Aspects of Compactification and Holography in String Theory

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Abstract

In this thesis, we explore several aspects of compactification and holography in open and closed string theories. We begin by reviewing the necessary background, tools and motivation for studying brane-worlds and noncommutative open string theory. We then give a short review of a large class of warped string geometries obtained via F-theory compactified on Calabi-Yau fourfolds. These theories upon reduction to 5 dimensions give consistent supersymmetric realizations of the Randall-Sundrum compactification scenario for brane-worlds. We also show how the AdS/CFT correspondence can be applied within the context of warped compactification to give a new perspective on the physics of gravitational collapse. Next, we construct chiral $\mathcal{N} = 1$ gauge theories in 4D by compactifying the 6D Blum-Intriligator $(1,0)$ theories of 5-branes at A_k singularities on T^2 with a nontrivial bundle of the global $U(1)$ symmetry of these theories. We end by investigating the complete phase diagram of the decoupled world-sheet theory of (P,Q) strings. These theories include 1+1 dimensional super Yang-Mills theory and non-commutative open string theory. We find that the system exhibits a rich fractal phase structure, including a cascade of alternating supergravity, gauge theory, and matrix string theory phases.

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In memory of my father

Wing Chung Chan

1948-2001

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Chapter 1

Introduction

In the pursuit of understanding quantum gravity, string theory has emerged as the leading candidate. It removes some divergences that emerge from applying quantum field theory to gravity by replacing fundamental point particles with one dimensional strings. All fundamental particles we observe today are manifestations of the different vibrational modes of the string. By quantizing the string, we have found five perturbative string theories which are labeled type I, type IIA, type IIB, heterotic $SO(32)$, and heterotic $E_8 \times E_8$. Moreover, the five theories are consistent for ten spacetime dimensions only.

As it turns out, strings are not the only fundamental objects in string theory. There are solitonic objects called D-p branes which are p+1 dimensional membranes on which open strings ends with Dirichlet boundary condition. D-branes were not discovered earlier because they are non-perturbative objects.

The picture that has emerged is that there is a theory whose moduli space is such that at five cusps, the theory is defined asymptotically by the five known perturbative string theories and at another cusp, it is defined by a mysterious theory, called M-theory, whose low energy limit is 11-d supergravity. All theories on this theory space are related to each other through dualities. For example, M-theory is S-dual to type

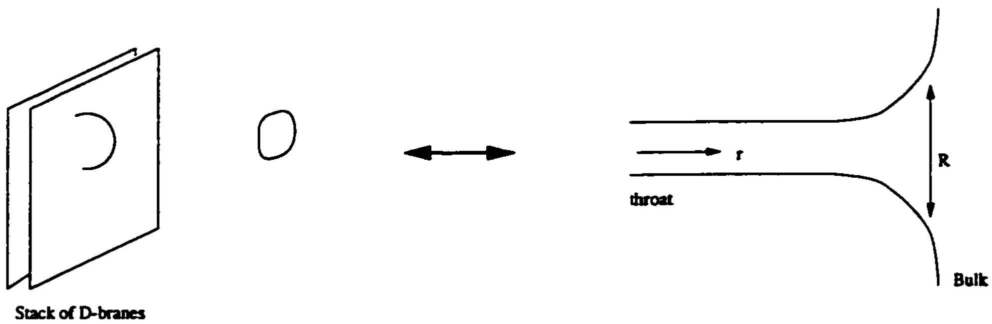

Figure 1.1: Two equivalent views of a stack of N D-branes in IIB string theory leading to the AdS/CFT correspondence. One view is through perturbative excitations. The other view is through backreaction on geometry.

IIA string theory. Type IIA string theory is only defined asymptotically for the string coupling constant going to zero. By S-duality, M-theory is type IIA string theory when the string coupling goes to infinity. The string coupling behaves as another dimension and in going from zero to infinity, it takes a ten dimensional theory into a eleven dimensional theory.

Each point on the theory space can have many solutions depending on the choice of compactification, sources, and fluxes. Due to duality, two different points on the theory space can also have some of the same solutions. Therefore, the solution space is very large and complex with solutions ranging from having 32 supercharges to non-supersymmetric solutions. The solution space is very likely to be made of many disjoint components of different dimensions.

The ultimate goal in string theory is to understand which point in the theory space and which solution of that theory leads to a description of our universe. This is a truly daunting task, but for a theory that purports to unify and explain all fundamental forces and more, we should expect nothing less.

Nevertheless, some progress have been made. One seminal work which has shaped much of recent activity in string theory is the discovery of the correspondence between gravity and gauge theory in one lower spacetime dimension. The AdS/CFT correspondence as conjectured by Maldacena [1] [2] [3] states that Type IIB string theory on $AdS_5 \times S^5$ is equivalent to $\mathcal{N} = 4$ supersymmetric Yang-Mills theory. Many evidence has since been found to corroborate this conjecture [4]. The motivation for the correspondence comes from considering IIB string theory in 10-d Minkowski space with a stack of N D3 branes. This system can be viewed in two ways as shown in Fig. 1.1. It can be studied through perturbative excitations. There are two types of perturbative excitations. Excitations of the closed strings which propagate through empty space correspond to excitations of the bulk. Excitations of the open strings which must end on the branes correspond to excitations of the branes. Now, consider the case when the energy of the system is less than the string tension, $\frac{1}{l_s}$, where l_s is the characteristic size of the string. Only massless string states are then excited. The effective action can be decomposed as

$$S = S_{bulk} + S_{branes} + S_{int} \tag{1.1}$$

where

- S_{bulk} is the action for IIB supergravity with gravity supermultiplet in 10-d plus higher derivative corrections.
- S_{branes} is the action for $\mathcal{N} = 4$ U(N) super Yang-Mills theory with $\mathcal{N} = 4$ vector supermultiplet in 3+1 d plus higher derivative corrections.
- S_{int} is the action for interactions between the brane and bulk modes.

We now take the limit as $l_s \to 0$ with all dimensionless parameters such as g_s and N fixed. It can be shown that S_{int} vanishes as well as the higher derivative corrections in S_{bulk} and S_{brane}. We are thus left with a decoupled system of a IIB supergravity in the bulk and a 4d gauge theory.

We will now study the system of a stack of N D3 branes in IIB string theory in 10-d Minkowski space in a different way. The D3 branes are massive, charged objects and will back-react on the geometry of the 10-d Minkowski space.

The D3 brane solution of supergravity has been worked out to be

$$ds^2 = f^{-\frac{1}{2}}(-dt^2 + dx_1^2 + dx_2^2 + dx_3^2 + dx_4^2) + f^{\frac{1}{2}}(dr^2 + r^2 d\Omega_5^2) \tag{1.2}$$

with

$$f = 1 + \frac{R^4}{r^4} \ , \ \ R^4 = 4\pi g_s \alpha'^2 N \tag{1.3}$$

where $\alpha' \equiv l_s^2$. For $r \gg R$, $f \to 1$ and the metric becomes asymptotic to 10d Minkowski space. For $r \ll R$, $f \to \frac{R^4}{r^4}$ and the metric becomes asymptotic to $AdS_5 \times S^5$. The energy of an object measured by an observer at infinity is related to the energy of the same object as measured by an observer at constant r via

$$E_\infty = f^{-\frac{1}{4}} E_r. \tag{1.4}$$

Thus an excitation of high energy as measured by a local observer at r will be seen by an observer at infinity to be a low energy excitation for r sufficiently small. We will now take the perspective of an observer at infinity and consider the system at low energy. Low energy excitations in the bulk (large r) cannot enter the throat region (small r) because the scattering cross section goes as $\sigma = \omega^3 R^8$ where ω is the energy of the excitation. An intuitive way of seeing that is noticing that R is roughly the gravitational size of the brane and $1/\omega \gg R$ is the wavelength of the excitation so the excitation cannot enter the throat. The excitations deep in the throat region (small r) which are not neccesarily of low energy as measured locally, cannot climb the gravitational potential to enter the bulk. As we take the limit of $\omega \to 0$, we are again left with a decoupled system of IIB supergravity in the bulk and IIB string theory in the throat.

We thus have two descriptions of a stack of N D3 branes in IIB string theory in 10-d Minkowski space studied in the same low energy limit. Both descriptions lead to a decoupled system where one of the decoupled theory is IIB supergravity in 10-d Minkowski space. The AdS/CFT conjecture is then the identification of the other decoupled theories from these two descriptions, namely, the identification of IIB string theory on $AdS_5 \times S^5$ with $\mathcal{N} = 4$ super Yang-Mills theory in Minkowski space.

The AdS/CFT correspondence is the first realization of holography in string theory. The idea of holography [5][6] loosely stated is that in a quantum gravitational system, the number of degrees of freedom of a region is bounded by the area of the boundary of the region. This is very different from what is expected in an ordinary quantum field theory where the number of degrees of freedom of a region scales as the volume of the region.

The motivation for the holographic principle comes from the entropy formula for a black hole, $S = \frac{A}{4G}$, where the entropy is proportional to the area of its event horizon and G is Newton's gravitational constant. Consider a region of space with boundary, B. Throw energy into this region of space until a black hole forms whose event horizon grows to become the boundary, B. In the process of throwing energy into the region, the entropy of the region is non-decreasing. Therefore, the entropy of the region must at all times be less than or equal to the entropy of the black hole whose event horizon is B.

In chapters 2 and 3, we will study warped compactifications using the ideas of the gravity/gauge theory correspondence coming from AdS/CFT and holography. Some phenomenological motivations for studying warped compactification comes from scenarios proposed in [7] which shed some light on 4-d localization of gravity and mass hierachy. The scenarios assume that we live on a 4-d submanifold such as a domain wall in higher dimensional space.

To set up the model, we assume that there is a 4-d domain wall called the Planck brane located at $x_5 = 0$ where gravity is localized. We also assume there is another domain wall located at $x_5 = r$ called the matter brane where the Standard Model fields are localized. The action for this system can be written as

$$S = \int d^5x\sqrt{-G}(R-\Lambda) + \int d^4x\sqrt{-g_M}(\mathcal{L}_M - V_M) + \int d^4x\sqrt{-g_{Pl}}(\mathcal{L}_{Pl} - V_{Pl}) \quad (1.5)$$

where g_M and g_{Pl} are the metrics of the domain walls induced by the 5-d metric, G. The most general ansatz for the 5-d metric preserving 4-d Poincare symmetry and thus permitting flat 4-d domain walls is the warped metric

$$ds^2 = e^{2A(x_5)}\eta_{\mu\nu}dx^\mu dx^\nu + dx_5^2. \quad (1.6)$$

Choosing $\Lambda < 0$, we find the solution

$$ds^2 = e^{-2k|x_5|}\eta_{\mu\nu}dx^\mu dx^\nu + dx_5^2 \quad (1.7)$$

where $k = \sqrt{\frac{-\Lambda}{12}}$, $V_{Pl} = -V_M = 12k$. Note that the tensions of the domain walls must be adjusted to keep them flat. The warped factor, $e^{-2k|x_5|}$, is sharply peaked at $x_5 = 0$. This leads to localization of gravity at the Planck brane at $x_5 = 0$. Also note that the 4-d Newton's constant

$$M_4^2 = \frac{M_5^3}{k}(1 - e^{-2kr}) \quad (1.8)$$

is finite so observers on the matter brane will see an effective 4-d gravity. If we compare the metric at the two branes, g_{Pl} and g_M, we get

$$g_{\mu\nu}^{SM} = e^{-2kr}g_{\mu\nu}^{Pl} \quad (1.9)$$

so an object of energy E at the Planck brane will be seen by an observer at the matter brane to have energy, Ee^{-kr}. The mass hierarchy between the Planck and TeV physics become much more reasonable. Of course, this does not solve the problem. However, instead of explaining a discrepancy of 10^{16} we have to explain 16 which is much more natural.

The model presented above is unphysical as it stands. The Planck and matter branes are placed in an ad hoc fashion with nothing to stabilize their positions. The warped factor having an absolute value sign has a cusp-like singularity and we don't know what those domain walls are. In chapter 2, we will show how this scenario can be realized in string theory following the work in [9]. In string theory, D-branes are natural candidates for the domain walls also known as braneworlds because gauge theory is confined to D-branes as excitations of open strings. In addition, a stack of D-branes exhibit non-abelian gauge symmetry which encompass that of the Standard Model.

In chapter 3, we will show how the AdS/CFT correspondence, when applied within the context of a warped compactification, can be used to give an interesting new perspective on the physics of gravitational collapse. We want to emphasize that in the original AdS/CFT correspondence, the bulk space is infinite and thus the boundary theory does not contain gravity. In the context of warped compactification, we introduce the Planck brane to make the bulk space finite and thus the boundary theory does include gravity as well. The 5-d dynamics in the bulk space will now tell us about the 4-d quantum field theory plus gravity. We will study the case of the gravitational collapse of colored star from a five dimensional perspective.

Any realistic model of our world must also contain chiral fermions. This is because our world appears asymmetric in left-handed and right-handed interactions. If we indeed live on a brane, that brane much exhibit chiral gauge symmetry. In chapter

4, we will show how to obtain a chiral gauge theory from compactification of a six dimensional conformal field theory. Although we are not working with string theory, this six dimensional theory can be imbedded in string theory as it comes from a decoupling limit of NS-5 branes. In short, we will construct chiral fermions from geometry.

The NS-5 brane is a solitonic brane of type IIB string theory. It is a magnetic source of the NS-NS 2-form. It was pointed out in [8] that by taking the decoupling limit of $g_s \to 0$ with M_s (mass of fundamental string) held fixed, the world-volume of the 5-brane results in a 6D theory which includes stringy excitations but without gravity. This theory has $\mathcal{N} = (1,1)$ supersymmetry. The infrared limit of this theory, with energies small compared to M_s, appear to be local quantum field theories. However, the full theory being stringy is not a local quantum field theory.

It was further pointed out in [39] that by starting with a stack of N NS-5 branes on an orbifold singularity and taking the same decoupling limit, one will obtain from the world-volumes of the 5-branes a stringy non-gravitational theory with $\mathcal{N} = (1,0)$ supersymmetry. Locally, the singularities of Calabi-Yau spaces look like ALE spaces of the form, $\mathbf{C}^2/\Gamma$. Γ is a discrete subgroup of $SU(2)$ and is classified by its correspondence with the simply-laced groups A_k, D_k, E_6, E_7, and E_8. We will take the orbifold to be, $\mathbf{C}^2/A_{k-1}$. In the infrared limit, one obtains a d=6 $\mathcal{N} = (1,0)$ supersymmetric Yang-Mills theory whose gauge group is $U(N)_1 \times ... \times U(N)_k$. In chapter 4, we will show how to obtain a chiral $\mathcal{N} = 1$ gauge theory in four dimensions by compactification in this framework. It follows the work in [10].

Another "principle" that we think is intrinsic to string theory is the breakdown of the notion of spacetime. Non-commutativity of spatial coordinates have been studied through non-commutative Yang-Mills theory. It can be obtained from the study of D-branes in a background magnetic field. By extension, we can ask about space-time non-commutativity, $[X^o, X^i] \neq 0$. This is a more radical idea because of problems with quantum mechanics. In quantum mechanics, time is a coordinate used to label evolution while spatial coordinates are operators. Nevertheless, the study of space-time non-commutativity can be obtained by studying string theory in a background electric field.

More specifically, consider a system of a bound state of a fundamental string with a D-string. The fundamental string dissolves into a unit of electric flux in the D-string. The open strings on the D-string have endpoints which are electrically charged. There is a competing force between the tension of the open string and the force exerted on the endpoints of the open string by the electric flux. Now, take the limit of string tension to infinity ($\alpha' \to 0$) and electric field to infinity ($E \to \infty$), such that $\alpha' E$ is held fixed. What results is a residual effective string tension. In the limiting process, G_N vanishes and gravity decouples. We are left with a theory of open strings without gravity. Moreover, the scale associated with the non-commutativity is the effective open string scale so space-time non-commutativity is intimately tied to stringy non-locality.

Because the non-commutative open string theory (NCOS) are non-gravitational, we can study its thermodynamics without worring about instabilites associated with gravitational systems in the infinite volume limit. By studying the thermodynamics of NCOS, we may have a better insight into its microscopic physics from criticalities such as phase transitions.

In chapter 5, we will use $SL(2, \mathbf{Z})$ S-duality and the AdS/CFT correspondence to map out the phase space of the decoupled worldsheet theory of a system of fundamental strings and D-strings. It follows the work in [11].

Chapter 2

Warped Compactification

2.1 Introduction

This chapter is intended to clarify the realization and interpretation of the Randall-Sundrum compactification scenario within string theory. In the model of [7], our 4-d world is extended with an extra direction r to a 5-d space-time with the warped metric

$$ds^2 = e^{2\sigma(r)}\,\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2 \tag{2.1}$$

with $\sigma(r) = -k|r|$. Even while the range of r is infinite, the warped form of the metric ensures that the effective volume of the extra direction is finite. As a result, matter particles sufficiently close to the domain wall region near $r = 0$ will experience ordinary 4-d gravity at long distances [7].

At first sight, this proposal seems like a rather drastic departure from the more conventional Kaluza-Klein framework. Indeed, in most works on string compactifications thus far, the four uncompactified directions and the compact manifold are assumed to form a simple direct product. Although it was realized for a long time that this basic KK set-up can be generalized to include the possibility of warped products, the physics of these more general scenarios is still largely unexplored.

A second important ingredient of the RS-scenario is that part or all of the observable matter may be thought of as confined to a 4-d sub-manifold of the higher dimensional space-time. A concrete theoretical realization of such world-branes are the D3-branes of IIB string theory, which confine open strings to their world-volume. D3-branes, however, do not bind 4-d gravity. Possible supersymmetric realizations of the Planck-brane, located around $r = 0$ in (2.1), are therefore rather expected to be found in the form of domain wall type configurations, or possible stringy generalizations thereof. Various attempts have been made to find smooth domain wall solutions of this type within 5-d gauged supergravity, but thus far without real success [12] [13].

There are several reasons for why this is indeed a hard problem. Even for a given compactification from 10 dimensions, it is an elaborate task to derive the dimensionally reduced theory. Thus far this has been done only for reductions over rather special symmetric 5-manifolds K_5 such as S^5 or S^5/Z_2, etc, and/or for special theories with extended supersymmetry. However, while it seems feasible to classify the possible types of supersymmetric solutions for each of these special dimensional reductions, there is no guarantee that they provide a general enough framework.

Instead of following the above procedure of (1) performing some special dimensional reduction to 5 dimensions and (2) looking for RS domain wall type solutions, it seems more practical to reverse the two steps. Since the scalar fields ϕ^a arise as moduli of some internal 5-d compact space K_5, *any* domain wall solution in 5-d gauged supergravity describes (upon lifting it back up to 10-dimensions) some specific warped compactification of the 10-d theory. It will therefore be much more general – and also easier – to *first* (a) identify a general class of warped compactifications of the 10-dimensional theory, and *then* (b) perform the same type of dimensional reduction from 10 to 5 dimensions. In the end, one can then hope to identify a class of 10-d

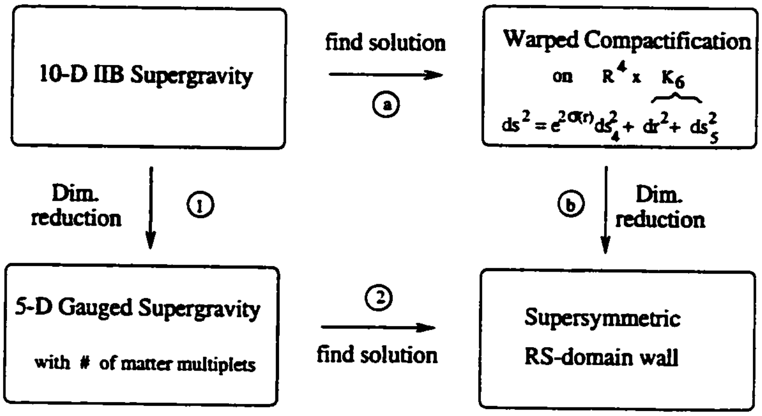

Figure 2.1: To identify supersymmetric RS-type geometries, we will follow the route IIB –> (a) –> (b) –> RS. It is still an open problem to find a direct construction of these geometries via the other route.

geometries for which the resulting dimensionally reduced solution has all the required properties.

As will be described below, such a class of warped IIB geometries indeed exists in the form of quite generic F-theory compactifications on Calabi-Yau four-folds.[1] These have been studied in some detail in the recent literature – a list of references include [15] [16][17][18] and [19] – and indeed none of our equations will be new. Given the current interest in the subject, however, it seems useful to collect some of the known facts about these compactifications, since it has not been generally appreciated that supersymmetric RS-geometries indeed exist in string theory, and furthermore that they are in fact quite generic.

Since all derivations are contained in existing papers, we will here only present the general form of the compactification geometry without any proof that it is really a supersymmetric solution to the 10-d equations of motion. This proof can however be quite directly extracted from the literature, in particular from the very clear discussion by Becker and Becker [15]. Their analysis was done in the context of M-theory

[1]Another special realisation of an RS geometry in terms of a toroidal type IIB orientifold compactification has been described in [14].

compactifications on C-Y four-folds. It can however be straightforwardly translated to the F-theory context by performing the T-duality transformation outlined in [17]. An explicit example of this T-duality transformation is discussed in [16].

Although the 10-d perspective will allow us to identify a large class of RS-type compactification geometries, their geometrical structure is rather involved. It is therefore not easy to *explicitly* perform the dimensional reduction of these solutions to 5-dimensions. We will nonetheless attempt to make this 5-d perspective as transparent as possible. In particular we will show that they indeed give rise to a 5-d metric of the generic form (2.1).

2.2 Warped Compactification in F-theory

In the papers of [24] [25], it was shown that for general conditions, supergravity does not permit warped compactification to Minkowski or deSitter space. As with all no-go theorems, one must examine their assumptions carefully to find a way to circumvent them. It was found that if one allowed for negative tension objects, one can evade the no-go theorem and indeed find warped compactifications of the Randall-Sundrum type. While pure supergravity does not have negative tension objects, they do exist in string theory. We will examine this in more detail below.

Let's begin with the IIB low energy effective action in the Einstein frame

$$S^E_{IIB} = \int d^{10}x\sqrt{-g}\left\{R - \frac{\partial_m\tau\partial^m\bar{\tau}}{2(\mathrm{Im}\,\tau)^2} - \frac{H\cdot\bar{H}}{12\mathrm{Im}\,\tau} - \frac{\tilde{F}_5^2}{4\cdot 5!}\right\} + \frac{1}{4i}\int\frac{C_4\wedge H\wedge\bar{H}}{\mathrm{Im}\,\tau} + S_{loc} \tag{2.2}$$

where

$$\tau = \tilde{\phi} + ie^{-\phi} \tag{2.3}$$
$$H^R = dC_2 \tag{2.4}$$
$$H^{NS} = dB_2 \tag{2.5}$$
$$H = H^R - \tau H^{NS} \tag{2.6}$$
$$\tilde{F}_5 = F_5 - \frac{1}{2}C_2 \wedge H^{NS} + \frac{1}{2}B_2 \wedge H^R \tag{2.7}$$

and the fields are listed below

(i)	the dilaton field	ϕ
(ii)	the RR-scalar or axion field	$\tilde{\phi}$
(iii)	the NSNS 3-form field strength	H^{NS}
(iv)	the RR 3-form field strength	H^R
(v)	the RR 5-form field strength	F_5

S_{loc} is the action of localized objects and we need to further impose the self-duality constraint

$$\tilde{F}_5 = *\tilde{F}_5 \tag{2.8}$$

on the equations of motion.

We want to study the general warped metric with the four dimensional transverse space having constant curvature, k, so we begin with the metric ansatz

$$ds^2_{10} = e^{2\alpha(y)} h_{\mu\nu} dx^\mu dx^\nu + e^{-2\alpha(y)} \tilde{g}_{mn} dy^m dy^n \tag{2.9}$$

where x^μ are the 4-dimensional coordinates of the constant curvature space and y^m are the coordinates on the compact manifold K_6.

The axion and dilaton will be allowed to vary over K_6. To maintain maximal symmetry of the transverse space with constant curvature, we must have

$$H = \frac{H_{mnp}(y)}{3!} dy^m \wedge dy^n \wedge dy^p \tag{2.10}$$

$$\tilde{F}_5 = \partial_m \beta(y)(1+*) dy^m \wedge dvol_4 \tag{2.11}$$

where $dvol_4$ is the volume element of the transverse space and $\tilde{F}_5$ explicitly satisfies the self duality condition and the Bianchi identity.

From the equation of motion and using the metric ansatz, one can show a constraint for flux/brane configurations that can give rise to warped solutions on compact manifolds [22]

$$\begin{aligned} \tilde{\nabla}^2 e^{4\alpha} &= \frac{e^{2\alpha} H_{mnp} \bar{H}^{mnp}}{12 \mathrm{Im}\tau} + e^{-6\alpha}(\partial_m \beta \partial^m \beta + \partial_m e^{4\alpha} \partial^m e^{4\alpha}) \\ &+ \frac{1}{4} e^{2\alpha} (T^m_m - T^\mu_\mu)^{loc} + 8k e^{2\alpha} \end{aligned} \tag{2.12}$$

where the tilde refers to the metric $\tilde{g}_{mn}$ and the third term comes from the energy-momentum tensor of the localized source. The integral of the LHS over the compact manifold K_6 vanishes whereas all the terms excluding the localized sources and curvature term are non-negative. Thus, in pure supergravity where there are no localized sources, only warped solutions with negative constant curvature transverse space are possible. However, if localized sources exist and have negative tension then we can have warped solutions with transverse spaces having non-negative constant curvature as we will now show.

Consider a p-brane that fills the four dimensional transverse space and is wrapped on a $p-3$ cycle, Σ, of manifold K_6. Assuming no fluxes along the brane, to leading order in α', the source action is

$$S_{loc} = -T_p \int_{R^4 \times \Sigma} d^{p+1}\eta \sqrt{-g} + \mu_p \int_{R^4 \times \Sigma} C_{p+1} \tag{2.13}$$

where T_p is the tension of the p-brane and μ_p is the coupling to the $p+1$ form RR potential, C_{p+1}. We will find that

$$(T^m_m - T^\mu_\mu)^{loc} = (7-p)T_p\delta(\Sigma). \tag{2.14}$$

So for $p < 7$, we require negative tension to give negative stress as required for warped compactification with transverse space having non-negative constant curvature. An example of such an object is the $O3$ plane which have a tension equal to $-\frac{1}{4}T_3$.

The D7 brane can also give rise to warped compactification even though the stress contribution is zero according to (2.14). This is because the Chern-Simons term in the action gives an induced D3 charge on the wrapped D7 brane. The induced charge will depend on the geometry of the compact manifold and it is more natural to study that using F-theory.

F-theory is a geometric language for describing compactifications of type IIB string theory, in which the expectation values of the dilaton and axion fields are allowed to vary non-trivially along the compactification manifold [20]. Compactifications of F-theory down to four-dimensions are specified by means of a Calabi-Yau four-fold that admit an elliptic fibration with a section. In other words, these are 8 dimensional compact manifolds K_8 that *locally* look like a product of a complex three-fold K_6 times a two-torus T^2. The two-torus will be taken to shrink to zero size. It can however be taken to change its shape when moving along the base K_6. In particular, it can have non-trivial monodromies around singular co-dimension 2 loci inside the K_6, where the elliptic fiber degenerates.

The four-fold K_8 is not the actual compactification geometry; rather it gives an economical way to characterize the compactification geometry as well as the expectation values of other fields. Moreover, due to the special geometric properties of the Calabi-Yau four-fold K_8 – vanishing first Chern class and $SU(4)$ holonomy – the associated IIB background by construction will preserve 4-d supersymmetry, at least at the classical and perturbative level.

The warped geometry of this type of F-theory compactifications has been derived in [16], by direct translation of the M-theory analysis of [15]. The full solution for the 10-dimensional IIB string metric, in the Einstein frame, takes the form (2.9). The shape of the warp-factor $e^{2\alpha}$ will depend on the detailed geometry of the CY four-fold K_8, as well as on other data such as the possible non-zero expectation values of other fields and the locations of the possible D-branes.

Besides the ten-dimensional space-time metric, the fields that can take non-trivial expectation values are the following: $\phi, \tilde{\phi}, H^{NS}, H^{R}$, and F_5. The expectation values of all these fields can be conveniently characterized in terms of the geometry of K_8.

In F-theory, τ becomes the modulus of the elliptic fibration. It parameterize the shape of the two-torus inside the K_8, describing the variation along the 6-d base manifold K_6 of the dilaton and axion fields, ϕ and $\tilde{\phi}$. As mentioned above, a key feature of F-theory is that this modulus in general has non-trivial monodromies around 4-d submanifolds inside K_6. These 4-d sub-manifolds are associated with the locations of D7-branes, of which the remaining 3+1-dimensions span the uncompactified space-time directions. In going around one of the D7-branes, the modulus field τ can pick up an $SL(2, \mathbf{Z})$ monodromy

$$\tau \to \frac{a\tau + b}{c\tau + d}, \tag{2.15}$$

which leaves the geometric shape of the two-torus fibre inside K_8 invariant, but nonetheless via (2.3) amounts to a non-trivial duality transformation of the IIB string theory. We thus notice that the dilaton and axion field are not smooth single-valued functions, but instead are multi-valued with branch cut singularities at the locations of the D7-branes. The full non-perturbative string theory, however, is expected to be well-behaved everywhere.

For the following, it will be convenient to combine the NSNS and RR three-from field strengths, H^{NS} and H^{R}, of the IIB supergravity into a single four-form field-strength G on K_8 as follows [19]. Let z and $\overline{z}$ denote the coordinates along the T^2 fiber. Then we can write

$$G = \frac{\pi}{i\tau_2}(H \wedge d\overline{z} - \overline{H} \wedge dz) \tag{2.16}$$

$$H = H^{R} - \tau H^{NS}\,; \qquad \overline{H} = H^{R} - \overline{\tau} H^{NS}. \tag{2.17}$$

For supersymmetric configurations, H defines an integral harmonic (1,2)-form on K_6 satisfying $H \wedge k = 0$ with k the Kähler class of K_6 [19]. It transforms under the $SL(2, Z)$ monodromy transformations (2.15) around the seven-branes as $H \to H/(c\tau + d)$. The field-strength G is invariant under these transformations.

An important aspect of F-theory compactifications is that they typically carry, via their non-trivial topology, an effective total D3-brane charge. The value of this charge is proportional to the Euler characteristic $\chi(K_8)$ of the original Calabi-Yau four-fold K_8. Here $\chi(K_8)$ is defined via

$$\frac{1}{24}\chi(K_8) = \int_{K_8} I_8(R) \tag{2.18}$$

where

$$I_8(R) = \frac{1}{192}\Big(\mathrm{tr}R^4 - \frac{1}{4}(\mathrm{tr}R^2)^2\Big) \tag{2.19}$$

with R the curvature two-form on K_8. Global tadpole cancellation, or conservation of the RR 5-form flux, requires that this charge must be canceled by other sources. These other sources come from possible non-zero fluxes of the NSNS or RR two-form fields, or from the explicit insertion of N D3-brane world branes, that is, D3-branes that span the 3+1-d uncompactified world but are localized as point-like objects inside the K_6. The number of such D3-branes is therefore not free, but completely determined via charge conservation. This global tadpole cancellation relation reads

$$N = \frac{1}{24}\chi(K_8) - \frac{1}{8\pi^2}\int_{K_8} G \wedge G. \tag{2.20}$$

Depending on the topology of K_8, N can reach values of up to 10^3 or larger. An example with $N = 972$, mentioned in [18], is provided by an elliptically fibered CY four-fold over $\mathbf{P}^3$. The Euler number $\chi(K_8)$ can be non-zero only if K_6 has a non-vanishing first Chern class, that is, provided the F-theory compactification makes use of a non-zero number of D7-branes.

The equation of motion for the warp factor $e^{2\alpha}$ obtained in [15] and [16] reads as follows

$$\Delta^{(8)}\, e^{-4\alpha} = *\, 4\pi^2 \left\{ I_8(R) - \frac{1}{8\pi^2} G \wedge G - \sum_{i=1}^{N} \delta^{(8)}(y - y_i) \right\} \tag{2.21}$$

where $\Delta^{(8)}$ denotes the Laplacian and $*$ the Hodge star on K_8. The points $y = y_i$ correspond to the location of the N D3-branes. Here, following [16], we have written the equation on the full 8-d manifold K_8, even though in F-theory the elliptic fiber T^2 inside K_8 has been shrunk to zero size. In this limit, the solution for α obtained via (2.21) only depends on the 6 coordinates y^{M} on K_6. Alternatively, using the analysis of [18], one may also first reduce the right-hand side to K_6, via integration over the T^2 fiber, and then solve the reduced equation to obtain the function $e^{-4\alpha}$ directly on K_6.

Finally, there is also an non-trivial expectation value for the self-dual RR five-form field strength, equal to [15] [16]

$$F_{\mu\nu\lambda\sigma\mathrm{M}} = \epsilon_{\mu\nu\lambda\sigma}\partial_{\mathrm{M}}\, e^{-4\alpha}. \tag{2.22}$$

We note that via (2.21) the D3-branes indeed form a source for this field strength, but that via (2.20) the total charge adds up to zero.

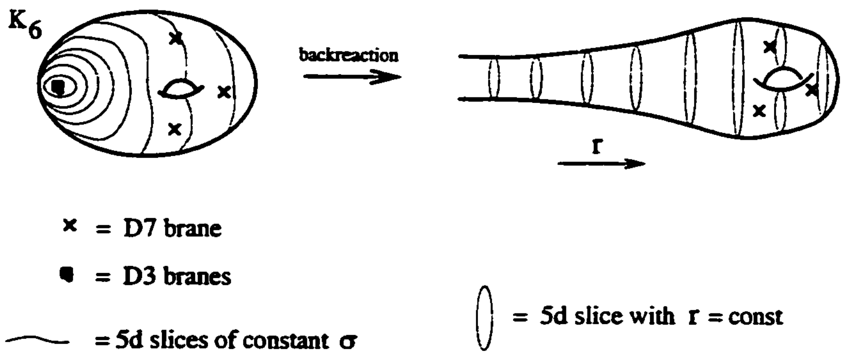

Figure 2.2: The contours with constant warp factor $e^{2\sigma}$ define a particular slicing of the 6-dimensional compactification manifold K_6, which can be used to represent K_6 as a one parameter flow along r of five-manifolds K_5. Upon dimensional reduction to 5 dimension, this geometry describes a one-sided RS-domain wall solution.

2.3 Shape of the warp factor

Let us summarize. Starting from the elliptically fibered CY four-fold K_8 we can extract a complete characterization of the warped compactification. First, since $K_8 \simeq K_6 \times T^2$, we obtain the metric g_{MN} on the base K_6, as well as the dilaton and axion via (2.3). We then deduce the form of the warp factor $e^{2\alpha}$ from (2.21), which incorporates the complete backreaction due to the G-flux and D3-branes. Finally from (2.9), we obtain the actual compactification geometry. Note that, as indicated in fig 2.2, the rescaling by $e^{-2\alpha}$ of g_{MN} in (2.9) may have a drastic effect on the shape of the compactification manifold, which indeed may look quite different from that of the original K_6. In particular, it is possible that near the locations of the D3-branes one of the internal directions may become non-compact.

We may formally solve the equation (2.21) via

$$e^{-4\alpha(y)} = e^{-4\alpha_0} + 4\pi^2 \int d^8y' \sqrt{g}\, \mathcal{G}(y,y') \Big[I_8(R(y')) - \frac{1}{8\pi^2} G \wedge G - \sum_{i=1}^{N} \delta^{(8)}(y'-y_i) \Big] \tag{2.23}$$

where $\mathcal{G}(y,y')$ denotes the Green function for $\Delta^{(8)}$. The term $e^{-4\alpha_0}$ parametrizes the constant zero mode of $e^{-4\alpha}$, which is not fixed by eqn (2.21). Note that for $e^{-4\alpha}$ to be everywhere positive, this constant $e^{-4\alpha_0}$ can not be arbitrarily small, since the second term on the r.h.s. of (2.23) can become negative. This implies that the warped 6-geometry automatically has a *minimal* volume[2] .

An interesting limiting case is when all $D3$ branes are concentrated in one point, say $y = y_0$. Close to this point, the warp function $\alpha(y)$ reduces to

$$\alpha(y) \simeq \log|y - y_0| + \text{const.} \tag{2.24}$$

Via (2.9) this describes the familiar semi-infinite near-horizon geometry of N D3-branes: $AdS_5 \times S^5$ with radius $R = \sqrt[4]{4\pi N g_s}$. (See fig 2.2.) Although the radial AdS_5 coordinate $r \simeq -R\log|y - y_0|$ runs over semi-infinite range, the compactification geometry (2.9) still gives rise to a 4-d Einstein action with a *finite* 4-d Newton constant $1/\ell_4^2$ equal to

$$\frac{1}{(\ell_4)^2} = \frac{1}{(\ell_{10})^8} \int_{K_6} \sqrt{g}\, e^{2\phi - 4\alpha} \tag{2.25}$$

with ℓ_{10} the 10-d Planck length.

2.4 Fixing of the moduli

Calabi-Yau compactification has a large number of moduli corresponding to deformations of the compact manifold consistent with CY conditions. These massless fields are the complex moduli, Kahler moduli, and axion-dilaton. However, the fluxes present in warped compactification fix many of these moduli. It should be noted that not all moduli can be fixed by the fluxes. The necessary and sufficient conditions leading to warped compactification are invariant under the rescaling,

$$\tilde{g}_{mn} \to \lambda^2 \tilde{g}_{mn}. \tag{2.26}$$

[2]We thank Sav Sethi for bringing this feature to our attention.

Thus, there will be at least one moduli that can not be fixed, namely the volume of the compact manifold.

We will proceed to fix the moduli by first fixing the flux in accord with the Dirac quantization,

$$\frac{1}{2\pi\alpha'}\int_{C_I} H^R \in 2\pi\mathbf{Z}, \quad \frac{1}{2\pi\alpha'}\int_{C_I} H^{NS} \in 2\pi\mathbf{Z} \tag{2.27}$$

where C_I form a homology basis for 3-cycles on K_6. The moduli will then adjust to minimize the F-terms arising from the Gukov-Vafa-Witten superpotential

$$W = \int \Omega \wedge H. \tag{2.28}$$

We will first work out the kinetic term for the moduli. They can be extracted from the quadratic order terms in the expansion of the Einstein-Hilbert term in (2.2) using a decomposition of the metric that displays the fluctuations of the geometrical moduli fields following [23]

$$ds^2 = e^{2\alpha(y)}e^{-6u(x)}g_{\mu\nu}dx^\mu dx^\nu + e^{-2\alpha(y)}e^{2u(x)}(\tilde{g}_{mn}(y) + T^I(x)\delta g_{Imn}(y)dy^m dy^n) \tag{2.29}$$

where $\tilde{g}^{mn}\delta g_{Imn} = 0$ so that only the fluctuations of e^{2u} scale the total volume. T^I are volume preserving and include the Kahler and complex structure moduli. If we define a complex field ρ such that $\mathrm{Im}\,\rho = e^{4u}$, then we find

$$\begin{aligned} S_{kinetic} &= \frac{1}{\kappa_4^2}\int d^4x\sqrt{-g_4}\Big(-3\frac{\partial_\mu\bar{\rho}\partial^\mu\rho}{|\rho-\bar{\rho}|^2} - \frac{\partial_\mu\bar{\tau}\partial^\mu\tau}{|\tau-\bar{\tau}|^2} \\ &\quad -\frac{1}{8V_\omega}\partial_\mu T^I\partial^\mu T^J \int d^6y\sqrt{\tilde{g}_6}e^{-4\alpha}\delta g_{Imn}\delta g_J^{\bar{m}\bar{n}}\Big) \end{aligned} \tag{2.30}$$

where $V_\omega \equiv \int d^6y\sqrt{\tilde{g}_6}e^{-4\alpha}$. These kinetic terms can be obtained from the Kahler potential

$$\begin{aligned} \kappa &= -3log(-i(\rho-\bar{\rho})) - log(-i(\tau-\bar{\tau})) \\ &\quad -log(-i\int d^6y e^{-4\alpha}\sqrt{\tilde{g}_6}) - log(-i\int e^{-4\alpha}\Omega\wedge\bar{\Omega}) \end{aligned} \tag{2.31}$$

where $\hat{g}(x,y) = \tilde{g}_{mn}(y) + T^I(x)\delta g^I_{mn}(y)$.

To find the moduli potential, we look for the dependence of the IIB supergravity action on the Calabi-Yau metric and the dilaton. We use the metric decomposition of (2.29) but setting $u = 0$ and $T^I = 0$ so that there are no dependence on x. The effective potential is computed from the $R, H \cdot \bar{H}$ and $\tilde{F}_5^2$ terms. The Einstein-Hilbert term gives

$$\int d^{10}x\sqrt{-g}R = \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}[-8(\nabla\alpha)^2 e^{4\alpha}]. \tag{2.32}$$

The $\tilde{F}_5^2$ term gives

$$\int d^{10}x\sqrt{-g}\frac{\tilde{F}_5^2}{4\cdot 5!} = \int d^4x\sqrt{-g_4}\int d^6y\sqrt{g_6}\frac{e^{-4\alpha}}{2}(\partial_m\beta)^2. \tag{2.33}$$

Use the relation $e^{4\alpha} = \beta$ and the Bianchi identity $dF_5 = 0$ in the form

$$\nabla^2 A = \frac{iH_{mnp} *_6 \bar{H}^{mnp}}{48\mathrm{Im}\,\tau} \tag{2.34}$$

to write the action for the effective potential as

$$S_{potential} = \int d^4\sqrt{-g_4}\int \frac{e^{4\alpha}}{2\mathrm{Im}\,\tau}H\wedge(*_6\bar{H} + i\bar{H}). \tag{2.35}$$

If we use the imaginary self and anti-self dual parts of the flux H,

$$H^\pm = \frac{1}{2}(H \pm i *_6 H), \quad *_6H^\pm = \mp iH^\pm \tag{2.36}$$

we can rewrite the potential as

$$S_{potential} = \int d^4x\sqrt{-g_4}\int \frac{e^{4\alpha}}{\mathrm{Im}\,\tau}H^+\wedge *_6\bar{H}^+. \tag{2.37}$$

By minimizing the potential, we can fix the moduli corresponding to the fields, τ, H^R, H^{NS} and F_5. We can check that this potential is indeed correct by showing it

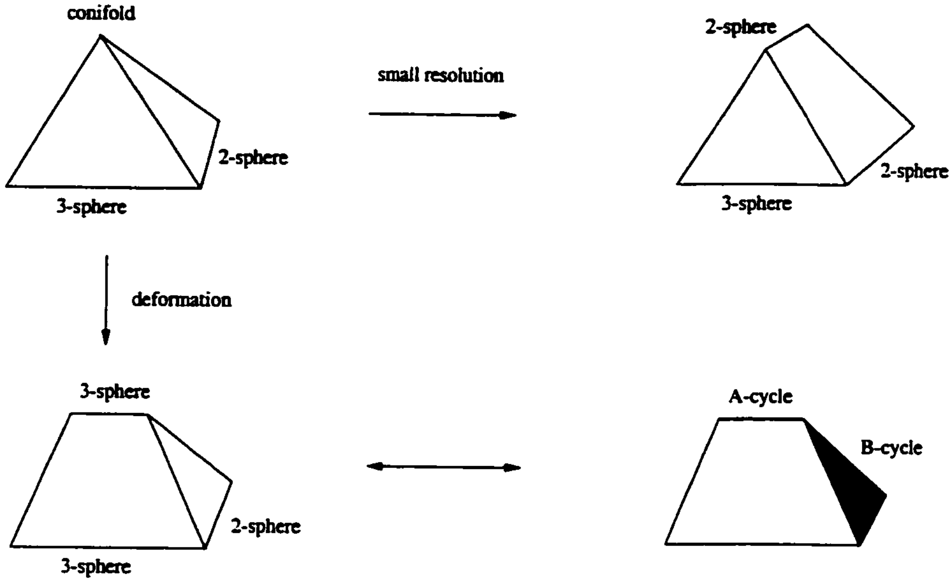

Figure 2.3: The conifold is a cone over $S^3 \times S^2$. The singularity at the apex can be resolved by small resolution or deformation by blowing up the S^2 or S^3 respectively. The deformed conifold has an A-cycle which is the blown-up S^3 and its dual B-cycle (shaded area) which intersects the A-cycle exactly once.

to be equivalent to the $\mathcal{N} = 1$ supergravity potential

$$V = \frac{1}{2\kappa_{10}^2} e^{\kappa} (G^{a\bar{b}} D_a W \overline{D_b W} - 3|W|^2) \tag{2.38}$$

where $D_a W = \partial_a W + W \partial_a \kappa$ and $G_{a\bar{b}} = \partial_a \partial_{\bar{b}} \kappa$ and the indices a, b are summed over superfields. The superpotential, $W = \int \Omega \wedge H$, is unmodified by the warped factor. It can also be shown [22] that $|D_\rho W|^2 - 3|W|^2 = 0$. This is indicative of a no-scale potential. The potential is then positive semi-definite and the minimum occurs when $D_i W = 0$, where i is a superfield moduli not including ρ.

To work out a specific example where we can obtain a large and stable hierarchy, we will follow [22]. Klebanov and Strassler [26] found smooth supergravity solutions with large relative warpings in the local vicinity of conifold points with fluxes. We'll examine their solution in some detail. A Calabi-Yau 3-fold is generally smooth with possibly some singularities that are generically conifolds. A conifold is a cone over a base which is $S^3 \times S^2$, see figure 2.3. The S^3 and S^2 cycles of the base shrink to zero size at the apex of the conifold which is the singular point. The equation of the conifold can be described as a locus in $\mathbf{C}^4$,

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = 0. \tag{2.39}$$

The singularity is located at the point, $(w_1, w_2, w_3, w_4) = 0$. The conifold singularity can be resolved in two ways. One way is to blow up the S^2 at the singularity and is called the small resolution. The other way is to blow up the S^3 at the singularity and is called a deformation. The latter will be relevant to us. The deformation of the conifold can be described as the locus,

$$w_1^2 + w_2^2 + w_3^2 + w_4^2 = \epsilon \tag{2.40}$$

where ϵ parameterize the size of the blown-up S^3 at what used to be the singularity.

We now introduce fluxes and see how they generate a potential to fix some of the moduli. There are two 3-cycles on the deformed conifold that are relevant to us. They are the blown-up 3-sphere, called the A-cycle, and its dual 3-cycle, called the B-cycle, which intersects the A-cycle exactly once, see figure 2.3. The Klebanov-Strassler solution corresponds to having M units of H^{RR} flux on the A-cycle and $-K$ units of H^{NS} flux on the B-cycle. M and K are not independent since they are related through the requirement of D3 charge conservation. The superpotential can be computed [22]

$$W = \int \Omega \wedge H \sim K\tau \int_A \Omega - M \int_B \Omega \tag{2.41}$$

where

$$\int_A \Omega = \epsilon \tag{2.42}$$

measures the size of the collapsing A-cycle. The dual B-cycle has the standard result

$$\int_B \Omega = \frac{\epsilon}{2\pi i} \ln \epsilon + holomorphic. \tag{2.43}$$

The potential is minimized when $D_\epsilon W = 0$. For a large hierarchy, we need to have $K/Mg_s \gg 1$ thus fixing

$$\epsilon \sim e^{-\frac{2\pi K}{Mg_s}}. \tag{2.44}$$

The warped factor can be computed by solving the differential equation (2.21). The hierarchy of the energy scale can also be estimated to be [22]

$$e^{\alpha_{min}} \sim \epsilon^{\frac{1}{3}} \sim e^{-\frac{2\pi K}{3Mg_s}}. \tag{2.45}$$

This result can be shown to be consistent with the holographic view.

The Klebanov-Strassler supergravity solution is holographically dual to a non-conformal $\mathcal{N} = 1$ gauge theory with gauge group $SU(MK + M) \times SU(MK)$. The renormalization group flow towards the infrared involves a cascade that takes place in steps with a ratio of energy scale $e^{-\frac{2\pi}{3Mg_s}}$. Each step has a strong coupling transition involving a Seiberg duality which lowers the rank of the larger of the two gauge groups by $2M$. The maximum number of such dualities is K so the full range of energy scales that takes place in the cascade is $e^{-\frac{2\pi K}{3Mg_s}}$.

The analysis of this section can also be carried out in F-theory compactification. One needs to embed the Klebanov-Strassler solution into an F-theory compactification by finding an elliptically fibered Calabi-Yau fourfold that admits a conifold singularity in its base. This is worked out in [22].

2.5 Reduction to 5 dimensions.

We would now like to show that these F-theory compactifications, upon performing a suitable dimensional reduction to five dimensions, reduce to supersymmetric RS domain wall solutions. To this end we will look for a specific coordinate system

$$y^M = (y^m, r) \tag{2.46}$$

where m now runs over 5 values, such that the 10-d metric takes the following form

$$ds^2_{RS} \;=\; e^{2\sigma(r)}\eta_{\mu\nu}\,dx^\mu dx^\nu + dr^2 + h_{mn}(y,r)dy^m dy^n \tag{2.47}$$

where ds^2_{RS} is related to the original IIB string metric (2.9) via the rescaling

$$ds^2_{RS} = e^{\phi/2}(V_5)^{1/4}\,ds^2_{IIB} \qquad\qquad V_5 = \frac{1}{(\ell_{10})^5}\int_{K_5}\sqrt{h}. \tag{2.48}$$

Here the prefactor in (2.48) is chosen such that ds^2_{RS} is the metric in the 5-d Einstein frame (where we have taken the 5-d Planck length ℓ_5 equal to the 10-d one). The 5-d slices of constant r define 5-d submanifolds K_5, on which $2\alpha(y) + \phi(y)/2 =$ constant; this correspondence guarantees that the warp-factor $e^{2\sigma}$ in ds^2_{RS} just depends on r and not on the remaining y^m's.

In this way we indeed obtain a solution that from the 5-d perspective looks just like an RS-type warped geometry. For large negative r, close to the D3-branes, the warp factor behaves like $e^{2\sigma} \simeq e^{-2|r|/R}$ with R the AdS-radius of the N D3-brane solution. On the other end, somewhere outside the throat region of the AdS-tube, near the 'equator' of the K_6, the warp factor $e^{2\sigma(r)}$ reaches some maximal value. Eventually, there is a boundary value for the coordinate r, which we can take to be $r = 0$, at which the transverse 5-manifold K_5 shrinks to zero size. However, as is clear from fig 2.2, this does not correspond to any singularity, but just to a smooth

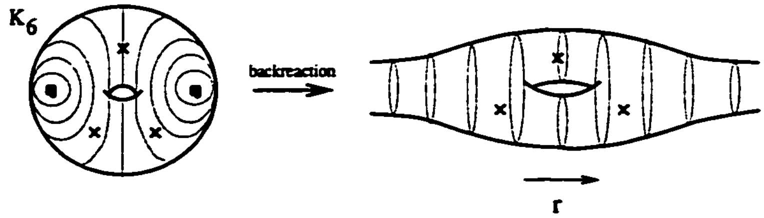

Figure 2.4: An RS domain wall in between two AdS-type regions can be obtained by starting with a $\mathbf{Z}_2$ symmetric 6-manifold K_6, in which the D3-branes are located at opposite image points under the $\mathbf{Z}_2$.

cap closing off the 6-d manifold K_6. This fact that r takes a maximal value $r = 0$ implies that the AdS-space is indeed compactified, in the sense that, relative to the 4-d space-time, it has a finite volume. It therefore produces to a finite 4-d Newton constant equal to (cf [7])

$$\frac{1}{(\ell_4)^2} = \frac{1}{(\ell_5)^3} \int_{-\infty}^{0} dr \, e^{2\sigma(r)}. \tag{2.49}$$

It is easily verified that via (2.48) and (2.47), this result coincides with the value (2.25) found via direct reduction from 10 dimensions.

Upon dimensional reduction, the metric h_{mn} of the internal K_5, as well as other fields such as the dilaton, all reduce to scalar fields that provide the matter multiplets of the 5-d supergravity. All these fields vary with the radial coordinate r, and since this radial flow is supersymmetric, it should in principle be described as some gradient flow driven by some appropriate superpotential. However, due to the rather complex geometrical structure of the typical F-theory compactification, it unfortunately seems impossibly hard to find the explicit form of this potential.

2.6 Discussion

In this note we have summarized the geometrical description of warped F-theory compactifications, and shown they can be used to obtain geometries very analogous to the RS-scenario. Due to the presence of the D7-branes in the F-theory geometry, the solutions are not completely smooth: the dilaton and axion field have isolated branch cut singularities at the D7-brane loci around which τ is multi-valued. The string theory, however, is well-defined in this background.

Our description of the solutions makes clear that the internal structure of the RS Planck brane is not that of an actual brane, but rather that of a compactification geometry (which however also contains the D7-branes). Consequently, the localized graviton zero mode of [7] is just the standard KK zero mode of the 10-d metric; because its wave-function is sharply peaked near the wall region, where the warp factor $e^{2\sigma}$ is maximal, the 4-d graviton indeed *looks like* some bound state. From the higher dimensional viewpoint, however, this RS-localization of gravity is not a new phenomenon.

The most interesting aspect of these warped string compactifications, is that there is no clear distinction between the low energy and extra dimensional physics. Kaluza-Klein excitations, when localized far inside the AdS region can describe particles with masses much smaller than the inverse size of the original K_6. While these particles naively look like new degrees of freedom arising from the presence of extra dimensions, the holographic AdS/CFT correspondence [1] tells us that they are in fact localized excitations of the low energy gauge theory. The same holds for string excited modes in this region. Hence via the holographic identification of the RG scale with a *real* extra dimension, the two usually separate stages of dimensional and low energy reduction should now be combined into one single procedure.

Finally, as a closely related point, we need to emphasize that the solutions as given here are generically unstable against small perturbations. The best way to understand this instability is via the RG language: in general there will exist relevant operators whose couplings, once turned on in the UV region, will quickly grow and typically produce some singularity that effectively closes off the AdS-tube [21]. In our way of obtaining the solutions, we did not immediately notice this instability, because we required that the original K_6 geometry and all other fields, except for the warp-factor and the 5-form flux, are smooth at the locations of the D3-branes. This requirement is special, however, and we should allow for deformations that may spoil this property.

In order to ensure that the perturbed geometry contains a substantial intermediate AdS-like region, we either need to fine-tune the UV initial conditions or introduce a symmetry that eliminates these unstable modes. In RG language, this means that the dual 4-d field theory should be made approximately conformal invariant over a large range of scales, separating the Planck scale from the scale set by the non-trivial gauge dynamics. Via the AdS/CFT dictionary, the problem of realizing an RS-geometry in string theory therefore is reduced back to the original problem it was designed to solve, namely how to generate a large gauge hierarchy. Or stated in more positive terms, in searching for realistic string compactification scenarios, the observed gauge hierarchy can be viewed as an indication that warped geometries of this type deserve serious attention.

Chapter 3

Gravitational Collapse via AdS/CFT

3.1 Introduction

In this chapter we show how the AdS/CFT correspondence, when applied within the context of a warped compactification, can be used to give an interesting new perspective on the physics of gravitational collapse.

Consider a 3+1-dimensional world in which all matter consists of some strongly coupled large N gauge theory with an AdS/CFT dual description. The gauge theory can be coupled to 3+1-d gravity, by assuming that it arises from a warped string compactification of the type described in the previous chapter, with a compact slice of the dual AdS space-time as part of its compactification geometry. The total 10-d target space Σ_{10} can be thought of as obtained by gluing together two parts

$$\Sigma_{10} = \Sigma_{IR} \cup \Sigma_{UV} \tag{3.1}$$

of the form

$$\Sigma_{IR} = M_5 \times K_5, \qquad \Sigma_{UV} = R^4 \times (K_6 - B_6) \tag{3.2}$$

with $\partial M_5 = R^4$ and $\partial B_6 = K_5$, where M_5 is a compact slice of AdS_5 and K_5 is a compact 5-manifold with the topology of a 5-sphere. Since both Σ_{IR} and Σ_{UV} have a common boundary, $R^4 \times K_5$, we can glue them together into one geodesically complete 10-d manifold Σ_{10}. We will assume that, by means of an appropriate set of fluxes and/or by other means, all compactification moduli are fixed and all moduli fields have become massive. In this case it is appropriate to call Σ_{UV} the "Planck brane" region, although it does not represent a proper string theoretic brane.

With a judicious choice of coordinates, the metric on Σ_{IR} takes the form of a trajectory of 9-d geometries

$$ds^2 = dr^2 + a^2(r) g_{\mu\nu} dx^\mu dx^\nu + h_{mn}(r, y) dy^m dy^n. \tag{3.3}$$

The r evolution has the holographic interpretation of an RG flow, and the cut in the 10-manifold thus corresponds to a separation of the effective 4-d degrees of freedom into a high and low energy sector separated by a cut-off scale set by the location r of the cut.

One of the most interesting aspect of this type of compactifications is that the classical higher dimensional theory incorporates several quantum effects of the effective 4-d theory:

(i) Via the open/closed string duality, all planar diagrams of the boundary gauge theory are included via classical effects in the bulk. In particular, the effective 4-d stress energy tensor that appears in the low energy 4-d Einstein equations includes a term that can be recognized as the contribution of the 4-d conformal anomaly.

(ii) The holographic correspondence identifies the warped extra dimension with 4-d energy scale, and the radial evolution in this direction thus represents a 4-d renormalization group flow. In a warped compactification, however, the extra dimension is truncated at a finite UV value of the warp-factor. This truncation is the holographic representation of the fact that the boundary theory possesses a UV cut-off at the 4-d Planck scale, the shortest possible 4-d distance scale. In a certain sense, the classical 5-d dynamics thus "knows" about (aspects of) 4-d quantum gravity!

Conversely, the boundary gauge theory also knows about aspects of quantum gravity in the higher dimensional bulk geometry. Namely when we heat up the 4-d gauge theory matter, its holographic dual takes the form of a 5-d AdS black hole, with Hawking temperature and Bekenstein-Hawking entropy in precise accordance with that of the 4-d matter. The 4-d dimensional reduction of the 5-d black hole thus gives the first explicit realization of the holographic principle of 't Hooft and Susskind, albeit within 5-d AdS space-time. It also partly demystifies it: one could simply view the thermal atmosphere surrounding the 5-d black hole as a direct spectral image of the 4-d thermal matter, where the radial location of each matter excitation determined its 4-d energy scale. The deepest statement of the AdS/CFT correspondence is not the kinematical representation of a 4-d theory as living in a 5-d AdS space-time, but rather the dynamical equivalence between the strongly coupled 4-d gauge theory and the 5-d gravitational dynamics.

In a warped compactification, the 4-d theory includes dynamical gravity. So it is natural to study the formation and behavior of 4-d black holes in this context. From the phenomenological viewpoint of the brane world model, these are 5-d black holes whose horizon intersects with the 4-d Planck bane.

These brane world black holes have been studied quite extensively in [27][28]. Rather than studying these black holes as static objects, we will instead focus our attention on the transition that takes place during the process of 4-d gravitational collapse, i.e. the transition from a bulk black hole to a brane world black hole. This transition is particularly fascinating, because it provides a direct link between the relatively well-understood holography of 5-d AdS bulk black holes and the much less well understood holography of black holes in 4-d flat space.

In outline, our plan is as follows. First we review how 4-d effective field theory arises from a holographic dimensional reduction of the higher dimensional theory inside the warped geometry. We then study the 4-d stress-energy distribution of a collapsing cloud, as the holographic dual of a 5-d (or in fact 10-d) black hole, localized inside the infrared part of the warped geometry, following a geodesic motion towards the "Planck brane". We then show how the presence of the Planck brane affects this motion, by inducing an effective 4-d gravitational force. We end with a number of qualitative and semi-quantitative remarks about how one may start to study the process of gravitational collapse from this higher dimensional perspective.

3.2 Stress Energy in a Warped compactification

Corresponding to the decomposition (3.2) of the 10-d geometry, we can write the 4-d low energy effective action $S_{\rm eff}$ as a sum of two terms, given by the integral of the 10-d supergravity lagrangian $\mathcal{L}_{sugra}$ over the two respective submanifolds:

$$S_{\rm eff}(g) = S_{UV}(g) + S_{IR}(g) \tag{3.4}$$

with

$$S_{IR} = \int_{\Sigma_{IR}} \mathcal{L}_{sugra}\,, \qquad\qquad S_{UV} = \int_{\Sigma_{UV}} \mathcal{L}_{sugra}. \tag{3.5}$$

Here g denotes the metric on the boundary $R^4 \times K^5$ that connects Σ_{UV} and Σ_{IR}; both action functionals are evaluated over the unique classical solution of the supergravity equations of motion as specified by the boundary condition g. (Recall here that the internal geometries of Σ_{UV} and Σ_{IR} are both compact manifolds with only one boundary, and therefore it is sufficient to specify the boundary values only; the normal derivatives are (uniquely) determined by integrating the classical equations of motion inwards.)

Since in the following we will be mostly interested in the 4-d effective gravitational dynamics, we will concentrate on the metric dependence of S and ignore all possible matter and moduli fields. In addition, we assume that the metric depends on the 4-d space-time coordinates only. At low energies compared to the size of K_6, the Σ_{UV} part of the effective action reduces to a local 4-d action functional, and therefore takes the general form

$$S_{UV}(g) = \int \sqrt{-g}\,(b_1 R + b_0). \tag{3.6}$$

This local action gives the leading contribution to the 4-d Einstein action. In the terminology of the brane world scenario, S_{UV} can be thought of as the action on the Planck brane; b_0, in particular, represents the brane tension.

The other contribution S_{IR} represents the quantum effective action of the conformal matter dual to the AdS bulk gravity. It can be split into a local term and a non-local term:

$$S_{IR} = S_{loc} + \Gamma \tag{3.7}$$

with

$$S_{loc} = \int \sqrt{-g}\,(c_1 R + c_0). \tag{3.8}$$

The holographic interpretation of this local term is that it represents the renormalization of the Newton and cosmological constant due to the quantum fluctuations of the conformal matter. As we will see shortly, the constants c_1 and c_0 are uniquely determined by the 5-d bulk Newton and cosmological constants.

For the field on the boundaries of the two halves to join smoothly, we need the condition [29]

$$\frac{\delta S_{\text{eff}}}{\delta g^{\mu\nu}} = 0. \tag{3.9}$$

This equation expresses the fact that the normal derivative of the metric must be continuous along the boundary that joins Σ_{UV} and Σ_{IR}. Inserting (3.6) and (3.7)-(3.8), this condition can be written in the form of a conventional 4-d Einstein equation

$$(R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R) - \frac{1}{2}\Lambda_4 g_{\mu\nu} = 16\pi G_4 \langle T_{\mu\nu}\rangle \tag{3.10}$$

Here G_4 and Λ_4 are the 4-d Newton and cosmological constants

$$\frac{1}{16\pi G_4} = b_1 + c_1\,, \qquad \frac{\Lambda_4}{16\pi G_4} = c_0 + b_0 \tag{3.11}$$

and we defined the expectation value of the stress energy tensor via

$$\frac{1}{\sqrt{-g}}\frac{\delta\Gamma}{\delta g^{\mu\nu}} = \langle T_{\mu\nu}\rangle. \tag{3.12}$$

For the Planck brane to be flat for $T_{\mu\nu} = 0$, we must fine-tune the brane tension b_0 to cancel the bulk contribution c_0, so that $\Lambda_4 = 0$.

We will use the Hamilton-Jacobi formalism to study the 5-d gravity equations of motion in ADM form. The ADM metric can be written as

$$ds^2 = N^2 dr^2 + g_{\mu\nu}(x,r)(dx^\mu + N^\mu dr)(dx^\nu + N^\nu dr) \tag{3.13}$$

where N is the lapse function and N^μ is the shift function. We are working in Euclidean signature and r plays the role of a euclidean time. The flow in the radial direction, r, will correspond to the holographic renormalization group flow. We can use diffeomorphism invariance to choose the gauge, $N = 1$ and $N^\mu = 0$. The canonical momentum dual to the 4-d metric is given by

$$\pi^{\mu\nu} = \frac{1}{\sqrt{-g}}\frac{\delta S_{IR}}{\delta g_{\mu\nu}} = c_0 g^{\mu\nu} + c_1(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) + T^{\mu\nu} \tag{3.14}$$

where we used eqns (3.7), (3.8) and (3.12). Geometrically, $\pi^{\mu\nu}$ is expressed in terms of the extrinsic curvature $K^{\mu\nu}$ via

$$\pi^{\mu\nu} = K^{\mu\nu} - Kg^{\mu\nu}$$

$$K^{\mu\nu} = \nabla^\mu \hat{n}^\nu + \nabla^\nu \hat{n}^\mu \tag{3.15}$$

with $\hat{n}^\mu$ the normal vector to the Planck brane.

We still need to impose the constraints from the equations of motion. The equations of motion for N^μ give the diffeomorphism constraints and the equation of motion for N gives the Hamilton constraint which provide the Hamilton-Jacobi equation.

Let us introduce the notation

$$\{S, S\} = \frac{1}{\sqrt{-g}}\left(\frac{1}{3}\left(g^{\mu\nu}\frac{\delta S}{\delta g^{\mu\nu}}\right)^2 - g^{\rho\mu}g^{\lambda\nu}\frac{\delta S}{\delta g^{\mu\nu}}\frac{\delta S}{\delta g^{\rho\lambda}}\right). \tag{3.16}$$

The Hamilton-Jacobi equation for the 5-d bulk gravity can then be written in the compact form,

$$\{S_{IR}, S_{IR}\} + \sqrt{-g}(R + \Lambda) = 0. \tag{3.17}$$

We now separate S_{loc} into terms with different scaling dimensions

$$S^{(0)}_{loc} = c_0 \int \sqrt{-g}, \qquad S^{(2)}_{loc} = c_1 \int \sqrt{-g}R \tag{3.18}$$

and impose the condition that all terms of a given scaling dimension sum to zero in the Hamilton-Jacobi equation. We get

$$\{S^{(0)}_{loc}, S^{(0)}_{loc}\} = -\sqrt{-g}\Lambda \implies \Lambda = -\frac{1}{3}c_0^2 \tag{3.19}$$

$$2\{S^{(2)}_{loc}, S^{(0)}_{loc}\} = -\sqrt{-g}R \implies c_1 = -\frac{3}{c_0}. \tag{3.20}$$

The next order relation

$$2\{S_{loc}^{(0)}, \Gamma\} + \{S_{loc}^{(2)}, S_{loc}^{(2)}\} = 0 \tag{3.21}$$

gives the conformal anomaly equation

$$\langle T\rangle = -\frac{\sqrt{3}}{(-\Lambda)^{3/2}} \left(R^2 - 3R^{\mu\nu}R_{\mu\nu}\right). \tag{3.22}$$

The conformal anomaly is a quantum effect in the 4-d gauge theory coming from a one loop diagram. Here we have reproduced the result from the 5-d perspective using classical dynamics. This illustrates how the classical 5-d dynamics in fact knows about the quantum effects of the 4-d system.

Summarizing, we see that the 4-d effective Einstein equations of motion (3.10) involve a stress-energy contribution that arises from "integrating out" the degrees of freedom that reside in Σ_{IR}. In most studies thus far, this stress-energy distribution was taken to be static. In the following section, we will consider a time-dependent solution to the 5-d equations of motion that in 4-d corresponds to a contracting cloud of matter.

3.3 5-d Perspective on a Contracting Cloud

Here we will describe how one can use the global conformal symmetry to obtain an explicit solution for a collapsing stress-energy distribution, as well as for the dual 5-d geometry. We will assume that the boundary has the topology of $S^3 \times R$, with metric

$$ds^2 = -d\tau^2 + R^2(d\theta^2 + \sin^2\theta\, d\Omega_2^2). \tag{3.23}$$

To introduce a finite stress energy, we begin with heating up the gauge theory matter, so that the dual 4+1-d geometry becomes that of an AdS-Schwarzschild black hole with metric

$$ds^2 = -\Big(1 + u^2 - \frac{\mu}{u^2}\Big)\, d\tau^2 + \frac{du^2}{1 + u^2 - \mu/u^2} + u^2(d\theta^2 + \sin^2\theta\, d\Omega_2^2). \tag{3.24}$$

Here the radius of curvature of AdS space has been set to 1 and the parameter μ is related to the mass M of the black hole via

$$\mu = \frac{8G_5 M}{3\pi}. \tag{3.25}$$

The temperature of the 3+1-d gauge theory matter, in units of the radius R of S^3, is given by

$$\beta_0 = \frac{2\pi u_0}{2u_0^2 + 1}, \qquad\qquad u_0^2 + 1 - \frac{\mu}{u_0^2} = 0. \tag{3.26}$$

Here u_0 denotes the location of the 5-d black hole horizon. The boundary stress-energy distribution corresponding to this solution can be calculated by using the holographic prescription discussed in the previous section [30]

$$T_{ab} = \mathrm{diag}\left[\rho,\, p\,,\, p\sin^2\theta,\, p\sin^2\theta\sin^2\phi\right]\,, \qquad p = \rho/3 \tag{3.27}$$

where $a, b = \{\tau, \theta, \phi, \psi\}$ and

$$\rho = \rho_0 + \rho_{vac} \qquad \rho_0 = \frac{M}{2\pi^2 R^3}, \qquad \rho_{vac} = \frac{3}{64\pi G_5 R}. \tag{3.28}$$

Here ρ_{vac} is a non-zero Casimir energy that arises from the finite curvature of the S^3.

We will now perform a conformal boost on the thermal plasma, setting it into a specific converging motion, so that all matter eventually gets concentrated inside a small region, say at $\theta = 0$. To find the appropriate form of the conformal boost, we note that after the coordinate transformation $t \pm r = tan(\frac{\tau\pm\theta}{2})$, the metric becomes

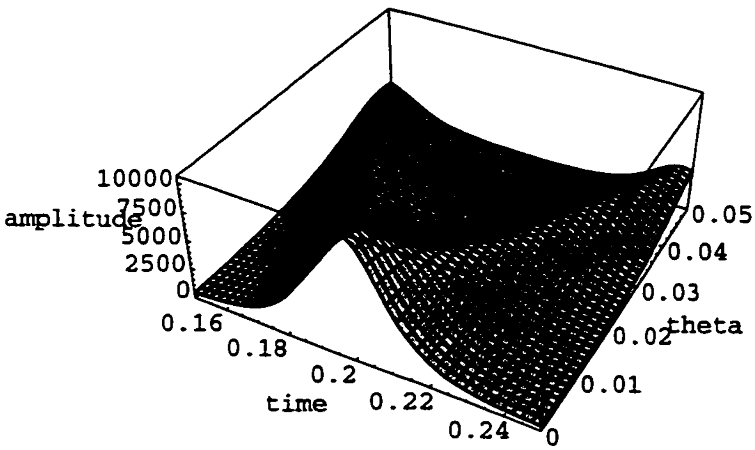

Figure 3.1: The amplitude of the energy density $T_{\tilde{\tau}\tilde{\tau}}$ is plotted as a function of time $\tilde{\tau}$ and angle $\tilde{\theta}$. We have set $\rho_0 = 0$ and $|a| = 10$.

conformal to Minkowski space. In Minkowski space coordinates, the special conformal transformations on the four vector $\vec{l} = (t, \vec{r})$ is given by

$$\vec{l}' = \frac{\vec{l} + \vec{b}l^2}{1 + 2\vec{b} \cdot \vec{l} + b^2 l^2}. \tag{3.29}$$

We will choose $\vec{b} = (a, 0, 0, 0)$. If we now go back to the metric on S^3, this special conformal transformation implies

$$\cot \tilde{\tau}_\pm = \cot \tau_\pm + a, \qquad \tau_\pm = \frac{1}{2}(\tau \pm \theta). \tag{3.30}$$

To obtain the transformed stress tensor, we apply this conformal transformation on the classical part (proportional to ρ_0) only, and not on the extra Casimir energy density ρ_{vac}. To see that this is the right prescription, recall that the transformation we are performing is in fact a combined coordinate and Weyl transformation, with the combined effect that the new metric is again the standard metric (3.23) on S^3.

Unlike the classical part, the Casimir energy contribution is not Weyl invariant, and in precisely such a way that the compensating Weyl transformation cancels the effect of the coordinate transformation. Concentrating on the classical part, we find that it takes the following form:

$$T_{\tilde{\tau}\tilde{\tau}} = \rho_0(W_+^2 + \frac{1}{3}W_-^2), \qquad T_{\tilde{\tau}\tilde{\theta}} = \frac{4}{3}\rho_0 W_+ W_-, \qquad T_{\tilde{\theta}\tilde{\theta}} = \rho_0(\frac{1}{3}W_+^2 + W_-^2), \tag{3.31}$$

$$W_\pm = \frac{F(\tilde{\tau}_+) \pm F(\tilde{\tau}_-)}{2}, \qquad F(\tilde{\tau}_\pm) = \frac{1}{\sin^2(\tilde{\tau}_\pm) + \left(\cos(\tilde{\tau}_\pm) - a\sin(\tilde{\tau}_\pm)\right)^2}. \tag{3.32}$$

It is easily verified that the energy density $T_{\tilde{\tau}\tilde{\tau}}$ indeed oscillates via a subsequent contracting and expanding motion towards the two poles, $\tilde{\theta} = 0$ and $\tilde{\theta} = \pi$, see figure 3.1. We will mostly be interested in the moment of maximal compression in the limit that a becomes large. Note that the energy density is quite highly concentrated in an annular form as it oscillates between the poles.

From the 5-d perspective, applying the conformal boost amounts to giving the AdS black hole an initial radial velocity towards the Planck brane. To write the isometry of AdS_5 that corresponds to the above special conformal transformation, we introduce the matrix

$$M = \begin{pmatrix} X_0 + X_1 & X_2 - X_3 \\ X_2 + X_3 & -X_0 + X_1 \end{pmatrix} \tag{3.33}$$

$$X_0 = \sqrt{1+u^2}\cos\tau, \qquad X_1 = u\cos\theta$$

$$X_2 = \sqrt{1+u^2}\sin\tau, \qquad X_3 = u\sin\theta$$

on which the above special conformal transformation acts via

$$M' = AMA^T, \qquad A = \begin{pmatrix} 1 & a \\ 0 & 1 \end{pmatrix}. \tag{3.34}$$

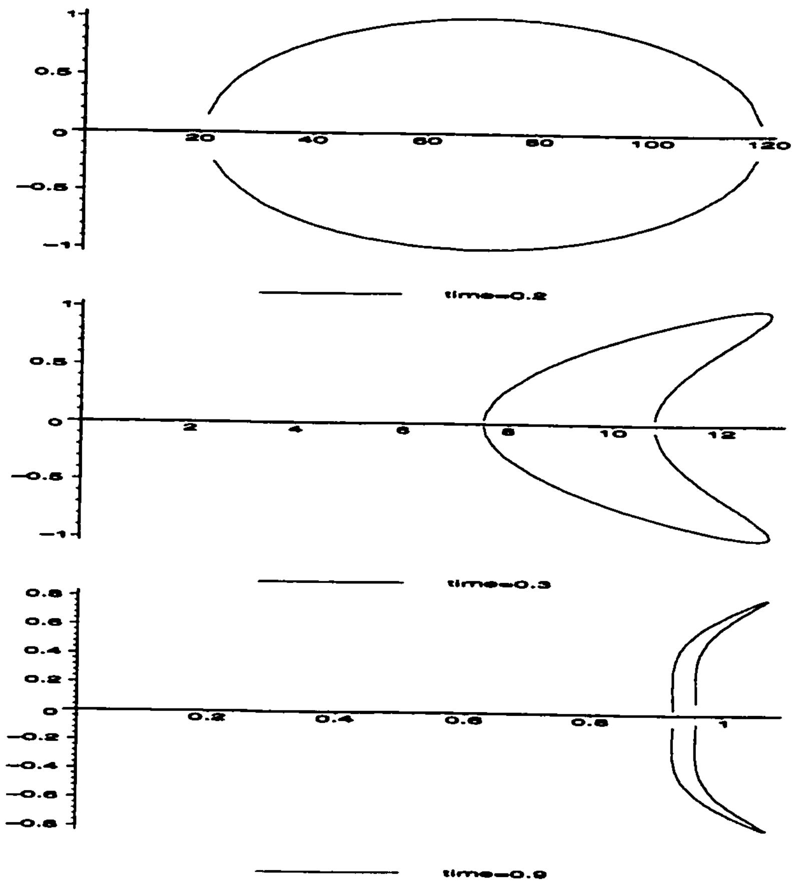

Figure 3.2: Location and shape of 5-d black hole horizon at times, $\tilde{\tau} = 0.2, 0.3, 0.9$. We have set $u_0 = 1$ and $a = 10$. The plot is in spherical polar coordinates where the right horizontal axis corresponds to $\tilde{\theta} = 0$ and the vertical axis corresponds to $\tilde{\theta} = \pi/2$.

In principle we can apply this transformation to the full black hole solution (3.24) and obtain a time-dependent metric describing a 5-d black hole oscillating back and forth towards the boundary. We will restrict our attention to the time-dependent location $(\tilde{u}, \tilde{\phi}, \tilde{\theta})$ of the horizon of this solution. Using the above characterization of the isometry, we obtain

$$\tilde{u} = |U|, \qquad U = \frac{a^2}{2}\left(u_0 \cos\theta + \sqrt{1+u_0^2}\cos\tau\right) \tag{3.35}$$

$$\tan\tilde{\theta} = \frac{u_0 \sin\theta}{U}, \qquad \tan\tilde{\tau} = \frac{2}{a} + \frac{\sqrt{1+u_0^2}}{U}\sin\tau \tag{3.36}$$

where we assumed that $a \gg 1$. See figure 3.2 for an evolution of the black hole horizon. When the black hole is at the center of the bulk space, it is moving at its fastest towards one of the poles and is shaped like a pancake. This is consistent with Lorentz contraction in its direction of motion as well as with the energy density of the boundary theory being concentrated in a ring around the equator of the boundary. As the black hole moves in towards the pole, the edge of the "pancake" black hole starts to curl inward pointing to the annular region on the boundary where the energy density has moved to. When the black hole is near its closest approach to the boundary, it is shaped like an elongated ellipsoid pointing at the pole where the energy density on the boundary is now concentrated.

It should be straightforward to extract the CFT stress-energy tensor from the time dependent bulk geometry, using the formalism of [30]. One expects to find the exact same stress-energy contribution as described above, plus the extra Casimir energy contribution coming from the conformal anomaly. We will not explicitly perform this calculation here.

3.4 5-d Perspective on 4-d Gravitational Collapse

We have described a 5-d black hole in pure AdS space in an oscillating trajectory, such that at the moment of maximal amplitude it represents a maximally compressed cloud of matter on the boundary. Due to the negative curvature of the AdS space-time, the black hole feels a force that pulls it away from the boundary. This corresponds to the tendency of the 4-d CFT cloud to disperse. Now what happens when we add 4-d gravity, that is, compactify the AdS-geometry with an effective Planck brane at some radial location $u = u_{\rm pl}$? We will see that an interesting 5-d picture emerges: as we show in the Appendix, the presence of the Planck brane produces a new 5-d gravitational force on the black hole that will counteract the acceleration due to the AdS curvature, and pulls it towards the Planck brane. This force is the 5-d manifestation of 4-d gravity, that will try to compress the cloud. Intuitively, we can view this new force as the attraction due to a "mirror mass" located at the opposite side of the Planck brane. Which of the two forces wins, the AdS acceleration or the 4-d gravity, or whether by chance they may balance each other, depends on the mass and distance of the black hole to the boundary, or equivalently, the mass and size of the cloud.

So we have set up a situation in which *in principle* we can start to study 4-d gravitational collapse from a higher dimensional perspective. *In practice*, however, this is a very complicated task, because the full dynamical equations of motion one would need to solve are highly non-linear. In the following we will contend ourselves with making a number of qualitative and semi-quantitative remarks.

(i) Before addressing the time-dependent problem, it is useful to first contemplate which possible time-independent solutions may exist. In principle we can imagine three different situations as indicated in fig 3.3.

First, it could happen that a bulk black hole feels a net zero force, due to cancellation of the AdS acceleration with the gravitational pull of the Planck brane. Assuming the black hole horizon does not intersect the Planck brane, this corresponds to a static cloud: a star made from CFT matter! Note however, that it is an unstable star: since (relative to the constant AdS acceleration) the 4-d gravitational pull gets stronger near the Plank brane, any perturbation of the center of mass will result in a falling

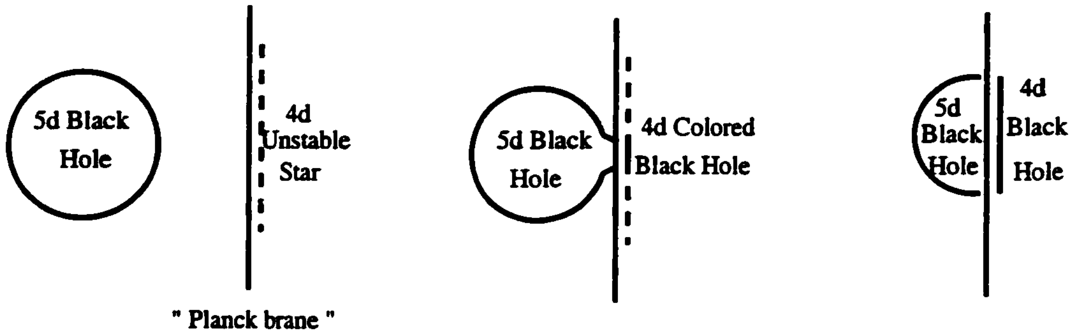

Figure 3.3: There are three static solutions where the gravitational attraction of the Planck brane is balanced by the acceleration of AdS space. If the 5d black hole doesn't intersect the Plank brane, an unstable 4d colored star forms. If the 5d black hole intersect the Planck brane but its center of mass still lies in the bulk, then a 4d colored black hole forms. The last solution is a collapse of the 5d black hole onto the Planck brane forming a 4d black hole.

motion either away or towards the Planck brane. In 4-d language: the CFT star is unstable against dispersion or gravitational collapse. Apart from this instability, it is clear that the 5-d black hole, though deformed by the presence of the Planck brane, will otherwise remain intact. The possible existence of these CFT stars is further supported by the known existence of classical colored stars [32], which are similar unstable solutions to the classical Yang-Mills-Einstein equations (which classically is also a conformally invariant interacting theory coupled to gravity).

As a second type of static situation, it is conceivable that the two forces balance for a 5-d black hole, with a center of mass still in the 5-d bulk, but with a horizon that already intersects the Planck brane. This is an unstable colored black hole: on the outside it looks like a star of CFT matter, while its center in fact contains a (potentially very) small black hole. Again, such solutions to classical YME equations are known to exist [34].

Finally, the 5-d black hole can collapse to a brane-world black hole. From the 4-d perspective this is just a 4-d black hole.

(ii) To decide conclusively whether one or more of the three situations depicted in fig 3.3 is actually realizable as an exact solution, one would need to solve the bulk 5-d Einstein equations in combination with the full non-linear equation of motion for the Planck brane. This last equation can be obtained from the effective 4-d Einstein equation (3.10), which can be recast, using (3.14)(3.15), as a condition that relates the intrinsic and extrinsic brane geometry as follows:

$$b_0 g^{\mu\nu} + b_1(R^{\mu\nu} - \frac{1}{2}Rg^{\mu\nu}) = K^{\mu\nu} - Kg^{\mu\nu}. \tag{3.37}$$

One possible procedure for trying to find exact solutions, which was followed with success in [28] to find a brane-world black hole in 3+1-d AdS, is to start with a static 5-d geometry of an accelerating black hole at a constant AdS radial position (held at a fixed distance from the boundary by means of a string), and then try to find a suitable trajectory for the Planck brane solving (3.37). For another approach see [36]. An interesting aspect of this situation, as emphasized in [31], is that it in fact describes a solution to the 4-d effective Einstein equations with a stress-energy source that includes the quantum corrections due to the conformal anomaly.

(iii) From the full 10-d perspective, the transition between the bulk and brane-world black hole is topologically described as follows. The bulk black hole dual to the thermal cloud has an 8-dimensional horizon with topology

$$H_8 = S^3 \times K_5. \tag{3.38}$$

It is entirely contained inside Σ_{IR}. During the gravitational collapse, the black hole horizon extends into the UV part of the compactification manifold, and eventually fully wraps Σ_{UV}. The horizon of the collapsed black hole has the topology

$$H_8 = H_{IR} \cup H_{UV} \tag{3.39}$$

with

$$H_{IR} = B_3 \times K_5\,, \qquad H_{UV} = S^2 \times (K_6 - B_6). \tag{3.40}$$

Both components are glued together at the common boundary $S^2 \times K_5$. The transition from (3.38) to (3.39)-(3.40) is a higher dimensional version of the inverse Gregory-Laflamme transition.

The entropy of both the bulk black hole and the collapsed black hole is in complete accordance with the 4-d interpretation. The 10-d Planck area of (3.38) is the 5-d Planck area of the S^3, which is known to match with the standard extensive entropy of the corresponding thermal 4-d CFT cloud. The leading contribution to the 10-d Planck area of the collapsed black hole horizon (3.39)-(3.40), on the other hand, comes from H_{UV} which essentially equals the 4-d Planck area of the S^2.

(iv) Turning to the time-dependent evolution, we can similarly imagine three basic types of dynamical behavior, separated by the static solutions described above. All these three types of behavior have indeed been found in numerical studies of the classical Einstein-Yang-Mills equations [32] [33] [34].

Firstly, the 5-d black hole may stay far enough away from the Planck brane so that the effective 4-d gravity remains too weak to counter-act the AdS acceleration. The 5-d black hole, and the corresponding CFT cloud, essentially follow the same trajectory as described in the previous section. Only now, they are slightly deformed by the presence of 4-d gravity. We call this the dispersion regime.

Secondly, the 5-d black hole can have a large enough initial velocity, so that it comes so close to the Planck brane that it collapses to a brane-world black hole. We will call this the regime of type I gravitational collapse: in the boundary theory, it represents the 4-d black hole formation in which almost all of the CFT matter ends up inside the horizon. This regime is separated from the dispersion regime by the process of unstable star formation.

Finally, one can imagine an intermediate regime in which the center of mass of the 5-d black hole is far enough from the boundary to be pulled away from it by the AdS acceleration, while the horizon still extends far enough to intersect with the Planck brane. From the 4-d perspective, this intersection looks like the formation of a small black hole inside of a CFT cloud. The cloud eventually disperses, leaving behind a small black hole. This type of phenomenon is indeed known to occur in classical Einstein-Yang-Mills theory, and is known as type II gravitational collapse. It is separated from the type I collapse regime by the formation of a colored black hole.

(v) We can make some rough estimates on the range of parameters in which these three types of behavior can occur. The motion described in the previous section is parametrized by two basic parameters: the mass density ρ_0 of the initial static background, and the boost parameter a. Let us first determine in which regime we expect type I collapse. For this it is sufficient to consider the linearized gravity equations.

At the moment of maximal contraction and for small θ, the energy density takes the form

$$\rho \simeq \frac{48\rho_0 a^2}{(4+a^4\theta^2)^2} \tag{3.41}$$

and the total energy at this moment equals (here R is the radius of the three sphere)

$$E_{tot} \simeq \frac{24\pi^2\rho_0 R^3}{a^2} \tag{3.42}$$

In the Newtonian approximation, the gravitational self-energy of this mass distribution is

$$V_{grav} \simeq -288\, G_4 \frac{\pi^3\rho_0^2\, R^5}{a^2} \tag{3.43}$$

This result is accurate for as long as the energy density and its backreaction on the geometry remain small. This assumption clearly breaks down when ρ_0 is taken so large that V_{grav} becomes of the same order as E_{tot}. This happens around

$$\rho_{0,\text{crit}} = \frac{1}{12\pi G_4 R^2} \simeq \frac{1}{\ell_{\text{pl}}^2 R^2}. \tag{3.44}$$

At this point the system becomes unstable against type I gravitational collapse. Note that this criterion of type I collapse is in fact, in leading order, independent of the conformal boost parameter a. This is somewhat counter-intuitive. The reason is that, although the cloud is highly compressed for large a, the total energy also scales inversely with a^2. Still it is clear that if a gets too large, the center of mass of the 5-d black hole will inevitably reach the Planck brane.

Type II collapse occurs when the 5-d black hole horizon reaches the Planck brane. In principle the location of Planck brane is dynamical, and reacts to the stress energy of the 5-d black hole. As a first approximation, however, we can assume that, for small enough 4-d density, it stays fixed at its vacuum location. To determine this location, we first note that the location u_0 of the original 5-d black hole (the one at rest at the AdS origin) is related to ρ_0 via

$$\rho_0 = \frac{3u_0^4}{16\pi G_5} \tag{3.45}$$

where we used that $u_0 \gg 1$. This u_0^4 scaling reflects that ρ_0 has dimension $1/(\text{lenght})^4$. Now it seems reasonable to identify the radial location u_{pl} of the Planck brane with the location of the horizon of a 5-d black hole corresponding to a 4-d CFT thermal gas with the 4-d Planck density. This gives

$$u_{\text{pl}} \simeq \frac{u_0}{(\rho_0)^{1/4}\ell_{\text{pl}}}. \tag{3.46}$$

Next, from (3.35) we see that, at the moment of maximal amplitude, the horizon of the boosted 5-d black hole reaches radial location $u_{\text{max}} \simeq a^2 u_0$. Thus the horizon will

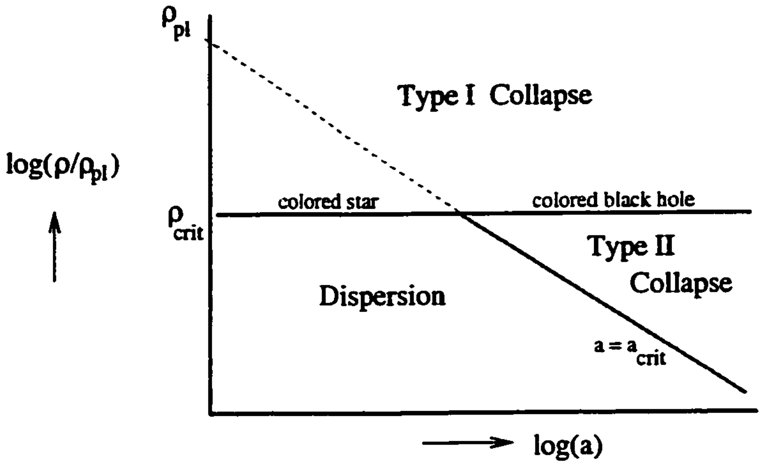

Figure 3.4: The phase diagram for gravitational collapse of 4d gauge field.

start to intersect the Planck brane around the critical value

$$a^2_{\rm crit} \simeq \frac{1}{(\rho_0)^{1/4}\ell_{\rm pl}}. \tag{3.47}$$

Assuming that ρ_0 is (sufficiently) below the critical value (3.44) for type I collapse, this critical value $a_{\rm crit}$ separates dispersion from type II collapse. It is interesting to note that the transition to type II collapse seems to occur while the maximal density at the center of the CFT cloud is $\rho_{\rm max} \simeq \rho_0 a^2_{\rm crit} \simeq (\rho_0)^{3/4}/\ell_{\rm pl}$, which is still much smaller than the Planck density. Hence our approximation of treating the Planck brane location as fixed should still be reasonably accurate in this regime.

(vi) Thus we arrive at an interesting phase diagram for gravitational collapse as indicated in fig 3.4. This phase diagram has the same structure as the one found from numerical study of the classical Einstein-Yang-Mills equations. Although a full analysis of the non-linear dynamics will produce some modification of the diagram, we expect that this will not change the main characteristic properties. The most interesting point of the phase diagram is the tri-critical point where all the three possible regimes meet: it corresponds to a 5-d static black hole solution whose horizon just barely touches the Planck brane. It is specified by $\rho_0 = \rho_{0,\text{crit}}$ and $a = a_{\text{crit}}$, and represents the beginning point of type II collapse. When a slightly exceeds its critical value, the collapse and subsequent dispersion is expected to produce a small 4-d black hole with a mass that scales with $a = a_{\text{crit}}$ with a universal exponent

$$m_{bh} \sim (a - a_{\text{crit}})^{\gamma}. \tag{3.48}$$

This is known as Choptuik scaling. Perhaps the 5-d perspective provides some new insight into the origin of this behavior. It is tempting to speculate that the exponent γ may be related to the cusp like shape of the 5-d horizon when it touches the Planck brane.

Chapter 4

Chiral Compactifications of 6D Conformal Theories

4.1 Introduction

In recent years it has been realized that many 3+1D gauge theories can be obtained as special low-energy limits of compactified 5+1D superconformal theories. Some of the known 5+1D theories are the $\mathcal{N} = (2,0)$ theory [37], the E_8 $\mathcal{N} = (1,0)$ theory [38] and the Blum-Intriligator (BI) [39] theories of N M5-branes at an A_{k-1} singularity.

Indeed, part of the appeal of these theories is that by compactification on $\mathbf{T}^2$ we can get various gauge theories in 3+1D at low-energy. Thus, $\mathcal{N} = 4$ SYM is obtained from the (2,0)-theory [37] and $\mathcal{N} = 2$ SYM with various matter content is obtained from the E_8 $\mathcal{N} = (1,0)$ theory [40, 41].

Starting with the 5+1D BI theories we can compactify on $\mathbf{S}^1$ to get, at low-energies, the $\mathcal{N} = 2$ quiver gauge theories with gauge group $SU(N)^k$ and bi-fundamental matter hypermultiplets [42]. One can also realize a mass to the hypermultiplets by using the global $U(1)$ symmetry of the BI theories. Turning on a small background

Wilson line for that $U(1)$ corresponds at low energy to turning on the mass [43].

In this chapter we will construct chiral 3+1D theories from the BI theories. As an intermediate step, we start with a 4+1D hypermultiplet. Given a hypermultiplet in 4+1D we can construct a low-energy chiral multiplet as follows. Let us take an infinite 5^{th} direction and let us give the fermions of the hypermultiplet a mass $m(x_5)$ that varies along the 5^{th} direction from $m = -\infty$ at $x_5 = -\infty$ to $m = \infty$ at $x_5 = \infty$ (see [44]). As we shall review below, if we also let the scalar fields have masses $\sqrt{m^2 \pm \frac{dm}{dx_5}}$ then in the remaining 4 dimensions $\mathcal{N} = 1$ supersymmetry is preserved and at low energies we get a chiral multiplet localized near the point where $m(x_5) = 0$. Thus, by varying the mass of the hypermultiplets in a 5D gauge theory along the 5^{th} direction, we can obtain, at low-energies, a chiral gauge theory in 4D.

A 5D gauge theory is only defined as a low-energy effective action. However, we can realize it as a 6D theory compactified on a circle. We would like to elevate the construction of chiral gauge theories to 6D. One motivation for that is that a 6D realization often provides insight into the strong coupling behavior of the theory. The 6D theories that we will use are the BI theories and the construction of chiral gauge theories from their compactifications is the purpose of this chapter.

The chapter is organized as follows. In section (2) we review the example of a 4+1D hypermultiplet. In section (3) we study the compactification of a general 5+1D theory. In section (4) we discuss the BI theories and their compactification.

4.2 A free hypermultiplet

In this section we will study a free hypermultiplet in 5+1D and 4+1D. The reason for studying this simple system is that it gives us an explicit realization of the mechanism which produces chiral matter in 3+1D. We will later apply the same type of compactification to obtain chiral matter in 3+1D starting from 5+1D theories.

We will show that a 4+1D hypermultiplet with a mass that varies along the 5th direction preserves $\mathcal{N}=1$ SUSY in 3+1D and gives rise to chiral multiplets. The 4+1D hypermultiplet with a varying mass can be obtained from a 5+1D hypermultiplet compactified on a circle and coupled to a background field.

1 A 5+1D chiral hypermultiplet

A convenient way of getting the quantum numbers of a 5+1D hypermultiplet is to start from 9+1D super Yang-Mills reduced to 5+1D. This theory comprises of a single multiplet under the $\mathcal{N}=(1,1)$ SUSY. However, under an $\mathcal{N}=(1,0)$ subgroup of the supersymmetry algebra it decomposes into a vector-multiplet and a hypermultiplet. The statements below follow easily by thinking about the system in this way.

A hypermultiplet in 5+1D (with $\mathcal{N}=(1,0)$ supersymmetry) contains 4 real scalars and one chiral fermion. It is convenient to decompose the components under the Lorentz group $SO(5,1)$, the R-symmetry group $SU(2)_R$ and the global flavor symmetry $SU(2)_F$. Under $SO(5,1)\times SU(2)_R\times SU(2)_F$ the SUSY generators Q^i_α transform as $(\underline{4},\underline{2},\underline{1})$. Note that both $\underline{4}$ and $\underline{2}$ are pseudo-real representations so one can add a reality condition to have 8 real SUSY generators. Here $i=1,2$ is an index of the $\underline{2}$ of $SU(2)_R$ and $\alpha=1\ldots4$ is an index of the $\underline{4}$ of $SO(5,1)$. We will assume that the hypermultiplet is charged under $SU(2)_F$. The fermions of the hypermultiplet transform as $(\underline{4},\underline{1},\underline{2})$ with an added reality condition. We will denote them by ψ^a_α with $a=1,2$ an index of $SU(2)_F$. The bosons transform as $(\underline{1},\underline{2},\underline{2})$ and will be denoted by ϕ^{ia}. Recall that the Dirac matrices, $\Gamma^\mu_{\alpha\beta}$ $(\mu=0\ldots5)$, of $SO(5,1)$ can be chosen to be anti-symmetric. In the rest of the chapter they will be anti-symmetric. We will also use the anti-symmetric ϵ_{ij} of the $\underline{2}$ of $SU(2)_R$ to lower and raise the indices $i,j=1,2$.

The reality conditions are,

$$(\phi^{ia})^\dagger = C_b{}^a C_j{}^i \phi^{jb}, \qquad (\psi^b_\beta)^\dagger = C_a{}^b C_\beta{}^\alpha \psi^a_\alpha, \tag{4.1}$$

where $C_b{}^a$, $C_j{}^i$ and $C_\beta{}^\alpha$ are the charge conjugation matrices of (respectively) $\underline{2}$ of $SU(2)_F$, $\underline{2}$ of $SU(2)_R$ and $\underline{4}$ of $SO(5,1)$.

The action is

$$S = \int d^6x \left(-\frac{1}{4}\epsilon_{ij}\epsilon_{ab}\partial_\mu\phi^{ia}\partial^\mu\phi^{jb} + \frac{1}{2}\epsilon_{ab}\psi^a_\alpha\Gamma^{\mu\alpha\beta}\partial_\mu\psi^b_\beta \right).$$

Our sign conventions are $\epsilon_{12} = \epsilon^{12} = 1$. The equations of motion derived from this action are

$$\Box\phi^{ia} = 0, \qquad \Gamma^{\mu,\alpha\beta}\partial_\mu\psi^a_\alpha = 0.$$

The supersymmetry transformations are:

$$\delta\phi^{ia} = 2\eta^{\alpha i}\psi^a_\alpha, \qquad \delta\psi^a_\alpha = \epsilon_{ij}\eta^{\beta i}\Gamma^\mu_{\alpha\beta}\partial_\mu\phi^{ja}.$$

2 A 4+1D massive hypermultiplet

Now we will consider a massive hypermultiplet in 4+1D. The quantum numbers, action and supersymmetry transformations of this can easily be obtained from the 5+1D case. We consider a 5+1D hypermultiplet with a specific x^5 dependence.

$$\begin{aligned} \phi^a(x, x^5) &= \phi^b(x)(e^{imx^5\tau^3})^a{}_b \\ \psi^a(x, x^5) &= \psi^b(x)(e^{imx^5\tau^3})^a{}_b \end{aligned} \tag{4.2}$$

Here x stands for x^0, x^1, x^2, x^3, x^4 and

$$\tau^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is inserted to give the right sign in the exponential. ϕ^1 and ϕ^2 must have different signs because of the reality condition (4.1). The quantum numbers are the same as in 5+1D. A 4+1D massive hypermultiplet contains 4 bosons ϕ^{ia} in the $(\underline{1}, \underline{2}, \underline{2})$ of $SO(4,1) \times SU(2)_R \times SU(2)_F$, where $SO(4,1)$ is the Lorentz group and $SU(2)_R$ and $SU(2)_F$ are the R-symmetry and flavor symmetry, respectively. It also has fermions $\psi^{\alpha a}$ in the $(\underline{4}, \underline{1}, \underline{2})$. Recall that the representation $\underline{4}$ of $SO(4,1)$ has an invariant anti-symmetric form $\epsilon_{\alpha\beta}$ which we will use to lower and raise indices. From the 5+1D point of view that is just Γ^5 which commutes with $SO(4,1)$ transformations. The action in 4+1D is obtained simply by plugging the fields in (4.2) into the 5+1D action.

$$\begin{aligned} S \;=\; & \int d^5x \Big(-\frac{1}{4}\epsilon_{ij}\epsilon_{ab}(\partial_\mu\phi^{ia}\partial^\mu\phi^{jb} + m^2\phi^{ia}\phi^{jb}) + \frac{1}{2}\epsilon_{ab}\psi^a_\alpha\Gamma^{\mu\alpha\beta}\partial_\mu\psi^b_\beta \\ & + \frac{1}{2}im\epsilon_{ab}(\tau^3)_c{}^b\psi^a_\alpha\Gamma^{5\alpha\beta}\partial_5\psi^c_\beta \Big) \end{aligned}$$

The equations of motion follow:

$$(\Box + m^2)\phi^{ia} = 0, \qquad \Gamma^{\mu,\alpha\beta}\partial_\mu\psi^a_\alpha + im(\tau^3)_b{}^a\Gamma^{5\alpha\beta}\psi^b_\alpha = 0.$$

The reality conditions on the fields are the same as in 5+1D, as is obvious from the way we obtained them. The SUSY transformations are obtained from the 5+1D transformations:

$$\begin{aligned} \delta\phi^{ia} &= 2\eta^{\alpha i}\psi^a_\alpha, \\ \delta\psi^a_\alpha &= \epsilon_{ij}\eta^{\beta i}\Gamma^\mu_{\alpha\beta}\partial_\mu\phi^{ja} + im\epsilon_{ij}\eta^{\beta i}\Gamma^5_{\alpha\beta}(\tau^3)^a_b\phi^{jb}. \end{aligned} \tag{4.3}$$

3 Variable mass

We will now discuss a reduction of the 4+1D massive hypermultiplet to 3+1D in a way that preserves half the supersymmetry (i.e. $\mathcal{N} = 1$ in 3+1D) and can produce chiral multiplets. This reduction was also discussed in [44]. We pick a spatial direction x^4 and let the mass vary as a function of x^4 only. Let this function be $m(x^4)$. In the previous subsection we wrote down the action and supersymmetry transformations for a massive hypermultiplet. The mass, m, was constant. The question is what action should we use when m is not constant. The only condition the new action must fulfill is that it reduces to the usual one when m is constant. However that only determines the action up to terms involving derivatives of m. Since we are interested in preserving some supersymmetry we will impose the condition that the action should be invariant under the transformations (4.3) for some η. Varying the above action, now with $m(x^4)$ a function, gives:

$$\delta(S) = \int d^5x\, m'(x^4)\epsilon_{ij}\epsilon_{ab}(\tau^3)_c{}^b\eta^{\gamma i}(i\Gamma^4\Gamma^5)_\gamma{}^\alpha\psi^a_\alpha\phi^{jc}.$$

Here $m'(x_4) \equiv dm/dx^4$. Let us try adding the following term to the Lagrangian:

$$L_{new} = \frac{1}{4}m'(x^4)\epsilon_{ab}(\tau^3)_c{}^b\epsilon_{ij}(\tau^3)_k{}^j\phi^{ia}\phi^{kc}.$$

The supersymmetry variation of this term is:

$$\delta(L_{new}) = \frac{1}{2}m'(x^4)\epsilon_{ab}(\tau^3)_c{}^b\epsilon_{ij}(\tau^3)_k{}^j 2\eta^{\alpha i}\psi^a_\alpha\phi^{kc}$$

We see that this term cancels $\delta(S)$ if

$$(\tau^3)_j{}^i\eta^{\alpha j} = \eta^{\gamma i}(i\Gamma^4\Gamma^5)_\gamma{}^\alpha \tag{4.4}$$

This equation breaks half the supersymmetry and leaves $\mathcal{N} = 1$ in 3+1D.

We thus conclude that a sensible action for a hypermultiplet with a varying mass is:

$$\begin{aligned} S &= \int d^5x\Big(-\frac{1}{4}\epsilon_{ij}\epsilon_{ab}(\partial_\mu\phi^{ia}\partial^\mu\phi^{jb} + m^2\phi^{ia}\phi^{jb} - m'(x^4)(\tau^3)_c{}^b(\tau^3)_k{}^j\phi^{ia}\phi^{kc}) \\ &\quad +\frac{1}{2}\epsilon_{ab}\psi^a_\alpha\Gamma^{\mu\alpha\beta}\partial_\mu\psi^b_\beta + \frac{1}{2}im\epsilon_{ab}(\tau^3)_c{}^b\psi^a_\alpha\Gamma^{5\alpha\beta}\partial_5\psi^c_\beta\Big). \end{aligned} \tag{4.5}$$

It preserves the supersymmetry transformations (4.3) when η solves (4.4). The equations of motion are:

$$\begin{aligned} (\Box + m(x^4)^2)\,\phi^{ia} - m'(x^4)(\tau^3)^i_j(\tau^3)^a_b\phi^{jb} &= 0, \\ \Gamma^{\mu,\alpha\beta}\partial_\mu\psi^a_\beta + im(x^4)(\tau^3)_b{}^a\Gamma^{5\alpha\beta}\psi^b_\beta &= 0. \end{aligned} \tag{4.6}$$

4 Chiral zero modes

As usual one can reduce the fields along the x^4 direction and find the modes seen from a 3+1D point of view. The x^4 direction is noncompact here. Later we will consider the compactified case. Let us find the massless modes in 3+1D. Since $\mathcal{N} = 1$ is preserved in 3+1D we know that the fields have to come in chiral multiplets. The bosons will have a 3+1D massless mode for every solution of (setting $y \equiv x^4$):

$$\left(-\frac{d^2}{dy^2} + m(y)^2\right)\phi^{ia} - m'(y)(\tau^3)_b{}^a(\tau^3)_j{}^i\phi^{jb} = 0.$$

The fermions will have zero modes for every solution of:

$$(i\Gamma^4\Gamma^5)_\alpha{}^\beta \frac{d}{dy}\psi^a_\beta + m(\tau^3)_b{}^a\psi^b_\alpha = 0.$$

We see that the bosonic equation is the square of the fermionic one in a certain sense and that the term proportional to $m'(y)$ is essential for this. The solution to the fermionic equation is:

$$\psi(y) = e^{-i\Gamma^4\Gamma^5\tau^3\int_0^y m(y')dy'}\psi_0.$$

Here we use matrix notation and suppress indices. Both matrices $(i\Gamma^4\Gamma^5)$ and τ^3 have eigenvalues $+1$ and -1. For the solution to be normalizable it is thus necessary that either:

$$\int_0^y m(y')dy' \to \infty \qquad \text{for } y \to \pm\infty.$$

or

$$\int_0^y m(y')dy' \to -\infty \qquad \text{for } y \to \pm\infty.$$

In the former case the solution is normalizable if ψ_0 has the same eigenvalue as $(i\Gamma^4\Gamma^5)$ and τ^3 and in the latter case the eigenvalues must be opposite. In both cases we end up with two chiral spinors in 3+1D which are related by the reality condition (4.1) leaving one independent chiral spinor.

The solution to the bosonic equation is:

$$\phi(y) = e^{-\tau_R^3 \tau_F^3 \int_0^y m(y')dy'} \phi_0,$$

where again we suppress indices. The τ^3 matrices are written with a subindex to distinguish the R-symmetry and the flavor-symmetry. There is a normalizable solution exactly in the same two cases of $\int_0^y m(y')dy'$ as above. In both cases there are two solutions which are related by the reality condition. So there is one massless complex boson in both cases. This one pairs up with the chiral fermion to give a massless $\mathcal{N} = 1$ chiral multiplet as we expect. (For a similar mechanism, see [45].)

The condition on $m(y)$ stated above implies in particular that $m(y)$ crosses zero at some point. A particular example of an $m(y)$ that obeys the condition is a function that goes to $-m_0$ for $y \to -\infty$, crosses zero and goes to m_0 for $y \to \infty$.

5 Flavor current multiplet

In subsection (2) above, we generated a 4+1D mass by reduction from 5+1D requiring that the fields have a specific x^5 behavior (4.2). If one just compactifies on a circle, the 4+1D theory will have a tower of Kaluza-Klein states with the lowest one being massless. The massless mode is the constant mode on the circle. The theory has a current, J_μ, associated with the $U(1)_F$ symmetry. We can introduce a background gauge field, A_μ, that couples to this current. Creating a Wilson line for the background gauge field, A_μ, around the circle is equivalent to changing the periodicity condition of the 5+1D hypermultiplet fields. They will be identified with themselves up to a $U(1)_F$ rotation. This gives them exactly the x^5 behavior of (4.2). In a circle compactification with a Wilson line for A_μ there will still be a Kaluza-Klein tower of states in 4+1D but their masses will be shifted with an amount proportional to the Wilson line. The $U(1)_F$ is part of an $SU(2)_F$ symmetry. The 5+1D hypermultiplet has a current J_μ^A ($A = 1, 2, 3$ is an index of the $\underline{\mathbf{3}}$ of $SU(2)$) associated with the

$SU(2)_F$ flavor symmetry. This Noether-current is easily found from the action. By applying supersymmetry transformations to the current one finds that it is part of the following supermultiplet:

$$\begin{aligned} J^A_\mu &= i\frac{1}{4}\epsilon_{ij}\epsilon_{ab}(\tau^A)_c{}^b(\phi^{jc}\partial_\mu\phi^{ia} - \partial_\mu\phi^{jc}\phi^{ia}) - i\frac{1}{2}\epsilon_{ab}(\tau^A)_c{}^b\Gamma^{\alpha\beta}_\mu\psi^a_\alpha\psi^c_\beta, \\ S^{jA}_\alpha &= i\epsilon_{ba}(\tau^A)_c{}^b\phi^{ja}\psi^c_\alpha, \\ D^{ijA} &= \frac{1}{2}i\epsilon_{ba}(\tau^A)_c{}^b\phi^{ic}\phi^{ja}. \end{aligned} \tag{4.7}$$

Note that D^{ijA} is symmetric in i and j. The SUSY transformations of these operators are:

$$\begin{aligned} \delta J^{\mu A} &= \epsilon_{ij}\eta^{\alpha i}(\Gamma^{\mu\nu})_\alpha{}^\beta\partial_\nu S^{jA}_\beta, \\ \delta S^{jA}_\beta &= \eta^{j\gamma}\Gamma^\mu_{\beta\gamma}J^A_\mu + \epsilon_{ki}\eta^{\gamma k}\Gamma^\mu_{\beta\gamma}\partial_\mu D^{ijA}, \\ \delta D^{ijA} &= \eta^{\alpha i}S^{jA}_\alpha + \eta^{\alpha j}S^{iA}_\alpha. \end{aligned}$$

In the transformation of J^A_μ the equation of motion for ψ_α was used.

Since a mass in 4+1D comes from the component A_5 along the circle, a mass varying in the x^4 direction comes from an A_5 which varies along x^4. In other words, there is a nonzero field strength F_{45}. The usual way of coupling A_μ to a theory is by adding

$$\int d^6x J_\mu A^\mu$$

to the action plus a term proportional to A^2 in order to preserve gauge invariance. In the action (4.5) the terms proportional to m and m^2 come from this coupling. What about the extra term needed for supersymmetry? We see that it is proportional to $D^{12(A=)3}$. Since $m'(x^4)$ is F_{45} we see that the extra term is just proportional to

$$\int d^6x F_{45} D^{12(A=)3}.$$

We will apply these observations to more general systems in the next section. The important point is that the deformation of the Lagrangian can be expressed in terms of the current J_μ and its superpartner D without referring to the specific fields of the theory.

4.3 Construction from 6D

We wish to analyze the situation starting from a general 5+1D theory. We start with a 5+1D theory with $\mathcal{N} = (1,0)$ supersymmetry and a global $U(1)$ symmetry and we compactify it on $\mathbf{T}^2$. We wish to put a background gauge field A_μ that is associated to the $U(1)$ symmetry along $\mathbf{T}^2$ such that the first Chern class will be $c_1 = n$. The question is how do we do it while preserving half the supersymmetry.

1 The current multiplet

The 5+1D theory has a current J_μ associated with the $U(1)$ symmetry. The current is a member of an $\mathcal{N} = (1,0)$ multiplet which also contains a fermionic partner S^i_α and a bosonic "D-term" partner D^{ij} as we saw in subsection (5) for the free hypermultiplet. Here, $i, j = 1, 2$ are $SU(2)_R$ symmetry indices and D^{ij} is symmetric. They satisfy:

$$\begin{aligned} \delta J^\mu &= \epsilon_{ij}\eta^{\alpha i}(\Gamma^{\mu\nu})_\alpha{}^\beta \partial_\nu S^j_\beta \\ \delta S^j_\beta &= \eta^{j\gamma}\Gamma^\mu_{\beta\gamma} J_\mu + \epsilon_{ki}\eta^{\gamma k}\Gamma^\mu_{\beta\gamma}\partial_\mu(D^{ij}) \\ \delta D^{ij} &= \eta^{\alpha i}S^j_\alpha + \eta^{\alpha j}S^i_\alpha \end{aligned} \tag{4.8}$$

We claim that compactifying on $\mathbf{T}^2$ and adding:

$$S_1 = -\int (A_4 J_4 + A_5 J_5 + iF_{45}D^{12} + \cdots) \tag{4.9}$$

to the action gives a supersymmetric theory with $\mathcal{N} = 1$ in the uncompactified 3+1D. The i in the second term is necessary to make the action real, since D^{12} is imaginary.

The $(\cdots)$ represent $O(A_\mu^2)$ terms that are dictated by $U(1)$ gauge invariance. For example, if under a local $U(1)$ transformation

$$\delta J_\mu = \partial_\mu \epsilon \Theta,$$

we have to add $\frac{1}{2}A_\mu A^\mu \Theta$ to the Lagrangian.

In order to see that $\mathcal{N}=1$ is unbroken we calculate the supersymmetry variation of S_1 using (4.8).

$$\begin{aligned} \delta S_1 &= \int A_\mu (\epsilon_{ij}\eta^{\alpha i}(\Gamma^{\mu\nu})_\alpha{}^\beta \partial_\nu S^j_\beta) + iF_{45}(\eta^{\alpha 1}S^2_\alpha + \eta^{\alpha 2}S^1_\alpha) \\ &= \int (F_{45}(\Gamma^{45})_\alpha{}^\beta \eta^{\alpha 2} + iF_{45}\eta^{\beta 2})S^1_\beta + (-F_{45}(\Gamma^{45})_\alpha{}^\beta \eta^{\alpha 1} + iF_{45}\eta^{\beta 1})S^2_\beta \end{aligned}$$

which is equal to zero if

$$(\Gamma^{45})_\alpha{}^\beta \eta^{\alpha 1} = i\eta^{\beta 1}, \qquad (\Gamma^{45})_\alpha{}^\beta \eta^{\alpha 2} = -i\eta^{\beta 2}.$$

These two equations are complex conjugate of each other. We see that we are left with $\mathcal{N}=1$ in 3+1D.

2 Example – a free hypermultipet

After compactification on a $\mathbf{T}^2$ to 3+1D we would like to know the masses of the fields. There will be a Kaluze-Klein tower of fields. In the low energy limit we are, of course, only interested in the massless fields. Let us go back to the free hypermultiplet and calculate the Kaluza-Klein masses. We need only do it for the fermions because of $\mathcal{N}=1$. The Dirac equation for the fermions reads

$$\Gamma^\mu \nabla_\mu \psi = 0$$

where $\nabla_\mu = \partial_\mu + iA_\mu$ is the covariant derivative with respect to the $U(1)$ symmetry. In our case the only nonzero components of A_μ are A_4 and A_5. In reducing to 3+1D ψ can be written as

$$\psi = \psi_L \phi_L + \psi_R \phi_R$$

where ψ_L, ψ_R are left- and righthanded spinors in 3+1D and ϕ_L, ϕ_R are left- and righthanded spinors on $\mathbf{T}^2$. Plugging into the Dirac equation we get the following formula for the mass m in 3+1D.

$$\begin{aligned}(\nabla_4\Gamma_4 + \nabla_5\Gamma_5)\phi_R &= m\phi_L \\ (\nabla_4\Gamma_4 + \nabla_5\Gamma_5)\phi_L &= -m^*\phi_R \end{aligned} \tag{4.10}$$

The mass m is a complex number. The physical mass is the absolute value of m. The phase can be transformed away by redefining ϕ_L, say. The phase would then show up in the couplings. In the free theory there is no meaning to them. We will just rotate the phase away for now and let m be real. We see that for $m \neq 0$, ϕ_L and ϕ_R come in pairs. This implies that in 3+1D ψ_L and ψ_R come in pairs of the same mass. This is as it should be, since a chiral spinor that is charged under a $U(1)$ symmetry cannot be massive. Both a lefthanded and a righthanded spinor are needed for a mass term. However for $m = 0$ there is no relation between a lefthanded solution and a righthanded one. For each solution of

$$(\nabla_4\Gamma_4 + \nabla_5\Gamma_5)\phi_R = 0$$

there is a massless righthanded fermion in 3+1D and for each solution of

$$(\nabla_4\Gamma_4 + \nabla_5\Gamma_5)\phi_L = 0$$

there is a massless lefthanded fermion in 3+1D.

Eq. (4.10) implies second order differential equations for ϕ_L and ϕ_R:

$$\begin{aligned}(\nabla_4^2 + \nabla_5^2 - F_{45})\phi_L &= -m^2\phi_L \\ (\nabla_4^2 + \nabla_5^2 + F_{45})\phi_R &= -m^2\phi_R \end{aligned} \tag{4.11}$$

These equations are the same as the ones determining the boson masses. It had to be so due to the supersymmetry. In these equations A_μ is a connection in a $U(1)$-bundle over $\mathbf{T}^2$ and $\phi_{L,R}$ are sections of this circle-bundle. The setup here is the same as a charged particle on a torus moving in a background magnetic field (Landau levels). For a general A_μ the eigenvalues m are not known, to our knowledge.

We can say more about the case of $m = 0$. Here we find the zero modes of the Dirac equation in 2 dimensions for respectively lefthanded and righthanded spinors. The number of those will depend on the gauge field A_μ but the difference between the number of lefthanded and righthanded zero modes is known as the index of the Dirac operator. It is equal to the first chern class, c_1, of the circle-bundle.

$$c_1 = \frac{1}{2\pi}\int_{\mathbf{T}^2} F_{45}$$

For a generic gauge field there will be $|c_1|$ solutions of one kind and 0 of the other kind. But for special gauge fields it could be different. An example of a special case is the case of $A_\mu = 0$. Here $c_1 = 0$. There is one zero mode of each chirality, namely the constant function. We thus conclude that in the theories under consideration the hypermultiplets will give rise to c_1 massless chiral multiplets. Even in the special cases mentioned above this will also be the case, since the couplings generically will lift the accidental pairs and still leave us with c_1 massless chiral multiplets.

Now we will consider the special case of constant F_{45}, where the problem has an explicit solution. Let the first chern class be $c_1 = n$. We will take the fields to obey the following boundary conditions.

$$\begin{aligned}
\phi(x_4, x_5 + 2\pi R_5) &= \phi(x_4, x_5) \\
\phi(x_4 + 2\pi R_4, x_5) &= e^{-in\frac{x_5}{R_5}}\phi(x_4, x_5)
\end{aligned}$$

Here ϕ denotes both ϕ_R and ϕ_L. The gauge field can be gauge transformed to the following form:

$$\begin{aligned}
A_4(x_4, x_5) &= a_4 \\
A_5(x_4, x_5) &= \frac{nx_4}{2\pi R_4 R_5} + a_5
\end{aligned}$$

Here a_4, a_5 are constants. On the plane they could be gauged to zero, but on the torus they are there in general. The eigenvalue equations (4.11) now read

$$\left[(\partial_4 + ia_4)^2 + \left(\partial_5 + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right]\phi \;=\; -m^2\phi,$$

where the $\pm$ refers to ϕ_R and ϕ_L, respectively. The periodicity conditions above imply that we can write ϕ as:

$$\phi(x_4, x_5) = \sum_{k=-\infty}^{\infty} e^{ik\frac{x_5}{R_5}}\phi_k(x_4) \qquad 0 \le x_4 \le 2\pi R_4, \tag{4.12}$$

with the boundary condition:

$$\phi_k(2\pi R_4) = \phi_{k+n}(0). \tag{4.13}$$

The equation for ϕ_k for $0 \le x_4 \le 2\pi R_4$ becomes

$$\left[(\partial_4 + ia_4)^2 + \left(i\frac{k}{R_5} + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right]\phi_k(x_4) = -m^2\phi_k(x_4). \tag{4.14}$$

Using the boundary condition (4.13) we can define n functions, $f_k, k = 0, 1, ..., n-1$ on the real line:

$$f_k(x_4) = \phi_{k+ln}(x_4 - 2\pi R_4 l) \qquad \text{for } 2\pi R_4 l \le x_4 \le 2\pi R_4(l+1).$$

It follows from (4.14) that f_k for $-\infty < x_4 < \infty$ obeys

$$\left[(\partial_4 + ia_4)^2 + \left(i\frac{k}{R_5} + i\frac{nx_4}{2\pi R_4 R_5} + ia_5\right)^2 \pm F_{45}\right] f_k(x_4) = -m^2 f_k(x_4). \tag{4.15}$$

Here $k = 0, 1, ..., n-1$ and $\pm$ still refers to the two chiralities. We are only interested in normalizable solutions. The norm square of a field ϕ in (4.12) is equal to the sum of the norm squares of the n functions on the real line, f_k. This means that the eigenvalues and eigenfunctions are exactly the normalizable solutions to (4.15).

To solve (4.15) we first redefine f_k by a phase to set a_4 to zero. This can now be done since x_4 runs over the real line. The equation becomes the eigenvalue problem for a one dimensional harmonic oscillator. The eigenvalues are:

$$m_j^2 = (j + \frac{1}{2} \mp \frac{1}{2})\frac{n}{\pi R_4 R_5} \qquad j = 0, 1, 2, \ldots$$

for each $k = 0, 1, ..., n-1$. We see that there is a n-fold degeneracy of all masses. There are n massless modes of one chirality and zero of the other. For the massive levels there is an equal number of solutions of each chirality. These features were general as discussed above and it is nice to see how it works in the special case of constant F_{45}.

We thus conclude that the free hypermultiplet compactified in this way produces n chiral multiplets with zero mass as well as a tower of nonchiral (double) multiplets $\Phi_j^{k,\pm}$ ($j = 1, \ldots$ and $k = 0, 1, \ldots, n-1$) with masses,

$$m_j^2 = \frac{jn}{\pi R_4 R_5}.$$

The superpotential therefore contains a term,

$$\sum_{k=1}^{n}\sum_{j=1}^{\infty}\left(\frac{jn}{\pi R_4 R_5}\right)^{1/2} \Phi_j^{k,+}\Phi_j^{k,-}.$$

3 σ-models

The previous example can be generalized to q hypermultiplets describing a low-energy σ-model with a hyper-Kähler target space, $\mathcal{M}$, of dimension $4q$. Let us also assume that $\mathcal{M}$ has a $U(1)$ isometry that is related to a hyper-Kähler moment map. Recall that a hyper-Kähler manifold has a $\mathbf{CP}^1$-family of complex structures and each complex structure has its own Kähler class. The collection of Kähler 2-forms can be written as:

$$\omega = \sum_{a=1}^{3} c_a \omega_a, \qquad \sum c_a^2 = 1.$$

Here, the ω_a's are (real) 2-forms and the c_a's are real coefficients. They satisfy,

$$g_{IK}\omega_a^{IJ}\omega_b^{KL} + g_{IK}\omega_b^{IJ}\omega_a^{KL} = 2\delta_{ab}g^{JL},$$

where g_{IJ} is the metric ($I, J, K = 1 \dots 4q$). A hyper-Kähler moment map is a $\mathbf{CP}^1$-family of functions on $\mathcal{M}$:

$$\mu = \sum_{a=1}^{3} c_a \mu_a.$$

They satisfy,

$$\omega_a^{IK}\partial_K \mu_b + \omega_b^{IJ}\partial_J \mu_a = 2\delta_{ab}\xi^I,$$

where ξ^I is the Killing vector for the $U(1)$ isometry.

Now, let us consider a 5+1D σ-model with target space $\mathcal{M}$ (the hypermultiplet moduli space). (See [46] and [47].) The $U(1)$ current is given by:

$$J_\mu = \xi_I \partial_\mu \phi^I.$$

The role of the triplet of operators D^{ij} from (4.8) is played by the triplet of moment maps μ_a ($a = 1 \dots 3$). When we compactify on $\mathbf{T}^2$, (4.9) becomes:

$$S_1 = -\int (A_4 J_4 + A_5 J_5 + iF_{45}\mu_1 + \cdots) \tag{4.16}$$

Let us discuss the low-energy description of this model. We wish to find the dimension of the moduli space of solutions to the scalar equations of motion. The kinetic part of the σ-model:

$$\int g_{i\bar{j}}(\phi, \overline{\phi})\overline{\partial}\phi^i \partial\overline{\phi}^{\bar{j}} + \int g_{i\bar{j}}(\phi, \overline{\phi})\partial\phi^i \overline{\partial\phi}^{\bar{j}},$$

leads to the following equations of motion:

$$0 = -\partial(g_{i\bar{j}}\overline{\partial}\phi^i) - \overline{\partial}(g_{i\bar{j}}\partial\phi^i) + \partial_{\bar{j}}g_{i\bar{k}}\partial\overline{\phi}^{\bar{k}}\overline{\partial}\phi^i + \partial_{\bar{j}}g_{i\bar{k}}\overline{\partial\phi}^{\bar{k}}\partial\phi^i.$$

We use the Kähler condition:

$$\partial_{\bar{j}}g_{i\bar{k}} = \partial_{\bar{k}}g_{i\bar{j}} = g_{i\bar{l}}\Gamma^{\bar{l}}_{\bar{j}\bar{k}}$$

and obtain:

$$(D\overline{D}\phi)^i = 0, \qquad \overline{D}D\overline{\phi}^{\bar{j}} = 0.$$

Here D is the covariant derivative:

$$(\overline{D}\phi)^i = \overline{\partial}\phi^i, \qquad (D\overline{D}\phi)^i = \partial\overline{\partial}\phi^i + \Gamma^i_{jk}\partial\phi^j\overline{\partial}\phi^k.$$

This implies:

$$\overline{\partial}\phi^i = 0, \qquad \partial\overline{\phi}^{\bar{j}} = 0.$$

The zero modes are thus holomorphic curves from $\mathbf{T}^2$ into the target space, as is well known. To incorporate the gauge field A_μ we replace ∂ and $\overline{\partial}$ with the $U(1)$-covariant derivative:

$$(\overline{D}\phi)^j = \overline{\partial}_{\bar{z}}\phi^j - iA_{\bar{z}}\xi^j.$$

Now let us fix the complex structure that corresponds to ω_1 (out of the 3 ω_a's). We can then express the Killing vector, ξ^j, in terms of μ_1 as:

$$\xi^j = g^{j\bar{k}}\partial_{\bar{k}}\mu_1. \tag{4.17}$$

The zero modes corresponding to (4.16) are easily seen to satisfy:

$$0 = \overline{\partial}_{\bar{z}}\phi^j - iA_{\bar{z}}\xi^j. \tag{4.18}$$

How many zero modes do we get? Let us assume that ϕ^j is a solution and study the linearized equation:

$$0 = \overline{\partial}_{\bar{z}}\delta\phi^j - iA_{\bar{z}}\partial_k\xi^j\delta\phi^k - iA_{\bar{z}}\partial_{\bar{k}}\xi^j\delta\overline{\phi}^{\bar{k}}.$$

Using (4.17) we see that:

$$\partial_{\bar{k}}\xi_{\bar{l}} = \partial_{\bar{l}}\xi_{\bar{k}},$$

but since ξ is assumed to be a Killing vector it must satisfy:

$$\partial_{\bar{k}}\xi_{\bar{l}} + \partial_{\bar{l}}\xi_{\bar{k}} = 2\Gamma^{\bar{j}}_{\bar{k}\bar{l}}\xi_{\bar{j}}$$

so

$$\partial_{\bar{k}}\xi_{\bar{l}} = \Gamma^{\bar{j}}_{\bar{k}\bar{l}}\xi_{\bar{j}}$$

Also,

$$\partial_{\bar{k}}g^{j\bar{l}} = -g^{j\bar{n}}\partial_{\bar{k}}g_{m\bar{n}}g^{m\bar{l}} = -g^{j\bar{n}}\Gamma^{\bar{l}}_{\bar{k}\bar{n}}$$

Therefore,

$$\partial_{\bar{k}}\xi^{j} = 0.$$

The linearized equations of motion are therefore:

$$0 = \overline{\partial}_{\bar{z}}\delta\phi^{j} - iA_{\bar{z}}\partial_{k}\xi^{j}\delta\phi^{k}.$$

To solve this we write the $2q \times 2q$ matrix with elements $A_{\bar{z}}\partial_{k}\xi^{j}$ as:

$$-iA_{\bar{z}}\partial_{k}\xi^{j} = (\Omega^{-1})^{l}_{k}\overline{\partial}_{\bar{z}}\Omega^{j}_{l},$$

where $\Omega(z,\bar{z}) \in GL(2q, \mathbf{C})$. We find that:

$$\overline{\partial}_{\bar{z}}(\Omega^{j}_{k}\delta\phi^{k}) = 0.$$

Thus $\Omega\delta\phi$ is a holomorphic section of a vector-bundle. Moreover, from the Killing vector equation:

$$\partial_{\bar{k}}\xi_{l} + \partial_{l}\xi_{\bar{k}} = 2\Gamma^{\bar{j}}_{\bar{k}l}\xi_{\bar{j}} + 2\Gamma^{j}_{\bar{k}l}\xi_{j} = 0.$$

We therefore find:

$$\partial_{l}\xi^{i} = \partial_{l}g^{i\bar{k}}\xi_{\bar{k}} + g^{i\bar{k}}\partial_{l}\xi_{\bar{k}} = -\Gamma^{i}_{lk}\xi^{k} - g^{i\bar{k}}\partial_{\bar{k}}\xi_{l}$$

Using (4.18) we can write:

$$(\Omega^{-1})^{l}_{k}\overline{\partial}_{\bar{z}}\Omega^{j}_{l} = -\Gamma^{j}_{kl}\overline{\partial}_{\bar{z}}\phi^{l} + iA_{\bar{z}}g^{j\bar{l}}\partial_{\bar{l}}\xi_{k}$$

Now $\delta\phi^j$ is a section of the pullback $\phi^*T\mathcal{M}$ of the tangent-bundle $T\mathcal{M}$ of $\mathcal{M}$ under the map $\phi : \mathbf{T}^2 \mapsto \mathcal{M}$. This vector-bundle has the connection $\Gamma^j_{kl}\overline{\partial}_{\bar{z}}\phi^l$. Thus, the vector-bundle V, of which $\Omega\delta\phi$ is a holomorphic section can be described as follows. Find $\widetilde{\Omega} \in GL(2q, \mathbf{C})$ such that:

$$(\widetilde{\Omega}^{-1})^l_k \overline{\partial}_{\bar{z}} \widetilde{\Omega}^j_l = iA_{\bar{z}} g^{j\bar{l}} \partial_{\bar{l}} \xi_k = iA_{\bar{z}} g^{j\bar{l}} \partial_{\bar{l}} \partial_k \mu_1.$$

Then, $\widetilde{\Omega}$ is a section of a principal bundle with the same structure group as V. This means the following: Let $\mathbf{T}^2$ be described by z, as we did, with

$$z \sim z + 1, \qquad z \sim z + \tau.$$

If s is a section of V then the boundary conditions on s are that $\widetilde{\Omega}(z, \bar{z})^{-1}s$ should be continuous.

The eigenvalues of the $GL(2q, \mathbf{C})$ matrix with elements $g^{j\bar{l}}\partial_{\bar{l}}\xi_k$ pulled back to $\mathbf{T}^2$ are constants, and therefore also integers. The fact that the invariant polynomial $P(\lambda) \equiv \det(g^{j\bar{l}}\partial_{\bar{l}}\xi_k - \lambda\delta^j_k)$ is constant follows from $\partial_{\bar{k}}\xi^l = 0$. It implies that $\partial_{\bar{k}}P(\lambda) = 0$. Thus $P(\lambda)$ is a holomorphic function. If $\mathcal{M}$ were compact this is enough. Even if it is not compact, it still follows that the pullback of $P(\lambda)$ to $\mathbf{T}^2$ is holomorphic and therefore constant. Thus, the vector-bundle V splits into a product: $\bigotimes_{i=1}^{2q} \mathcal{O}(n\lambda_i)$ where λ_i are the eigenvalues of $P(\lambda)$. They must therefore be integers.

4 Coupling to a vector multiplet

Now let us start with a 5+1D hypermultiplet in the representation $\underline{\mathbf{N}}$ ($\underline{\bar{\mathbf{N}}}$) of $SU(N)$ and couple it to a 5+1D $SU(N)$ vector-multiplet. Although this is a nonrenormalizable interaction, we can think of it as the low-energy description of a sector of one of the little-string theories of [48]. The 5+1D coupling of the vector-multiplet to the hypermultiplet preserves $SU(2)_R \times U(1)_F$. Out of the two chiral fermions ψ^a_α $(a = 1, 2)$ one transforms in the $\underline{\mathbf{N}}$ of $SU(N)$ and the other transforms in the $\underline{\bar{\mathbf{N}}}$ of $SU(N)$.

Let us classically reduce, as before, on T^2 with a global $U(1)$ background field with first Chern class $c_1 = n$. The hypermultiplet gives rise to n chiral multiplets $\Phi_0^{(k),+}$ ($k = 1 \dots n$) in the $\underline{\mathbf{N}}$ of $SU(N)$ as well as a tower of massive multiplets $\Phi_j^{(k),\pm}$ ($j = 1 \dots$) where $\Phi_j^{(k),+}$ is in the $\underline{\mathbf{N}}$ of $SU(N)$ and $\Phi_j^{(k),-}$ is in the $\underline{\bar{\mathbf{N}}}$. Their masses are given by the superpotential,

$$\sum_{k=1}^{n}\sum_{j=1}^{\infty}\left(\frac{jn}{\pi R_4 R_5}\right)^{1/2}\Phi_j^{(k),+}\Phi_j^{(k),-}.$$

The 5+1D vector-multiplet gives rise to an $\mathcal{N} = 1$ vector-multiplet in 3+1D and a chiral multiplet Φ_{ad} in the adjoint representation of $SU(N)$. There is also a Yukawa coupling of the fields Φ_{ad}, $\Phi_{j+1}^{(k),-}$ and $\Phi_j^{(k),+}$.

4.4 Compactifying the BI theory

We will now construct a specific example that produces chiral matter in 3+1D by compactifying the Blum-Intriligator (BI) theories [39].

1 Preliminaries

Compactifying the BI theory of N M5-branes at an A_{k-1} singularity on S^1 of radius R one obtains a low-energy description given by a gauge theory with gauge group

$$SU(N)_1 \times SU(N)_2 \times \cdots \times SU(N)_k.$$

The sub-indices are added for purposes of identification. There are also hypermultiplets in the $(\overline{N}_i, N_{i+1})$ representation (with $k+1 \equiv 1$). On top of that there are $(k-1)$ more $U(1)$ vector multiplets. The scalar components set the coupling constants of the k $SU(N)$ gauge groups. These coupling constants, g_i, ($i = 1 \dots k$) satisfy

$$\sum_{i=1}^{k}\frac{1}{g_i^2} = \frac{1}{R}.$$

If we compactify on another $\mathbf{S}^1$ of radius R' we obtain a 3+1D gauge theory at low-energies. The $(k-1)$ $U(1)$ vector multiplets that set the gauge couplings decouple and the gauge couplings become background parameters. The interacting gauge theory has a gauge group $SU(N)^k$ and $(\overline{N}_i, N_{i+1})$ hypermultiplets. The coupling constants and θ-angles are set by $(k-1)$ background parameters (originating from the original $(k-1)$ $U(1)$ vector multiplets) and subject to the condition that

$$\sum_{i=1}^{k} \tau_i = i\frac{R'}{R}, \qquad \tau_i \equiv \frac{\theta_i}{2\pi} + \frac{8\pi i}{g_i^2}$$

2 Adding the background $U(1)$ field

Now we take a specific 5+1D theory – the BI theory. Also, let the complex structure τ of $\mathbf{T}^2$ become very large. We can take $\mathbf{T}^2 = \mathbf{S}^1 \times \mathbf{S}^1$ with one $\mathbf{S}^1$ of radius R_4 and the other with radius $R_5 \ll R_4$. We can first reduce the theory along R_5. The holonomy $W(x_4) = \int_0^{2\pi R_5} A_5(x_4, x_5)dx_5$ varies from 0 to $2\pi n$ as x_4 varies from 0 to $2\pi R_4$.

For a fixed x_4, the reduction of the BI theory along $\mathbf{S}^1$ with Wilson line $W(x_4)$ was studied in [49, 43]. For small $W(x_4)$ and at low energies $0 \leq E \ll R_5^{-1}$ the theory is described by an effective 4+1D Lagrangian which is the quiver theory of [42] of N D4-branes at an A_{k-1} singularity but such that the hypermultiplets have a mass $m = W(x_4)R_4^{-1}$. For generic x_4 the mass is of the order of R_4^{-1}. There are n values of x_4 for which $W(x_4)$ is a multiple of 2π and in the vicinity of those points the mass m varies from a small negative to a small positive value. According to the discussion in subsection (3), the 3+1D low-energy description contains a chiral multiplet for every time the mass crosses zero. Note that the term $F_{56}D^{12}$ in (4.9) becomes the term proportional to dm/dx^4 in (4.6). In subsection (3) the 4^{th} direction (counting from $0 \ldots 4$) was infinite and there was a continuum of massive modes with arbitrarily low mass. In our case the 4^{th} direction is compact and therefore we expect a discrete spectrum with the first level of order R_4^{-1}. The chiral mode is likely to remain massless because of arguments similar to those of [50].

The low-energy description in 3+1D will therefore contain n chiral multiplets for each hypermultiplet of the quiver theory. We obtain an $SU(N)^k$ vector multiplets of $\mathcal{N}=2$ supersymmetry together with n copies of chiral multiplets (of $\mathcal{N}=1$ supersymmetry) in the $(\overline{N}_i, N_{i+1})$ representations, for each $i=1\dots k$. The $\mathcal{N}=2$ vector multiplets should be decomposed into $\mathcal{N}=1$ vector multiplets and chiral multiplets in the adjoint representation of the fields.

Let us now discuss the issue of whether the adjoint multiplets have a superpotential or not. On the face of it, the adjoint mutliplets can receive a mass term. In the limit that we have been using, $R_4 \gg R_5$, the mass term, if it exists, might be of the order of R_5^{-1}. However, the 6D origin of the expectation value of the chiral multiplets is the expectation values for the $k(N-1)$ tensor multiplets of the 6D theory. Specifically, let Φ be the scalar of one of those tensor multiplets and let B_{45} be the component of the anti-self-dual tensor field corresponding to it. We can set $\phi = 4\pi^2(\Phi + iB_{45})R_4R_5$. In the limit that $\Phi R_4 R_5$ is large, we can trust the 6D low-energy description of the Coulomb branch of the BI theory and dimensionally reduce the 6D low-energy effective action to 4D on $\mathbf{T}^2$ with twists. Because of the periodicity $\phi \sim \phi + 2\pi i$, a superpotential for ϕ has to have the form $\sum a_n e^{-n\phi}$. We recognize this as the contribution of instantons made from strings of the 6D BI-theory wrapped on $\mathbf{T}^2$. To determine whether such instantons contribute to the superpotential we have to count the zero modes of the fermions in the low-energy effective action that describes the world-sheet of the string. The world-sheet theory that lives on the string of the BI theory can be deduced by dimensionally reducing the theory that lives on the M2-brane and an A_{k-1} singularity on a segment between two M5-branes, setting the boundary conditions appropriately. It seems that the 1+1D effective theory always has a supermultiplet of $\mathcal{N} = (2,2)$ supersymmetry which comprises of 4 scalars (describing transverse motion of the string inside the 5+1D space) and fermions that are uncharged under $U(1)$. Because they are uncharged, and because it is only the interaction with this global $U(1)$ that breaks the supersymmetry into $\mathcal{N} = 1$ in 3+1D, the instanton will have twice as many fermionic zero modes than required for a superpotential. It will therefore not contribute to a superpotential.

4.5 Discussion

We argued that chiral gauge theories can be realized as a low-energy limit of certain compactifications of 6D conformal field theories. There are several issues that we have not addressed in this chapter. In section (2) we argued that the particular compactification of the BI theory that we studied gives an $SU(N)^k$ gauge vector multiplets of $\mathcal{N} = 2$ supersymmetry together with n copies of chiral multiplets (of $\mathcal{N} = 1$ supersymmetry) in the $(\overline{N}_i, N_{i+1})$ representations, for each $i = 1 \ldots k$. Some questions for further study would be:

- Do the adjoint chiral multiplets get a mass term?
- Can we realize the compactifications in an M-theory setting? That is, can we find a supergravity solution with M5-brane whose low-energy is described by the compactifications we considered?
- In that case, are these models dual to other chiral gauge field constructions similar to those in [51, 52, 53] or chiral F-theory compactifications [54] (and see also [55] and refs. therein)? Are they dual to the new compactifications discovered in [16]?

Chapter 5

Duality Cascade and Oblique Phases in NCOS

5.1 Introduction

Duality is a powerful tool for analyzing dynamical aspects of field theories and string theories. Some dualities are exact. They assert that two seemingly different theories are physically equivalent. String theory with sufficiently many unbroken supersymmetries are believed to exhibit such exact duality relations, in the form of discrete symmetry relations acting on the space of couplings. These symmetries, and the fact that a large class of field theories can be formulated as decoupling limits of string theories, have been used to derive many examples of field theory dualities, as well as dual correspondences between field theories and supergravity theories. Similar considerations led to the understanding of duality relations among less familiar theories such as non-commutative gauge theories, non-commutative open string theories, and little string theories.

In this article, we consider the duality relations of non-commutative open string theories [56, 57, 58] in 1+1 dimensions with 16 supercharges. These theories can be formulated as a decoupling limit of bound states of D-strings and fundamental strings in type IIB string theory. The relevant duality relations follow from the $SL(2, \mathbf{Z})$ S-duality symmetry of type IIB string theory and the gauge theory/supergravity correspondence applied to (P, Q) strings. Our aim is to use a combination of both these dualities to gain insight into the thermodynamic properties of the theory. Various authors have also considered aspects of S-duality in the context of NCOS theories [59, 60, 61, 62, 63] and their supergravity duals [64, 65].

NCOS theory is closely related to ordinary 1+1-d super Yang-Mills theory in at least two different ways. First, for rational value of its string coupling G_o^2, it's known to be S-dual to ordinary SYM theory with a non-zero electric flux, which therefore provides the proper ultraviolet definition of the theory. On the other hand, like any other known open string theory, NCOS theory reduces, at scales sufficiently below its string scale, to an effective low energy gauge theory. Near the NCOS string scale, however, this gauge theory description breaks down, and the system undergoes a phase transition into an effective matrix string theory phase [66, 67]. The formation of the matrix strings can be viewed as an ionization process of the non-commutative open strings, that escape via the Coulomb branch from the bound state with the D-strings.

On general grounds, different dual descriptions are never simultaneously weakly coupled, since two distinct weakly coupled theories are manifestly inequivalent. This means that for given temperature and couplings, one can expect that, among the set of theories related by duality, there typically exists one preferred description which is most weakly coupled. In this chapter, we will use this intuition to map out the complete phase diagram of the 1+1-dimensional NCOS theory. The main new conclusions of our study are the following:

- In most studies done so far of the thermodynamics of 1+1-dimensional NCOS theory, the effective open string coupling constant was assumed to be fixed at some rational value $G_o^2 = P/Q$. We will find, however, that G_o^2 can be any real positive number. This allows us to view NCOS theory as a continuous non-commutative deformation of ordinary gauge theory.

- Upon systematic consideration of the role of full $SL(2,\mathbf{Z})$ duality structure and the AdS/CFT correspondence, a remarkably elaborate phase structure emerges. Various $SL(2,\mathbf{Z})$ dual descriptions become preferred in disjoint regions of the phase diagram parameterized by temperature T and the NCOS coupling constant G_o^2. These regions form a complicated fractal pattern.

- As a function of T, the theory can go through a cascade of alternating supergravity, gauge theory, and matrix string theory phases. The cascade proceeds via a series of $SL(2,\mathbf{Z})$ S-duality transformations, and depends sensitively on P and Q. In particular, we find that the system may undergo a sequence of successive ionization and recombination transitions.

The fractal pattern seen in the phase diagram closely resembles the phase structure found for the supergravity duals of non-commutative Yang-Mills theories on a torus [68, 69]. There, the role of $SL(2,\mathbf{Z})$ was played by the Morita/T-duality group. The duality cascade, which involved only the supergravity descriptions, were found not to give rise to any observable thermodynamic effects, simply because the area of the horizon in Einstein frame is invariant under T-duality transformations. Here, the duality cascade will act among the gauge theory, matrix theory, and supergravity phases, giving rise to thermodynamically observable cross-over effects.

The organization of this chapter is as follows. In section 2, we collect various preliminary facts regarding NCOS theory as decoupling limit of string theory and the form of the $SL(2, \mathbf{Z})$ S-duality transformations. In section 3, we analyze the role of supergravity dual and the Hagedorn transition for the theory corresponding to a given set of charges P and Q. In section 4, we describe how the various dual descriptions fit together to form a continuous, though fractal, phase diagram. We conclude in section 5.

5.2 Preliminaries

1 Parameters of 1+1-d NCOS theory

In this subsection we introduce the parameter space of 1+1-d non-commutative open string theory.

Since 1+1-d NCOS theory is an interacting theory of open strings, it is specified by an open string coupling constant, G_o, and by the string tension, α'_{eff}. In addition, we can introduce $U(Q)$ Chan-Paton factors, as well as turn on a discrete electric flux. In two dimensions, this flux behaves like a discrete θ parameter [70]

$$\theta = \frac{2\pi P}{Q} \tag{5.1}$$

where P is an integer ranging from zero to $Q-1$. In the language of the underlying IIB string theory, P and Q count the number of fundamental and Dirichlet strings, respectively, that make up the bound state [71].

Just as in ordinary open string theory, we can expect that, in a suitable low energy regime, NCOS theory reduces to 1+1-d super Yang-Mills theory with $U(Q)$ gauge symmetry. The dimensionful gauge coupling g_{YM} is related to the NCOS coupling and string length via[1]

$$g_{YM}^2 = \frac{G_o^2}{\alpha'_{eff}} \,. \tag{5.2}$$

[1]Here and hereafter, we will ignore constant numerical factors of order one.

This infrared gauge theory should not be confused with the S-dual Yang-Mills theory in the ultraviolet of NCOS frequently discussed in the literature [72, 73, 74].

Since g^2_{YM}, α'_{eff}, and G^2_o are related by (5.2), any two out of the three can be taken as the parameters defining the theory. For our purposes it will be convenient to think of NCOS theory as a modification of super Yang-Mills theory in the ultraviolet, induced by an irrelevant space-time non-commutativity perturbation. To emphasize this point of view, we will choose our parameters to be g^2_{YM} and G^2_o. We will see later that, contrary to some claims in the literature [73, 74], G^2_o can in fact take on arbitrary positive real values, which from the gauge theory perspective sets the scale of the non-commutativity parameter, in units set by g^2_{YM}. Ordinary super Yang-Mills theory is recovered in the limit $G^2_o \rightarrow 0$, with g_{YM} held fixed.

To summarize, the set of independent parameters that we will use to parameterize non-commutative open strings in 1+1 dimensions are the following:

$$\{g^2_{YM}, \quad G^2_o, \quad P, \quad Q\} \ . \tag{5.3}$$

For the purpose of studying the action of $SL(2, \mathbf{Z})$ duality group on these parameters, it will sometimes be useful to separate the factor N which is the greatest common divisor of P and Q and write

$$P = Np, \qquad Q = Nq \tag{5.4}$$

where p and q are relatively prime integers.

2 SYM Decoupling limit of (P,Q) strings

Here we recall the decoupling limit of the (P, Q) string theory that produces 1+1-dimensional SYM theory, and introduce its supergravity dual.

Starting from the world sheet theory of a (P, Q) string bound state in IIB string theory, we can consider the limit

$$g_s \to 0 \tag{5.5}$$

while focusing on physics taking place at energy scale or temperature of order

$$T^2 \sim g_{YM}^2 = \frac{g_s}{\alpha'} . \tag{5.6}$$

In this limit, the world sheet theory reduces to 1+1 dimensional super Yang-Mills theory with gauge coupling g_{YM}, gauge group $U(Q)$ and P units of electric flux.

To formulate the corresponding near horizon supergravity geometry, we can start from the full IIB supergravity solution of the (P, Q) string obtained by Schwarz in [75]. This solution is parameterized by the asymptotic values of the string coupling and axion field, and by the two quantized charges P and Q. Applying the scaling limit (5.5), while focusing on the range of radial coordinates parameterized by $U = r/\alpha'$ with r as in [75], gives

$$ds^2 = \alpha' \left(\frac{U^3}{g_{YM}\sqrt{Q}}(-dt^2 + dx^2) + \frac{g_{YM}\sqrt{Q}}{U^3}(dU^2 + U^2 d\Omega_7^2) \right) \tag{5.7}$$

$$e^\phi = \frac{g_{YM}^3 \sqrt{Q}}{U^3} \tag{5.8}$$

$$\chi = \frac{P}{Q} \tag{5.9}$$

$$B_{NS} = 0 \tag{5.10}$$

$$B_{RR} = \frac{\alpha' U^6}{g_{YM}^4 Q} . \tag{5.11}$$

The metric, the dilaton, and the two-form fields are exactly the same as the near horizon limit of the $(0, Q)$ string [76]. The effect of the non-vanishing electric flux manifests itself only in the constant axion background (5.9).

The dual supergravity description of the (P, Q) gauge theory is valid in the regime of couplings and scales where both the string coupling e^ϕ and the curvature of the near-horizon geometry, measured in string units, remain small.

3 The NCOS decoupling limit of (P,Q) strings

Here we describe the decoupling limit of (P, Q) string theory that produces 1+1-dimensional NCOS theory, and introduce its supergravity dual. A new element in our discussion is that, as a result of including the IIB axion field, the NCOS coupling G_o^2 is incorporated as a continuous free parameter.

1+1-dimensional NCOS theory arises from the world-sheet theory on the (P, Q) string bound state in IIB theory upon taking the decoupling limit

$$g_s \to \infty, \qquad g_s^2\,\alpha' \ \text{ fixed.} \tag{5.12}$$

This limit is S-dual to the SYM limit (5.5). The NCOS parameters α'_{eff} and G_o^2 are related to the IIB parameters via

$$g_s^2\,\alpha' = G_o^4\,\alpha'_{eff}, \qquad \alpha'\mathrm{tr}F = 1 - \frac{\alpha'}{2\alpha'_{eff}}, \tag{5.13}$$

where $F = \epsilon^{01}F_{01}$ is the $U(Q)$ Yang-Mills field strength on the D-string worldsheet. In the limit (5.12), tr$F$ is automatically tuned to approach its critical value $\alpha'\mathrm{tr}F = 1$, at precisely such a rate that its electrostatic force counteracts the infinite fundamental open string tension, so as to produce a finite effective tension α'_{eff} of the NCOS strings.

To see this explicitly, consider the Born-Infeld effective lagrangian of the D-string bound state (omitting all fields except the 1+1-d gauge field)

$$\mathcal{L} = -\frac{1}{g_s\alpha'}\mathrm{tr}\sqrt{(1-(\alpha' F)^2)} + \chi\mathrm{tr}F. \tag{5.14}$$

Here we included the topological term associated with the constant axion field χ. The compactness of the gauge group implies that the $U(1)$ part of the electric field

$P = \text{tr}E$ where E, defined as the canonical conjugate to the gauge field $E = \frac{\partial \mathcal{L}}{\partial \dot{A}}$, takes on integer values only. Inverting the relation

$$P - \chi Q = \frac{\alpha' \text{tr} F}{g_s \sqrt{(1 - (\alpha' F)^2)}} \tag{5.15}$$

reveals that the field strength indeed becomes near-critical in the NCOS limit (5.12)

$$\text{tr}F \simeq 1 - \frac{Q^2}{2g_s^2 (P - \chi Q)^2}. \tag{5.16}$$

Furthermore, from the relations (5.13) defining the NCOS parameters, we read off that

$$G_o^2 = \frac{Q}{|P - \chi Q|}. \tag{5.17}$$

As claimed, the effective coupling G_o^2 can thus indeed attain arbitrary real, positive values. As we will see shortly, the existence of more general NCOS theories with a continuously varying coupling also naturally arises from S-duality symmetry of the underlying IIB string theory.

The full supergravity solution dual to the 1+1-d non-commutative open string theory, for arbitrary values of the parameters $\left\{ g_{YM}, G_o, P, Q \right\}$, follows from the general expression obtained by Schwarz in [75], by applying the NCOS scaling limit

(5.12). One finds

$$ds^2 = \alpha' \left(1+\frac{U^6G_o^4}{g_{YM}^6Q}\right)^{1/2}\left(\frac{U^3}{g_{YM}\sqrt{Q}}(-dx_0^2+dx_1^2)+\frac{g_{YM}\sqrt{Q}}{U^3}(dU^2+U^2d\Omega_7^2)\right), \quad (5.18)$$

$$e^\phi = \frac{g_{YM}^3\sqrt{Q}}{U^3}\left(1+\frac{U^6G_o^4}{g_{YM}^6Q}\right), \quad (5.19)$$

$$\chi = \frac{g_{YM}^2PQ+G_o^2(G_o^2P+Q)U^6}{Q(g_{YM}^6Q+G_o^4U^6)}, \quad (5.20)$$

$$B_{NS} = -\frac{\alpha' G_o^2U^6}{g_{YM}^4Q}, \quad (5.21)$$

$$B_{RR} = \frac{\alpha'(Q+G_o^2P)U^6}{g_{YM}^4Q^2}. \quad (5.22)$$

Here, relative to the notation used in [75], we made the identifications

$$g_s^2 = \frac{G_o^6}{\alpha' g_{YM}^2}, \qquad \chi_\infty = \frac{P}{Q}+\frac{1}{G_o^2}, \qquad r^2 = \frac{\alpha' G_o^2U^2}{g_{YM}^2}. \quad (5.23)$$

This dual supergravity description of (P,Q) NCOS theory is valid in the regime of couplings and scales where both e^ϕ and the curvature of the near-horizon geometry remain small.

4 SL(2,Z) duality of NCOS theory

In this subsection, we describe how $SL(2,\mathbf{Z})$ S-duality transformations act on the NCOS data g_{YM}^2, G_o^2, P, and Q, and write the supergravity data in a more manifestly S-duality covariant form.

Before taking any decoupling limit, the (P,Q) strings are permuted by the $SL(2,\mathbf{Z})$ S-duality symmetry of the IIB theory, via

$$\begin{pmatrix}\tilde{P}\\ \tilde{Q}\end{pmatrix} = \begin{pmatrix}a & b\\ c & d\end{pmatrix}\begin{pmatrix}P\\ Q\end{pmatrix}, \quad (5.24)$$

which leaves $N = \gcd(P, Q)$ invariant. The string coupling g_s and axion χ transform via

$$\tilde{\lambda} = \frac{a\lambda + b}{c\lambda + d}, \qquad \lambda = \chi + \frac{i}{g_s}. \tag{5.25}$$

Evidently, S-duality does not preserve the super Yang-Mills decoupling limit; instead it is mapped onto the NCOS limit.

The NCOS limit, on the other hand, is in general preserved. The axion field χ_∞ transforms like

$$\tilde{\chi}_\infty = \frac{a\chi_\infty + b}{c\chi_\infty + d} \tag{5.26}$$

in the scaling limit. For generic values of χ_∞, the $SL(2, \mathbf{Z})$ transformed value is again finite, and thus the transformed theory is also a regular NCOS theory. Using the relations (5.13) and (5.17) with the IIB parameters, a straightforward calculation shows that the $SL(2, \mathbf{Z})$ transformation law for the NCOS parameters reads

$$\begin{pmatrix} \tilde{P} \\ \tilde{Q} \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} P \\ Q \end{pmatrix} \tag{5.27}$$

$$\tilde{g}^2_{YM} = \frac{g^2_{YM}(cP + dQ)^3}{Q^3}, \tag{5.28}$$

$$\tilde{G}^2_o = \frac{(cP + dQ)(cQ + (cP + dQ)G^2_o)}{Q^2}. \tag{5.29}$$

These formulas closely resemble the Morita duality transformations of the parameters of non-commutative Yang-Mills theory [68, 77].

Note that, in the special case that χ and G^2_o are rational, there is always one particular $SL(2, Z)$ transformation for which the denominator in (5.26) vanishes, which implies that the transformed theory has $\tilde{G}^2_o = 0$. Hence in this case the NCOS theory can be mapped back onto a commutative SYM theory. Conversely, this means that any NCOS theory with rational G^2_o has a precise field theoretic definition via this equivalent SYM gauge theory. NCOS theories with irrational G^2_o, on the other hand, do not have such a UV definition; they need to be defined via the corresponding IIB decoupling limit (5.12).

In order to make the $SL(2,\mathbf{Z})$ multiplet structure of the supergravity background more manifest, it is convenient to go to the Einstein frame, where the metric becomes

$$ds^2 = l_p^2 \left(\frac{U^3}{g_{YM}^3\sqrt{Q}}\right)^{1/2} \left(\frac{U^3}{g_{YM}\sqrt{Q}}(-f(U)dx_0^2 + dx_1^2) + \frac{g_{YM}\sqrt{Q}}{U^3}(f^{-1}(U)dU^2 + U^2 d\Omega_7^2)\right). \tag{5.30}$$

Here we have included the thermal factor

$$f(U) = 1 - \frac{U_0^6}{U^6}\,. \tag{5.31}$$

This solution describes a non-extremal black string with a horizon located at $U = U_0$. It should therefore be interpreted as the supergravity dual of generic NCOS theory at a finite temperature T_0. The relation between U_0 and T_0 can be determined by analytically continuing the solution to Euclidean signature $t = i\tau$ and looking at the metric in the U and τ coordinates near $U = U_0$. Introducing the coordinate

$$\rho^2 \sim 1 - \frac{U_0^6}{U^6} \tag{5.32}$$

the metric near $\rho = 0$ take the form

$$ds^2 \sim (d\rho^2 + \frac{9U_0^4}{g_{YM}^2 Q}\rho^2 d\tau^2) \tag{5.33}$$

from which we infer that (up to factors of order one), the temperature T_0 equals

$$T_0 \simeq \frac{U_0^2}{g_{YM}\sqrt{Q}}\,. \tag{5.34}$$

This is the standard UV/IR relation for D1-branes [78].

Since temperature is a physical notion independent of the S-duality orbit, let us now choose to parameterize the radial coordinate by

$$T = \frac{U^2}{g_{YM}\sqrt{Q}}\,, \tag{5.35}$$

and introduce the $SL(2,\mathbf{Z})$ invariant combination

$$\gamma^2 = \frac{g_{YM}^2}{Q^3}. \qquad (5.36)$$

Replacing g_{YM}^2 and U by γ^2 and T, the supergravity solution in Einstein frame becomes

$$ds^2 = l_p^2 \left(\frac{T}{\gamma}\right)^{1/4} \left(T^2(-dt^2 + dx^2) + \frac{1}{4T^2} dT^2 + d\Omega_7^2\right) \qquad (5.37)$$

$$\begin{pmatrix} B_{NS} \\ B_{RR} \end{pmatrix} = l_p^2 \begin{pmatrix} -G_o^2 \\ 1 + \frac{G_o^2 P}{Q} \end{pmatrix} \left(\frac{T^3}{\gamma Q}\right) \qquad (5.38)$$

$$e^\phi = \frac{\gamma^3 Q^4 + G_o^4 T^3}{\gamma^{3/2} Q^2 T^{3/2}} \qquad (5.39)$$

$$\chi = \frac{\gamma^3 P Q^4 + G_o^2 (G_o^2 P + Q) T^3}{Q(\gamma^3 Q^4 + G_o^4 T^3)}. \qquad (5.40)$$

As expected, we see that the Einstein frame metric is S-duality invariant. The two-form fluxes and the axion and dilaton fields, on the other hand, transform covariantly.

In the following, we will identify the dilaton profile (5.39) as a function of the radial coordinate T with the actual effective string coupling, in the given (P, Q) frame, as a function of the physical temperature. The justification for this identification is that, at temperature T_0, most of the degrees of freedom of the supergravity can be thought of as being localized near the black string horizon at $T = T_0$. Indeed, provided we are in a regime where the supergravity description is valid, we can identify the thermodynamic entropy density of the NCOS theory with the Bekenstein-Hawking entropy of the black string

$$s = \frac{S}{V} = \frac{A_H}{l_p^8 V} = \frac{T^2}{\gamma}. \qquad (5.41)$$

This confirms that the NCOS matter can be thought of as forming the thermal atmosphere of the black string, and that the strength of interactions is governed by the effective string coupling e^ϕ close to the horizon at $T = T_0$.

5.3 Phases of NCOS theory

In the previous section, we formulated NCOS both as a decoupled theory on a brane and as a supergravity dual. As we emphasized in the introduction, these two formulations of the same theory should complement one another, in the sense that depending on the circumstances, one or the other should single itself out as the preferred description of the system. Let us address this issue concretely by first fixing all of the parameters g^2_{YM}, G^2_o, P and Q, while varying the temperature.

For starters, the value of the dilaton needs to stay small in order for the theory to be weakly coupled. From the form of the dilaton given in (5.39), we find that this restricts the temperature to take value in the range $e^\phi \ll 1$, or

$$Q^2\left(\frac{1-\sqrt{1-4G_o^4}}{2G_o^4}\right) \ll \left(\frac{T}{\gamma}\right)^{3/2} \ll Q^2\left(\frac{1+\sqrt{1-4G_o^4}}{2G_o^4}\right) . \qquad (5.42)$$

It is clear that G^2_o must be less than $1/2$ for this region to exist, and that the region is bounded below at $T^{3/2} = \gamma^{3/2}Q^2$. We will analyze what happens outside this range of temperatures in the following section.

Let us now explore the full range of validity of the (P,Q) supergravity description. The general criteria for the effectiveness of supergravity description [76] dictate that the curvature radius as measured in the string frame metric should be large compared to the string length. The curvature radius of the background (5.37) can be estimated from the radius of the 7-sphere forming the black string horizon. Comparing this radius with the string length, we thus deduce that the supergravity approximation breaks down in the region

$$\frac{R^2}{\alpha'} = \left(\frac{T}{\gamma}\right)^{1/4} e^{-\phi/2} \ll 1 . \qquad (5.43)$$

It can be seen that this region is completely contained inside of the range (5.42). Both regions are indicated in figure 5.1, for values of G^2_o ranging from $-1/2$ to $1/2$.

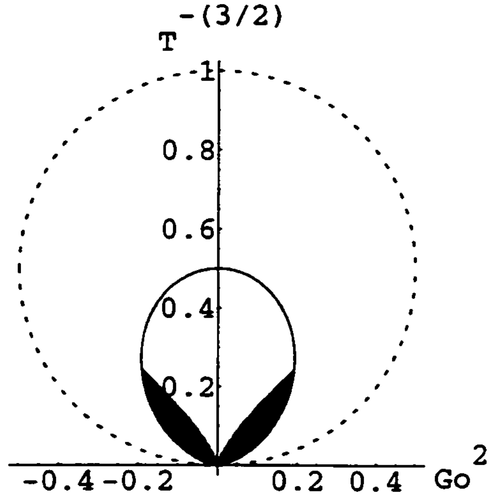

Figure 5.1: This figure indicates the regimes of validity of the three possible phases of NCOS theory, for given charges P and Q: (i) the supergravity phase, inside the black dashed circle, outside the red line, (ii) the gauge theory phase, inside the red line, and (iii) the matrix string phase, the green shaded region.

For the vertical coordinate in the figure we have chosen $T^{-3/2}$, so that the $e^{\phi} = 1$ boundary (5.42) takes the form of a circle, indicated by the black dashed line. The regime (5.43) is bounded by the red solid line, so that the supergravity description is valid in the region inside the black dotted line and outside the red solid line. Note that the ultraviolet is at the lower end of the figure.

Since the curvature radius of the supergravity is small inside the red circle, we can expect that the dual gauge theory description may take over in this region. This is most easily verified at the special vertical line at $G_o^2 = 0$, where the non-commutativity parameter is turned off. This line corresponds to ordinary 1+1 dimensional super Yang-Mills theory. The red line intersects the $G_0^2 = 0$ axis at $T^2 = g_{YM}^2/Q$, which is indeed exactly the point where the 't Hooft coupling of the gauge theory is of order one. Moreover, since SYM theory in 1+1 dimensions is super-renormalizable, the gauge theory description remains valid for arbitrarily high energies; this is indicated in the figure 5.1 by the fact that the red circle is touching the abscissa.

Away from $G_o^2 = 0$, the effect of non-commutativity should manifest itself. Specifically, when starting from the infra-red (from above in the figure), we expect that when the temperature T reaches the scale set by the NCOS string tension

$$T = \frac{1}{\sqrt{\alpha'_{eff}}} = \sqrt{\frac{g_{YM}^2}{G_o^2}} \tag{5.44}$$

the theory must undergo a Hagedorn transition. Beyond this temperature, the system is most accurately described by a matrix string theory (MST) phase [66], that is a sigma model on $\mathbf{R}^8/S_N$ describing the eigenvalue dynamics of the matrix scalar fields of the SYM model. These matrix strings can be thought of as ionized NCOS strings, that due to the thermal fluctuations have managed to escape the D-string bound state via the Coulomb branch. A concrete quantitative check of this physical picture is provided by the fact that the effective tension of a long fundamental string that escapes to infinity in the supergravity geometry (5.18) coincides with the tension of the NCOS strings:

$$\frac{1}{2\alpha'_{eff}} = \frac{1}{\alpha'}(\sqrt{g^s_{00}g^s_{11}} + B_{NS}) = \frac{g_{YM}^2}{2G_o^2}. \tag{5.45}$$

The range of temperatures in which this phase dominates is illustrated by the green shaded region in figure 5.1.

At even higher temperatures, one hits the boundary of the red region where the supergravity description again becomes valid. At that temperature, the long NCOS strings recombine with the D-string due to the strong gravitational attraction caused by the black hole geometry of the supergravity dual [79]. The sequence of phases

$$\text{SUGRA} \quad \rightarrow \quad \text{NCOS} \quad \rightarrow \quad \text{MST} \quad \rightarrow \quad \text{SUGRA} \tag{5.46}$$

going up in temperature was also described in [66].

5.4 SL(2,Z) duality cascades

Our remaining task now is to describe what happens outside the black circle in figure 5.1. Clearly, since the effective string coupling is getting large there, we can expect that the system goes over into another S-dual regime. A small subtlety is that, because of the non-trivial axion background, a simple inversion will not necessarily map strong coupling to weak coupling. More general $SL(2,\mathbf{Z})$ transformations may be needed.

To address this issue in a systematic way, we will take advantage of the survey of $SL(2,\mathbf{Z})$ duality transformations of NCOS theory given in section 2. To begin, it will turn out to be convenient to exploit the S-duality equivalence and combine all theories into one single parameterization. The most convenient choice is to take as the base theory, the system with charges

$$P = 0, \qquad Q = N \tag{5.47}$$

and couplings

$$g^2_{YM}, \qquad G^2_o, \tag{5.48}$$

and parameterize all the dual theories by the element

$$\Lambda = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbf{Z}) \tag{5.49}$$

which maps it to the system with couplings and charges

$$\tilde{g}^2_{YM} = d^3 g^2_{YM}, \qquad \tilde{G}^2_o = d(c + dG^2_o), \qquad \tilde{P} = bN, \qquad \tilde{Q} = dN\,. \tag{5.50}$$

In other words, we can use g^2_{YM} to set the scale, and G^2_o as the data parameterizing the $SL(2,\mathbf{Z})$ equivalence class, and c and d as the data parameterizing the specific elements of the $SL(2,\mathbf{Z})$ orbit.

In terms of these data, the dilaton profile (5.39) as a function of temperature takes the following form

$$e^{\phi} = (c + dG_o^2)^2 \left(\frac{T^{3/2}}{\gamma^{3/2}N^2}\right) + d^2 \left(\frac{\gamma^{3/2}N^2}{T^{3/2}}\right) . \tag{5.51}$$

At this point, it is convenient to introduce the dimensionless parameters

$$x = G_o^2, \qquad y = \frac{\gamma^{3/2}N^2}{T^{3/2}} \tag{5.52}$$

to quantify the coupling and the temperature, respectively. Note, as we did in section 4, that y scales like $T^{-3/2}$ so small y corresponds to large temperature. In terms of x and y, the dilaton profile (5.52) becomes

$$e^{\phi} = \frac{(c + dx)^2}{y} + d^2 y . \tag{5.53}$$

Our task is to determine, for given value of the parameters x and y, which effective theory provides the best description of the system. As a first step in this procedure, we will identify the pair of integers (c, d) which minimizes the string coupling (5.53) at each given point in the (x, y)-plane. For this purpose, it is helpful to first draw the locus on the (x, y)-plane for which $e^{\phi} = 1$ for all possible integers (c, d). These loci are circles and are illustrated in figure 5.2.a.

The circle corresponding to $(c, d) = (0, 1)$ is the one drawn earlier in figure 5.1; the rest are its generalizations to other values of (c, d). Inside each of the circles, the corresponding string coupling $g_s = e^{\phi}$ is smaller than 1. None of the circles overlap, so if a point (x, y) happens to be inside a (c, d) circle, the most weakly coupled theory is the one labeled by (c, d). There are some points, however, which are not covered by the circles. Here we can not apply $SL(2, \mathbf{Z})$ duality to make the string coupling less than one. However, since we are only interested in identifying the dual theory which minimizes the dilaton, we can take the freedom to extend the circular regions in such a way that adjoining (c, d) cells have the same value of the dilaton along

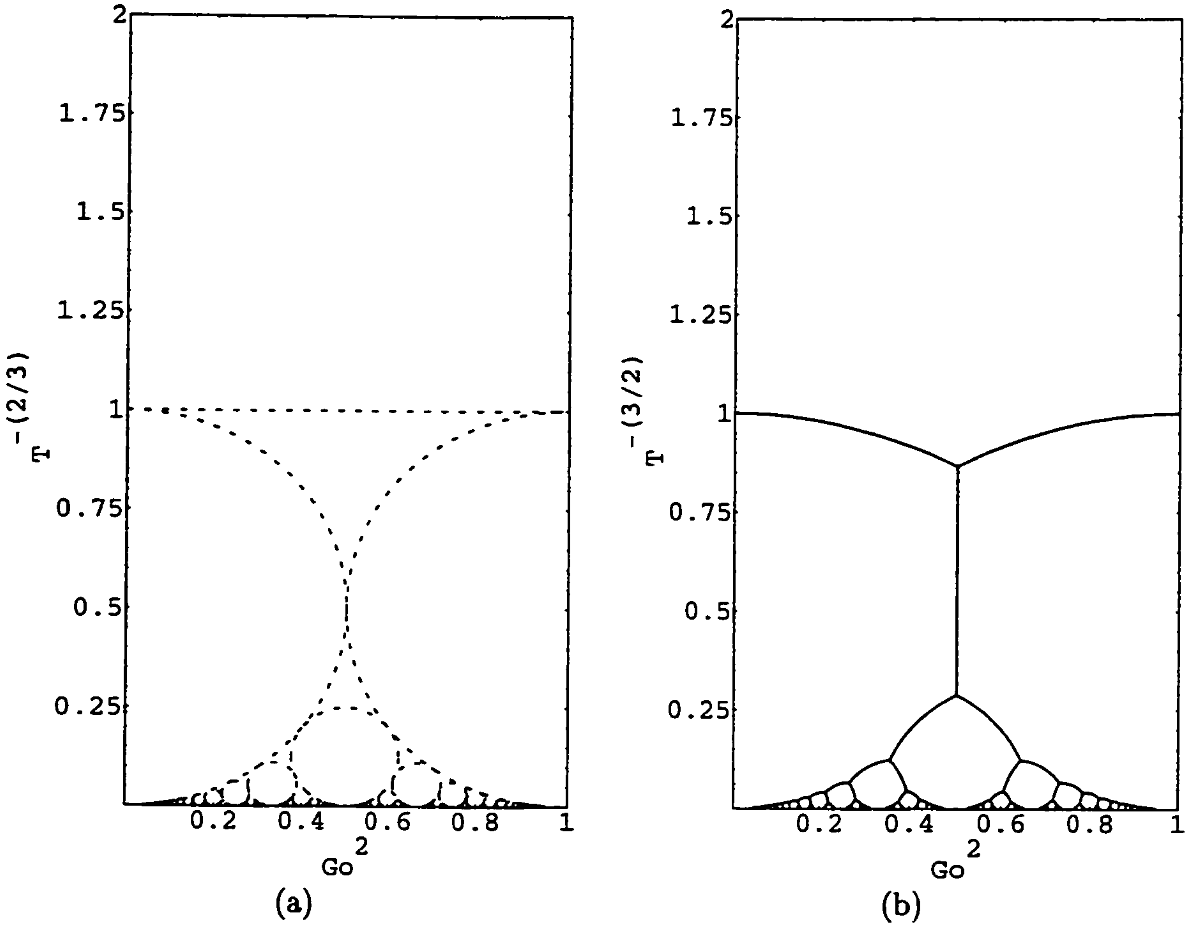

Figure 5.2: In the left figure, we have indicated the circles in the (x, y) plane inside of which $e^{\phi} < 1$ for some integers (c, d). In the right figure, these regions are extended, such that the adjoining (c, d) cells have the same value for the dilaton along the boundary. Inside each cell, one unique (c, d) description minimizes the dilaton.

the boundary. The resulting (c, d) cells, which now fill the entire (x, y) plane, are illustrated in figure 5.2.b.

These (c, d) regions on the phase diagram have an identical structure to that found in [69] in the context of non-commutative gauge theory on a torus, where the role of $SL(2, \mathbf{Z})$ was played by the Morita equivalence relation. As was emphasized in [69], these phase structures also bear very interesting resemblance to the phase structure of lattice spin models with θ parameters considered in [80, 81]. Similar structures have also appeared in the context of dissipative Hofstadter model [82] and quantum Hall systems [83].

The analysis of the phase structure in each of the (c, d) cells will closely parallel our earlier discussion in section 3. In particular, we expect that within each of the cells, we can identify three different regions, corresponding to the effective gauge theory phase, matrix string phase, and the supergravity phase. The respective ranges of validity of these different phases are summarized in the phase diagram displayed in figure 5.3.

Let us highlight some of the features of this phase diagram.

- The vertical axis is proportional to $T^{-3/2}$, so that the ultraviolet corresponds to the bottom, and the infrared to the upper end of the figure. Each vertical slice corresponds to the $(0, N)$ NCOS theory with given G_o^2.

- For every rational value of G_o^2, there is a (c, d) cell touching the horizontal axis at $y = 0$. At this point

$$\tilde{G}_o^2 = d(c + dG_o^2) = 0, \tag{5.54}$$

  it corresponds to an ordinary super Yang-Mills theory, with gauge group $U(dN)$ and electric flux $P = bN$.

- Starting from a given SYM gauge theory with E-flux, one can flow upwards in y

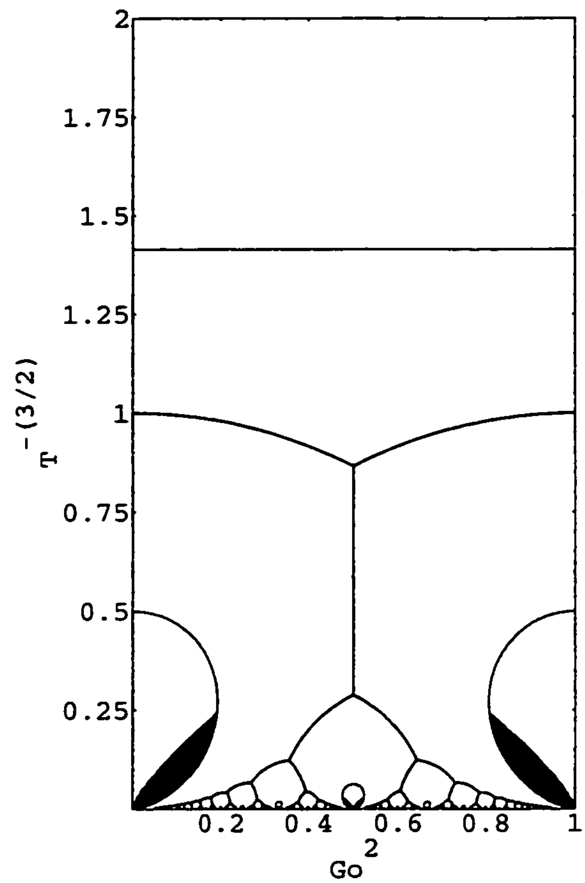

Figure 5.3: The phase diagram that combines all possible phases of SYM/NCOS theory in 1+1 dimensions for fixed $N = \gcd(P, Q)$. A unique (P, Q) theory provides the most weakly coupled description inside each fundamental domain. Each fundamental domain is further divided into the supergravity phase (outside the red circle), the field theory phase (inside the red circle), and the coexisting matrix string phase (shaded green region).

toward the infrared. It is possible that the system then crosses over into another (c, d) cell, and reaches another effective gauge theory phase. This effective gauge theory is deformed with an irrelevant non-commutative perturbation, proportional to the effective tension of the corresponding NCOS phase. This sequence of phases has been described in [66]. What we see here, however, is that the sequence of phases does not necessarily stop here: by going further toward the infrared, one can potentially cross many more (c, d) cells before reaching the deep infrared region at $y = \infty$.

- The number of $SL(2, \mathbf{Z})$ transformations involved in the flow from the UV to the IR depends sensitively on the rationality of G_o^2. Irrational values of G_o^2 require infinitely many $SL(2, \mathbf{Z})$ transformation in the ultraviolet region, as indicated by the fractal phase pattern illustrated in figure 5.3.

- Since the ionization/recombination phase transition associated with the Hagedorn scale takes place in each of the (c, d) cells, the system can undergo these transitions multiple times as the temperature is varied monotonically.

As a concrete illustration of this type of duality cascade, let us consider a given theory with parameters

$$P = 0, \qquad Q = N, \qquad G_o^2 = \frac{1}{n_1 - \frac{1}{n_2}} \tag{5.55}$$

corresponding in the ultra-violet to a $U(N(n_1 n_2 - 1))$ super Yang-Mills theory with $N n_1$ units of electric flux. The SYM degrees of freedom, however, are weakly coupled in the far UV only. In particular, since electric flux creates a mass gap in two dimensions, one expects that towards the infra-red, the $U(Q)$ gauge symmetry gets broken to $U(N)$. Ultimately, the system will flow towards $U(N)$ matrix string theory.

It is instructive to trace all the intermediate phases between the UV gauge theory phase and the IR matrix string phase. They are listed in figure 5.4, where we have also indicated the behavior of the entropy in all different phases. It will turn out that the phases are well separated provided that $n_2 \gg Nn_1 \gg N^2 \gg 1$.

Starting from the infra-red, flowing down towards the UV, the system first follows the successive phases outlined in [66] and [76]. Continuing further towards the UV, however, the theory again enters a supergravity phase. At the point where $e^{\phi} = 1$, we now need to apply the S-duality transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & n_1 \end{pmatrix}, \tag{5.56}$$

connecting to an NCOS theory with charges (N, Nn_1). Then, after a similar sequence of supergravity, gauge theory, and matrix string theory phases, the duality cascade continues via the S-duality transformation

$$\begin{pmatrix} 0 & 1 \\ -1 & n_2 \end{pmatrix} \tag{5.57}$$

which finally takes us to a commutative theory with charges $(P, Q) = (Nn_1, N(n_1 n_2 - 1))$.

5.5 Conclusions

In this article, we investigated the full phase structure of non-commutative open string theories in 1+1 dimensions and found a rich fractal structure closely resembling the phase structure of non-commutative gauge theory on a torus. The most striking conclusion of our analysis is that, with increasing temperature, the system can undergo multiple transitions between alternating gauge theory, matrix string theory, and supergravity phases.

A comment is in order regarding the nature of the thermodynamic transitions at the various phase boundaries. Since we are working in 1+1 dimensions, strictly

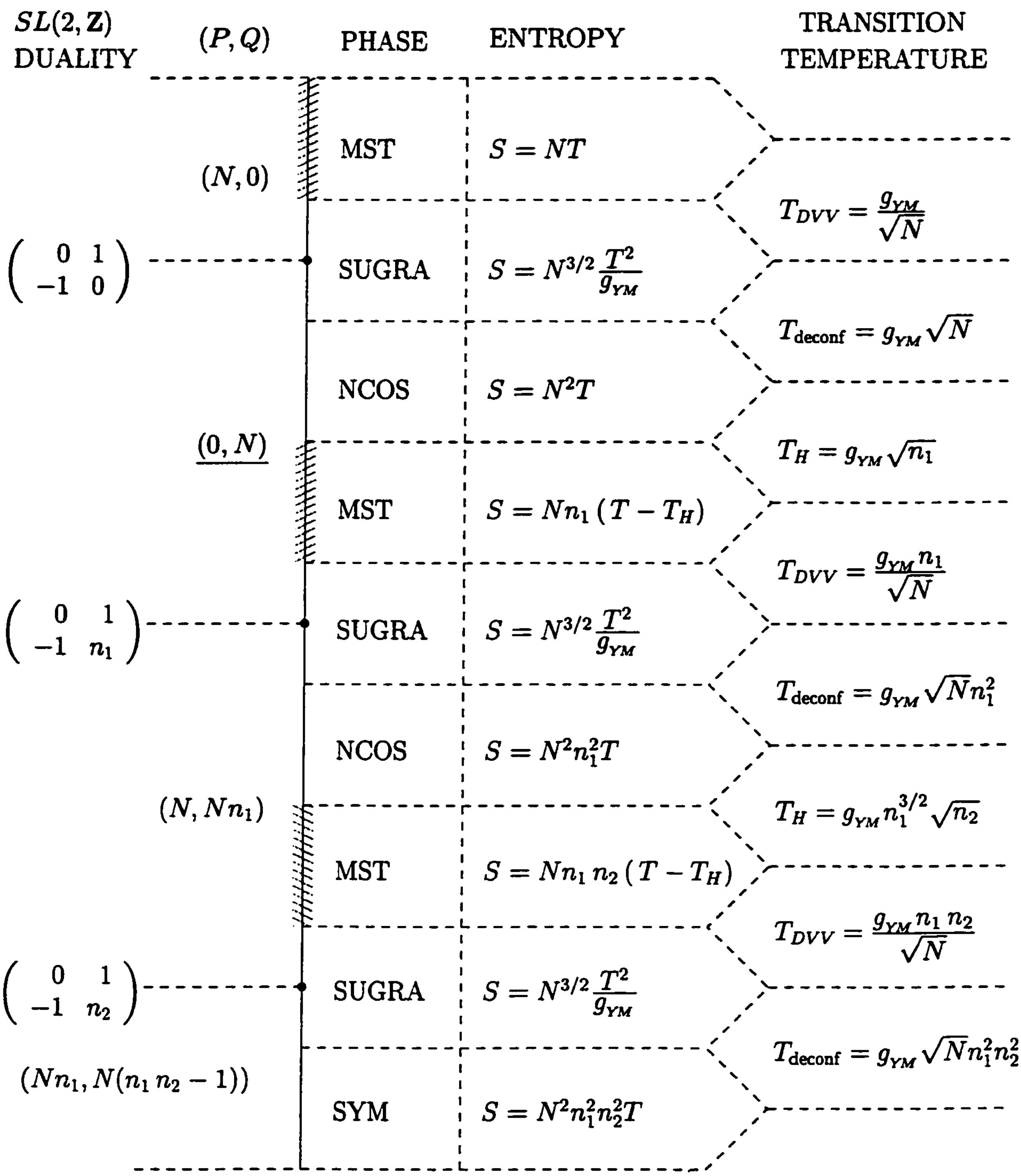

Figure 5.4: An overview of the duality cascade and all the intermediate phases for the special case of a $(0, N)$ NCOS theory with $G_o^2 = \frac{1}{n_1 - \frac{1}{n_2}}$ with $n_2 \gg N n_1 \gg N^2 \gg 1$. We have also given the qualitative behavior of the entropy, and the transition temperatures. The entropy is maximal in the ultraviolet (bottom of the figure), and decreases monotonically towards the infrared (top of the figure).

speaking there should not be any phase transition, unless possibly when we take the large N limit. At large but finite N, it is more appropriate to refer to the transitions between the different phases as "crossovers." It is conceivable, however, that in the $N \to \infty$ limit, some of the crossovers (in particular the Hagedorn transition) may actually become true phase transitions. Even without sharp transitions, however, the duality cascade described in this article should have many observable consequences.

For rational values of G_o^2, the thermodynamics described here is that of ordinary super Yang-Mills theory with some electric flux. It would be interesting to see if it is possible to reproduce some of our results more directly via other techniques. For example, it may be possible to find signatures of these phase transitions in the behavior of the thermal partition function of the gauge theory or of the two-point function of the stress energy tensor [84]. Perhaps a computation on the lattice or DLCQ methods [85, 86] can provide some useful insights.

Much of the $SL(2, \mathbf{Z})$ structure of NCOS theory in 1+1 dimension can rather straightforwardly be generalized to higher dimensions. There are several important differences, however. In 3+1 dimensions, for example, g_{YM}^2 is a freely adjustable dimensionless quantity. Therefore, the full phase diagram will be three dimensional, parameterized by g_{YM}^2, G_o^2, and T (measured in units of the NCOS string length). A preliminary study indicates that the cross sections with constant g_{YM}^2 display an analogous structure as describe here, whereas the cross sections for constant G_0^2 look similar to the phase diagram described in [79] for the case of small G_o^2. It should be instructive to map out the full multi-dimensional phase structure also for these higher dimensional cases.

Another interesting open question about the 1+1-dimensional case is what happens in the case of irrational G_o^2. In this case the system does not have any known UV definition. Nonetheless, at any finite temperature, it can be approximated to arbitrary precision by a sequence of rational G_o^2 theories, which do have a UV description. It would be very interesting to find out whether the irrational theory allows for an independent UV fixed point description.

Appendix A

5-d View of 4-d Linearized Gravity

It is instructive to see how the presence of the 4-d gravitational force is reproduced from the holographic 5-d point of view. In the linearized approximation, it is useful to think of the 5-d black hole as a collection of mass points, so that we can compute the 5-d curvature by superposing all the small metric deformations due to the presence of each separate mass point. In this appendix, we show how, in this linearized approximation, the 5-d black hole will feel an effective 4-d gravitational force. Our analysis will be somewhat schematic; a more complete treatment of the linearized equations of motion in the RS model, that includes a careful discussion on the gauge fixing as well as the contribution of the different polarizations of $h_{\mu\nu}$, is given in [35].

Consider the minimal RS model with the AdS metric

$$ds^2 = dr^2 + e^{-2r} dx^\mu dx_\mu \qquad \text{(A.1)}$$

with $r \geq 0$; here $r = 0$ is the location of the Planck brane with action (3.6), with fine-tuned tension so that the vacuum solution is 4-d Poincare invariant. Introduce a stress-energy source localized at some location $r = r_0$ in the bulk. The linearized 5-d

Einstein equation then looks like (suppressing the indices)

$$\left(-\frac{1}{2}\partial_r^2 + 2 + \frac{1}{2}e^{2r}\nabla^2\right)h(x,r) = \kappa_5\delta(r-r_0)T(x), \tag{A.2}$$

with $\kappa_5 = 16\pi G_5$. The presence of the localized stress-energy source at $r = r_0$ leads to the following (dis)continuity equations for the metric fluctuation $h(x,r)$:

$$h(r_{0+}) = h(r_{0-}) \qquad \partial_r h(r_{0+}) = \partial_r h(r_{0-}) - 2\kappa_5 T. \tag{A.3}$$

Working in the low energy regime $p^2 \ll e^{-2r}$, we can write the general solution to (A.2) in the region $r > r_0$ as

$$h(x,r) \simeq e^{-2r}h_+(x) - \frac{1}{4}\nabla^2 h_+(x) \tag{A.4}$$

with $h_+(x)$ to be determined. Similarly we find for $r < r_0$

$$h(x,r) \simeq e^{-2r}h_-(x) - \frac{1}{4}\nabla^2 h_-(x) + \frac{\kappa_4}{2}e^{2r}\nabla^2 h_-(x). \tag{A.5}$$

Here the last term on the right-hand side, though seemingly subleading, is needed to be able to satisfy the boundary condition at the Planck brane (using the fine-tuned value of the brane tension)

$$\partial_r h(0) = \frac{\delta S_{UV}}{\delta h} = -2h(0) + b_1\nabla^2 h(0). \tag{A.6}$$

Solving this boundary condition fixes

$$2\kappa_4 = \frac{1}{2} + b_1. \tag{A.7}$$

Having the solutions for $r > r_0$ and $r < r_0$, we can match them via the (dis)continuity conditions (A.3) at the location of the stress-energy source. Keeping only leading terms, this gives

$$\begin{aligned} e^{-2r_0} h_+ &= e^{-2r_0} h_- + \frac{\kappa_4}{2} e^{2r_0} \nabla^2 h_- \\ e^{-2r_0} h_+ &= e^{-2r_0} h_- - \frac{\kappa_4}{2} e^{2r_0} \nabla^2 h_- + \kappa_5 T. \end{aligned} \tag{A.8}$$

Subtracting the two equations results in

$$\kappa_4 e^{2r_0} \nabla^2 h_-(x) = \kappa_5 T(x) \tag{A.9}$$

which we recognize as the 4-d linearized Einstein equation for the reflected wave, with 4-d Newton constant $16\pi G_N = \kappa_5/\kappa_4$. It explicitly shows that a 5-d stress energy source $T(x)$ located at distance $r = r_0$ from the Planck brane acts as an effective 4-d stress energy source $e^{-2r_0}T(x)$, as expected from the relative red-shift between its location and that of the Planck brane. Plugging the solution to eqn (A.9) into the above expressions for $h(x,r)$ gives the complete 5-d linearized back-reaction due to the source $T(x)$. The perturbation $h(x,r)$ represents a new (relative to that in pure AdS space) component to the 5-d gravitational force, equal to that of the effective 4-d gravity, that pulls the mass distribution towards the Planck brane.

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