards the intersection with the $D4_1$-brane. Similarly, the inflow from the $D6_1$-brane bulk into the intersection with the $D6_2$-brane turns into an inflow from the $D4_1$-brane into the intersection with the $D6_2$-brane. This is all consistent with the interpretation of an inflow for a continuous $D6_2$-$D6_1/D4_1$ intersection.

Turning now to the cone construction and keeping track of the orientations, see figure 3b, the compactification of the 10d theory on $\mathbf{S}^6$ now produces a 4d theory with one chiral fermion in the $(\square_1, \overline{\square}_2)$, from the $D6_1$-$D6_2$ intersection, and one in the $(\overline{\square}_1, \square_2)$, from the $D6_2$-$D4_1$ intersection. The complete spectrum is therefore non-chiral, again in analogy with the cone construction for the open heterotic string in section 2.1.2.

The cone construction hence describes an interesting dynamical cobordism in a theory with a non-trivial sector of 4d fermions, but ultimately a non-chiral one. In appendix A we quickly describe other variants with a similar set of ingredients,[10] again leading to non-chiral theories upon the cone construction. These examples illustrate that getting chirality in the cone construction is thus highly non-trivial. In the next section we will identify a key property underlying the non-chirality in these examples, and will overcome it and obtain a new large class of constructions of boundary configurations for genuine 4d chiral theories.

3.2 Chiral cones from branes at singularities

A main reason why the above constructions ultimately lead to non-chiral theories upon the cone construction is that the gauge groups extend in more than one dimensions away from the tip of the cone (i.e. the boundary of the 4d fermion). This implies that the two copies of chiral fermions arising over the base of the cone are charged under the same gauge factors and lead to a non-chiral configuration.

Fortunately, there are several ways to obtain 4d chiral fermions from D-branes (see [93] for a review). In addition to intersecting D6-branes (or their mirror realization, magnetized-branes), it can be achieved using D3-branes at singularities [100–103] (see [104] for model building applications and [93] for review). Hence, it is natural to resolve the above problems by using 4d chiral fermions from D-branes of lower dimension, specifically D3-branes at singularities, as we explore in this section. Incidentally, the resulting construction bears some analogies with the 6d setup discussed in section 2.2.3.

[10] It is also possible to define semi-infinite D6-branes by allowing them to end on D8-branes. In general, such configurations eventually fail to produce boundary configurations for chiral theories because the gauge groups on D6-branes are linked to those of the D8-brane on which they end. Hence, a configuration of intersecting D6-branes ending on a D8-brane fails to produce chirality because all gauge factors become identified, collapsing the bifundamental fermion onto some non-chiral representation.

3.2.1 Example: cone over the dP_0 theory

For concreteness, we illustrate the main construction in an explicit example, known as the dP_0 theory, leaving the general construction for section 3.2.2. To build the dP_0 theory, consider a stack of D3-branes at the tip of a $\mathbf{C}^3/\mathbf{Z}_3$ orbifold singularity, with the generator $\theta \in \mathbf{Z}_3$ acting on the $\mathbf{C}^3$ coordinates as

$$\theta : (z_1, z_2, z_3) \to (e^{2\pi i/3} z_1, e^{2\pi i/3} z_2, e^{-4\pi i/3} z_3)\,. \tag{3.7}$$

We choose the orbifold action on the Chan-Paton indices in N copies of the regular representation

$$\gamma_\theta = \mathrm{diag}(\mathbf{1}_N, e^{2\pi i/3}\mathbf{1}_N, e^{4\pi i/3}\mathbf{1}_N)\,. \tag{3.8}$$

The resulting 4d $\mathcal{N} = 1$ gauge theory on the D3-branes[11] has gauge group and chiral multiplet content given by

$$\begin{gathered} \mathrm{SU}(N)_0 \times \mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \\ 3\,[\,(\square_0, \overline{\square}_1, \mathbf{1}) + (\mathbf{1}, \square_1, \overline{\square}_2) + (\overline{\square}_0, \mathbf{1}, \square_2)\,]\,, \end{gathered} \tag{3.9}$$

and there is a cubic superpotential which we skip for the moment. Note that the cubic anomalies cancel, as expected as a consequence of twisted RR tadpole cancellation [106], automatically satisfied for the regular representation.

Let us now perform a cone construction using the above configuration. Consider $\mathbf{C}^3/\mathbf{Z}_3 \times \mathbf{R}$, where the $\mathbf{R}$ corresponds to one of the directions along the D3-branes, say x^3. The full 7d space can be regarded as $\mathbf{R}^7$ modded out by a $\mathbf{Z}_3$ quotient acting on the first 6 real coordinates and leaving x^3 invariant. This can be regarded as a real cone over a 6d base $\mathbf{S}^6/\mathbf{Z}_3$, where the $\mathbf{S}^6$ (of radius R) is described as

$$|z_1|^2 + |z_2|^2 + |z_3|^2 + (x^3)^2 = R^2\,, \tag{3.10}$$

and the generator of $\mathbf{Z}_3$ acts on it as in (3.7). Hence, we can regard the configuration as a 4d compactification of type IIB theory on $\mathbf{S}^6/\mathbf{Z}_3$, in which the size of the internal space runs along a 4d spacetime coordinate. Namely, it corresponds to a dynamical cobordism in the spirit in [6, 7, 19, 20], a connection which we make more explicit in a related class of constructions in section 3.4.

Let us now give some more details about the content of this 4d theory. On the $\mathbf{S}^6$ there are two fixed points corresponding to $z_i = 0$, $x^3 = \pm R$, at each of which there is a system of D3-branes at a local $\mathbf{C}^3/\mathbf{Z}_3$ singularity, leading to the 4d chiral spectrum (3.9). Due to the different orientation (since increasing the radial coordinate corresponds to increasing or decreasing x^3 at the two points, respectively), what we have is a system of D3-branes and anti-D3-branes. The 4d chiral spectrum arising at this point is thus again given by a copy of (3.9), and a second copy with 4d fermions of the opposite chirality [104] (recall that we abuse language with the use of susy jargon, cf. footnote 7). In contrast with the

[11] We are removing the U(1) factors as they are made massive by Stückelberg couplings, which in this case are required for the 4d Green-Schwarz mechanism [105].

previous section, the gauge groups now arising at the two points are two independent sets, and therefore the fermions at the two points are charged in conjugate bifundamentals but under *different* sets. Hence, the resulting 4d theory is chiral, and the cone construction, regarded as dynamical cobordism in the 4d theory, provides a boundary configuration for a genuinely 4d chiral theory.

The fact that the two stacks correspond to D3-brane / anti-D3-brane pairs suggests that the dynamical mechanism that explains the gapping of the 4d chiral degrees of freedom corresponds to a brane-antibrane annihilation process. As expected, this is beyond the regime admitting a weakly coupled description, or even a field theory description, since open string tachyon condensation can be properly described only in string field theory. Let us note that however, in analogy with the 6d example in section 2.2.3, it is possible to provide a simpler effective description of the resulting boundary conditions. Indeed, the boundary conditions at the tip of the cone amount to an exchange of left- and right chiralities, with a simultaneous exchange of the two singularities and their corresponding gauge sectors, i.e. a $\mathbf{Z}_2$ outer automorphism symmetry of the theory, which is a symmetry of the underlying geometry of the base of the cone. Overall, the mechanism is a close cousin of that in the bubble of nothing in [17], when regarded from the 10d perspective, in which identical sectors preserving different supersymmetries annihilate against each other at the ETW brane.

Let us also mention that, although the 4d theory under discussion is highly non-supersymmetric, the final running solution describing the dynamical cobordism is supersymmetric, as it secretly corresponds to the system of D3-branes at an orbifold of flat space. Similar to the discussion in section 2.3, the non-trivial spacetime gradients in the non-compact dimensions make the running solution supersymmetric. This is not captured by the effective lower-dimensional theory, because the energy scale of the gradients is comparable to that set by the size of the internal dimension, so it is manifest only in the full theory. The fact that dynamical cobordism solutions may enjoy more supersymmetry than the effective theory is familiar from several other examples, see e.g. [84].

3.2.2 Generalization

The above example admits a straightforward generalization to a large class of configurations. As discussed in [71, 72], there are large classes of CY3 singularities $\mathbf{X}_6$ built as cones over 5d geometries $\mathbf{Y}_5$, for which the corresponding gauge theory on D3-brane probes can be identified. In particular, for toric singularities $\mathbf{X}_6$ there is a specific dictionary via dimer diagrams (a.k.a. brane tilings) [107–110](see [111] for a review), allowing to read out the gauge theory from geometric data, and vice versa, which has been extensively exploited in holography [108] (see also e.g. [112–115]) and model building, see e.g. [116, 117]. We will discuss this specific dictionary in section 3.3.2, but it is not necessary in this section, where we keep the discussion general.

We hence consider a system of D3-branes at a (not necessarily toric) CY3 singularity $\mathbf{X}_6$, leading to a 4d chiral gauge theory with group G (a product of unitary factors) and 4d chiral fermions in a representation $\mathcal{R}$. Let us now consider the 7d geometry $\mathbf{X}_7 = \mathbf{X}_6 \times \mathbf{R}$, with $\mathbf{R}$ parameterized by one of the coordinates along the D3-branes, say x^3. Let us write the metric as

$$ds_7^2 = (dx^3)^2 + dr'^2 + r'^2 ds_{\mathbf{Y}_5}^2 \,, \tag{3.11}$$

with $r' > 0$. Defining polar coordinates in the (r', x^3) 2-plane, i.e. $r' = r\cos\theta$, $x^3 = r\sin\theta$, we have

$$ds_7^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta ds_{\mathbf{Y}_5}^2)\,, \tag{3.12}$$

which describes the 7d geometry as a real cone, with radial coordinate r and base geometry $\mathbf{Y}_6$, given by the suspension of $\mathbf{Y}_5$, i.e. the fibration of $\mathbf{Y}_5$ over a segment, parametrized by $\theta \in [-\pi/2, \pi/2]$, with the fiber collapsed to a point over the two endpoints. The locus $r' = 0$ and arbitrary x^3 is a real line of singularities locally identical to $\mathbf{X}_6$, located at $\theta = \pm\pi/2$ and arbitrary r in polar coordinates in $\mathbf{X}_7$.

Hence, we have a 4d theory (in the directions 012 and r) obtained by compactification of type IIB theory on $\mathbf{Y}_6$, with D3-branes and antibranes located at the points $\theta = \pm\pi/2$, respectively, in the internal space. There is a 4d gauge group $G \times G$, and 4d chiral fermions in the representation $(\mathcal{R}, \mathbf{1}) + (\mathbf{1}, \mathcal{R})$. This leads to a large class of 4d chiral theories for which the above construction produces boundary configurations described as dynamical cobordisms to nothing.

It is interesting to point out that the boundary condition effectively exploits a combination of chirality flip and the $\mathbf{Z}_2$ outer automorphism exchanging the two gauge theories. This resembles the behaviour in the bubble of nothing in [17], when regarded from the 10d perspective (it is also reminiscent of the folding trick used to define boundary states in 2d theories).

Despite its appeal, the explicit presence of branes and antibranes in the configuration, equivalently of two copies of the gauge sector (with opposite chiralities) in the 4d theory, makes this construction less enticing. In the next section we will present a variation, which improves on this respect.

3.3 Boundaries from $\mathbf{Z}_2$ quotients

In this section we build on the construction in the previous section to obtain new classes of boundary configurations for chiral 4d theories. They are inspired in $\mathbf{Z}_2$ quotients used in the construction of barely G_2 holonomy spaces [55–57], which we review next.

3.3.1 D6-branes at G_2 holonomy 7d geometries

Given a CY3 $\mathbf{X}_6$, which can be compact or not in this general discussion, we consider the quotient $\mathbf{X}_7 = (\mathbf{X}_6 \times \mathbf{R})/\mathbf{Z}_2$, with the generator $R \in \mathbf{Z}_2$ acting as $x^3 \to -x^3$ on the coordinate parametrizing $\mathbf{R}$, and as an antiholomorphic action on $\mathbf{X}_6$. Specifically, the action on the Kähler form J and holomorphic 3-form Ω are

$$R(J) = -J, \quad R(\Omega) = \overline{\Omega}\,. \tag{3.13}$$

The resulting $\mathbf{X}_7$ is a 7d barely G_2 holonomy space with covariantly constant 3-form

$$\varphi_3 = J dx^3 + \mathrm{Re}\,(\Omega)\,, \tag{3.14}$$

which is clearly invariant under the action of R. The term *barely* reflects the fact that the actual holonomy is an $\mathrm{SU}(3) \ltimes \mathbf{Z}_2$ subgroup of G_2. This kind of construction has been exploited in the M-theory lifts of type IIA configurations in [118].

We are interested to consider type IIA models on $\mathbf{X}_6$ supporting 4d chiral fermions. So we consider stacks of N_a D6-branes wrapped on special lagrangian 3-cycles Π_a of $\mathbf{X}_6$, corresponding to intersecting brane models [78, 90–92, 119], see [93, 120] for review. Note that, although in the compact setup these models are non-supersymmetric unless O6-planes are introduced [78, 119], for non-compact CY threefolds the additional freedom in RR tadpole cancellation allows for supersymmetric models with D6-branes wrapped on compact 3-cycles [121], so we focus on the latter setup. The supersymmetric 3-cycles are defined by the condition that they satisfy the special lagrangian conditions

$$J\big|_{\Pi_a} = 0, \quad \mathrm{Im}\,(\Omega)\big|_{\Pi_a} = 0\,. \tag{3.15}$$

Equivalently, the 3-cycles are calibrated with respect to the 3-form $\mathrm{Re}\,(\Omega)$ [122, 123].

As is familiar, the 4d $\mathcal{N}=1$ spectrum contains, an $\mathrm{SU}(N_a)$ gauge groups on each D6-brane stack (the U(1) factors are generically massive due to Stückelberg couplings), and their intersections lead to a net number $I_{ab} = [\Pi_a]\cdot[\Pi_b]$ of chiral multiplets in the bifundamental $(\square_a, \overline{\square}_b)$ representation.

It is now easy to check that the above supersymmetric 3-cycle conditions in $\mathbf{X}_6$ are invariant under $\mathbf{Z}_2$ action R, so each individual 3-cycle Π_a is either invariant under R, or exchanged with another supersymmetric 3-cycle, denoted by $\Pi_{a'}$. Starting with a $\mathbf{Z}_2$ invariant set of D6-branes wrapped on such supersymmetric 3-cycles, they descend to D6-branes wrapped on supersymmetric coassociative 4-cycles in the G_2 geometry $\mathbf{X}_7$. Specifically, namely they are calibrated with respect to the 4-form $*_{7d}\varphi_3$.

Given one such D6-brane configuration, there is a spectrum of 4d chiral fermions localized on real lines (parametrized by x^3) in $\mathbf{X}_7$, as follows. In the covering space of the $\mathbf{Z}_2$ quotient, we have the 4d chiral theory described above. The effect of the $\mathbf{Z}_2$ action on this theory is as follows: a generic 3-cycle Π_a in $\mathbf{X}_6$, at a location x^3 in $\mathbf{R}$, is mapped to the image 3-cycle $\Pi_{a'}$ at the location $-x^3$ in $\mathbf{R}$, and such that the fundamental $\square_a$ is mapped to the *same* representation $\square_{a'}$, because this is an orbifold, rather than an orientifold, projection. Hence, at a point x^3 in $\mathbf{R}$, the effect of the $\mathbf{Z}_2$ projection is not felt locally, and the local 4d spectrum we obtain is as described in the previous paragraphs. However, at the image point $-x^3$ in the double cover, the degrees of freedom are not independent, but are a mere image of them. In particular notice that remarkably a 4d chiral multiplet Φ_{ab} in a bifundamental $(\square_a, \overline{\square}_b)$ of $\mathrm{SU}(N_a)\times\mathrm{SU}(N_b)$ at a location x^3 is related to a bifundamental chiral multiplet $\Phi_{b'a'}$ in the $(\overline{\square}_{a'}, \square_{b'})$ of the image group[12] $\mathrm{SU}(N_{a'})\times\mathrm{SU}(N_{b'})$ at the location $-x^3$, namely the $(\overline{\square}_a, \square_b)$ of $\mathrm{SU}(N_a)\times\mathrm{SU}(N_b)$. The two sets of degrees of freedom seem to be in different (in particular conjugate) representations of the gauge group, which would make the $\mathbf{Z}_2$ identification impossible. However, we should notice that the $\mathbf{Z}_2$ generator R acts on x^3 as a parity operation, thus flipping the 4d chirality of the corresponding fermion, and this precisely compensates the conjugation of the gauge representation. In other words, the identification by the full orbifold action is $\Phi_{ab}\leftrightarrow\overline{\Phi_{b'a'}}$, which implies the chirality flip and the conjugation of quantum numbers. Hence the action of the $\mathbf{Z}_2$ quotient is consistent and defines a consistent identification of degrees of freedom in the spectrum. In short, we get one copy of the 4d

[12]For groups mapped to themselves under the $\mathbf{Z}_2$ action, we postpone the discussion to the explicit examples in later sections.

chiral gauge theory in the $\mathbf{Z}_2$ quotient. This construction holds the key to the removal of the doubling of degrees of freedom encountered in section 3.2.

We would now like to obtain models of 4d chiral theories by taking local models of intersecting D6-branes on a non-compact Calabi-Yau $\mathbf{X}_6$, and carry out a 7d cone construction involving the extra x^3 coordinate. However, obtaining a global 7d cone is possible only if $\mathbf{X}_6$ is a cone itself. We present one particular explicit example in appendix B, based on a model in [68]. However, such explicit examples are scarce, due to the familiar difficulties to build special lagrangian 3-cycles in general CY3s. Hence, in the following section, we instead turn to the implementation of the above construction in systems of D3-branes at singularities, where the cone structure is built in from the beginning, and so it leads to a large class of explicit examples.

3.3.2 The type IIB picture

A large class of models of supersymmetric local intersecting D6-brane models [121], namely D6-branes on compact 3-cycles on non-compact CY3 geometries, can be obtained as the mirror of the systems of D3-branes at toric CY3 singularities, mentioned in section 3.2.2, which are efficiently studied using dimer diagrams (a.k.a. brane tilings) [107–110](see [111] for a review). In particular, the mirror map between D3-branes at toric CY3 singularities and local intersecting D6-brane models can be carried out systematically via the explicit map in [109]. This map allows to perform a construction similar to that in the previous section, but in systems of D3-branes at conical CY3 singularities. This setup will be best suited to subsequently perform a Chiral Cone construction and lead to boundary configurations for large classes of 4d chiral theories.

Consider a system of D3-branes at a CY3 toric singulary $\mathbf{X}_6$. We momentarily focus on regular D3-brane systems (i.e. all gauge factors have the same rank), although in later discussions we will allow for fractional branes (i.e. anomaly-free rank assignments for the different nodes). As explained, the 4d $\mathcal{N} = 1$ gauge theory is efficiently encoded in dimer diagrams, as we will make explicit in concrete examples in section 3.3.3.

The mirror geometry $\tilde{\mathbf{X}}_6$ is constructed as a base $\mathbf{C}$ parametrized by a coordinate z, over which we fiber a $\mathbf{C}^*$, with the fiber degenerating at $z = 0$, and a Riemann surface Σ, with various 1-cycles C_i degenerating at various points z_i on the base, as explained later. Namely, the geometry is described as

$$uv = z, \quad P(w_1, w_2) = z\,, \tag{3.16}$$

where u, v parametrize the $\mathbf{C}^*$ fiber, an the second equation describes Σ, with $P(w_1, w_2)$ the Newton polynomial of the toric geometry. The compact special lagrangian 3-cycles wrapped by the D6-branes mirror to the D3-branes in some node i of the quiver are obtained as follows: one takes a segment on the base joining $z = 0$ and the degeneration point z_i of some 1-cycle $C_i \subset \Sigma$, and fibers the $\mathbf{S}^1 \subset \mathbf{C}^*$ times $C_i \subset \Sigma$. The result is a set of topological 3-spheres $\mathbf{S}^3_i$, shown in [109] to lead to the intersections and worldsheet instantons to yield the spectrum and interactions of the original D3-brane theory. The mirror geometry is hence basically controlled by the geometry of the fiber Σ and its set of degenerating 1-cycles. The construction of this mirror Riemann surface and the 1-cycles wrapped by the D6-branes will be carried out in explicit examples using the procedure in [109].

One may thus carry out the $(\tilde{X}_6 \times \mathbf{R})/\mathbf{Z}_2$ quotient on the type IIA system with intersecting D6-branes, but this does not allow for a cone construction because $\tilde{X}_6$ is not conical. So the strategy is to return to the original picture of D3-branes at the cone $\mathbf{X}_6$, and consider now the space $(\mathbf{X}_6 \times \mathbf{R})/\mathbf{Z}_2$, where the generator $R \in \mathbf{Z}_2$ acts as $x^3 \to -x^3$ in $\mathbf{R}$ and as a $\mathbf{Z}_2$ involution on $\mathbf{X}_6$ (also denoted by R, with abuse of language) corresponding to the antiholomorphic action in the type IIA mirror geometry $\tilde{\mathbf{X}}_6$, just mentioned. The action of R on the gauge theory can be read as an action on the dimer diagram, via the explicit mirror map. As we will show in explicit examples, it also corresponds to an antiholomorphic $\mathbf{Z}_2$ action on $\mathbf{X}_6$, so that the full quotient preserves half of the supersymmetries. The resulting 7d space thus has G_2 holonomy, and presumably corresponds to the mirror of the type IIA G_2 manifold in the sense of [124]. The fixed loci are orbifold 5-planes, wrapped on special lagrangian 3-cycles in $\mathbf{X}_6$ and sitting at $x^3 = 0$, where they define a boundary configuration in the quotient.[13]

Compared with the models in section 3.2, the effective boundary condition for the 4d theory involves a chirality flip and a $\mathbf{Z}_2$ action on the gauge theory, such that the combined action is a symmetry. The $\mathbf{Z}_2$ action on the gauge theory can in general be a combination of inner and outer automorphisms of the different gauge factors.

The different $\mathbf{Z}_2$ actions on dimer diagrams were studied, in the context of orientifold quotients,[14] in [127], and correspond to reflections leaving fixed points or fixed lines in the dimer diagram. As will be clear from the examples in section 3.3.3, the antiholomorphic involutions in the type IIA mirror correspond to actions with fixed lines. This allows to efficiently describe the effect of the $\mathbf{Z}_2$ involution R in large classes of dimer gauge theories. We will present specific examples in later sections.

The construction in the IIB side allows for a cone construction because $\mathbf{X}_6$ is a cone, hence so is $(\mathbf{X}_6 \times \mathbf{R})/\mathbf{Z}_2$. As in section 3.2.2, we take the metric in $\mathbf{X}_6 \times \mathbf{R}$

$$ds_7^2 = (dx^3)^2 + dr'^2 + r'^2 ds^2_{\mathbf{Y}_5} \,, \tag{3.17}$$

where $\mathbf{X}_6$ is written as a real cone over the 5d base $\mathbf{Y}_5$. Using polar coordinates in the (r', x^3) 2-plane, i.e. $r' = r\cos\theta$, $x^3 = r\sin\theta$, we have

$$ds_7^2 = dr^2 + r^2(d\theta^2 + \cos^2\theta ds^2_{\mathbf{Y}_5}) \,, \tag{3.18}$$

which describes the 7d geometry as a real cone over the base geometry $\mathbf{Y}_6$, given by the suspension of $\mathbf{Y}_5$. The real line of singularities locally identical to $\mathbf{X}_6$ is the locus $r' = 0$ and arbitrary x^3, equivalently $\theta = \pm\pi/2$ and arbitrary r.

Performing the $\mathbf{Z}_2$ quotient, the coordinate r is invariant, while we have a non-trivial quotient $\theta \to -\theta$. This means that the quotient geometry is of the kind (3.18), with the restriction $\theta \in [0, \pi/2)$. We thus have a real cone over the 6d space given by a quotient of the suspension of $\mathbf{Y}_5$. The locus corresponding to the singularity $\mathbf{X}_6$ is now given by just $\theta = \pi/2$.

[13]These orbifold 5-planes are S-dual to configurations of $\mathrm{O5}^-$-planes with D5-branes on top [125], which have been exploited to define boundary configurations in compactifications [18] and in holographic setups [43, 126]. It would be interesting to explore further connections with these setups.

[14]The reason why orientifold actions appear as the relevant quotients in our context is because the orbifold includes a parity flip in the direction x^3, which acts on the fermions by conjugation of quantum numbers (equivalently, by a chirality flip, as befits to the definition of a boundary condition), an operation which, for actions preserving Poincaré invariance, arises only in orientifold quotients.

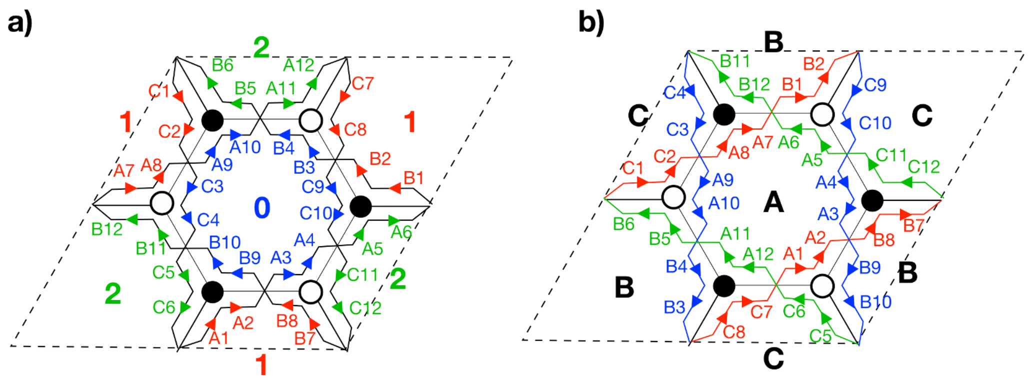

Figure 4. a) The dimer diagram for the dP_0 theory, with its set of zig-zag paths. b) The mirror Riemann surface obtained via the untwisting procedure in [109].

This implies that, in the 4d theory obtained by reducing on the 6d base $\mathbf{Y}_6$ of the cone, there is a single copy of the 4d $\mathcal{N} = 1$ gauge theory corresponding to the systems of D3-branes at the tip of a local $\mathbf{X}_6$ singularity. The cone construction for $\mathbf{X}_7$ is hence a Chiral Cone construction, providing a boundary configuration for this 4d $\mathcal{N} = 1$ gauge theory, coupled to gravity. The resulting 4d configuration describes a running solution in which the scalar corresponding to the $\mathbf{Y}_6$ size varies along the direction r. It corresponds to a dynamical cobordism in which at a finite spacetime distance point $r = 0$ the scalar blows up, the internal space shrinks to zero size and spacetime ends. This solution is discussed in more detail in section 3.4.

As already mentioned, one should regard the configuration as a local description of a possibly more involved global solution, which moderates the asymptotic growth of $\mathbf{Y}_6$ e.g. to a constant size, in analogy with Witten's bubble of nothing.

3.3.3 Examples

The $\mathbf{dP_0}$ theory. In order to illustrate the above construction in practice, we consider a few illustrative examples. Let us consider the dP_0 theory, which is obtained from D3-branes at a $\mathbf{C}^3/\mathbf{Z}_3$ singularity (i.e. a complex cone over $dP_0 = \mathbf{P}_2$), already appeared in section 3.2.1. Because this is an orbifold of flat space, the gauge theory can be determined using standard worldsheet techniques for the open string sectors. The geometry is toric, hence the theory has a dimer diagram description shown in figures 4a, 5a. The gauge theory is given by

$$\begin{gathered} \mathrm{SU}(N)_0 \times \mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \\ 3\left[\,(\square_0, \overline{\square}_1, \mathbf{1}) \,+\, (\mathbf{1}, \square_1, \overline{\square}_2) \,+\, (\overline{\square}_0, \mathbf{1}, \square_2)\,\right] \\ W = \epsilon_{ijk} X_{01}^i X_{12}^j X_{20}^k \,, \end{gathered} \tag{3.19}$$

where the bifundamental fields in the superpotential have subindices indicating the gauge representation (in a hopefully self-explanatory way) and a superindex labelling the three copies of each field (and which correspond to the three complex coordinates $(z^1, z^2, z^3) \in \mathbf{C}^3$). Also, the trace in the superpotential is implicit here and in what follows. For completeness, we display in figure 4b the mirror Riemann surface obtained via the untwisting procedure in [109].

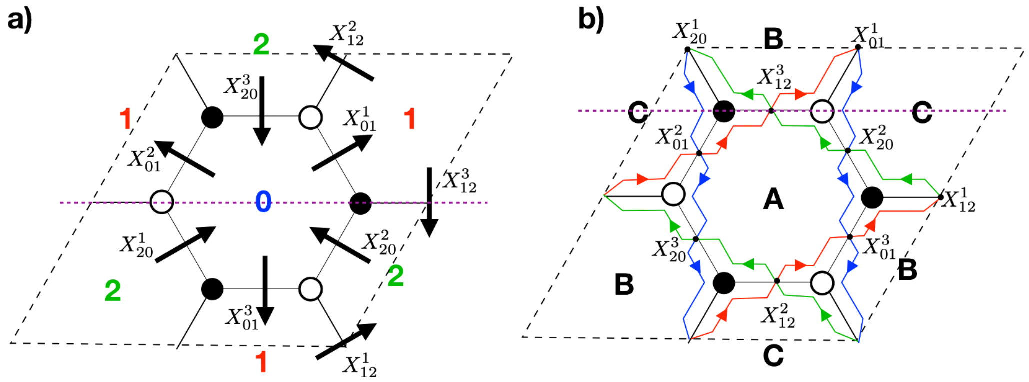

Figure 5. a) The dimer diagram for the dP_0 theory, with arrows indicating the bifundamental chiral multiplets. We also display the $\mathbf{Z}_2$ involution as a reflection with respect to the fixed line, indicated as a dashed violet line. b) The mirror Riemann surface with the D6-brane cycles in colors, and with the location of the bifundamental chiral multiplets at their intersections. The action of the $\mathbf{Z}_2$ involution matching that in figure a) corresponds to an antiholomorphic action, of the kind studied in section 3.3.1.

In figure 5a we have indicated the $\mathbf{Z}_2$ involution required to build the 7d geometry $\mathbf{X}_7$. Specifically, the gauge group $\mathrm{SU}(N)_0$ is mapped to itself, while $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_2$ are exchanged. On the bifundamental matter, we have the action

$$R: \quad X_{01}^1 \leftrightarrow \overline{X}_{20}^2, \quad X_{20}^1 \leftrightarrow \overline{X}_{01}^2 \\ X_{12}^1 \leftrightarrow \overline{X}_{12}^2, \quad X_{01}^3 \leftrightarrow \overline{X}_{20}^3\,, \tag{3.20}$$

with X_{12}^3 being mapped to its conjugate. As mentioned above, it is easy to check that in the mirror geometry, this action corresponds to an antiholomorphic involution of the kind discussed in section 3.3.1. The action on the coordinate z on the base and the coordinates u, v in the $\mathbf{C}^*$ fiber in (3.16) is $z \to \overline{z}$, $u, v \to \overline{u}, \overline{v}$, while the action to the Riemann surface in shown in figure 5b. The action is thus antiholomorphic, and acts as (3.13) on the mirror geometry $\tilde{\mathbf{X}}_6$.

The action (3.20) on the fields can be translated into a geometric action on the type IIB $\mathbf{C}^3/\mathbf{Z}_3$ geometry. Recall that the coordinates of the transverse space to the D3-branes can be constructed as the gauge invariant mesons of the quiver theory. For $\mathbf{C}^3/\mathbf{Z}_3$ we specifically have

$$z^1 = X_{01}^1 X_{12}^1 X_{20}^1, \quad z^2 = X_{01}^2 X_{12}^2 X_{20}^2, \quad z^3 = X_{01}^3 X_{12}^3 X_{20}^3\,, \tag{3.21}$$

modulo F-term relations. Using (3.20), this corresponds to

$$R: (z_1, z_2, z_3) \to (\overline{z}_2, \overline{z}_1, \overline{z}_3)\,. \tag{3.22}$$

As anticipated, the antiholomorphic action on the CY threefold $\mathbf{C}^3/\mathbf{Z}_3$, together with $x_3 \mapsto -x_3$, defines a 7*d* G_2 orbifold, which thus preserves half of the supersymmetries. Following the above Chiral Cone construction, we have a boundary configuration for a 4d $\mathcal{N} = 1$ chiral theory with spectrum (3.19) coupled to gravity.

The $\mathbf{F_0}$ theory. Let us quickly go through another example, the $\mathbf{F}_0$ theory, obtained from D3-branes at the CY3 singularity given by the complex cone over $\mathbf{F}_0$. This is equivalent to a

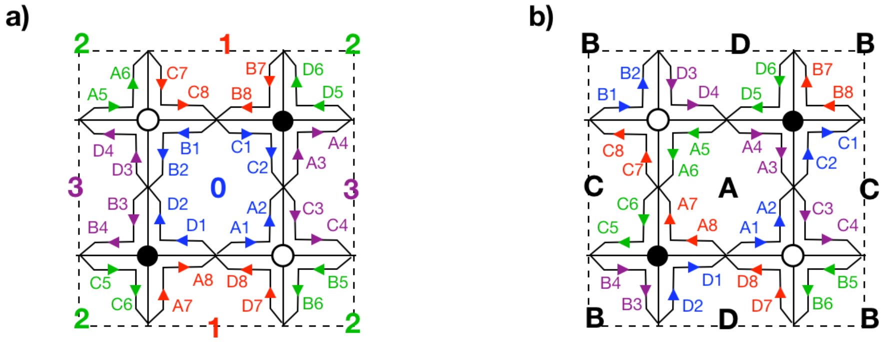

Figure 6. a) The dimer diagram for the $\mathbf{F}_0$ theory, with its set of zig-zag paths. b) The mirror Riemann surface obtained via the untwisting procedure in [109].

$\mathbf{Z}_2$ orbifold of the conifold $\{xy - zw = 0 \,|\, x, y, z, w \in \mathbf{C}\}$, whose generator θ acts as

$$\theta : (x, y, z, w) \to (-x, -y, -z, -w)\,. \tag{3.23}$$

The geometry is toric, hence the theory has a dimer diagram description[15] shown in figures 6a, 7a. The gauge theory is

$$\begin{gathered} \mathrm{SU}(N)_0 \times \mathrm{SU}(N)_1 \times \mathrm{SU}(N)_2 \times \mathrm{SU}(N)_3 \\ 2\,[\,(\square_0, \overline{\square}_1) + (\square_1, \overline{\square}_2) + (\square_2, \overline{\square}_3) + (\square_3, \overline{\square}_0)\,]\,. \end{gathered} \tag{3.24}$$

There is also a quartic superpotential which can be read easily from the dimer diagram and which we skip. For completeness, we display in figure 6b the mirror Riemann surface obtained via the untwisting procedure in [109].

In figure 7a we have indicated the $\mathbf{Z}_2$ involution required to build the 7d geometry $\mathbf{X}_7$. Specifically, the gauge groups $\mathrm{SU}(N)_1$ and $\mathrm{SU}(N)_3$ are exchanged, while $\mathrm{SU}(N)_0$ and $\mathrm{SU}(N)_2$ are mapped to themselves. The action on the bifundamental matter is

$$\begin{aligned} R: \quad & X^1_{01} \leftrightarrow \overline{X}^2_{30}, \quad X^1_{12} \leftrightarrow \overline{X}^2_{23} \\ & X^1_{23} \leftrightarrow \overline{X}^2_{12}, \quad X^1_{30} \leftrightarrow \overline{X}^2_{01}\,. \end{aligned} \tag{3.25}$$

In the mirror geometry this action corresponds to an antiholomorphic involution of the kind discussed in section 3.3.1, see figure 7b for its restriction to the mirror Riemann surface. The action on the fields allows to easily obtain the geometric action on the type IIB geometry. The coordinates of the parent conifold are constructed as the gauge invariant mesons

$$\begin{aligned} x &= X^1_{01} X^1_{12} X^1_{23} X^1_{30}, & y &= X^2_{01} X^2_{12} X^2_{23} X^2_{30} \\ z &= X^1_{01} X^2_{12} X^1_{23} X^2_{30}, & w &= X^2_{01} X^1_{12} X^2_{23} X^1_{30}, \end{aligned} \tag{3.26}$$

modulo F-term relations. Using (3.26), the action is

$$R\,:\,(x, y, z, w) \,\to\, (\overline{y}, \overline{x}, \overline{z}, \overline{w})\,, \tag{3.27}$$

which again defines a 7d G_2 orbifold, preserving half of the supersymmetries. Following the above Chiral Cone construction, we have a boundary configuration for a 4d $\mathcal{N} = 1$ chiral theory with spectrum (3.24) coupled to gravity.

[15] As discussed in [128] the theory actually has two toric phases, related by Seiberg duality [129, 130].

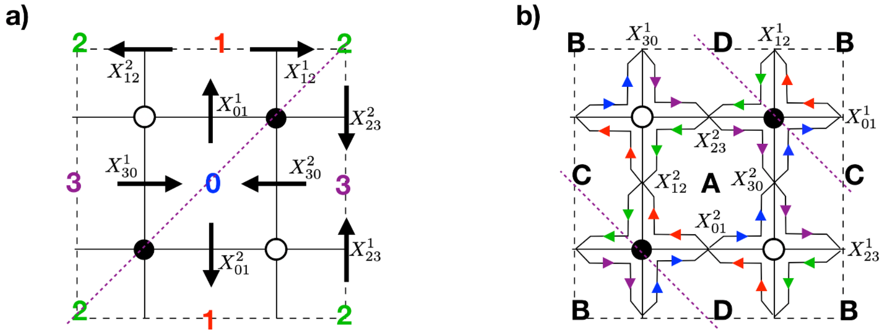

Figure 7. a) The dimer diagram for the $\mathbf{F}_0$ theory, with arrows indicating the bifundamental chiral multiplets. We also display the $\mathbf{Z}_2$ involution as a reflection with respect to the fixed line, indicated as a dashed violet line. b) The mirror Riemann surface with the D6-brane cycles in colors, and with the location of the bifundamental chiral multiplets at their intersections. The action of the $\mathbf{Z}_2$ involution matching that in figure a) corresponds to an antiholomorphic action, of the kind studied in section 3.3.1.

3.3.4 Deformation fractional branes

In the constructions we have encountered so far, the mechanism gapping the chiral non-anomalous set of fermions is not amenable to a simple field theoretical analysis, as it either involves transitions between tensor and hypermultiplets in 6d, or tachyon condensation processes. In this section we describe a large class of examples, based on the construction of 7d cones over D3-branes in (orbifolded) CY3 singularities, admitting a simple description in terms of supersymmetric gauge theory dynamics. In this sense, they provide a realization, in theories coupled to gravity, of the symmetric mass generation field theory mechanisms in [14, 15], also [16] for a review.

The models are based on the use of fractional branes. In the previous discussion, we have considered regular D3-branes at the CY3 singularity $\mathbf{X}_6$, which correspond (in the toric setup) to all gauge factors having equal rank. They describe systems where the D3-branes can move off the singular point into the CY3 bulk. The resulting gauge theories are 4d $\mathcal{N}=1$ SCFT's, dual to $AdS_5 \times \mathbf{Y}_5$ [71, 72].

By fractional branes we mean general rank assignments constrained by anomaly cancellation.[16] The resulting gauge theories are no longer exactly conformal and, as explained in [114], the different kinds of fractional branes can be classified according to their infrared behaviour: (1) $\mathcal{N}=2$ fractional branes have exact Coulomb branches, describing the motion of D3-branes along a complex plane of singularities, and their gravity duals include enhançon singularities [132]; (2) Deformation fractional branes have strong infrared gauge dynamics generating a mass gap, and their gravity duals correspond to a complex deformation of the original singularity [112], generalizing the mechanism in [133]; (3) DSB branes have strong infrared dynamics breaking supersymmetry [113–115] and producing runaway behaviours [114, 134].[17]

[16] By this we mean that the anomalies are cancelled even when additional brane antibrane pairs are introduced. This is equivalent to requiring cancellation of the underlying RR tadpoles, which is in general a stronger condition than mere anomaly cancellation [131].

[17] The introduction of orientifolds provides a further class fractional branes, which break supersymmetry with a presumably stable vacuum [127, 135, 136]; note that these have been conjectured in [137] not to admit an AdS-like holographic dual due unstabilities upon the addition of regular D3-branes.

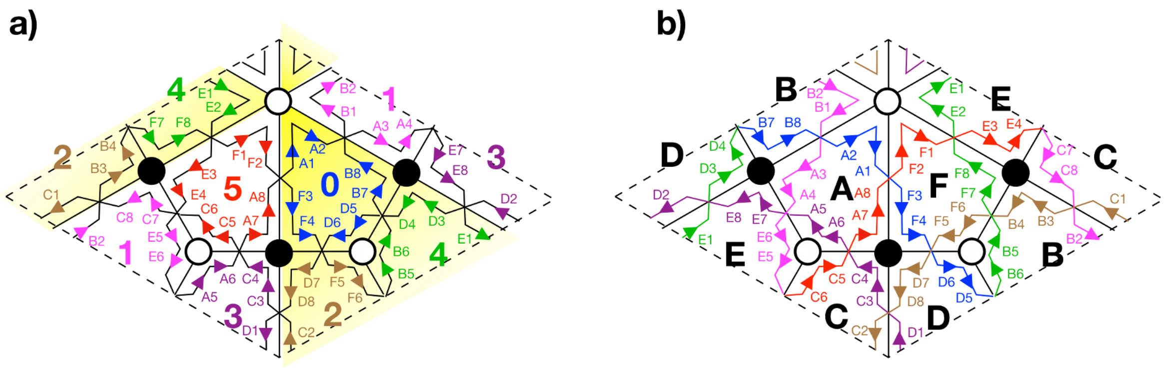

Figure 8. a) The dimer diagram for the dP_3 theory, with its set of zig-zag paths. The yellow area corresponds to the set of faces covered by the deformation fractional branes. b) The mirror Riemann surface obtained via the untwisting procedure in [109].

For our discussion, $\mathcal{N}=2$ fractional branes tend to correspond to non-chiral theories, while DSB branes actually have no stable vacuum, hence we focus on deformation fractional branes. The most familiar realization corresponds to the fractional brane of the conifold theory [133], but it is easy to use dimer techniques to generate large classes of examples [112].

For concreteness, we focus on a deformation fractional brane in the complex cone over dP_3, see [112] for details. The dimer diagram if shown in figures 8a, 9a. For completeness, we display in figure 8b the mirror Riemann surface obtained via the untwisting procedure in [109].

As explained above, we allow for general ranks, compatible with anomaly cancellation. The most general solution for the rank vector is

$$\vec{N} = N(1,1,1,1,1,1) + M(1,0,1,0,1,0) + P_1(1,0,0,1,0,0) + P_2(0,1,0,0,1,0)\,. \qquad (3.28)$$

The set of N corresponds to regular D3-branes, the sets of P_1 or P_2 corresponds to two deformations to conifold singularities, and the set of M corresponds to a deformation directly to a smooth geometry. Although the construction of the orbifolded 7d cone can be carried out in more general cases, we focus on the simple choice $N = P_1 = P_2 = 0$. The gauge theory is given by

$$\begin{gathered} \mathrm{SU}(M)_0 \times \mathrm{SU}(M)_2 \times \mathrm{SU}(M)_4 \\ (\Box_0, \mathbf{1}, \overline{\Box}_4) + (\mathbf{1}, \overline{\Box}_2, \Box_4) + (\overline{\Box}_0, \Box_2, \mathbf{1}) \\ W = X_{04} X_{42} X_{20}\,. \end{gathered} \qquad (3.29)$$

In figure 9a we have indicated the $\mathbf{Z}_2$ involution required to build the 7d geometry $\mathbf{X}_7$. Although the involution can be determined in general, we just specify the action on those fields corresponding to the deformation fractional branes. Specifically, the gauge group $\mathrm{SU}(M)_0$ is mapped to itself, while $\mathrm{SU}(M)_2$ and $\mathrm{SU}(M)_4$ are exchanged. On the bifundamental matter, X_{42} is invariant while X_{04} and X_{20} are exchanged. As in previous examples, this action corresponds to an antiholomorphic involution in the mirror geometry.

Following the by now familiar Chiral Cone construction, we have a boundary configuration for the 4d $\mathcal{N}=1$ chiral theory (3.29) coupled to gravity. It is now easy to check the gauge field theory mechanism by which this gauge theory is gapped, as we quickly sketch below,

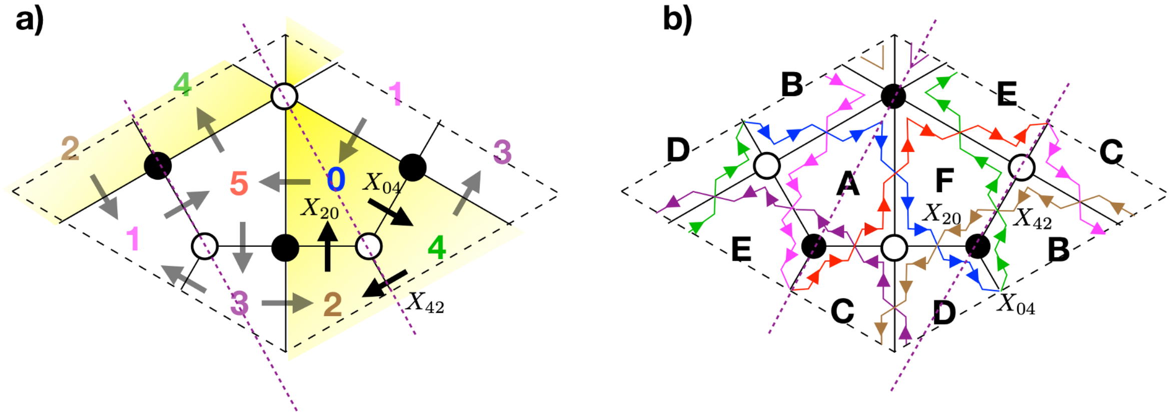

Figure 9. a) The dimer diagram for the dP_3 theory, with arrows indicating the bifundamental chiral multiplets. We have highlighted the faces and arrows present for the deformation fractional branes. We also display the $\mathbf{Z}_2$ involution as a reflection with respect to the fixed line, indicated as a dashed violet line. b) The mirror Riemann surface with the D6-brane cycles in colors, and with the location of the bifundamental chiral multiplets at their intersections, and the antiholomorphic $\mathbf{Z}_2$ action.

see [112] for more detailed discussion. As the configuration approaches the boundary of spacetime i.e. the tip of the cone, the wrapped cycles are shrinking, hence the gauge factors are driven to strong coupling. Consider for simplicity that we consider one of the gauge factors, say $SU(M)_0$, to have a larger dynamical scale than the other two. Then, as we run to the infrared, we account for its strong gauge dynamics, taking $SU(M)_2$ and $SU(M)_4$ as global flavour symmetries. The $SU(M)_0$ theory has $N_f = N_c$, so that it confines and has a quantum deformed mesonic moduli space. Equivalently, we may replace it by the corresponding Seiberg dual, which has trivial gauge group, and a set of mesons M_{42} in the $(\overline{\square}_2, \square_4)$. The gauge theory is thus

$$\begin{gathered} SU(M)_2 \times SU(M)_4 \\ (\overline{\square}_2, \square_4) + (\square_2, \overline{\square}_4) \\ W = M_{42} X_{24} \,. \end{gathered} \tag{3.30}$$

Namely, the composite mesons transform in the conjugate representation of the (spectator) bifundamental X_{24}, and in fact they get a mass term via the superpotential coupling. Hence the original chiral non-anomalous set of fermions is gapped, in the spirit of symmetric mass generation. In the language of the original gauge theory (3.29), the boundary condition at the ETW brane relates the bifundamental X_{24} with the value of a composite degree of freedom of X_{40} and X_{02}. Similar statements hold if we consider some other of the original gauge factors to confine first; actually, due to the cyclic symmetry of the theory, we expect that all gauge factors confine at the same scale and there is a combination of the above phenomenon taking place for each gauge factor. The whole gauge theory analysis is efficiently encoded in simple manipulations in the dimer diagram, as shown in [110, 112].

In the context of holography, the above kind of behaviour was argued in [112] to occur for any deformation fractional brane. Namely, the set of gauge factors in the fractional D3-brane gauge theory has a subsector that confines and develops a quantum deformed mesonic moduli space, resulting in a gapping of the remaining theory. In the gravity dual, this is described by a generalization of [133], in which the quantum deformed moduli space describes the complex deformation of $\mathbf{X}_6$, by replacing its singular tip by a 3-cycle whose size is fixed by 3-form fluxes dual to the number of fractional brane number M.

There exist systematic techniques to build gauge theories with fractional deformation branes and study the complex deformations in the corresponding toric geometries [110]. Hence, it is straightforward to use these tools to construct further explicit examples. We hope that the above discussion suffice to illustrate the main points, and refrain from further discussions.

3.4 The dynamical cobordism

In this section we describe explicitly the Cone Constructions as solutions of the theory after compactification on the base of the cone, and show that they correspond to dynamical cobordisms in the precise sense of [7]. Namely, the solutions describe a running scalar which attains infinite field space distance at a curvature singularity at finite spacetime distance, at which spacetime ends.

The discussion holds for the general class of cone constructions, in particular also the 6d examples in section 2. Nevertheless, for the sake of concreteness we focus on the particular class of 4d theories arising from D3-branes at general CY3 singularities, in sections 3.2 and 3.3. We moreover treat the D3-branes as probes, hence the solution is mainly associated to the compactification over the 6d geometry $\mathbf{Y}_6$ given by the suspension of the base $\mathbf{Y}_5$ of the CY3 cone $\mathbf{X}_6$.

3.4.1 Dimensional reduction from D to n dimensions

We start with the $D-$dimensional Einstein-Hilbert action:

$$S_D = \frac{1}{2}\int d^Dx\sqrt{-G_D}R_D\,, \tag{3.31}$$

and consider the following ansatz for compactification on a p-dimensional space $\mathbf{Y}_p$ parametrized by y^i, $i = n+1,\ldots,n+p = D$:

$$ds^2_D = e^{2\alpha\rho(x)}g_{\mu\nu}dx^\mu dx^\nu + e^{2\beta\rho(x)}g_{ij}dy^i dy^j\,, \tag{3.32}$$

where α and β are constant coefficients. The relation among the D-, $n-$ and p-dimensional curvature scalars and the scalar ρ is

$$\begin{aligned} R_D = e^{-2\alpha\rho}\Big\{&R_n + R_p e^{2(\alpha-\beta)\rho} - 2\left[(\alpha-\beta)(n-1)+\beta(D-1)\right]\Delta\rho - \\ &-\Big[(\alpha-\beta)^2(n-2)(n-1) + 2\beta(\alpha-\beta)(n-2)(D-1) + \beta^2(D-1)(D-2)\Big]\,|d\rho|^2\Big\}\,. \end{aligned} \tag{3.33}$$

Using this and the relation between determinants of the metrics, the action S_D (3.31) becomes:

$$S_D = \frac{1}{2}\int_{\mathbf{Y}_p}\int_{\mathcal{M}_n} d^py\, d^nx\,\sqrt{-g_n}\,\sqrt{g_p}\,e^{[(n-2)\alpha+p\beta]\rho}\left\{R_n + R_p\,e^{2(\alpha-\beta)\,\rho} - C_1\Delta\rho - C_2|d\rho|^2\right\},$$

with

$$\begin{aligned} C_1 &= 2\left[(\alpha-\beta)(n-1)+\beta(n+p-1)\right] \\ C_2 &= (\alpha-\beta)^2(n-2)(n-1) + 2\beta(\alpha-\beta)(n-2)(n+p-1) + \beta^2(n+p-2)(n+p-1)\,, \end{aligned} \tag{3.34}$$

Redefining to the Einstein frame (we exclude $n \neq 2$ in what follows) and to canonically normalized scalar kinetic term with the conditions

$$\beta = -\frac{n-2}{p}\alpha, \quad \alpha^2 = \frac{p}{(n-2)(n+p-2)}, \tag{3.35}$$

and integrating by parts the Beltrami operator term, we obtain

$$S_D = \frac{1}{2}\int_{\mathbf{Y}_p}\int_{\mathcal{M}_n} d^p y\, d^n x \sqrt{-g_n}\sqrt{g_p}\left\{R_n + R_p\, e^{2\frac{p+n-2}{p}\alpha\rho} - |d\rho|^2\right\}. \tag{3.36}$$

3.4.2 Compactification on the base of the cone

At this point we specialize to the setup of 4d solutions in type IIB string theory, hence we set $D = 10$ and $p = 6$, so the action (3.36) becomes:

$$S_{10} = \frac{1}{2}\int_{\mathbf{Y_6}}\int_{\mathcal{M}_4} d^6 y\, d^4 x\, \sqrt{-g_4}\sqrt{g_6}\left\{R_4 + R_6\, e^{\pm 2\sqrt{2/3}\rho} - |d\rho|^2\right\}. \tag{3.37}$$

where the two signs correspond to the two solutions for α in (3.35).

For constant curvature compact space, for instance $\mathbf{S}^6/\mathbf{Z}_k$, arising in singularities from orbifolds of flat space (e.g. the dP_0 example in sections 3.2.1, 3.3.3), it is straightforward to integrate over the internal space and get

$$S_4 \propto \frac{\mathcal{V}_{\mathbf{S}^6}}{2k}\int d^4x\sqrt{-g_4}\left\{R_4 - |d\rho|^2 + \frac{30}{R_0^2}e^{\pm 2\sqrt{2/3}\rho}\right\}. \tag{3.38}$$

where R_0 is the radius of the covering $\mathbf{S}^6$, and $\mathcal{V}_{\mathbf{S}^6} = \frac{16}{15}\pi^3 R_0^6$ is its volume. More in general, we want to consider the configurations in sections 3.2.2, 3.3, so we consider compactification on a 6d geometry $\mathbf{Y}_6$ (eventually the base of a 7d cone) given by a suspension of a 5d geometry $\mathbf{Y}_5$ (eventually the base of the 6d CY3 cone) over the segment $\theta \in [-\pi/2, \pi/2]$, namely recalling (3.12), (3.18),

$$ds^2_{\mathbf{Y}_6} = d\theta^2 + \cos^2\theta\, ds^2_{\mathbf{Y}_5}. \tag{3.39}$$

Hence we may express the action in terms of the geometric properties of $\mathbf{Y}_5$. In particular, using the relation of curvatures

$$R_{\mathbf{Y_6}} = \frac{1}{\cos^2\theta}\left[R_{\mathbf{Y_5}} - 10\cos^2\theta - 20\sin^2\theta\right], \tag{3.40}$$

and integrating over θ in (3.37) we obtain:[18]

$$\begin{aligned} S_{10} &= \frac{8}{15}\int_{\mathbf{Y_5}}\int_{\mathcal{M}_4} d^5y d^4x\sqrt{-g_4}\sqrt{g_{\mathbf{Y_5}}}\left\{R_4 - |d\rho|^2 + \frac{5}{4}\left(-12 + R_{\mathbf{Y_5}}\right)e^{\pm 2\sqrt{2/3}\rho}\right\}. \\ &= \frac{8}{15}Vol_{\mathbf{Y_5}}\int_{\mathcal{M}_4} d^4x\sqrt{-g_4}\left[R_4 - |d\rho|^2 - 15e^{\pm 2\sqrt{2/3}\rho}\right] + \frac{2}{3}\cdot A\int_{\mathcal{M}_4} d^4x\sqrt{-g_4}e^{\pm 2\sqrt{2/3}\rho}, \end{aligned} \tag{3.41}$$

where $A = \int_{\mathbf{Y_5}} d^5y\sqrt{g_{\mathbf{Y_5}}}R_{\mathbf{Y_5}}$.

[18] The expressions correspond to the simple compactification of section 3.2.2. The $\mathbf{Z}_2$ quotient for models of section 3.3 lead to additional simple factors of 2.

3.4.3 Dynamical cobordism and scaling relations

We can now match the 10d metric with the compactification ansatz (3.32):

$$ds_{10}^2 = e^{2\alpha\rho} ds_4^2 + e^{2\beta\rho} R_0^2 ds_6^2 = \left(dx^0\right)^2 + \left(dx^1\right)^2 + \left(dx^2\right)^2 + dr^2 + r^2 ds_{\mathbf{Y_6}}^2, \tag{3.42}$$

where R_0 is a reference value for the internal space size.

The matching of the internal part leads to the profile of the breathing mode $\rho(r)$

$$R_0^2 e^{2\beta\rho} ds_6^2 = r^2 ds_{\mathbf{Y_6}}^2 \quad \Longrightarrow \quad \rho(r) = \frac{1}{\beta} \log\left(\frac{r}{R_0}\right). \tag{3.43}$$

Comparing the non-compact directions, we can extract the 4d metric. Using the above scalar profile and the Einstein frame condition, we obtain

$$ds_4^2 = e^{6\log(r/R_0)} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2 + dr^2\right]. \tag{3.44}$$

This can be recast in the standard local dynamical cobordism form in [7], as follows. We redefine coordinates to encode the physical spacetime distance, via

$$dy^2 = \left(\frac{r}{R_0}\right)^6 dr^2 \quad \Longrightarrow \quad y = \int_0^r \left(\frac{\tilde{r}}{R_0}\right)^3 d\tilde{r} = \frac{r^4}{4R_0^3} \sim r^4. \tag{3.45}$$

In terms of the new coordinate the metric (3.44) takes the form:

$$ds_4^2 = e^{\frac{3}{2}\log(4y/R_0)} \left[(dx^1)^2 + (dx^2)^2 + (dx^3)^2\right] + dy^2. \tag{3.46}$$

The metric and scalar profiles obey the local dynamical cobordism ansatz in [7]

$$ds^2 = e^{-2\sigma(y)} ds_{n-1}^2 + dy^2, \quad \sigma(y) \simeq \frac{2}{\delta^2} \log y, \quad \rho(y) \simeq -\frac{2}{\delta} \log y, \tag{3.47}$$

with a critical exponent given by

$$\delta = \frac{2}{3}\sqrt{6}, \tag{3.48}$$

Note that this agrees with the scaling of the potential in (3.41) as $V \sim \exp(\delta\phi)$ [7]. As also shown in this reference, the quantity δ also controls the scalings of the field space distance $\mathcal{D}$ and the curvature R with the spacetime distance Δ via

$$\Delta \sim e^{-\frac{\delta}{2}\mathcal{D}}, \quad |R| \sim e^{-\delta\mathcal{D}}. \tag{3.49}$$

This result confirms that our cones constructions can be regarded as dynamical cobordism solutions of the theory obtained upon compactification on the base of the cone. We emphasize that similar computations lead to this conclusion also for other setups, such as the 6d examples in section 2, or the alternative 4d setup in appendix B.

4 Conclusions

In this paper we have studied the construction of boundary configurations for several large classes of 6d and 4d chiral theories arising from string theory, hence including gravity. The boundary configurations are mostly constructed using cones over codimension 1 slices at which some interesting physics takes place, such as chirality changing phase transitions associated to the ending of some lower-dimensional brane with chiral worldvolume theory, or to some $\mathbf{Z}_2$ quotient leading to reduced but non-trivial supersymmetry at the tip of the cone.

Interestingly, the physical mechanisms associated to the boundary configuration often admit a field theory interpretation, albeit at strong coupling. For example, in the 6d cases it is associated to the transitions trading one 6d $\mathcal{N} = 1$ tensor multiplet for 29 hypermultiplets, which does not admit a lagrangian description, while in the 4d case they are often related to confinement and pairing up of fundamental chiral multiplets with composite mesons, as in the case of deformation fractional branes.

This work is a useful stepping stone in the general program of building boundary configurations for general chiral theories coupled to gravity. Clearly, there remain many important challenges in this plan, for instance:

- Most prominently, an open question is the definition of boundary configurations for the chiral 10d string theories. One interesting possibility is to exploit their realization as the endpoint of closed string tachyon condensation of higher-dimensional supercritical strings [11–13, 138–141], to allow for some version of the cone construction described in our examples.
- In the 6d setup we have found boundary configurations involving chirality changing phase transitions. It would be interesting to build explicit examples of boundary configurations for 4d chiral theories from 4d chirality changing phase transitions as well.
- In the context of 4d dimensional examples, we have managed to provide boundary configurations for bulk chiral theories with additional $\mathbf{Z}_2$ symmetries, either under exchange of whole gauge sectors, or as involutions of a given quiver gauge theories. It would be interesting to explore boundary conditions for general 4d theories, possibly not enjoying such symmetries.
- The connection of our cone constructions with those involved in the derivation of SymTFTs, discussed in section 2.4 seems to provide a new interesting tool to analyze the topological properties of chirality changing phase transitions. One may hope to use these tools to gain a better understanding of the basic (yet highly non-trivial) 6d transition turning one tensor multiplet into 29 hypermultiplets.

We hope to come back to these and other questions in the coming future.

Acknowledgments

We are pleased to thank Matilda Delgado, Bjoern Friedrich, Iñaki García-Etxebarria, Arthur Hebecker, Jesús Huertas, Luis Ibáñez, Fernando Marchesano, Miguel Montero, Irene Valenzuela, Johannes Walcher and Xingyang Yu for useful discussions. R.A. thanks the CERN

Theory Physics Department for hospitality during the completion of this work and the ERC Starting Grant QGuide-101042568 - StG 2021 for supporting this stay. C. W. thanks the hospitality of the Department of Mathematical Sciences of Durham University, where part of this work was carried out. This work is supported through the grants CEX2020-001007-S, PID2021-123017NB-I00 and ATR2023-145703 funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe.

A More systems of intersecting D6-branes with boundaries

In this appendix we consider configurations of intersecting semi-infinite D6-branes with boundaries defined by NS5-branes, generalizing those in section 3.1. We will similarly find that, due to the presence of additional emitted D4-branes, the cone construction leads to dynamical cobordisms for theories which are ultimately non-chiral. These extra examples thus confirm that this class of construction is not optimal to obtain boundary configurations for chiral theories.

We would like to consider a configuration similar to that in section 3.1, but with two semi-infinite D6-brane stacks, each ending on an NS5-brane. Because the number of D6-branes emitted by an NS5-brane is determined by the Romans mass m, we have to consider the two stacks to have the same number of D6-branes $N_1 = N_2 = m$. Actually, this example is part of a slightly more general class, in which we consider stacks of half D6-branes on both sides of each NS5-branes, with the numbers differing by m units consistently with the Freed-Witten effect on the NS5-branes. Namely, we take a set of D6-branes along 0123 456, split in two semi-infinite stacks of half-D6-branes by an NS5-brane (dubbed $\mathrm{NS5}_1$) at 012 456 at $x^3 = 0$, so we have N_1 $\mathrm{D6}_1$- and N_1' $\mathrm{D6}_1'$-branes at $x^3 > 0$ and $x^3 < 0$ respectively, with $N_1 - N_1' = m$. Similarly, we introduce another set of D6-branes along 0123 789, split in two semi-infinite stacks of half-D6-branes by an NS5-brane (dubbed $\mathrm{NS5}_2$) at 012 789 at $x^3 = 0$, so we have N_2 $\mathrm{D6}_2$- and N_2' $\mathrm{D6}_2'$-branes at $x^3 > 0$ and $x^3 < 0$ respectively, with $N_2 - N_2' = m$.

Configuration of intersecting NS5-branes are often very non-trivial (see e.g. [142]), hence we will thus regulate our setup by taking one of the NS5-branes (e.g. the $\mathrm{NS5}_1$) at a nonzero value[19] of $x^3 = \epsilon > 0$, see figure 10.

We now have four sets of gauge fields on the $\mathrm{D6}_1$-, $\mathrm{D6}_1'$-, $\mathrm{D6}_2$- and $\mathrm{D6}_2'$-branes. Also, we have a 4d intersection of the $\mathrm{D6}_1$- and $\mathrm{D6}_2$-branes on the half-space along 012 and at $x^3 > \epsilon$, giving a 4d chiral fermion in the bifundamental $(\square_1, \mathbf{1}; \overline{\square}_2, \mathbf{1})$, which ends at the $\mathrm{NS5}_1$-brane at $x^3 = \epsilon$, an intersection of the $\mathrm{D6}_1'$- and $\mathrm{D6}_2$- branes on the space along 0123 and the segment $0 < x^3 < \epsilon$, giving a 4d chiral fermion in the $(\mathbf{1}, \square_1'; \overline{\square}_2, \mathbf{1})$, which ends at the $\mathrm{NS5}_2$-brane at $x^3 = 0$, and a 4d intersection of the $\mathrm{D6}_1'$- and $\mathrm{D6}_2'$-branes on the half-space along 012 and at $x^3 < 0$, giving a 4d chiral fermion in the bifundamental $(\mathbf{1}, \square_1'; \mathbf{1}, \overline{\square}_2')$.

As in section 3.1, the discontinuity of the chiral fermion spectrum and the apparent mismatch of anomalies across the NS5-branes indicates that the configuration is missing extra ingredients. These are again given by emitted D4-branes stretching between the NS5- and the D6-branes, as follows. Using by now familiar arguments, there are D4-branes (dubbed

[19] The situation with $\epsilon < 0$ can be studied similarly, and leads to slightly different intermediate spectra. This indicates that the limit $\epsilon \to 0$ is presumably not smooth, signalling a possibly non-lagrangian strongly coupled theory for the coincident NS5-brane case.

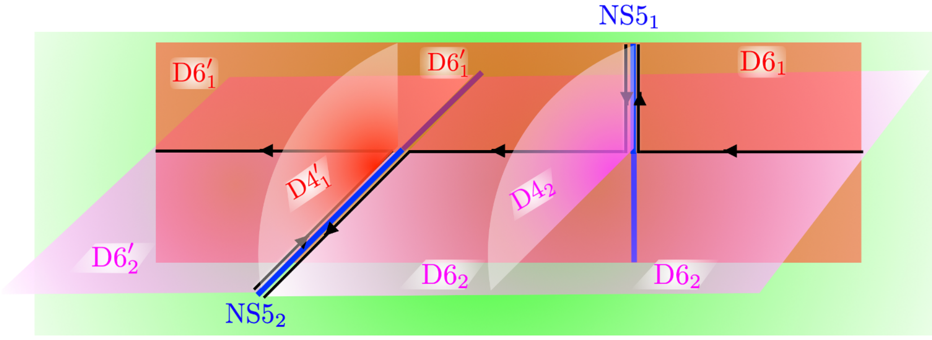

Figure 10. Stacks of semi-infinite D6-branes separated by (slightly separated) NS5-branes, and intersecting over 4d. We have depicted the D4$'_1$-branes stretched between the D6$'_1$-branes and the NS5$_2$-brane, and the D4$_2$-branes stretched between the D6$_2$-branes and the NS5$_1$-brane. We have indicated with black lines and arrows the location of 4d chiral fermions in the configuration. They complete continuous paths, displaying the consistency with anomaly inflow from the bulk of the D6/D4-branes.

D4$_2$-branes) attached to the D6$_2$-branes and ending on the NS5$_1$-brane, and D4$'_1$-branes attached to the D6$'_1$-branes and ending on the NS5$_2$-brane. As explained, we should think about the D4-branes as carrying the same worldvolume gauge group as the D6-branes to which they are attached (hence the similar notation). The final configuration is depicted in figure 10, and shows the extra D4-branes guarantee the continuity of the 4d chiral fermion content, indicated by black lines with arrows.

It is now easy to consider the limit $\epsilon \to 0$ (or very small, for that matter) and to carry out the cone construction on this configuration, to obtain a dynamical cobordism of the 4d theory upon compactification on $\mathbf{S}^6$. It is clear that, as in the example in section 3.1, the continuity of the lines supporting the 4d fermions implies that, in the cone perspective, for each line of incoming chiral fermions there is an outgoing line of fermions with the same quantum numbers. Hence there is a doubling of the spectrum of the 4d theory making it non-chiral.

We hope this example suffices to illustrate this is a general pattern for this class of configurations, as mentioned in the main text.

B Chiral cone constructions with intersecting D6-branes

In this appendix we describe an example of a 7d G_2 holonomy space $\mathbf{X}_7$ given by a cone over a base $\mathbf{Y}_6$ with D6-branes wrapped on non-compact associative 4-cycles intersecting at points over $\mathbf{Y}_6$, and leading to a 4d chiral non-anomalous spectrum. The example is based on geometries constructed in [143] and considered in the holographic context in [68] (see also [144]). In our context, the system provides an explicit example realizing the ideas in section 3.3.1 (albeit for a genuine, rather than barely, G_2 holonomy cone). We start with a review of the geometry, referring the reader to these works for further details.

The 7d space $\mathbf{X}_7$ is a cone over a 6d base $\mathbf{Y}_6$ given by the coset $\mathrm{SU}(2)^3/\mathrm{SU}(2)$, where the quotient is by the diagonal subgroup. The space $\mathbf{Y}_6$ is topologically $\mathbf{S}^3 \times \mathbf{S}^3$, and the metric is induced from the round metric in the parent $SU(2)^3$ space, namely

$$ds^2 = dr^2 + \frac{r^2}{9}({\omega_a}^2 + \tilde{\omega}_a^2 - \omega_a \tilde{\omega}_a)\,, \tag{B.1}$$

where ω_a, $\tilde{\omega}_a$, $a = 1, 2, 3$, are left invariant 1-forms of the two $\mathbf{S}^3$'s, when regarded as SU(2) groups. The metric (B.1) has G_2 holonomy [143]. The space $\mathbf{Y}_6$ admits an almost complex structure defined in terms of the complex frames

$$\eta_a = \omega_a + e^{2\pi i/3}\tilde{\omega}_a\,, \tag{B.2}$$

in terms of which the metric reads $ds^2 = dr^2 + r^2 \eta_a \overline{\eta}_a$. One can define the (non-closed) Kähler and holomorphic forms

$$\begin{aligned} \omega &\equiv \frac{i}{2}\eta_a \wedge \overline{\eta}_a, & d\omega &= -3\mathrm{Im}\,\Omega \\ \Omega &\equiv \eta_1 \wedge \eta_2 \wedge \eta_3, & d\mathrm{Re}\,\Omega &= -2\omega \wedge \omega\,. \end{aligned} \tag{B.3}$$

The associative 3-form and coassociative 4-forms in $\mathbf{X}_7$ are given in terms of these by

$$\varphi = r^2 dr \wedge \omega - r^3 \mathrm{Im}\,\Omega, \quad *\varphi = r^3 \mathrm{Re}\,\Omega \wedge dr + \frac{r^4}{2}\omega \wedge \omega\,. \tag{B.4}$$

The metric is invariant under the order 6 permutation group of the 3 $\mathbf{S}^3$'s in the parent space. This is generated by the order 2 exchange α of the two $\mathbf{S}^3$'s in $\mathbf{Y}_6$, acting as $\alpha : \omega_a \leftrightarrow \tilde{\omega}_a$, and the order 3 cyclic permutation β of the three $\mathbf{S}^3$'s in the parent space, acting as $\beta : \omega_a \to -\tilde{\omega}_a\,,\ \tilde{\omega}_a \to \omega_a - \tilde{\omega}_a$. Equivalently, the actions are $\alpha : \eta_a \to \tau\overline{\eta}_a$ and $\beta : \eta_a \to \tau\eta_a$. The action over the forms (B.4) is $\alpha : \varphi \to -\varphi$, leaving $*\varphi$ invariant, while β leaves both φ and $*\varphi$ invariant. These actions can be extended to actions on the full cone $\mathbf{X}_7$, for which, with a slight abuse of notation, we use the same names.

These symmetries are extremely useful to build supersymmetric cycles. As is well-known, it is notoriously difficult to construct calibrated submanifolds in G_2 manifolds, but, in analogy with the (similarly difficult construction of special lagrangian 3-cycles in CY3 spaces) particular examples can be obtained as the fixed point set of certain $\mathbf{Z}_2$ involutions. In particular, one can build supersymmetric 4-cycles in G_2 manifolds as the fixed point set under a $\mathbf{Z}_2$ action which flips the sign of φ and leaves $*\varphi$ invariant. This is precisely the way α acts on $\mathbf{X}_7$, hence the fixed point set of α in $\mathbf{X}_7$ provides a supersymmetric 4-cycle in the cone geometry. Clearly, in the base $\mathbf{Y}_6$, it corresponds to the diagonal $\mathbf{S}^3$ in the $\mathbf{S}^3 \times \mathbf{S}^3$, hence in the full cone $\mathbf{X}_7$ we get a cone over that $\mathbf{S}^3$. Moreover, we can obtain other 4-cycles as images of the previous one under the action of β; or equivalently, as the fixed point set of the actions $\beta\alpha\beta^{-1}$ and $\beta^2\alpha\beta^{-2}$. Clearly, by symmetry, they are cones over $\mathbf{S}^3$'s, obtained as diagonal combinations of consecutive $\mathbf{S}^3$'s in the parent $SU(2)^3$ space.

Let us denote these 3-cycles in $\mathbf{Y}_6$ as Q_i, $i = 0, 1, 2$. In the symplectic basis of 3-homology $[A]$, $[B]$ of $\mathbf{Y}_6 = \mathbf{S}^3 \times \mathbf{S}^3$, their homology classes can be expressed as

$$[Q_0] = [A] + [B], \quad [Q_1] = -[A], \quad [Q_2] = -[B]\,. \tag{B.5}$$

This means that it is possible to wrap an equal number N of D6-branes on each of these 3-cycles on $\mathbf{Y}_6$ in a way compatible with RR tadpole cancellation, since the total homology class vanishes. In the full 7d geometry, the D6-branes stretch also in the radial direction, namely they span the directions 012 and the 4-cycles given by the cones over Q_i.

We can now regard this configuration in the spirit of the cone construction, as a 4d theory obtained upon compactifying 10d type IIA on $\mathbf{Y}_6$ with the wrapped D6-branes, namely an intersecting D6-brane brane model [90–92] (see [93] for a review). The compactification space is not Calabi-Yau, but the determination of the spectrum is topological. Noticing that the intersection numbers of the 3-cycles are $= [Q_0] \cdot [Q_1] = [Q_1] \cdot [Q_2] = [Q_2] \cdot [Q_0] = 1$ (and zero for those others not fixed by antisymmetry of the intersection product), we obtain a 4d gauge group, matter content, and superpotential

$$\begin{gathered} \mathrm{SU}(N)^3 \\ (\square, \overline{\square}, \mathbf{1}) + (\mathbf{1}, \square, \overline{\square}) + (\overline{\square}, \mathbf{1}, \square) \\ W \sim X_{01} X_{12} X_{20} \,, \end{gathered} \tag{B.6}$$

where the superpotential can be shown to arise from worldsheet instantons, and has a trace over color indices that we leave implicit. Note that we have removed the U(1) factors, as the diagonal combination simply decouples, and the two other combinations are made massive by Stückelberg couplings.

In the spirit of the Chiral Cone construction, the full configuration in the cone $\mathbf{X}_7$ describes a running solution of this theory, in which the scalar describing the size of $\mathbf{Y}_6$ runs along the radial coordinate of the cone, and diverges at finite distance in spacetime, corresponding to the shrinking of $\mathbf{Y}_6$ at the tip of the cone. Hence, it fits the picture of a dynamical cobordism [6, 7, 19, 20], as emphasized in the main text and shown in section 3.4, hence providing a boundary configuration for the chiral theory.

Incidentally, we note that the gauge theory (B.6) is the same as the theory (3.29) realized on deformation branes of the complex cone over dP_3, despite being a completely different context. This allows us to borrow the discussion at the end of section 3.3.4 regarding the field theory analysis explaining how the chiral non-anomalous theory gets gapped. As in that discussion, the strong coupling gauge dynamics nicely dovetails the fact that the wrapped 3-cycles are shrinking as one approaches the boundary of spacetime, driving the gauge factors to strong coupling.

Let us finish with the observation in [68] that the type IIA configuration admits a lift to M-theory, in which the D6-branes are fully geometrized (locally as $\mathbf{C}^2/\mathbf{Z}_N$ singularities). The final configuration is M-theory on an 8d Spin(7) holonomy cone, and the gauge theory discussed above arises from the structure of codimension-4 singularities and the enhancements at the intersections of the corresponding singular loci.

Data Availability Statement. This article has no associated data or the data will not be deposited.

Code Availability Statement. This article has no associated code or the code will not be deposited.

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3
Topological Defects

This chapter contains the article:

- *Topological defects in K3 sigma models*,
  R. Angius, Stefano Giaccari, Roberto Volpato
  JHEP 07 (2024) 111 arXiv:2402.08719 - Inspire

Received: *March 3, 2024*
Accepted: *June 17, 2024*
Published: *July 11, 2024*

Topological defects in K3 sigma models

Roberta Angius,[a] **Stefano Giaccari**[b] **and Roberto Volpato**[b]

[a]*Instituto de Física Teórica IFT-UAM/CSIC,*
C/ Nicolás Cabrera 13-15, Campus de Cantoblanco, 28049 Madrid, Spain
[b]*Dipartimento di Fisica e Astronomia 'Galileo Galilei', Università di Padova*
& INFN, sez. di Padova,
Via Marzolo 8, 35131, Padova, Italy

E-mail: roberta.angius@csic.es, stefano.giaccari@pd.infn.it, volpato@pd.infn.it

Abstract: We consider the topological defect lines commuting with the spectral flow and the $\mathcal{N} = (4,4)$ superconformal symmetry in two dimensional non-linear sigma models on K3. By studying their fusion with boundary states, we derive a number of general results for the category of such defects. We argue that while for certain K3 models infinitely many simple defects, and even a continuum, can occur, at generic points in the moduli space the category is actually trivial, i.e. it is generated by the identity defect. Furthermore, we show that if a K3 model is at the attractor point for some BPS configuration of D-branes, then all topological defects have integral quantum dimension. We also conjecture that a continuum of topological defects arises if and only if the K3 model is a (possibly generalized) orbifold of a torus model. Finally, we test our general results in a couple of examples, where we provide a partial classification of the topological defects.

Keywords: Conformal Field Models in String Theory, D-Branes, Discrete Symmetries, Global Symmetries

ArXiv ePrint: 2402.08719

https://doi.org/10.1007/JHEP07(2024)111

Contents

1 Introduction

Defects in a quantum field theory (QFT) or in statistical lattice models and quantum spin systems are broadly defined as inhomogeneities localized on submanifolds of positive codimension, and appear in a variety of physical settings, with the codimension 1 case playing the distinctive role of interfaces between different theories. A more specific notion is the one of topological defects, which are assumed to be invariant under continuous deformations as long as the deformations do not move the inhomogeneities past other defects or field operator insertions. Historically, in particular in the context of $2d$ QFT, topological defect lines (TDLs) have been studied in Conformal Field Theories (CFTs), where conformal invariance provides a very restrictive framework, even more so for Rational CFTs (RCFTs), because of their connection to Boundary Conformal Field Theories (BCFTs), twisted boundary conditions, and orbifolds [1–10]. However most recent developments stem from the realization that, in QFT with generic dimension d, topological defects provide a natural generalization of the notion of group symmetry, with the group elements corresponding to topological defects of codimension 1 across which the value of field operators jumps by the respective symmetry actions. This picture has naturally led to considering both "higher-form symmetries", realized by topological defects of higher codimension (see e.g. [11]), and "non-invertible symmetries", corresponding to topological defects which cannot be fused with another topological defect to produce the trivial defect. In two dimensions, the group structure is in the latter case generalized by the one of fusion categories [12–15]. The classical example of fusion category symmetry is the Ising one, defined by three simple TDLs: the trivial defect, the $\mathbb{Z}_2$ Ising symmetry and the "Kramers-Wannier" duality defect in the Ising model [16–19], which relates spin correlators in the Ising model at some inverse temperature to disorder/twist correlators at the Kramers-Wannier dual inverse temperature. TDLs have been widely studied in RCFTs [1–10], gapped boundaries of $(2+1)D$ topological field theories, and anyon chains, and found to encode nontrivial topological information about a theory, such as constraints on the operator spectrum of CFTs and renormalization group flows in gauge theories [20]. Non-invertible topological defects can also be considered in higher-dimensional theories where they also provide relevant information about the dynamics, so that they are by now considered a tool of paramount importance in unraveling the non-perturbative aspects of Quantum Field Theory.

One of the main hurdles in advancing our knowledge is the fairly limited number of theories where the fusion category of defects or at least a subset thereof are known. This is true even in the simple framework of 2d CFTs. While in rational CFTs some general techniques have been developed, the much broader realm of non rational theories is largely unexplored. In fact, topological interfaces, which in general separate possibly different theories, have been studied in a very limited number of examples, in particular for the free boson compactified on a circle and on an orbifold thereof [21–23] and for d-dimensional torus models, where they are assumed to preserve a $u(1)^{2d}$ current algebra [24].

In this article, we consider topological defects in two-dimensional superconformal field theories (SCFT) arising as supersymmetric non-linear sigma models with target space a K3 surface. More precisely, we will focus on the defects that preserve the full $\mathcal{N} = (4,4)$ superconformal symmetry at central charge $c = \bar{c} = 6$, and that are invariant under the spectral flow transformations that relate the different (NS-NS, NS-R, R-NS, R-R) sectors of the theory. Non-linear sigma models on K3 (or K3 models, for short), provide the simplest examples of Calabi-Yau compactifications in type II string theory. A generic K3 model is not a rational CFT, and it cannot be solved exactly. Nevertheless, due to the large amount of space-time and worldsheet supersymmetries, many general results about these models are known, such as the geometry of the moduli space, the elliptic genus (which is the same for every K3 model), the spectrum of short $\mathcal{N}=(4,4)$ representations, and even the finite groups of symmetries at each point in the moduli space [25–27]. For these reasons, K3 models represent the ideal framework to understand topological defects in a non-rational CFT, besides the torus models examples. An interesting analysis of the topological defects in some non-rational K3 models, using a different approach, recently appeared in [28]; we will comment about the relationship with this article in the conclusions (see point 4 in section 7).

Because K3 models are not rational with respect to the $\mathcal{N}=(4,4)$ superconformal algebra, one can expect infinitely many distinct simple defects. In particular, we will see some examples of K3 models where a continuum of simple non-invertible topological defects arise, a phenomenon that has already been observed in orbifolds of torus models [15, 22, 23]. This implies that, strictly speaking, we are putting ourselves outside of the mathematical framework of fusion categories, at least in its most restrictive definitions. Nevertheless, we will assume that the basic properties of fusion categories still hold for the defects we consider. In particular, we will require that the fusion of any two defects is a superposition of finitely many simple defects. We denote by $\mathsf{Top}_{\mathcal{C}}$ the fusion category (in a broad sense) of topological defects in a K3 model $\mathcal{C}$ preserving the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow. We will only focus on certain properties of this category, in particular on its fusion ring, on the behaviour of defects when moved past local operators, and on the fusion with boundary states. We will mostly ignore the detailed properties of the fusion matrices. The main constraints on the topological defects in $\mathsf{Top}_{\mathcal{C}}$ come from considering the fusion with boundary states representing 1/2 BPS D-branes, i.e. preserving half of the space-time supersymmetry of type II superstring compactified on $\mathcal{C}$. The idea of studying defects in the presence of boundaries is a very natural one, and has been explored in a number of articles [29–37]. The 1/2 BPS D-branes we are interested in are always charged with respect to the $\mathrm{U}(1)^{24}$ gauge group of R-R ground fields of the theory. One can argue that fusion with defects in $\mathsf{Top}_{\mathcal{C}}$ preserves the set of such boundary states. This means that each defect $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$ can be associated with a $\mathbb{Z}$-linear map (endomorphism) $\mathsf{L}\in\mathrm{End}(\ {}^{4,20}_{R-R})$ on the even unimodular lattice ${}^{4,20}_{R-R}$ of D-brane R-R charges. The endomorphism L is further constrained by the requirement that the defect cannot mix R-R ground fields belonging to different representations of the $\mathcal{N}=(4,4)$ algebra. The map $\mathsf{Top}_{\mathcal{C}}\to\mathrm{End}(\ {}^{4,20}_{R-R})$ gives rise to a ring homomorphism from the fusion ring of $\mathsf{Top}_{\mathcal{C}}$ the ring $\mathrm{End}(\ {}^{4,20}_{R-R})$ (or rather the subring of endomorphisms satisfying suitable properties), so that properties of the former ring can be deduced by studying the latter. One can argue that the property of a defect $\mathcal{L}$ to be preserved by some marginal deformation of the model depends only on L.

Using this simple idea, we are able to derive several properties of topological defects in generic K3 models (see in particular section 3.2). We argue that $\mathsf{Top}_{\mathcal{C}}$ is trivial (i.e. the only simple defect is the identity) in most K3 models $\mathcal{C}$, except a subset with null measure in the moduli space (Claim 3). While the map $\mathsf{Top}_{\mathcal{C}}\to\mathrm{End}(\ {}^{4,20}_{R-R})$ is not injective, we can show that an endomorphism L proportional to the identity can only be associated to a superposition of n copies of the identity defect. Given a point $\mathcal{C}$ in the moduli space of K3 models, we will spell out some necessary conditions for $\mathsf{Top}_{\mathcal{C}}$ to be an integral category, i.e. such that the quantum dimensions of all topological defects are integral (Claim 2). In particular, this condition is satisfied for the points in the moduli space of K3 models that are attractor points for some 1/2 BPS configuration of D-branes [38–41]. We also derive some weaker constraints on the quantum dimensions that are valid everywhere in the moduli space.

While we know some examples of K3 models $\mathcal{C}$ where $\mathsf{Top}_{\mathcal{C}}$ contains a continuum of topological defects, Claim 3 implies that this cannot be the generic situation. It is natural to ask for a characterization of K3 models where such a continuum exists. In section 4.5, we conjecture that this only happens for (generalised) orbifolds of torus models.

The main limitations in our approach comes from the fact that the map $\mathsf{Top}_{\mathcal{C}} \to \mathrm{End}(\ ^{4,20}_{R-R})$ is, in general, neither injective, nor surjective. In particular, whenever $\mathsf{Top}_{\mathcal{C}}$ admits a continuum of defect $\mathcal{L}_\theta$, parametrised by some real parameter(s) θ, all such defects $\mathcal{L}_\theta$ are necessarily mapped to the same endomorphism L. As for surjectivity, while we are able to put some constraints on the endomorphisms $\mathsf{L} \in \mathrm{End}(\ ^{4,20}_{R-R})$ that arise from a defect $\mathcal{L}$, we cannot determine precisely what the image of the map $\mathsf{Top}_{\mathcal{C}} \to \mathrm{End}(\ ^{4,20}_{R-R})$ is.

The article is structured as follows. In section 2, we review some basic facts about topological defects in two dimensional CFTs, and fix the notation that we will use in the rest of the paper. Section 3 is the core of the article: after reviewing the main properties of K3 models, we describe and prove the main results of our work in sections 3.2 and 3.3. We stress that the proofs are on a physics level of rigour, as they are based on various assumptions about K3 models and boundary states that are not mathematically rigorous. This is why we prefer to call such statements 'Claim' rather than 'Theorem'. In section 4 we focus on K3 models that can be described as torus orbifolds. We show that, in general, they admit a continuum of topological defects in Top. In section 4.5, we conjecture that the converse might be true: generalised torus orbifolds are actually the *only* K3 models for which such a continuum exists. In sections 5 and 6, we describe some topological defects in $\mathsf{Top}_{\mathcal{C}}$ in a couple of interesting K3 models. While in none of these two models we were able to determine precisely the category $\mathsf{Top}_{\mathcal{C}}$, the examples are useful both to confirm some of the general arguments of section 3.2, and were used in the proof of some of the claims. Finally, in section 7 we describe some avenues for future investigation. Various technical details of our calculations are relegated in the appendices.

2 Generalities on topological defects in 2D CFT

In this section we give a simple and concise review about defects in two dimensional CFTs. We refer to [13, 16, 42] for more detailed information.

Usually, when we talk about a generic CFT, we characterize it by specifying the whole set of *local operators* and their corresponding OPEs. However, it is well known that in many cases further extended objects associated with non-local operators can also exist in the theory. Such objects encode additional properties of the quantum field theory that are not visible at the level of the spectrum, and they are easily understood in the language of the *defects*.

In a generic QFT defined over a d-dimensional spacetime $\mathcal{M}_d$, such extended objects can be described through operators $D_a\left(\mathcal{M}^{d-q}\right)$ supported on $(d-q)-$dimensional submanifolds of $\mathcal{M}_d$, with $q < d$. These operators are called *topological* in the sense that small deformations of their support manifold, which do not cross other operators of the theory, do not affect the physical observables.

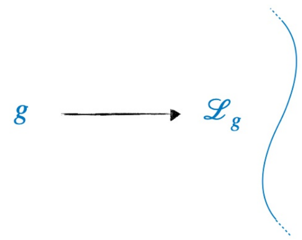

Figure 1. Invertible TDL in 2d associated to the element $g \in G$.

In the limit where the support manifolds of two distinct defects D_a and D_b overlap, the generalized OPE between the corresponding operators defines a *fusion algebra* among defects of the form:

$$D_a\left(\mathcal{M}^{d-q}\right) \times D_b\left(\mathcal{M}^{d-q}\right) = \sum_c N_{ab}^c D_c\left(\mathcal{M}^{d-q}\right). \tag{2.1}$$

The simplest example of defects we can encounter in a QFT are the *invertible defects*, which encode information about the standard and higher-form *global symmetries* owned by the theory. More specifically, let G be the group of p-form global symmetries in our QFT, then we can associate to each element $g \in G$ a $(d-p-1)-$dimensional topological defect D_g such that the induced fusion algebra (2.1) satisfies the same group multiplication law as G:

$$D_g\left(\mathcal{M}^{d-p-1}\right) \times D_{g'}\left(\mathcal{M}^{d-p-1}\right) = D_{g''}\left(\mathcal{M}^{d-p-1}\right), \qquad g'' = gg'. \tag{2.2}$$

The name *invertible* for this class of defects comes from the fact that for each of them, i.e. D_g, there exists a second defect $D_{g^{-1}}$ such that their fusion produces the trivial defect D_e associated with the identity operator:

$$D_g \times D_{g^{-1}} = D_{g^{-1}} \times D_g = D_e, \qquad D_e \mapsto \mathbb{I}. \tag{2.3}$$

The above definitions apply to any QFT of generic dimension d, where topological defects supported on submanifolds of different codimensions may be present. In the rest of the discussion we will focus our attention on 2D QFT, where topological defects have support on oriented $1-$dimensional manifolds (*lines*) of the 2d spacetime. For this reason, we will refer to them as *Topological Defect Lines* (TDLs) and denote them with the notation $\mathcal{L}$.

In particular, if G is the symmetry group of our CFT, we can associate to each element $g \in G$ an invertible TDL $\mathcal{L}_g$.

By definition of global symmetry, the elements of the group G define a non-trivial action on the bulk operators:

$$g: \quad \mathcal{O}_i(x_i) \quad \mapsto \quad \rho(g)\cdot\mathcal{O}_i(x_i) \tag{2.4}$$

while they leave the correlators invariant:

$$\forall g \in G \quad \longrightarrow \quad \langle \prod_i \mathcal{O}_i(x_i) \rangle = \langle \prod_i \left(\rho(g)\cdot\mathcal{O}_i(x_i)\right) \rangle. \tag{2.5}$$

We can represent the action (2.4) in the language of topological defects through the loop contraction of the TDL $\mathcal{L}_g$ encircling the bulk operator $\mathcal{O}_\psi(x_i)$ as depicted on the left side

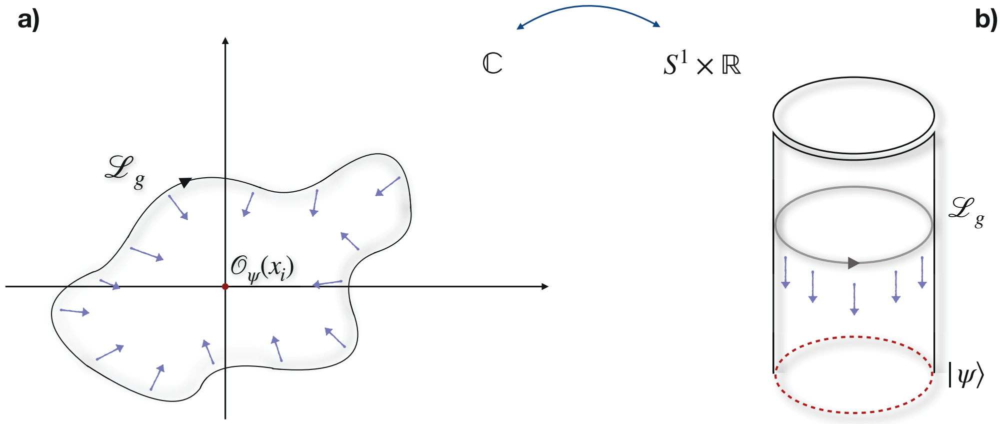

Figure 2. **(a)** Action of $\mathcal{L}_g$ on the bulk operator $\mathcal{O}_\psi(x_i)$. **(b)** Action of $\mathcal{L}_g$ on the asymptotic state $|\psi\rangle$.

of figure 2. This mechanism is also valid for general TDLs that are not associated with a global symmetry group. From now on we will use the hat notation $\hat{\mathcal{L}}$ to denote the extended operator supported on the line $\mathcal{L}$.

Invertible defects associated with the elements of a continuous global symmetry group G offer the simplest explicit construction of an extended operator supported on an oriented line. In this case the Noether's theorem provides us a set of conserved currents $J_{(r)}$, such that:

$$\langle d{*}J_{(r)}(x)\dots\rangle = 0, \tag{2.6}$$

where the dots denote any operator insertion away from the point x. Exponentiating the integral of the Noether's currents on the support line $\mathcal{L}$:

$$\hat{\mathcal{L}}_g = e^{i\alpha^{(r)}\int_{\mathcal{L}} {*}J_{(r)}}. \tag{2.7}$$

we get the extended operator $\hat{\mathcal{L}}_g$ associated with the element $g \in G$ specified by the group parameters $\left\{\alpha^{(r)}\right\}$. The operator $\hat{\mathcal{L}}_g$ is topological due to the conservation law in (2.6).

Similarly, we can define extended operators associated with elements of discrete symmetry groups satisfying the same above proprieties.

Beyond these objects, we can equip our theory with other point like operators where TDLs can terminate or join. In the first case we can associate to each TDL $\mathcal{L}$ the space $\mathcal{H}_{\mathcal{L}}$ of possible point like operators on which the line $\mathcal{L}$ can end. If $\mathcal{H}_{\mathcal{L}} \neq \emptyset$, then the line $\mathcal{L}$ is said *endable*, and the point like operators of $\mathcal{H}_{\mathcal{L}}$ are called *defect operators*.

2.1 Defining properties of topological defect lines

Let us now focus on the case of topological defects $\mathcal{L}(\gamma)$ supported on lines γ in unitary Euclidean two dimensional CFTs. In the same line of the Introduction, we can think about *topological defects* as a generalization of *global symmetries*. As we spell out below, the set of such defects is equipped with a composition law (2.1) that is generally non-invertible. This means that one cannot define on this set the standard group structure, as for ordinary symmetries, but rather a fusion category.

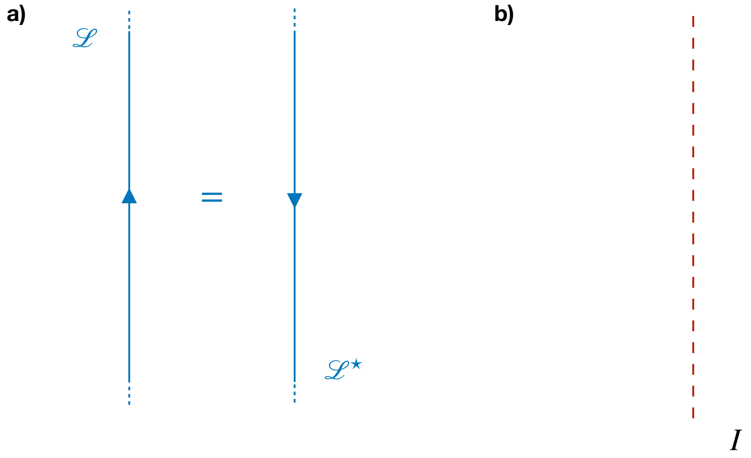

Figure 3. **(a)** Involution map $\mathcal{L} \mapsto \mathcal{L}^\star$. **(b)** Identity operator in the trivial line.

In this subsection we will summarize the basic properties that we need in order to specify this structure.

As stated above, topological defect lines (TDLs) $\mathcal{L}$ represent the fundamental objects of the structure. They are supported on oriented lines γ on the worldsheet. On the set of distinct defects $\{\mathcal{L}\}$ is defined an involution $\mathcal{L} \mapsto \mathcal{L}^\star$, shown in figure 3(a), that corresponds to inverting the orientation of the support. A defect $\mathcal{L}$ is unoriented if $\mathcal{L}^\star = \mathcal{L}$.

Correlation functions with the insertions of topological defects are invariant under deformation of the support line, as long as the line is not moved past another operator insertion. We will say that a defect $\mathcal{L}$ and a local operator ϕ are 'transparent' to each other, or that $\mathcal{L}$ preserves ϕ, if the support $\mathcal{L}$ can be moved past the support of ϕ without changing any correlation function. In general, the holomorphic and anti-holomorphic stress tensors $T(z)$ and $T(z)$ are always preserved by any topological defects. In the following sections, we will consider topological defects that preserve all supercurrents generating the $\mathcal{N} = (4,4)$ superconformal algebra of a K3 sigma model.

The set of TDLs in a 2d QFT always includes the *Identity* defect $\mathcal{I}$, typically represented by the dotted lines as depicted in figure 3 (b). The insertion or removal of the identity defect does not change any correlation function; equivalently, $\mathcal{I}$ is transparent to all local operators of the theory.

Let us consider the two-dimensional CFT on the cylinder $S^1 \times \mathbb{R}$. A TDL $\mathcal{L}$ inserted along the circle S^1 defines a linear operator $\mathcal{L}: \mathcal{H} \to \mathcal{H}$ on the Hilbert space $\mathcal{H}$ of states on S^1 (see figure 2.b). Equivalently, we can consider a closed line $\mathcal{L}$ encircling the insertion point of a local operator $\phi(z,z)$, corresponding to a state $\phi \in \mathcal{H}$. By shrinking the circle around the point z we obtain a new local operator $(\mathcal{L}\phi)(z,z)$.[1]

On the other hand, inserting the defect line $\mathcal{L}$ along the Euclidean time direction $\mathbb{R}$ of $S^1 \times \mathbb{R}$, correspond to modifying the space of states $\mathcal{H}$. We denote by $\mathcal{H}_\mathcal{L}$ the new Hilbert space of states on the circle S^1 that are 'twisted' by the defect $\mathcal{L}$. For example, if $\mathcal{L}$ is an invertible defect associated with a symmetry $g \in G$, then $\mathcal{H}_\mathcal{L}$ is simply the *g-twisted sector* of the theory.

[1]To be precise, the definition of $\mathcal{L}$ on the sphere might differ from the definition of the cylinder by a phase, see section 2.4 in [13]. In this case, we reserve the notation $\mathcal{L}$ for the operator defined on the cylinder.

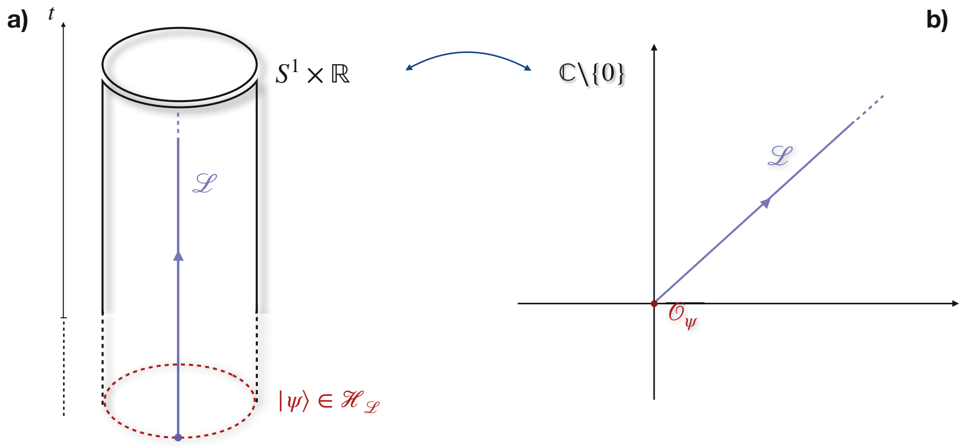

Figure 4. An *endable* TDL in the cylinder (figure a) produces a non-local operator in the complex plane (figure b).

The state/operator correspondence defined by the conformal mapping from the cylinder $S^1 \times \mathbb{R}$ to the plane $\mathbb{C}\backslash\{0\}$, allows us to identify every state $|\psi\rangle \in \mathcal{H}_\mathcal{L}$ on the cylinder with a non-local defect operator $\mathcal{O}_\psi$ on the plane, i.e. an operator attached to an outgoing defect $\mathcal{L}$, as shown in figure 4. Using this identification, we will refer to $\mathcal{H}_\mathcal{L}$ indifferently as the space of $\mathcal{L}$-twisted states or as the space of $\mathcal{L}$ defect operators. When $\mathcal{L} = \mathcal{I}$ we recover the ordinary space $\mathcal{H}_\mathcal{I} = \mathcal{H}$ of local point-like operators of the CFT.

If the defect $\mathcal{L}$ is transparent for a certain set of local (anti-)holomorphic operators, then the space $\mathcal{H}_\mathcal{L}$ is a representation of the corresponding (anti-)chiral algebra. In particular, $\mathcal{H}_\mathcal{L}$ is always a representations of the holomorphic and antiholomorphic Virasoro algebra. More generally, the space $\mathcal{H}_\mathcal{L}$ is a $\mathcal{L}$-twisted representation of the algebra of local operators on the cylinder. This means that for every $\phi \in \mathcal{H}$, $\phi(z,\bar{z})$ defines a linear operator on the Hilbert space $\mathcal{H}_\mathcal{L}$ that obeys the OPE relations with other local operators. By $\mathcal{L}$-twisted, we mean that correlation functions on $S^1 \times \mathbb{R}$ with the insertion of a point operator $\phi(z,\bar{z})$ and a defect line $\mathcal{L}$ along $\mathbb{R}$ are not quite periodic as z moves around S^1, but have some non-trivial discontinuities at the support of the defect that depend on $\mathcal{L}$. A defect $\mathcal{L}$ is *simple* if the space $\mathcal{H}_\mathcal{L}$ is irreducible as a twisted representation of the algebra of local operators.

When the CFT is defined on a torus $S^1 \times S^1$, the modular S-transformation exchanges the insertion of a line $\mathcal{L}$ along the 'space' circle with the insertion along the Euclidean 'time' circle. This establishes a relation between the linear operator $\hat{\mathcal{L}}$ on $\mathcal{H}$ and the twisted space $\mathcal{H}_\mathcal{L}$.

The 2-point correlation functions on the sphere with two defect operators $\phi(z,\bar{z}), \psi(w,\bar{w})$ connected by a defect line $\mathcal{L}$ define a natural non-degenerate bilinear pairing (ϕ, ψ) between $\mathcal{H}_\mathcal{L}$ and $\mathcal{H}_{\mathcal{L}^\star}$. The bilinear pairing is related to the hermitian product on the Hilbert space $\mathcal{H}_\mathcal{L}$ by a anti-linear involution $\iota : \mathcal{H}_\mathcal{L} \to \mathcal{H}_{\mathcal{L}^\star}$, such that $\langle \phi_1 | \phi_2 \rangle = (\iota(\phi_1), \phi_2)$.

The set of TDLs is endowed with the algebraic structure of a generally non-commutative (semi-)ring defined by the two operations of *direct sum* (or superposition) $(+)$ and *fusion* $(\triangleright)$.

The *direct sum* allows to associate to each pair of topological defects $\mathcal{L}_a$ and $\mathcal{L}_b$ a third defect $\mathcal{L}_a + \mathcal{L}_b$ such that:

$$\mathcal{H}_{\mathcal{L}_a + \mathcal{L}_b} = \mathcal{H}_{\mathcal{L}_a} \oplus \mathcal{H}_{\mathcal{L}_b}. \tag{2.8}$$

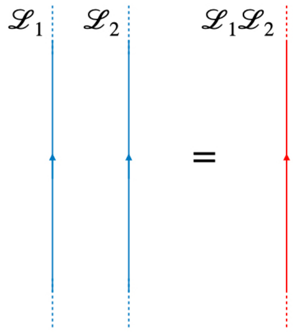

Figure 5. Fusion of two TDLs.

This first operation is associative and commutative. Unitary CFTs are expected to be *semi-simple*, i.e. every defect $\mathcal{L}$ can be written as a superposition of *simple* defects

$$\mathcal{L} = \sum_{\text{simple } \mathcal{L}_i} n_i \mathcal{L}_i \,, \tag{2.9}$$

for some non-negative multiplicities $n_i \in \mathbb{Z}_{\geq 0}$. Correspondingly, each reducible $\mathcal{H}_\mathcal{L}$ can be decomposed into a direct sum of irreducible components $\mathcal{H}_\mathcal{L} = \oplus_i n_i \mathcal{H}_{\mathcal{L}_i}$.

If there are no additional intermediate insertions between two TDLs $\mathcal{L}_1$ and $\mathcal{L}_2$, we can define the *fusion* deforming one defect into the other, as shown in figure 5. The resulting object is again a topological defect line denoted as:

$$\mathcal{L}_1 \triangleright \mathcal{L}_2 = \mathcal{L}_1 \mathcal{L}_2 . \tag{2.10}$$

Fusion defines a notion of tensor product between defect operator spaces

$$\mathcal{H}_{\mathcal{L}_1} \otimes \mathcal{H}_{\mathcal{L}_2} = \mathcal{H}_{\mathcal{L}_1 \mathcal{L}_2} \tag{2.11}$$

This second operation is associative but it is in general not commutative. The fusion of two simple defects $\mathcal{L}_i$ and $\mathcal{L}_j$ is not necessarily simple, so that one has a decomposition

$$\mathcal{L}_i \mathcal{L}_j = \sum_{\text{simple } \mathcal{L}_k} N_{ij}^k \mathcal{L}_k \,, \tag{2.12}$$

for some fusion coefficients $N_{ij}^k \in \mathbb{Z}_{\geq 0}$. Formally, this means that the set of defects has the structure of a fusion ring, with the simple defects playing the role of a distinguished basis.

Sets of defects that also satisfy commutativity for the fusion form commutative rings.

The *trivial line* $\mathcal{I}$ is the *neutral element* under fusion

$$\mathcal{L}\mathcal{I} = \mathcal{I}\mathcal{L} = \mathcal{L} . \tag{2.13}$$

In analogy with the construction of the spaces $\mathcal{H}_\mathcal{L}$, we can consider k parallel defect lines $\mathcal{L}_1, \ldots, \mathcal{L}_k$ inserted in the time-like direction on the cylinder. The corresponding Hilbert space $\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$ of states on S^1 is identified, via the state/operator correspondence, with the space of *k-junction* operators. As a convention, we choose that all the involved lines are outgoing from the junction. Once again, $\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$ are genuine representations of the chiral and anti-chiral algebras preserved by the defects, and a suitable twisted representation of the algebra of local operators.

By moving the parallel lines $\mathcal{L}_1,\ldots,\mathcal{L}_k$ on the cylinder $S^1\times\mathbb{R}$ very close to each other, we get the identification

$$\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}\cong\mathcal{H}_{\mathcal{L}_1\cdots\mathcal{L}_k}\cong\mathcal{H}_{\mathcal{L}_1}\otimes\cdots\otimes\mathcal{H}_{\mathcal{L}_k}\,, \tag{2.14}$$

between the k-junction space $\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$ and the space of defect operators of the fusion $\mathcal{L}_1\cdots\mathcal{L}_k$.

The subspace $V_{\mathcal{L}_1,\ldots,\mathcal{L}_k}\subset\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$ of states with conformal weights $(0,0)$ correspond to junction operators that are themselves topological, i.e. such that the junction point can be moved without changing a correlation function (as long as the insertion point is not moved past the support of some other operator). We can restrict ourselves to consider the correlation functions where all the k-junctions with $k>1$ are topological, because all the other junction operators can be obtained by suitable OPE with local operators.

For a simple defect $\mathcal{L}$, the only topological $\mathcal{L}$-twisted operator is the vacuum operator for $\mathcal{L}=\mathcal{I}$, so that

$$\dim V_{\mathcal{L}}=\begin{cases}1 & \text{for } \mathcal{L}=\mathcal{I}\\ 0 & \text{for simple } \mathcal{L}\neq\mathcal{I}\ .\end{cases} \tag{2.15}$$

For $k>1$, a topological operator $u\in V_{\mathcal{L}_1,\ldots,\mathcal{L}_k}$ can also be interpreted as a linear map $u:\mathcal{H}_{\mathcal{L}_k^\star}\to\mathcal{H}_{\mathcal{L}_1,\ldots,\mathcal{L}_{k-1}}$ that is a homomorphism of twisted representations of the algebra of local operators. In particular, $V_{\mathcal{L},\mathcal{L}^\star}=\mathrm{Hom}(\mathcal{H}_{\mathcal{L}},\mathcal{H}_{\mathcal{L}})$ has always dimension at least 1, because it contains (multiples) of the identity map $\mathbf{1}_{\mathcal{L}}:\mathcal{H}_{\mathcal{L}}\to\mathcal{H}_{\mathcal{L}}$. A defect $\mathcal{L}$ is simple if and only if $\dim V_{\mathcal{L},\mathcal{L}^\star}=1$. Furthermore, for two simple defects $\mathcal{L}$ and $\mathcal{L}'$,

$$\dim V_{\mathcal{L}',\mathcal{L}^\star}=\begin{cases}1 & \text{if } \mathcal{L}=\mathcal{L}'\\ 0 & \text{otherwise}\end{cases}.$$

As for topological 3-junctions, one can prove that if $\mathcal{L}_i$, $\mathcal{L}_j$, and $\mathcal{L}_k$ are simple defects, then the dimension of topological junction operators is exactly the fusion coefficient

$$\dim V_{\mathcal{L}_i,\mathcal{L}_i,\mathcal{L}_k^\star}=N_{ij}^k\ . \tag{2.16}$$

This fits with the idea that $V_{\mathcal{L}_i,\mathcal{L}_i,\mathcal{L}_k^\star}$ is the space $\mathrm{Hom}(\mathcal{H}_{\mathcal{L}_k},\mathcal{H}_{\mathcal{L}_i\mathcal{L}_j})$ of morphisms from $\mathcal{H}_{\mathcal{L}_k}$ to $\mathcal{H}_{\mathcal{L}_i\mathcal{L}_j}=\oplus_l N_{ij}^l\mathcal{H}_{\mathcal{L}_l}$. Notice that for $\mathcal{L}$ simple, one can think of the identity 2-junction $1\in V_{\mathcal{L},\mathcal{L}^\star}$ as a 3-junction with the identity defect. As a consequence, $\mathcal{L}$ is simple if and only if $\dim V_{\mathcal{L},\mathcal{L}^\star,\mathcal{I}}=1$, i.e. if and only if $\mathcal{I}$ appears with multiplicity 1 in the fusion of $\mathcal{L}$ with its dual

$$\mathcal{L}\mathcal{L}^\star=\mathcal{I}+\ldots\ . \tag{2.17}$$

Here, $\ldots$ denotes a sum with non-negative multiplicities over simple defects distinct from the identity.

At this point it is important to emphasize that the set of topological defect lines equipped by fusion multiplication in general does not form a group. The reason is the absence of an inverse element associated to each TDL. Only a subclass of all possible TDLs admit an

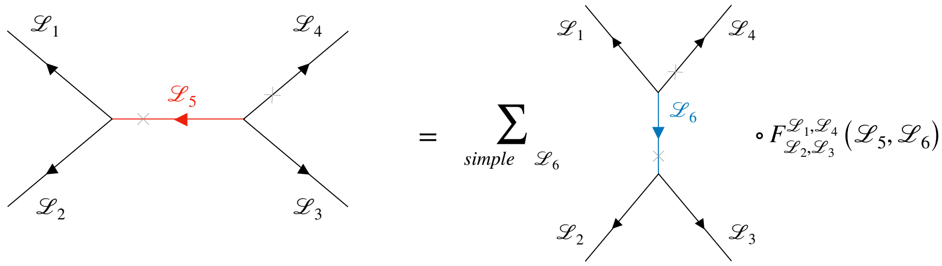

Figure 6. Equivalent configurations under the action of the fusion matrices $F^{\mathcal{L}_1,\mathcal{L}_3}_{\mathcal{L}_2,\mathcal{L}_4}$.

inverse under fusion. They are the *invertible TDLs*, and they form a group with respect to this operation.

The remaining TDLs are called *non-invertible* and, equipped with the fusion multiplication and the direct sum, they form a more complicated algebraic structure named *Fusion Ring*.

A fundamental quantity that we can associate with each TDL $\mathcal{L}$ is the *quantum dimension* $\langle\mathcal{L}\rangle \equiv \langle\mathcal{L}\rangle_{S^1\times\mathbb{R}}$, defined as the vacuum expectation value of a defect $\mathcal{L}$ wrapping the circle S^1 on the cylinder:

$$\langle\mathcal{L}\rangle \equiv \langle\mathcal{L}\rangle_{S^1\times\mathbb{R}} := \langle 0|\mathcal{L}|0\rangle. \tag{2.18}$$

Using the modular invariance properties of the partition function with the defect $\mathcal{L}$ inserted, it is easy to prove that for unitary theories with a unique vacuum the quantum dimension is bounded from below:

$$\langle\mathcal{L}\rangle \geq 0. \tag{2.19}$$

Such constraint is more restrictive when we consider a unitary, compact CFT, where the condition becomes:

$$\langle\mathcal{L}\rangle \geq 1. \tag{2.20}$$

Notice that the quantum dimension is also the absolute value of the vacuum expectation value $\langle\mathcal{L}\rangle_{S^2}$ on the sphere, defined by considering an loop encircling only the vacuum on S^2

$$|\langle\mathcal{L}\rangle_{S^2}| = \langle\mathcal{L}\rangle\,,$$

but in general the phase might be different. The quantum dimensions provide a 1-dimensional representation of the fusion ring, so that, in particular,

$$\langle\mathcal{L}_i\rangle\langle\mathcal{L}_j\rangle = \sum_k N_{ij}^k \langle\mathcal{L}_k\rangle\,, \tag{2.21}$$

for any simple $\mathcal{L}_i, \mathcal{L}_j, \mathcal{L}_k$. Together with the condition $\langle\mathcal{L}\rangle \geq 1$, this means that for every simple $\mathcal{L}_i$, $\mathcal{L}_j$ of finite quantum dimension, there are only finitely many non-zero fusion coefficients N_{ij}^k. In this article, we only consider defects with finite quantum dimension; see [22] for a discussion about more general possibilities.

The topology of a network of defects can be modified using the fusion rules shown in figure 6, where the fusion matrices $F^{\mathcal{L}_1,\mathcal{L}_4}_{\mathcal{L}_2,\mathcal{L}_3}(\mathcal{L}_5,\mathcal{L}_6)$ map the topological junctions $V_{\mathcal{L}_1,\mathcal{L}_2,\mathcal{L}_5^\star} \otimes$

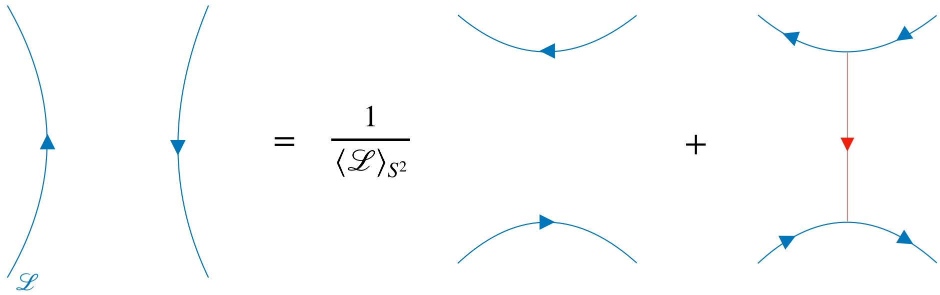

Figure 7. Equivalent configurations under the action of the fusion move. In the second term on the right: the red defect is the superposition (with multiplicity) of the simple defects $\mathcal{L}_k \neq \mathcal{I}$ appearing in the fusion $\mathcal{L}\mathcal{L}^\star$; the topological 3-junction operators are given by the images of $\mathbf{1}\otimes\mathbf{1}$ by matrices

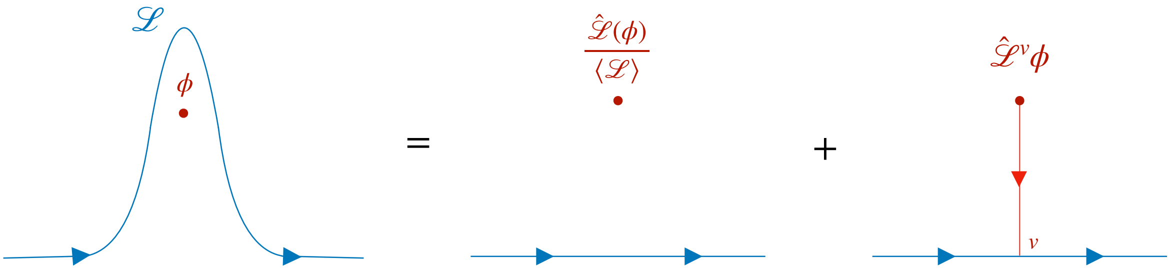

Figure 8. Moving of a simple defect line through the local operator $\phi(z,\bar{z})$. The defect in the second term on the right is a superposition of all simple defect lines $\mathcal{L}_i \neq \mathcal{I}$ appearing in the fusion $\mathcal{L}\mathcal{L}^\star = \mathcal{I} + \ldots$. The factor $1/\langle\mathcal{L}\rangle$ is determined by observing that the defect must be transparent to the vacuum operator and to its descendants.

$V_{\mathcal{L}_5,\mathcal{L}_3,\mathcal{L}_4}$ in $V_{\mathcal{L}_1,\mathcal{L}_6,\mathcal{L}_4} \otimes V_{\mathcal{L}_2,\mathcal{L}_3,\mathcal{L}_6^\star}$. In particular, $F^{\mathcal{L},\mathcal{L}^\star}_{\mathcal{L}^\star,\mathcal{L}}(\mathcal{I},\mathcal{I})$ maps $\mathbf{1}\otimes\mathbf{1}$ to $\frac{1}{\langle\hat{\mathcal{L}}\rangle_{S^2}}\mathbf{1}\otimes\mathbf{1}$, see figure 7.

This can be used to determine how the correlation function is modified when a simple defect $\mathcal{L}$ is moved past a local operator $\phi(z,z)$, see figure 8. In particular, $\mathcal{L}$ is transparent to ϕ if and only if $\mathcal{L}(\phi) = \langle\mathcal{L}\rangle\phi$, because in this case one can prove that the term $\mathcal{L}^v(\phi)$ in figure 8 vanishes. We will use this property repeatedly in the following.

Formally, the properties of TDLs described in this section can be formulated in terms of a *fusion category*, where the defects $\mathcal{L}$ are the objects and the topological junctions $u \in V_{\mathcal{L}_1,\mathcal{L}_2^\star} \equiv \mathrm{Hom}(\mathcal{L}_2,\mathcal{L}_1)$ are the morphisms. Sometimes in the definition of fusion categories one requires that the number of simple objects is finite. This condition might be in general violated when a CFT is not rational with respect to the chiral and anti-chiral algebra preserved by the topological defects, as will be the case in this article.

3 Topological defects in K3 models

In this section, after a short review of non-linear sigma models on K3, we discuss the general properties of topological defects in such models.

3.1 General properties of K3 models

Let us review some of the main properties of supersymmetric non-linear sigma models on K3 (or K3 models, for short), and fix our notation for the rest of the article. See [25, 27] for more information.

All K3 models contain a holomorphic and an anti-holomorphic copy of the small $\mathcal{N}=4$ superconformal algebra at central charge $c=6$ [43–45]. The bosonic subalgebra of $\mathcal{N}=4$ is generated by the stress-tensor and the three currents in a $\widehat{su}(2)_1$ current algebra. The SU(2) group generated by the zero modes of such holomorphic currents is the R-symmetry group, and the four supercurrents belong to two R-symmetry doublets. The unitary representations of the $\mathcal{N}=4$ algebra at $c=6$ can be labeled by a pair (h,q), where h is the conformal weight and $q\in\{0,1/2\}$ the SU(2) highest weight of the ground states in the representation; similarly, we will denote by $(h,q;h,q)$ the representations of the full (holomorphic and antiholomorphic) $\mathcal{N}=(4,4)$ algebra at $(c,c)=(6,6)$. The Neveu-Schwarz (NS) version of the $\mathcal{N}=4$ algebra contains two unitary short (BPS) representations $(h,q)=(0,0)$ and $(h,q)=(\frac{1}{2},\frac{1}{2})$ and infinitely many long (massive) representations $(h,0)$ with $h>0$. The spectral flow automorphism relating the NS and the Ramond (R) versions of $\mathcal{N}=4$ maps the NS shorts representations $(0,0)$ and $(\frac{1}{2},\frac{1}{2})$ to the short R representations $(\frac{1}{4},\frac{1}{2})$ and $(\frac{1}{4},0)$, respectively, and the long $(h,0)$ with $h>0$ to the long R representation $(h+\frac{1}{4},\frac{1}{2})$.

Every K3 model contains a single $(0,0;0,0)$ representation of $\mathcal{N}=(4,4)$ (the vacuum and its descendants), and 20 $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ representations. There are no fields in the $(\frac{1}{2},\frac{1}{2};0,0)$ or $(0,0;\frac{1}{2},\frac{1}{2})$; such holomorphic and anti-holomorphic free fermions are a characteristic feature of supersymmetric sigma models on T^4, where they are part of a 'large' $\mathcal{N}=(4,4)$ superconformal algebra at $(c,c)=(6,6)$. Besides these short-short (BPS) representations, a K3 model contains infinitely many different short-long, long-short and long-long representations, with multiplicities depending on the particular model. It is believed that the $\mathcal{N}=4$ superconformal algebra is the full chiral algebra of a generic K3 model, that is therefore not a rational CFT. At special points in the moduli space of K3 models, the chiral algebra might be extended so that the CFT is rational.

K3 models are invariant under spectral flow exchanging the NS-NS sector and R-R sector. This implies that the R-R sector contains a single $(\frac{1}{4},\frac{1}{2};\frac{1}{4},\frac{1}{2})$ and 20 $(\frac{1}{4},0;\frac{1}{4},0)$ representations, as well as infinitely many $(h,\frac{1}{2};h,\frac{1}{2})$, with $h+h>1/2$. The four R-R ground fields of $(\frac{1}{4},\frac{1}{2};\frac{1}{4},\frac{1}{2})$ transform in a $(\mathbf{2},\mathbf{2})$ representation of the holomorphic and anti-holomorphic $\mathrm{SU}(2)\times\mathrm{SU}(2)$ R-symmetry group; the OPE with such fields generate the spectral flow between the NS-NS and the R-R sectors. For this reason, we will refer to such fields as the *spectral flow generators*.

It is often useful to think of the K3 model as the internal CFT in a six dimensional compactification of type IIA superstring on $\mathbb{R}^{1,5}\times K3$. In this case, one should also consider the NS-R and R-NS sectors of the K3 sigma model, tensored with the corresponding sectors of the space-time and superghost CFTs, and with the correct GSO projections. The NS-R and R-NS sectors are also related to the NS-NS and R-R sectors by spectral flow, so that they contain a single $(\frac{1}{4},\frac{1}{2};0,0)$ and $(0,0;\frac{1}{4},\frac{1}{2})$ representations. The corresponding physical string states are the space-time gravitinos, whose zero modes is associated with the space-time $\mathcal{N}=(1,1)$ supersymmetry in six dimensions.

In general, sigma models on K3 and on T^4 are the only known (and, conjecturally, the only consistent) unitary SCFTs with $\mathcal{N}=(4,4)$ superconformal algebra at $(c,c)=(6,6)$ and whose spectrum is invariant under spectral flow.

Every K3 model admits deformations that preserve the full $\mathcal{N}=(4,4)$ superconformal symmetry, corresponding to an 80-dimensional space of exactly marginal operators contained in the 20 NS-NS $(\frac{1}{2},\frac{1}{2};\frac{1}{2},\frac{1}{2})$ representations of $\mathcal{N}=(4,4)$. There are compelling arguments suggesting that conformal perturbation by any such operator converges in a neighborhood of each model; we will assume that this is true. The corresponding 80-dimensional moduli space $\mathcal{M}_{K3}$ of K3 models is given by a quotient

$$\mathcal{M}_{K3}=O(4,20,\mathbb{Z})\backslash\mathcal{T}_{K3} \tag{3.1}$$

where $O(4,20,\mathbb{Z})$ is the integral orthogonal group. Here, the Teichmüller space $\mathcal{T}_{K3}$ is an open subset in the Grassmannian

$$\mathcal{T}_{K3}\subset O(4,20,\mathbb{R})/(O(4)\times O(20)) \tag{3.2}$$

parametrising positive definite four-dimensional subspaces within the real space $\mathbb{R}^{4,20}$ with signature $(4,20)$. The moduli space $\mathcal{M}_{K3}$ admits the following physical interpretation: $O(4,20,\mathbb{Z})$ is the T-duality group, and can be identified with the group of automorphisms of the lattice ^{4,20} of D-brane charges, which is an even unimodular lattice of signature $(4,20)$. One can think of this lattice (or rather its dual) as being embedded in the 24-dimensional real space V of (CPT self-conjugate) Ramond-Ramond ground states with conformal weights $h=h=\frac{1}{4}$:

$$^{4,20}\subset V:=\{\text{R-R ground states}\}\cong\ ^{4,20}\otimes\mathbb{R}\cong\mathbb{R}^{4,20}\ . \tag{3.3}$$

The space V contains a positive definite subspace

$$V\supset\ :=\{\text{spectral flow generators}\}\cong\mathbb{R}^{4,0}, \tag{3.4}$$

spanned by the four spectral flow generators, i.e. the R-R ground states in a $(\frac{1}{4},\frac{1}{2};\frac{1}{4},\frac{1}{2})$ representation of $\mathcal{N}=(4,4)$. The orthogonal complement $^{\perp}\subset V$ is the space of R-R ground states in the 20 $\mathcal{N}=(4,4)$ representations $(\frac{1}{4},0;\frac{1}{4},0)$

$$V\supset\ ^{\perp}:=\{\text{states in }(\frac{1}{4},0;\frac{1}{4},0)\text{ representations}\}\cong\mathbb{R}^{0,20}\ . \tag{3.5}$$

Then, $\mathcal{M}_{K3}$ is essentially the Grassmannian of the four-dimensional subspaces $\cong\mathbb{R}^{4,0}$ within $V\cong\ ^{4,20}\otimes\mathbb{R}$, modulo lattice automorphisms (T-dualities) $O(4,20,\mathbb{Z})\cong O(\ ^{4,20})$. To be precise, one needs to exclude some points in this Grassmannian, where the CFT is believed to be inconsistent [25]. From a string theoretical point of view, these are the points in the moduli space of type IIA superstrings where some D-brane becomes exactly massless, so that the perturbative description breaks down even at small string coupling.

Henceforth, we will denote by

$$\mathcal{C}_{\Pi}:=\text{K3 model corresponding to }\ \subset V\cong\ ^{4,20}\otimes\mathbb{R} \tag{3.6}$$

the K3 model corresponding to a choice of $\subset V$, i.e. to a point in the Grassmannian $\mathcal{T}_{K3}$.

3.2 Symmetries and topological defects

The main focus of this article are the topological defects in non-linear sigma models of K3. A generic topological defect commutes only with the chiral and anti-chiral Virasoro algebra, i.e. it is 'transparent' only to the holomorphic and antiholomorphic stress-energy tensor $T(z)$ and $T(z)$. In this article, we will focus on a subcategory of topological defects $\mathcal{L}$ that satisfy some further constraints, namely:

1. They commute with the full $\mathcal{N}=(4,4)$ superconformal algebra, i.e. they are transparent to all supercurrents and to the $\widehat{su}(2)_1$ R-symmetry currents, besides the stress-energy tensor;

2. They commute with spectral flow generators. This implies that they are transparent to the four R-R ground fields in the $(\frac{1}{4},\frac{1}{2};\frac{1}{4},\frac{1}{2})$ representation of $\mathcal{N}=(4,4)$. When the K3 model is the internal SCFT in a full type IIA compactification, we also require the defect to be transparent with respect to the NS-R and R-NS ground fields corresponding to the space-time gravitini. These fields generate the purely holomorphic or purely anti-holomorphic spectral flows.

For a K3 model $\mathcal{C}\equiv\mathcal{C}_\Pi$, corresponding to a choice of a four dimensional positive definite subspace $\ \subset {}^{4,20}\otimes\mathbb{R}$, we denote by

$$\mathsf{Top}_\mathcal{C}\equiv\mathsf{Top}_\Pi \tag{3.7}$$

the category of topological defects of $\mathcal{C}_\Pi$ satisfying the properties 1 and 2.

Properties 1 and 2 lead to a number of important consequences:

- As explained in section 2, each topological defect $\mathcal{L}$ is associated with a linear operator

$$\mathcal{L}:\mathcal{H}\to\mathcal{H} \tag{3.8}$$

  on the Hilbert space of states on the circle S^1 (or, equivalently, the space of local point-like operators) of the K3 model $\mathcal{C}$. The action of $\mathcal{L}$ is defined by inserting a defect $\mathcal{L}$ along the circle S^1 on a cylinder $S^1\times\mathbb{R}$. Because of the properties 1 and 2, this operator commutes with the $\mathcal{N}=(4,4)$ algebra and the spectral flow. Therefore, it maps $\mathcal{N}=4$ primaries into $\mathcal{N}=4$ primaries in the same representation. Furthermore, because the defect $\mathcal{L}$ is transparent with respect to the spectral flow generators, once the action of $\mathcal{L}$ is known on one of the sectors ($\mathcal{H}_{NS-NS}$, $\mathcal{H}_{R-R}$, $\mathcal{H}_{R-NS}$ or $\mathcal{H}_{NS-R}$), then it is uniquely determined in all the other sectors as well.

- In general, the properties of topological defects in fermionic CFTs are more complicated than the ones in purely bosonic ones. In particular, fermionic CFTs can contain topological defects $\mathcal{L}$ of 'q-type', that admit topological 2-junctions $\psi\in V_{\mathcal{L},\mathcal{L}^\star}$ with odd fermion number $(-1)^{F_L+F_R}=-1$, see [46, 47]. However, none of the topological defects in the category Top is of q-type. Indeed, in the K3 models we consider, the holomorphic and anti-holomorphic fermion numbers $(-1)^{F_L}$ and $(-1)^{F_R}$ can be identified with the central $\mathbb{Z}_2$ elements in the holomorphic and anti-holomorphic SU(2) R-symmetry group.

The topological defects $\mathcal{L}\in\mathsf{Top}$ are transparent to the $\widehat{su}(2)_1$ currents in the $\mathcal{N}=(4,4)$ algebra, and, as a consequence, they commute with the SU(2) R-symmmetry groups generated by their zero modes, and therefore with the fermion numbers $(-1)^{F_L}$ and $(-1)^{F_R}$. Furthermore, all spaces of defect and junction operators are representations of the $\mathcal{N}=(4,4)$ superconformal algebra, and the fermion number of each state is determined by their SU(2) R-symmetry charges. In particular, topological operators with $h=0=h$ have zero R-symmetry charges, and therefore they are always bosons.

In fact, each $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$ induces a topological defect in the purely bosonic CFT $\mathcal{C}^{bos}$ obtained by a type 0 GSO projection, i.e. by including only the NS-NS and R-R fields of the K3 model $\mathcal{C}$ with positive fermion number. Furthermore, all properties of the topological defect $\mathcal{L}$ in the original supersymmetric model $\mathcal{C}$ are completely determined in terms of the action of the operator $\mathcal{L}$ in the bosonic model $\mathcal{C}^{bos}$. Note, however, that the definition of the category $\mathsf{Top}_{\mathcal{C}}$ is much more natural in the supersymmetric setup. In particular, we do not know whether the condition of preserving the $\mathcal{N}=(4,4)$ superconformal algebra admits an equivalent formulation in the bosonic model $\mathcal{C}^{bos}$.

- With each defect $\mathcal{L}$ is associated a $\mathcal{L}$-twisted space of states $\mathcal{H}_{\mathcal{L}}$, with different sectors $\mathcal{H}_{\mathcal{L}}^{NS-NS}$, $\mathcal{H}_{\mathcal{L}}^{R-R}$, $\mathcal{H}_{\mathcal{L}}^{R-NS}$, $\mathcal{H}_{\mathcal{L}}^{NS-R}$. When $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$, each of these sectors decomposes into representations of the $\mathcal{N}=(4,4)$ superconformal algebra. Furthermore, the sectors are related to each other by spectral flow.

Topological defects that are invertible form the group of symmetries of the K3 model. In [26], all group of symmetries commuting with the $\mathcal{N}=(4,4)$ algebra and the spectral flow generators have been classified. In particular, consider a K3 model $\mathcal{C}_\Pi$ in the moduli space $\mathcal{M}_{K3}$, corresponding to the choice of the positive definite four-dimensional subspace of spectral flow generators in the space of RR ground fields $V\cong {}^{4,20}\otimes\mathbb{R}$. Then, the group G_Π of symmetries of $\mathcal{C}_\Pi$ satisfying 1 and 2 is isomorphic to the subgroup Stab( ) of $O({}^{4,20})\cong O(4,20,\mathbb{Z})$ fixing pointwise [26].

Let us revisit the argument that led to this result, and then discuss to what extent such argument can be generalized to the case of topological defects. Every symmetry $g\in G_\Pi$ maps 1/2 BPS boundary states to 1/2 BPS boundary states, and therefore maps the lattice ^{4,20} of RR charge vectors into itself. The map must be linear and preserve the bilinear form of the lattice — indeed, the bilinear form is a Witten index counting the Ramond ground states for open strings suspended between two D-branes, and is invariant under the action of G_Π. Therefore, every $g\in G_\Pi$ induces an automorphism of the lattice ^{4,20}. Furthermore, by property 2, the induced action on RR ground states must act trivially on the spectral flow generators in . We conclude that there is a homomorphism $\rho: G_\Pi\to \mathrm{Stab}(\ )\subset O({}^{4,20})$. Then one proves that such homomorphism is both injective and surjective. To show surjectivity, one notices that $O({}^{4,20})\cong O(4,20,\mathbb{Z})$ is the T-duality group, and Stab( ) is a subgroup of dualities mapping the model $\mathcal{C}_\Pi$ into itself, i.e. self-dualities. But all self-dualities are symmetries of the model, so they must correspond to some $g\in G_\Pi$. As for injectivity, let $K_\Pi\subseteq G_\Pi$ be the kernel of ρ. Then K_Π acts trivially on ^{4,20}, and, by linearity, on all RR ground states. But the RR ground states in the twenty $(\frac{1}{4},0;\frac{1}{4},0)$ representations of $\mathcal{N}=4$ are related by spectral flow to the 80-dimensional space of exactly marginal operators in the

NS-NS sector. Thus, the symmetries in K_Π act trivially on all such exactly marginal operators, and as a consequence they are not broken by any deformation of the model. Given that the moduli space $\mathcal{M}_{K3}$ is connected, we conclude that the kernel of ρ is the same group $K_\Pi \equiv K$ for all K3 models. At this point, one just needs to consider a simple example of non-linear sigma model on K3, where the group K can be explicitly described — for example the model considered in section 5. It turns out that K is trivial in that model, and therefore is trivial everywhere in the moduli space $\mathcal{M}_{K3}$. We conclude that $\rho : G_\Pi \to \mathrm{Stab}(\ )$ is an isomorphism. In [26], it was then proved that every group of the form Stab( ) is isomorphic to a subgroup of the Conway group Co_0, the group of automorphisms of the Leech lattice , fixing a sublattice of   of rank at least 4. All subgroups of Co_0 that are lattice stabilizers were classified in [48].

Let us now discuss how a similar argument could be generalized to a classification of topological defects $\mathcal{L} \in \mathsf{Top}$. As described in section 3.3, the fusion of a boundary state $||\alpha\rangle\rangle$ and a defect $\mathcal{L}$ yields a new boundary state $||\mathcal{L}\alpha\rangle\rangle$. In particular, the defects $\mathcal{L} \in \mathsf{Top}$ preserve the space-time supersymmetry, so they map 1/2 BPS D-branes into 1/2 BPS D-branes, and RR charge vectors to RR charge vectors. Therefore, we have a map

$$\begin{aligned} \mathsf{Top}_\Pi &\to \mathrm{End}(\ ^{4,20}) \\ \mathcal{L} &\mapsto \mathsf{L} \end{aligned} \tag{3.9}$$

that assigns a $\mathbb{Z}$-linear function $\mathsf{L} :\ ^{4,20} \to\ ^{4,20}$ to each defect $\mathcal{L} \in \mathsf{Top}_\Pi$. This map is compatible with fusion product, i.e. it gives rise to a ring homomorphism from the fusion ring of Top_Π to $\mathrm{End}(\ ^{4,20})$. The extension of L by linearity to the real space of R-R ground fields $V \cong\ ^{4,20} \otimes \mathbb{R}$ coincides with the restriction $\mathcal{L}_{|V} \in \mathrm{End}_\mathbb{R}(V)$ of the linear operator $\mathcal{L}$ to V,

$$\mathcal{L}_{|V} : V \to V \ . \tag{3.10}$$

To summarize:

(a) The restriction $\mathcal{L}_{|V} : V \to V$ maps $\ ^{4,20} \subset V$ into $\ ^{4,20}$, i.e. it is the extension by $\mathbb{R}$-linearity of some lattice endomorphism $\mathsf{L} :\ ^{4,20} \to\ ^{4,20}$.

Henceforth, we use the symbol L to denote both maps $\mathsf{L} :\ ^{4,20} \to\ ^{4,20}$ and $\mathcal{L}_{|V} : V \to V$. Because $\mathcal{L}$ commutes with the $\mathcal{N} = (4,4)$ algebra, it cannot mix RR ground fields in different representations. Furthermore, the condition that the spectral flow generators are transparent with respect to $\mathcal{L}$ implies that the map $\mathcal{L}$ acts on them in the same way as on the vacuum, i.e. by multiplication by the quantum dimension $\langle \mathcal{L} \rangle \geq 1$. The following property then follows:

(b) $\mathsf{L} : V \to V$ is block-diagonal with respect to the orthogonal decomposition $V = \ \oplus\ ^\perp$, i.e. $\mathsf{L}(\ ) \subseteq$ and $\mathsf{L}(\ ^\perp) \subseteq\ ^\perp$. Furthermore, the restriction $\mathsf{L}_{|\Pi}$ is proportional to the identity $\mathsf{L}_{|\Pi} = \langle \mathcal{L} \rangle \mathrm{id}_\Pi$, where $\langle \mathcal{L} \rangle \geq 1$ is the quantum dimension of the defect.

It is useful to introduce the real vector space

$$B^{4,20}(\mathbb{R}) := \left\{ \begin{pmatrix} d \cdot \mathbf{1}_{4\times 4} & 0 \\ 0 & b_{20\times 20} \end{pmatrix} \mid d \in \mathbb{R}, b_{20,20} \in Mat_{20\times 20}(\mathbb{R}) \right\} \tag{3.11}$$

of block diagonal real 24×24 matrices, with a 4×4 upper left-corner proportional to the identity and an unconstrained 20×20 lower-right block, and its subset

$$B^{4,20}_+(\mathbb{R}) := \left\{ \begin{pmatrix} d \cdot \mathbf{1}_{4\times 4} & 0 \\ 0 & b_{20\times 20} \end{pmatrix} \mid d \geq 1, b_{20,20} \in Mat_{20\times 20}(\mathbb{R}) \right\} \subset B^{4,20}(\mathbb{R}) \ . \tag{3.12}$$

Upon choosing a suitable orthonormal basis of V, compatible with the splitting $V \cong \quad \oplus \quad ^{\perp}$, the space of linear maps $V \to V$ that satisfy property (b) can be identified with $B_{+}^{4,20}(\mathbb{R})$. Furthermore, if we define

$$B_{\Pi}^{4,20}(\mathbb{Z}) := \mathrm{End}(\quad ^{4,20}) \cap B^{4,20}(\mathbb{R}), \qquad B_{\Pi,+}^{4,20}(\mathbb{Z}) := \mathrm{End}(\quad ^{4,20}) \cap B_{+}^{4,20}(\mathbb{R}), \tag{3.13}$$

then the maps $V \to V$ satisfying both properties (a) and (b) can be identified with $B_{\Pi,+}^{4,20}(\mathbb{Z})$. Then the image of the map (3.9) is actually contained in $B_{\Pi,+}^{4,20}(\mathbb{Z})$, so that we can restrict the target and consider the map

$$\begin{aligned} \mathsf{Top}_{\Pi} &\to B_{\Pi,+}^{4,20}(\mathbb{Z}) \subset \mathrm{End}(\quad ^{4,20}) \\ \mathcal{L} &\mapsto \mathsf{L} \end{aligned} \tag{3.14}$$

which gives rise to a homomorphism of semirings. As the notation suggests, the intersections $B_{\Pi}^{4,20}(\mathbb{Z})$ and $B_{\Pi,+}^{4,20}(\mathbb{Z})$ depend on the way the four-dimensional space is embedded in $^{4,20} \otimes \mathbb{R}$, i.e. on the point on the moduli space $\mathcal{M}_{K3}$.

It is plausible that the maps L satisfy some further constraints associated with unitarity. Let $\mathcal{L}$ be a *simple* defect, so that the Hilbert space $\mathcal{H}_{\mathcal{L}\mathcal{L}^\star}$ admits an orthogonal decomposition as

$$\mathcal{H}_{\mathcal{L}\mathcal{L}^\star} \cong \mathcal{H} \oplus \mathcal{H}_{\mathcal{L}\mathcal{L}^\star - \mathcal{I}},$$

where $\mathcal{H}_{\mathcal{L}\mathcal{L}^\star - \mathcal{I}}$ is a sum (with suitable multiplicities) of simple defect spaces $\mathcal{H}_{\mathcal{L}_i}$ with $\mathcal{L}_i \neq \mathcal{I}$. Suppose that, in a correlation function, we move the support of the defect line $\mathcal{L}$ past the insertion point z of a point-like operator $\phi(z)$, with $\phi \in \mathcal{H}$. Then, $\phi(z)$ gets replaced with the sum of an operator $\frac{1}{\langle \mathcal{L} \rangle} \mathcal{L}(\phi) \in \mathcal{H}$ plus (possibly vanishing) contributions from each of the components of $\mathcal{H}_{\mathcal{L}\mathcal{L}^\star - \mathcal{I}}$ (see figure 8 in section 2). This move defines a linear map $\mathcal{H} \to \mathcal{H}_{\mathcal{L}\mathcal{L}^\star}$. In a unitary CFT, it is natural to expect such a map to be an isometry; we call this assumption a *strong unitarity hypothesis*. Because for a simple defect, the contribution $\frac{1}{\langle \mathcal{L} \rangle} \mathcal{L}(\phi) \in \mathcal{H}$ is orthogonal to the contributions from $\mathcal{H}_{\mathcal{L}\mathcal{L}^\star - \mathcal{I}}$, we get as a consequence

$$\frac{\|\mathcal{L}(\phi)\|^2}{|\langle \mathcal{L} \rangle|^2} \leq \|\phi\|^2, \qquad \forall \phi \in \mathcal{H}, \tag{3.15}$$

where the equality holds if and only if all the other contributions vanish. We call the condition (3.15) the *weak unitarity hypothesis*. Notice that if (3.15) holds for all *simple* defects $\mathcal{L}$, then it must hold for all superpositions as well. While we do not know any counterexample to these hypotheses,[2] we are not aware of any general proof either. A proof of (3.15) was given in [22] (see proposition 8), under certain conditions on $\mathcal{L}$ and ϕ. In particular, eq. (3.15) holds for all Verlinde lines in unitary rational CFT. Unfortunately, K3 models are not rational with respect to the $\mathcal{N} = (4,4)$ algebra and we were not able to prove that such conditions are satisfied for all $\mathcal{L} \in \mathsf{Top}$. If (3.15) holds, an immediate consequence is the following:

[2] There are well-known counterexamples in non-unitary theories though. For example, the Lee-Yang model with central charge $-\frac{22}{5}$ admits a simple defect $\mathcal{L}$ of quantum dimension $\langle \mathcal{L} \rangle = \frac{\sqrt{5}-1}{2}$ where one of the eigenvalues of $\mathcal{L}$ is $\frac{\sqrt{5}+1}{2} > \langle \mathcal{L} \rangle$ [13].

(c) (Assuming (3.15) holds.) The operatorial norm $\|\mathsf{L}\| := \sup_{0\neq v\in V} \frac{\|\mathsf{L}v\|}{\|v\|}$ equals $\langle\mathcal{L}\rangle$. Here, $\|v\|$ denotes the *Euclidean* norm on V.

Let us define the 'bounded' sets

$$B^{4,20}_{+,b}(\mathbb{R}) = \left\{ \begin{pmatrix} d\cdot\mathbf{1}_{4\times4} & 0 \\ 0 & b_{20\times20} \end{pmatrix} \mid d\geq 1, b_{20,20}\in Mat_{20\times20}(\mathbb{R}),\ \|b_{20,20}\|\leq d \right\} \subset B^{4,20}_{+}(\mathbb{R})\,, \quad (3.16)$$

and

$$B^{4,20}_{\Pi,+,b}(\mathbb{Z}) = \mathrm{End}(\quad^{4,20}) \cap B^{4,20}_{+,b}(\mathbb{R})\ . \quad (3.17)$$

If property (c) holds we can further restrict the target of the map (3.14) to

$$\begin{aligned} \mathsf{Top}_\Pi &\to B^{4,20}_{\Pi,+,b}(\mathbb{Z}) \subset \mathrm{End}(\quad^{4,20}) \\ \mathcal{L} &\mapsto \mathsf{L}\ . \end{aligned} \quad (3.18)$$

The map (3.18) still gives rise to a homomorphism of semirings. Notice that, for any given real number $d>0$, there are finitely many maps $\mathsf{L}\in B^{4,20}_{\Pi,+,b}(\mathbb{Z})$ with quantum dimension $\langle\mathcal{L}\rangle\leq d$. We will not use property (c) to prove any of the claims in the rest of the paper.

The set $B^{4,20}_{\Pi,+}(\mathbb{Z})$ contains the group Stab($) \subset O(\quad^{4,20})$ of lattice automorphisms fixing . In fact, Stab( ) can be characterized as the subset of invertible elements in $B^{4,20}_{\Pi,+}(\mathbb{Z})$, this follows immediately by noticing that if $\mathsf{L}\in\mathrm{End}(\quad^{4,20})$ admits a multiplicative inverse $\mathsf{L}^{-1}\in\mathrm{End}(\quad^{4,20})$, then both L and L^{-1} are in $O(\quad^{4,20})$.

Therefore, the semiring homomorphism (3.14) (or (3.18), if (3.15) holds) can be understood as an extension of the group isomorphism $\rho: G_\Pi\to$ Stab( ).

Unfortunately, in general we expect the homomorphism (3.14) to be neither injective nor surjective. As for surjectivity, recall that a topological defect $\mathcal{L}$ is invertible if and only if its quantum dimension is $\langle\mathcal{L}\rangle=1$. On the other hand, it is quite easy to construct elements of $B^{4,20}_{\Pi,+}(\mathbb{Z})$ that have dimension $d=1$ but are not invertible, and therefore are not in the image of ρ. Of course, one could put further restrictions on $B^{4,20}_{\Pi,+}(\mathbb{Z})$ by simply excluding such elements. However, we have no guarantee that *all* the elements of $B^{4,20}_{\Pi,+}(\mathbb{Z})$ with dimension larger than 1 are associated with topological defects.

As for injectivity, we can try to run an argument analogous to the one used in [26] to classify the symmetry groups G_Π. Let K_Π denote the subcategory of topological defects $\mathcal{L}$ of the model $\mathcal{C}_\Pi$ preserving $\mathcal{N}=(4,4)$ and spectral flow, and such that the corresponding maps L are proportional to the identity on V, i.e. such that

$$\rho(\mathsf{K}_\Pi) \subseteq \{\mathsf{L}=d\cdot\mathrm{id}_V,\ d\geq1\} \subset B^{4,20}_{\Pi,+}(\mathbb{Z})\ . \quad (3.19)$$

We can prove the following:

Claim 1. *For all K3 models $\mathcal{C}_\Pi$ the category K_Π of defects preserving $\mathcal{N}=(4,4)$ and spectral flow, and acting by multiplication by some $d\in\mathbb{R}$ on the space of RR ground fields V is generated by the trivial defect*

$$\mathsf{K}_\Pi = \{d\mathcal{I},\ d\in\mathbb{N}\}\ . \quad (3.20)$$

Proof. Let $\mathcal{L}$ be a defect in K_Π. Because $\mathcal{L}$ is transparent to the spectral flow generators, the real number d must be the quantum dimension $d = \langle\mathcal{L}\rangle$. As a consequence, all R-R ground fields are transparent to $\mathcal{L}$. This implies that $\mathcal{L}$ is transparent to all exactly marginal operators of $\mathcal{C}_\Pi$, and it cannot be lifted by any deformation of the model. Because the moduli space of K3 is connected, this means that the category K_Π is the same for all K3 models. Therefore, it is sufficient to determine $\mathsf{K}_\Pi \equiv \mathsf{K}$ in a specific model. In section 5, we will show that in a certain torus orbifold the defects of K_Π are necessarily superpositions of d copies of the identity defect (see Claim 7); in particular, d is a natural number. □

This result generalizes the analogous statement for symmetry groups that the kernel of ρ is trivial. However, in the case of defects, this is not sufficient to conclude that ρ is injective. If g, h are elements of the group G_Π, then $\rho(g) = \rho(h)$ implies that $\rho(gh^{-1}) = \rho(g)\rho(h)^{-1} = 1$, and therefore $gh^{-1} \in \ker\rho$ must be the identity and $g = h$. But if $\mathcal{L}$ and $\mathcal{L}'$ are non-invertible defects, the fact that $\rho(\mathcal{L}) = \rho(\mathcal{L}')$ does not imply that $\mathcal{L}$ and $\mathcal{L}'$ can be obtained from each other by fusion with a defect in K. Indeed, in sections 4, 5 and 6 we will see examples of continuous families of distinct defects $\mathcal{L}_\theta$, all with the same image L.

The possible quantum dimensions of defect $\mathcal{L} \in \mathsf{Top}_\Pi$ are strongly constrained by properties (a) and (b). In section 3.3 we will prove the following:

Claim 2. *The quantum dimension $\langle\mathcal{L}\rangle$ of a defect $\mathcal{L} \in \mathsf{Top}_\Pi$ is an algebraic integer of degree at most* 6*. Furthermore, if* $\;\cap\;{}^{4,20} \neq 0$*, then $\langle\mathcal{L}\rangle$ is integral for all $\mathcal{L} \in \mathsf{Top}_\Pi$.*

We recall that an algebraic integer is the root of a monic polynomial $p(x)$ with integral coefficients, and its degree is d if any such $p(x)$ has degree at least d. We do not know whether the upper bound on the degree is sharp. A slightly weaker necessary condition for the quantum dimension to be integral is given in proposition 4 in section 3.3.

The condition $\;\cap\;{}^{4,20} \neq 0$ has a nice physical interpretation. Let $v \neq 0$ be a primitive vector in $\;\cap\;{}^{4,20}$. Because the subspace   is positive definite, the vector v has positive norm $v^2 > 0$. From the viewpoint of type IIA superstring, a primitive vector $v \in\;{}^{4,20}$ with $v^2 > 0$ represents the charge of a BPS D0-D2-D4-brane configuration.[3] The mass of such a BPS configuration depends on the moduli, and is proportional to v_Π^2, where v_Π and $v_\perp$ are the orthogonal projections of v along   and ${}^\perp$, so that $v^2 = v_\Pi^2 - v_\perp^2$. An attractor point in the moduli space for the BPS state with charge v is a point where the BPS mass $v_\Pi^2 = v^2 + v_\perp^2$ is minimized, and this happens if and only if $v \in$ [38–41]. Thus, the points in the moduli space where $\;\cap\;{}^{4,20} \neq 0$ are exactly the attractor points for some BPS brane configurations. Claim 2 then implies that whenever the K3 model $\mathcal{C}$ is 'attractive', all topological defects $\mathcal{L} \in \mathsf{Top}_\mathcal{C}$ have integral quantum dimension.

By combining Claims 1 and 2, we show that generically, i.e. outside of a subset of null measure in the moduli space $\mathcal{M}_{K3}$, the category Top_Π is essentially trivial:

Claim 3. *For a generic K3 sigma model $\mathcal{C}_\Pi$, the only topological defects in Top_Π are integral multiples of the identity.*

[3]While the charges of BPS D-branes span the lattice ^{4,20}, a *generic* vector $v \in\;{}^{4,20}$ is the charge of a system of branes and anti-branes that is not by itself BPS.

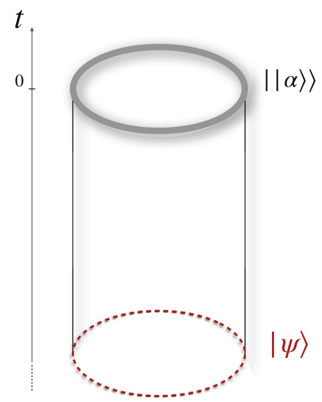

Figure 9. Amplitude $\langle\langle\alpha||\psi\rangle$ in the half-cylinder $S^1\times\mathbb{R}_{\leq 0}$ with boundary $||\alpha\rangle\rangle$ at $t=0$.

See section 3.3 for the proof. As a consequence, for any non-trivial topological defect $\mathcal{L}\in\mathsf{Top}_\Pi$ there is a deformation of the K3 model $\mathcal{C}_\Pi$ that lifts it. This is in particular true for the K3 sigma models allowing for a non-trivial Stab(Π), whose elements can in general be characterized as the invertible topological defect lines in Top_Π. We stress that our result does not exclude that the subset of the moduli space where Top is non-trivial is dense in $\mathcal{M}_{K3}$.

3.3 Action on D-branes

Let us consider the set of BPS boundary states in a K3 model $\mathcal{C}_\Pi$, corresponding to D-branes in the full string theory that are 1/2-BPS, i.e. that preserve 8 out of 16 space-time supersymmetries.

As in the previous section, we denote by V the 24-dimensional real space of RR ground states that are CPT self-conjugate. The hermitian form on $\mathcal{H}$ induces a positive definite bilinear form on V.

With each 1/2-BPS boundary state $||\alpha\rangle\rangle$ is associated a 24-dimensional charge vector $q_\alpha\in V^*$. The pairing of q_α with a R-R ground state $\psi\in V$ is given by the amplitude

$$q_\alpha(\psi):=\langle\langle\alpha||\psi\rangle\in\mathbb{R}\,, \tag{3.21}$$

on a half-cylinder

$$S^1\times\mathbb{R}_{\leq 0}=\{(x,t)\,|\,x\in\mathbb{R}/2\pi\mathbb{Z},\ t\leq 0\} \tag{3.22}$$

where $|\psi\rangle$ is the asymptotic state at $t\to-\infty$ and α the boundary condition at $t=0$, see figure 9.

Geometrically, if S is the target K3 surface, we can identify V with the even real cohomology $H^{even}(S,\mathbb{R})$ (in fact, RR ground fields correspond to harmonic forms on S), and the lattice of D-brane charges with $H_{even}(S,\mathbb{Z})$, the integral even homology. We can define a bilinear form (q_α,q_β) on the lattice of D-brane charges, corresponding to the Mukai pairing on $H_{even}(S,\mathbb{Z})$:

$$(q_\alpha,q_\beta):=\mathrm{Tr}_{\mathcal{H}^{\rm open}_{R,\alpha,\beta}}(q^{H_{\rm open}}(-1)^F)=\langle\langle\alpha||(-1)^{F_L+1}\tilde{q}^{H_{closed}}||\beta\rangle\rangle_{R-R} \tag{3.23}$$

Physically, the bilinear form is given by a string amplitude on a *bounded* cylinder $S^1\times[0,1]$ with boundaries α and β, describing a loop of Ramond open strings with periodic conditions

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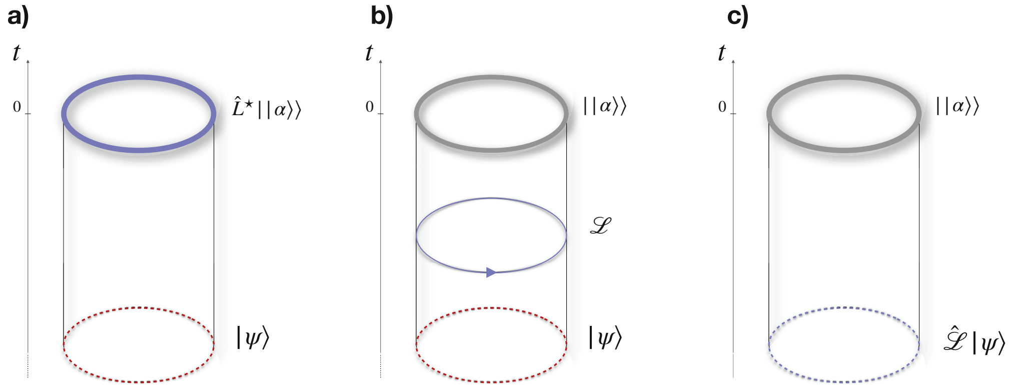

Figure 10. Amplitude $\langle\langle\alpha|\hat{\mathcal{L}}|\psi\rangle$ in the half-cylinder $S^1\times\mathbb{R}_{\leq 0}$ with insertion of a TDL $\mathcal{L}\in Top$.

for the fermions. This amplitude reduces to an index counting Ramond ground states in the space $\mathcal{H}^{\text{open}}_{R,\alpha,\beta}$ of open strings with Chan-Paton factors α–β. In particular, (q_α,q_β) is just an integral (in fact, even) number independent of the size of the cylinder. With respect to this pairing, the RR charge lattice is isomorphic to the (unique) even unimodular lattice with signature $(4,20)$.

In the closed string channel, the amplitude gets contributions only from the Ramond-Ramond ground fields propagating between the corresponding boundary states, with the insertion of a left-moving fermion number $(-1)^{F_L+1}$. The latter acts by $+1$ on Π (i.e. on the spectral flow generators in the $(\frac{1}{4},\frac{1}{2};\frac{1}{4},\frac{1}{2})$ representation of $\mathcal{N}=(4,4)$) and by -1 on $\Pi^\perp$ (i.e. on states in $(\frac{1}{4},0;\frac{1}{4},0)$), and can be used to define a non-degenerate bilinear form with signature $(4,20)$ on V

$$(\psi,\psi'):=-\langle\psi|(-1)^F|\psi'\rangle\ . \tag{3.24}$$

Let us focus on the dual of the charge lattice, i.e. the lattice $\Gamma^{4,20}\subset V$ of states with respect to which the charge of any boundary state is integral

$$\Gamma^{4,20}:=\{\Psi\in V\,|\,\langle\langle\alpha|\Psi\rangle\in\mathbb{Z},\ \forall||\alpha\rangle\rangle\}\ . \tag{3.25}$$

Since the charge lattice is self-dual, $\Gamma^{4,20}$ is again an even unimodular lattice of signature $(4,20)$ with respect to the bilinear form (3.24).

Let us now consider, as above, a half-cylinder amplitude $\langle\langle\alpha|\hat{\mathcal{L}}|\psi\rangle$ for a boundary α and a state $\psi\in V$, with the insertion of a topological defect line $\mathcal{L}\in\mathsf{Top}$ wrapping once along the circle S^1.

By moving the line $\mathcal{L}$ along the cylinder to $t\to-\infty$ or $t\to 0$, $\langle\langle\alpha|\hat{\mathcal{L}}|\psi\rangle$ can be interpreted as either the RR charge of the D-brane α with respect to $\hat{\mathcal{L}}|\psi\rangle$, or as the charge of $\mathcal{L}^\star||\alpha\rangle\rangle$ with respect to $|\psi\rangle$. Therefore,

$$q_\alpha(\hat{\mathcal{L}}\psi)=\langle\langle\alpha|\hat{\mathcal{L}}|\psi\rangle=q_{\hat{\mathcal{L}}^\star\alpha}(\psi)\ . \tag{3.26}$$

As in the previous section, we denote by $\mathsf{L}:=\hat{\mathcal{L}}_{|V}$, the restriction of $\hat{\mathcal{L}}$ to V. In particular, if $\Psi\in\Gamma^{4,20}\subset V$, then

$$q_\alpha(\mathsf{L}\Psi)=q_{\hat{\mathcal{L}}^\star\alpha}(\Psi)\in\mathbb{Z}, \tag{3.27}$$

for all $||\alpha\rangle\rangle$. It follows that $\mathsf{L}(\quad)\in\ ^{4,20}$, i.e. $\mathcal{L}$ gives rise to an endomorphism $\mathsf{L}\in\mathrm{End}_{\mathbb{Z}}(\ ^{4,20})$ of the unimodular lattice ^{4,20}

$$\mathsf{L}:\ ^{4,20}\to\ ^{4,20}$$
$$\mapsto\mathsf{L}(\quad)\,.$$

The operator $\mathcal{L}$ commutes with the left-moving fermion number $(-1)^{F_L}$

$$[(-1)^{F_L},\mathcal{L}]=0\,, \tag{3.28}$$

so that the linear map L preserves the orthogonal decomposition $V=\quad\oplus\ ^{\perp}$. In particular, it is block diagonal with respect to an orthonormal basis $\{\psi_1,\ldots,\psi_{24}\}\subset V$ satisfying

$$\psi_1,\ldots,\psi_4\in\quad,\qquad\psi_5,\ldots,\psi_{24}\in\ ^{\perp}\,. \tag{3.29}$$

Finally, we notice that on a spectral flow generators $\psi\in\quad$, L acts by

$$\mathsf{L}(\psi)=\langle\mathcal{L}\rangle\psi\,,\qquad\forall\psi\in\quad\subset V\,, \tag{3.30}$$

i.e. the 4×4 upper-left block of L is just $\langle\mathcal{L}\rangle$ times the identity. This argument leads to the conditions (a) and (b) described in section 3.2.

An easy but powerful consequence of this construction is the following:

Proposition 4. *Let $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}_\Pi}$, and suppose there is some $\quad\in\ ^{4,20}$, $\quad\neq0$, such that $\mathsf{L}(\quad)=\langle\mathcal{L}\rangle\quad$, where $\langle\mathcal{L}\rangle$ is the quantum dimension of $\mathcal{L}$. Then, $\langle\mathcal{L}\rangle$ is integral. In particular, if for a certain model $\mathcal{C}_\Pi$, one has $\quad\cap\ ^{4,20}\neq0$, then all $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}_\Pi}$ have integral quantum dimension.*

Indeed, without loss of generality, we can assume the vector $\quad\in\ ^{4,20}$ is primitive (i.e., not an integral multiple of a shorter vector in the lattice). Then, because $\langle\mathcal{L}\rangle\quad\in\ ^{4,20}$, it must necessarily be an integral multiple of $\quad$.

More generally, because the matrix L representing the action of $\mathcal{L}$ on the space of RR ground fields is an integral 24×24 matrix, it satisfies $p(\mathsf{L})=0$ where $p(x)$, the characteristic polynomial, is a monic polynomial of degree 24 with integral coefficients. This implies that the quantum dimension $\langle\mathcal{L}\rangle$, which is an eigenvalue of L, is also a root of the same monic polynomial, i.e. it is an algebraic integer. Let $d=\langle\mathcal{L}\rangle$ be an algebraic integer, and let $r(x)$ be the minimal polynomial for d, i.e. the least degree monic polynomial with integral coefficient having d as a root. Since d has multiplicity at least 4 as an eigenvalue, it follows that $r(x)^4$ divides $p(x)$, so that $r(x)$ must have degree at most 6. We conclude that:

Proposition 5. *The quantum dimension $\langle\mathcal{L}\rangle$ of any defect $\mathcal{L}\in\mathsf{Top}$ is an algebraic integer of degree at most* 6.

This result, together with Claim 1, leads us to a proof of Claim 3.

Proof of Claim 3. Consider a K3 model $\mathcal{C}_\Pi$. Let $\ _1,\ldots,\ _{24}$ be generators of the lattice $^{4,20}\subset V$, let $d:=\langle\mathcal{L}\rangle$ be the quantum dimension of a defect $\mathcal{L}\in\mathsf{Top}_\Pi$, and let $\mathbb{Q}[d]$ be the extension of the field $\mathbb{Q}$ by the algebraic integer d. Then, for a generic model $\mathcal{C}_\Pi$, we expect the scalar products $\langle\psi|\ _1\rangle,\ldots,\langle\psi|\ _{24}\rangle$, with some spectral flow generator $\psi\in\quad$, to generate a space of maximal rank 24 over the field extension $\mathbb{Q}[d]$. Indeed, because the set of algebraic numbers has measure zero in $\mathbb{R}$, for a generic model $\mathcal{C}_\Pi$ every non-trivial linear relation among $\langle\psi|\ _1\rangle,\ldots,\langle\psi|\ _{24}\rangle$ will contain some transcendental coefficient. Let us define the matrix elements L_{ij} by $\mathsf{L}|\ _i\rangle = \sum_j \mathsf{L}_{ij}|\ _j\rangle$, so that $\mathsf{L}_{ij}\in\mathbb{Z}$. Using

$$d\langle\psi|\ _i\rangle = \langle\psi|\mathsf{L}|\ _i\rangle = \sum_{j=1}^{24}\mathsf{L}_{ij}\langle\psi|\ _j\rangle\,, \tag{3.31}$$

we get

$$\sum_j (d\delta_{ij} - \mathsf{L}_{ij})\langle\psi|\ _j\rangle = 0 \qquad \forall i = 1,\ldots,24\;. \tag{3.32}$$

Because $d\delta_{ij} - \mathsf{L}_{ij}\in\mathbb{Q}[d]$ and $\langle\psi|\ _j\rangle$ are linearly independent over $\mathbb{Q}[d]$, it follows that

$$d\delta_{ij} - \mathsf{L}_{ij} = 0 \qquad \forall i,j = 1,\ldots,24\;. \tag{3.33}$$

This means that L is d times the identity, and by proposition 4 d is integral. By Claim 1, this means that $\mathcal{L} = d\mathcal{I}$, and we conclude. □

3.4 Defects as boundaries in the doubled theory and other approaches

In this section we discuss other approaches to topological defects that are not used elsewhere in this article.

First of all, a topological defect in a CFT $\mathcal{C}$ can be described as a boundary state in the doubled theory $\mathcal{C}\times\mathcal{C}^*$, where $\mathcal{C}^*$ is obtained from $\mathcal{C}$ by worldsheet parity. In particular, when $\mathcal{C}$ is a sigma model on K3, the doubled theory $\mathcal{C}\times\mathcal{C}^*$ is a sigma model on the Cartesian product of two K3 surfaces. This CFT contains a $\mathcal{N}=(4,4)$ superconformal algebra at central charge $c = c = 12$. Notice that the $\mathcal{N}=4$ algebra at $c=12$ contains a copy of $\widehat{su}(2)_2$ at level 2, rather than 1.

The space of R-R ground fields of weight $h = h = \frac{1}{2}$ of $\mathcal{C}\times\mathcal{C}^*$ has dimension $24^2 = 576$, and is isomorphic to $V\otimes V^*\cong \mathrm{End}_{\mathbb{R}}(V)$. Similarly, the lattice of RR charges is the tensor product[4] $\ ^{4,20}\otimes(\ ^{4,20})^*\cong\mathrm{End}_{\mathbb{Z}}(\ ^{4,20})$. Therefore, for each given $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$ one can naturally identify $\mathsf{L}\in\mathrm{End}_{\mathbb{Z}}(\ ^{4,20})$ with the RR charge of the corresponding boundary state in $\mathcal{C}\times\mathcal{C}^*$. It follows that property (a) is just quantization of RR charges in the doubled theory.

In order to understand the analogue of property (b), let us consider the decomposition of the space

$$V\otimes V^*\cong V\otimes V = (\ \oplus\ ^\perp)\otimes(\ \oplus\ ^\perp) \tag{3.34}$$

into representations of the $\mathcal{N}=(4,4)$ algebra. There are three kinds of (short) Ramond representations of $\mathcal{N}=4$ at $c=12$ with $h=\frac{1}{2}$, that are labeled by the highest weight (charge) $q\in\{0,1/2,1\}$ with respect to the $\widehat{su}(2)_2$ algebra. In particular, the subspace $\ ^\perp\otimes\ ^\perp$ is contained in $20^2 = 400$ representations of $\mathcal{N}=(4,4)$ with holomorphic and antiholomorphic charges $q = q = 0$; the 80-dimensional subspaces $\ \otimes\ ^\perp$ and $\ ^\perp\otimes\ $ are contained in 40

[4]Actually, because $\ ^{4,20}$ is unimodular, the non-degenerate bilinear form defines isomorphisms $\ ^{4,20}\cong(\ ^{4,20})^*$ and $V\cong V^*$.

$\mathcal{N}=(4,4)$-representations with $(q,q)=(\frac{1}{2},0)$ and $(q,q)=(0,\frac{1}{2})$, respectively; finally, the 16-dimensional subspace $\otimes$ decomposes into one $\mathcal{N}=(4,4)$-representation with $(q,q)=(1,1)$ (9 RR ground states), one representation with $(q,q)=(1,0)$ (3 states), one with $(q,q)=(0,1)$ (3 states) and one with $(q,q)=(0,0)$ (one state). This gives an orthogonal decomposition

$$V\otimes V^* = \ _{0,0}\oplus \ _{\frac{1}{2},0}\oplus \ _{0,\frac{1}{2}}\oplus \ _{1,0}\oplus \ _{0,1}\oplus \ _{1,1}\,, \tag{3.35}$$

where $_{q,\tilde{q}}$ is the subspace of RR ground states of $\mathcal{C}\times\mathcal{C}^*$ contained in the (q,q) representation of $\mathcal{N}=(4,4)$. Here,

$$\dim \ _{0,0}=401\,,\quad \dim \ _{\frac{1}{2},0}=\dim \ _{0,\frac{1}{2}}=80\,,\quad \dim \ _{1,0}=\dim \ _{0,1}=3\,,\quad \dim \ _{1,1}=9\;.$$

It is clear that property (b) for $\mathcal{L}\in\mathsf{Top}^*_{\mathcal{C}}$ is equivalent to the condition the corresponding boundary state in $\mathcal{C}\times\mathcal{C}^*$ is charged only under the 401 RR ground states in $_{0,0}$, and neutral with respect to the other ones. In particular, the condition that the restriction of L to is proportional to the identity implies that the map L commutes with the left- and right-moving SU(2) R-symmetry groups; in turn, this means that the corresponding boundary state in $\mathcal{C}\times\mathcal{C}^*$ can only be charged under the singlet in $\otimes$. In other words, the space $B^{4,20}(\mathbb{R})$ can be identified with the 401-dimensional subspace $_{0,0}$ of R-R ground states of $\mathcal{C}\times\mathcal{C}^*$.

This construction also provides some intuition as for why, for a generic K3 sigma model $\mathcal{C}$, the only simple defect in $\mathsf{Top}_{\mathcal{C}}$ is the identity. Indeed, while there are infinitely many boundary states in $\mathcal{C}\times\mathcal{C}^*$, whose RR charges span the whole lattice $^{4,20}\otimes(\ ^{4,20})^*$, it is not obvious at all that *any* of these lattice vectors is contained in the subspace $_{0,0}\equiv B^{4,20}(\mathbb{R})$. Let us elaborate on this point in more detail. We know that in any K3 model $\mathcal{C}$ there is the identity defect $\mathcal{I}$, and the charge of the corresponding boundary state in the doubled theory is an element $\mathsf{1}\in\ ^{4,20}\otimes(\ ^{4,20})^*$ lying in the 401-dimensional subspace $_{0,0}$ with $(q,q)=(0,0)$. Under the identification $^{4,20}\otimes(\ ^{4,20})^*\cong\mathrm{End}(\ ^{4,20})$, $\mathsf{1}$ simply corresponds to the identity map. Therefore, the intersection

$$(\ ^{4,20}\otimes(\ ^{4,20})^*)\cap\ _{0,0}=\mathrm{End}(\ ^{4,20})\cap B^{4,20}(\mathbb{R})=B^{4,20}_{\Pi}(\mathbb{Z}) \tag{3.36}$$

is always at least one-dimensional. Now, deformations of the K3 model correspond to $O(4,20,\mathbb{R})$ transformations of $V\cong\ \oplus\ ^{\perp}$ within $^{4,20}\otimes\mathbb{R}$. While V is an irreducible representation of the orthogonal group $O(4,20,\mathbb{R})$ (the vector representation), the tensor product $V\otimes V^*\cong V^{\otimes 2}$ decomposes as

$$V^{\otimes 2}=\mathbf{1}\oplus\wedge^2V\oplus\mathrm{Sym}^2_0V\,,$$

where $\mathbf{1}$, $\wedge^2V$, and Sym^2_0V are, respectively, the trivial, the anti-symmetric and the traceless symmetric representations of $O(4,20,\mathbb{R})$. Under the identification $V\otimes V^*\cong\mathrm{End}_{\mathbb{R}}(V)$, clearly the trivial representation $\mathbf{1}$ corresponds to the identity map, i.e. to the charge $\mathsf{1}$ of the identity defect $\mathcal{I}$. Because it is invariant under $O(4,20,\mathbb{R})$ transformations, the charge $\mathsf{1}\in\ ^{4,20}\otimes(\ ^{4,20})^*$ is always contained in $_{0,0}$. This fits with the obvious fact that the identity defect is not lifted by any deformation of the K3 model. On the other hand, both irreducible representations $\wedge^2V$, and Sym^2_0V have non-trivial intersection with $_{0,0}$ and with other $_{q,\tilde{q}}$. Thus, a generic $O(4,20,\mathbb{R})$ transformation will mix any vector in the 400-dimensional space

$$(\ _{0,0}\cap\mathsf{1}^{\perp})\subset\wedge^2V\oplus\mathrm{Sym}^2_0V$$

with vectors in the other components $\ _{q,\tilde{q}}$. Therefore, generically, we do not expect any other lattice vector in $\ ^{4,20}\otimes(\ ^{4,20})^*$ to lie along $\ _{0,0}\cap 1^\perp$. This means that the intersection (3.36) is generically one-dimensional, and contains only the maps L that are multiple of the identity. By claim 1, we conclude that $\mathcal{L}$ is a superposition of identity defects.

We stress that this result is not in contrast with the possibility that Top_Π is non-trivial in a subset of null measure in the moduli space — in fact, we know that there are families of K3 models with non-trivial symmetry group G_Π. Indeed, one can tune a $O(4,20,\mathbb{R})$ rotation so that the space $\ _{0,0}$ contains a non-trivial sublattice of $\ ^{4,20}\otimes(\ ^{4,20})^*$. In fact, this happens whenever $\ _{0,0}\cap 1^\perp$ has a non-trivial intersection with the rational space $(\ ^{4,20}\otimes(\ ^{4,20})^*)\otimes\mathbb{Q}\subset V\otimes V^*$. Starting from such a point in the moduli space $\mathcal{M}_{K3}$ of K3 models, corresponding to some $\ \subset V$, and acting by a *rational* orthogonal transformation $O(4,20,\mathbb{Q})$, one gets another point in $\mathcal{M}_{K3}$ with non-trivial intersection, and therefore potentially non-trivial defects. Thus, the subset of $\mathcal{M}_{K3}$ where the category Top is potentially non-trivial is dense in $\mathcal{M}_{K3}$ (though we stress that, apart from invertible symmetries, we are not able to prove that such potential defects actually exist). Our argument, however, shows that such a subset cannot contain a full open neighborhood of any point in the moduli space, and therefore has zero measure. Indeed, a *real* $O(4,20,\mathbb{R})$ transformation of $\ \subset V$, even an infinitesimal one, will generically lift any intersection.[5]

There is one point in our argument where we have been slightly naive. In general, boundary states in $\mathcal{C}\times\mathcal{C}^*$ correspond to defects in $\mathcal{C}$ that are conformal, but not necessarily topological, i.e. they preserve a particular linear combination of the holomorphic and anti-holomorphic stress energy tensor, but not $T(z)$ and $T(z)$ separately. The topological defects in Top correspond to a particular choice of gluing conditions for boundary states in $\mathcal{C}\times\mathcal{C}^*$, where the two copies of the holomorphic $\mathcal{N}=4$ algebra are identified with the two anti-holomorphic copies up to an automorphism exchanging them (i.e., they are 'permutation branes'). In our treatment, we are ignoring this restriction, and therefore what we get is an upper bound on the actual number of topological defects. Boundary states satisfying the correct gluing conditions are in one-to-one correspondence with topological defect operators $\mathcal{L}:\mathcal{H}\to\mathcal{H}$ commuting with the $\mathcal{N}=(4,4)$ algebra. The requirement that the defect preserves the spectral flow puts further constraints on the R-R charge of the corresponding boundary states. In particular, the boundary state must preserve space-time supersymmetry, so that it must carry a non-zero R-R charge in $\ ^{4,20}\otimes(\ ^{4,20})^*$. The argument above then shows that, outside of a null measure subset in the moduli space, all operators $\mathcal{L}$ corresponding to a defect $\mathcal{L}\in\mathsf{Top}_\Pi$ must act on the space of R-R ground fields as a multiple of the identity. Then, one needs to use claim 1 to conclude that all such defects $\mathcal{L}$ are multiples of the trivial one.

Let us now comment on other common approaches to determine topological defects in two dimensional conformal field theories. One simple and powerful method pioneered by Petkova and Zuber in [8] consists in imposing the analogue of the Cardy condition for boundary states. In practice, a generic defect $\mathcal{L}$ preserving the chiral and antichiral algebra $\mathcal{A}\times\mathcal{A}$ is

[5]The subset of $\mathcal{M}_{K3}$ with non-trivial Top_Π, however, might contain submanifolds of strictly positive codimension, corresponding to rotations of $\ $ by proper subgroups of the real group $O(4,20,\mathbb{R})$. Again, this is compatible with the known fact that there are families of K3 models with the same non-trivial group of symmetries.

parametrized in terms of the (unknown) eigenvalues of $\mathcal{L}$ on the primary fields. One can then compute the torus $\mathcal{L}$-twined partition function, obtained by inserting the $\mathcal{L}$-line along the 'space' direction of the worldsheet torus, as a function of these parameters. A modular S-transformation relates this $\mathcal{L}$-twined partition function to the $\mathcal{L}$-twisted one, where the defect runs along the 'time' direction. Because the $\mathcal{L}$-twisted partition function is just a trace over the $\mathcal{L}$-twisted space $\mathcal{H}_\mathcal{L}$, it must decompose into characters of the $\mathcal{A}\times\mathcal{A}$ algebra with non-negative integral coefficients. Imposing such property on the latter coefficients gives a set of quantization conditions on the unknown parameters of the defect $\mathcal{L}$, whose solutions form a lattice. In rational CFTs, where there is a finite number of primary fields and of unknown parameters, this method is very effective and very often allows one to determine all possible topological defects in the theory.[6]

The main difficulty in applying this method to study $\mathsf{Top}_\mathcal{C}$ in a K3 model $\mathcal{C}$ is that the theory is not rational with respect to the $\mathcal{N}=(4,4)$ algebra, so that there infinitely many primary operators and unknown parameters. It is known that the character of the $\mathcal{N}=4$ algebra (in particular, for the BPS representations) are mock modular forms [43–45], and their S-tranformations give rise to a continuum of massive characters. As is well-known from the study of boundary states in non-rational theories, imposing Cardy-like conditions to these cases is technically very difficult.

Despite these obstacles, modular properties of torus amplitudes have been successfully used in the past to get information about the action of finite symmetry groups on the space $\mathcal{H}$ (or, at least, on some subspace) of states of a K3 model. In particular, when $\mathcal{L}$ is an invertible defect preserving the superconformal algebra, one can consider the $\mathcal{L}$-twined elliptic genus, which is a (weakly) holomorphic Jacobi function for a certain subgroup of the modular group $\mathrm{SL}(2,\mathbb{Z})$. Because the space of such Jacobi functions is finite dimensional, the problem is once again reduced to determining a finite number of unknown coefficients. In particular, these methods were applied in the context of Moonshine conjectures for K3 models [49–54].

One crucial step in these methods is the fact that, for an invertible symmetry of order n, the $\mathcal{L}$-twined genus is modular with respect to a level n congruence subgroup of $\mathrm{SL}(2,\mathbb{Z})$. This is a consequence of the fusion relation $\mathcal{L}^n=\mathcal{I}$. Furthermore, all possible orders n of these symmetries can be determined by the general classification theorem of [26]. Unfortunately, when $\mathcal{L}$ is an unknown non-invertible defect in some K3 model, we do not have any information about the relations in the fusion ring generated by $\mathcal{L}$. In principle, the decomposition of $\mathcal{L}^k$ might involve new simple defects at every power k. For this reason, one cannot predict what the modular properties of the $\mathcal{L}$-twined genus are. In fact, we hope that the methods developed in the present article will provide enough information about the fusion ring of $\mathsf{Top}_\mathcal{C}$ so as to make these methods effective.

Finally, when the CFT $\mathcal{C}$ can be obtained as a IR fixed of a RG flow from a Landau-Ginzburg model, there are well-developed techniques to study its topological defects and their fusion with supersymmetric boundary conditions, relying in particular on matrix

[6] More generally, in order to determine all simple defects in a rational CFT, one needs to impose a Cardy-like condition to all possible products $\mathcal{L}_1\mathcal{L}_2^\star$, for all pairs of simple defects $\mathcal{L}_1$, $\mathcal{L}_2$. In the doubled theory, the Petkova-Zuber method just corresponds to imposing the Cardy condition for open strings stretched between any pair of boundary states.

factorization [55–61]. It would be interesting to apply these methods to K3 models and use them to verify and extend the general results of our article.

4 Topological defects in torus orbifolds

Many interesting examples of K3 models can be described in terms of orbifolds of supersymmetric sigma models on T^4 by some finite group of symmetries. In this section, we will show that for all such K3 models $\mathcal{C}$ the category $\mathsf{Top}_{\mathcal{C}}$ contains some family of simple defects $\mathcal{L}_\theta$ parametrized by continuous parameters θ. This result is an immediate generalization of known properties of the orbifold $S^1/\mathbb{Z}_2$ of a single free boson on a circle [15, 22, 23, 62].

4.1 Supersymmetric sigma models on T^4

A supersymmetric sigma model $\mathcal{T}$ on T^4 contains four holomorphic and four anti-holomorphic $u(1)$ currents $i\partial X^k$, $i\partial X^k$, $k=1,\dots,4$, as well as their superpartners, the four holomorphic and anti-holomorphic free fermions ψ^k, ψ^k with weights $(h,h)=(\frac{1}{2},0)$ and $(0,\frac{1}{2})$, respectively. The full current algebra is $\widehat{so}(4)_1\oplus u(1)^4$, where the six currents of $\widehat{so}(4)_1\cong\widehat{su}(2)_1\oplus\widehat{su}(2)_1$ are given by normal-ordered products $:\!\psi\psi\!:$ of pairs of free fermions. These fields generate the full chiral and anti-chiral algebra of the T^4 model at a generic point in the moduli space — the algebra can be enhanced at subloci of positive codimension.

The NS-NS primary operators with respect to this algebra are the vertex operators $V_\lambda(z,z)$ labeled by vectors

$$\lambda\equiv(\lambda_L,\lambda_R)=(\lambda_L^1,\dots,\lambda_L^4;\lambda_R^1,\dots,\lambda_R^4)\in\mathbb{R}^8\,, \tag{4.1}$$

with conformal weights $h=\frac{\lambda_L\cdot\lambda_L}{2}$ and $h=\frac{\lambda_R\cdot\lambda_R}{2}$. For each given T^4 model, the allowed vectors $\lambda\in\mathbb{R}^8$ are the points of a 8-dimensional lattice $\;^{4,4}$ (the Narain lattice) that is even and unimodular with respect to the bilinear form

$$(\lambda,\mu):=\lambda_L\cdot\mu_L-\lambda_R\cdot\mu_R\,,\qquad \lambda,\mu\in\;^{4,4}\,, \tag{4.2}$$

with signature $(4,4)$. Notice that all even unimodular lattices $\;^{4,4}$ with signature $(4,4)$ are isomorphic to each other. There is a (non-unique) choice of basis $\{n_1,\dots,n_4,w_1,\dots,w_4\}$ for $\;^{4,4}$ with respect to which the bilinear form is

$$(n_i,n_j)=0=(w_i,w_j)\,,\qquad (n_i,w_j)=\delta_{ij}\;. \tag{4.3}$$

The vector $\lambda\in\mathbb{R}^8$ represents the weights of the corresponding primary state $|\lambda\rangle$ with respect to the $\mathrm{U}(1)^4\times\mathrm{U}(1)^4$ group generated by the four bosonic currents $i\partial X^k$, $i\partial X^k$. In other words, the invertible defects

$$W_\theta\,,\qquad\qquad \theta\in(\;^{4,4}\otimes\mathbb{R})/\;^{4,4}\cong(\mathbb{R}/\mathbb{Z})^8\,, \tag{4.4}$$

corresponding to the $\mathrm{U}(1)^8$ symmetries, commute with the full chiral algebra and act by

$$W_\theta|\lambda\rangle=e^{2\pi i(\theta,\lambda)}|\lambda\rangle\,, \tag{4.5}$$

on the primary state λ. Geometrically, $\mathrm{U}(1)^8 \cong \mathrm{U}(1)^4 \times \mathrm{U}(1)^4$ is the product of the $\mathrm{U}(1)^4$ group of translations along the four direction of the torus T^4, times the $\mathrm{U}(1)^4$ group of translations along the T-dual torus.

The RR sector representations of the chiral algebra are labeled by the same vectors $\lambda \in \ ^{4,4}$. For each such $\lambda \in \ ^{4,4}$, there are 16 degenerate ground states with conformal weights $h = \frac{\lambda_L \cdot \lambda_L}{2} + \frac{1}{4}$ and $h = \frac{\lambda_R \cdot \lambda_R}{2} + \frac{1}{4}$, which form an irreducible representation of the Clifford algebra of zero modes of the 8 fermions $\psi^k \psi^k$, $k = 1, \ldots, 4$. With respect to the holomorphic and antiholomorphic $\widehat{so}(4)_1 \cong \widehat{su}(2)_1 \oplus \widehat{su}(2)_1$ algebras and the corresponding $\mathrm{Spin}(4) \cong \mathrm{SU}(2) \times \mathrm{SU}(2)$ groups, the 16 ground states decompose into four irreducible representations with weights $(s_1, s_2; s_1, s_2)$

$$\left(\frac{1}{2},0;\frac{1}{2},0\right), \qquad \left(\frac{1}{2},0;0,\frac{1}{2}\right), \qquad \left(0,\frac{1}{2};\frac{1}{2},0\right), \qquad \left(0,\frac{1}{2};0,\frac{1}{2}\right) . \tag{4.6}$$

The chiral algebra of the T^4 model contains several copies of the small $\mathcal{N} = 4$ superconformal algebra, corresponding to a choice of $\widehat{su(2)}_1$ current algebra (the R-symmetry algebra) inside $\widehat{so(4)}_1$; we choose a small $\mathcal{N} = (4,4)$ once and for all. The four R-R ground states with $\lambda = 0$ (i.e. conformal weights $h = h = \frac{1}{4}$) that are charged with respect to both the holomorphic and antiholomorphic $\widehat{su}(2)_1$ algebra are the spectral flow generators, and belong to a single $(\frac{1}{4}, \frac{1}{2}; \frac{1}{4}, \frac{1}{2})$ representation of $\mathcal{N} = (4,4)$. The remaining 12 RR ground states belong to two $(\frac{1}{4}, \frac{1}{2}; \frac{1}{4}, 0)$, two $(\frac{1}{4}, 0; \frac{1}{4}, \frac{1}{2})$, and four $(\frac{1}{4}, 0; \frac{1}{4}, 0)$ representations of $\mathcal{N} = (4,4)$.

For a generic T^4 model $\mathcal{T}$, the group $G_{\mathcal{T}}$ of symmetries preserving the small $\mathcal{N} = (4,4)$ algebra and the spectral flow generators is

$$G_{\mathcal{T}} \cong \mathrm{U}(1)^8 \rtimes \mathbb{Z}_2 , \tag{4.7}$$

where the $\mathbb{Z}_2$ symmetry $\mathcal{R}$ is the coordinate reflection

$$\partial X^k \to -\partial X^k , \qquad \partial X^k \to -\partial X^k \qquad \psi^k \to -\psi^k , \qquad \psi^k \to -\psi^k . \tag{4.8}$$

The fusion of $\mathcal{R}$ with W_θ is

$$\mathcal{R} W_\theta = W_{-\theta} \mathcal{R} . \tag{4.9}$$

At special loci in the moduli space, $G_{\mathcal{T}}$ can be enhanced to a larger group. In general, the group $G_{\mathcal{T}}$ is always a group extension

$$1 \longrightarrow \mathrm{U}(1)^8 \longrightarrow G_{\mathcal{T}} \longrightarrow G^0_{\mathcal{T}} \longrightarrow 1 , \tag{4.10}$$

where $G^0_{\mathcal{T}} \subset O(\ ^{4,4}) \cong O(4,4,\mathbb{Z})$ is a finite group of lattice automorphisms, that always contains the reflection (4.8) as a central element. Physically, $O(\ ^{4,4}) \cong O(4,4,\mathbb{Z})$ is the group of torus T-dualities, and $G^0_{\mathcal{T}}$ is the group self-dualities of the model $\mathcal{T}$. See [63] for a complete classification of the groups $G^0_{\mathcal{T}}$.

The orbifold of a T^4 model $\mathcal{T}$ by a non-anomalous finite group of symmetries $H \subset G_{\mathcal{T}}$ is again a $\mathcal{N} = (4,4)$ superconformal field theory at $c = c = 6$, so it is a sigma model on either K3 or T^4. In this case, the elements $g \in G_{\mathcal{T}}$ that normalize H, i.e. such that $gHg^{-1} = H$, commute with $\sum_{h \in H} \mathcal{L}_h$ and therefore give rise to invertible defects in the orbifold theory

$\mathcal{T}/H$. On the other hand, if $g \in G_{\mathcal{T}}$ does not commute with $\sum_{h\in H}\mathcal{L}_h$, then it can induce a non-invertible defect in $\mathcal{T}/H$. In the next sections, we will see some examples where H is cyclic and the orbifold $\mathcal{T}/H$ is a K3 model.

If H is a finite non-anomalous subgroup of $\mathrm{U}(1)^8$, then the orbifold $\mathcal{T}/H$ is again a torus model. If the symmetry group of $\mathcal{T}$ is the generic $G_{\mathcal{T}} = \mathrm{U}(1)^8 \rtimes \mathbb{Z}_2$, then every $g \in G_{\mathcal{T}}$ normalizes[7] $H \subset \mathrm{U}(1)^8$, and therefore necessarily induces an invertible symmetry in $\mathcal{T}/H$. On the other hand, if one starts from a model $\mathcal{T}$ with non-trivial $G^0_{\mathcal{T}}$, then one can get non-invertible defects in infinitely many different torus models $\mathcal{T}/H$, by varying the orbifold subgroups $H \subset \mathrm{U}(1)^8$. This is the higher dimensional analogue of a phenomenon observed for a single free boson on S^1 [23, 24]: at the self-dual radius R_{sd} there is a $\mathrm{SU}(2)\times\mathrm{SU}(2)$ group of symmetries commuting with the Virasoro algebra; by taking orbifolds by suitable non-anomalous $H = \mathbb{Z}_N \times \mathbb{Z}_M$ groups, one can get S^1 models at every rational multiple $\frac{N}{M}R_{sd}$ of the self-dual radius, so that in every such model there are non-invertible topological symmetries labeled by a pair $(g,h) \in \mathrm{SU}(2)\times\mathrm{SU}(2)$ [23]. A simple T^4 model example is given by the product $\mathcal{T} = (S^1)^4$ of four S^1 free bosons at the self-dual radius (this corresponds to the model denoted as A_1^4 in [63]). The model $\mathcal{T}$ is self-dual under T-duality along each of the four circles; furthermore, T-dualities in an even number of direction preserve the $\mathcal{N} = (4,4)$ algebra and spectral flow, so that they correspond to elements $g \in G_{\mathcal{T}}$. By taking orbifolds by finite subgroups $H \subset \mathrm{U}(1)^8$, one can get infinitely many T^4 models with non-invertible topological defects induced by g. We stress that while the subset of such T^4 models is dense in the moduli space, it is still of measure zero.

4.2 Continuous defects in $T^4/\mathbb{Z}_2$

Given any torus model $\mathcal{T}$, the orbifold $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ by the $\mathbb{Z}_2$ symmetry (4.8) is a non-linear sigma model on K3.

A torus orbifold $\mathcal{C} = \mathcal{T}/\mathbb{Z}_2$ always admits an invertible defect Q (the quantum symmetry) acting by -1 on the twisted sector and trivially on the untwisted one. Besides Q, it also contains some invertible simple defects that are induced by the simple defects W_θ of the torus model $\mathcal{T}$ that commute with $\mathcal{R}$, i.e such that $\theta \equiv -\theta \mod {}^{4,4}$. More precisely, with each defect $W_{\frac{\lambda}{2}}$, $\lambda \in {}^{4,4}/2\,{}^{4,4}$ of $\mathcal{T}$, are associated two defects $\eta_{\frac{\lambda}{2}}, \eta'_{\frac{\lambda}{2}} \equiv Q\eta_{\frac{\lambda}{2}}$ of $\mathcal{C}$. In particular, for $\lambda = 0$, one has $\eta_0 = I$ (the identity defect) and $\eta'_0 = Q$.

While the defects $W_{\frac{\lambda}{2}}$, $\lambda \in {}^{4,4}$, generate an abelian group of symmetries $\mathbb{Z}_2^8 \subset \mathrm{U}(1)^8$ of the torus model $\mathcal{T}$, the group generated by the $\eta_{\lambda/2}$ is a non-abelian extension of $\mathbb{Z}_2^8$ by a central $\mathbb{Z}_2$ generated by the quantum symmetry Q. Consider a basis $\{n_1,\ldots,n_4,w_1,\ldots,w_4\}$ for ^{4,4} as in (4.3). The defects η_{n_i}, η_{w_i} and Q obey the following relations

$$\eta^2_{\frac{n_i}{2}} = I = \eta^2_{\frac{w_i}{2}}\,, \qquad Q^2 = 1\,, \qquad \eta_{\frac{n_i}{2}}Q = Q\eta_{\frac{n_i}{2}}\,, \qquad \eta_{\frac{w_i}{2}}Q = Q\eta_{\frac{w_i}{2}} \tag{4.11}$$

If $\lambda = \sum_i (a_i n_i + b_i w_i)$, $a_i, b_i \in \mathbb{Z}/2\mathbb{Z}$, is an element of ${}^{4,4}/2\,{}^{4,4}$, then we define

$$\eta_{\frac{\lambda}{2}} = \eta_{\frac{1}{2}\sum_i(a_i n_i + b_i w_i)} := \eta_{\frac{a_1 n_1}{2}} \cdots \eta_{\frac{a_4 n_4}{2}} \eta_{\frac{b_1 w_1}{2}} \cdots \eta_{\frac{b_4 w_4}{2}}\,, \qquad \eta'_{\frac{\lambda}{2}} := Q\eta_{\frac{\lambda}{2}}\,. \tag{4.12}$$

[7] Notice in particular that $\mathcal{R}h\mathcal{R} = h^{-1}$ for all $h \in \mathrm{U}(1)^8$.

We have the fusion rules

$$Q^2 = I\,, \qquad \eta_{\frac{\lambda}{2}} Q = Q \eta_{\frac{\lambda}{2}}\,, \qquad (\eta_{\frac{\lambda}{2}})^2 = Q^{(\lambda,\lambda)/2} \tag{4.13}$$

which imply

$$\eta^2_{\frac{n_i}{2}} = I = \eta^2_{\frac{w_i}{2}}\,, \qquad \eta_{\frac{\lambda}{2}} \eta_{\frac{\mu}{2}} = Q^{(\lambda,\mu)} \eta_{\frac{\mu}{2}} \eta_{\frac{\lambda}{2}} \tag{4.14}$$

Different choices of the basis $\{n_1,\ldots,n_4,w_1,\ldots,w_4\}$ just lead to exchanging $\eta_{\frac{\lambda}{2}} \leftrightarrow \eta'_{\frac{\lambda}{2}}$ for some values of θ — this is just a relabeling of the defects. Altogether, these invertible elements have group-like fusion rules, corresponding to an extraspecial group 2^{1+8} [64]

$$1 \longrightarrow \langle Q\rangle \cong \mathbb{Z}_2 \longrightarrow 2^{1+8} \longrightarrow \mathbb{Z}_2^8 \longrightarrow 1\ . \tag{4.15}$$

Besides invertible defects, torus orbifolds $\mathcal{T}/\mathbb{Z}_2$ always contain a continuum of defects T_θ, parametrized by $\theta \in ((\mathbb{R}/\mathbb{Z})^8)/\pm 1$, that preserve the $\mathcal{N}=(4,4)$ algebra and the spectral flow generators, and that are induced by the $\mathcal{R}$-invariant superposition $W_\theta + W_{-\theta}$ of topological defects of the torus model $\mathcal{T}$. The defects T_θ have dimension 2 and satisfy the fusion rules

$$T_\theta T_{\theta'} = T_{\theta+\theta'} + T_{\theta-\theta'}\,,$$

While $W_\theta + W_{-\theta}$ are clearly a superposition, the T_θ are actually simple for generic values of θ. The only exceptions are when θ is one of the $\mathcal{R}$-fixed points $\theta = \frac{\lambda}{2} \in \frac{1}{2}\ {}^{4,4}/\ {}^{4,4}$, and in this case they are superpositions

$$T_{\frac{\lambda}{2}} = \eta_{\frac{\lambda}{2}} + Q\eta_{\frac{\lambda}{2}}\ . \tag{4.16}$$

Notice that $T_0 = \mathcal{I} + Q$, so that

$$(T_\theta)^2 = T_0 + T_{2\theta} = \mathcal{I} + Q + T_{2\theta}\,, \tag{4.17}$$

that implies that T_θ is unoriented, $(T_\theta)^\star = T_\theta$. The fusion with the invertible defects is

$$QT_\theta = T_\theta Q = T_\theta\,, \qquad \eta_{\frac{\lambda}{2}} T_\theta = T_\theta \eta_{\frac{\lambda}{2}} = T_{\theta+\frac{\lambda}{2}}\,, \tag{4.18}$$

where in the last identity, one uses $\frac{\lambda}{2} \equiv -\frac{\lambda}{2} \mod {}^{4,4}$, so that $T_{\theta+\frac{\lambda}{2}} = T_{\theta-\frac{\lambda}{2}}$.

According to [15], the operators T_θ associated with these defects act on the untwisted sector as the operators $W_\theta + W_{-\theta}$ in the original theory, while they annihilate the twisted sector. On the RR ground states, all T_θ act by twice the identity on the states in the untwisted sector, and annihilate all the states in the twisted sector.

If $2\theta \in (\ {}^{4,4} \otimes \mathbb{R})/\ {}^{4,4}$ is a $\mathcal{R}$-fixed point, i.e. if $2\theta \equiv -2\theta \mod {}^{4,4}$, then $\theta = \frac{\lambda}{4}$ for some $\lambda \in\ {}^{4,4}$, and the fusion product (4.17) becomes

$$(T_{\frac{\lambda}{4}})^2 = \mathcal{I} + Q + \eta_{\frac{\lambda}{2}} + Q\eta_{\frac{\lambda}{2}}\ . \tag{4.19}$$

The right-hand side is just the sum over all invertible defects in the order 4 group generated by Q and $\eta_{\frac{\lambda}{2}}$ (this is either $\mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$, depending on whether $(\eta_{\frac{\lambda}{2}})^2$ equals $\mathcal{I}$ or Q, i.e. if (λ,λ) equals 0 or 2 mod 4). Therefore, $T_{\frac{\lambda}{4}}$ is a duality defect, providing an equivalence of our theory to the orbifold by this subgroup.

4.3 Topological defects in $T^4/\mathbb{Z}_N$ models

Let us generalize the construction of the previous sections to the case of a K3 model $\mathcal{C}$ obtained as the orbifold $\mathcal{T}/\mathbb{Z}_N$ of a torus model $\mathcal{T}$ by a cyclic group $\langle g\rangle \cong \mathbb{Z}_N$.

In order for $\mathcal{T}/\mathbb{Z}_N$ to be a K3 model, all holomorphic and anti-holomorphic fields of spin 1/2 must be projected out. Because the $\mathrm{U}(1)^4\times\mathrm{U}(1)^4$ group generated by W_θ acts trivially on such fields, this implies that g must be a lift of an automorphism of the Narain lattice $\;^{4,4}$. Furthermore, because the eight fermions transform in the same representation as $\;^{4,4}\otimes\mathbb{R}$, we require g to have no fixed vectors in such representation. Notice that, for $N>2$, the group $G_\mathcal{T}$ preserving the small $\mathcal{N}=(4,4)$ algebra and spectral flow contains such lifts only at special loci in the moduli space of torus models. All such models and symmetries were classified in [63].

The quantum symmetry Q has order N and acts by multiplication by a phase $e^{\frac{2\pi i k}{N}}$ on the g^k-twisted sector. The orbifold procedure can be described as the gauging of the finite group $\langle g\rangle$, and defect line Q^k is interpreted as a Wilson line associated with the 1-dimensional representation ρ_k of $\langle g\rangle$, where $\rho_k(g)=e^{\frac{2\pi i k}{N}}$. In particular, operators that were local in the original torus model $\mathcal{T}$ and transforming in the ρ_k representation of $\langle g\rangle$ become operators in the defect space $\mathcal{H}_{Q^k}$ in the orbifold $\mathcal{C}$.

For each $\theta\in\;^{4,4}\otimes\mathbb{R}/\;^{4,4}$, there is a continuum of topological defects $T_\theta\in\mathsf{Top}_\mathcal{C}$ of dimension N induced by a superposition of defects $W_{g^k(\theta)}$ of the torus model $\mathcal{T}$

$$W_\theta+W_{g(\theta)}+\ldots+W_{g^{N-1}(\theta)}\longrightarrow T_\theta\ . \tag{4.20}$$

The defect T_θ only depends on the orbit of θ with respect to the action of $\langle g\rangle$, and its dual is $(T_\theta)^\star=T_{-\theta}$. For generic θ, when the $\mathbb{Z}_N$-orbit has N distinct elements $g^k(\theta)$, T_θ is simple, while it decomposes into a sum of N/d simple defects when θ has a non-trivial stabiliser $\mathbb{Z}_{N/d}\subset\mathbb{Z}_N$. More precisely, when the stabiliser subgroup $Stab(\theta)$ is $\langle g^d\rangle\cong\mathbb{Z}^{N/d}$ for some $d|N$, there are N/d simple defects $\eta_\theta, Q\eta_\theta, Q^2\eta_\theta,\ldots,Q^{N/d-1}\eta_\theta$ of dimension d induced by

$$\sum_{k=0}^{d-1}W_{g^k(\theta)}\longrightarrow\eta_\theta \tag{4.21}$$

and we have a decomposition

$$T_\theta=\eta_\theta+Q\eta_\theta+\ldots+Q^{N/d-1}\eta_\theta\ . \tag{4.22}$$

This decomposition can be understood as follows: when $g^d(\theta)=\theta$, each defect space $\mathcal{H}_{W_{g^k(\theta)}}$ in the torus model, with $k=0,\ldots,d-1$, carries a non-trivial representation of $\langle g^d\rangle$, and operators in $\mathcal{H}_{W_{g^k(\theta)}}$ with different g^d-charge must belong to different (non-isomorphic) irreducible modules of the orbifold theory $\mathcal{C}$. The OPE with defect operators in $\mathcal{H}_Q$ modifies the g^d-charge, and therefore maps each of these N/d modules into one another. Therefore, the defect lines for these modules can be written as $Q^k\eta_\theta\equiv\eta_\theta Q^k$, $k=0,\ldots,N/d-1$. We notice that operators in $Q^{N/d}$ have trivial g^d charge, so that we have the fusion rule

$$Q^{N/d}\eta_\theta=\eta_\theta Q^{N/d}=\eta_\theta\ . \tag{4.23}$$

This fusion rule also implies that the linear operator $\eta_\theta:\mathcal{H}\to\mathcal{H}$ associated with η_θ must annihilate all g^k-twisted states, unless k is a multiple of d. Similarly, the fusion rule

$$QT_\theta=T_\theta Q=T_\theta\,, \tag{4.24}$$

implies that T_θ annihilates all g^k-twisted sectors, for all $k\neq 0\mod N$.

The fusion rules are

$$T_\theta T_{\theta'} = \sum_{k=0}^{N-1} T_{\theta + g^k(\theta')} \,, \tag{4.25}$$

and, in particular, for generic θ,

$$T_\theta T_{-\theta} = T_0 + \sum_{k=1}^{N-1} T_{\theta - g^k(\theta)} = 1 + Q + \ldots + Q^{N-1} + \sum_{k=1}^{N-1} T_{(1-g^k)(\theta)} \ . \tag{4.26}$$

Finally, because the reflection (4.8) commutes with every automorphism of the lattice $\ ^{4,4}$, we have that $\mathcal{R}$ is also an invertible topological defect of the orbifold model $\mathcal{C}$.

The elements θ that are stabilised by g are all the solutions $x \in (\ ^{4,4} \otimes \mathbb{R})/\ ^{4,4}$ of the equation

$$(1-g)(x) \equiv 0 \quad \text{mod} \quad ^{4,4} \ . \tag{4.27}$$

Because by hypothesis g has no fixed vectors on $\ ^{4,4} \otimes \mathbb{R}$ then

$$1 + g + \ldots g^{N-1} = 0 \,, \tag{4.28}$$

and $(1-g)$ is invertible, with inverse

$$(1-g)^{-1} = -\frac{1}{N}(g + 2g^2 + \ldots + (N-1)g^{N-1}) \,, \tag{4.29}$$

as can be easily verified. Thus, the fixed vectors x are all elements of

$$((1-g)^{-1}\ ^{4,4})/\ ^{4,4} \subseteq \left(\frac{1}{N}\ ^{4,4}\right)/\ ^{4,4} \ . \tag{4.30}$$

The number of distinct points in this quotient is given by $\det(1-g)$ and can be easily computed once the eigenvalues of g are known.

Consider for example the case when N is prime. The analysis in [63] shows that the possible values are $N = 2, 3, 5$ (for $N = 5$, the symmetry g has no geometric interpretation as an automorphism of the target T^4 torus.). In this case, for any $\theta \in (\ ^{4,4} \otimes \mathbb{R})/\ ^{4,4}$, either the orbit of θ contains N distinct elements, or θ is fixed by the full group $\langle g \rangle$. For each N, the eigenvalues of g are all primitive N-th roots of unity with the same multiplicity [63]. Therefore, the number $\det(1-g)$ of distinct points in the quotient $((1-g)^{-1}\ ^{4,4})/\ ^{4,4}$ equals 2^8 for $N = 2$, 3^4 for $N = 3$ and 5^2 for $N = 5$. The corresponding simple defects η_x are invertible, and together with Q they generate some non-abelian central extension of $\mathbb{Z}_2^8$, $\mathbb{Z}_3^4$ and $\mathbb{Z}_5^2$ called extraspecial groups 2^{1+8}, 3^{1+4} and 5^{1+2} for $N = 2$, $N = 3$ and $N = 5$, respectively:

$$1 \longrightarrow \langle Q \rangle \cong \mathbb{Z}_N \longrightarrow N^{1+k} \longrightarrow \mathbb{Z}_N^k \to 1 \ . \tag{4.31}$$

In particular, two lines η_x and $\eta_{x'}$ do not necessarily commute

$$\eta_x \eta_{x'} = Q^{c_g(x,x')} \eta_{x'} \eta_x \ . \tag{4.32}$$

The 2-cocycle $c_g(x,x')$ (4.32) characterizing the central extension N^{1+k} are determined by the 't Hooft anomaly in the $\mathrm{U}(1)^8$ symmetry of the torus model, see [65–68].

Similar arguments hold when N is not prime. The possible values of N, in this case, are $4,6,8,10,12$ (they are the values $o(\pm g_0)$ in table 2 of [63]; the eigenvalues of g when acting on the space $\ ^{4,4}\otimes\mathbb{R}$ are denoted by $\pm\zeta_L,\pm\zeta_R,\pm\zeta_L^{-1},\pm\zeta_R^{-1}$ in the same table). For each $d|N$, consider the element g^d. If 1 is an eigenvalue of g^d, then the points $x\in(\ ^{4,4}\otimes\mathbb{R})/\ ^{4,4}$ stabilized by g^d form a continuum, corresponding to the g^d-fixed subspace of $\ ^{4,4}\otimes\mathbb{R}$. On the other hand, suppose that none of the eigenvalues of g^d equals 1, so that $(1-g^d)$ is invertible. In this case, the g^d fixed vectors $x\in\ ^{4,4}\otimes\mathbb{R}/\ ^{4,4}$ are the $\det(1-g^d)$ elements in the quotient $((1-g^d)^{-1}\ ^{4,4})/\ ^{4,4}$. When $d=1$, the corresponding defects η_x are invertible, and form a group that is a central extension of some $\mathbb{Z}_N^k$ by the quantum symmetry Q.

Consider the case where g has *prime* order N, and $(1-g)$ is invertible. Let $x\in(\ ^{4,4}\otimes\mathbb{R})/\ ^{4,4}$, $x\neq 0$, be a non-trivial fixed point of g, i.e. such that $x\equiv g(x) \mod\ ^{4,4}$; this is the image $x=(1-g)^{-1}\lambda$ of some vector $\lambda\in\ ^{4,4}$. If we apply the operator $(1-g)^{-1}$ once again to x, we obtain a vector

$$v:=(1-g)^{-1}x\,, \tag{4.33}$$

with some special properties. In particular, we get

$$(1-g^k)v=(1+g+\ldots+g^{k-1})(1-g)(1-g)^{-1}x=kx \mod\ ^{4,4}\,. \tag{4.34}$$

Let us define the simple defect

$$\mathcal{N}_x:=\mathcal{R}T_v=\mathcal{R}\sum_{k=0}^{N-1}W_{g^k(v)}=\mathcal{R}\sum_{k=0}^{N-1}W_{v-kx}\,. \tag{4.35}$$

We get

$$\begin{aligned}\mathcal{N}_x^2=\mathcal{R}T_v\mathcal{R}T_v=T_{-v}T_v=\sum_{k=0}^{N-1}T_{v-g^k(v)}=\sum_{k=0}^{N-1}T_{kx}\\ =(1+\eta_x+\ldots+\eta_{(N-1)x})(1+Q+\ldots+Q^{N-1})\,,\end{aligned} \tag{4.36}$$

and

$$\mathcal{N}_x\eta_x=\mathcal{R}T_v\eta_x=\mathcal{R}T_{v+x}=\mathcal{R}\sum_{k=0}^{N-1}W_{v-kx+x}=\mathcal{R}\sum_{k=0}^{N-1}W_{v-(k-1)x}=\mathcal{N}_x \tag{4.37}$$

Therefore, $\mathcal{N}_x$ is a duality defect for the abelian group of order N^2 generated by η_x and Q. This group could be isomorphic to either $\mathbb{Z}_N\times\mathbb{Z}_N$ or $\mathbb{Z}_{N^2}$, depending on the norm of x. This argument shows that the torus orbifold $\mathcal{T}/\langle g\rangle$ is self-orbifold with respect to any such group of symmetries.

More generally, one expects a continuum of defects in $\mathsf{Top}_\mathcal{C}$ whenever $\mathcal{C}$ can be described as an orbifold of a torus model $\mathcal{T}$ by some (possibly non-abelian) symmetry group G of $\mathcal{T}$. Such defects are induced by superpositions $\sum_{g\in G}W_{g(\theta)}$ of topological defects of torus models, and are simple for generic values of θ.

4.4 An example of torus orbifold $T^4/\mathbb{Z}_4$

Let us consider an example of the general theory described in the previous section. We consider a torus model $\mathcal{T}$ with a large group $\mathrm{U}(1)^8 \rtimes G^0_{\mathcal{T}}$ of symmetries commuting with a small $\mathcal{N}=(4,4)$ superconformal algebra, where $G^0_{\mathcal{T}}$ has order 192. This is the torus model described in section 4.4.1 of [63], and contains a $\widehat{so}(8)_1$ chiral and anti-chiral algebra. The $\mathbb{Z}_2$ orbifold by $\mathcal{R}$ (the centre of $G^0_{\mathcal{T}}$) gives the K3 model with the 'largest symmetry group' [69, 70] that we will consider in section 5.

The winding-momentum lattice $\ ^{4,4}$ of this model is given by the columns of the following matrix

$$\frac{1}{\sqrt{2}}\begin{pmatrix} 2 & 0 & 0 & -1 & 1 & 1 & 1 & 1 \\ 0 & 2 & 0 & -1 & 1 & 0 & 0 & 1 \\ 0 & 0 & 2 & -1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \tag{4.38}$$

In each column, the first four coordinates are the eigenvalues $\lambda^1_L,\ldots,\lambda^4_L$ of the zero modes of $\partial X^1,\ldots,\partial X^4$, while the last four are the eigenvalues $\lambda^1_R,\ldots,\lambda^4_R$ of the zero modes of $\partial X^1,\ldots,\partial X^4$.

Notice that the vertex operators $V_\lambda(z)$ corresponding to the first four columns are holomorphic currents ($h=\frac{\lambda_L\cdot\lambda_L}{2}=1$, $h=\frac{\lambda_R\cdot\lambda_R}{2}=0$), which together with ∂X^k generate the $\widehat{so}(8)_1$ current algebra.

Let us focus on a symmetry $g\in G^0_{\mathcal{T}}$ of order $N=4$, acting by

$$\begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix} \tag{4.39}$$

on $\ ^{4,4}\otimes\mathbb{R}$. One can easily verify that this is a lattice automorphism, $g\in O(\ ^{4,4})$, and that $\det(1-g)=64\neq 0$. This is a symmetry in the class $-4A$, in the notation of [63], and the orbifold $\mathcal{T}/\langle g\rangle$ is again the K3 model with largest symmetry group that we will describe in section 5 (see section 6 of [69]).

Let us consider the g-fixed points in $\ ^{4,4}\otimes\mathbb{R}/\ ^{4,4}$. As explained in the previous section they correspond to the points

$$((1-g)^{-1}\ ^{4,4})/\ ^{4,4}\cong\mathbb{Z}_2\times\mathbb{Z}_2\times\mathbb{Z}_4\times\mathbb{Z}_4 \tag{4.40}$$

with generators

$$y_1 = \begin{pmatrix} 0 \\ 1 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad y_2 = \begin{pmatrix} -1/2 \\ -1/2 \\ -1/2 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_1 = \begin{pmatrix} 3/2 \\ 3/2 \\ 0 \\ 0 \\ 1 \\ 0 \\ 0 \\ 0 \end{pmatrix} \qquad u_2 = \begin{pmatrix} 3/2 \\ 0 \\ 1 \\ 1/2 \\ 3/2 \\ 1/2 \\ 1/2 \\ 1/2 \end{pmatrix}, \tag{4.41}$$

where y_1 and y_2 have order 2, while u_1 and u_2 have order 4 (modulo $\ ^{4,4}$).

The invertible topological defects W_x of the torus model $\mathcal{T}$, where $x \in ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4}$ are g-fixed points, induce invertible defects $\eta_x, Q\eta_x, Q^2\eta_x, Q^3\eta_x$ in the K3 model $\mathcal{T}/\langle g\rangle$, where Q is the quantum symmetry of order 4. The group they generate is a central extension of $((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4} \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4$ by $\langle Q\rangle \cong \mathbb{Z}_4$. In order to determine the central extension in detail, one needs to know the 't Hooft anomaly for the abelian group

$$H = \langle g\rangle \times ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4} \cong \mathbb{Z}_4 \times \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_4 \times \mathbb{Z}_4 \,. \tag{4.42}$$

The anomaly is encoded in a cohomology class $[\omega] \in H^3(H, \mathrm{U}(1))$, with representative a 3-cocycle $\omega : H \times H \times H \to \mathrm{U}(1)$. It is known that

$$H^3(\mathbb{Z}_2^2 \times \mathbb{Z}_4^3, \mathrm{U}(1)) \cong \mathbb{Z}_2^{18} \times \mathbb{Z}_4^7 , \tag{4.43}$$

and a basis of generators can be found, for example, in [71]. In order to determine which class in $H^3(H, \mathrm{U}(1))$ is relevant in this case, one can just consider, for each $k \in H$, the failure of the level matching condition for the k-twisted sector of $\mathcal{T}$. In particular, if $k \in H$ has order $o(k)$, the level matching condition is satisfied if the spin $spin(k)$ (i.e. the difference $h - h$ of conformal weights) of the k-twisted states take values in $\frac{1}{o(k)}\mathbb{Z}$; in this case, the restriction of the anomaly class $[\omega]$ to the cyclic group $\langle k\rangle$ is trivial. More generally, if the restriction of $[\omega]$ to $\langle k\rangle$ is non-trivial, one has the following relation [16, 65–67, 72]

$$e^{2\pi i o(k) spin(k)} = \prod_{i=1}^{o(k)-1} \omega(k, k^i, k) , \tag{4.44}$$

between the spin and the cocycle ω. If we know the spin of the k-twisted sectors for all $k \in H$, we can use (4.44) to determine the class $[\omega]$. In fact, for elements $x \in ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4} \subset H$ the spin of the x-twisted ground state is

$$\mathrm{spin}(x) = \frac{(x,x)}{2} \,. \tag{4.45}$$

On the other hand, the cohomology class $[\omega]$ is trivial when restricted to the cyclic subgroup $\langle g\rangle \subset H$ (see [63]). More generally, the restriction of $[\omega]$ to any cyclic group of the form $\langle gx\rangle \subset H$ is trivial for all $x \in ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4}$, because g and gx are conjugate within the group G. These data are sufficient to determine the class $[\omega]$ uniquely.

Once the cocycle ω representing the 't Hooft anomaly of H is known, one can determine the central extension of $((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4}$ by the quantum symmetry Q that is induced on the orbifold theory $\mathcal{T}^4/\langle g\rangle$. In particular, the commutation relations are

$$\eta_{y_1}\eta_{y_2} = Q^2\eta_{y_2}\eta_{y_1}\,, \qquad \eta_{u_1}\eta_{u_2} = Q\eta_{u_2}\eta_{u_1}\,, \qquad \eta_{y_j}\eta_{u_k} = Q^2\eta_{u_k}\eta_{y_j}\,, \qquad j,k=1,2\ .$$

Furthermore, while η_{y_1} and η_{y_2} can be chosen of order 2

$$\eta_{y_1}^2 = 1 = \eta_{y_2}^2\,, \tag{4.46}$$

the symmetries η_{u_1} and η_{u_2} have order 8

$$\eta_{u_1}^4 = Q^2 = \eta_{u_2}^4\,, \qquad Q^4 = 1\ . \tag{4.47}$$

The orbifold theory $\mathcal{T}/\langle g\rangle$ contains a continuum of topological defects T_θ of dimension 4, with $\theta \in (\ {}^{4,4}\otimes\mathbb{R})/\ {}^{4,4}$, that are simple for generic θ. These defects, as well as the defects of the form Q^kT_θ, $k=0,1,2,3$, are induced by the superpositions $\sum_{j=0}^3 W_{g^j(\theta)}$ in the torus model

$$W_\theta + W_{g(\theta)} + W_{g^2(\theta)} + W_{g^3(\theta)} \longrightarrow T_\theta,\ QT_\theta,\ Q^2T_\theta,\ Q^3T_\theta\ . \tag{4.48}$$

When $\theta \equiv x \in ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4}$ is one of the g-fixed points, $T_\theta \equiv T_x$ is not simple, but becomes a superposition of four invertible defects

$$T_x = \eta_x + Q\eta_x + Q^2\eta_x + Q^3\eta_x\,, \tag{4.49}$$

and satisfies $QT_x = T_x$.

An intermediate case occurs when θ is fixed by g^2, but not by g. This happens when θ is of the form $\theta = (\theta_L, 0)$, so that $g(\theta) = -\theta$. In this case, T_θ decomposes as a sum

$$T_\theta = \xi_\theta + Q\xi_\theta\,, \tag{4.50}$$

where ξ_θ and $Q\xi_\theta$ are dimension two defects induced by $W_\theta + W_{-\theta}$ on $\mathcal{T}$

$$W_\theta + W_{-\theta} \longrightarrow \xi_\theta, Q\xi_\theta\ . \tag{4.51}$$

The defects ξ_θ, where $\theta = (\theta_L, 0)$, are simple for generic θ_L, while they decompose as $\xi_x = \eta_x + Q\eta_x$ when $x \equiv (\theta_L, 0)$ is fixed by g. Furthermore, the ξ_θ are always unoriented and satisfy

$$Q^2\xi_\theta = \xi_\theta Q^2 = \xi_\theta\,, \qquad \xi_\theta\xi_{\theta'} = \xi_{\theta+\theta'} + \xi_{\theta-\theta'}\ . \tag{4.52}$$

As described in the previous section, for each g-fixed vector $x \in ((1-g)^{-1}\ {}^{4,4})/\ {}^{4,4}$, we can consider $v = (1-g)^{-1}x$. There are two cases to be considered, depending on whether x is fixed by g^2 or not. In the first case, x is a linear combination

$$x = a_1y_1 + a_2y_2 + 2b_1u_1 + 2b_2u_2\,, \tag{4.53}$$

with $a_1, a_2, b_1, b_2 \in \{0,1\}$. Notice the x has always order 2 in this case, $2x \in {}^{4,4}$, and $g(x) = -x \mod {}^{4,4}$. In this case, we have

$$v = a_1 \begin{pmatrix} 0 \\ 12 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + a_2 \begin{pmatrix} -1/4 \\ -1/4 \\ -1/4 \\ -1/4 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b_1 \begin{pmatrix} 3/4 \\ 3/4 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} + b_2 \begin{pmatrix} -1/2 \\ 0 \\ 0 \\ -1/2 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix} \tag{4.54}$$

so that $g(v) = -v$ and $g^2(v) = v$. Thus, in the orbifold theory $\mathcal{T}/\langle g \rangle$ there are two dimension 2 unoriented simple topological defects, ξ_v and $Q\xi_v$, induced by the defect $W_v + W_{-v}$ of $\mathcal{T}$, as in (4.51). The fusion of ξ_v and $Q\xi_v$ with themselves gives

$$\xi_v^2 = (Q\xi_v)^2 = \xi_0 + \xi_x = 1 + Q^2 + \eta_x + Q^2\eta_x \ . \tag{4.55}$$

Furthermore, because $v + x \equiv -v \mod {}^{4,4}$, one gets

$$\xi_v \eta_x = \eta_x \xi_v = \xi_v \ . \tag{4.56}$$

Therefore, both ξ_v and $Q\xi_v$ are duality defects for the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group generated by η_x and Q^2.

In the second case, where x is not fixed by g^2, we have

$$x = a_1 y_1 + a_2 y_2 + c_1 u_1 + c_2 u_2 \,, \tag{4.57}$$

where $a_1, a_2, c_1, c_2 \in \{0,1\}$, with at least one among c_1 and c_2 being odd. In this case, x has order 4, i.e. $4x \in {}^{4,4}$, and $v := (1-g)^{-1}x$ is not fixed by either g or g^2. Thus, the orbifold theory $\mathcal{T}/\langle g \rangle$ contains four dimension 4 defects $Q^k T_v$, $k = 0,1,2,3$, induced by $\sum_{k=0}^{3} W_{g^k(v)}$, as in (4.48). The treatment is analogous to the case described in the previous section, where the order N of g was a prime number. One defines

$$\mathcal{N}_x := \mathcal{R} T_v \,, \tag{4.58}$$

and the resulting topological defect $\mathcal{N}_x$ is unoriented and satisfies

$$\mathcal{N}_x^2 = (1 + Q + Q^2 + Q^3)(1 + \eta_x + \eta_{2x} + \eta_{3x}) \tag{4.59}$$

Thus, it is a duality defect for the $\mathbb{Z}_2 \times \mathbb{Z}_8$ group with generators η_x of order 8 and $Q\eta_x^2$ of order 2.

4.5 K3 models with continuous defects: a conjecture

In the previous sections, we have seen that all K3 models $\mathcal{C}$ that can be described as torus orbifolds contain a continuous family of topological defects $\mathcal{L}_\lambda \in \mathsf{Top}_{\mathcal{C}}$, that preserve the $\mathcal{N} = (4,4)$ algebra and spectral flow, and that are simple for generic values of λ. On the other hand, in claim 3 we argued that for a generic K3 model $\mathsf{Top}_{\mathcal{C}}$ is trivial, and in particular has no such family of defects. Thus, it is natural to look for a characterisation of the K3 models admitting such continuous families. We propose the following conjecture:

Conjecture 6. *Let $\mathcal{C}$ be a K3 model that admits a continuous family of topological defects $\mathcal{L}_\lambda \in \mathsf{Top}_\mathcal{C}$, preserving the $\mathcal{N}=(4,4)$ superconformal algebra and spectral flow, and simple for all λ except possibly a zero measure set. Then, $\mathcal{C}$ is the (generalised) orbifold of a torus model $\mathcal{T}$.*

Let us recall the notion of generalised orbifold [16, 61, 73]. In a 'standard' orbifold of a CFT $\mathcal{C}$ by a finite group of symmetries G, one can consider the superposition $A=\sum_{g\in G}\mathcal{L}_g$ of all invertible defects in the group. Then, the Hilbert space of local operators in the orbifold theory $\mathcal{C}/G$ is just the subspace of G-invariant operators in $\mathcal{H}_A$, and their correlation functions can be obtained as correlation functions in the original theory with an insertion of a fine enough network of defects of type A. This construction can be generalised to any topological defect A, for which there exist some 'multiplication' and 'co-multiplication' maps, i.e. topological junction operators $\mu:\mathcal{H}_A\otimes\mathcal{H}_A\to\mathcal{H}_A$ and $\mu:\mathcal{H}_A\to\mathcal{H}_A\otimes\mathcal{H}_A$ satisfying some suitable conditions — essentially, that the associator of the multiplication map is trivial, and that $\mu\circ\mu$ is the identity. When such properties are satisfied, one can define a new consistent CFT, the generalised orbifold $\mathcal{C}/A$, whose space of local operators is a suitable subspace of $\mathcal{H}_A$, and whose correlation functions are obtained by inserting networks of defects A in the $\mathcal{C}$ correlators, with the operators μ and μ inserted at trivalent junctions. Furthermore, the generalised orbifold procedure is always reversible: if $\mathcal{C}'=\mathcal{C}/A$ is obtained as a generalised orbifold of some CFT $\mathcal{C}$, then $\mathcal{C}=\mathcal{C}'/A'$ is a generalised orbifold of $\mathcal{C}'$.

Let us now sketch an argument for a possible proof of conjecture 6. According to [15], the presence of a continuum of operators $\mathcal{L}_\lambda$ is related to the presence of a conserved current $j\in\mathcal{H}_{\mathcal{L}_\lambda\mathcal{L}_\lambda^\star}$:

$$j:=J(z)dz+J(z)dz\,, \tag{4.60}$$

where $\partial J(z)=0=\partial J(z)$. This current is such that an infinitesimal deformation $\mathcal{L}_{\lambda+\delta\lambda}$ of the defect $\mathcal{L}_\lambda$ can be obtained as

$$\mathcal{L}_{\lambda+\delta\lambda}=\mathcal{L}_\lambda e^{i\delta\lambda\int j}\,, \tag{4.61}$$

where $j\in\mathcal{H}_{\mathcal{L}_\lambda\mathcal{L}_\lambda^\star}\cong\mathcal{H}_{\mathcal{L}_\lambda}\otimes\mathcal{H}^*_{\mathcal{L}_\lambda}$ is interpreted as a linear operator from $\mathcal{H}_{\mathcal{L}_\lambda}$ to itself, and the integration $\int j$ is over the support of the defect $\mathcal{L}_\lambda$.

In the case we are interested in, since $\mathcal{L}_\lambda$ and $\mathcal{L}_{\lambda+\delta\lambda}$ are defects in $\mathsf{Top}_\mathcal{C}$, then the current j must preserve the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow. In particular, $J(z)$ and $J(z)$ must be neutral with respect to the holomorphic and anti-holomorphic $\widehat{su}(2)_1$ R-symmetry, and must be the supersymmetric descendants of holomorphic and anti-holomorphic spin 1/2 fields. This implies that the NS-NS sector of $\mathcal{H}_{\mathcal{L}_\lambda\mathcal{L}_\lambda^\star}$ contains two $\mathcal{N}=(4,4)$ BPS representation $(\frac{1}{2},\frac{1}{2};0,0)$ and $(0,0;\frac{1}{2},\frac{1}{2})$ containing the currents $J(z)$ and $J(z)$. As a matter of fact, because the ground states of these representations are a $\widehat{su}(2)_1$ doublet on *complex* spin 1/2 fields, and because $(\mathcal{L}_\lambda\mathcal{L}_\lambda^\star)^\star=\mathcal{L}_\lambda\mathcal{L}_\lambda^\star$, there must be 4 holomorphic and 4 anti-holomorphic Majorana spin 1/2 fields (in other words, one needs to include also the CPT conjugates of these BPS representations). The OPE of the four holomorphic spin 1/2 fields is very constrained, and only the vacuum operator can appear in a singular term. We can take $\mathcal{L}_{orb}\subseteq\mathcal{L}_\lambda\mathcal{L}_\lambda^\star$ to be the smallest unoriented ($\mathcal{L}_{orb}^\star=\mathcal{L}_{orb}$) defect containing I, with $\mathcal{L}_{orb}\subset(\mathcal{L}_{orb})^2$, and such that $\mathcal{H}_{\mathcal{L}_{orb}}$ contains the four holomorphic and antiholomorphic spin 1/2 fields and is closed under OPE. Then $\mathcal{H}_{\mathcal{L}_{orb}}$ contains a holomorphic and a antiholomorphic copy of the algebra generated by $\mathcal{N}=4$ and the free fermions. This is just the chiral algebra of a generic T^4 model, i.e. the algebra of four free bosons and four free fermions. Furthermore, $\mathcal{H}_{\mathcal{L}_{orb}}$ decomposes into (possibly twisted) representations of this chiral and anti-chiral algebra. What remains to prove is that $A:=\mathcal{L}_{orb}$ (or possibly some extension of $\mathcal{L}_{orb}$), satisfies all properties such that the generalised orbifold $\mathcal{C}/A$ is well-defined. It might be possible to prove this fact using properties of the category of (twisted) representations of the chiral algebra of T^4. If the orbifold $\mathcal{C}/A$ is consistent, then it contains four free fermions and four free bosons and has central charges $c=6=\bar{c}$, so that it is necessarily a sigma model $\mathcal{T}$ on T^4. By reversibility of the orbifold procedure, we conclude that $\mathcal{C}$ is a generalised orbifold of the torus model $\mathcal{T}$.

In the examples of continuous defects discussed in the previous sections, $\mathcal{L}_{orb}$ was given by $I+Q+\ldots+Q^{N-1}$, and indeed the orbifold of $\mathcal{C}$ by $\mathcal{L}_{orb}$ gives back $\mathcal{T}$.

5 An example: a K3 model with $\mathbb{Z}_2^8:M_{20}$ symmetry group

In this section, we discuss the topological defects $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$ preserving the $\mathcal{N}=(4,4)$ superconformal algebra (SCA) in a particular K3 sigma model $\mathcal{C}_{GTVW}$, the 'most symmetric model' of [69] (see also [70]). The bosonic parts of the holomorphic and anti-holomorphic chiral algebras of this model are isomorphic to $\mathcal{A}:=(\widehat{su}(2)_1)^6$ (six copies of the $su(2)$ current algebra at level 1). Recall that $\widehat{su}(2)_1$ has two irreducible representations with conformal weights 0 and 1/4, and the fusion ring of $\widehat{su}(2)_1$ is the group ring of the cyclic group $\mathbb{Z}_2$ of order two, which we represent as the additive group with elements $0,1$. Each irreducible $\mathcal{A}$-module is labeled by an element in $\mathbb{Z}_2^6$, as $M_{a_1,\ldots,a_6}$, $a_i\in\{0,1\}$.

The NS-NS sector of $\mathcal{C}_{GTVW}$ is given by the sum

$$\bigoplus_{[a_1,\ldots,a_6;b_1,\ldots,b_6]\in A_{NS-NS}} M_{a_1,\ldots,a_6}\otimes M_{b_1,\ldots,b_6}\,, \tag{5.1}$$

where the set $A_{NS-NS}\subset\mathbb{Z}_2^6\times\mathbb{Z}_2^6$ is given by the disjoint union

$$A_{NS-NS}=A_{(NS-NS)^+}\sqcup A_{(NS-NS)^-} \tag{5.2}$$

of a 'bosonic' subset (corresponding to the subsector with positive fermion number)

$$A_{(NS-NS)^+}=\{[a_1,\ldots,a_6;b_1,\ldots,b_6]\in\mathbb{Z}_2^6\times\mathbb{Z}_2^6\,|\,a_i=b_i,\quad \sum_i a_i\equiv 0\mod 2\} \tag{5.3}$$

and a 'fermionic' one

$$A_{(NS-NS)^-}=\{[a_1,\ldots,a_6;b_1,\ldots,b_6]\in\mathbb{Z}_2^6\times\mathbb{Z}_2^6\,|\,a_i=b_i+1,\quad \sum_i a_i\equiv 0\mod 2\}\,. \tag{5.4}$$

Similarly, the R-R sector is a sum over the modules whose labels take values in the set $A_{R-R}=(A_{R-R})^+\sqcup(A_{R-R})^-$ with

$$A_{(R-R)^+}=\{[a_1,\ldots,a_6;b_1,\ldots,b_6]\in\mathbb{Z}_2^6\times\mathbb{Z}_2^6\,|\,a_i=b_i,\quad \sum_i a_i\equiv 1\mod 2\} \tag{5.5}$$

and

$$A_{(R-R)^-} = \{[a_1,\ldots,a_6;b_1,\ldots,b_6] \in \mathbb{Z}_2^6 \times \mathbb{Z}_2^6 \,|\, a_i = b_i + 1, \quad \sum_i a_i \equiv 1 \mod 2\} \,. \tag{5.6}$$

The bosonic CFT $\mathcal{C}^{bos}_{GTVW}$ that contains only the states in the $(NS-NS)^+$ and $(R-R)^+$ sectors of the SCFT $\mathcal{C}_{GTVW}$ is just the diagonal modular invariant of the $(\widehat{su}(2)_1)^{\oplus 6}$ algebra.

The K3 model $\mathcal{C}_{GTVW}$ can also be defined as the orbifold $\mathcal{T}/\mathbb{Z}_2$ of a particular torus model $\mathcal{T}$ by the $\mathbb{Z}_2$ symmetry $\mathcal{R}$ reflecting all torus coordinates, see section 4. Alternatively, it can be described as a $\mathcal{T}/\mathbb{Z}_4$ orbifold of the same torus model $\mathcal{T}$ by a symmetry of order 4, as described in section 4.4. See also [69] for more details on this construction.

This model contains several different copies of the $\mathcal{N}=(4,4)$ superconformal algebra, related to each other by symmetries of the CFT. We focus on one of them, such that the $\widehat{su}(2)_1$ subalgebra of the (anti-)holomorphic $\mathcal{N}=4$ is identified with the first factor in the (anti-)chiral algebra $(\widehat{su}(2)_1)^6$. The four supercurrents of the holomorphic $\mathcal{N}=(4,4)$ are suitably chosen ground states in the $M_{1,1,1,1,1,1} \otimes M_{0,0,0,0,0,0}$ module of $\mathcal{A}\times\mathcal{A}$. The space of ground states of this module is isomorphic to a tensor product $(\mathbb{C}^2)^{\otimes 6}$, and the $\mathcal{N}=4$ supercurrents can be nicely described in terms of a quantum error correcting code [70]. The full (bosonic and fermionic) chiral algebra of the theory is generated by the bosonic $(\widehat{su}(2)_1)^6$ and one of the supercurrents.

The 24 (real) Ramond-Ramond states with conformal weight $h = h = \frac{1}{4}$ generating the space V are the ground states in the six $\mathcal{A}\times\mathcal{A}$ representations

$$\begin{aligned}&[1,0,0,0,0,0;1,0,0,0,0,0]\,,\ [0,1,0,0,0,0;0,1,0,0,0,0]\,,\ [0,0,1,0,0,0;0,0,1,0,0,0]\\ &[0,0,0,1,0,0;0,0,0,1,0,0]\,,\ [0,0,0,0,1,0;0,0,0,0,1,0]\,,\ [0,0,0,0,0,1;0,0,0,0,0,1]\end{aligned} \tag{5.7}$$

each one containing 4 ground states; we will call any such a set of four states a *tetrad*. In particular, the first tetrad, i.e. the ground states in $[1,0,0,0,0,0;1,0,0,0,0,0]$, are the spectral flow generators, spanning $\subset V$.

The full group of symmetry of $\mathcal{C}_{GTVW}$ is $(\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) \rtimes S^6$, where the two $\mathrm{SU}(2)^6$ subgroups are generated by the zero modes of the holomorphic and anti-holomorphic currents, while S^6 is the group of permutations of the six $\widehat{su}(2)_1$ factors in $(\widehat{su}(2)_1)^6$ acting diagonally on the holomorphic and anti-holomorphic currents. To be precise, such a group does not act faithfully on $\mathcal{H}$, because a certain subgroup $Z_0 \subset \mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$ acts trivially on all the states of the theory. Indeed, let

$$t_i, t_i\,, \qquad i = 1,\ldots,6, \tag{5.8}$$

be the generators of the centre $\mathbb{Z}_2^6 \times \mathbb{Z}_2^6$ of $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$, where t_i and t_i act on the representation $M_{a_1,\ldots,a_6} \otimes M_{b_1,\ldots,b_6}$ by multiplication by $(-1)^{a_i}$ and $(-1)^{b_i}$, respectively. Then, the subgroup

$$Z_0 = \{\prod_{i=1}^{6} (t_i t_i)^{r_i} \;\; | \sum_{i=1}^{6} r_i \in 2\mathbb{Z}\} \cong \mathbb{Z}_2^5 \tag{5.9}$$

acts trivially on all states of the theory. Thus, the group acting faithfully is $((\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6)/Z_0) \rtimes S^6$. In particular, the quotient $(\mathbb{Z}_2^6 \times \mathbb{Z}_2^6)/Z_0 \cong \mathbb{Z}_2^7$ is the group of symmetries commuting with the full bosonic chiral algebra $\mathcal{A}$, and we can take

$$t_1,\ldots,t_6,t_1\,, \tag{5.10}$$

as representatives of the generators (modulo Z_0).

In [69] it was shown that the subgroup $G_{GTVW}\subset((\mathrm{SU}(2)^6\times\mathrm{SU}(2)^6)/Z_0)\rtimes S^6$ fixing the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow generators is finite and isomorphic to a split extension $\mathbb{Z}_2^8:M_{20}$ of the Mathieu group M_{20} by $\mathbb{Z}_2^8$. In turn, M_{20} is a split extension $\mathbb{Z}_2^4:A_5$ of the alternating group A_5 by $\mathbb{Z}_2^4$. See appendix A for a description of the generators. The intersection

$$G_{GTVW}\cap(\mathbb{Z}_2^6\times\mathbb{Z}_2^6)/Z_0=\{\prod_{i=2}^{6}t_i^{r_i}\ \mid\sum_{i=2}^{6}r_i\in 2\mathbb{Z}\}\cong\mathbb{Z}_2^4\,, \tag{5.11}$$

is the subgroup of symmetries commuting with both $\mathcal{N}=(4,4)$ superconformal algebra and with the bosonic current algebra $(\widehat{su}(2)_1)^6\times(\widehat{su}(2)_1)^6$.

5.1 Topological defects

Let us now consider the possible topological defects $\mathcal{L}\in\mathsf{Top}_{GTVW}\equiv\mathsf{Top}_{\mathcal{C}_{GTVW}}$ preserving the $\mathcal{N}=(4,4)$ algebra and spectral flow. The following result was used in section 3.2 to prove that an analogous statement holds for the category of defects Top_Π of any K3 models $\mathcal{C}_\Pi$ (see claim 1).

Claim 7. *The only defects $\mathcal{L}\in\mathsf{Top}_{GTVW}$ that are transparent to all 24 R-R operators in V are integral multiples of the identity defect.*

To prove this statement, we notice that if $\mathcal{L}$ is transparent to the $\mathcal{N}=(4,4)$ algebra and to all R-R operators in V, then it is also transparent to all the operators that can be obtained from their OPE. Thus, we just need to prove that one can obtain *all* operators in $\mathcal{H}$ by OPE of R-R operators in V and their $\mathcal{N}=(4,4)$ descendants. In fact, by taking the OPE of the four operators the i-th tetrad, $i=1,\ldots,6$, one can obtain the currents in the holomorphic and anti-holomorphic i-th $\widehat{su}(2)_1$ factor. Thus, in this way we can generate the full chiral and antichiral algebra $\mathcal{A}\times\mathcal{A}$. Furthermore, the fusion rules between $\mathcal{A}$ modules imply that the OPE of the $\mathcal{A}\times\mathcal{A}$ primary fields in the six modules (5.7), together with the ones in the $[1,1,1,1,1,1;0,0,0,0,0,0]$ and $[0,0,0,0,0,0;1,1,1,1,1,1]$ modules, where the supercurrents live, generate every other $\mathcal{A}\times\mathcal{A}$ primary operator in the spectrum, and we conclude.

Some examples of defects in Top_{GTVW} are as follows:

1. *Invertible defects.* We have one simple defect $\mathcal{L}_g$ for each $g\in G_{GTVW}\cong\mathbb{Z}_2^8:M_{20}\cong\mathbb{Z}_2^8:(\mathbb{Z}_2^4:A_5)$.

2. *Defects induced by the torus orbifold description.* Since the model $\mathcal{C}_{GTVW}$ can be defined as a torus orbifold $\mathcal{T}/\mathbb{Z}_2$, it inherits all defects from the torus model $\mathcal{T}$. In particular, as described in section 4, Top_{GTVW} contains a continuum of non-invertible defects T_θ, $\theta\in(\ ^{4,4}\otimes\mathbb{R})/\ ^{4,4}$ of dimension 2 that are simple for generic values of θ. One can identify the first two tetrads in (5.7) as the RR ground states in the untwisted sector

of the orbifold, and the last four tetrads as the twisted sector. Then, all operators T_θ act on the first two tetrads by multiplication by 2, while they annihilate the last four tetrads. Notice that symmetries in G_{GTVW} permute the second tetrad with the last four, while keeping the first tetrad of spectral flow generators fixed; therefore, they change the identification of the tetrads with the twisted or untwisted sector of the orbifold. This means that fusion of T_θ with invertible defects in Top_{GTVW} provide further sets of continuous defects of dimension 2. Any such defect acts by multiplication by 2 on the spectral flow generators (the first tetrad) and on one more tetrad (the untwisted sector RR ground states), while annihilating the remaining four tetrads (the twisted sector).

The model $\mathcal{C}_{GTVW}$ can also be described as a $T^4/\mathbb{Z}_4$ orbifold. In fact, this is exactly the model described in section 4.4. We denote by Q_4 the quantum symmetry of order 4 in this torus orbifold description. Among the elements of the group G_{GTVW}, the quantum symmetries Q_4 can be characterised by their eigenvalues on the 24-dimensional space V of RR ground states (± 1 with multiplicity 4 each, and $\pm i$ with multiplicity 8 each). From the $T^4/\mathbb{Z}_4$ description, one can deduce that Top_{GTVW} must contain a continuum of topological defects $T_\theta^{(4)}$ of order 4 and a continuum of topological defects ξ_θ of order 2. The former act on the first tetrad of RR ground states by multiplication by 4, while they annihilate all other 5 tetrads. The latter act by multiplication by 2 on two tetrads (including the first), while they annihilate the remaining four tetrads.

3. *Verlinde lines for* $(\widehat{su}(2)_1)^{\oplus 6}$. Since this SCFT is rational with respect to the chiral algebra $\mathcal{A}$, it is useful to consider the topological defects that preserve the whole algebra $\mathcal{A}$. For rational CFTs, there is a finite number of such defects. In particular, for a bosonic CFT corresponding to the diagonal invariant of $\mathcal{A}$, simple topological defects preserving $\mathcal{A}$ are completely classified, and are given by the Verlinde line defects. In general, Verlinde lines are in one-to-one correspondence with representations of the algebra $\mathcal{A}$, and their fusion ring is the same as the fusion ring of $\mathcal{A}$ representations. Let us consider the simple topological defects of the bosonic CFT $\mathcal{C}^{bos}_{GTVW}$ that preserve the $(\widehat{su}(2)_1)^{\oplus 6}$ chiral and anti-chiral algebras. They are 2^6 Verlinde lines, one for each $(\widehat{su(2)}_1)^{\oplus 6}$ representation, and obey group-like $\mathbb{Z}_2^6$ fusion rules; this means that they are all invertible, so that they form a $\mathbb{Z}_2^6$ group of symmetries. The lifts of such symmetries from the bosonic CFT $\mathcal{C}^{bos}_{GTVW}$ to the SCFT $\mathcal{C}_{GTVW}$, together with the fermion number, generate the full group $(\mathbb{Z}_2^6 \times \mathbb{Z}_2^6)/Z_0 \cong \mathbb{Z}_2^7$ of symmetries fixing all $(\widehat{su}(2)_1)^{\oplus 6}$. Notice that not all Verlinde defects preserve the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow generators — $\mathcal{A}$ only contains the even subalgebra of $\mathcal{N}=4$, but not the supercurrent. In fact, only the subgroup (5.11) of $(\mathbb{Z}_2^6 \times \mathbb{Z}_2^6)/Z_0$ is contained in G_{GTVW} and therefore gives rise to defects in Top_{GTVW}. In particular, t_1, t_1, and $t_1 \cdots t_6$ can be identified, respectively, with the left-moving and right-moving fermion number, and with the $\mathbb{Z}_2$ symmetry acting by -1 on all R-R states. In conclusion, Verlinde lines only provide topological defects that are generated by ordinary symmetries.

4. *Lines preserving some 'large' chiral algebra.* There are two ways we can generalize the construction of Verlinde lines in the previous point. One is to consider topological

defects that act on the $(\widehat{su}(2)_1)^{\oplus 6}$ chiral and anti-chiral algebra by some non-trivial automorphism. The full analysis is performed in appendix C. The final outcome is that, besides the invertible defects, there are a few more duality defects (see the next point), related to the fact that the SCFT Top_{GTVW} is self-orbifold $\mathsf{Top}_{GTVW} = \mathsf{Top}_{GTVW}/H$ with respect to certain subgroups H of the subgroup $G_{GTVW} \cap (\mathbb{Z}_2^6 \times \mathbb{Z}_2^6)/Z_0$ of symmetries fixing all the $(\widehat{su}(2)_1)^{\oplus 6}$ currents. Actually, it turns out that the duality defects obtained in this way are elements in the family of continuous defects T_θ, at special values of θ. The possibility that some T_θ are duality defects was discussed in section 4.

The other possibility to discover new objects in Top_{GTVW} is to consider the defects that preserve some chiral algebra $\mathcal{B}$ that is smaller than the full chiral algebra of the theory, but such that the theory is still rational with respect to $\mathcal{B}$. There are many simple subalgebras of the bosonic $(\widehat{su}(2)_1)^{\oplus 6}$ whose representation theory are well known, such as, for example, products of affine subalgebras $h \subset (\widehat{su}(2)_1)^{\oplus 6}$ and cosets $(\widehat{su}(2)_1)^{\oplus 6}/h$. The main difficulty with this approach is to find the defects that preserve the $\mathcal{N}=4$ supercurrents. We were not able to find any new defects in Top_{GTVW} using this technique.

5. *Duality defects.* By definition, a duality defect N is such that the fusion with the reversed orientation defect $N^\star$ is a superposition of invertible defects

$$N^\star N = \sum_{h\in H} \mathcal{L}_h\,, \qquad N\mathcal{L}_h = N\,, \tag{5.12}$$

for some group H of symmetries. Duality defects occur when a model $\mathcal{C}$ is self-orbifold with respect to the group H, i.e. if the orbifold theory $\mathcal{C}/H$ is a consistent CFT isomorphic to $\mathcal{C}$. This means that there is an isomorphism between the space $\mathcal{H}$ of local operators of $\mathcal{C}$ and the one of $\mathcal{C}/H$ such that correlation functions are the same. In this case, moving a local operator of $\mathcal{C}$ through N gives the corresponding operator in $\mathcal{C}/H$. In particular, when $N = N^\star$ is unoriented, and H is an abelian group, the category generated by N and $\mathcal{L}_h$, $h \in H$, is called a Tambara-Yamagami (TY) category, with fusions

$$N^2 = \sum_{h\in H} \mathcal{L}_h\,, \qquad \mathcal{L}_h N = N = N\mathcal{L}_h\ . \tag{5.13}$$

If $N \in \mathsf{Top}_{GTVW}$ is a duality defect, then the corresponding H must be an subgroup of G_{GTVW}, the group of symmetries acting trivially on the $\mathcal{N}=(4,4)$ superconformal algebra and spectral flow. Notice, however, that the converse is not true in general: even if a certain subgroup $H \subset G_{GTVW}$ is such that $\mathcal{C}_{GTVW}/H \cong \mathcal{C}_{GTVW}$, the corresponding duality defect N is not necessarily transparent to the $\mathcal{N}=(4,4)$ algebra and spectral flow, and in this case it is not in Top_{GTVW} (see appendix B for an example).

Let us discuss the possible abelian groups H for which $\mathcal{C}_{GTVW}/H$ is isomorphic to $\mathcal{C}_{GTVW}$. One necessary condition is that the orbifold model contains bosonic chiral and antichiral algebras isomorphic to $(\widehat{su}(2)_1)^{\oplus 6}$. We can separate the subgrops $H \subset G_{GTVW}$

into two classes, depending on whether they act trivially on the chiral and anti-chiral $\mathcal{A}\times\mathcal{A}$ or not.

In the first case, H must be a subgroup of (5.11), i.e. the intersection of G_{GTVW} and the centre $(\mathbb{Z}_2^6\times\mathbb{Z}_2^6)/Z_0$. Furthermore, $\frac{\hat{N}}{\sqrt{|H|}}$ must act by an automorphism on $(\widehat{su}(2)_1)^{\oplus 6}$. All such defects $N\in\mathsf{Top}$ are discussed in appendix C. The results are as follows. We find three kind of duality defects (up to conjugation in G_{GTVW}): $\mathcal{N}_{ijk}$ for all $2\leq i<j<k\leq 6$, with $\mathcal{N}_{ijk}^2=\mathcal{I}+\mathcal{L}_{t_it_j}+\mathcal{L}_{t_jt_k}+\mathcal{L}_{t_it_k}$; $\mathcal{N}_{ij,kl}$ for all pairwise distinct $i,j,k,l\in\{2,\dots,6\}$, with $\mathcal{N}_{ij,kl}^2=\mathcal{I}+\mathcal{L}_{t_it_j}+\mathcal{L}_{t_kt_l}+\mathcal{L}_{t_it_jt_kt_l}$; and $\mathcal{N}_{23456}$ where $\mathcal{N}_{23456}^2$ equals to the superposition of the 16 invertible defects in $G_{GTVW}\cap(\mathbb{Z}_2^6\times\mathbb{Z}_2^6)/Z_0\cong\mathbb{Z}_2^4$.

Let us now consider the case where H acts non-trivially on the chiral algebra. In order for $\mathcal{C}/H$ to be isomorphic to $\mathcal{C}$, there must be holomorophic currents in some twisted sector $\mathcal{H}_h$, for some $h\in H$.

By a direct calculation of the h-twisted partition function, we found that the group G_{GTVW} admits only three conjugacy classes of elements h such that $\mathcal{H}_h$ contains holomorphic currents. One class is given by symmmetries of the form $t_it_jt_kt_l$, $2\leq i<j<k<l\leq 6$, that are contained in the group (5.11). Each of them can be interpreted as the quantum symmetry in a description of the model as a torus orbifold $T^4/\mathbb{Z}_2$. This means that the orbifold of the CFT $\mathcal{C}_{GTVW}$ by $\mathbb{Z}_2$ group generated by any such symmetries is a torus model, and in particular cannot be isomorphic to $\mathcal{C}_{GTVW}$ itself. However, there could be larger abelian groups containing symmetries of the form $t_it_jt_kt_l$, under which $\mathcal{C}$ is self-orbifold. In fact, we already found some of these abelian groups among the ones acting trivially on currents.

The second class are given by symmetries Q_4 of order four, such as, for example

$$(1\underline{1}\underline{x}xxx;\ 111111)(1)^6 \tag{5.14}$$

in the notation of appendix A; the third class is related to the second by exchanging the action on the holomorphic and anti-holomorphic sectors. The elements Q_4 in either the second or the third conjugacy class are quantum symmetries in a description of the model $\mathcal{C}_{GTVW}$ as a $\mathbb{Z}_4$ torus orbifold $T^4/\mathbb{Z}_4$. As in the previous case, this means that the orbifold of $\mathcal{C}$ by $\langle Q_4\rangle\cong\mathbb{Z}_4$ is a torus model (more precisely, the model described in section 4.4), and therefore is not isomorphic to $\mathcal{C}_{GTVW}$. On the other hand, from the results in section 4.4, we know that there are duality defects $\mathcal{N}_x\in\mathsf{Top}_{GTVW}$ of order 4 for groups of the form $\mathbb{Z}_2\times\mathbb{Z}_8$ generated by Q_4 and by some other symmetry η_x of order 8. In principle, there could be even larger abelian groups with respect to which $\mathcal{C}$ is self-orbifold, but we did not attempt a full classification.

5.2 D-branes and RR charges

Let us describe the lattice [4,20] of RR charges of the K3 model $\mathcal{C}_{GTVW}$, expressed in the orthonormal basis $\{|1,i\rangle,|2,i\rangle,|3,i\rangle,|4,i\rangle\}_{i=1,\dots,6}$ of the space V of R-R ground fields that is described in appendix A. In particular, for each fixed $i=1,\dots,6$, the subset $\{|1,i\rangle,|2,i\rangle,|3,i\rangle,|4,i\rangle\}$ corresponds to a 'tetrad' of states, belonging to one of the six representations of $\mathcal{A}\times\mathcal{A}$ listed in eq. (5.7). The first tetrad ($i = 1$) is given by the spectral flow generators and spans the subspace $\subset V$, while $^{\perp}$ is spanned by the remaining tetrads with $i = 2, \ldots, 6$.

In order to find ^{4,20} the most direct way would be to find an explicit description of a suitable set of 24 boundary states. However, we will consider a simpler method, that uses the group of symmetries G_{GTVW}. The outcome of this analysis is that the only even unimodular lattice $\subset V$ with signature $(4, 20)$ that is invariant under the action of the G_{GTVW} on V, as described in appendix A, is the one spanned by the columns of the following matrix:

$$\frac{1}{\sqrt{8}}\begin{pmatrix}
8&4&4&4&4&4&4&2&4&4&4&2&4&2&2&2&4&2&2&2&2&0&0&1\\
0&4&0&0&0&0&0&2&0&0&0&2&0&2&0&0&0&0&0&2&0&0&0&1\\
0&0&4&0&0&0&0&2&0&0&0&2&0&0&2&0&0&2&0&0&0&0&0&1\\
0&0&0&4&0&0&0&2&0&0&0&2&0&0&0&2&0&0&2&0&0&0&0&1\\
0&0&0&0&4&0&0&2&0&0&0&0&0&2&2&2&0&2&2&2&2&0&0&1\\
0&0&0&0&0&4&0&2&0&0&0&0&0&2&0&0&0&0&2&0&0&0&0&1\\
0&0&0&0&0&0&4&2&0&0&0&0&0&0&2&0&0&0&0&2&0&0&0&1\\
0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&2&0&2&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&4&0&0&2&0&2&2&2&0&2&2&2&2&2&2&1\\
0&0&0&0&0&0&0&0&0&4&0&2&0&2&0&0&0&2&0&0&0&2&0&1\\
0&0&0&0&0&0&0&0&0&0&4&2&0&0&2&0&0&0&2&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&2&0&0&0&2&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&4&2&2&2&0&0&0&0&2&2&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&2&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&0&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&4&2&2&2&2&2&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&2&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&0&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&2&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&0&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&2&1\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&1
\end{pmatrix} \tag{5.15}$$

This is a slight modification (necessary because of the different signature) of the basis of vectors of the Leech lattice in [74]. It is easy to verify by a direct calculation that the lattice generated by these vectors is invariant under the group G_{GTVW}, and that it is even and unimodular with respect to the diagonal metric $\eta = \mathrm{diag}(1, 1, 1, 1, -1, \ldots, -1)$ of signature $(4, 20)$.

The proof that the lattice with these properties is unique (up to $O(4) \times O(20)$ transformations that do not affect the splitting $V = \quad \oplus \quad^{\perp}$) is as follows. It is known from [26] that the group of symmetries $G_{\mathcal{C}}$ of any K3 model $\mathcal{C}$ is isomorphic to a subgroup G_{Λ} of the Conway group $O(\quad) \cong Co_0$, the group of automorphisms of the Leech lattice . Furthermore, if $^{G} \subset \quad^{4,20}$ and $^{G} \subset$ denote the sublattices of G-fixed vectors and $_{G} := \quad^{4,20} \cap (\quad^{G})^{\perp}$ and $_{G} := \quad \cap (\quad^{G})^{\perp}$ their orthogonal complement, then there is an isomorphism $_{G} \cong \quad_{G}$ (reversing the sign of the quadratic form) that is compatible with the action of $G_{\mathcal{C}} \cong G_{\Lambda}$. The relevant subgroups of Co_0 were classified by Höhn and Mason in [48], and G_{GTVW} appears as group 99 in their table. In particular, up to conjugation in Co_0, [48] shows that there is a unique sublattice $_{G} \cong \quad_{G}$ for this group; this means that $\cap \quad^{\perp} \cong \quad_{G}$ is unique up to $O(20)$ transformations of $^{\perp}$. As described, for example, in [75] (see also [48]), the genus of the primitive sublattice $_{G} \subset \quad^{4,20}$ determines the genus of its orthogonal complement ^{G} in the even unimodular lattice ^{4,20}. We checked, with some computer aid, that such a genus contains a unique isomorphism class of lattices; this means that $\cap \cong {}^G$ is also uniquely determined up to $O(4)$ transformations. Therefore, for any choice of even unimodular lattice $\subset V$ invariant under G_{GTVW}, there is a primitive embedding of ${}^G \oplus {}_G$ into . Different choice of the lattice would lead to primitive embedddings that are not related by either $O({}^G)\times O({}_G)$ or $O(\ )$ automorphisms (the latter can thought of as changes of basis in a given lattice). The possible equivalence classes of such primitive embeddings, up to $O({}^G)\times O({}_G)$ and $O(\ )$ transformations, are described by proposition 2.1 of [48], which is a reformulation of propositions 1.4.1 and 1.6.1 of [75]. In particular, the fact that for group 99 in [48] the index i_G equals 1, implies that there is a unique class of such embeddings, and this concludes the proof. We refer to [48, 75] for more information about embeddings of primitive sublattices into even unimodular lattices, and in particular for the meaning of the index i_G.

Notice that the first four vectors in the lattice basis

$$\frac{8}{\sqrt{8}}|1,1\rangle\,,\qquad \frac{4}{\sqrt{8}}(|1,1\rangle+|2,1\rangle)\,,\qquad \frac{4}{\sqrt{8}}(|1,1\rangle+|3,1\rangle)\,,\qquad \frac{4}{\sqrt{8}}(|1,1\rangle+|4,1\rangle)\,, \tag{5.16}$$

are contained in the subspace $\subset V$ of spectral flow operators. By Claim (2), this implies that all topological defects $\mathcal{L}\in \mathsf{Top}_{GTVW}$ in this model have integral quantum dimension.

As described in section 3.2, with each $\mathcal{L}\in \mathsf{Top}_{GTVW}$ is associated a lattice endomorphism $\mathsf{L}:\ {}^{4,20}\to\ {}^{4,20}$, which is contained in the intersection $B^{4,20}_{\Pi}(\mathbb{Z})=\mathrm{End}(\ {}^{4,20})\cap B^{4,20}(\mathbb{R})$ of $\mathrm{End}(\ {}^{4,20})$ with the 401-dimensional real space $B^{4,20}(\mathbb{R})$ of block diagonal matrices defined in eq. (3.11).

Using some computer aid [76], we found that for the model $\mathcal{C}_{GTVW}$ the $\mathbb{Z}$-module $B^{4,20}_{\Pi}(\mathbb{Z})$ has maximal rank, and computed a set of 401 generators $\{\mathsf{L}_i\}_{i=1,\ldots,401}$, so that any such map L can be written as

$$\mathsf{L}=\sum_i k_i\mathsf{L}_i\,,\qquad k_i\in\mathbb{Z}\ . \tag{5.17}$$

We stress that not all elements $\mathsf{L}\in B^{4,20}_{\Pi}(\mathbb{Z})$ are expected to correspond to actual topological defects in Top_{GTVW}. The $\mathbb{Z}$-module $B_{4,20}(\mathbb{Z})$ contains a submodule,

$$B^{4,20}_{\Pi,inv}(\mathbb{Z})\subset B^{4,20}_{\Pi}(\mathbb{Z}) \tag{5.18}$$

generated by all maps L_g induced by invertible defects $\mathcal{L}_g$, $g\in G_{GTVW}$. Notice that, since the 24-dimensional representation of G_{GTVW} is faithful, $B^{4,20}_{\Pi,inv}(\mathbb{Z})$ is isomorphic to the group ring $\mathbb{Z}[G_{GTVW}]$. We found that $B^{4,20}_{\Pi,inv}(\mathbb{Z})$ has also maximal rank, has index 2^{168} in $B^{4,20}_{\Pi}(\mathbb{Z})$, and that for all $\mathsf{L}\in B^{4,20}_{\Pi}(\mathbb{Z})$ one has $4\cdot\mathsf{L}\in B^{4,20}_{\Pi,inv}(\mathbb{Z})$. In particular,

$$B^{4,20}_{\Pi}(\mathbb{Z})/B^{4,20}_{\Pi,inv}(\mathbb{Z})\cong\mathbb{Z}_4^8\oplus\mathbb{Z}_2^{152}\ . \tag{5.19}$$

In fact, all the examples of topological defects $\mathcal{L}\in\mathsf{Top}_{GTVW}$ described in this section, including the simple non-invertible ones, correspond to elements L in $B^{4,20}_{\Pi,inv}(\mathbb{Z})$ (recall that the map $\mathcal{L}\to\mathsf{L}$ is not injective). Unfortunately, we were not able to determine which elements (if any) $\mathsf{L}\in B^{4,20}_{\Pi}(\mathbb{Z})$, $\mathsf{L}\notin B^{4,20}_{\Pi,inv}(\mathbb{Z})$, are actually induced by topological defects in Top_{GTVW}.

It is interesting to consider the action on RR charges of the duality defects $\mathcal{N}_{ijk}$, described in appendix C. For example, the map N_{256} related to the defect $\mathcal{N}_{256}$ is

$$\mathsf{N}_{256} = \begin{pmatrix}
2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&2&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&-1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&-1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&1&1&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&1&-1&1&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&1&1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&1&1&-1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&-1&1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&-1&1&-1&1&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix} \tag{5.20}$$

While $\mathsf{N}_{256} \in B^{4,20}_{\Pi,inv}(\mathbb{Z})$, it cannot be written as a sum of the form $\mathsf{L}_g + \mathsf{L}_h$, for some $g,h \in G_{GTVW}$: one always needs an integral linear combination of (at least) four L_g, $g \in G_{GTVW}$ with (at least) one *negative* coefficient. This can be proved, for example, by noticing that each L_g acts by a signed permutation in this basis, so that each row or column in L_g has always a single non-zero entry, equal to ± 1. Thus, any sum $\mathsf{L}_g + \mathsf{L}_h$ has at most two non-zero entries in each row or column, so it cannot be equal to N_{256}. This example shows that the image of Top_{GTVW} in End$(\;^{4,20})$ by $\mathcal{L} \to \mathsf{L}$ is strictly larger than the image of the subcategory generated only by invertible defects.

Finally, let us show an example of a deformation of this model lifting all topological defects in Top_{GTVW}. Let us relabel $\{|\psi_1\rangle, \dots, |\psi_{24}\rangle\}$ the ordered basis of R-R ground states, where $|\psi_{4k-3}\rangle, \dots, |\psi_{4k}\rangle$ are the states in the k-th tetrad

$$|\psi_{4k-3}\rangle = |1,k\rangle, \qquad |\psi_{4k-2}\rangle = |2,k\rangle, \qquad |\psi_{4k-1}\rangle = |3,k\rangle, \qquad |\psi_{4k}\rangle = |4,k\rangle.$$

We denote by $\psi_1, \dots, \psi_{24}$ the corresponding operators. The exactly marginal operators are the supersymmetric descendants of NS-NS operators of weights $h = h = \frac{1}{2}$, obtained from OPE of one of the spectral flow operators $\psi_1, \dots, \psi_4$ and one of the other 20 RR ground fields $\psi_5, \dots, \psi_{24}$. Thus, we can label a basis of such NS-NS operators by $\chi_{i,j}$, $i = 1, \dots, 4$, $j = 5, \dots, 24$. Each infinitesimal deformation is generated by a linear combination χ of the $\chi_{i,j}$. A defect $\mathcal{L}$ is preserved by the deformation generated by χ if and only if χ is transparent with respect to the defect $\mathcal{L}$. This is equivalent to saying that the operator $\mathcal{L}$ acts on the state $|\chi\rangle$ in the same way as on the vacuum, i.e.

$$\mathcal{L}|\chi\rangle = \langle\mathcal{L}\rangle|\chi\rangle \ . \tag{5.21}$$

Let us consider the following linear combination

$$|\chi\rangle = |\chi_{1,5}\rangle + \pi|\chi_{1,6}\rangle + \pi^2|\chi_{1,7}\rangle + \ldots + \pi^{19}|\chi_{1,24}\rangle = \sum_{j=5}^{24} \pi^{j-5}|\chi_{1,j}\rangle\,, \tag{5.22}$$

where $\pi = 3.14\ldots$. Because $\mathcal{L}$ is transparent to the spectral flow operators $\psi_1,\ldots,\psi_4$, the action of $\mathcal{L}$ on $\chi_{i,j}$ is the same as on ψ_j

$$\mathcal{L}|\chi_{i,j}\rangle = \sum_{k=5}^{24} \mathsf{L}_{jk}|\chi_{i,k}\rangle \ .$$

Thus, for the deformation (5.22), the condition (5.21) reads

$$\sum_{k,j=5}^{24} \pi^{j-5}\mathsf{L}_{jk}|\chi_{1,k}\rangle = \langle\mathcal{L}\rangle \sum_{k=5}^{24} \pi^{k-5}|\chi_{1,k}\rangle,$$

which is equivalent to

$$\sum_{j=5}^{24} \pi^{j-k}\frac{\mathsf{L}_{jk}}{\langle\mathcal{L}\rangle} = 1, \qquad\qquad \forall k = 5,\ldots,24 \ . \tag{5.23}$$

It is now sufficient to observe that, in this basis, the matrix $\frac{1}{\langle\mathcal{L}\rangle}\mathsf{L}_{jk}$ has rational entries, so that the only way this relation can be satisfied is

$$\frac{\mathsf{L}_{jk}}{\langle\mathcal{L}\rangle} = \begin{cases} 1 & \text{for } j = k \\ 0 & \text{for } j \neq k \ . \end{cases} \tag{5.24}$$

Thus, the only topological defects that are preserved by the deformation χ are the ones that are proportional to the identity, in agreement with the general statements in section 3. The same result would hold for any linear combination $|\chi\rangle = \sum_{j=5}^{24} \alpha_j|\chi_{i,j}\rangle$ such that the ratios α_j/α_k are irrational for all $j \neq k$.

6 Another example: the Gepner model $(1)^6$

The second example of K3 model we will analyse is the Gepner model $(1)^6$, that was considered in [26]. In general, Gepner models are obtained by considering products of $\mathcal{N} = 2$ minimal models, each one having central charge $c_k = \frac{3k}{k+2}$ with $k \in \mathbb{N}$, and then taking an orbifold that projects on the states carrying integral total U(1) R-charge. When the total central charge is $c = 6$ and the spectrum is invariant under spectral flow, then the $\mathcal{N} = (2,2)$ superconformal algebra is enhanced to $\mathcal{N} = (4,4)$ and the CFT is always a non-linear sigma model on K3.

In particular, as the name suggests, the $(1)^6$ model $\mathcal{C}_{(1)^6}$ is obtained by taking six copies of the $k = 1$ minimal model with $c_{k=1} = 1$. We refer to [26] for a description of this CFT as a Gepner model. Here, we notice that this particular Gepner model admits different descriptions that might be more useful to study symmetries, defects and D-branes.

6.1 The $(1)^6$ Gepner model as a free scalar CFT

Let us describe the K3 model $\mathcal{C}_{(1)^6}$ as a rational CFT. Both chiral and anti-chiral superalgebras $\mathcal{A}$ and $\mathcal{A}$ contain[8] the product of six copies of the $\mathcal{N}=2$ superconformal algebra at $c=1$. It is known (see appendix D) that the bosonic subalgebra of $\mathcal{N}=2$ at $c=1$ can be completely described in terms of a chiral free boson on a circle of suitable radius. This means that the K3 model $\mathcal{C}_{(1)^6}$ is generated by 6 holomorphic and six antiholomorphic chiral free bosons

$$i\partial X^k(z)\,, \qquad\qquad i\partial X^k(z)\,, \qquad\qquad k=1,\ldots,6\,,$$

together with the vertex operators $V_\lambda(z,z)\sim :e^{i(\vec{\lambda}_L\cdot\vec{X}_L(z)+\vec{\lambda}_R\cdot\vec{X}_R(\bar{z}))}:$ of conformal weights $(h_L,h_R)=(\frac{\vec{\lambda}_L^2}{2},\frac{\vec{\lambda}_L^2}{2})$, where $\lambda\equiv(\vec{\lambda}_L,\vec{\lambda}_R)$ takes values in a suitable integral (odd) lattice $\quad\in\mathbb{R}^{6,6}$. Let $\{e_1,\ldots,e_6,e_1,\ldots,e_6\}$ denote an orthonormal basis of $\mathbb{R}^{6,6}$, so that $\lambda_L=\sum_i^6\lambda_i e_i$ and $\lambda_R=\sum_k^6\lambda_k e_k$. The lattice is given by the union $\quad = \quad ^{NS-NS}\cup \quad ^{R-R}$ of a NS-NS and a R-R component.

The chiral and anti-chiral algebras $\mathcal{A}$ and $\mathcal{A}$ are generated by the currents $i\partial X^k$ and $i\partial X^k$, and by purely holomorphic (respectively, anti-holomorphic) vertex operators $V_{(\vec{\lambda}_L,0)}(z)$ (respectively, $V_{(0,\vec{\lambda}_R)}(z)$) with $\vec{\lambda}_L$ takes values in a suitable six dimensional positive definite lattice $\quad_{\mathcal{A}}\subset\mathbb{R}^6$, such that

$$(\quad^{NS-NS}\cap\mathbb{R}^{6,0})\cong\quad_{\mathcal{A}}\,,\qquad(\quad^{NS-NS}\cap\mathbb{R}^{0,6})\cong\quad_{\mathcal{A}}(-1)\,,\tag{6.1}$$

where $\quad_{\mathcal{A}}(-1)$ denotes the lattices with opposite quadratic form. In particular, the $(\mathcal{N}=2)^6\subset\mathcal{A}$ algebra can be described as the lattice super vertex operator algebra (SVOA) associated with the odd lattice $(\sqrt{3}\mathbb{Z})^6\equiv(\sqrt{3}\mathbb{Z})\oplus\ldots\oplus(\sqrt{3}\mathbb{Z})$ (six copies), so that

$$(\sqrt{3}\mathbb{Z})^6\subset\quad_{\mathcal{A}}\,.$$

The $(\mathcal{N}=2)^6$ algebra is not the full chiral algebra $\mathcal{A}$ of the model $\mathcal{C}_{(1)^6}$: one needs to extend this SVOA by two additional holomorphic currents $V_{\pm(\lambda_L,0)}(z)$, corresponding to the lattice vectors

$$\pm\vec{\lambda}_L=\pm\left(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right)\subset\mathbb{R}^6\,.$$

The full chiral and antichiral algebras $\mathcal{A}$ and $\mathcal{A}$ of the theory are therefore both isomorphic to the lattice SVOA associated with the integral odd lattice

$$\quad_{\mathcal{A}}=\{\frac{n}{\sqrt{3}}(1,1,1,1,1,1)+\sqrt{3}(x_1,\ldots,x_6)\,|\,n,x_1,\ldots,x_6\in\mathbb{Z}\}\subset\mathbb{R}^6\,.\tag{6.2}$$

The bosonic chiral algebra is the lattice VOA associated with the even lattice

$$\quad^{bos}_{\mathcal{A}}=\{\frac{n}{\sqrt{3}}(1,1,1,1,1,1)+\sqrt{3}(x_1,\ldots,x_6)\,|\,n,x_1,\ldots,x_6\in\mathbb{Z},\ \sum_i x_i\in2\mathbb{Z}\}\subset\quad_{\mathcal{A}}\,.\tag{6.3}$$

Therefore, the NS-NS lattice $\quad^{NS-NS}$ contains $\quad\oplus\quad(-1)$ as a sublattice, corresponding to the chiral and anti-chiral algebra $\mathcal{A}\times\mathcal{A}$.

[8]Notice that this is not the full chiral algebra of this CFT.

The representations of the algebra $\mathcal{A}\times\mathcal{A}$ are labeled by cosets $[v]: v+(\ {}_{\mathcal{A}}\oplus\ {}_{\mathcal{A}}(-1))$, for suitable $v\in\mathbb{R}^{6,6}$ so that $\ ^{NS-NS}$ and $\ ^{R-R}$ decompose as unions of cosets

$$\ ^{NS-NS}=\bigcup_{[v]\in A_{(NS-NS)}} v+(\ {}_{\mathcal{A}}\oplus\ {}_{\mathcal{A}}(-1)) \tag{6.4}$$

and

$$\ ^{R-R}=\bigcup_{[v]\in A_{(R-R)}} v+(\ {}_{\mathcal{A}}\oplus\ {}_{\mathcal{A}}(-)) \tag{6.5}$$

In particular, the NS-NS sector contains the representations

$$A_{(NS-NS)}=\Big\{[\tfrac{1}{\sqrt{3}}(a_1,\ldots,a_6;a_1,\ldots,a_6)]\ \mid\ a_i\in\mathbb{Z}/3\mathbb{Z},\,,\ \sum_{i=1}^{6}a_i\in 3\mathbb{Z}\Big\}$$

while the R-R sector includes

$$A_{(R-R)}=\Big\{[\tfrac{1}{\sqrt{3}}(a_1,\ldots,a_6;a_1,\ldots,a_6)]\ \mid\ a_i\in\frac{1}{2}+\mathbb{Z}/3\mathbb{Z},\,,\ \sum_{i=1}^{6}a_i\in 3\mathbb{Z}\Big\}\ .$$

Notice that we have the identifications

$$[v]\sim[v+\frac{1}{\sqrt{3}}(1,1,1,1,1,1;1,1,1,1,1,1)]$$

and the fusion rules

$$[v]\times[v']=[v+v']$$

In this description, the $\mathcal{N}=4$ superconformal algebra at $c=6$ is given as follows. The $su(2)_1$ R-symmetry algebra is generated by

$$J^3(z)=\frac{1}{2\sqrt{3}}\sum_{k=1}^{6}i\partial X^k(z)\,,\qquad J^{\pm}(z)=V_{(\pm\frac{1}{\sqrt{3}},\ldots,\pm\frac{1}{\sqrt{3}};0,\ldots,0)}(z) \tag{6.6}$$

while the supercurrents are given by

$$G^{\pm}(z)=\sqrt{\frac{2}{3}}\sum_{k=1}^{6}V_{\pm\sqrt{3}e_k}(z)\,,\qquad G^{\pm}(z)=\sqrt{\frac{2}{3}}\sum_{k=1}^{6}V_{\mp\sqrt{3}e_k\pm(\frac{1}{\sqrt{3}},\ldots,\frac{1}{\sqrt{3}};0,\ldots,0)}(z) \tag{6.7}$$

The 24 R-R ground states are given by V_λ with the following vectors $\lambda\in\ ^{R-R}$:

$$\pm\frac{1}{2\sqrt{3}}(1,\ldots,1;1,\ldots,1)\qquad \pm\frac{1}{2\sqrt{3}}(1,\ldots,1;-1,\ldots,-1)\qquad \text{4 spectral flow generators}$$
$$\frac{1}{2\sqrt{3}}(1,1,1,-1,-1,-1;1,1,1,-1,-1,-1)\ \text{and permutations}\qquad \text{20 operators}\ .$$

The symmetries of the model preserving the $\mathcal{N}=(4,4)$ algebra form a group

$$G_{(1)^6}\cong\mathbb{Z}_3^4\rtimes A_6\,, \tag{6.8}$$

i.e. the extension by $\mathbb{Z}_3^4$ of the alternating group A_6 of even permutations of six objects [26]. Here, the normal subgroup $\mathbb{Z}_3^4$ is the group of CFT symmetries acting trivially on the chiral and anti-chiral algebras $\mathcal{A}\times\mathcal{A}$. It is generated by elements $t_1^{m_1}\cdots t_6^{m_6}$ acting by

$$\prod_{k=1}^{6} t_k^{m_k}(V_\lambda)=e^{\frac{2\pi i}{3}\sum_{k=1}^6 m_k a_k}V_\lambda\,,\qquad \lambda\in\frac{1}{\sqrt{3}}(a_1,\ldots,a_6;a_1,\ldots,a_6)+(\quad_{\mathcal{A}}\oplus\quad_{\mathcal{A}}(-1))\,, \tag{6.9}$$

where $m_1,\ldots,m_6\in\mathbb{Z}/3\mathbb{Z}$ satisfy the condition $\sum_{k=1}^6 m_k\equiv 0 \mod 3$. Notice that, because of the condition $\sum_{k=1}^6 a_k\equiv 0 \mod 3$, we have that the action is trivial whenever $m_1=\ldots=m_6$

$$t_1t_2t_3t_4t_5t_6=t_1^2t_2^2t_3^2t_4^2t_5^2t_6^2=1\ . \tag{6.10}$$

The alternating group A_6 acts by even permutations σ on the left- and right-moving $u(1)$ currents, and transforms the vertex operators accordingly:

$$i\partial X^k\mapsto i\partial X^{\sigma(k)}\,,\qquad i\bar\partial X^k\mapsto i\bar\partial X^{\sigma(k)}\,,\qquad V_{\sum_k(\lambda_k e_k+\tilde\lambda_k\tilde e_k)}\mapsto \pm V_{\sum_k(\lambda_k e_{\sigma(k)}+\tilde\lambda_k\tilde e_{\sigma(k)})}\,,$$

where the sign in front of each V_λ must be chosen in such a way that the OPE between vertex operators is preserved.

The full group of symmetries of the CFT is $[(\mathrm{SU}(2)\times\mathrm{U}(1)^5)\times(\mathrm{SU}(2)\times\mathrm{U}(1)^5)]\rtimes S_6$, where the two $\mathrm{SU}(2)\times\mathrm{U}(1)^5$ factors are generated by the zero modes of the holomorphic and anti-holomorphic currents. The permutation group S_6 acts simultaneously on each of the $\mathrm{U}(1)^5$ factors as in the 5-dimensional irreducible representation.

6.2 The $(1)^6$ model as a torus orbifold $T^4/\mathbb{Z}_3$

The $\mathcal{C}_{(1)^6}$ Gepner model can also be described in terms of a torus orbifold $T^4/\mathbb{Z}_3$. In fact, there are actually many different ways to obtain this model from an orbifold. One simple way to prove that $\mathcal{C}_{(1)^6}$ is a torus orbifold is to notice that the symmetry group $\mathbb{Z}_3^4:A_6$ contains some symmetry Q of order 3 (e.g. the element of $\mathbb{Z}_3^4$ with $m_1=m_2=m_3=1$ and $m_4=m_5=m_6=0$) whose trace on the 24 dimensional representation of RR ground states V is -3. The orbifold of $\mathcal{C}_{(1)^6}$ by Q gives a sigma model $\mathcal{T}:=\mathcal{C}_{(1)^6}/\langle Q\rangle$ on T^4, as can be checked by verifying that the elliptic genus of $\mathcal{C}_{(1)^6}/\langle Q\rangle$ is 0 [77]. By reversibility of the orbifold procedure, because $\mathcal{T}=\mathcal{C}_{(1)^6}/\langle Q\rangle$ with $\langle Q\rangle\cong\mathbb{Z}_3$, we must conclude that the K3 model $\mathcal{C}_{(1)^6}$ is an orbifold $\mathcal{C}_{(1)^6}=\mathcal{T}/\langle g\rangle$ of the T^4 model $\mathcal{T}$ by a cyclic group $\langle g\rangle\cong\mathbb{Z}_3$, and Q is the 'quantum symmetry' acting trivially on the untwisted sector and multiplying by $e^{\frac{2\pi ik}{3}}$ the g^k-twisted sector.

Knowing the action of Q on the states of $\mathcal{C}_{(1)^6}$ allows us to identify the untwisted sector and the g^k twisted sector in the bosonic description of the previous section:

$$\text{untwisted sector: } x_1+x_2+x_3\equiv 0 \mod 3\,,$$

$$g^k\text{twisted sector: } x_1+x_2+x_3\equiv k \mod 3\ .$$

It is easy to construct the sigma model $\mathcal{T}$ on T^4 corresponding to the orbifold $\mathcal{T}=\mathcal{C}_{(1)^6}/\langle Q\rangle$. We know that the symmetry Q can be written as

$$V_\lambda\mapsto e^{2\pi i(\delta_L\cdot\lambda_L-\delta_R\cdot\lambda_R)}V_\lambda=V_\lambda \tag{6.11}$$

where

$$\delta = \frac{1}{2\sqrt{3}}(4,4,4,0,0,0;0,0,0,0,0,0) \tag{6.12}$$

is a vector such that $3\delta \in \quad$ and $\delta_L^2 - \delta_R^2 \in 2\mathbb{Z}$. The untwisted sector of $\mathcal{C}_{(1)^6}/\langle Q\rangle$ contains the currents $i\partial X^i(z), i\partial X^i(z)$, as well as the vertex operators V_λ, $\lambda \in \ {}^Q$, where

$${}^Q = \{\lambda \in \quad | \, \delta_L \cdot \lambda_L - \delta_R \cdot \lambda_R \in \mathbb{Z}\} \ . \tag{6.13}$$

The full orbifold $\mathcal{C}_{(1)^6}/\langle Q\rangle$ is generated by the $i\partial X^i(z), i\partial X^i(z)$ currents together with the vertex operators V_λ with λ taking values in the extended lattice

$${}' = \ {}^Q \cup (\delta + \ {}^Q) \cup (2\delta + \ {}^Q) \ . \tag{6.14}$$

Let us consider the holomorphic fields of this orbifold. As expected for a supersymmetric sigma model on T^4, there are four holomorphic fields of weight 1/2, namely

$$\chi_1^\pm(z) \sim V_{\pm(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0,0,0;0,\ldots,0)} \qquad\qquad \chi_2^\pm(z) \sim V_{\pm(0,0,0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};0,\ldots,0)} \ .$$

By taking normal ordered products of pairs of these four fermions, one obtains the 6 currents of a 'fermionic' $so(4)_1 = su(2)_1 \oplus su(2)_1$ algebra, where one of the $su(2)_1$ is the R-symmetry of the $\mathcal{N}=4$ superconformal algebra with currents (6.6), and the second commuting $su(2)_1$ algebra is generated by

$$i\sum_{k=1}^{3}(\partial X^k - \partial X^{k+3}) \,, \qquad V_{\pm(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}},-\frac{1}{\sqrt{3}};0,\ldots,0)}(z) \ .$$

Besides the $so(4)_1$ currents that can be obtained from the OPE of the spin 1/2 fields, the orbifold $\mathcal{C}_{(1)^6}/\langle Q\rangle$ contains 16 additional spin 1 fields, that can be arranged into two commuting algebras

$$\begin{array}{lll} \frac{i}{\sqrt{2}}(\partial X^1 - \partial X^2)\,, & \frac{i}{\sqrt{2}}(\partial X^2 - \partial X^3) & \\ V_{\pm(\frac{-2}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0,0,0;0,\ldots,0)}(z) & V_{\pm(\frac{1}{\sqrt{3}},\frac{-2}{\sqrt{3}},\frac{1}{\sqrt{3}},0,0,0;0,\ldots,0)}(z)\,, & V_{\pm(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-2}{\sqrt{3}},0,0,0;0,\ldots,0)}(z) \end{array}$$

and

$$\begin{array}{lll} \frac{i}{\sqrt{2}}(\partial X^4 - \partial X^5)\,, & \frac{i}{\sqrt{2}}(\partial X^5 - \partial X^6) & \\ V_{\pm(0,0,0,\frac{-2}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};0,\ldots,0)}(z) & V_{\pm(0,0,0,\frac{1}{\sqrt{3}},\frac{-2}{\sqrt{3}},\frac{1}{\sqrt{3}};0,\ldots,0)}(z)\,, & V_{\pm(0,0,0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{-2}{\sqrt{3}};0,\ldots,0)}(z). \end{array}$$

These two commuting algebras have dimension 8 and rank 2, and this is enough to conclude that they are both isomorphic to $su(3)_1$. Thus, the orbifold model $\mathcal{C}_{(1)^6}/\langle Q\rangle$ has both a chiral and anti-chiral algebras isomorphic to $(su(3)_1)^2$, besides the $so(4)_1$ from the free fermions. For this reason, we denote this torus model as $\mathcal{T}_{A_2^2}$, so that

$$\mathcal{C}_{(1)^6}/\langle Q\rangle = \mathcal{T}_{A_2^2} \ . \tag{6.15}$$

The $(su(3)_1)^2$ currents include the superconformal descendants $i\partial Z_i^\pm$ of the free fermions $\chi_i^\pm$. The scalar fields Z_1^+, Z_2^+ can be identified with the complex coordinates on the target torus T^4, with Z_1^-, Z_2^- their complex conjugate. Notice that the currents $i\partial Z_i^\pm$ are linear combinations of the $i\partial X^k$ and of suitable vertex operators V_λ; this means that one cannot give the X^k any geometric interpretation as the coordinates on the target torus.

As a CFT, any supersymmetric non-linear sigma model $\mathcal{T}_{A_2^2}$ on T^4 can be described as the product $\mathcal{T}_{A_2^2} = F(4)\otimes F(4)\times \mathcal{T}_{A_2^2}^{bos}$ of a theory $F(4)\otimes F(4)$ of four chiral and four antichiral free fermions with central charges $(2,2)$, times the bosonic sigma model $\mathcal{T}_{A_2^2}^{bos}$ on the same torus T^4, with central charges $(4,4)$. Let us focus on the bosonic sigma model factor, whose fields are characterised by having non-singular OPE with all the chiral and anti-chiral free fermions (i.e. they commute with all the free fermion modes). This is an honest bosonic CFT, with modular invariant partition function. It contains the chiral algebra $(su(3)_1)^2$ — the free fermions have zero charge with respect to this algebra, so their OPE with the currents is non-singular. More generally, every V_λ with λ orthogonal to the vectors

$$\pm\Big(\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},0,0,0;0,\ldots,0\Big), \qquad \pm\Big(0,0,0,\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}};0,\ldots,0\Big)$$

and their anti-holomorphic counterparts, belong to the bosonic sigma model.

As we show below, the bosonic sigma model $\mathcal{T}_{A_2^2}^{bos}$ on T^4 is just the product $\mathcal{T}_{A_2^2}^{bos} = \mathcal{T}_{A_2}^{bos}\otimes \mathcal{T}_{A_2}^{bos}$ of two isomorphic bosonic sigma models $\mathcal{T}_{A_2}^{bos}$ on T^2, each one corresponding to the diagonal modular invariant for the algebra $su(3)_1$. Recall that $su(3)_1$ admits three different modules, which we denote by $M_{\mathbf{1}}$ (the vacuum), $M_{\mathbf{3}}$ and $M_{\bar{\mathbf{3}}}$; the latter modules are charge conjugate of each other, and their highest weight vector has weight $2/3$. Therefore, the spectrum of the diagonal modular invariant is

$$(M_{\mathbf{1}}\otimes\overline{M}_{\mathbf{1}})\oplus(M_{\mathbf{3}}\otimes\overline{M}_{\mathbf{3}})\oplus(M_{\bar{\mathbf{3}}}\otimes\overline{M}_{\bar{\mathbf{3}}}) \ .$$

More precisely, the first $\mathcal{T}_{A_2}^{bos}$ is generated by the first (holomorphic and anti-holomorphic) $su(3)_1$ factor, by the 9 non-holomorphic vertex operators of conformal weights $(2/3,2/3)$

$$\begin{array}{lll}
V_{(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)} & V_{(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} & V_{(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} \\
V_{(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} & V_{(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)} & V_{(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} \\
V_{(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} & V_{(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} & V_{(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)}
\end{array}$$

that are the ground states of the $M_{\mathbf{3}}\otimes\overline{M}_{\mathbf{3}}$ modules, and by the 9 vertex operators V_λ with the opposite signs for λ, namely

$$\begin{array}{lll}
V_{-(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)} & V_{-(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} & V_{-(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} \\
V_{-(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} & V_{-(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)} & V_{-(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} \\
V_{-(\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0;0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0)} & V_{-(0,\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0;\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0)} & V_{-(\frac{-1}{\sqrt{3}},0,\frac{1}{\sqrt{3}},0,0,0;\frac{1}{\sqrt{3}},\frac{-1}{\sqrt{3}},0,0,0,0)}
\end{array}$$

that are the ground states of the charge conjugate module $M_{\bar{\mathbf{3}}}\otimes\overline{M}_{\bar{\mathbf{3}}}$. The second $\mathcal{T}_{A_2}^{bos}$ torus model is generated in a similar way, by exchanging the first free set of bosonic coordinates X^1, X^2, X^3 with the second set X^4, X^5, X^6.

The analysis in this section is sufficient to identify $\mathcal{T}_{A_2^2}$ with the supersymmetric T^4 sigma model denoted by A_2^2 in section 4.4.4 of [63]. Geometrically, the target space is the product $T^2\times T^2$ of two tori $T^2=\mathbb{C}/(\mathbb{Z}+e^{\frac{2\pi i}{3}}\mathbb{Z})$ with a suitable B-field.

The K3 model $\mathcal{C}_{(1)^6}$ can be obtained by the orbifold of $\mathcal{T}_{A_2^2}$ by a symmetry g of order 3. More precisely, g belongs the class denoted by 3A in [63], and corresponds to a geometric rotation obtained by multiplying the complex coordinates (z_1,z_2) parametrizing the first and second torus $T^2=\mathbb{C}/(\mathbb{Z}+e^{\frac{2\pi i}{3}}\mathbb{Z})$ by $(z_1,z_2)\mapsto(e^{\frac{2\pi i}{3}}z_1,e^{-\frac{2\pi i}{3}}z_2)$. Indeed, only for this class of symmetries the orbifold $\mathcal{T}_{A_2^2}/\langle g\rangle$ contains 6 R-R ground states in the untwisted sector and 9+9 in the two twisted sectors, as expected for the model we are considering. The g-invariant untwisted sector contains a subalgebra $\widehat{su}(2)_1\oplus\widehat{u}(1)$ from the 'fermionic' $\widehat{so}(4)_1$ algebra of $\mathcal{T}_{A_2^2}$, as well as subalgebras $\widehat{u}(1)^2\subset\widehat{su}(3)_1$ for each of the factors in the bosonic $\widehat{su}(3)_1$. Altogether, we get $\widehat{su}(2)_1\oplus\widehat{u}(1)^5$, which is the current algebra of $\mathcal{C}_{(1)^6}=\mathcal{T}_{A_2^2}/\langle g\rangle$ — no further currents come from the twisted sectors.

The full symmetry group of the $\mathcal{T}_{A_2^2}$ torus model preserving the small $\mathcal{N}=(4,4)$ algebra and the spectral flow generators is $\mathrm{U}(1)^8\rtimes G_0$ where G_0 is a group of order 36 acting by automorphisms on the winding-momentum lattice. The subgroup commuting with g is $\langle g\rangle\times(\mathbb{Z}_3^4\rtimes\mathbb{Z}_6)$, where $\mathbb{Z}_3^4\subset\mathrm{U}(1)^8$ and $\mathbb{Z}_6$ is generated $\mathcal{R}:(z_1,z_2)\mapsto(-z_1,-z_2)$ and by a non-geometric symmetry of order 3 that rotates only the holomorphic currents $(\partial Z_1,\partial Z_2)\mapsto(e^{\frac{2\pi i}{3}}\partial Z_1,e^{-\frac{2\pi i}{3}}\partial Z_2)$ while keeping the anti-holomorphic currents $(\partial Z_1,\partial Z_2)$ fixed. These torus symmetries induce a group of symmetries of the orbifold $\mathcal{C}_{(1)^6}=\mathcal{T}_{A_2^2}/\langle g\rangle$ isomorphic to $3^{1+4}\rtimes\mathbb{Z}_6$, obtained from $\langle g\rangle\times(\mathbb{Z}_3^4\rtimes\mathbb{Z}_6)$ by first quotienting out $\langle g\rangle$, and then taking a non-trivial central extension by the quantum symmetry $\langle Q\rangle\cong\mathbb{Z}_3$, as described in section 4.

6.3 Symmetries and topological defects

After describing in some detail the model $\mathcal{C}_{(1)^6}$, we are now ready to discuss the topological defects in $\mathsf{Top}_{(1)^6}$ that preserve the $\mathcal{N}=(4,4)$ algebra and the spectral flow. Such a category includes:

1. *Invertible defects.* There is one defect $\mathcal{L}_g\in\mathsf{Top}_{(1)^6}$ for each $g\in G_{(1)^6}\cong\mathbb{Z}_3^4\rtimes A_6$.

2. *Defects acting by automorphisms on the chiral algebra.* The Verlinde lines preserving the full chiral and anti-chiral algebras $\mathcal{A}\times\mathcal{A}$ must be in one-to-one correspondence with the representations of $\mathcal{A}\times\mathcal{A}$ and with the same fusion rules. This implies that the Verlinde lines are all invertible, and generate the subgroup $\mathbb{Z}_2^4\subset G_{(1)^6}$.

   It is more interesting to consider topological lines acting on $\mathcal{A}\times\mathcal{A}$ by algebra automorphisms that fix the $\mathcal{N}=(4,4)$ subalgebra, and do *not* lift to CFT symmetries. The analysis is very similar to the one in appendix C, so we just summarize the main points. Recall that the full group of symmetries of the CFT is $((\mathrm{SU}(2)\times\mathrm{U}(1)^5)^2)\rtimes S_6)$. Up to conjugation by such CFT symmetries, we can focus on defects that act on the chiral algebra $\mathcal{A}$ by an *outer* automorphism ρ_L, while acting trivially on $\mathcal{A}$. Outer automorphisms preserving the $\mathcal{N}=4$ subalgebra correspond to *even* permutations in A_6; it is sufficient to consider a representative ρ_L for each even conjugacy class in S_6, i.e. for each even cycle shape. There are five possible cycle shapes, corresponding to the partitions 1+1+2+2, 1+1+1+3, 3+3, 2+4, 1+5. Let $\mathcal{L}_{(\rho_L,1)}$ denote one of these

Partition	ρ_L	$\langle\mathcal{N}\rangle$	H	Generators of H
1+1+2+2	$(ij)(kl)$	3	$\mathbb{Z}_3\times\mathbb{Z}_3$	$t_it_j^{-1},\quad t_kt_l^{-1}$
1+1+1+3	(ijk)	3	$\mathbb{Z}_3\times\mathbb{Z}_3$	$t_it_j^{-1},\quad t_it_jt_k$
3+3	$(ijk)(lmn)$	3	$\mathbb{Z}_3\times\mathbb{Z}_3$	$t_it_j^{-1}t_l^{-1}t_m,\quad t_it_jt_k$
2+4	$(ij)(klmn)$	9	$\mathbb{Z}_3^4$	All $\prod_k t_k^{m_k}$
1+5	$(ijklm)$	9	$\mathbb{Z}_3^4$	All $\prod_k t_k^{m_k}$

Table 1. For each partition of $\{1,\ldots,6\}$, we report the cycle shape of a generic automorphism $\rho_L\in A_6$, the dimension $\langle\mathcal{N}\rangle$ of the duality defect $\mathcal{N}$ acting on $\mathcal{A}\times\mathcal{A}$ by automorphisms (ρ'_L,ρ'_R) as in (6.17), the group $H\subseteq\mathbb{Z}_3^4$ appearing in (6.18), and the generators of H in the form (6.9).

defects. Then, the $\mathcal{A}\times\mathcal{A}$ representation labeled by $a_1,\ldots,a_6\in\mathbb{Z}/3\mathbb{Z}$ is annihilated by the operator $\mathcal{L}_{(\rho_L,1)}$ unless it satisfies

$$(a_{\rho_L(1)},\ldots,a_{\rho_L(6)})=(a_1,\ldots,a_6)+n(1,\ldots,1)\,, \tag{6.16}$$

for some $n\in\mathbb{Z}/3\mathbb{Z}$. For a given $(\rho_L,1)$, the number of simple defects $\mathcal{L}_{(\rho_L,1)}$ equals the number of representations satisfying (6.16), and can be obtained from each other by conjugation $\mathcal{L}_g\mathcal{L}_{(\rho_L,1)}\mathcal{L}_g^\star$ by invertible defects $\mathcal{L}_g$ with $g\in\mathbb{Z}_3^4\subset G_{(1)^6}$. Let us also notice that for every invertible defect $\mathcal{L}_g$ with $g\in\mathbb{Z}_3^4\subset G_{(1)^6}$ and acting trivially on all representations satisfying (6.16), one has $\mathcal{L}_g\mathcal{L}_{(\rho_L,1)}=\mathcal{L}_{(\rho_L,1)}\mathcal{L}_g=\mathcal{L}_{(\rho_L,1)}$.

For all $\mathcal{L}_{(\rho_L,1)}$ we can find an invertible $\mathcal{L}_g$ such that $\mathcal{N}=\mathcal{L}_g\mathcal{L}_\rho$ is unoriented — this implies that $\mathcal{N}$ acts on the chiral and antichiral algebras by involutions (ρ'_L,ρ'_R),

$${\rho'_L}^2=1={\rho'_R}^2\,,\qquad \rho'_L\rho'_R=\rho_L\ . \tag{6.17}$$

Furthermore, because $\mathcal{N}^2$ acts trivially on the chiral and antichiral algebra, it must be a superposition of Verlinde lines, i.e. invertible defects in $\mathbb{Z}_3^4\subset G_{(1)^6}$. This implies that all the $\mathcal{N}$ obtained in this way are duality defects for some subgroup $H\subseteq\mathbb{Z}_3^4$

$$\mathcal{N}^2=\sum_{h\in H}\mathcal{L}_h\,, \tag{6.18}$$

of order $|H|=\langle\mathcal{N}\rangle^2$. The basic information about these duality defects is contained in table 1. This analysis implies that the K3 model $\mathcal{C}_{(1)^6}$ is self-orbifold with respect to the groups $H\subseteq\mathbb{Z}_3^4$ in this table.

3. *Defects induced by the description as a torus orbifold $T^4/\mathbb{Z}_3$.* We have seen in section 6.2 that $\mathcal{C}_{(1)^6}=\mathcal{T}_{A_2^2}/\langle g\rangle$ for a suitable torus symmetry g of order 3 preserving the small $\mathcal{N}=(4,4)$ superconformal algebra. From the general analysis of section 4, this implies $\mathsf{Top}_{(1)^6}$ contains a continuum of topological defects T_θ of quantum dimension 3, that are induced by superpositions $W_\theta+W_{g(\theta)}+W_{g^2(\theta)}$ of torus model defects, and that are simple for generic $\theta\in{}^{4,4}/({}^{4,4}\otimes\mathbb{R})$. When $x\in{}^{4,4}/({}^{4,4}\otimes\mathbb{R})$ is a g-fixed point, i.e. $g(x)=x \mod {}^{4,4}$, the defect T_x decomposes as a superposition $\eta_x+Q\eta_x+Q^2\eta_x$, where Q is a quantum symmetry of order 3 and η_x are invertible defects. The invertible defects η_x obtained in this way, together with Q, generate an extraspecial group 3^{1+4},

that must be a subgroup of $G_{(1)^6}$. Indeed, the quantum symmetry Q can be identified, for example, with a symmetry $t_1t_2t_3 = t_4t_5t_6$ acting as in (6.9), and the subgroup of $G_{(1)^6}$ commuting with Q is isomorphic to $3^{1+4} \rtimes \mathbb{Z}_6$. The generators of the latter group are induced by the symmetries of the torus model $\mathcal{T}$ that commute with g. The group $3^{1+4} \rtimes \mathbb{Z}_6$ contains the subgroup $\mathbb{Z}_3^4$ fixing the chiral and anti-chiral algebra $\mathcal{A} \times \mathcal{A}$ — indeed, the index of $3^{1+4} \rtimes \mathbb{Z}_6$ in $G_{(1)^6}$ is 20, which is not divisible by 3. However, none of the elements η_x are contained in this $\mathbb{Z}_3^4$ group, because they all act non-trivially on some holomorphic or antiholomorphic operator.

4. *Duality defects.* We do not attempt a classification of all abelian groups $H \subset G_{(1)^6}$ with respect to which $\mathcal{C}_{(1)^6}$ is self-orbifold, i.e $\mathcal{C}/\langle H \rangle \cong \mathcal{C}_{(1)^6}$, but just mention a few examples. When H leaves invariant the full chiral and anti-chiral algebra, then the corresponding duality defect $\mathcal{N}$ must act on $\mathcal{A} \times \mathcal{A}$ by automorphisms. All cases where $\mathcal{N}$ preserves the $\mathcal{N} = (4,4)$ algebra were considered in point 2 above. As discussed in 4, the continuum of defects induced by the torus orbifold construction contains also the duality defects $\mathcal{N}$ for the order 9 groups H generated by η_x and Q. Notice that the element η_x, and therefore the group H, does not act trivially on $\mathcal{A} \times \mathcal{A}$, so that $\mathcal{N}$ is not one of the defects considered in point 2 above.

Let us describe the lattice ^{4,20} of R-R charges in this model. A set of boundary states generating the full lattice was determined in [26], using the Gepner model description. The same lattice can be also obtained in a way similar as section 5.2, using arguments based purely on the symmetry group $G_{(1)^6}$ of the model and on lattice theory. In particular, $G_{(1)^6}$ correspond to group 101 in the list of [48], and their results imply that there is a unique point in the moduli space of K3 models with this symmetry group. It is easier to describe ^{4,20} as a complex lattice in $\mathbb{C}^{2,10}$, the complex vector space with indefinite sesquilinear form

$$\langle z, w \rangle = z_1 w_1 + z_2 w_2 - z_3 w_3 - \ldots - z_{12} w_{12} \,,$$

of signature $(2,10)$, where $z \equiv (z_1, \ldots z_{12}) \in \mathbb{C}^{2,10}$ and $w \equiv (w_1, \ldots w_{12}) \in \mathbb{C}^{2,10}$. Using the notation $\omega := e^{\frac{2\pi i}{3}}$ and $\theta = \omega - \omega = i\sqrt{3}$, we find that $^{4,20} \subset \mathbb{C}^{2,10}$ is generated by the rows of the following matrix

$$\frac{\sqrt{2}}{3}\begin{pmatrix}
3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
3\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\theta & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\theta\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\omega & 3\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\omega & 0 & 3\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\omega & 0 & 0 & 3\omega & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -3 & 0 & 0 & 0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3\omega & 0 & 0 & 0 & 3\omega & 0 & 0 & 0 & 0 & 0 & 0 \\
\theta\omega & -\theta\omega & -\theta\omega & -\theta\omega & \theta\omega & -\theta\omega & \theta\omega & 0 & 0 & 0 & 0 & 0 \\
-\theta\omega & -\theta\omega & \theta\omega & -\theta\omega & -\theta\omega & -\theta\omega & 0 & \theta\omega & 0 & 0 & 0 & 0 \\
-\theta\omega & 0 & 0 & \theta\omega & -\theta\omega & -\theta\omega & 0 & 0 & \theta\omega & 0 & 0 & 0 \\
0 & -\theta\omega & \theta\omega & 0 & \theta\omega & -\theta\omega & 0 & 0 & 0 & \theta\omega & 0 & 0 \\
\theta\omega & 0 & -\theta\omega & \theta\omega & 0 & -\theta\omega & 0 & 0 & 0 & 0 & \theta\omega & 0 \\
0 & \theta\omega & \theta\omega & \theta\omega & \theta\omega & 0 & 0 & 0 & 0 & 0 & 0 & \theta\omega \\
\theta & -\theta & -\theta & -\theta & \theta & -\theta & \theta & 0 & 0 & 0 & 0 & 0 \\
-\theta & -\theta & \theta & -\theta & -\theta & -\theta & 0 & \theta & 0 & 0 & 0 & 0 \\
-\theta & 0 & 0 & \theta & -\theta & -\theta & 0 & 0 & \theta & 0 & 0 & 0 \\
0 & -\theta & \theta & 0 & \theta & -\theta & 0 & 0 & 0 & \theta & 0 & 0 \\
0 & \theta & \theta & \theta & \theta & 0 & 0 & 0 & 0 & 0 & 0 & \theta \\
0 & 2\omega-\omega & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 4
\end{pmatrix} \tag{6.19}$$

The image of this complex lattice with respect to the standard map

$$\mathbb{C}^{2,10} \in (z_1,\dots,z_{12}) \mapsto (\Re(z_1),\mathrm{Im}(z_1),\dots,\Re(z_{12}),\mathrm{Im}(z_{12})) \in \mathbb{R}^{4,20},$$

defines a real even unimodular lattice of signature $(4,20)$, as can be checked by a direct computation. The construction is analogous to the definition of the complex Leech lattice (see for example chapter 7 section 8 in [74]), with some modifications due to the different signature. The first two columns in (6.19) correspond to complex coordinates z_1,z_2 for the space $\subset V$ of spectral flow generators, while the remaining 10 columns are complex coordinates $z_3,\dots,z_{12}$ for its orthogonal complement. Notice that first four vectors (rows) in the lattice basis, namely

$$\begin{array}{l}
(3 \quad 0 \quad 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
(3\omega \quad 0 \quad 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
(0 \quad 3\theta \quad 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0) \\
(0 \quad 3\theta\omega\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0\ 0)
\end{array}$$

are contained in the subspace . By claim 2, the existence of such charges imply that all defects $\mathcal{L} \in \mathsf{Top}_{(1)^6}$ have integral quantum dimension.

The coordinates $z_3,\dots,z_{12}$ are a complex basis for the space $V \cap\ ^\perp$ of R-R ground states in the $(\frac{1}{4},0;\frac{1}{4},0)$ representation on $\mathcal{N}=(4,4)$. In particular, the basis consists of the 20 highest weight vectors for the chiral and anti-chiral algebra $\mathcal{A}\times\mathcal{A}$ labeled by

$$[a_1,\dots,a_6] = \left[+\frac{1}{2},+\frac{1}{2},+\frac{1}{2},-\frac{1}{2},-\frac{1}{2},-\frac{1}{2}\right] \qquad \text{and permutations.}$$

Let us denote by $|i,j,k\rangle$, $1 \le i < j < k \le 6$, the highest weight vector with the + signs in positions i,j,k, i.e. in the $\mathcal{A}\times\mathcal{A}$ representation $[a_1,\dots,a_6]$ with

$$a_n = \begin{cases} +1/2 & \text{if } n \in \{i,j,k\} \\ -1/2 & \text{if } n \notin \{i,j,k\} \ . \end{cases}$$

This is a complex basis, since the CPT conjugate of $[a_1,\dots,a_6]$ is the opposite $[-a_1,\dots,-a_6]$. We identify the states $|i,j,k\rangle$ with the complex coordinates $z_3, z_3^*, \dots, z_{12}, z_{12}^*$ as follows:

$$\begin{array}{ll|ll} z_3 \to |1,2,3\rangle & z_3^* \to |4,5,6\rangle & z_8 \to |1,3,4\rangle & z_8^* \to |2,5,6\rangle \\ z_4 \to |1,2,4\rangle & z_4^* \to |3,5,6\rangle & z_9 \to |1,5,6\rangle & z_9^* \to |2,3,4\rangle \\ z_5 \to |1,2,5\rangle & z_5^* \to |3,4,6\rangle & z_{10} \to |1,3,5\rangle & z_{10}^* \to |2,4,6\rangle \\ z_6 \to |1,4,6\rangle & z_6^* \to |2,3,5\rangle & z_{11} \to |1,3,6\rangle & z_{11}^* \to |2,4,5\rangle \\ z_7 \to |1,4,5\rangle & z_7^* \to |2,3,6\rangle & z_{12} \to |1,2,6\rangle & z_{12}^* \to |3,4,5\rangle \end{array} \tag{6.20}$$

The group of invertible defect in $\mathsf{Top}_{(1)^6}$ is $G_{(1)^6} \cong \mathbb{Z}_3^4 \rtimes A_6$. All such symmetries act trivially on the spectral flow operators, and therefore on the complex coordinates z_1, z_2. A permutation $\sigma \in A_6$ acts in the obvious way on the states $|i,j,k\rangle$

$$|i,j,k\rangle \mapsto |\sigma(i),\sigma(j),\sigma(k)\rangle \ . \tag{6.21}$$

Using the correspondence (6.20), it is immediate to derive the action of σ on the coordinates $z_3, z_3^*, \dots, z_{12}, z_{12}^*$.

The elements of the subgroup $\mathbb{Z}_3^4$ are labeled by $(m_1,\dots,m_6) \in (\mathbb{Z}/3\mathbb{Z})^6$, with the condition $\sum_i m_i \equiv 0 \mod 3$ and modulo $(1,1,1,1,1,1)$. They multiply the states $|i,j,k\rangle$, and therefore complex coordinates $z_3,\dots,z_{12}$, by some cubic roots of unity, determined by

$$|i,j,k\rangle \mapsto \exp\left(\frac{2\pi i}{3}\left[\sum_{n\in\{i,j,k\}} m_n - \sum_{n\notin\{i,j,k\}} m_n\right]\right)|i,j,k\rangle \ . \tag{6.22}$$

It is easy to check that these transformations of the space $\mathbb{C}^{2,10}$ correspond to automorphisms of the lattice spanned by the rows of 6.19.

As for the model $\mathcal{C}_{GTVW}$, we computed a set of generators for the intersection

$$B^{4,20}_{(1)^6}(\mathbb{Z}) := \mathrm{End}(\ ^{4,20}) \cap B^{4,20}(\mathbb{R}) \,,$$

between the $\mathbb{Z}$-linear endomorphisms of the lattice $\ ^{4,20}$ and the 401-dimensional real space $B^{4,20}(\mathbb{R})$ of block diagonal 24×24 matrices, with a upper left 4×4 block proportional to the identity, and an unconstrained lower right 20×20 block, see (3.11). As in the previous example, the $\mathbb{Z}$-module $B^{4,20}_{(1)^6}(\mathbb{Z})$ has maximal rank 401, i.e. equal to the dimension of $B^{4,20}(\mathbb{R})$.

We also computed the submodule

$$B^{4,20}_{(1)^6,inv}(\mathbb{Z}) \subset B^{4,20}_{(1)^6}(\mathbb{Z})$$

generated by the L where $\mathcal{L}$ is a superposition of *invertible* defects in $\mathsf{Top}_{(1)^6}$. As in the previous example, we find that the submodule $B^{4,20}_{(1)^6,inv}(\mathbb{Z})$ has full rank 401, with quotient

$$B^{4,20}_{(1)^6}(\mathbb{Z}) / B^{4,20}_{(1)^6,inv}(\mathbb{Z}) \cong \mathbb{Z}_3^{104} \ . \tag{6.23}$$

For all the defects $\mathcal{L}\in\mathsf{Top}_{(1)^6}$ that are discussed above, the corresponding map L is contained in $B^{4,20}_{(1)^6}(\mathbb{Z})$, i.e. it can be written as an integral linear combination $\mathsf{L}=\sum_g n_g\mathsf{L}_g$ of maps L_g related to invertible defects $\mathcal{L}_g$, $g\in G_{(1)^6}$, for some $n_g\in\mathbb{Z}$. However, some L cannot be written as linear combinations $\sum_g n_g\mathsf{L}_g$ with *non-negative* integral coefficients $n_g\in\mathbb{Z}_{\geq 0}$. An example is given by the duality defect $\mathcal{N}_{(12)(34)}$ considered in point 2 above, that acts on the chiral algebra $\mathcal{A}$ by a permutation with cycle shape (12)(34), while preserving the anti-chiral algebra $\bar{\mathcal{A}}$. The square of $\mathcal{N}_{(12)(34)}$ is a superposition of 9 invertible defects in the $\mathbb{Z}_3\times\mathbb{Z}_3$ group generated by $t_1t_2^2$ and $t_3t_4^2$

$$\mathcal{N}^2_{(12)(34)}=(\mathcal{I}+\mathcal{L}_{t_1t_2^2}+\mathcal{L}_{t_1^2t_2})(\mathcal{I}+\mathcal{L}_{t_3t_4^2}+\mathcal{L}_{t_3^2t_4})\ . \tag{6.24}$$

This determines (up to a sign) the action of $\mathcal{N}_{(12)(34)}$ on the basis (6.20) of RR ground states in the $(\frac{1}{4},0;\frac{1}{4},0)$ representation of $\mathcal{N}=(4,4)$, and therefore on the space $\mathbb{C}^{2,10}$ with coordinates $z_1,\ldots,z_{12}$. In particular, the action is via the diagonal matrix

$$\mathsf{N}_{(12)(34)}=\begin{pmatrix} 3&0&0&0&0&0&0&0&0&0&0&0\\ 0&3&0&0&0&0&0&0&0&0&0&0\\ 0&0&-3&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0&0 \end{pmatrix} \tag{6.25}$$

where the minus sign is fixed by the requirement that this linear map induces an endomorphisms of the lattice of D-brane charges. On the other hand, all invertible defects $\mathcal{L}_g$ act on the basis (6.20) by multiplication by some cubic root of unity followed by a permutation. This means that in the corresponding matrix L_g the sum of the entries in each row or column is a cubic root of unity. Because it is impossible to obtain -3 as a sum over three cubic roots of unity, it follows that $\mathsf{N}_{(12)(34)}$ cannot be written as a sum of three matrices $\mathsf{L}_{g_1}+\mathsf{L}_{g_2}+\mathsf{L}_{g_3}$ for any $g_1,g_2,g_3\in G_{(1)^6}$.

As for the GTVW model, it is easy to exhibit a deformation of the model that lifts all non-trivial topological defects $\mathcal{L}\in\mathsf{Top}_{(1)^6}$. It is sufficient to consider a basis $\chi_{i,j}$, $i=1,\ldots,4$, $j=5,\ldots,24$ of the 80-dimensional space of exactly marginal operators with respect to which all the linear operators $\mathcal{L}$ are represented by matrices with rational entries — or, more generally, entries in some algebraic number field. For example, one could take the $|\chi_{ij}\rangle$ to be the states related by spectral flow to the R-R $|i,j,k\rangle$ in (6.20). With respect to this basis, the operators $\mathcal{L}$ are represented by matrices with entries in the algebraic number field $\mathbb{Q}[\omega]$, where $\omega=e^{2\pi i/3}$. Then, one can take a linear combination $\chi=\sum_{j=5}^{24}\alpha_j\chi_{1,j}$ whose coefficients have transcendental ratios α_j/α_k, for example $\alpha_j=\pi^{j-5}$ as in (5.22). The same argument as in section 5.2 shows that the only defects that satisfy (5.21), and are therefore preserved by the deformation χ, are the ones proportional to the identity.

7 Conclusions

In this article, we discuss some general properties of the categories $\mathsf{Top}_{\mathcal{C}}$ of topological defects preserving the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow in a supersymmetric non-linear sigma model on K3 $\mathcal{C}$. In particular, we focus on the fusion of such topological defects with boundary states corresponding to BPS D-branes, and define a homomorphism from the fusion ring of Top to the ring of $\mathbb{Z}$-linear endomorphisms on the lattice of D-brane charges. This construction, together with some standard assumptions about the moduli space of K3 models, allows us to derive a number of general properties of the topological defects in $\mathsf{Top}_{\mathcal{C}}$. For example, we show that the set of K3 models where Top is not trivial (i.e., where there are simple defects distinct from the identity) has zero measure in the moduli space. Furthermore, we provide some restrictions on the possible quantum dimensions of the defects in Top and a sufficient condition on the model $\mathcal{C}$ (or rather on the corresponding point in the moduli space) for all quantum dimensions to be integral. Finally, we apply our methods in a couple of well understood examples of K3 models.

There are many directions of investigation that would be interesting to pursue:

1. One of the main tools in our analysis is the map $\mathsf{Top}\to\mathrm{End}(\ ^{4,20})$ that associates with each topological defect $\mathcal{L}$ a $\mathbb{Z}$-linear endomorphism L of the lattice $\ ^{4,20}$ of RR charges of the model. In section 3.2 we provided some conditions that must be satisfied by the endomorphisms L that are associated with some defect. It is clear, though, that in general these conditions are not sharp enough to identify the image of the $\mathsf{Top}\to\mathrm{End}(\ ^{4,20})$. Can we refine them?

2. Similarly, we know that the map $\mathsf{Top}\to\mathrm{End}(\ ^{4,20})$ is in general not injective. In fact, in the two K3 models $\mathcal{C}$ that we studied, we found some families of defects $\mathcal{L}_\theta\in\mathsf{Top}_{\mathcal{C}}$ depending on real parameters θ, corresponding to the same $\mathsf{L}\in\mathrm{End}(\ ^{4,20})$. On the other hand, there are some cases (for example, when L is a multiple of the identity) where we were able to prove that L is the image of a single $\mathcal{L}$. Can we determine for which L the corresponding defect $\mathcal{L}$ is unique?

   Let us give some more insight into this question. It is known that each topological defect $\mathcal{L}$ in the CFT $\mathcal{C}$ can be described as a boundary state in the double theory $\mathcal{C}\times\mathcal{C}$, see section 3.4. In this description, it is natural to identify $\mathsf{L}\in\mathrm{End}(\ ^{4,20})$ with the R-R charge $\mathsf{L}\in\ ^{4,20}\otimes(\ ^{4,20})^*$ of the boundary state in the double theory. Thus, a continuum of defects for L in the model $\mathcal{C}$ corresponds to a moduli space of D-branes with a given charge L in the double theory. In particular, the twisted conserved currents implementing the deformations of the topological defect (see the discussion in section 4.2 and references therein) should correspond to suitable massless modes for open strings with both ends on the D-brane in the double theory. It is natural to wonder whether there can be examples where there is a *discrete* set of defects $\mathcal{L}\in\mathsf{Top}$ with the same image L via the map $\mathsf{Top}\to\mathrm{End}(\ ^{4,20})$. They would correspond to a discrete set of different BPS D-branes carrying the same R-R charge in some non-linear sigma model on $K3\times K3$.

As a second observation, we notice that a given topological defect $\mathcal{L}$ is preserved by a marginal deformation of the K3 model if and only if the defect is 'transparent' for the corresponding exactly marginal operator. This property is completely encoded in the form of L. As a consequence, *all* defects $\mathcal{L}$ with the same L are either all lifted or all preserved by a certain deformation. Thus, for each L, one can identify some connected families of K3 models $\mathcal{C}$ in the moduli space, with the property that $\mathsf{Top}_{\mathcal{C}}$ contains a set of defects $\mathcal{L}$ with action L on the D-brane charges. These arguments suggest that the cardinality of this set of defects might only depend on the connected family, and not on the particular K3 model.

3. In section 4.2 we conjectured that $\mathsf{Top}_{\mathcal{C}}$ contains a continuum of defects only if $\mathcal{C}$ is a (possibly generalised) orbifold of a torus model. It would be very interesting to either prove this conjecture or to find a counterexample. Notice that both models studied in our article are actually torus models. In this respect, it would be very useful to study an example of K3 model that is *not* a torus model. Unfortunately, while torus orbifolds are necessarily a zero measure set in the moduli space of K3 models (they are a discrete union of subloci of dimension at most 16), finding such an example of K3 model that is reasonably under control (e.g. it is rational) is not an easy task. In particular, we do not know of any simple criterion to determine whether a given K3 model admits or not a description as a generalised orbifold of a torus model.

4. In [28], the authors study the topological defects in various CY manifolds, including some K3 models. The idea is to classify the topological defects for some rational models (Gepner models), and then consider deformations by exactly marginal operators that preserve some of the defects. In this way, one can obtain information about topological defects in models that are (probably) not rational. This method is very effective in establishing the existence of topological defects in a neighborhood of known rational CFTs. In contrast, our approach, based on the action of topological defects on D-brane charges, allows us to provide information about possible defects even at points in the moduli space that are far from any (known) rational point. On the other hand, while our methods are effective in putting restrictions on properties of putative defects, they are not sufficient to prove the existence of such defects without an explicit description of the model. It would be very interesting to combine the two approaches: one could study the topological defects in many Gepner models, and then use our methods to determine precisely the sublocus of the moduli space where the topological defect is preserved. Furthermore, by considering the points in the moduli space at the intersection of several such loci, one might be able to establish the existence of K3 models with large categories of topological defects.

5. Given a symmetry g of a K3 model $\mathcal{C}$, one can define a twining genus $\phi_g(\tau,z)$ by computing the elliptic genus of the model $\mathcal{C}$ with the insertion of the topological defect $\mathcal{L}_g$ along one of the cycles of the torus. In the same spirit, one can define a twining genus $\phi_{\mathcal{L}}(\tau,z)$ for any topological defect $\mathcal{L}\in\mathsf{Top}_{\mathcal{C}}$. On general grounds, one expects

such functions to be holomorphic and modular, in a suitable sense.[9] Twining genera play a prominent role in the calculation of the microstates degeneracy for 1/4-BPS black hole in string compactifications on K3 and on orbifolds thereof (CHL models). See for example [54, 78–88]. They are also the main characters in the 'moonshine' conjectures for string theory on K3 [49–53, 89, 90].

The study of twining genera related to topological defects, rather than symmetries, opens a number of new paths for the research in these subjects.

6. A mysterious correspondence has been observed [91] between symmetries of K3 sigma models and automorphisms of a certain $\mathcal{N}=1$ supersymmetric vertex operator algebra $V^{s\natural}$ (a holomorphic superconformal field theory, in physics parlance) with central charge 12. In particular, most (though, probably, not all) of the twining genera ϕ_g of K3 models can be exactly reproduced by certain g-twisted supertraces in $V^{s\natural}$. It is reasonable to expect the SCFT $V^{s\natural}$ to admit suitable topological defects that preserve the $\mathcal{N}=1$ superconformal symmetry — to the best of our knowledge, such defects have not been studied yet. While one cannot define boundary states in a purely holomorphic theory, we do expect some modification of our methods to apply to this case as well. We are planning to describe such methods in a forthcoming paper [92]. It is natural to wonder whether the K3-VOA correspondence extends to the case of defects.

Acknowledgments

The research of S.G. and R.V. has been supported by a BIRD-2021 project (PRD-2021) and by the PRIN Project n. 2022ABPBEY, "Understanding quantum field theory through its deformations". The research of R.A. has been supported by the grants CEX2020001007-S and PID2021-123017NB-I00, funded by MCIN/AEI/10.13039/501100011033 and by ERDF A way of making Europe. S.G. and R.V. would like to thank the Pollica Physics Center for support and hospitality during the programme "New connections between Physics and Number Theory" where work on this paper was undertaken, and to thank the participants to the programme for stimulating discussions.

A Generators of $\mathbb{Z}_2^8 : M_{20}$

In this section, we provide an explicit description of the group of symmetries $G_{GTVW} \cong \mathbb{Z}_2^8 : M_{20}$ of the K3 model $\mathcal{C}_{GTVW}$ considered in section 5; see [69] and [70] for a derivation. Recall that the full symmetry group of the CFT is $(\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) : S_6$, where the $\mathrm{SU}(2)^6$ factors are generated by the zero modes of the holomorphic and antiholomorphic currents, while $G_{GTVW} \cong \mathbb{Z}_2^8 : M_{20}$ is the subgroup that preserves the $\mathcal{N}=(4,4)$ superconformal algebra and the spectral flow. Every element of $(\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) : S_6$ can be written as $(A_1, \ldots, A_6\ B_1, \ldots, B_6)\pi$ where A_i and B_i are SU(2) matrices, and $\pi \in S_6$ is a permutation. As discussed in section 5, $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$ does not act faithfully on the operators of the CFT. There is a subgroup $Z_0 \cong \mathbb{Z}_2^5$ of the center $\mathbb{Z}_2^6 \times \mathbb{Z}_2^6$ of $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$ that acts trivially on all states of the

[9] More precisely, we expect them to be weak Jacobi forms of weight 0 and index 1 with respect to a suitable subgroup of $\mathrm{SL}(2, \mathbb{Z})$.

theory (see eq. (5.9)). This means that there is an ambiguity in writing the symmetries as $(A_1,\dots,A_6\ B_1,\dots,B_6)\pi$, as one can multiply by any element in $Z_0 \cong \mathbb{Z}_2^5$.

The SU(2) matrices that one needs to write all elements of $\mathbb{Z}_2^8 : M_{20}$ are

$$\mathsf{1} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad x = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \qquad y = \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}, \qquad z = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \tag{A.1}$$

and their opposite matrices

$$\underline{\mathsf{1}} = \begin{pmatrix} -1 & 0 \\ 0 & -1 \end{pmatrix}, \qquad \underline{x} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \qquad \underline{y} = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \qquad \underline{z} = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix}. \tag{A.2}$$

Notice that x,y,z obey the quaternionic relations $x^2 = y^2 = z^2 = -1$, $xy = z = -yx$. Finally we need the matrices

$$= \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix}, \qquad {}^\dagger = \begin{pmatrix} \frac{1+i}{2} & \frac{1+i}{2} \\ \frac{1-i}{2} & \frac{1-i}{2} \end{pmatrix} \tag{A.3}$$

that satisfy ${}^3 = -1$, so that ${}^6 = 1$ and ${}^\dagger = -{}^2$. Furthermore, $x\ {}^\dagger = z$, $y\ {}^\dagger = x$, $z\ {}^\dagger = y$. The permutations π will be denoted by their non-trivial cycles.

In this notation, the generators of $\mathbb{Z}_2^8 : M_{20}$ are as follows:

- Generators in the $\mathbb{Z}_2^6 \times \mathbb{Z}_2^6$ subgroup of $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$:

$$t_{23} = (1\underline{1}\underline{1}111\ 111111)(1)^6 \tag{A.4}$$
$$t_{24} = (1\underline{1}1\underline{1}11\ 111111)(1)^6 \tag{A.5}$$
$$t_{25} = (1\underline{1}11\underline{1}1\ 111111)(1)^6 \tag{A.6}$$
$$t_{26} = (1\underline{1}111\underline{1}\ 111111)(1)^6 \tag{A.7}$$

  These generators form a $\mathbb{Z}_2^4$ group, with $\mathbb{Z}_2^4 \subset \mathbb{Z}_2^8 \subset \mathbb{Z}_2^8 : M_{20}$, that contains 10 elements t_{ij}, $2 \le i < j \le 6$ with non-trivial SU(2) factors $A_i = A_j = \underline{1}$, and 5 elements t_{ijkl}, $2 \le i < j < k < l \le 6$, with non-trivial SU(2) factors $A_i = A_j = A_k = A_l = \underline{1}$.

- Further generators of the $\mathbb{Z}_2^8$ normal subgroup of $\mathbb{Z}_2^8 : M_{20}$

$$s_1 = (\mathsf{1}\mathsf{1}\underline{x}\,\underline{x}\,\underline{x}\,\underline{x};\ \mathsf{1}\mathsf{1}xxxx)(1)^6 \tag{A.8}$$
$$s_2 = (\mathsf{1}\mathsf{1}\underline{y}\,\underline{y}\,\underline{y}\,\underline{y};\ \mathsf{1}\mathsf{1}yyyy)(1)^6 \tag{A.9}$$
$$s_3 = (\mathsf{1}xz\underline{y}\,\underline{x}\mathsf{1};\ \mathsf{1}xzyx\mathsf{1})(1)^6 \tag{A.10}$$
$$s_4 = (\mathsf{1}zy\underline{x}\,\underline{z}\mathsf{1};\ \mathsf{1}zyxz\mathsf{1})(1)^6 \tag{A.11}$$

- Generators of the $\mathbb{Z}_2^4$ subgroup of $M_{20} \cong \mathbb{Z}_2^4.A_5$ (modulo elements of the form t_{ijkl})

$$v_1 = (\mathsf{1}\mathsf{1}xxxx;\ \mathsf{1}\mathsf{1}zzzz)(1)^6 \tag{A.12}$$
$$v_2 = (\mathsf{1}\mathsf{1}yyyy;\ \mathsf{1}\mathsf{1}xxxx)(1)^6 \tag{A.13}$$
$$v_3 = (\mathsf{1}yy\mathsf{1}xz;\ \mathsf{1}xx\mathsf{1}zy)(1)^6 \tag{A.14}$$
$$v_4 = (\mathsf{1}xyz\mathsf{1}x;\ \mathsf{1}zxy\mathsf{1}z)(1)^6 \tag{A.15}$$

  Altogether, the t_{ij}, $s_1,\dots,s_4$, and $v_1,\dots,v_4$ generate the subgroup of $\mathbb{Z}_2^8 : M_{20}$ contained in $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$.

- Generators of the quotient A_5 of $M_{20} \cong \mathbb{Z}_2^4.A_5$

$$p_1 = (111111;\ 111111)(34)(56) \tag{A.16}$$

$$p_2 = (111111;\ 111111)(35)(46) \tag{A.17}$$

$$p_3 = (11\ \ ^{\dagger}\ \ 11;\ 11\ \ ^{\dagger}\ \ 11)(25)(34) \tag{A.18}$$

Action on the currents. The 18 left-moving currents of $su(2)^6$ can be denoted by a pair $(\vec{\sigma}\cdot\vec{n},i)_L$, where $i=1,\dots,6$ denotes the $su(2)_1$ factors, and $\vec{\sigma}\cdot\vec{n}$ is a traceless hermitian matrix, written a linear combination of Pauli matrices with coefficients $\vec{n}=(n_1,n_2,n_3)\in\mathbb{R}^3$, $\vec{n}\cdot\vec{n}=1$. The action of $g=(A_,\dots,A_6;B_1,\dots,B_6)\pi$ on $(\vec{\sigma}\cdot\vec{n},i)_L$ is

$$g\cdot(\vec{\sigma}\cdot\vec{n},i)_L = (A_{\pi(i)}\vec{\sigma}\cdot\vec{n}A^{\dagger}_{\pi(i)},\pi(i))_L \tag{A.19}$$

where we make the permutation act before the $\mathrm{SU}(2)^6\times\mathrm{SU}(2)^6$ adjoint action. Similarly, the action on the right-moving currents is

$$g\cdot(\vec{\sigma}\cdot\vec{n},i)_R = (B_{\pi(i)}\vec{\sigma}\cdot\vec{n}B^{\dagger}_{\pi(i)},\pi(i))_R\ . \tag{A.20}$$

Action on the RR ground states. An orthonormal basis of the 24-dimensional space of RR ground states is given by six 'tetrads' of states $|1,i\rangle$, $|2,i\rangle$, $|3,i\rangle$, $|4,i\rangle$, $i=1,\dots,6$. One can assign with each element $|a,i\rangle$ in a tetrad a 2×2 matrix U_a, $a=1,2,3,4$, as follows

$$|1,i\rangle \longrightarrow U_1 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & 1\\ -1 & 0\end{pmatrix} = \frac{x}{\sqrt{2}} \tag{A.21}$$

$$|2,i\rangle \longrightarrow U_2 = \frac{1}{\sqrt{2}}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} = \frac{1}{\sqrt{2}} \tag{A.22}$$

$$|3,i\rangle \longrightarrow U_3 = \frac{1}{\sqrt{2}}\begin{pmatrix}i & 0\\ 0 & -i\end{pmatrix} = -\frac{z}{\sqrt{2}} \tag{A.23}$$

$$|4,i\rangle \longrightarrow U_4 = \frac{1}{\sqrt{2}}\begin{pmatrix}0 & i\\ i & 0\end{pmatrix} = \frac{y}{\sqrt{2}}, \tag{A.24}$$

in such a way that $\langle a,i|b,j\rangle = \delta_{ij}\,\mathrm{Tr}(U_a^{\dagger}U_b)$. The action of $g=(A_1,\dots,A_6;B_1,\dots,B_6)\pi$ on $|a,i\rangle\equiv(U_a,i)$, $a=1,\dots,4$, $i=1,\dots,6$ is

$$g\cdot(U_a,i) = (A_{\pi(i)}U_aB^{t}_{\pi(i)},\pi(i))\ . \tag{A.25}$$

B A duality defect not in $\mathbf{Top}_{GTVW}$

Consider the K3 model $\mathcal{C}_{GTVW}$ described in section 5 and let $H\subset G_{GTVW}$ the abelian group $H\cong\mathbb{Z}_2$ generated by the symmetry $g=t_5t_6$. By computing its partition function, one can show that the orbifold $\mathcal{C}/H$ is a consistent K3 model with the same bosonic chiral and anti-chiral algebras and representations as $\mathcal{C}$; therefore, the two CFTs must be isomorphic $\mathcal{C}\cong\mathcal{C}/H$. In particular, all holomorphic and anti-holomorphic currents are g-invariant; the g-twisted sector contains no further currents. Thus, we expect a topological duality defect $\mathcal{N}$ such that

$$\mathcal{N}^2 = \mathcal{I}+\mathcal{L}_g\,, \qquad \mathcal{L}_g\mathcal{N} = \mathcal{N}\mathcal{L}_g = \mathcal{N}\ . \tag{B.1}$$

In this section, we describe the duality defect $\mathcal{N}$ and show that $\mathcal{N} \notin \mathsf{Top}_{GTVW}$, and in particular that $\mathcal{N}$ does not preserve the $\mathcal{N}=(4,4)$ superconformal algebra.

The fusion rules for $\mathcal{N} \equiv \mathcal{N}_{56}$ imply that $\mathcal{N}$ must act with eigenvalues $\pm\sqrt{2}$ on the states that are fixed by g, while eigenstates of g with eigenvalue -1 must be in the kernel. On the other hand, these topological defects $\mathcal{N}$ cannot commute with the whole $(\widehat{su}(2)_1)^{\oplus 6}$ algebra — we have already identified all such defects, and they are all invertible.

What is the possible action on the chiral algebra?

- Consider moving a defect $\mathcal{N}$ through local holomorphic operators in $\mathcal{A}$, we obtain a map from $\mathcal{A}$ to itself in a way compatible with OPE. This means that the defect should act on $\mathcal{A}$ by an automorphism of order 2.
- The action of $\mathcal{N}$ on $\mathcal{A}$ should not lift to a symmetry of the whole CFT. Indeed, if this was the case, then for a suitable CFT symmetry h, the fusion $\mathcal{L}_h\mathcal{N}$ would act trivially on $\mathcal{A}$. But the only simple defects with this property are the Verlinde lines for the algebra $\mathcal{A}$ and all such defects are invertible. It would follow that $\mathcal{N}$ is a superposition of invertible defects, which cannot be true for a duality defect.
- Because $\mathcal{N}^2$ acts trivially on the g-invariant space of states $\mathcal{H}^g$, then $\mathcal{N}$ acts by permutations on the set of $\mathcal{A}\times\mathcal{A}$ representations contained in $\mathcal{H}^g$. In particular, it cannot map any such representation into one that is not contained in $\mathcal{H}^g$.

The group of outer automorphisms of the chiral and antichiral algebra is the $S_6\times S_6$ permutation group acting separately on the holomophic and anti-holomorphic currents. Only the diagonal $S_6 \subset S_6\times S_6$ lifts to a symmetry of the whole theory; the other elements of do not preserve the set of $\mathcal{A}\times\mathcal{A}$ representations in the space of states $\mathcal{H}$. If instead we consider the subgroup of $S_6\times S_6$ that preserve only the set of representations contained in the subspace $\mathcal{H}^g \subset \mathcal{H}$ of g-invariant states, then there is one non-trivial choice (modulo the diagonal S_6 symmetry): the involution exchanging the 5-th and the 6-th $\widehat{su}(2)_1$ components on the (say) holomorphic side, while keeping the anti-holomorphic side fixed.

Therefore, modulo automorphisms that lift to CFT symmetries, this is the only possible action of $\mathcal{N}$ on the algebra $(\widehat{su}(2)_1)^6$.

We can calculate the $\mathcal{N}$-twined partition functions

$$Z^{NSNS}_{1,\mathcal{N}} = \mathrm{Tr}_{NS-NS}(\mathcal{N} q^{L_0-\frac{c}{24}} q^{\bar{L}_0-\frac{\bar{c}}{24}}), \quad Z^{\widetilde{NSNS}}_{1,\mathcal{N}} = \mathrm{Tr}_{NS-NS}((-1)^{F+\tilde{F}}\mathcal{N} q^{L_0-\frac{c}{24}} q^{\bar{L}_0-\frac{\bar{c}}{24}}) \tag{B.2}$$

$$Z^{RR}_{1,\mathcal{N}} = \mathrm{Tr}_{R-R}(\mathcal{N} q^{L_0-\frac{c}{24}} q^{\bar{L}_0-\frac{\bar{c}}{24}}), \qquad Z^{\widetilde{RR}}_{1,\mathcal{N}} = \mathrm{Tr}_{R-R}((-1)^{F+\tilde{F}}\mathcal{N} q^{L_0-\frac{c}{24}} q^{\bar{L}_0-\frac{\bar{c}}{24}}) \tag{B.3}$$

using the characters of $\widehat{su}(2)_1$, namely

$$\mathrm{ch}_{1,0}(\tau,z) = \frac{\theta_3(2\tau,2z)}{\eta(\tau)}, \qquad \mathrm{ch}_{1,\frac{1}{2}}(\tau,z) = \frac{\theta_2(2\tau,2z)}{\eta(\tau)}, \tag{B.4}$$

from which

$$\mathrm{ch}_{1,0}(\tau,0) = \frac{\theta_3(2\tau,0)}{\eta(\tau)} = \frac{\eta(2\tau)^5}{\eta(\tau)^3\eta(4\tau)^2}, \qquad \mathrm{ch}_{1,\frac{1}{2}}(\tau,0) = \frac{\theta_2(2\tau,0)}{\eta(\tau)} = 2\frac{\eta(4\tau)^2}{\eta(2\tau)\eta(\tau)}. \tag{B.5}$$

The S-transformations are

$$\mathrm{ch}_{1,0}(-\tfrac{1}{\tau},0)=\frac{1}{\sqrt{2}}(\mathrm{ch}_{1,0}(\tau,0)+\mathrm{ch}_{1,\frac{1}{2}}(\tau,0))\,,\quad \mathrm{ch}_{1,\frac{1}{2}}(-\tfrac{1}{\tau},0)=\frac{1}{\sqrt{2}}(\mathrm{ch}_{1,0}(\tau,0)-\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)) \tag{B.6}$$

Recall $\mathcal{N}$ acts on the t_5t_6 invariant states by $\sqrt{2}$ times the permutation of the fifth and sixth holomorphic $\widehat{su}(2)_1$ components. It follows that the $\mathcal{N}$-twined partition function are

$$\begin{aligned}
Z_{1,\mathcal{N}}^{NSNS}=\sqrt{2}\sum_{\substack{a_1,\dots,a_4\in\{0,\frac{1}{2}\}\\ \sum a_i\in\mathbb{Z}}}&\Big[\prod_{i=1}^4|\mathrm{ch}_{1,a_i}(\tau,0)|^2\,(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2)\\
&+\prod_{i=1}^4\mathrm{ch}_{1,a_i}(\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}-a_i}(\tau,0)}(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2)\Big]\\
Z_{1,\mathcal{N}}^{NSNS}=\sqrt{2}\sum_{\substack{a_1,\dots,a_4\in\{0,\frac{1}{2}\}\\ \sum a_i\in\mathbb{Z}}}&\Big[\prod_{i=1}^4|\mathrm{ch}_{1,a_i}(\tau,0)|^2\,(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2)\\
&-\prod_{i=1}^4\mathrm{ch}_{1,a_i}(\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}-a_i}(\tau,0)}(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2)\Big]\\
Z_{1,\mathcal{N}}^{RR}=\sqrt{2}\sum_{\substack{a_1,\dots,a_4\in\{0,\frac{1}{2}\}\\ \sum a_i\in\frac{1}{2}+\mathbb{Z}}}&\Big[\prod_{i=1}^4|\mathrm{ch}_{1,a_i}(\tau,0)|^2\,(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2)\\
&+\prod_{i=1}^4\mathrm{ch}_{1,a_i}(\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}-a_i}(\tau,0)}(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2)\Big]\\
Z_{1,\mathcal{N}}^{RR}=\sqrt{2}\sum_{\substack{a_1,\dots,a_4\in\{0,\frac{1}{2}\}\\ \sum a_i\in\frac{1}{2}+\mathbb{Z}}}&\Big[\prod_{i=1}^4|\mathrm{ch}_{1,a_i}(\tau,0)|^2\,(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2)\\
&-\prod_{i=1}^4\mathrm{ch}_{1,a_i}(\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}-a_i}(\tau,0)}(\mathrm{ch}_{1,0}(2\tau,0)\overline{\mathrm{ch}_{1,\frac{1}{2}}(\tau,0)}^2+\mathrm{ch}_{1,\frac{1}{2}}(2\tau,0)\overline{\mathrm{ch}_{1,0}(\tau,0)}^2)\Big]
\end{aligned}$$

By a direct calculation we obtain

$$Z_{1,\mathcal{N}}^{RR}=(16+36q+96q^2+\dots)+O(q^1)\ . \tag{B.7}$$

This shows that $\mathcal{N}$ does not preserve any holomorphic supercurrent. Indeed, if a holomorphic supercurrent were preserved by $\mathcal{N}$, then $Z_{1,\mathcal{N}}^{RR}$ would receive contributions only from the RR ground states, and it would be a constant in q.

C Defects acting by automorphisms of the chiral algebra

In this section we classify the topological defects $\mathcal{L}\in\mathsf{Top}_{GTVW}$ of the model $\mathcal{C}_{GTVW}$ of section 5, such that $\mathcal{L}$ acts on all holomorphic fields generating the bosonic chiral algebra $\mathcal{A}\cong(\widehat{su}(2)_1)^6$ by an algebra automorphism ρ_L times the quantum dimension $\langle\mathcal{L}\rangle$. This implies that when a defect line $\mathcal{L}$ is moved past the insertion point of one of the holomorphic currents $j(z)$, the latter gets simply transformed into the current $\rho_L(j(z))$. Similarly, we require the defect to act on the anti-holomorphic fields by a (possibly different) automorphism ρ_R. The group of automorphisms of $(\widehat{su}(2)_1)^6$ is $\mathrm{SO}(3)^6 \rtimes S_6$, where $\mathrm{SO}(3)^6$ are the inner automorphisms generated by the zero modes of the currents; note that the center $\mathbb{Z}_2^6$ of $\mathrm{SU}(2)^6$ acts trivially on the algebra itself. We still require ρ_L and ρ_R to act trivially on the $\mathcal{N}=(4,4)$ superconformal algebra; this condition constrains ρ_L and ρ_R to be in a finite subgroup of $\mathrm{SO}(3)^6 \rtimes S_6$. If the automorphisms (ρ_L, ρ_R) extends to a symmetry of the CFT, then there is an invertible defect $\mathcal{L}_g$, for some $g \in G_{GTVW} \cong \mathbb{Z}_2^8 : M_{20}$, such that the fusion $\mathcal{L}_g \mathcal{L}$ acts trivially on the whole $\mathcal{A} \otimes \mathcal{A}$. The only simple defects with this property are the Verlinde lines discussed in section 5.1, where it is shown that they are all invertible. This means that, in this case, $\mathcal{L}$ is just a superposition of invertible defects.

Therefore, in order to get some new defects, we have to require that (ρ_L, ρ_R) does *not* lift to a symmetry of the CFT. Now, inner automorphisms $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$ are generated by the current zero modes, and therefore always define CFT symmetries. The group of outer automorphisms of $\mathcal{A} \times \mathcal{A}$ is $S_6 \times S_6$, and in this case, only the diagonal $S_6^{\mathrm{diag}} \subset S_6 \times S_6$, permuting the holomorphic and antiholomorphic $\widehat{su}(2)_1$ factors in the same way, lifts to a CFT symmetry. Thus, the defects acting by algebra automorphisms, modulo fusion with invertible defects from the left or from the right, correspond to non-trivial double cosets in

$$\Big((\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) \rtimes S_6^{\mathrm{diag}}\Big) \backslash \Big((\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) \rtimes (S_6 \times S_6)\Big) / \Big((\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) \rtimes S_6^{\mathrm{diag}}\Big) \ . \tag{C.1}$$

For each such coset, we can always choose a representative (ρ_L, ρ_R) with $\rho_R = 1$. Furthermore, we can choose one ρ_L for each S_6-conjugacy class, i.e. for each possible cycle shape.

Finally, we require ρ_L to preserve the $\mathcal{N}=4$ superconformal algebra and the spectral flow generators. This implies that the induced permutation must be contained in the $A_5 \subset S_6$ subgroup of *even* permutations fixing the first $\widehat{su}(2)_1$ factor. There are only three possible non-trivial cycle shapes, corresponding to the partitions 1+1+1+3, 1+1+2+2 and 1+5. Let us consider each of these three cases in detail.

Partition 1+1+1+3. It is known (see appendix A), that G_{GTVW} contains a symmetry

$$g_3 = p_1 \circ p_3 = (1 1 \quad {}^{\dagger} 1 1;\ 1 1 \quad {}^{\dagger} 1 1)(265) \,, \tag{C.2}$$

where we denote the elements of $(\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6) \rtimes S_6^{\mathrm{diag}}$ by $(A_1, \ldots, A_6; B_1, \ldots, B_6)\pi$, with $A_i, B_i \in \mathrm{SU}(2)$ and $\pi \in S_6^{\mathrm{diag}}$, and where

$$= \begin{pmatrix} \frac{1-i}{2} & \frac{1+i}{2} \\ -\frac{1-i}{2} & \frac{1+i}{2} \end{pmatrix} \in \mathrm{SU}(2) \ . \tag{C.3}$$

This means that if we define (ρ_L, ρ_R) as

$$\rho_L = (1 1 \quad {}^{\dagger} 1 1)(265) \,, \qquad \rho_R = 1 \ .$$

the left automorphism ρ_L acts on the holomorphic algebra in the same way as the symmetry element g_3. This ensures that the holomorphic $\mathcal{N}=4$ supercurrents are invariant under ρ_L. However, ρ_R is not the same as for g_3, and in particular the left- and right-moving

$\widehat{su}(2)_1$ factors are permuted in different ways. This means that the pair (ρ_L, ρ_R) cannot be extended to a symmetry of the whole CFT.

More explicitly, any such symmetry would have to map any field in the representation $[110000;110000]$, for example, to a field in a representation $[100010;110000]$; however, while the former field is in the spectrum of the theory, the latter is not. This implies that there cannot be any *invertible* defect acting by (ρ_L, ρ_R) on the chiral and antichiral algebras. However, there is no obstruction to having a non-invertible defect with such an action. Indeed, a non-invertible defect $\mathcal{L}$, when circling a field in the $[110000;110000]$ representation, can simply annihilate it. More generally, the operator $\mathcal{L}$ associated with any such defect needs to annihilate any field in a representation $[a_1 \ldots, a_6; b_1 \ldots b_6]$ such that $[\rho_L(a_1 \ldots, a_6); \rho_R(b_1 \ldots b_6)]$ is not in the spectrum. The NS-NS representations $[a_1 \ldots, a_6; b_1 \ldots b_6]$ that are not necessarily annihilated by $\mathcal{L}$ are the ones satisfying

$$a_2 = a_5 = a_6\,, \qquad b_2 = b_5 = b_6\,, \tag{C.4}$$

and can be grouped into four sets:

$$\begin{aligned}
{}_1 &= \{[000000;000000],\ [111111;000000],\ [000000;111111],\ [111111;111111]\}\,,\\
{}_2 &= \{[101000;101000],\ [010111;101000],\ [101000;010111],\ [010111;010111]\}\,,\\
{}_3 &= \{[100100;100100],\ [011011;100100],\ [100100;011011],\ [011011;011011]\}\,,\\
{}_4 &= \{[001100;001100],\ [110011;001100],\ [001100;110011],\ [110011;110011]\}\ .
\end{aligned}$$

The four members in each set are related to each other by the action of the $\mathcal{N} = (4,4)$ supercurrents. Therefore, the requirement that the $\mathcal{N} = (4,4)$ algebra is invariant under $\mathcal{L}$ implies that the action of $\mathcal{L}$ is the same on all representations in the same set $_i$. We conclude that the action of $\mathcal{L}$ on the NS-NS sector depends only on four parameters $\alpha_1, \ldots, \alpha_4$ as

$$\mathcal{L} = \sum_{i=1}^{4} \alpha_i P^i_{\rho_L,\rho_R}\ . \tag{C.5}$$

Here, for each $i = 1, \ldots, 4$, $P^i_{\rho_L,\rho_R}$ acts by the automorphism (ρ_L, ρ_R) on the representations in the set $_i$, while it annihilates all fields in the representations in $_j$ for $j \neq i$.

Requiring $\mathcal{L}$ to commute with the spectral flow operators fixes the action of $\mathcal{L}$ on the R-R sector to be of the same form (C.5) with the same parameters α_i, where now each $P^i_{\rho_L,\rho_R}$ is non-zero only on the set of representations $_i$ with

$$\begin{aligned}
{}_1 &= \{[100000;100000],\ [011111;100000],\ [100000;011111],\ [011111;011111]\}\,,\\
{}_2 &= \{[001000;001000],\ [110111;001000],\ [001000;110111],\ [110111;110111]\}\,,\\
{}_3 &= \{[000100;000100],\ [111011;000100],\ [000100;111011],\ [111011;111011]\}\,,\\
{}_4 &= \{[101100;101100],\ [010011;101100],\ [101100;010011],\ [010011;010011]\}\ .
\end{aligned}$$

The four parameters $\alpha_1, \ldots, \alpha_4$ are constrained by the Cardy-like conditions that are obtained by considering the torus partition function with the defect line $\mathcal{L}$ wrapping one of the cycles. Notice that if $(\alpha_1, \alpha_2, \alpha_3, \alpha_4)$ correspond to a consistent defect $\mathcal{L}$ acting by (ρ_L, ρ_R) on the chiral and antichiral algebras, then also $(\alpha_1, -\alpha_2, -\alpha_3, \alpha_4)$, $(\alpha_1, \alpha_2, -\alpha_3, -\alpha_4)$, and

$(\alpha_1, -\alpha_2, \alpha_3, -\alpha_4)$ correspond to consistent defects, since they can be obtained by fusion of $\mathcal{L}$ with the invertible defects acting trivially on all the currents and all the $\mathcal{N}=(4,4)$ supercurrents (in particular, fusion with the symmetries t_2t_3, t_2t_4 and t_3t_4 gives all such defects; fusion with the symmetries t_2t_5, t_2t_6, and t_5t_6 leaves each of these defects invariant).

Let us consider the possible fusion products of the simple defect $\mathcal{L}$ with $\alpha_1 = \ldots = \alpha_4$. The dual defect $\mathcal{L}^\star$ acts on the space of states by the adjoint operator

$$\mathcal{L}^\star = \sum_{i=1}^{4} \alpha_i^* P^i_{\rho_L^{-1}, \rho_R^{-1}} \tag{C.6}$$

so that the product $\mathcal{L}\mathcal{L}^\star$ acts by

$$\mathcal{L}\mathcal{L}^\star = \sum_{i=1}^{4} |\alpha_i|^2 P^i \,, \tag{C.7}$$

where P^i is the projector on the representations in the set ${}_i$. On the other hand, we know that $\mathcal{L}\mathcal{L}^\star = \mathcal{I} + \ldots$ acts trivially on the whole algebra $\widehat{su}(2)_1^6 \oplus \widehat{su}(2)_1^6$ as well as on the $\mathcal{N}=(4,4)$ algebra. This means that it must be a superposition of the invertible defects $\mathcal{L}_{t_it_j}$ generating the $\mathbb{Z}_2^4$ subgroup of $\mathbb{Z}_2^8 : M_{20}$ that is contained in the centre of $\mathrm{SU}(2)^6 \times \mathrm{SU}(2)^6$. Furthermore, $\mathcal{L}\mathcal{L}^\star$ must annihilate the representations $[a_1, \ldots, b_6]$ of the chiral algebra that do not satisfy (C.4), and must be a sum with positive coefficients of the projectors P_i. The only possibility is

$$\mathcal{L}\mathcal{L}^\star = \mathcal{I} + \mathcal{L}_{t_2t_5} + \mathcal{L}_{t_2t_6} + \mathcal{L}_{t_5t_6} \,, \tag{C.8}$$

so that

$$|\alpha_i| = \langle \mathcal{L} \rangle = \langle \mathcal{L}^\star \rangle = 2 \ .$$

The torus partition function with the insertion of $\mathcal{L}$ is

$$Z_{\mathcal{L}}(\tau) = \mathrm{Tr}(\mathcal{L} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}) = \sum_{i=1}^{4} \alpha_i \, \mathrm{Tr}(P^i_{\rho_L, \rho_R} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}) \,, \tag{C.9}$$

where

$$\mathrm{Tr}(P^i_{\rho_L, \rho_R} q^{L_0 - \frac{c}{24}} \bar{q}^{\bar{L}_0 - \frac{\bar{c}}{24}}) = \sum_{[a_1, \ldots, b_6] \in \Omega_i} \Big[\mathrm{ch}_{a_1}(\tau, 0)\, \mathrm{ch}_{a_2}(3\tau, 0)\, \mathrm{ch}_{a_3}(\tau, \tfrac{1}{6})\, \mathrm{ch}_{a_4}(\tau, \tfrac{1}{6}) \\ \overline{(\mathrm{ch}_{b_1}(\tau, 0)\, \mathrm{ch}_{b_2}(\tau, 0)^3\, \mathrm{ch}_{b_3}(\tau, 0)\, \mathrm{ch}_{b_4}(\tau, 0))} \Big]$$

Let us calculate $Z_{\mathcal{L}}$ with the ansatz

$$\alpha_1 = \alpha_2 = \alpha_3 = \alpha_4 = \langle \mathcal{L} \rangle \tag{C.10}$$

Using the $su(2)$ characters, $Z_{\mathcal{L}}(\tau)$ takes the form:

$$\begin{aligned} Z_{\mathcal{L}}(\tau) = \frac{\langle \mathcal{L} \rangle}{\eta(\tau)^6 \eta(\tau)^3 \eta(3\tau)} \Bigg\{ & \left(\overline{\theta_3(2\tau)^6} + \overline{\theta_2(2\tau)^6} \right) \left[\theta_3(2\tau)\theta_3(6\tau)\theta_3\left(2\tau, \frac{1}{3}\right)^2 + \theta_2(2\tau)\theta_2(6\tau)\theta_2\left(2\tau, \frac{1}{3}\right)^2 \right] + \\ & + \left(\overline{\theta_3(2\tau)^2\theta_2(2\tau)^4} + \overline{\theta_2(2\tau)^2\theta_3(2\tau)^4} \right) \left[2\left(\theta_3\left(2\tau, \frac{1}{3}\right) \theta_2\left(2\tau, \frac{1}{3}\right) \right) (\theta_3(2\tau)\theta_2(6\tau) + \theta_2(2\tau)\theta_3(6\tau)) + \right. \\ & \left. + \left(\theta_3(2\tau)\theta_3(6\tau)\theta_2\left(2\tau, \frac{1}{3}\right)^2 + \theta_2(2\tau)\theta_2(6\tau)\theta_3\left(2\tau, \frac{1}{3}\right)^2 \right) \right] \Bigg\} . \end{aligned}$$

Applying the transformation $\tau \mapsto -1/\tau$ we get:

$$Z^{\mathcal{L}}(\tau) = \frac{\langle \mathcal{L}\rangle q^{1/18}}{32\overline{\eta(\tau)^6}\eta(\tau)^3\eta(\tau/3)} \left\{ \left[\left(\theta_3\left(\frac{\tau}{2}\right)\theta_3\left(\frac{\tau}{6}\right)\theta_4\left(\frac{\tau}{2};\frac{\tau}{6}\right)^2 + \theta_4\left(\frac{\tau}{2}\right)\theta_4\left(\frac{\tau}{6}\right)\theta_3\left(\frac{\tau}{2};\frac{\tau}{6}\right)^2 \right) + \right.\right.$$
$$+2\left(\theta_3\left(\frac{\tau}{2};\frac{\tau}{6}\right)\theta_4\left(\frac{\tau}{2};\frac{\tau}{6}\right)\right)\left(\theta_3\left(\frac{\tau}{2}\right)\theta_4\left(\frac{\tau}{6}\right)+\theta_4\left(\frac{\tau}{2}\right)\theta_3\left(\frac{\tau}{6}\right)\right)\Bigg] \left(\overline{\theta_3\left(\frac{\tau}{2}\right)^4\theta_4\left(\frac{\tau}{2}\right)^2} + \overline{\theta_4\left(\frac{\tau}{2}\right)^4\theta_3\left(\frac{\tau}{2}\right)^2} \right) +$$
$$\left. + \left[\theta_3\left(\frac{\tau}{2}\right)\theta_3\left(\frac{\tau}{6}\right)\theta_3\left(\frac{\tau}{2};\frac{\tau}{6}\right)^2 + \theta_4\left(\frac{\tau}{2}\right)\theta_4\left(\frac{\tau}{6}\right)\theta_4\left(\frac{\tau}{2};\frac{\tau}{6}\right)^2 \right] \left(\overline{\theta_3\left(\frac{\tau}{2}\right)^6} + \overline{\theta_4\left(\frac{\tau}{2}\right)^6} \right) \right\},$$

which series expansion in q and q is:

$$Z^{\mathcal{L}} = \frac{\langle \mathcal{L}\rangle}{q^{1/4}\overline{q}^{1/4}} \left[\left(\frac{q^{1/6}}{2} + \frac{3}{2}q^{1/2} + 2q^{2/3} + 3q^{5/6} + 6q + 9q^{7/6} + 12q^{4/3} + \frac{39}{2}q^{3/2} + 28q^{5/3} + 36q^{11/6} + \right.\right.$$
$$\left. +54q^2 + \frac{149}{2}q^{13/6} + 96q^{7/3} + \frac{273}{2}q^{5/2} + 182q^{8/3} + 234q^{17/6} + O\left(q^3\right) \right) + q^{1/2}\left(6q^{1/6} + 40q^{1/3} + \right.$$
$$+82q^{1/2} + 128q^{2/3} + 196q^{5/6} + 320q + 524q^{7/6} + 776q^{4/3} + 1098q^{3/2} + 1616q^{5/3} + 2320q^{11/6} +$$
$$\left.\left. +O\left(q^2\right)\right) + q\left(33q^{1/6} + 160q^{1/3} + 355q^{1/2} + 548q^{2/3} + 838q^{5/6} + O(q)\right) + O\left(\overline{q}^{3/2}\right)\right].$$

As predicted above, we find that such expansion produces only integer non-negative coefficients if $\langle \mathcal{L}\rangle = 2$.

Notice that the automorphism (ρ_L, ρ_R) has order 3, so that $\mathcal{L}^2$ acts on the chiral algebra by $(\rho_L^2, \rho_R^2) = (\rho_L^{-1}, \rho_R^{-1})$ (the same as $\mathcal{L}^\star$), while $\mathcal{L}^3$ acts trivially on the chiral algebra. This suggests

$$\mathcal{L}^2 = 2\mathcal{L}^\star, \tag{C.11}$$

and

$$\mathcal{L}^3 = 2\mathcal{L}\mathcal{L}^\star = 2(\mathcal{I} + \mathcal{L}_{t_2t_5} + \mathcal{L}_{t_2t_6} + \mathcal{L}_{t_5t_6}), \tag{C.12}$$

which fits with the quantum dimensions.

We notice that if $\mathcal{L}_{p_1}$ is the invertible defect corresponding to the symmetry

$$p_1 = (111111;\ 111111)(34)(56)$$

of order 2, then $\mathcal{L}_{p_1}\mathcal{L}\mathcal{L}_{p_1}$ acts by $(\rho_L^{-1}, \rho_R^{-1})$ on the chiral algebra. This suggests that

$$\mathcal{L}_{p_1}\mathcal{L}\mathcal{L}_{p_1} = \mathcal{L}^\star\ . \tag{C.13}$$

As a consequence, if we define

$$\mathcal{N}_{256} := \mathcal{L}\mathcal{L}_{p_1}, \tag{C.14}$$

then $\mathcal{N}_{256}$ is unoriented $\mathcal{N}_{256} = \mathcal{N}_{256}^\star$, and

$$\mathcal{N}_{256}^2 = \mathcal{I} + \mathcal{L}_{t_2t_5} + \mathcal{L}_{t_2t_6} + \mathcal{L}_{t_5t_6}\ . \tag{C.15}$$

This means that $\mathcal{N}_{256}$ is the duality defect related to the fact that the theory is self-orbifold with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ group of symmetries with generators t_2t_5 and t_2t_6. Similarly, by conjugating by invertible defects, for every $1<i<j<k\leq 6$ one can find duality defects $\mathcal{N}_{ijk}$ of order 2 such that

$$\mathcal{N}_{ijk}^2=\mathcal{I}+\mathcal{L}_{t_it_j}+\mathcal{L}_{t_jt_k}+\mathcal{L}_{t_it_k}\ . \tag{C.16}$$

Partition 1+1+2+2. Let us now consider the class of defects acting by

$$\rho_L=(\mathbf{111111})(34)(56)\,,\qquad \rho_R=1\,, \tag{C.17}$$

on the chiral and anti-chiral algebra. We denote such defects by $\mathcal{L}_a^{(34)(56)}$, $a=1,2,\ldots$. Because ρ_L act in the same way as $p_1\in G_{GTVW}$ on the chiral algebra, $\mathcal{L}_a^{(34)(56)}$ must preserve the $\mathcal{N}=(4,4)$ SCA. The only representations where $\mathcal{L}_a^{(34)(56)}$ can be non-zero are the ones satisfying

$$a_3=a_4\,,\qquad a_5=a_6\,,\qquad b_3=b_4\,,\qquad b_5=b_6\ . \tag{C.18}$$

Once again, we arrange such representations (in the NS-NS sector) in sets

$$\begin{aligned}
{}_1&=\{[000000;000000],\ [111111;000000],\ [000000;111111],\ [111111;111111]\}\,,\\
{}_2&=\{[110000;110000],\ [001111;110000],\ [110000;001111],\ [001111;001111]\}\,,\\
{}_3&=\{[001100;001100],\ [110011;001100],\ [001100;110011],\ [110011;110011]\}\,,\\
{}_4&=\{[000011;000011],\ [111100;000011],\ [000011;111100],\ [111100;111100]\}\,,
\end{aligned}$$

so that

$$\mathcal{L}_a^{(34)(56)}=\sum_{i=1}^4\alpha_i P^i_{(34)(56)}\ .$$

Because there are 4 parameters α_i, we expect (at most) 4 simple defects of this kind. Let us denote by $\mathcal{L}:=\mathcal{L}_1^{(34)(56)}$ one of these defects. Then, by fusion of $\mathcal{L}^{(34)(56)}$ with the invertible defects $\mathcal{L}_{t_2t_3}$, $\mathcal{L}_{t_2t_5}$, $\mathcal{L}_{t_3t_5}$, we find three more simple defects $\mathcal{L}_2^{(34)(56)}$, $\mathcal{L}_3^{(34)(56)}$, $\mathcal{L}_4^{(34)(56)}$ whose parameters α_i differ only by signs. The product $\mathcal{L}\mathcal{L}^\star=\mathcal{L}^\star\mathcal{L}=\mathcal{I}+\ldots$ acts trivially on the whole chiral and antichiral algebra, as well as on the $\mathcal{N}=(4,4)$ superconformal algebra, and annihilate the representations that do not satisfy (C.18). This leads to

$$\mathcal{L}\mathcal{L}^\star=\mathcal{I}+\mathcal{L}_{t_3t_4}+\mathcal{L}_{t_5t_6}+\mathcal{L}_{t_3t_4t_5t_6}\ . \tag{C.19}$$

It follows that

$$|\alpha_1|=|\alpha_2|=|\alpha_3|=|\alpha_4|=|\mathcal{L}|=2\ . \tag{C.20}$$

The torus partition function is

$$Z_{\mathcal{L}}(\tau)=\mathrm{Tr}(\mathcal{L}q^{L_0-\frac{c}{24}}q^{\bar{L}_0-\frac{\bar{c}}{24}})=\sum_{i=1}^4\alpha_i\,\mathrm{Tr}(P^i_{\rho_L,\rho_R}q^{L_0-\frac{c}{24}}q^{\bar{L}_0-\frac{\bar{c}}{24}})$$

where

$$\mathrm{Tr}(P^i_{(34)(56)} q^{L_0-\frac{c}{24}} \bar{q}^{\bar{L}_0-\frac{\bar{c}}{24}}) = \sum_{[a_1,\ldots,b_6]\in\Omega_i} \Big[\mathrm{ch}_{a_1}(\tau,0)\,\mathrm{ch}_{a_2}(\tau,0)\,\mathrm{ch}_{a_3}(2\tau,0)\,\mathrm{ch}_{a_5}(2\tau,0)$$

$$\overline{(\mathrm{ch}_{b_1}(\tau,0)\,\mathrm{ch}_{b_2}(\tau,0)\,\mathrm{ch}_{b_3}(\tau,0)^2\,\mathrm{ch}_{b_5}(\tau,0)^2)}\Big]$$

Using the ansatz $\alpha_i = \langle \mathcal{L}^{(34)(56)} \rangle$, the partition function with a $\mathcal{L}^{(34)(56)}$ insertion can be written in terms of the Theta functions as:

$$Z_{\mathcal{L}^{(34)(56)}}(\tau) = \frac{\langle \mathcal{L}^{(34)(56)} \rangle}{\overline{\eta(\tau)^6 \eta(\tau)^2 \eta(2\tau)^2}} \Big\{ \Big(\overline{\theta_3(2\tau)^6} + \overline{\theta_2(2\tau)^6}\Big) \Big(\theta_3(2\tau)^2\theta_3(4\tau)^2 + \theta_2(2\tau)^2\theta_2(4\tau)^2\Big) +$$
$$+ \Big(\overline{\theta_2(2\tau)^2\theta_3(2\tau)^4} + \overline{\theta_3(2\tau)^2\theta_2(2\tau)^4}\Big)\Big(\theta_2(2\tau)^2\theta_3(4\tau)^2 + \theta_3(2\tau)^2\theta_2(4\tau)^2\Big) +$$
$$+2\Big(\overline{\theta_3(2\tau)^4\theta_2(2\tau)^2} + \overline{\theta_3(2\tau)^2\theta_2(2\tau)^4}\Big)\Big(\theta_3(2\tau)^2\theta_3(4\tau)\theta_2(4\tau) + \theta_2(2\tau)^2\theta_3(4\tau)\theta_2(4\tau)\Big)\Big\}.$$

Upon an S-transformation, we obtain the twisted function $Z^{\mathcal{L}^{(34)(56)}}(\tau) = Z_{\mathcal{L}^{(34)(56)}}(-1/\tau)$ in the form:

$$Z^{\mathcal{L}^{(34)(56)}}(\tau)$$
$$= \frac{\langle \mathcal{L}^{(34)(56)} \rangle}{32\overline{\eta(\tau)^6\eta(\tau)^2\eta(\tau/2)}} \Bigg\{ \Bigg(\overline{\theta_3\left(\frac{\tau}{2}\right)^6} + \overline{\theta_4\left(\frac{\tau}{2}\right)^6}\Bigg) \Bigg(\theta_3\left(\frac{\tau}{2}\right)^2 \theta_3\left(\frac{\tau}{4}\right)^2 + \theta_4\left(\frac{\tau}{2}\right)^2 \theta_4\left(\frac{\tau}{4}\right)^2\Bigg)$$
$$+ \Bigg(\overline{\theta_4\left(\frac{\tau}{2}\right)^2 \theta_3\left(\frac{\tau}{2}\right)^4} + \overline{\theta_3\left(\frac{\tau}{2}\right)^2 \theta_4\left(\frac{\tau}{2}\right)^4}\Bigg) \Bigg[\Bigg(\theta_4\left(\frac{\tau}{2}\right)^2 \theta_3\left(\frac{\tau}{4}\right)^2 + \theta_3\left(\frac{\tau}{2}\right)^2 \theta_4\left(\frac{\tau}{4}\right)^2\Bigg) +$$
$$+2\Bigg(\theta_3\left(\frac{\tau}{4}\right)\theta_4\left(\frac{\tau}{4}\right)\Bigg)\Bigg(\theta_3\left(\frac{\tau}{2}\right)^2 + \theta_4\left(\frac{\tau}{2}\right)^2\Bigg)\Bigg]\Bigg\},$$

whose expansion in q and q and is:

$$Z^{\mathcal{L}^{(34)(56)}}(\tau)$$
$$= \frac{\langle \mathcal{L}^{(34)(56)} \rangle}{q^{1/8}\bar{q}^{1/4}} \Bigg[\Bigg(\frac{1}{2} + 5q^{1/2} + \frac{47q}{2} + 75q^{3/2} + \frac{403q^2}{2} + 501q^{5/2} + 1158q^3 + 2502q^{7/2} + \frac{10309}{2}q^4 +$$
$$+10228q^{9/2} + O\left(q^5\right)\Bigg) + \bar{q}^{1/2}\left(6 + 32q^{1/4} + 128q^{3/8} + 60q^{1/2} + 192q^{3/4} + 512q^{7/8} + 282q +\right.$$
$$\left.+672q^{5/4} + 1792q^{11/8} + 900q^{3/2} + 1984q^{7/4} + 5120q^{15/8} + O\left(q^2\right)\right) + \bar{q}\left(33 + 128q^{1/4} + 512q^{3/8} +\right.$$
$$\left.+330q^{1/2} + 768q^{3/4} + 2048q^{7/8} + 1551q + 2688q^{5/4} + 7168q^{11/8} + O\left(q^{3/2}\right)\right) + O\left(\bar{q}^{3/2}\right)\Bigg]$$

The minimal integer value required for the quantum dimension of $\mathcal{L}^{(34)(56)}$ in order to have just positive integer coefficients in this expansion is $\langle \mathcal{L}^{(34)(56)} \rangle = 2$. The three additional simple defects, obtained by fusion with $\mathcal{L}_{t_2t_3}, \mathcal{L}_{t_2t_5}$ and $\mathcal{L}_{t_3t_5}$, act on the representations $_i$ through the operators $\pm 2P^i_{(34)(56)}$, where the minus sign appears for $i = 2,3$, or $i = 2,5$, or $i = 3,5$, respectively. In all such cases, the q-expansion of the resulting $\mathcal{L}$-twisted partition functions have positive integer coefficients, as expected.

Because the coefficients α_i are real and (ρ_L, ρ_R) has order two, we find that $\mathcal{L}^\star = \mathcal{L}$ is unoriented, and

$$\mathcal{L}^2 = \mathcal{I} + \mathcal{L}_{t_3t_4} + \mathcal{L}_{t_5t_6} + \mathcal{L}_{t_3t_4t_5t_6} \ . \tag{C.21}$$

This implies that $\mathcal{L}^{(34)(56)}$ coincides with the duality defect $\mathcal{N}_{34,56} \equiv \mathcal{L}^{(34)(56)}$ related to the fact that the theory is self-orbifold with respect to the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup of $\mathbb{Z}_2^8 : M_{20}$ generated by $t_3 t_4$ and $t_5 t_6$. Similarly, for every choice of pairwise distinct $i,j,k,l \in \{2,\ldots,6\}$, we have duality defects $\mathcal{N}_{ij,kl} \equiv \mathcal{N}_{kl,ij}$ of dimension 2 such that

$$\mathcal{N}_{ij,kl}^2 = \mathcal{I} + \mathcal{L}_{t_i t_j} + \mathcal{L}_{t_k t_l} + \mathcal{L}_{t_i t_j t_k t_l} \ . \tag{C.22}$$

Notice that, in a suitable description of the model $\mathcal{C}_{GTVW}$ as a torus orbifold $T^4/\mathbb{Z}_2$, the symmetries $t_i t_j$, $t_k t_l$ and $t_i t_j t_k t_l$ can be identified, respectively, with $\eta_{\frac{\lambda}{2}}$, $Q\eta_{\frac{\lambda}{2}}$, and Q, where $\eta_{\frac{\lambda}{2}}$ is induced by the half-periods along one of the directions of T^4, and Q is the quantum symmetry of the orbifold, see sections 4.2. This suggests that $\mathcal{N}_{ij,kl}$ can be identified with the topological defect $T_{\frac{\lambda}{4}}$ in eq. (4.19).

Partition 1+5. Finally, let us consider the class of defects that act by the automorphism

$$\rho_L = (11 \qquad {}^{\dagger}11)(23645) \,, \qquad \rho_R = 1 \tag{C.23}$$

of order 5, preserving the $\mathcal{N} = (4,4)$ supercurrents (in our conventions, the permutation acts after the $\mathrm{SU}(2)^6$ transformation). The only possible for a simple defect $\mathcal{L}^{(23645)}$ in this class is as follows (we set $\mathcal{L} \equiv \mathcal{L}^{(23645)}$ in this part)

$$\mathcal{L} = \langle \mathcal{L} \rangle P_{(23645)} \,, \tag{C.24}$$

where $P_{(23645)}$ acts by the automorphism (ρ_L, ρ_R) on the four NS-NS representations

$$[000000;000000], \ [111111;000000], \ [000000;111111], \ [111111;111111]$$

that satisfy

$$a_2 = a_3 = a_4 = a_5 = a_6 \,, \qquad b_2 = b_3 = b_4 = b_5 = b_6 \tag{C.25}$$

and annihilates any other representation in the NS-NS sector. The characters contributing to torus NS-NS partition function are:

$$Z_{\mathcal{L}^{(23645)}}(\tau) = \langle \mathcal{L} \rangle \sum_{a,b=0}^{1} \mathrm{ch}_a(\tau,0)\, \mathrm{ch}_a(5\tau,0) \overline{\mathrm{ch}_b(\tau,0)}^6 \,, \tag{C.26}$$

which expressions in terms of the $su(2)$ characters is:

$$Z_{\mathcal{L}^{(23645)}}(\tau) = \langle \mathcal{L} \rangle \frac{1}{\overline{\eta(\tau)^6}} \frac{1}{\eta(\tau)\eta(5\tau)} \left(\overline{\theta_3(2\tau)}^6 + \overline{\theta_2(2\tau)}^6 \right) \left(\theta_3(2\tau)\theta_3(10\tau) + \theta_2(2\tau)\theta_2(10\tau) \right)$$

The twined partition function, obtained from the previous one by modular transformation, takes the form:

$$Z^{\mathcal{L}^{(23645)}}(\tau) = \frac{\langle \mathcal{L} \rangle}{16 \overline{\eta(\tau)}^6 \eta(\tau)\eta(\tau/5)} \left(\overline{\theta_3\left(\frac{\tau}{2}\right)}^6 + \overline{\theta_4\left(\frac{\tau}{2}\right)}^6 \right) \left(\theta_3\left(\frac{\tau}{2}\right) \theta_3\left(\frac{\tau}{10}\right) + \theta_4\left(\frac{\tau}{2}\right) \theta_4\left(\frac{\tau}{10}\right) \right),$$

with corresponding series expansion given by:

$$Z^{\mathcal{L}^{(23645)}}(\tau)=\frac{\langle\mathcal{L}\rangle}{q^{1/20}\bar{q}^{1/4}}\Bigg[\Bigg(\frac{1}{4}+\frac{3}{4}q^{1/5}+q^{3/10}+q^{2/5}+q^{1/2}+\frac{7}{4}q^{3/5}+3q^{7/10}+\frac{13}{4}q^{4/5}+4q^{9/10}+\frac{11}{2}q+$$
$$+7q^{11/10}+\frac{19}{2}q^{6/5}+11q^{13/10}+\frac{55q^{7/5}}{4}q^{7/5}+18q^{3/2}+\frac{83}{4}q^{8/5}+26q^{17/10}+\frac{129}{4}q^{9/5}+39q^{19/10}+O\left(q^2\right)\Bigg)+$$
$$+\bar{q}^{1/2}\left(15+45q^{1/5}+60q^{3/10}+60q^{2/5}+60q^{1/2}+105q^{3/5}+180q^{7/10}+195q^{4/5}+240q^{9/10}+O\left(q\right)\right)+$$
$$+\bar{q}\left(\frac{129}{2}+\frac{387}{2}q^{1/5}+258q^{3/10}+258q^{2/5}+258q^{1/2}+\frac{903}{2}q^{3/5}+774q^{7/10}+\frac{1677}{2}q^{4/5}+O\left(q^{9/10}\right)\right)$$
$$+O(\bar{q}^{3/2})\Bigg]$$

From such expansion we can extract the condition for the minimal value of the quantum dimension at $\langle\mathcal{L}\rangle=4$.

The product $\mathcal{L}\mathcal{L}^\star$ must act trivially on the whole chiral and antichiral algebra and annihilate any representation that does not satisfy (C.25). Because the defect with smallest quantum dimension satisfying these properties has dimension 16, this implies $\langle\mathcal{L}\rangle=4$ and

$$\mathcal{L}\mathcal{L}^\star=\mathcal{I}+\sum_{2\le i<j\le 6}\mathcal{L}_{t_it_j}+\sum_{i=2}^{6}\mathcal{L}_{t_it_2t_3t_4t_5t_6}\ . \tag{C.27}$$

The defects $\mathcal{L}^{(26534)}$, $\mathcal{L}^{(24356)}$, $\mathcal{L}^{(25463)}\equiv\mathcal{L}^\star$ can be obtained by conjugation $\mathcal{L}_g\mathcal{L}^{(23645)}\mathcal{L}_g^\star$ with suitable invertible defects $\mathcal{L}_g$, with $g\in\mathbb{Z}_2^8:M_{20}$, so that all such defects have dimension 4, and are the unique simple defects acting with the given automorphism on the chiral algebra.

Let $p_2\in\mathbb{Z}_2^8:M_{20}$ be the order 2 symmetry acting by $p_2=(111111;\ 111111)(35)(46)$, so that $\mathcal{L}^\star=\mathcal{L}_{p_2}\mathcal{L}\mathcal{L}_{p_2}$. Then $(\mathcal{L}_{p_2}\mathcal{L})^\star=\mathcal{L}_{p_2}\mathcal{L}$ is unoriented, has dimension 4, and

$$(\mathcal{L}_{p_2}\mathcal{L})^2=\mathcal{I}+\sum_{2\le i<j\le 6}\mathcal{L}_{t_it_j}+\sum_{i=2}^{6}\mathcal{L}_{t_it_2t_3t_4t_5t_6}\ . \tag{C.28}$$

This means that $\mathcal{N}_{23456}:=\mathcal{L}_{p_2}\mathcal{L}$ is the duality defect related to the fact that the theory is self-orbifold under the $\mathbb{Z}_2^4$ group of symmetries generated by t_it_j, $2\le i<j\le 6$.

D The $\mathcal{N}=2$ $c=1$ superconformal algebra as a free boson.

The $\mathcal{N}=2$ superconformal algebra at central charge $c_k=\frac{3k}{k+2}$, $k\in\mathbb{N}$, can be described in terms of a coset $\frac{\widehat{su}(2)_k\oplus\widehat{u}(1)_4}{\widehat{u}(1)_{2k+4}}$, which provides the bosonic subalgebra of $\mathcal{N}=2$. In the particular case of $k=1$, the bosonic subalgebra of the $\mathcal{N}=2$ algebra, simplifies and becomes essentially the $u(1)_{12}$ algebra [93–95]. Here, $u(1)_{2l}$ denotes[10] the $c=1$ chiral algebra generated by a single chiral free boson $i\partial X_L(z)=\sum_n\alpha_n z^{-n-1}$ (whose modes generate the Heisenberg algebra), together with the holomorphic vertex operators $V_{n\sqrt{2l}}(z)\sim:e^{in\sqrt{2l}X_L(z)}:$, $n\in\mathbb{Z}$, of α_0-eigenvalue $n\sqrt{2l}$ and conformal weight n^2l. In other words, this is the lattice VOA

[10] Our normalization differs by a factor 2 from the conventions in [96]: the algebra $u(1)_{2l}$ in this paper is denoted as $u(1)_l$ in [96].

associated with the lattice $\sqrt{2l}\mathbb{Z}$. The irreducible representations $M^l_{[x]}$ of $u(1)_{2l}$ are labeled by $[x]\in\mathbb{Z}/2l\mathbb{Z}$, and the characters are given by

$$K^{2l}_{[x]}(\tau,z)=\frac{\sum_{Q\in\frac{x}{2l}+\mathbb{Z}} q^{lQ^2}y^Q}{\eta(\tau)}\,,\qquad y=e^{2\pi iz}\,, \tag{D.1}$$

where Q is the α_0-eigenvalue divided by $\sqrt{2l}$ (this strange normalization is justified below). In particular, the ground state of $M^l_{[x]}$ has conformal weight $h=\frac{x^2}{4l}$ and charge $Q=\frac{x}{2l}$, where $x\in\mathbb{Z}$ is a representative of $[x]\in\mathbb{Z}/2l\mathbb{Z}$ in the range $-l+1\leq x\leq l$.

The $\mathcal{N}=2$ superconformal algebra at $c=1$ is obtained by adjoining the bosonic algebra $u(1)_{12}$ (i.e., $l=6$) with its module $M^6_{[6]}$, which contains the two supercurrents and the other fermionic fields. The charge Q is the U(1) R-charge with the standard normalization such that the supercurrents have charge $\pm1/2$. In general, the $\mathcal{N}=2$ modules are given by sums $M^6_{[x]}\oplus M^6_{[x+6]}$ of $u(1)_{12}$ modules, and the representations are NS or Ramond depending on whether $[x]$ is even or odd, respectively. In other words, the $\mathcal{N}=2$ algebra at $c=1$ can be identified with the lattice SVOA related with the *odd* lattice $\sqrt{3}\mathbb{Z}$. The representations of this superalgebra are given by $R_{[x]}=M^6_{[x]}\oplus M^6_{[x+6]}$ where now $x\in\mathbb{Z}/6\mathbb{Z}$, and correspond to the lattice cosets $\frac{x}{2\sqrt{3}}+\sqrt{3}\mathbb{Z}\subset\frac{1}{2\sqrt{3}}\mathbb{Z}$.

Gepner models are usually described in terms of the coset algebra $(su(2)_1\oplus u(1)_4)/u(1)_6$, whose representations are labeled as $[l,m,s]\equiv[1-l,m+3,s+2]$, $l=0,1$, $m\in\mathbb{Z}/6\mathbb{Z}$, $s\in\mathbb{Z}/4\mathbb{Z}$ with $l+m+s\equiv 0 \mod 2$. As mentioned above, the algebra $(su(2)_1\oplus u(1)_4)/u(1)_6$ is isomorphic to $u(1)_{12}$, and the respective representations can be identified as follows:

$$\begin{aligned}
[0,m,0]&\equiv M^6_{[m]} && m\in\{0,\pm2\}\,,\\
[0,m,2]&\equiv M^6_{[m+6]} && m\in\{0,\pm2\}\\
[0,m,1]&\equiv M^6_{[m]} && m\in\{3,\pm1\}\,,\\
[0,m,-1]&\equiv M^6_{[m+6]} && m\in\{3,\pm1\}\ .
\end{aligned}$$

In terms of representations of the $\mathcal{N}=2$ superconformal algebra, one has the following identifications

$$\begin{aligned}
[0,m,0]\oplus[0,m,2]&\equiv R_{[m]} && m\in\{0,\pm2\}\,,\\
[0,m,1]\oplus[0,m,-1]&\equiv R_{[m]} && m\in\{3,\pm1\}\ .
\end{aligned}$$

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4

Global summary of results: discussion and conclusions

In this thesis, we explored various aspects and implications of emergent symmetries throughout the string theory moduli space, aiming to identify features that may signal fundamental principles underlying a complete theory of quantum gravity. Our investigation proceeded along two complementary directions. First, we adopted a bottom-up approach, examining how the absence of global symmetries, specifically the requirement that all cobordism classes of admissible compactification backgrounds in string theory are trivial, imposes constraints on the behavior of effective field theories (EFTs) near infinite-distance loci in moduli space. Second, we employed a worldsheet microscopic approach, using exact two-dimensional conformal field theory (CFT) techniques to probe the interior of the moduli space of string theory compactifications. Focusing on non-linear sigma models (NLSMs) with K3 surfaces as target spaces, we studied the emergence of both standard and generalized symmetries within the corresponding worldsheet theories.

We now provide a concise summary of the central points and principal results presented in each article of the compendium.

In Chapter 2, we studied the definition and various realizations of the Cobordism Conjecture within the framework of effective field theories through Dynamical Cobordism solutions. These configurations implement codimension-1 End of The World (ETW) boundaries in spacetime, realized as spacetime-dependent solutions to the equations of motion.These solutions exhibit a metric singularity at a finite spacetime distance, where scalar fields diverge towards infinite distances in moduli or field space. At this singularity, certain internal cycles of the compact geometry shrink to zero size, providing a dynamical realization of a cobordism to nothing configuration.

In the first article, we explore timelike linear dilaton backgrounds in supercritical string theories as time-dependent Dynamical Cobordisms in string theory. These configurations exhibit spacelike singularities at finite time, interpreted as boundaries marking the beginning of time. The study proposes that these singularities correspond microscopically to regions of (a strong coupling version of) closed tachyon condensation. We argue that this beginning of time is closely related to (and shares the same scaling behaviour as) the bubbles of nothing obtained in a weakly coupled background with lightlike tachyon condensation.

In the second article, we investigate the role of small black holes in probing infinite distances within moduli spaces. We show that in 4 theories with scalar potentials that grow fast enough at infinity, it becomes energetically prohibitive for scalar fields to diverge at the black hole core. Consequently, such small black holes either puff up into regular black holes or exhibit runaway behavior. A critical exponent is derived to characterize the conditions under which these "small black hole explosions" occur, analyzed from both four-dimensional perspectives and two-dimensional theories obtained after an truncation. This two-dimensional framework facilitates a unified treatment of fluxes, domain walls, and black holes, resolving an apparent puzzle in the expression of their potentials in the 4 $\mathcal{N} = 2$ gauged supergravity context. The study also uncovers that many known regular black hole solutions in the literature are incomplete, due to Freed-Witten anomalies, and require the emission of strings by the black hole. From the two-dimensional viewpoint, small black holes correspond to dynamical cobordisms, with their cores representing ETW branes. Explosions of small black holes signify obstructions to completing these Dynamical Cobordisms. We study the implications for the Cobordism Distance Conjecture, which states that in any theory there should exist Dynamical Cobordisms accessing all possible infinite distance limits in scalar field space. The realization of this principle using small black holes leads to non-trivial constraints on the 4d scalar potential of any consistent theory.

In the third article, we initiate the construction of a new class of Dynamical Cobordism solutions describing networks of intersecting ETW branes in theories with multiple scalars. These solutions allow to probe new infinite-distance limits of the moduli/field space where multiple scalars diverge simultaneously. Within this description many physically interesting examples are included, such as intersections of Witten's bubbles of nothing in toroidal compactifications, generalizations in compactifications on products of spheres, and possible flux dressings thereof that allow to include charged objects. These solutions can be regarded as a mere superposition of ETW branes, in the sense that their source terms are localized on the individual codimension-1 ETW branes.

In the fourth article, we begin the exploration of the infinite distance singularities in the complex structure moduli space of Calabi-Yau fourfolds compactifications in M-theory with a four-form flux turned on, which is described in terms of normal intersecting divisors classified by asymptotic Hodge theory. We provide spacetime realizations for these loci in terms of networks of intersecting codimension-1 ETW branes classified by specific critical exponents which encapsulate the relevant information of the asymptotic Hodge structure characterizing the corresponding divisors. In order to match the leading behaviour of the flux potential given by the asymptotic growth of the Hodge norm, the required spacetime solutions for intersecting ETW branes are more general than those considered in the previous work and we have provided the explicit construction of such generalization, by relaxing the constraint of conformally flat ansatz in the solutions considered in prior works.

Finally, in the fifth article we do a step forward in the construction of ETW configurations addressing the problem to build boundaries in chiral Quantum Gravity theories. We initiate the construction in a large classes of examples in 6d and 4d chiral theories arising from string theory compactifications. The main tool used to obtain these configurations is the Chiral Cone Construction. It essentially consists in considering a dimensional localized chiral field theory in dimensional spacetime and constructing on top of it a (1) dimensional boundary regarding the local transverse space as a cone over a ( ) dimensional base. The compactification over the base of the cone leads to a dimensional potentially chiral theory with a (1) dimensional ETW boundary sitting at the tip of the cone. Interestingly, the Cone Construction suggests a direct link with a Dynamical Cobordism in the compactified theory, where a scalar running along the radial direction of the cone and controlling the size of the ( )-dimensional base blows up to infinite field theory distance at a finite spacetime distance, producing a singularity at which spacetime ends. Beyond this simple geometric construction, in order to get a proper chiral theory in d-dimensions we need to require some special physics at the boundary: in the first class of 6 models described in the paper this involves chirality changing phase transitions, in the second class of 4 models it involves fixed planes under involutions.

In Chapter 3, which contains a single article, we discussed some general properties of the categories $_{\mathcal{C}}$ of topological defects in supersymmetric non-linear sigma model on K3 $\mathcal{C}$ that preserve the $\mathcal{N} = (4\ 4)$ superconformal algebra and the spectral flow. In particular, we focus on the fusion of such topological defects with boundary states corresponding to BPS D-branes, and define a homomorphism from the fusion ring of to the ring of -linear endomorphisms on the lattice of D-brane charges, that generalize the one constructed in [35] for standard symmetries. This construction, together with some standard assumptions about the moduli space of K3 models, allows us to derive a number of general properties for the category of such defects. We shown that the set of K3 models where is not trivial (i.e., where there are simple defects distinct from the identity) has zero measure in the moduli space and corresponds to points where periods are algebraic. We provided some restrictions on the possible quantum dimensions of the defects in and we prove that if a K3 model is at the attractor point for some BPS configuration of D-branes, then all topological defects have integral quantum dimension. In this work, we addressed the initial step of one of our initial goal: to understand the No Global Symmetry Conjecture in relation to generalized symmetries. Specifically, we developed techniques to detect such symmetries within the worldsheet theory. The subsequent step involves investigating the fate of these topological defects in the target spacetime. Usually, when a two-dimensional conformal field theory (CFT) possesses a group of standard global symmetries, these symmetries impose selection rules on string amplitudes that persist at all orders in perturbation theory. This persistence corresponds to the presence of gauge symmetries in the spacetime theory. However, this process has not an obvious extension for non-invertible defects. Recent studies [58,59] have shown that the selection rules associated with non-invertible symmetries typically hold only at tree level (i.e., on the sphere) but are in general broken at higher orders in the topological expansion. It would be interesting to continue investigating if the existence of these defects in the worldsheet has same meaning for the spacetime physics.

Motivated by the intriguing observation that all symmetry groups of K3 sigma models preserving the $\mathcal{N} = (4\ 4)$ superconformal algebra and spectral flow can be realized as subgroups of the Conway group , the automorphism group of the Leech lattice, we initiated the study of the category of topological defects that preserve supersymmetry in the vertex operator superalgebra [60, 61]. Building upon the work of Gaberdiel, Hohenegger, and Volpato [35], who demonstrated that supersymmetry-preserving automorphisms of K3 sigma models correspond to four-plane-preserving subgroups of , and the findings of Duncan and Mack-Crane, who showed that the twining genera of K3 non-linear sigma models are reproduced by corresponding twining genera in [62], we conjecture a correspondence between four-plane-preserving topological defect lines (TDLs) in and supersymmetry-preserving TDLs in K3 non-linear sigma models. This conjecture extends the known correspondence between symmetry groups to the level of tensor category symmetries, suggesting a deeper structural relationship between the two theories.

Symmetries have consistently played a central role in guiding the exploration of the string theory moduli space. At asymptotic boundaries, bottom-up approaches such as dynamical cobordism have described the behavior of scalar fields and predicted the presence of End of the World branes, offering valuable insights into the structure of infinite distance limits. Conversely, in the interior of moduli space, top-down conformal field theory analysis have enabled the study of generalized symmetries and deepened our understanding of the associated worldsheet theories. We conclude by emphasizing that symmetries have had a profound impact thus far. Given their crucial role in constraining and uncovering the underlying physics and geometry of the moduli space, we believe they will continue to be instrumental in future explorations, potentially unveiling deeper structures and guiding principles within string theory and related frameworks.

5

Resumen global de los resultados: discusión y conclusiones

En esta tesis, exploramos varios aspectos e implicaciones de las simetrías emergentes a lo largo del espacio de módulos de la teoría de cuerdas, con el objetivo de identificar características que puedan señalar principios fundamentales subyacentes a una teoría completa de la gravedad cuántica. Nuestra investigación se desarrolló en dos direcciones complementarias. En primer lugar, adoptamos un enfoque ascendente, examinando cómo la ausencia de simetrías globales, en particular el requisito de que todas las clases de cobordismo de los fondos de compactificación admisibles en teoría de cuerdas sean triviales, impone restricciones sobre el comportamiento de las teorías efectivas de campo (EFTs) cerca de los lugares de distancia infinita en el espacio de módulos. En segundo lugar, empleamos un enfoque microscópico desde la hoja-mundo, utilizando técnicas exactas de teoría conforme de campos (CFT) bidimensional para explorar el interior del espacio de módulos de las compactificaciones de la teoría de cuerdas. Centrándonos en modelos sigma no lineales (NLSMs) con superficies K3 como espacios objetivo, estudiamos la aparición de simetrías estándar y generalizadas dentro de las teorías de hoja-mundo correspondientes.

A continuación, proporcionamos un resumen conciso de los puntos centrales y los principales resultados presentados en cada artículo del compendio.

En el Capítulo 2, estudiamos la definición y varias realizaciones de la Conjetura del Cobordismo en el marco de teorías efectivas de campo a través de soluciones de Cobordismo Dinámico. Estas configuraciones implementan fronteras de fin del mundo (ETW) de codimensión 1 en el espacio-tiempo, realizadas como soluciones dependientes del espacio-tiempo a las ecuaciones de movimiento. Estas soluciones exhiben una singularidad métrica a distancia finita en el espacio-tiempo, donde los campos escalares divergen hacia distancias infinitas en el espacio de campos o de módulos. En esta singularidad, ciertos ciclos internos de la geometría compacta se encogen a tamaño cero, proporcionando una realización dinámica de una configuración de cobordismo hacia la nada.

En el primer artículo, exploramos fondos con dilatón lineal temporal en teorías de cuerdas supercríticas como cobordismos dinámicos dependientes del tiempo. Estas configuraciones presentan singularidades espaciales a tiempo finito, interpretadas como fronteras que marcan el inicio del tiempo. El estudio propone que estas singularidades corresponden microscópicamente a regiones de condensación de taquiones cerrados (en una versión de acoplamiento fuerte). Argumentamos que este comienzo del tiempo está estrechamente relacionado con (y comparte el mismo comportamiento de escalamiento que) las burbujas de la nada obtenidas en un fondo de acoplamiento débil con condensación de taquiones lumínicos.

En el segundo artículo, investigamos el papel de los agujeros negros pequeños en la exploración de distancias infinitas dentro del espacio de módulos. Mostramos que en teorías en 4 con potenciales escalares que crecen lo suficientemente rápido en el infinito, se vuelve energéticamente prohibitivo que los campos escalares diverjan en el núcleo del agujero negro. En consecuencia, estos agujeros negros pequeños o bien se inflan convirtiéndose en agujeros negros regulares, o exhiben un comportamiento de fuga. Derivamos un exponente crítico que caracteriza las condiciones bajo las cuales ocurren estas "explosiones de agujeros negros pequeños", analizadas tanto desde perspectivas en cuatro dimensiones como en teorías bidimensionales obtenidas tras una truncación sobre . Este marco bidimensional facilita un tratamiento unificado de flujos, paredes de dominio y agujeros negros, resolviendo un aparente enigma en la expresión de sus potenciales en el contexto de supergravedad gauged $\mathcal{N} = 2$ en 4 . El estudio también revela que muchas soluciones de agujeros negros regulares conocidas en la literatura están incompletas debido a anomalías de Freed-Witten y requieren la emisión de cuerdas por parte del agujero negro. Desde el punto de vista bidimensional, los agujeros negros pequeños corresponden a cobordismos dinámicos, con sus núcleos representando branas ETW. Las explosiones de agujeros negros pequeños significan obstrucciones para completar estos cobordismos dinámicos. Estudiamos las implicaciones para la Conjetura de la Distancia del Cobordismo, la cual afirma que en cualquier teoría deben existir cobordismos dinámicos que accedan a todos los posibles límites de distancia infinita en el espacio de campos escalares. La realización de este principio mediante agujeros negros pequeños impone restricciones no triviales sobre el potencial escalar en 4d de cualquier teoría consistente.

En el tercer artículo, iniciamos la construcción de una nueva clase de soluciones de cobordismo dinámico que describen redes de intersecciones de branas ETW en teorías con múltiples escalares. Estas soluciones permiten explorar nuevos límites de distancia infinita del espacio de módulos/campos donde múltiples escalares divergen simultáneamente. Esta descripción incluye muchos ejemplos físicamente interesantes, como intersecciones de burbujas de la nada de Witten en compactificaciones toroidales, generalizaciones en compactificaciones sobre productos de esferas, y posibles versiones con flujo que permiten incluir objetos cargados. Estas soluciones pueden considerarse como una superposición de branas ETW, en el sentido de que sus términos fuente están localizados en cada una de las branas ETW de codimensión 1.

En el cuarto artículo, comenzamos la exploración de las singularidades de distancia infinita en el espacio de módulos de estructura compleja de compactificaciones de cuatro-plegados de Calabi-Yau en M-teoría con flujo de cuatro-forma activado, descritas en términos de divisores normales intersectantes clasificados por la teoría de Hodge asintótica. Proporcionamos realizaciones espacio-temporales de estos loci en términos de redes de branas ETW de codimensión-1 intersectantes clasificadas por exponentes críticos específicos que encapsulan la información relevante de la estructura de Hodge asintótica que caracteriza a los divisores correspondientes. Para coincidir con el comportamiento dominante del potencial de flujo dado por el crecimiento asintótico de la norma de Hodge, las soluciones espacio-temporales requeridas para las branas ETW intersectantes son más generales que las consideradas en trabajos previos, y proporcionamos la construcción explícita de dicha generalización, relajando la condición del ansatz conforme plano en las soluciones consideradas anteriormente.

Finalmente, en el quinto artículo damos un paso adelante en la construcción de configuraciones ETW abordando el problema de construir fronteras en teorías de gravedad cuántica quirales. Iniciamos la construcción en una gran clase de ejemplos en teorías quirales de 6d y 4d provenientes de compactificaciones de teoría de cuerdas. La herramienta principal utilizada para obtener estas configuraciones es la Construcción del Cono Quiral. Consiste esencialmente en considerar una teoría de campos quiral localizada en dimensiones dentro de un espacio-tiempo de dimensiones, y construir sobre ella una frontera de dimensión (1) considerando el espacio transversal local como un cono sobre una base de dimensión ( ). La compactificación sobre la base del cono conduce a una teoría potencialmente quiral en dimensiones con una frontera ETW de dimensión (1) ubicada en la punta del cono. De manera interesante, la Construcción del Cono sugiere un vínculo directo con un cobordismo dinámico en la teoría compactificada, donde un escalar que corre a lo largo de la dirección radial del cono y controla el tamaño de la base de dimensión ( ) diverge a distancia infinita en el espacio de teoría de campos a una distancia finita en el espacio-tiempo, produciendo una singularidad donde el espacio-tiempo termina. Más allá de esta construcción geométrica simple, para obtener una teoría verdaderamente quiral en dimensiones se requiere imponer una física especial en la frontera: en la primera clase de modelos de 6 descritos en el artículo esto implica transiciones de fase que cambian la quiralidad, y en la segunda clase de modelos de 4 implica planos fijos bajo involuciones .

En el Capítulo 3, que contiene un solo artículo, discutimos algunas propiedades generales de las categorías $_{\mathcal{C}}$ de defectos topológicos en modelos sigma no lineales supersimétricos sobre K3 $\mathcal{C}$ que preservan el álgebra superconforme $\mathcal{N} = (4\ 4)$ y el flujo espectral. En particular, nos centramos en la fusión de dichos defectos topológicos con estados de frontera que corresponden a D-branas BPS, y definimos un homomorfismo del anillo de fusión de al anillo de endomorfismos -lineales sobre la red de cargas de D-branas, que generaliza el construido en [35] para simetrías estándar. Esta construcción, junto con ciertos supuestos estándar sobre el espacio de módulos de modelos K3, nos permite derivar una serie de propiedades generales para la categoría de dichos defectos. Mostramos que el conjunto de modelos K3 donde no es trivial (es decir, donde hay defectos simples distintos de la identidad) tiene medida cero en el espacio de módulos y corresponde a puntos donde los períodos son algebraicos. Proporcionamos algunas restricciones sobre las posibles dimensiones cuánticas de los defectos en y demostramos que si un modelo K3 está en el punto atractor para alguna configuración BPS de D-branas, entonces todos los defectos topológicos tienen dimensión cuántica entera. En este trabajo abordamos el primer paso de uno de nuestros objetivos iniciales: entender la Conjetura de No Simetrías Globales en relación con las simetrías generalizadas. En particular, desarrollamos técnicas para detectar tales simetrías dentro de la teoría de hoja-mundo. El paso siguiente implica investigar el destino de estos defectos topológicos en el espacio-tiempo objetivo. Normalmente, cuando una teoría conforme de campos (CFT) bidimensional posee un grupo de simetrías globales estándar, estas simetrías imponen reglas de selección sobre las amplitudes de cuerdas que persisten en todos los órdenes en teoría de perturbaciones. Esta persistencia corresponde a la presencia de simetrías gauge en la teoría del espacio-tiempo. Sin embargo, este proceso no tiene una extensión obvia para defectos no invertibles. Estudios recientes [58, 59] han mostrado que las reglas de selección asociadas a simetrías no invertibles suelen mantenerse sólo a nivel de árbol (es decir, sobre la esfera) pero generalmente se rompen en órdenes superiores en la expansión topológica. Sería interesante seguir investigando si la existencia de estos defectos en la hoja-mundo tiene un significado para la física del espacio-tiempo.

Motivados por la intrigante observación de que todos los grupos de simetría de modelos sigma sobre K3 que preservan el álgebra superconforme $\mathcal{N} = (4\;4)$ y el flujo espectral pueden realizarse como subgrupos del grupo de Conway , el grupo de automorfismos de la red de Leech, iniciamos el estudio de la categoría de defectos topológicos en el superálgebra de operadores de vértice que preservan la supersimetría [60, 61]. Basándonos en el trabajo de Gaberdiel, Hohenegger y Volpato [35], quienes demostraron que los automorfismos que preservan la supersimetría de los modelos sigma sobre K3 corresponden a subgrupos preservadores de planos de dimensión cuatro de , y en los hallazgos de Duncan y Mack-Crane, quienes mostraron que los géneros de entrelazamiento de los modelos sigma no lineales sobre K3 son reproducidos por los géneros de entrelazamiento correspondientes en [62], conjeturamos una correspondencia entre líneas de defecto topológicas (TDLs) en que preservan planos de dimensión cuatro y las TDLs que preservan la supersimetría en modelos sigma no lineales sobre K3. Esta conjetura extiende la correspondencia conocida entre grupos de simetría al nivel de simetrías de categorías tensoriales, sugiriendo una relación estructural más profunda entre ambas teorías.

Las simetrías han desempeñado constantemente un rol central en la exploración del espacio de módulos de la teoría de cuerdas. En las fronteras asintóticas, los enfoques de tipo "bottom-up", como el cobordismo dinámico, han descrito el comportamiento de los campos escalares y predicho la presencia de branas del fin del mundo, proporcionando valiosas ideas sobre la estructura de los límites a distancia infinita. Por el contrario, en el interior del espacio de módulos, los análisis de teoría conforme de campos desde una perspectiva "top-down" han permitido estudiar simetrías generalizadas y profundizar en la comprensión de las teorías de la hoja de mundo asociadas. Concluimos enfatizando que las simetrías han tenido hasta ahora un impacto profundo. Dado su rol crucial en restringir y revelar la física y la geometría subyacentes del espacio de módulos, creemos que seguirán siendo fundamentales en futuras exploraciones, con el potencial de descubrir estructuras más profundas y principios rectores dentro de la teoría de cuerdas y marcos relacionados.