

On the Bethe states of the one-dimensional supersymmetric $t - J$ model with generic open boundaries

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ABSTRACT: By combining the algebraic Bethe ansatz and the off-diagonal Bethe ansatz, we investigate the supersymmetric $t - J$ model with generic open boundaries. The eigenvalues of the transfer matrix are given in terms of an inhomogeneous $T - Q$ relation, and the corresponding eigenstates are expressed in terms of nested Bethe states which have well-defined homogeneous limit. This exact solution provides basis for further analyzing the thermodynamic properties and correlation functions of the model.

KEYWORDS: Bethe Ansatz, Lattice Integrable Models

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Contents

1	Introduction	1
2	Integrability of the model	2
3	Nested algebraic Bethe ansatz	5
4	Reduced spectrum problem	7
5	Nested inhomogeneous $T - Q$ relation	11
6	Concluding remarks	12

1 Introduction

The $t - J$ model is one of the cornerstones in the study of high- T_c superconductivity [1], which is a large- U limit of the single-band Hubbard model [2–5]. The Hamiltonian of the model have played essential roles in theoretical study of strongly correlated copperoxide based materials [6]. In general, the Hamiltonian of the supersymmetric $t - J$ model with the general boundary interaction terms is given by

$$\begin{aligned}
 H = & -t \sum_{\alpha,j=1}^{L-1} \mathcal{P} \left[c_{j,\alpha}^+ c_{j+1,\alpha} + c_{j+1,\alpha}^+ c_{j,\alpha} \right] \mathcal{P} + J \sum_{k=1}^{L-1} \left[\mathbf{S}_k \cdot \mathbf{S}_{k+1} - \frac{1}{4} n_k n_{k+1} \right] + \sum_{l=1}^{L-1} n_l + n_{l+1} \\
 & - \mu \hat{N} + \xi_1 n_1 + 2h_1^z S_1^z + 2h_1^- S_1^- + 2h_1^+ S_1^+ + \xi_L n_L + 2h_L^z S_L^z + 2h_L^- S_L^- + 2h_L^+ S_L^+, \quad (1.1)
 \end{aligned}$$

where t is the nearest neighbor hopping of electrons and J is the antiferromagnetic exchange; L is the total number of lattice sites; the operators $c_{j,\sigma}$ and $c_{j,\sigma}^+$ are the annihilation and creation operators of the electron with spin $\sigma = \pm 1$ on the lattice site j , which satisfies anticommutation relations, i.e., $\{c_{i,\sigma}^+, c_{j,\tau}\} = \delta_{i,j} \delta_{\sigma,\tau}$. There are only three possible states at the lattice site i due to the factor $\mathcal{P} = (1 - n_{j,-\sigma})$ ruled out double occupancies; the operator $n_j = \sum_{\sigma=\pm} n_{j,\sigma}$ means the total number operator on site j and $n_{j,\sigma} = c_{j,\sigma}^+ c_{j,\sigma}$; μ is the chemical potential and $\hat{N} = \sum_{j=1}^L n_j$; $\xi_{1,L}$ are the boundary chemical potentials; $h_{1,L}^z$ and $h_{1,L}^\pm$ are the boundary fields; the spin operators $S^- = \sum_{j=1}^L S_j^-$, $S^+ = \sum_{j=1}^L S_j^+$ and $S^z = \sum_{j=1}^L S_j^z$, form the $su(2)$ algebra and can be expressed by

$$S_j^- = c_{j,1}^+ c_{j,-1}, \quad S_j^+ = c_{j,-1}^+ c_{j,1}, \quad S_j^z = \frac{1}{2} (n_{j,1} - n_{j,-1}). \quad (1.2)$$

It is well-known that the one-dimensional $t - J$ model is integrable at the supersymmetric point $J = \pm 2t$ [7–9], and the model with the periodic boundary condition or the diagonal

boundaries has been studied by employing many Bethe ansatz methods [10–20]. For the non-diagonal boundary case, the nested algebraic Bethe ansatz method doesn't work since the $U(1)$ symmetry is broken. With the help of the off-diagonal Bethe ansatz [21–27], the exact energy spectrum of the one-dimensional supersymmetric $t - J$ model with unparallel boundary fields has been obtained [28]. However, the eigenstates (or Bethe states) which have played important roles in applications of the model are still missing.

In this paper, we study the supersymmetric $t - J$ model with generic integrable boundary conditions in grading: bosonic, fermionic and fermionic (BFF). By combining the graded nested algebraic Bethe ansatz and off-diagonal Bethe ansatz, we obtain the Bethe states which have well-defined homogeneous limit and the corresponding eigenvalues of the transfer matrix of the model. Numerical results for the small size systems suggest that the spectrum obtained by the nested Bethe ansatz equations (BAEs) is complete.

The paper is organized as follows. In section 2, the associated graded R -matrix and corresponding generic integral non-diagonal boundary reflection matrices are introduced. In section 3, by using the graded algebraic Bethe ansatz, we derive the eigenvalues of the transfer matrix of the system which related with the eigenvalues of the nested transfer matrix. In section 4, the eigenvalues of the nested transfer matrix are derived by off-diagonal Bethe ansatz, and the Bethe states are also be given. In section 5, we construct the nested inhomogeneous $T - Q$ relation and the nested Bethe ansatz equations of the supersymmetric $t - J$ model. Section 6 contains our results and give some discussions.

2 Integrability of the model

In this paper we consider $J = 2t = 2$ which corresponds to the supersymmetric and integrable point [29]. The integrability of the model is associated with the rational R -matrix $R(u)$ given by

$$R_{12}(u) = \left(\begin{array}{c|c|c} u + \eta & & \\ \hline & u & \eta \\ \hline & & u \\ \hline \eta & & u - \eta \\ \hline & & & u & -\eta \\ \hline & & & & u \\ \hline & \eta & & -\eta & u \\ \hline & & & & & u - \eta \end{array} \right). \tag{2.1}$$

The R -matrix $R(u)$ possesses the following properties

Initial condition: $R_{12}(0) = \eta P_{12},$ (2.2)

Unitarity relation: $R_{12}(u)R_{21}(-u) = \rho_1(u) \times \text{id},$ (2.3)

Crossing Unitarity relation: $R_{12}^{st_1}(-u + \eta)R_{21}^{st_1}(u) = \rho_2(u) \times \text{id}.$ (2.4)

Here P_{12} is the graded permutation operator with the definition

$$P_{\beta_1\beta_2}^{\alpha_1\alpha_2} = (-1)^{p(\alpha_1)p(\alpha_2)} \delta_{\alpha_1\beta_2} \delta_{\beta_1\alpha_2}, \tag{2.5}$$

$p(\alpha_i)$ is the Grassmann parities which is one for fermions and zero for bosons. Here, we choose BFF grading which means $p(1) = 0$, $p(2) = p(3) = 1$ and $R_{21}(u) = P_{12}R_{12}(u)P_{12}$, st_i denotes the super transposition in the i -th space ($(A^{st})_{ij} = A_{ji}(-1)^{p(i)[p(i)+p(j)]}$) and ist_i denotes the inverse super transposition. The functions $\rho_1(u)$ and $\rho_2(u)$ are given by

$$\rho_1(u) = -(u - \eta)(u + \eta), \quad \rho_2(u) = -u(u - \eta). \quad (2.6)$$

Here and below we adopt the standard notations: for any matrix $A \in \text{End}(\mathbf{V})$, A_j is a super embedding operator in the Z_2 graded tensor space $\mathbf{V} \otimes \mathbf{V} \otimes \cdots$, which acts as A on the j -th space and as identity on the other factor spaces. For $R \in \text{End}(\mathbf{V} \otimes \mathbf{V})$, R_{ij} is a super embedding operator of R in the Z_2 graded tensor space, which acts as identity on the factor spaces except for the i -th and j -th ones. The super tensor product of two operators are defined through $(A \otimes B)_{\beta\delta}^{\alpha\gamma} = (-1)^{[p(\alpha)+p(\beta)]p(\gamma)} A_{\beta}^{\alpha} B_{\delta}^{\gamma}$. (For further details we refer the reader to [30]).

The R -matrix is an even operator (i.e., the parities of the non-zero matrix elements R_{bd}^{ac} of the R -matrix satisfies $p(a) + p(b) + p(c) + p(d) = 0$) and satisfies the graded quantum Yang-Baxter equation (QYBE)

$$R_{12}(u - v) R_{13}(u) R_{23}(v) = R_{23}(v) R_{13}(u) R_{12}(u - v). \quad (2.7)$$

In terms of the matrix entries, it reads

$$\begin{aligned} & R(\lambda - u)_{\beta_1\beta_2}^{\alpha_1\alpha_2} R(\lambda)_{\gamma_1\beta_3}^{\beta_1\alpha_3} R(u)_{\gamma_2\gamma_3}^{\beta_2\beta_3} (-1)^{(p(\beta_1)+p(\gamma_1))p(\beta_2)} \\ &= R(u)_{\beta_2\beta_3}^{\alpha_2\alpha_3} R(\lambda)_{\beta_1\gamma_3}^{\alpha_1\beta_3} R(\lambda - u)_{\gamma_1\gamma_2}^{\beta_1\beta_2} (-1)^{(p(\alpha_1)+p(\beta_1))p(\beta_2)}. \end{aligned} \quad (2.8)$$

Let us now introduce the reflection matrix $K^-(u)$ and its dual one $K^+(u)$. The former satisfies the graded reflection equation (RE) [31]

$$\begin{aligned} & R_{12}(u_1 - u_2) K_1^-(u_1) R_{21}(u_1 + u_2) K_2^-(u_2) \\ &= K_2^-(u_2) R_{12}(u_1 + u_2) K_1^-(u_1) R_{21}(u_1 - u_2), \end{aligned} \quad (2.9)$$

and the latter satisfies the dual RE which take the form [32]

$$\begin{aligned} & R_{12}(u_2 - u_1) K_1^+(u_1) \tilde{R}_{21}(-u_1 - u_2)^{ist_1, st_2} K_2^+(u_2) \\ &= K_2^+(u_2) \tilde{R}_{12}(-u_1 - u_2)^{ist_1, st_2} K_1^+(u_1) R_{21}(u_2 - u_1), \end{aligned} \quad (2.10)$$

where

$$\tilde{R}_{21}(u)^{ist_1, st_2} = \left([\{R_{21}^{-1}(u)\}^{ist_2}]^{-1} \right)^{st_2}, \quad (2.11)$$

$$\tilde{R}_{12}(u)^{ist_1, st_2} = \left([\{R_{12}^{-1}(u)\}^{st_1}]^{-1} \right)^{ist_1}. \quad (2.12)$$

For our case, the dual reflection equation (2.10) reduces to

$$\begin{aligned} & R_{12}(u_2 - u_1) K_1^+(u_1) R_{21}(-u_1 - u_2 + \eta) K_2^+(u_2) \\ &= K_2^+(u_2) R_{12}(-u_1 - u_2 + \eta) K_1^+(u_1) R_{21}(u_2 - u_1). \end{aligned} \quad (2.13)$$

In this paper we consider the generic non-diagonal K -matrices $K^-(u)$

$$K^-(u) = \begin{pmatrix} \zeta + (2c-1)u & 0 & 0 \\ 0 & \zeta - u & 2c_1u \\ 0 & 2c_2u & \zeta + u \end{pmatrix} \equiv \begin{pmatrix} k_{11}^- & 0 & 0 \\ 0 & k_{22}^- & k_{23}^- \\ 0 & k_{32}^- & k_{33}^- \end{pmatrix}. \quad (2.14)$$

Here the four boundary parameters c , c_1 , c_2 and ζ are not independent with each other, and satisfy a constraint

$$c^2 = c_1c_2 + c.$$

The dual non-diagonal reflection matrix $K^+(u)$ is given by

$$K^+(u) = K^-(-u + \eta/2) \Big|_{(\zeta, c, c_1, c_2) \rightarrow (\zeta', c', c'_1, c'_2)} \equiv \begin{pmatrix} k_{11}^+ & 0 & 0 \\ 0 & k_{22}^+ & k_{23}^+ \\ 0 & k_{32}^+ & k_{33}^+ \end{pmatrix}, \quad (2.15)$$

with the constraint

$$c'^2 = c'_1c'_2 + c'.$$

In order to show the integrability of the system, we first introduce the ‘‘row-to-row’’ monodromy matrices $T_0(u)$ and $\hat{T}_0(u)$

$$T_0(u) = R_{0L}(u - \theta_L)R_{0L-1}(u - \theta_{L-1}) \cdots R_{01}(u - \theta_1), \quad (2.16)$$

$$\hat{T}_0(u) = R_{10}(u + \theta_1)R_{20}(u + \theta_2) \cdots R_{L0}(u + \theta_L), \quad (2.17)$$

where $\{\theta_j, j = 1 \cdots L\}$ are the inhomogeneous parameters and L is the number of sites. The one-row monodromy matrices are the 3×3 matrices in the auxiliary space 0 and their elements act on the quantum space $\mathbf{V}^{\otimes L}$. The tensor product is in the graded space, so we can write

$$\begin{aligned} \left\{ [T(u)]^{ab} \right\}_{\beta_1 \cdots \beta_L}^{\alpha_1 \cdots \alpha_L} &= R_{0N}(u)_{c_L \beta_L}^{a \alpha_L} \cdots R_{0j}(u)_{c_j \beta_j}^{c_{j+1} \alpha_j} \cdots R_{01}(u)_{b \beta_1}^{c_2 \alpha_1} \\ &\times (-1)^{\sum_{j=2}^L (p(\alpha_j) + p(\beta_j)) \sum_{i=1}^{j-1} p(\alpha_i)}. \end{aligned} \quad (2.18)$$

For the system with open boundaries, we need to define the double-row monodromy matrix

$$\mathbb{T}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u), \quad (2.19)$$

which satisfies the similar relation as (2.9), in terms of matrix entries, they are

$$\begin{aligned} R(u - \lambda)_{b_1 b_2}^{a_1 a_2} \mathbb{T}(u)_{c_1}^{b_1} R(u + \lambda)_{c_2 d_1}^{b_2 c_1} \mathbb{T}(\lambda)_{d_2}^{c_2} (-1)^{(p(b_1) + p(c_1))p(b_2)} \\ = \mathbb{T}(\lambda)_{b_2}^{a_2} R(u + \lambda)_{b_1 c_2}^{a_1 b_2} \mathbb{T}(u)_{c_1}^{b_1} R(u - \lambda)_{d_2 d_1}^{c_2 c_1} (-1)^{(p(b_1) + p(c_1))p(c_2)}. \end{aligned} \quad (2.20)$$

Then the transfer matrix of the system is constructed as

$$t(u) = \text{str}_0 \{ K_0^+(u) \mathbb{T}_0(u) \} = \sum_{\alpha=1}^3 (-1)^{p(\alpha)} [K_0^+(u) \mathbb{T}_0(u)]_{\alpha\alpha}. \quad (2.21)$$

By using the (2.8), (2.9) and (2.10), we can prove the commutativity of $t(u)$. (For further details about the commuting transfer matrix with boundaries for graded case, we refer the reader to [14, 32, 33]). The Hamiltonian (1.1) can be constructed by taking the derivative of the logarithm of the transfer matrix $t(u)$ of the system

$$H = -\frac{\eta}{2} \frac{\partial \ln t(u)}{\partial u} \Big|_{u=0, \{\theta_j=0\}} + \frac{\eta(2c-1)}{2\zeta} + \frac{\zeta'}{(c'-1/2)\eta - \zeta'} - \mu\hat{N} + L - 1, \quad (2.22)$$

with the parameters chosen as follows:

$$\begin{aligned} \xi_1 &= -\frac{\eta}{2\zeta}(1-2c), & h_1^z &= -\frac{\eta}{2\zeta}, & h_1^- &= -\frac{\eta}{2\zeta}c_2, & h_1^+ &= -\frac{\eta}{2\zeta}c_1, & \xi_L &= \frac{(c'-1/2)\eta}{(c'-1/2)\eta - \zeta'}, \\ h_L^z &= \frac{-\eta/2}{(c'-1/2)\eta - \zeta'}, & h_L^- &= \frac{-\eta c_2'/2}{(c'-1/2)\eta - \zeta'} & \text{and} & & h_L^+ &= \frac{-\eta c_1'/2}{(c'-1/2)\eta - \zeta'}. \end{aligned}$$

3 Nested algebraic Bethe ansatz

The block-diagonal structure of the K -matrix (2.14) permits us to use the nested algebraic Bethe ansatz to construct the associated Bethe state and obtain the eigenvalue as follows. We first represent the double-row monodromy matrix $\mathbb{T}_0(u) = T_0(u)K_0^-(u)\hat{T}_0(u)$ in the form

$$\mathbb{T}_0(u) = \begin{pmatrix} A(u) & B_1(u) & B_2(u) \\ C_1(u) & D_{11}(u) & D_{12}(u) \\ C_2(u) & D_{21}(u) & D_{22}(u) \end{pmatrix}. \quad (3.1)$$

Then the transfer matrix can be expressed by

$$t(u) = \left[k_{11}^+(u)A(u) - \sum_{i,j=1}^2 k_{i+1,j+1}^+(u)D_{ji}(u) \right], \quad (3.2)$$

where k_{ij}^\pm is the K^\pm matrix element in the i th row and j th column.

Now we use the graded version of the nested algebraic Bethe ansatz method to obtain the eigenvalues of the transfer matrix (3.2). For this purpose, we first define the reference state $|\Psi_0\rangle$ as

$$|\Psi_0\rangle = \bigotimes_{j=1}^L |0\rangle_j, \quad |0\rangle_j = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}. \quad (3.3)$$

From the relations (2.21), (3.1) and (3.3), the elements of matrix $\mathbb{T}_0(u)$ acting on the reference state $|\Psi_0\rangle$ give rise to

$$\begin{aligned} A(u) |\Psi_0\rangle &= k_{11}^-(u)a_0(u) |\Psi_0\rangle, \\ D_{11}(u) |\Psi_0\rangle &= \left\{ \frac{\eta}{2u+\eta} k_{11}^-(u)a_0(u) + \left[k_{22}^-(u) - \frac{\eta}{2u+\eta} k_{11}^-(u) \right] b_0(u) \right\} |\Psi_0\rangle, \\ D_{22}(u) |\Psi_0\rangle &= \left\{ \frac{\eta}{2u+\eta} k_{11}^-(u)a_0(u) + \left[k_{33}^-(u) - \frac{\eta}{2u+\eta} k_{11}^-(u) \right] b_0(u) \right\} |\Psi_0\rangle, \\ D_{12}(u) |\Psi_0\rangle &= k_{23}^-(u)b_0(u) |\Psi_0\rangle, \\ D_{21}(u) |\Psi_0\rangle &= k_{32}^-(u)b_0(u) |\Psi_0\rangle, \\ B_i(u) |\Psi_0\rangle &\neq 0, \quad C_i(u) |\Psi_0\rangle = 0, \quad i = 1, 2, \end{aligned} \quad (3.4)$$

where

$$b_0(u) = \prod_{j=1}^L (u - \theta_j)(u + \theta_j), \quad a_0(u) = b_0(u + \eta). \quad (3.5)$$

The operators $B_1(u)$ and $B_2(u)$ acting on the reference state give nonzero values, and can be regarded as the creation operators of the eigenstates of the system. Following the procedure of the nested algebraic Bethe ansatz, the eigenstates of the transfer matrix can be constructed as

$$|u_1, \dots, u_M; \mathcal{F}\rangle = B_{a_1}(u_1)B_{a_2}(u_2) \dots B_{a_M}(u_M) \mathcal{F}^{a_1 a_2 \dots a_M} |\Psi_0\rangle, \quad (3.6)$$

where we have used the convention that the repeated indices indict the sum over the values 1,2, and $\mathcal{F}^{a_1 \dots a_n}$ is a function of the spectral parameters u_j . Moreover, the coefficients $\mathcal{F}^{a_1 \dots a_n}$ are actually the vector components of the nested Bethe state (see below (4.27)). As the transfer matrix (3.2) acting on the assumed states (3.6), we should exchange the positions of the operators $A(u)$, $D_{ij}(u)$ and the operators $B_{a_j}(u_j)$. With the help of the reflection equation (2.20) and the Yang-Baxter equation (2.8), we can derive commutation relations

$$B_i(u)B_j(v) = -B_k(v)B_l(u) \frac{r_{lk}^{ij}(u-v)}{u-v+\eta}, \quad (3.7)$$

$$\begin{aligned} A(u)B_j(v) &= \frac{(u-v-\eta)(u+v)}{(u+v+\eta)(u-v)} B_j(v)A(u) - \frac{\eta}{u+v+\eta} B_i(u)\tilde{D}_{ij}(v) \\ &\quad + \frac{2v\eta}{(u-v)(2v+\eta)} B_j(u)A(v), \end{aligned} \quad (3.8)$$

$$\begin{aligned} \tilde{D}_{ij}(u)B_k(v) &= \frac{r_{ef}^{id}(u+v+\eta)r_{kj}^{fg}(u-v)}{(u+v+\eta)(u-v)} B_d(v)\tilde{D}_{eg}(u) + \eta \frac{r_{ej}^{id}(2u+\eta)}{(2u+\eta)(u-v)} B_d(u)\tilde{D}_{ek}(v) \\ &\quad - \frac{2v\eta}{2u+\eta} \frac{r_{kj}^{id}(2u+\eta)}{(2v+\eta)(u+v+\eta)} B_d(u)A(v), \end{aligned} \quad (3.9)$$

where $r_{ij} = u + \eta \mathbb{P}_{ij}$, $\mathbb{P}_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = (-1)^{p(\alpha_1)p(\beta_2)} \delta_{\alpha_1 \beta_2} \delta_{\beta_1 \alpha_2}$ with the grading $p^{(1)} = p^{(2)} = 1$, and

$$\tilde{D}_{ij}(u) = D_{ij}(u) - \delta_{ij} \frac{\eta}{2u+\eta} A(u). \quad (3.10)$$

Acting the transfer matrix $t(u)$ on the state $|\Psi\rangle$ and repeatedly using the commutation relations (3.8) and (3.9), we obtain

$$t(u) |u_1, \dots, u_M; \mathcal{F}\rangle = \Lambda(u) |u_1, \dots, u_M; \mathcal{F}\rangle + \text{unwanted terms}, \quad (3.11)$$

where the corresponding eigenvalue $\Lambda(u)$ is

$$\begin{aligned} \Lambda(u) &= \left[-\frac{\eta}{2u+\eta} \sum_{i=2}^3 k_{ii}^+(u) + k_{11}^+(u) \right] k_{11}^-(u) a_0(u) \prod_{i=1}^M \frac{(u-u_i-\eta)}{(u-u_i)} \frac{(u+u_i)}{(u+u_i+\eta)} \\ &\quad - b_0(u) \prod_{i=1}^M \frac{1}{(u-u_i)(u+u_i+\eta)} \hat{\Lambda}(u, \{u_j\}), \end{aligned} \quad (3.12)$$

and $\hat{\Lambda}(u, \{u_j\})$ is the eigenvalue of the nested transfer matrix $\hat{t}(u, \{u_j\})$ given by

$$\hat{t}(u, \{u_j\}) = \frac{2u}{2u + \eta} \text{tr}_0 \left[\bar{K}_0^+(u) r_{01}(u + u_1 + \eta) \dots r_{0M}(u + u_M + \eta) \right. \\ \left. \times \bar{K}_0^-(u) r_{M0}(u - u_M) \dots r_{10}(u - u_1) \right], \quad (3.13)$$

$$\bar{K}^+(u) = \begin{pmatrix} k_{22}^+(u) & k_{23}^+(u) \\ k_{32}^+(u) & k_{33}^+(u) \end{pmatrix}, \quad (3.14)$$

$$\bar{K}^-(u) = \frac{2u + \eta}{2u} \begin{pmatrix} k_{22}^-(u) - \frac{\eta}{2u + \eta} k_{11}^-(u) & k_{23}^-(u) \\ k_{32}^-(u) & k_{33}^-(u) - \frac{\eta}{2u + \eta} k_{11}^-(u) \end{pmatrix}, \quad (3.15)$$

namely,

$$\hat{t}(u, \{u_j\}) |\mathcal{F}\rangle = \hat{\Lambda}(u, \{u_j\}) |\mathcal{F}\rangle. \quad (3.16)$$

The vector components $\{F^{a_1 \dots a_n}\}$ allow us to reconstruct the associated Bethe state (3.6), while the eigenvalue $\hat{\Lambda}(u, \{u_j\})$ gives rise to the associated eigenvalue (3.12) of the transfer matrix $t(u)$ of the model. We shall determine the eigenvalue $\hat{\Lambda}(u, \{u_j\})$ and the corresponding eigenstate $|\mathcal{F}\rangle$ in the next section. The condition that the unwanted terms should be zero gives rise to that the M Bethe roots must satisfy the associated Bethe ansatz equations (BAEs)

$$1 = \frac{K^{(1)}(u_k) a_0(u_k) Q^{(1)}(u_k - \eta)}{(2u_k + \eta) b_0(u_k) \hat{\Lambda}(u_k, \{u_j\})}, \quad k = 1 \dots M, \quad (3.17)$$

where

$$Q^{(1)}(u) = \prod_{i=1}^M (u - u_i)(u + u_i + \eta), \quad (3.18)$$

$$K^{(1)}(u) = \left((2 - 4c')u^2 + 2\zeta'u - \eta\zeta' - \frac{1}{2}\eta^2 + \eta^2 c' \right) (\zeta + (2c - 1)u). \quad (3.19)$$

Some remarks are in order. It is easy to check that the nested Bethe state $|u_1, \dots, u_M; \mathcal{F}\rangle$ given by (3.6) and the eigenvalue $\Lambda(u)$ given by (3.12) both have well-defined homogeneous limit (i.e., $\theta_j \rightarrow 0$). This implies that in the homogeneous limit, the resulting Bethe states and the eigenvalue give rise to the eigenstate and the corresponding eigenvalue of the super $t - J$ model described by the Hamiltonian (1.1).

4 Reduced spectrum problem

In the previous section, we have reduced searching eigenstates of the original transfer matrix $t(u)$ (2.21) into the spectrum problem (3.16) of the nested transfer matrix $\hat{t}(u, \{u_j\})$ given by (3.13). Now, we are in the position to calculate the eigenvalue $\hat{\Lambda}(u, \{u_j\})$ and the corresponding eigenstate $|\mathcal{F}\rangle$ of the nested transfer matrix $\hat{t}(u, \{u_j\})$ which allows us to reconstruct the Bethe state (3.6) of the supersymmetric $t - J$ model. Because the reflection matrices (3.14) and (3.15) have the off-diagonal elements. The traditional algebraic Bethe ansatz is invalid [22] due to the fact that the system doesnot have the obvious reference

state. Thanks to the works [34–37], we can solve the spectrum problem (3.16) as follows. For simplicity, let $\lambda = u + \frac{1}{2}\eta$ and $\lambda_j = u_j + \frac{1}{2}\eta$. We recognize the $\hat{t}(u, \{\lambda_j\})$ as the transfer matrix of the open spin-1/2 XXX chain of length M with non-diagonal boundary terms. Following the procedure in [34]

$$\begin{aligned}\hat{t}(\lambda, \{\lambda_j\}) &= \frac{2\lambda - \eta}{2\lambda} \bar{t}(\lambda, \{\lambda_j\}) \\ &= \frac{2\lambda - \eta}{2\lambda} \text{Tr}_{\bar{0}} \left\{ \bar{K}_0^+(\lambda) \bar{T}_0(\lambda, \{\lambda_j\}) \bar{K}_0^-(\lambda) \hat{\bar{T}}_0(\lambda, \{\lambda_j\}) \right\},\end{aligned}\quad (4.1)$$

where

$$\begin{aligned}\bar{T}_0(\lambda, \{\lambda_j\}) &= r_{\bar{0}1}(\lambda + \lambda_1) \dots r_{\bar{0}M}(\lambda + \lambda_M), \\ \hat{\bar{T}}_0(\lambda, \{\lambda_j\}) &= r_{M\bar{0}}(\lambda - \lambda_M) \dots r_{1\bar{0}}(\lambda - \lambda_1),\end{aligned}\quad (4.2)$$

$$\bar{K}^+(\lambda) = \begin{pmatrix} \zeta' + \lambda - \eta & 2c'_1(-\lambda + \eta) \\ 2c'_2(-\lambda + \eta) & \zeta' - \lambda + \eta \end{pmatrix},\quad (4.3)$$

$$\bar{K}^-(\lambda) = \begin{pmatrix} -\lambda + 1/2\eta + \zeta - c\eta & 2c_1\lambda \\ 2c_2\lambda & \lambda + 1/2\eta + \zeta - c\eta \end{pmatrix}.\quad (4.4)$$

We have checked that (4.4) is the solution of the normal RE of the following form

$$\begin{aligned}r_{12}(u_1 - u_2) \bar{K}_1^-(u_1) r_{21}(u_1 + u_2) \bar{K}_2^-(u_2) \\ = \bar{K}_2^-(u_2) r_{12}(u_1 + u_2) \bar{K}_1^-(u_1) r_{21}(u_1 - u_2),\end{aligned}\quad (4.5)$$

and that (4.3) satisfies the dual one. The r -matrix possesses the properties

$$\text{Initial condition:} \quad r_{12}(0) = -\eta \bar{P}_{12},\quad (4.6)$$

$$\text{Unitary relation:} \quad r_{12}(\lambda) r_{21}(-\lambda) = \bar{\rho}_1(\lambda) \times \text{id},\quad (4.7)$$

$$\text{Crossing Unitary relation:} \quad r_{12}^{t_1}(\lambda) r_{21}^{t_1}(-\lambda + 2\eta) = \bar{\rho}_2(\lambda) \times \text{id},\quad (4.8)$$

$$\text{PT-symmetry:} \quad r_{21}(\lambda) = r_{12}^{t_1 t_2}(\lambda).\quad (4.9)$$

Here $r_{21}(\lambda) = \bar{P}_{12} r_{12}(\lambda) \bar{P}_{12}$ and $\bar{P}_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = \delta_{\alpha_1 \beta_2} \delta_{\beta_1 \alpha_2}$. The functions $\bar{\rho}_1(\lambda)$ and $\bar{\rho}_2(\lambda)$ are given by

$$\bar{\rho}_1(\lambda) = -(\lambda - \eta)(\lambda + \eta), \quad \bar{\rho}_2(\lambda) = -\lambda(\lambda - 2\eta).\quad (4.10)$$

From the definition (4.1), we know that the eigenvalue $\bar{\Lambda}(\lambda)$ of the transfer matrix $\bar{t}(\lambda, \{\lambda_j\})$ is a polynomial of λ and satisfies the relations:

$$\text{Crossing symmetry:} \quad \bar{\Lambda}(\lambda) = \bar{\Lambda}(-\lambda + \eta),\quad (4.11)$$

$$\text{Asymptotic behavior:} \quad \bar{\Lambda}(\lambda) \sim (-2 - 4c_1 c'_2 - 4c'_1 c_2) u^{2M+2}, \quad \lambda \rightarrow \infty,\quad (4.12)$$

and

$$\bar{\Lambda}(\lambda_j) \bar{\Lambda}(\lambda_j + \eta) = \frac{\Delta_q(\lambda_j)}{(\eta - 2\lambda_j)(\eta + 2\lambda_j)}, \quad j = 1, \dots, M,\quad (4.13)$$

where

$$\begin{aligned} \Delta_q(\lambda) &= (2\eta + 2\lambda)(2\eta - 2\lambda) \\ &\times (\xi'^2 - (1 + 4c'_1 c'_2)\lambda^2) \left((\xi + \frac{1}{2}\eta - c\eta)^2 - (1 + 4c_1 c_2)\lambda^2 \right) \\ &\times \prod_{j=1}^M (\lambda + \lambda_j - \eta)(\lambda - \lambda_j - \eta)(\lambda - \lambda_j + \eta)(\lambda + \lambda_j + \eta). \end{aligned} \quad (4.14)$$

Some special points can also be calculated directly by using the properties of the r -matrix and the reflection matrices $\bar{K}^{(\pm)}(u)$ as:

$$\bar{\Lambda}(0) = \prod_{l=1}^M \bar{\rho}_1(\lambda_l) \text{tr}\{\bar{K}^+(0)\} \bar{K}^-(0) \times \text{id}, \quad (4.15)$$

$$\bar{\Lambda}(\eta) = \prod_{l=1}^M \bar{\rho}_2(\lambda_l + \eta) \text{tr}\{\bar{K}^-(\eta)\} \bar{K}^+(\eta) \times \text{id}. \quad (4.16)$$

It is remarked that the above relations were derived independently by the Separation of Variables [38]. These conditions (4.11)–(4.16) allow us to construct the eigenvalue $\hat{\Lambda}(\lambda)$ in terms of an inhomogeneous $T - Q$ relation as [34, 35]

$$\begin{aligned} \hat{\Lambda}(\lambda) &= \frac{(2\lambda - \eta)}{2\lambda} \bar{\Lambda}(\lambda) \\ &= \frac{(2\lambda - 2\eta)}{2\lambda} K^{(2)}(\lambda) \bar{a}(\lambda) \frac{Q^{(2)}(\lambda + \eta)}{Q^{(2)}(\lambda)} + K^{(3)}(\lambda) \bar{d}(\lambda) \frac{Q^{(2)}(\lambda - \eta)}{Q^{(2)}(\lambda)} \\ &\quad + (2\lambda - \eta)(2\lambda - 2\eta) \bar{a}(\lambda) \bar{a}(-\lambda + \eta) \frac{h}{Q^{(2)}(\lambda)}, \end{aligned} \quad (4.17)$$

where

$$\bar{a}(\lambda) = \prod_{j=1}^M (\lambda + \lambda_j - \eta)(\lambda - \lambda_j - \eta), \quad (4.18)$$

$$\bar{d}(\lambda) = \prod_{j=1}^M (\lambda - \lambda_j)(\lambda + \lambda_j), \quad (4.19)$$

$$\begin{aligned} K^{(2)}(\lambda) &= (-\sqrt{1 + 4(c'^2 - c')}\lambda + \zeta') \\ &\quad \times (\sqrt{1 + 4(c^2 - c)}\lambda + \zeta + 1/2\eta - c\eta), \end{aligned} \quad (4.20)$$

$$\begin{aligned} K^{(3)}(\lambda) &= (\sqrt{1 + 4(c'^2 - c')}(\lambda - \eta) + \zeta') \\ &\quad \times (\sqrt{1 + 4(c^2 - c)}(-\lambda + \eta) + \zeta + 1/2\eta - c\eta), \end{aligned} \quad (4.21)$$

$$h = \frac{1}{2} \left(-1 - 2(c'_1 c_2 + c'_2 c_1) + \sqrt{(1 + 4c'_1 c'_2)(1 + 4c_1 c_2)} \right), \quad (4.22)$$

$$Q^{(2)}(\lambda) = \prod_{j=1}^M (\lambda - w_j)(\lambda + w_j - \eta) \stackrel{\text{def}}{=} \prod_{j=1}^M \left(\lambda - \nu_j - \frac{1}{2}\eta \right) \left(\lambda + \nu_j - \frac{1}{2}\eta \right). \quad (4.23)$$

Such parametrization obviously satisfies the crossing symmetry (4.11), asymptotic behavior (4.12), production identity (4.13) and the values of the special points (4.15) and (4.16). To ensure $\hat{\Lambda}(\lambda)$ to be a polynomial, the residues of $\hat{\Lambda}(\lambda)$ at the poles w_j must vanish, i.e., the M Bethe roots must satisfy the BAEs

$$(2w_j - 2\eta)K^{(2)}(w_j)\bar{a}(w_j)Q^{(2)}(w_j + \eta) + 2w_jK^{(3)}(w_j)\bar{d}(w_j)Q^{(2)}(w_j - \eta) + 2w_j(2w_j - \eta)(2w_j - 2\eta)\bar{a}(w_j)\bar{a}(-w_j + \eta)h = 0. \quad (4.24)$$

Now, we construct the eigenstates $|\mathcal{F}\rangle$ of the nested transfer matrix $\hat{t}(\lambda)$. Following the ideas in [36, 37], we first introduce two transformation matrices $g^{(\pm)}$

$$g^{(-)} = \begin{pmatrix} -1 & \frac{2c_1}{1-\sqrt{1+4c_1c_2}} \\ 1 & \frac{-2c_1}{1+\sqrt{1+4c_1c_2}} \end{pmatrix}, \quad g^{(+)} = \begin{pmatrix} -m & \sqrt{1+mn}-1 \\ -m & -\sqrt{1+mn}-1 \end{pmatrix}, \quad (4.25)$$

where $m = \frac{-4c_1c_2 - (2c_1c_2' - 2c_1'c_2)\sqrt{1+4c_1c_2} + 2c_1c_2' + 2c_1'c_2}{(1+2c_1c_2' + 2c_1'c_2)(\sqrt{1+4c_1c_2} + 1)}$ and $n = \frac{-4c_1c_2 - (2c_1'c_2 - 2c_1c_2')\sqrt{1+4c_1c_2} + 2c_1'c_2 + 2c_1c_2'}{(1+2c_1'c_2' + 2c_1c_2)(\sqrt{1+4c_1c_2} - 1)}$. The gauge matrices diagonalize the nested K -matrix $\bar{K}^-(\lambda)$ given by (4.3) and the matrix $g^{(-)}\bar{K}^+(\lambda)\{g^{(-)}\}^{-1}$ respectively, namely,

$$\begin{aligned} & g^{(+)}\{g^{(-)}\bar{K}^+(\lambda)\{g^{(-)}\}^{-1}\}\{g^{(+)}\}^{-1} \\ &= -\frac{1+2c_1c_2'+2c_1'c_2}{\sqrt{1+4c_1c_2}} \begin{pmatrix} \sqrt{1+mn}(\lambda-\eta) - \frac{\sqrt{1+4c_1c_2}}{1+2c_1c_2'+2c_1'c_2}\xi' & 0 \\ 0 & -\sqrt{1+mn}(\lambda-\eta) - \frac{\sqrt{1+4c_1c_2}}{1+2c_1c_2'+2c_1'c_2}\xi' \end{pmatrix}, \\ & g^{(-)}\bar{K}^-(\lambda)\{g^{(-)}\}^{-1} = \begin{pmatrix} 1/2\eta + \xi - c\eta + \lambda\sqrt{1+4c_1c_2} & 0 \\ 0 & 1/2\eta + \xi - c\eta - \lambda\sqrt{1+4c_1c_2} \end{pmatrix}. \end{aligned} \quad (4.26)$$

With the gauge transformation, we can introduce the gauged monodromy matrix $\mathbb{U}(\lambda)$

$$\mathbb{U}(\lambda) = g^{(+)}\bar{T}(\lambda)\{g^{(-)}\bar{K}^-(\lambda)\{g^{(-)}\}^{-1}\}\hat{T}(\lambda)\{g^{(+)}\}^{-1} = \begin{pmatrix} \mathbb{A}(\lambda) & \mathbb{B}(\lambda) \\ \mathbb{C}(\lambda) & \mathbb{D}(\lambda) \end{pmatrix}.$$

Then it was shown in [36, 37] that the eigenstate $|\mathcal{F}\rangle$ in (3.16) can be expressed as

$$|\mathcal{F}\rangle = \bigotimes_{j=1}^M \{g_{(j)}^{(-)}\}^{-1} \prod_{j=1}^M \mathbb{B}(w_j)|0\rangle = \sum_{a_i=1,2} \mathcal{F}^{a_1 a_2 \dots a_M} |a_1, \dots, a_M\rangle, \quad (4.27)$$

where the reference state $|0\rangle$ is

$$|0\rangle = \bigotimes_{j=1}^M |1\rangle_j, \quad |1\rangle_j = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad (4.28)$$

provided that the parameters $\{w_j|j=1, \dots, M\}$ satisfy the BAEs (4.24). The corresponding vector components $\{\mathcal{F}^{a_1 a_2 \dots a_M}\}$ allow us to reconstruct the eigenstates $|u_1, \dots, u_M; \mathcal{F}\rangle$ given by (3.6) of the original system.¹

¹We have numerically checked, for small-site cases (such as $L=2,3$), that the states constructed by (3.6) with vector components $\{\mathcal{F}^{a_1 a_2 \dots a_M}\}$ given by (4.27) give rise to the complete set of eigenstates of the transfer matrix $t(u)$ given by (2.21), provided that the parameters $\{u_j\}$ and $\{v_j\}$ (or $\{w_j\}$) satisfy the BAEs (5.2)–(5.3).

5 Nested inhomogeneous $T - Q$ relation

Now we are ready to write out the eigenvalues $\Lambda(u)$ of the transfer matrices $t(u)$ in terms of some inhomogeneous $T - Q$ relation with the help of (3.12) and (4.17) as²

$$\begin{aligned} \Lambda(u) &= \frac{1}{(2u + \eta)} K^{(1)}(u) a_0(u) \frac{Q^{(1)}(u - \eta)}{Q^{(1)}(u)} \\ &\quad - \frac{(2u - \eta)}{(2u + \eta)} K^{(2)} \left(u + \frac{1}{2}\eta \right) b_0(u) \frac{Q^{(1)}(u - \eta) Q^{(2)}(u + \frac{3}{2}\eta)}{Q^{(1)}(u) Q^{(2)}(u + \frac{1}{2}\eta)} \\ &\quad - K^{(3)}(u + \frac{1}{2}\eta) b_0(u) \frac{Q^{(2)}(u - \frac{1}{2}\eta)}{Q^{(2)}(u + \frac{1}{2}\eta)} - 2u(2u - \eta) b_0(u) \frac{hQ^{(1)}(u - \eta)}{Q^{(2)}(u + \frac{1}{2}\eta)}, \end{aligned} \quad (5.1)$$

where the $2M$ Bethe roots must satisfy the BAEs (3.17) and (4.24), namely,

$$\begin{aligned} 1 + \frac{(2\nu_l - \eta) K^{(2)}(\nu_l + \frac{1}{2}\eta) Q^{(1)}(\nu_l - \eta) Q^{(2)}(\nu_l + \frac{3}{2}\eta)}{(2\nu_l + \eta) K^{(3)}(\nu_l + \frac{1}{2}\eta) Q^{(1)}(\nu_l) Q^{(2)}(\nu_l - \frac{1}{2}\eta)} \\ = -h(2\nu_l)(2\nu_l - \eta) \frac{Q^{(1)}(\nu_l - \eta)}{K^{(3)}(\nu_l + \frac{1}{2}\eta) Q^{(2)}(\nu_l - \frac{1}{2}\eta)}, \quad l = 1 \dots M, \end{aligned} \quad (5.2)$$

$$1 = (2u_k - \eta) \frac{K^{(2)}(u_k + \frac{1}{2}\eta) b_0(u_k) Q^{(2)}(u_k + \frac{3}{2}\eta)}{K^{(1)}(u_k) a_0(u_k) Q^{(2)}(u_k + \frac{1}{2}\eta)}, \quad k = 1 \dots M. \quad (5.3)$$

In the homogeneous limit, the corresponding $T - Q$ relation and associated BAEs become (5.1) and (5.2)–(5.3) by setting $\theta_j = 0$, $j = 1, \dots, N$. Therefore the energy of the Hamiltonian (1.1) reads

$$\begin{aligned} E &= -\frac{\eta}{2} \frac{\partial \ln \Lambda(u)}{\partial u} \Big|_{u=0, \{\theta_j\}=0} + \frac{\eta(2c - 1)}{2\zeta} + \frac{\zeta'}{(\zeta' - 1/2)\eta - \zeta'} - \mu M + L - 1 \\ &= -\sum_{k=1}^M \frac{\eta^2}{u_k(u_k + \eta)} - \mu M, \end{aligned} \quad (5.4)$$

where the $2M$ parameters $\{u_j | j = 1, \dots, M\}$ and $\{v_j | j = 1, \dots, M\}$ satisfy the resulting BAEs (5.2) and (5.3). Here we present the results for the $L = 2$ and $L = 3$ cases: the numerical solutions of the BAEs are shown in table 1 and table 2, which indicated that the eigenvalues are identical with the results we get from the exact diagonalization of the Hamiltonian (1.1). Numerical results for the small-site cases suggest that the spectrum obtained by the nested BAEs (5.2)–(5.3) is complete.

²Although the inhomogeneous $T - Q$ relation given by (5.1) is different from that obtained in [28], each of them gives rise to the complete set of eigenvalues of the transfer matrix. The $T - Q$ relation (5.1) takes advantage over one in [28] is that it leads to a simple form (4.27) of Bethe states of the reduced spectrum problem (3.16).

u_1	u_2	ν_1	ν_2	E_n	n
$-0.1000 - 1.6602i$	$0.1004 - 0.0000i$	$0.1264 + 3.3108i$	$0.1264 - 3.3108i$	-5.312156	1
$-0.1000 - 0.2048i$	$0.1004 + 0.0000i$	$0.8095 - 3.4060i$	$0.8095 + 3.4060i$	-4.555656	2
$0.1005 + 0.0000i$	—	$0.0000 - 3.3070i$	—	-3.325040	3
$-0.1000 - 1.6539i$	$-0.1000 - 0.2053i$	$0.0255 + 3.3085i$	$0.0255 - 3.3085i$	-3.218186	4
$-0.1000 - 3.7095i$	$-0.1000 - 0.1000i$	$0.0000 - 4.6812i$	$0.1496 - 0.0000i$	-1.996355	5
$-0.1000 - 2.3555i$	—	$0.0000 - 3.7060i$	—	-1.992804	6
$-0.1000 + 0.2040i$	—	$-0.0000 + 3.3148i$	—	-1.225154	7
—	—	—	—	0	8
$-0.1000 - 0.0999i$	—	$-0.1496 - 0.0000i$	—	0.001822	9

Table 1. Solutions of BAEs (5.2) and (5.3) where $L = 2$ with the parameters $\eta = 0.2, \mu = 2, \zeta = 0.1, c = 0.1, c_1 = -0.5, \zeta' = -0.5, c' = -0.3$ and $c'_1 = -0.7$ for the case of E_n is the corresponding eigenenergy. The energy E_n calculated from (5.4) is the same as that from the exact diagonalization of the Hamiltonian (1.1).

6 Concluding remarks

In this paper, we have studied the one-dimensional supersymmetric $t - J$ model with the most generic integrable boundary condition, which is described by the Hamiltonian (1.1) and the corresponding integrable boundary terms are associated with the most generic non-diagonal K -matrices given by (2.14)–(2.15). By combining the algebraic Bethe ansatz and the off-diagonal Bethe ansatz, we construct the eigenstates of the transfer matrix in terms of the nested Bethe states given by (3.6) and (4.27), which have well-defined homogeneous limit. The corresponding eigenvalues are given in terms of the inhomogeneous $T - Q$ relation (5.1) and the associated BAEs (5.2)–(5.3). The exact solution of this paper provides basis for further analyzing the thermodynamic properties and correlation functions of the model. These are under investigation and results will be reported elsewhere.

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u_1	u_2	u_3	ν_1	ν_2	ν_3	E_n	n
0.1183 - 0.0000i	0.8870 + 0.0000i	0.4749 - 0.0000i	-0.0000 + 2.1856i	1.5149 - 4.0404i	1.5149 + 4.0404i	-7.229050	1
0.1073 - 0.0000i	-0.1000 - 0.1218i	-0.1000 + 2.3587i	0.0000 - 0.1091i	0.9332 - 4.1573i	0.9332 + 4.1573i	-5.595946	2
0.1183 - 0.0000i	0.4290 - 0.0000i	-	0.8379 - 3.4029i	0.8379 + 3.4029i	-	-5.210221	3
1.4165 - 0.0000i	-0.3183 - 0.0000i	-	0.9773 - 3.3657i	0.9773 + 3.3657i	-	-5.079400	4
0.1182 - 0.0000i	0.4997 - 0.0000i	-0.1000 - 0.0701i	-0.0000 + 1.9912i	1.4407 - 4.0634i	1.4407 + 4.0634i	-4.495822	5
-0.1000 + 0.0701i	0.1183 + 0.0000i	0.8474 + 0.0000i	0.0000 - 2.1041i	1.4824 - 4.0500i	1.4824 + 4.0500i	-4.426045	6
0.0005 - 0.0704i	0.0005 + 0.0704i	-0.1000 + 2.3565i	-0.0000 + 0.0684i	0.9328 - 4.1576i	0.9328 + 4.1576i	-4.253561	7
1.4210 + 0.0000i	0.4273 + 0.0000i	-	0.9950 - 3.3623i	0.9950 + 3.3623i	-	-4.166597	8
-0.1000 - 0.1216i	0.1073 + 0.0000i	-	0.0000 - 3.8739i	0.0000 - 0.1092i	-	-3.598898	9
-0.1000 + 0.0701i	0.8813 - 0.0000i	0.4760 + 0.0000i	1.5100 - 4.0418i	1.5100 + 4.0418i	0.0000 + 2.1735i	-3.484233	10
0.1183 + 0.0000i	-	-	0.0000 + 3.3063i	-	-	-3.061679	11
-0.1000 - 0.1733i	-0.1000 - 2.8910i	-	-0.0000 + 4.5408i	-0.0000 - 0.0711i	-	-2.996075	12
-0.1000 - 0.0601i	-0.1000 + 2.3432i	-0.1000 - 0.3120i	0.9298 - 4.1596i	0.9298 + 4.1596i	0.1995 - 0.0000i	-2.682705	13
-0.1000 + 0.0700i	0.1183 + 0.0000i	-	0.8135 - 3.4056i	0.8135 + 3.4056i	-	-2.378366	14
0.0004 - 0.0699i	0.0004 + 0.0699i	-	0.0000 - 3.8760i	0.0000 - 0.0680i	-	-2.256786	15
0.4178 - 0.0000i	-	-	0.0000 - 3.2852i	-	-	-2.154975	16
1.7975 - 0.0000i	-	-	0.0000 - 2.9281i	-	-	-2.011140	17
-0.1000 - 0.0577i	-0.1000 + 0.1731i	-0.1000 + 4.5434i	0.1682 - 0.0768i	0.1682 + 0.0768i	0.0000 - 5.7342i	-1.997344	18
-0.1000 - 0.0700i	-0.6290 + 0.0000i	-	0.8334 + 3.4033i	0.8334 - 3.4033i	-	-1.464602	19
-0.1000 + 0.0700i	1.4155 + 0.0000i	-	0.9731 - 3.3664i	0.9731 + 3.3664i	-	-1.333802	20
-0.1000 + 0.1730i	-	-	0.0000 + 0.0709i	-	-	-0.998177	21
-0.1000 - 0.0577i	-0.1000 + 2.8833i	-	0.1777 + 0.0000i	-0.0000 - 4.5376i	-	-0.995481	22
-0.1000 - 0.0601i	-0.1000 + 0.3107i	-	-0.0000 - 3.8889i	-0.1993 - 0.0000i	-	-0.686247	23
-	-	-	-	-	-	0	24
-0.1000 + 0.1731i	-0.1000 - 0.0577i	-	0.1682 - 0.0768i	0.1682 + 0.0768i	-	0.001770	25
-0.1000 + 0.0700i	-	-	0.0000 + 3.3113i	-	-	0.684187	26
-0.1000 - 0.0577i	-	-	0.1776 + 0.0000i	-	-	1.000607	27

Table 2. Solutions of BAEs (5.2) and (5.3) where $L = 3$ with the parameters $\eta = 0.2, \mu = 2, \zeta = 0.1, c = 0.1, c_1 = -0.5, \zeta' = -0.5, c' = -0.3$ and $c'_1 = -0.7$ for the case of E_n is the corresponding eigenenergy. The energy E_n calculated from (5.4) is the same as that from the exact diagonalization of the Hamiltonian (1.1).

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