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Applications of Gauge/Gravity Duality to QCD and Heavy Ion Collisions

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ACADEMIC DISSERTATION

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Abstract

The description of quarks and gluons, using the theory of quantum chromodynamics (QCD), has been known for a long time. Nevertheless, many fundamental questions in QCD remain unanswered. This is mainly due to problems in solving the theory at low energies, where the theory is strongly interacting. AdS/CFT is a duality between a specific string theory and a conformal field theory. Duality provides new tools to solve the conformal field theory in the strong coupling regime. There is also some evidence that using the duality, one can get at least qualitative understanding of how QCD behaves at strong coupling. In this thesis, we try to address some issues related to QCD and heavy ion collisions, applying the duality in various ways.

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Helsinki, April 2008

Touko Tahkokallio

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List of included publications

I. K. Kajantie, T. Tahkokallio and J. T. Yee,
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Author's contribution

In the first paper, the present author did most of the calculations and graphical presentation of the results. The results were analyzed and the paper written jointly with the present author, Jung-Tay Yee and Keijo Kajantie.

In the second paper, the author discovered the exact solution using Mathematica and carried out most of the calculations. The results were analyzed and paper written together with Keijo Kajantie.

In the third and fourth paper, the exact solutions were also discovered by the author. Also in these cases, the analyzing of the results was done jointly among the collaborators; the present author, Keijo Kajantie and Jorma Louko. In the third paper, one of the main ideas, the role of vacuum energy, was discovered by the author.

Chapter 1

Introduction

The fundamental theory describing quarks and gluons is quantum chromodynamics (QCD). The first hints of QCD arose in the sixties, when particle physicists tried to understand the origin of large and ever-growing number of particles observed in particle accelerators. The variety of these particles, hadrons, could be explained when one assumed that they were built from more fundamental particles, quarks and gluons, which interacted very strongly with each other.

Even though the formulation of QCD has been known for a long time, many questions still remain unanswered. This is mainly because the theory is very difficult to solve. The standard method of quantum field theories, i.e. perturbation theory, works only in the regime of high energy QCD. This is due to the flow of the QCD coupling constant — when particles interact with high energies, the interactions between particles are weak, and the expansion as a power series in the coupling constant is possible. However, when energies become smaller, the strength of the interaction grows and the problem becomes non-perturbative. Therefore, for example, the confinement of quarks inside hadrons cannot be understood in terms of perturbative calculations.

There exist different methods for approaching the non-perturbative part of QCD. The most used approach is lattice QCD — for a review on the subject, see e.g. [1]. In lattice QCD, one studies the theory on a discrete set of space-time points and uses computers to obtain numerical results to wide range of different problems. Other methods include effective theories, like chiral perturbation theory (see [2] for a review), where the degrees of freedom are no longer quarks and gluons but small-mass pseudoscalar mesons, the effective degrees of freedom of QCD in the low-energy domain.

Prior to the discovery of QCD, particle physicists found that many strange properties of these hadrons could be explained if one assumed that the particles were not point-like, but like strings with a finite length [3, 4, 5]. However, when the theory describing strings was quantized, it become clear that the theory had much more structure than what was needed to describe just the observed particles. The spectrum included, for example, a massless spin-2 particle — the graviton. Also it was found out that the theory was not consistent in four dimensions, but needed extra dimensions to be well defined. However, these observations were not fatal for the theory — on the contrary — string theory has been an extremely studied branch of theoretical physics since. It is a theory for quantum gravity and also naturally includes the structure necessary to produce standard model -like theories, therefore it is even today our best candidate for unified quantum description of gravity and the standard model.

The string-like properties of hadrons can be quantitatively understood in the language of QCD, even though analytic calculations remain out of reach. The endpoints of the strings can be understood as quarks. In QCD, quarks are electrically charged, but in addition have a new kind of charge, the color charge. The color force between two color charges, is mediated through gluons. The color force holds quarks tightly close to each other. When one tries to pull the quarks from each other, the energy stored in the color field becomes greater and concentrates on a line between two quarks, forming the so called flux-tube. From the QCD perspective, the flux-tube is effectively the explanation for the stringy properties of hadrons. It acts like a spring between two quarks, resisting when stretched.

The stringy behavior of hadrons remained as an inspiration for some phenomenological models of strong interactions, such as the Lund string model [6], but QCD soon established its status as the correct theory describing strong interactions. However, the paths of these theories started to converge again, when Juan Maldacena [7] in 1998 proposed a very interesting conjecture stating that type IIB string theory on an $\mathrm{AdS}_5 \times \mathbf{S}^5$ background is dual to 3+1 dimensional $\mathcal{N} = 4$ supersymmetric $\mathrm{SU}(N)$ Yang-Mills theory. The duality was further formulated in papers [8] and [9]. It states that these two very different theories should be in fact equivalent, in the sense that there is a dictionary between the two theories — one can calculate something on the string side and predict something on the field theory side, and vice versa.

Maldacena's original conjecture is just one example of the so called AdS/CFT dualities — or gauge/gravity dualities more generally. AdS/CFT duality relates a gravity theory on an anti de Sitter (AdS) spacetime to a conformal field theory (CFT) living on the boundary of the spacetime. This duality has been an extremely active subject of research ever since its discovery. Even though the conjecture has not been rigorously proven, it has been verified in a huge number of different tests, see e.g. [10, 11, 12, 13] or [14] for a review.

String theory is known to be notoriously difficult to quantize in curved backgrounds, such as the anti de Sitter space. However, in the limit when the radius of the anti de Sitter spacetime $\mathcal{L}$ is much bigger than the string length $\sqrt{\alpha'}$ and the number of colors N goes to infinity, string theory can be approximated using supergravity, which is a classical theory that is much easier to deal with. On the field theory side, the relation $\mathcal{L}^2 \gg \alpha'$ translates to $\lambda = g_{YM}^2 N \gg 1$, where λ is the 't Hooft coupling. It is the effective coupling constant of the theory, when $N \to \infty$. This gives an opportunity to study the strong coupling regime of 3+1 dimensional $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory, using methods of classical gravity.

The applications of this revolutionary duality to QCD are mainly threefold: first, even though $\mathcal{N} = 4$ supersymmetric Yang-Mills theory is a theory different from QCD, it still shares some properties with QCD and one can argue that in certain situations these theories are close enough so that one can use the results of the duality to understand QCD at strong coupling. Second, it seems that the gauge/gravity duality can be expanded to a variety of directions. This gives a possibility to search for a field theory that is closer to QCD and that has a gravity dual. One could even try to go a step further and try to find an exact dual string description of QCD. Third, one can also take the gauge/gravity picture as a motivating framework to construct phenomenological models in extra-dimensions which qualitatively produce some features of QCD.

In summary, the gauge/gravity duality offers variety of methods to study phenomena in QCD at strong coupling. Also, it offers an intriguing possibility to find and exact string description of QCD.

1.1 Organization of the thesis

The thesis consists of four articles and of an introductory part, divided in four Chapters. The introductory part is intended as an overview of some of the essential tools to study strongly interacting field theories, and QCD in particular, using gauge/gravity dualities.

In the second Chapter of the introductory part, we present the AdS/CFT duality and discuss how it can be used to study phenomena in the boundary theory.

In the third Chapter, we present the hydrodynamical description of hot strongly interacting matter in the heavy ion collision context. The rest of the Chapter three considers the application of AdS/CFT to heavy ion collisions. Finally, the Chapter four is the summary.

The four articles provide the core part of this thesis. In the first article, a phenomenological holographic model is constructed to describe some thermodynamical properties of QCD. In the second paper, an exact five dimensional time-dependent gravity solution is presented and the corresponding time-dependent behavior of the boundary theory is studied. In the third and in the fourth article, we consider a $1+1$ dimensional boundary theory having a particular time-dependent flow and present the exact time-dependent gravity solution which correspond to this flow. In the third article, the boundary flow is a $1+1$ dimensional version of the Bjorken flow. In the fourth article, the flow on the boundary is of the most general type of conformal flow in $1+1$ dimensions.

Chapter 2

AdS/CFT duality

In this chapter, we introduce AdS/CFT duality and review some properties of black holes. We also introduce the anti de Sitter black hole solution.

2.1 Black holes

Black holes have been in the center of interest in theoretical physics for a long time. The reason for this is not just that black holes are mysterious and intriguing objects, but that they are objects of extremes and their description pushes our theories to their limits. Also, to describe a black hole, one does not need just general relativity, but also quantum physics and thermodynamics. Therefore to understand truly the dynamics related to black holes one would need the full quantum theory of gravity. Nevertheless much can be learnt just by using more standard theories, such as general relativity and quantum field theory.

2.1.1 Black holes and the Schwarzschild solution

Black holes arise as solutions to theories of gravity. These spherically symmetric solutions (for non-rotating black holes) describe the structure of the curved spacetime around a black hole. For example, the four dimensional asymptotically flat black hole solution, known better as the Schwarzschild solution, can be written:

$$ds^2 = -\left(1 - \frac{2MG_4}{r}\right) dt^2 + \frac{dr^2}{\left(1 - \frac{2MG_4}{r}\right)} + r^2 \left(d\theta^2 + \sin^2\theta d\phi^2\right), \quad (2.1)$$

The parameter M is the mass of the black hole and G_4 denotes the four dimensional Newton's constant. When $r \to \infty$ the metric (2.1) reduces to the flat Minkowski metric. At the Schwarzschild radius, $r_s = 2MG_4$, the metric (2.1) diverges. This singularity is, however, only a coordinate singularity and all curvature invariants are finite at $r = r_s$.

At the Schwarzschild radius lies the event horizon. The event horizon is the boundary of a black hole — it is a spacelike hypersurface separating spacetime points that are connected to infinity by a timelike path from those that are not. Therefore nothing can escape from the black hole, once it is inside the event horizon. At $r = 0$ lies a true curvature singularity and the curvature invariants, like the Kretschmann scalar $R_{\mu\nu\alpha\gamma}R^{\mu\nu\alpha\gamma}$, diverge at this spacetime point.

In the seventies, it was found that the properties of black hole can be described using laws of thermodynamics [15, 16, 17, 18, 19]. Black holes are systems that can be in general described using only few variables, namely using mass M and angular momentum and charge parameters J and Q. The situation is similar to thermodynamics, where complex systems can be described using only few state-variables like pressure, temperature etc. Indeed, it was found in [18, 19] that black holes radiate and have well defined temperature. Also black holes have an entropy that is proportional to the area of the event horizon [15, 16, 17]. There exists even a consistent formulation of the zeroth, first, second and third laws of thermodynamics in the context of black hole system, see for example [20, 21, 22].

To study thermodynamics related to black holes, one can use the Euclidean path integral approach [23, 24]. In this approach, one calculates the thermal partition function of quantum gravity, summing over all geometries with an Euclidean time coordinate that has the period $\beta = 1/T$. The partition function can be approximated using the saddle point approach:

$$Z = \int Dg e^{-S_{EH}} \simeq e^{-S_{EH}(on-shell)}, \tag{2.2}$$

where the Euclidean version of the Einstein-Hilbert action should be evaluated using the classical black hole solution, with an imaginary time $\gamma = it$. The partition function is related to the free energy F of the system by the simple thermodynamical formula:

$$\beta F = -\log Z. \tag{2.3}$$

The easiest way to calculate the temperature of the black hole is to consider the Euclidean black hole metric. Consider as an example the metric

(2.1) near the Schwarzschild radius $r_s = 2MG_4$:

$$ds^2 \simeq \frac{r - r_s}{r_s} d\gamma^2 + \frac{r_s}{r - r_s} dr^2 + r^2 d\Omega_2^2. \tag{2.4}$$

If one now makes the coordinate transformation:

$$r = \frac{4r_s^2 + \rho^2}{4r_s}, \tag{2.5}$$

one obtains that the γ, ρ -part of the metric becomes:

$$ds^2 = \frac{\rho^2}{4r_s^2} d\gamma^2 + d\rho^2. \tag{2.6}$$

This metric has a conical singularity unless γ has a period $\beta = 4\pi r_s$, and thus $\beta/2r_s = 2\pi$. Therefore, one obtains that the temperature of the Schwarzschild black hole is $T_H = 1/\beta = 1/(8\pi MG_4)$.

Also, from the partition function one can directly calculate the entropy of the black hole using the thermodynamical formula: $S = \frac{\partial}{\partial T}(T \log Z)$. This entropy is the same as the Bekenstein-Hawking entropy which can be generally written as

$$S_{BH} = \frac{A}{4G_d}, \tag{2.7}$$

in d dimensions. In this formula, the area A denotes the area of the event horizon and G_d is the d-dimensional Newton's constant.

2.1.2 The anti de Sitter black hole

The metric (2.1) describes a black hole in flat spacetime. Consider now a black hole in a five dimensional spacetime with the negative cosmological constant $\Lambda = -6/\mathcal{L}^2$. In this case, the black hole solution is the anti de Sitter black hole:

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} + 1 - \frac{\mu}{r^2}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{\mathcal{L}^2} + 1 - \frac{\mu}{r^2}\right)} + r^2 d\Omega_3^2. \tag{2.8}$$

The parameter $\mathcal{L}$ denotes the curvature radius of the anti de Sitter space. The Ricci scalar is $R = -20/\mathcal{L}^2$, for all solutions of the five dimensional Einstein equations in vacuum with $\Lambda = -6/\mathcal{L}^2$.

The metric (2.8) has also a curvature singularity at $r = 0$ and a horizon at

$$r_h^2 = \frac{\mathcal{L}^2}{2}\left(\sqrt{1 + \frac{4\mu}{\mathcal{L}^2}} - 1\right). \tag{2.9}$$

In the limit $r \to \infty$, the metric (2.8) reduces asymptotically to anti de Sitter spacetime with the boundary $\mathbf{R} \times \mathbf{S}^3$, where $\mathbf{S}^3$ is the three-sphere.

Following the arguments used in the last section, one can calculate the temperature of the AdS black hole. One finds that in this case,

$$T_H = \frac{2\pi r_h}{1 + 2r_h^2/\mathcal{L}^2}. \tag{2.10}$$

The parameter μ is related to the mass M of the black hole through the relation[1] :

$$\mu = \frac{8G_5}{3\pi} M. \tag{2.11}$$

One can consider further the limit $M \to \infty$. In this limit, the black hole horizon becomes larger and appears locally as flat. As discussed in [25], the conformal boundary at $r \to \infty$ is in this case $\mathbf{R}^{1,3}$. The metric in the large mass limit therefore reads:

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - \frac{\mu}{r^2}\right) dt^2 + \frac{dr^2}{\left(\frac{r^2}{\mathcal{L}^2} - \frac{\mu}{r^2}\right)} + \frac{r^2}{\mathcal{L}^2} \sum_{i=1}^{3} dx_i^2, \tag{2.12}$$

and now

$$T_H = \frac{\pi \mathcal{L}^2}{r_h}, \quad \text{where} \quad r_h^2 = \mathcal{L}\mu^{1/2}. \tag{2.13}$$

If one further makes the coordinate transformation,

$$r = \frac{\mathcal{L}^2}{z}\sqrt{1 + \frac{z^4}{z_0^4}}, \quad \text{where} \quad z_0 = \frac{\sqrt{2}\,\mathcal{L}^2}{r_h}, \tag{2.14}$$

one obtains the AdS black hole metric in Fefferman-Graham coordinates [26],

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{\left(1 - \frac{z^4}{z_0^4}\right)^2}{1 + \frac{z^4}{z_0^4}} dt^2 + \left(1 + \frac{z^4}{z_0^4}\right) \sum_{i=1}^{3} dx_i^2 + dz^2\right]. \tag{2.15}$$

2.2 AdS/CFT duality

In this section, we are going to briefly introduce the main points of AdS/CFT duality, related to the thesis. For more detailed reviews on the subject, see, e.g., [14, 27, 28, 29, 30].

[1] This can be verified by evaluating the Euclidean version of the action, and by using the formula $E = M_{BH} = T^2 \frac{\partial}{\partial T} \log Z$.

The most important form of AdS/CFT duality states that type IIB string theory on $\mathrm{AdS}_5 \times \mathbf{S}^5$ background[2] is dual to $\mathcal{N} = 4$ supersymmetric SU(N) Yang-Mills theory in 3+1 dimension [7, 8, 9]. Duality means that these two theories are equivalent and one can use gauge theory to calculate processes in the string theory side and vice versa.

Most of practical calculations in AdS/CFT, and calculations relevant to this thesis, are done in the supergravity approximation of string theory. Supergravity can be obtained from string theory when considering the propagation of strings in background of massless fields $(g_{MN}, B_{MN}, \phi, \ldots)$, i.e. when considering the propagation of strings in the "condensate" of its own massless modes. When requiring conformal invariance at quantum level, one finds that these background fields must obey certain equations — the supergravity equations of motion. These equations can be derived as an expansion in α', when $\alpha' \to 0$, encoding the long-wavelength limit of string theory.

One can also derive supergravity equations of motion from an effective action — the supergravity action. Consider the following part of the bosonic sector of IIB supergravity [3] action in the Einstein frame:

$$S_{IIB} = \frac{1}{16\pi G_{10}} \int d^{10}x \sqrt{-g} \left[R - \frac{1}{2}(\nabla\phi)^2 - \frac{1}{4\cdot 5!} F_5^2 + \ldots \right]. \tag{2.16}$$

In the action (2.16), ϕ is the dilaton field and F_5 the five-form field strength, for which one further has to impose the self-duality condition $F_5 = *F_5$.

The equations of motion derived from this action have a solution which describes a 3+1 dimensional brane embedded in ten dimensional spacetime. The metric part of the solution reads:

$$ds^2 = H^{-1/2}(r) \left[-f(r)dt^2 + \sum_{i=1}^{3} dx_i^2 \right] + H^{1/2} \left[f^{-1}(r)dr^2 + r^2 d\Omega_5^2 \right], \tag{2.17}$$

with

$$H(r) = 1 + \frac{\mathcal{L}^4}{r^4} \qquad f(r) = 1 - \frac{r_0^4}{r^4}. \tag{2.18}$$

This solution describes the classical geometry of a stack of non-extremal D3-branes, i.e., thermally excited D3-branes (see e.g. [31, 32]). The metric (2.17) has a horizon at $r = r_0$.

[2] In the more general case, the spacetime needs to be only asymptotically (when $z \to 0$) AdS space. Also the compact part can differ from $\mathbf{S}^5$. Therefore, more generally the spacetime needs to be asymptotically $\mathrm{AdS}_5 \times \mathrm{X}^5$ spacetime, where X^5 is some other compact manifold. For simplicity, we only consider here the case where X^5 is $\mathbf{S}^5$.

[3] Type IIB supergravity is the effective long-wavelength description of type IIB string theory.

Consider now the limit $r \ll \mathcal{L}$. The metric in this region can be approximated by the metric

$$ds^2 = \frac{r^2}{\mathcal{L}^2}\left[-\left(1-\frac{r_0^4}{r^4}\right)dt^2 + \sum_{i=1}^{3} dx_i^2\right] + \frac{\mathcal{L}^2}{r^2}\frac{dr^2}{\left(1-\frac{r_0^4}{r^4}\right)} + \mathcal{L}^2 d\Omega_5. \qquad (2.19)$$

One notices that this metric is a product of AdS black hole in the large mass limit (2.12) and of the five sphere $\mathbf{S}^5$.

It is convenient to make a further coordinate transformation $r = \mathcal{L}^2/z$. The anti de Sitter part of the metric becomes now:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-(1-\frac{z^4}{z_0^4})dt^2 + d\vec{x}^2 + \frac{dz^2}{(1-\frac{z^4}{z_0^4})}\right]. \qquad (2.20)$$

In this coordinate system, the AdS black hole has a horizon at $z = z_0$ and the temperature $T_H = 1/\pi z_0$.

The AdS/CFT conjecture was motivated by the observation that there seems to be two distinct ways to describe the stack of N coincident $D3$-branes. On one hand, the stack of D-branes can be described using supergravity, when N is large and one considers the system at low energies, i.e., one considers only the massless excitations of the closed string spectrum. The near horizon region of the metric (2.17) can be described using $\mathrm{AdS}_5 \times \mathbf{S}^5$ metric as shown above. On the other hand, a D-brane is an endpoint for the open strings and these open strings describe the excitations of the brane. The massless spectrum of open string oscillations living on the D3-brane worldvolume is that of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions. This led Maldacena originally to propose the conjecture [7].

The Bekenstein-Hawking entropy of the black hole can be calculated from the area of the horizon, i.e., from the volume of the spacelike hypersurface with $z = z_0$ and $t = \text{const}$:

$$S_{BH} = \frac{A}{4G_{10}}, \qquad (2.21)$$

where G_{10} is the ten dimensional Newton constant and $G_{10} = 8\pi^6\alpha'^4 g_s^2$. For the metric (2.20), the area A of the horizon is

$$A = \int d\Omega_5 dx_1 dx_2 dx_3 \sqrt{-\gamma}\,_{|z=z_0} = \mathrm{Vol}[\mathbf{S}^5] \times \frac{V_3\mathcal{L}^3}{z_0^3} = \frac{\pi^3 V_3 \mathcal{L}^8}{z_0^3}. \qquad (2.22)$$

Here V_3 denotes the volume in x_1, x_2, x_3 -directions and γ is the determinant of the induced metric γ_{ij} on the corresponding hypersurface. AdS/CFT also provides the relation $\mathcal{L}^4 = 4\pi N g_s \alpha'^2$ [32]. Using this formula and the result that $z_0 = 1/\pi T$, one obtains:

$$S = \frac{\pi^2}{2} N^2 V_3 T^4. \tag{2.23}$$

The value of the Euclidean effective supergravity action I, with the black hole solution (2.20), can be identified with the free energy F of the system according to the formula $I = F/T$, as discussed in the section (2.1). Therefore the entropy of the system can also be calculated using the standard thermodynamical relation $S = -\partial F/\partial T$.

In terms of AdS/CFT, this implies that the free energy calculated from the supergravity action, can be identified with the free energy of the thermal $\mathcal{N} = 4$ gauge theory [25]. Therefore the entropy of the black hole can be interpreted as the entropy of thermal CFT matter, with the temperature T. This simple calculation gives directly the entropy of $\mathcal{N} = 4$ gauge theory (2.23) in the strong coupling regime.

The main content of AdS/CFT duality can be expressed in terms of the fundamental relation:

$$Z_{CFT}[\phi_0] = \int \mathcal{D}\mathcal{O} e^{iS_{CFT}[\mathcal{O}] + i\int d^4x \phi_0(x)\mathcal{O}(x)} = Z_{string}[\phi(x, z \to 0) = \phi_0(x)]. \tag{2.24}$$

On the left hand side one has the generating functional of the CFT — $\mathcal{O}(x)$ represents the operators of the theory and the sources are denoted by ϕ_0. The expectation value of the operator $\langle \mathcal{O} \rangle$ can be obtained in the usual way varying the generating functional with respect to the source terms ϕ_0:

$$\langle \mathcal{O}(x) \rangle = \frac{\int \mathcal{D}\mathcal{O}\, \mathcal{O} e^{iS_{CFT}[\mathcal{O}]}}{\int \mathcal{D}\mathcal{O} e^{iS_{CFT}[\mathcal{O}]}} \tag{2.25}$$

On the right hand side of the equation (2.24) one has the partition function of string theory. As discussed, it can be approximated using the supergravity action $Z_{string} \simeq e^{iS_{Sugra}}$, evaluated with the classical solution of the supergravity equations of motion, with the boundary conditions that the bulk fields $\phi(x, z)$ reduce on the boundary $z \to 0$ to the sources $\phi_0(x)$.

The effective action is:

$$S_{\text{eff}} = \int d^{10}x \sqrt{-g}\, \mathcal{L}_{Sugra}\left[\phi_c(x), \partial_\mu \phi_c(x), \ldots\right], \tag{2.26}$$

where the index c denotes that the action is to be evaluated with the classical values of the fields. One can also do the Kaluza-Klein reduction of the fields over the $\mathbf{S}^5$-part to obtain a five-dimensional action [14, 33, 34].

To get the action as a functional of the sources only, one needs to integrate over the z-coordinate also. One therefore proceeds to solve the z-dependence of the fields and then integrates over the coordinate. In this way one obtains the supergravity action as a functional of the boundary values of the fields only. After this, it is, at least in principle, straightforward to do the variation with respect to the sources $\phi_0(x)$ on the right hand side of the equation (2.24) and obtain the result for the CFT correlation function.

In the course of the integration, one finds, however, that the integration gives an infinite result — this is because there is a $1/z^2$ factor in front of the metric and the integration starts from $z = 0$. The way out of this is that one needs to regulate the integral, considering the boundary to be at some finite $z = \epsilon$. After the regularization, one introduces covariant counter terms to the action that together with the original action give a finite result, when calculating the limit $\epsilon \to 0$. This procedure is called the holographic renormalization.

2.3 Holographic renormalization

As described in the last section, according to the AdS/CFT prescription, the expectation value of an operator in the boundary theory can be calculated by varying the on-shell action with respect to the boundary value of the dual bulk field. For the stress-energy tensor, the dual field is the bulk metric. Therefore, one can determine the boundary stress-energy tensor by calculating the functional derivative of the on-shell gravitational action with respect to the boundary metric.

Consider the five dimensional gravity action:

$$S_{\text{gr}} = \frac{1}{16\pi G_5}\left(\int_M d^5x\sqrt{-G}\,(R-2\Lambda) - 2\int_{\partial M} d^4x\sqrt{-\gamma}K\right). \tag{2.27}$$

The metric $\gamma_{\mu\nu}$ is the induced metric on the boundary of the manifold M and $K = \gamma^{\mu\nu}K_{\mu\nu}$ is the trace of the second fundamental form $K_{\mu\nu}$. This boundary term guarantees that the variational problem is well defined [23].

From the action, one can derive the Einstein equations, with the cosmological constant $\Lambda = -6/\mathcal{L}^2$, for the five dimensional metric G_{MN}:

$$R_{MN} - \frac{1}{2}R\,G_{MN} - \frac{6}{\mathcal{L}^2}G_{MN} = 0. \tag{2.28}$$

As discussed, the anti de Sitter spacetime is a solution to the equation (2.28),

$$ds^2 = G_{MN}dx^M dx^N = \frac{\mathcal{L}^2}{z^2}\left[\eta_{\mu\nu}dx^\mu dx^\nu + dz^2\right]. \tag{2.29}$$

On the boundary [4], the metric (2.29) corresponds to an empty spacetime, $\langle T_{\mu\nu}\rangle = 0$, with the Minkowski metric $\eta_{\mu\nu}$.

However, one can also consider the case when the boundary metric is something more complicated, say $g^{(0)}_{\mu\nu}$, and the expectation value of the boundary energy-momentum tensor is also non-vanishing. In this case, the bulk metric changes correspondingly.

In the general case, one can write the bulk metric in Fefferman-Graham coordinate system as follows:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[g_{\mu\nu}(x,z)dx^\mu dx^\nu + dz^2\right]. \tag{2.30}$$

The function $g_{\mu\nu}(x,z)$ can be now expanded as a power series in z [5]:

$$g_{\mu\nu}(x,z) = g^{(0)}_{\mu\nu}(x) + g^{(2)}_{\mu\nu}(x)z^2 + g^{(4)}_{\mu\nu}(x)z^4 + \ldots \tag{2.31}$$

The expansion (2.31) can be inserted to Einstein equations and the coefficients $g^{(n)}_{\mu\nu}$ can be solved order by order in z.

One finds, in general dimensions, that all functions $g^{(n)}_{\mu\nu}$ can be expressed in terms of the boundary metric $g^{(0)}_{\mu\nu}$ and $g^{(d)}_{\mu\nu}$ only, where d is the dimension of the boundary theory. The functions $g^{(0)}_{\mu\nu}$ and $g^{(d)}_{\mu\nu}$ serve as the boundary conditions for the problem and when these are given, the bulk metric can be constructed order by order in z.

The function $g^{(d)}_{\mu\nu}$ is related to the expectation value of the energy-momentum tensor $\langle T_{\mu\nu}\rangle$. Using AdS/CFT duality (2.24), one can write that [35]:

$$\langle T_{\mu\nu}\rangle = -\frac{2}{\sqrt{-\det g^{(0)}}}\frac{\delta S_{\mathrm{gr}}}{\delta g^{(0)\mu\nu}}. \tag{2.32}$$

However, using simply the gravity action (2.27) here, one finds that $\langle T_{\mu\nu}\rangle$ diverges. These divergences correspond to the ultraviolet divergences in the

[4] The Greek indexes denote the boundary coordinates $\mu,\nu = 0,..,3$, where as the Latin indexes denote the bulk coordinates $M,N = 0,..,4$.

[5] In the expansion, there is in addition a possibility for a logarithmic term, but we will neglect it in here for simplicity. See [35], for more on the subject.

field theory. As is well known on the field theory side, one needs to first renormalize the theory, in order to get finite results. This is the case in the gravity side also — one needs to first regulate the gravity action (2.27), before applying the formula (2.32).

The renormalization is done in following way: first one regulates the action by changing the lower limit of the integration from $z = 0$ to $z = \epsilon$. The boundary term is, correspondingly, evaluated at $z = \epsilon$. Then one adds to the action (2.27) counterterms which are localized on the hypersurface $z = \epsilon$ and which are constructed in such way that they cancel the divergences of the original action in the $\epsilon \to 0$ limit.

For example, in four dimensions, the counterterm for the gravity action (2.27) is:

$$S_{\text{ct}} = \frac{\mathcal{L}}{8\pi G_5} \int_{\partial M,\, z=\epsilon} d^4x \sqrt{-\gamma} \left(\frac{1}{4} R + \frac{3}{\mathcal{L}^2}, \right) + c \log \epsilon \cdot \frac{\mathcal{L}}{8\pi G_5} \int d^4x \sqrt{-g^{(0)}}, \tag{2.33}$$

where R is the Ricci scalar constructed from the induced metric $\gamma_{\mu\nu}$ on the $z = \epsilon$ boundary and c is a coefficient that will cancel the logarithmic divergence in the action (2.27).

The renormalized gravity action is therefore

$$S_{\text{gr,ren}} = (S_{\text{gr}} + S_{\text{ct}})\,. \tag{2.34}$$

One can now safely calculate the expectation value of the boundary energy-momentum tensor, using the formula:

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \lim_{\epsilon \to 0} \frac{-2}{\sqrt{-\det g(x,\epsilon)}} \frac{\delta S_{\text{gr,ren}}}{\delta g^{\mu\nu}(x,\epsilon)}. \tag{2.35}$$

After a long calculation [35], one finds that in four dimensions,

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{\mathcal{L}^3}{4\pi G_5} \left[g^{(4)}_{\mu\nu} - \frac{1}{8} g^{(0)}_{\mu\nu} \left(\text{Tr}\, [g^{(2)}]^2 - \text{Tr}\, [(g^{(2)})^2] \right) - \frac{1}{2} \left((g^{(2)})^2 \right)_{\mu\nu} + \frac{1}{4} g^{(2)}_{\mu\nu} \text{Tr}\, g^{(2)} \right].$$

From the gravitational point of view, $\langle T_{\mu\nu} \rangle_{\text{ren}}$ can be understood as the Brown-York quasi-local energy-momentum tensor describing the gravitational energy on the boundary of the spacetime [36], [13].

If the boundary metric $g^{(0)}_{\mu\nu}$ is flat, then $g^{(2)}_{\mu\nu}$ is identically zero in four dimensions. In this case one obtains simply that:

$$\langle T_{\mu\nu} \rangle_{\text{ren}} = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu}. \tag{2.36}$$

The $g^{(2)}_{\mu\nu}$ dependent part describes the conformal anomaly of the theory which vanishes if the boundary metric is flat.

The previous analysis can be done also in other dimensions, see again [35] for details. In the case of $1+1$ dimensional boundary theory, or $2+1$ dimensional bulk, the corresponding relation is [35]:

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \frac{\mathcal{L}}{8\pi G_3}\left[g^{(2)}_{\mu\nu} - g^{(0)}_{\mu\nu}\text{Tr}\,[g^{(2)}]\right], \quad \text{where } \ \mu,\nu = 0,1,2 \tag{2.37}$$

or correspondingly,

$$\langle T_{\mu\nu}\rangle_{\text{ren}} = \frac{\mathcal{L}}{8\pi G_3} g^{(2)}_{\mu\nu}, \quad \text{if } \ g^{(0)}_{\mu\nu} \ \text{ is flat.} \tag{2.38}$$

In this way one can construct the bulk metric if the boundary metric and the expectation value of the energy-momentum tensor on the boundary are known. Or vice versa, if one knows the bulk metric, one can extract the boundary $\langle T_{\mu\nu}\rangle$.

The holographic renormalization method can be applied to other fields in the bulk as well. After renormalizing the corresponding supergravity action, using similar counterterm methods, one can extract the renormalized expectation value of the dual operator on the boundary, using the formula (2.24) [37, 38].

In AdS/CFT the extra dimensional coordinate z corresponds to the energy-scale of the boundary theory. The $z = \epsilon$ regulator in the gravity side can be understood as a UV cut-off on the boundary. The region $z \sim 0$ corresponds to the UV-region of the theory and the region deep in the AdS-space describes the IR-region. Similarly, if the anti de Sitter space contains a black hole which has a horizon at some fixed $z = z_0$, this corresponds to thermal field theory, with a temperature $T = 1/\pi z_0$. The deeper the horizon is in the AdS space, the smaller is the temperature T.

The fact that in AdS/CFT the boundary is defined to be at $z = 0$ ensures that gravity cannot propagate from the bulk to the boundary. This is due to the infinite redshift factor $\mathcal{L}/z$ in front of the metric. A bulk mode, with an energy E_0 at $z = z_s$ would have energy $E = \frac{z}{z_s}E_0$ at z. One observes that when $z \to 0$, $E \to 0$.

However, it is possible to consider AdS/CFT duality with a boundary at $z = \epsilon$, i.e., at the regulated surface, but now keeping the regulator ϵ finite. In this case the gravity action does not diverge and there is no need for the local counterterms. One finds that the boundary theory now includes also gravity and that it is induced from the bulk. This framework is known as the cut-off AdS/CFT. More discussion on this can be found e.g. in [39, 40]. For example, the Randal-Sundrum model [41, 42] and other brane world scenarios can also be understood in terms of the cut-off AdS/CFT description.

2.4 $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality

As discussed, the most explored realization of gauge/gravity duality is the duality between string theory in $\mathrm{AdS}_5 \times \mathbf{S}^5$ spacetime and 3+1 dimensional $\mathcal{N} = 4$ super Yang-Mills theory. However, gauge/gravity duality can be realized in many other cases also.

In this section, we briefly discuss the duality between type IIB string theory on $\mathrm{AdS}_3 \times \mathbf{S}^3 \times \mathrm{M}^4$ and a specific two dimensional conformal field theory, conjectured already in [7]. In here M^4 is a four dimensional compact manifold. See, e.g., [43] for a review on the subject. The duality between type IIB string theory in $\mathrm{AdS}_5 \times \mathbf{S}^5$ background and the $\mathcal{N} = 4$ SYM theory was originally motivated by the two distinct ways that could be used to describe a stack of N D3-branes. In the same way, one can motivate the $\mathrm{AdS}_3/\mathrm{CFT}_2$ duality by studying a system of N_5 D5-branes and N_1 D1-branes.

The interest in the D1-D5 system arose before the discovery of AdS/CFT duality itself, in the context of black hole physics. One of the deep mysteries in theoretical physics has been the Bekenstein-Hawking entropy for the black hole $S_{BH} = A/4G$ — entropy of a black hole is proportional to its area. This implies that the number of different possible microstates that create the entropy is proportional to $g \sim e^{A/4G}$. Physicists have been ever since pondering about this mysterious result. Standard field theory intuition would say that the entropy of a system should be extensive and the number of microstates describing the system should behave as $g \sim e^{cV}$, where V is the volume of the system and c is some constant. String theory, being a theory of quantum gravity, should be able to reproduce the black hole entropy from first principles.

In the D1-D5 system, D5-branes are extended along the dimensions $x_0, x_1, x_2, x_3, x_4, x_5$ and D1 branes along x_0, x_1. Here x_0 denotes the time direction. The directions x_2, x_3, x_4, x_5, that label the M^4 part, are now considered to be compact. Also the x_1 direction is now compactified as S^1. The system seems, therefore, as a point like one for an observer living in the five dimensions x_0, x_6, x_7, x_8, x_9. It is a string theory construction of a five dimensional black hole.

This system can be described in two complementary ways — using the supergravity description of the near horizon region of the branes or using the theory of massless open string modes on the intersection of D1 and D5 branes. The 1+1 dimensional CFT arising on this intersection is a very complicated theory and still not well understood [44]

The entropy describing this system has been calculated first in [45] using the conformal field theory description, defined by the low energy limit of the open string modes on the branes. One finds that

$$S = 2\pi\sqrt{N_1 N_5 N_m}, \tag{2.39}$$

when $N_1, N_5, N_m \gg 1$, Here, the parameter N_m describes the momentum number along the S_1 direction. One should note that this black hole is still an extremal black hole, even though it has non-zero entropy due to the momentum N_m.

The D1-D5 system has also a supergravity description. One finds that the solution has a horizon and one can calculate what is its area. Using the Bekenstein-Hawking formula for the entropy one finds that the entropy from the gravity calculation matches the result of (2.39). Therefore the result (2.39) is a string theory derivation of the black hole entropy.

Consider the case, where $N_m = 0$. The metric describing the D1-D5 system is now:

$$ds^2 = \Big(H_1(r)H_5(r)\Big)^{-1/2}\left[-dt^2 + dx_1^2\right] + \Big(H_1(r)H_5(r)\Big)^{1/2}\left[dr^2 + r^2 d\Omega_3^2\right], \tag{2.40}$$

where

$$H_1(r) = 1 + \frac{r_1^2}{r^2}, \qquad H_2(r) = 1 + \frac{r_5^2}{r^2}, \tag{2.41}$$

and

$$r_1^2 = \frac{16\pi^4 g_s N_1 \alpha'^3}{V}, \qquad r_5^2 = g_s N_5 \alpha'. \tag{2.42}$$

Here V is the volume of the M^4 part [46] which has been integrated out.

Consider now the limit $r^2 \ll r_1^2, r_5^2$, i.e., the region near the wrapped branes. The metric (2.40) in this limit takes the form:

$$ds^2 = \frac{r^2}{\mathcal{L}^2}\left[-dt^2 + dx_1^2\right] + \frac{\mathcal{L}^2}{r^2}dr^2 + \mathcal{L}^2 d\Omega_3^2, \tag{2.43}$$

which can be recognized as the $\mathrm{AdS}_3 \times \mathbf{S}^3$ metric with an AdS radius $\mathcal{L}^2 = r_1 r_5 = 16\pi^4 g_s^2 \alpha'^4 N_1 N_5 / V$.

Similarly, considering the near-horizon region of the non-extremal D1-D5 system, one ends up with the metric [47]:

$$ds^2 = -\left(\frac{r^2}{\mathcal{L}^2} - M\right) dt^2 + \frac{dr^2}{r^2/\mathcal{L}^2 - M} + r^2 d\phi^2 + \mathcal{L}^2 d\Omega_3^2. \qquad (2.44)$$

This is the product of the non-spinning BTZ black hole and three-sphere $\mathbf{S}^3$. The BTZ black hole is the anti de Sitter black hole in three dimensions and the dimensionless parameter M is proportional to the mass of the black hole [48].

2.5 AdS/CFT and quantum phenomena in the boundary theory

One of the most striking features of AdS/CFT duality is that although string theory can be approximated using only classical gravity, one can make predictions about quantum properties of the boundary theory. The simplest example of this is, of course, the possibility to calculate correlation functions on the boundary by varying the classical effective supergravity action (2.24). A more non-trivial example of this property is the conformal anomaly that we will consider next.

2.5.1 Conformal anomaly

In AdS/CFT correspondence, the boundary theory is a conformal field theory and therefore, in the case of a flat boundary metric, the expectation value of the trace of the energy-momentum tensor is zero. However, when the background metric for the conformal field theory is curved, the former is not generally true — one obtains a non-trivial contribution to the trace of the energy-momentum tensor due to the curved background metric [49].

The general formula for the conformal anomaly can be expressed in terms of the curvature invariants, constructed from the four dimensional metric $g^{(0)}_{\mu\nu}$:

$$\langle T^{\mu}_{\ \mu} \rangle = \alpha \Box R + \beta \left(R_{\rho\sigma} R^{\rho\sigma} - \frac{1}{3} R^2 \right). \qquad (2.45)$$

In the case of $\mathcal{N} = 4$ theory, the parameters α and β are:

$$\alpha = 0 \quad \text{and} \quad \beta = \frac{N^2 - 1}{32\pi^2}. \qquad (2.46)$$

These results have been also reproduced using the AdS/CFT duality [12].

The conformal anomaly arises when one renormalizes the quantum energy-momentum tensor on a curved background. In the flat case, one finds that a quantum field theory action is formally divergent — this is because there *ab initio* is no upper limit on the energy of the field modes. The standard way to proceed is to renormalize the theory.

In the case of curved background metric, the renormalization is, however, more difficult. This is because the UV-divergences obtain a contribution from the curved background metric. The expectation value of the renormalized energy-momentum tensor can be calculated from the quantum effective action W,

$$\langle T_{\mu\nu}\rangle_{\rm ren} = -\frac{2}{\sqrt{-g}}\frac{\delta W}{\delta g^{\mu\nu}}, \tag{2.47}$$

where W is constructed from the renormalized action on the curved spacetime [49]. If one computes $\langle T_{\mu\nu}\rangle_{\rm ren}$ on a curved background metric and compares the result to the flat case, one generally finds a finite difference between the results.

2.5.2 Casimir energy

In the case of conformal anomaly, the contribution to $\langle T_{\mu\nu}\rangle_{\rm ren}$ due to the non-trivial background metric, was induced because of the difference in local physics, namely in the UV-part of the spectrum. The contributions which arise in this way can be expressed in terms of geometrical quantities, i.e., in terms of curvature invariants. However, a non-trivial background spacetime can also give another kind of contribution to the expectation value of the energy-momentum tensor — this contribution can appear if the global structure or the topology of the spacetime is non-trivial. The corrections coming from the global properties are usually finite and cannot be expressed in terms of local geometrical quantities. These corrections are inherited from differences in the IR-physics and typically are more subtle to calculate.

Probably the most striking feature of these quantum field theory properties on a non-trivial background[6] is that corrections to $\langle T_{\mu\nu}\rangle_{\rm ren}$, compared to the Minkowski spacetime, can appear even in the case of flat spacetime if the global structure is non-trivial. Maybe the simplest example of this is the Casimir energy [50].

[6] In the literature, the formalism that deals with the problems discussed in this subsection is usually called the *quantum field theory in curved spacetime*. Nevertheless, in this Thesis we try to avoid this name, because the formalism gives also examples where a flat spacetime can induce new quantum effects.

In the Casimir effect, one has two parallel planes in a flat spacetime. These planes give non-trivial boundary conditions to the quantum fields; the fields must vanish at the planes. The planes force the field modes to form a discrete set in the direction orthogonal to the planes.

For example, if the planes are at distance L from each other, orthogonal to the direction x^1, one obtains that in d-dimensions:

$$\langle T^\mu_{\ \nu}\rangle_{\text{Casimir}} = \langle\, 0_{\text{L}}\, |T^\mu_{\ \nu}|\, 0_{\text{L}}\rangle - \langle\, 0\, |T^\mu_{\ \nu}|\, 0\rangle \sim \frac{c}{L^d}\begin{pmatrix} 1 & 0 & 0 & \dots & 0 \\ 0 & -(d-1) & 0 & \dots & 0 \\ 0 & 0 & 1 & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & 1 \end{pmatrix},$$

where c is a positive coefficient, depending on the dimension of the spacetime and the number of degrees of freedom associated with the field configuration between the planes. For example, for a scalar field in $3+1$ dimensions one obtains that $c = \pi^2/1440$.

In the case of Casimir energy associated with electromagnetic fields, the Casimir energy has been experimentally verified [51, 52]. In this case conducting metal plates provide a suitable physical realization of boundary conditions.

In the expression for the Casimir energy, $\epsilon_{\text{L}} = -\langle T^0_{\ 0}\rangle_{\text{Casimir}}$ denotes the physical vacuum energy between the planes for an observer which would observe zero vacuum energy in the Minkowksi space. One finds that the physical quantum vacuum is altered $|\,0\rangle \to |\,0_{\text{L}}\rangle$ due to non-trivial boundary conditions. This observable difference in the vacuum energy density is called the Casimir energy. Notice that $\langle T^\mu_{\ \nu}\rangle_{\text{Casimir}}$ is traceless in all dimensions.

2.5.3 Quantum fields in Milne spacetime

Let us consider closer a more specific example of quantum field theory on a non-trivial background, a scalar field in 1+1 dimensional Milne spacetime with the metric

$$ds^2 = -d\tau^2 + \tau^2 d\eta^2. \tag{2.48}$$

If the timelike coordinate τ is interpreted as the cosmic time, this spacetime is a 1+1 dimensional Robertson-Walker spacetime with the scale factor $a(\tau) = \tau$ and a singularity at $\tau = 0$. Transforming,

$$t = \tau\cosh\eta, \quad x = \tau\sinh\eta, \tag{2.49}$$

one obtains simply:

$$ds^2 = -dt^2 + dx^2, \quad \text{where} \quad 0 < t < \infty, \ -t < x < t. \tag{2.50}$$

However, even though the metric is simply Minkowskian, the coordinates do not cover the entire spacetime, but only a wedge of the Minkowski spacetime, the region corresponding to the future light-cone of an observer sitting at the point $t = 0, x = 0$. Thus an observer in the Milne coordinates τ, η perceives universe to expand, starting at $\tau = 0$.

Let us now consider a 1+1 dimensional scalar field action:

$$S = \int d^2x\sqrt{-g}\left(-\frac{1}{2}g^{\mu\nu}\nabla_\mu\phi\nabla_\nu\phi - \frac{1}{2}m^2\phi^2\right). \tag{2.51}$$

From the action one can derive the field equations and solve them. Using the solution, one can construct the energy-momentum tensor and calculate its expectation value. One finds that there are, however, two different sets of solutions to the field modes [49]. The first set of solutions describes the modes which have a positive frequency with respect to the time coordinate τ (and t). These modes define the adiabatic vacuum $|0\rangle$, or the Minkowski vacuum. However, the other set of solutions describes field modes that have a positive frequency with respect to the conformal time coordinate ρ, which is related to the Milne time coordinate τ through the relation $e^{c\rho} = c\tau$, where c is some constant [7]. This other set of modes define the conformal vacuum $|\tilde{0}\rangle$.

One expects that an observer which does not observe any vacuum energy density in the adiabatic vacuum, would observe non-zero vacuum energy density in the Milne conformal vacuum. Indeed, one finds that the difference is non-zero [53]:

$$\langle\,\tilde{0}\,|T^\mu_\nu|\,\tilde{0}\rangle - \langle\,0\,|T^\mu_\nu|\,0\rangle = \frac{1}{24\pi\tau^2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{2.52}$$

Interestingly, the coordinate system used usually in the context of ultrarelativistic heavy ion collisions is very similar to the Milne metric. In fact, if one considers a 1+1 dimensional analogy of a heavy ion collision, the metric is exactly the Milne metric. In a heavy ion collision, instead of a big-bang at $t = 0$ there is a *little bang*, i.e. the explosion from the collision. Therefore, the question rises: should one get in this case also similar quantum effects due to the non-trivial choice of the vacuum state? Indeed, using

[7] In the conformal coordinate system, the metric is $ds^2 = e^{2c\rho}(-d\rho^2 + c^{-2}d\eta^2)$, and $-\infty < \rho < \infty$.

gauge/gravity duality, one observes that this seems to be the case, at least in the 1+1 dimensional case [54].

2.6 AdS/QCD

AdS/CFT duality gives a prescription to calculate correlation functions of $\mathcal{N}=4$ theory in terms of supergravity. However, as one knows $\mathcal{N}=4$ theory is not a correct theory to describe the observed particles in real world. Even though $\mathcal{N}=4$ Yang-Mills theory can be used as a model, when describing some finite temperature QCD phenomena, one would like to have something more.

As discussed in the introduction, one can modify AdS/CFT duality to obtain theories with less supersymmetry, broken conformal symmetry etc, see for example [55, 56, 57, 58]. In this way, one can can obtain boundary theories that resemble QCD more than the $\mathcal{N}=4$ theory. One can also try to directly find a string background which would describe full QCD (some attempts in this direction have been made, e.g., in [59, 60]).

2.6.1 AdS/QCD models

One can also consider a more phenomenological viewpoint in the study of QCD. One can assume that in similar fashion to AdS/CFT, there could be a corresponding dual gravity theory for large N QCD and that a relation, similar to (2.24), would hold also in this case. One can then ask what properties should the gravity theory have, in order to produce the observed behavior of QCD in the field theory side? For example, one can ask what should be the field content of the gravity theory to give the required QCD operators on the boundary or what should the structure of the metric be.

This phenomenological viewpoint was first considered in [61] and in [62]. These holographic models are generally referred to using term AdS/QCD. A motivation for the holographic description was discussed in [63]. In this paper, Son and Stephanov showed that in the effective field theory description of hadrons and mesons and in the limit of infinitely many hidden local symmetries, a holographic description arises naturally in the continuum limit.

In the simplest case, the holographic model of QCD has the following 5-dimensional action:

$$S=\int d^4xdz\sqrt{-g}\mathrm{Tr}\left[-|D\Phi|^2+3|\Phi|^2-\frac{1}{4g_5^2}\left(F_L^2+F_R^2\right)\right], \qquad (2.53)$$

where $D_\mu \Phi = \partial_\mu \Phi(z) - iA_{L,\mu}\Phi(z) - iA_{R,\mu}\Phi(z)$ and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i[A_\mu, A_\nu]$, $A_\mu = A^a_\mu T^a$ for A_L and A_R.

In the model, one also assumes the metric to be of the form

$$ds^2 = w(z)^2 \left(\eta_{\mu\nu} dx^\mu dx^\nu + dz^2 \right). \tag{2.54}$$

The simplest choice for the function $w(z)$ is,

$$w(z) = \frac{\mathcal{L}^2}{z^2}, \quad \text{where} \quad 0 < z < z_m. \tag{2.55}$$

The coordinate z is restricted to the interval $0 < z < z_m$, as e.g. in [61]. The z_m denotes the position of the IR-brane which functions as an IR-cutoff scale for the theory.

In this model, the gauge fields A^μ_L and A^μ_R are dual to the operators $\bar{q}_L \gamma^\mu q_L$ and $\bar{q}_R \gamma^\mu q_R$ in QCD. The scalar $\Phi(z)$ is correspondingly dual to the operator $\bar{q}_R q_L$. The correlation functions for the operators can be now calculated in similar fashion to AdS/CFT duality varying the on-shell action with respect to the boundary value of the bulk fields.

The AdS/QCD models [61, 62], and their successors (see for example [64, 65, 66, 67]) encode surprisingly well some properties of QCD. For example, fitting the three free parameters of the model in [61], gave seven observed quantities, such as ρ meson mass, with a surprisingly good accuracy of $\mathcal{O}(10\%)$.

However, one of the problems of these models was that they predicted a meson mass spectrum which grows with respect to the excitation number n as:

$$M_n^2 \sim n^2. \tag{2.56}$$

In reality, the mass spectrum should behave as $M_n^2 \sim n$. Using the phenomenological approach, one can add further terms to the action and try to remedy the situation. Indeed, in [68] a dilaton, with a certain profile was added, and the behavior:

$$M_{n,S}^2 \sim (n + S), \tag{2.57}$$

was obtained. The parameter S describes the spin excitation. Therefore the model [68] also correctly produced the linear Regge behavior for the meson masses.

The AdS/QCD models seem to capture surprisingly well many phenomena relevant to QCD. Still, these models are quite *ad hoc*; for example, the action of the model is postulated and not derived from some supergravity theory. Nevertheless, the models seem to suggest that there might be some truth hidden in the five dimensional gravity description of QCD.

Chapter 3

QCD and heavy ion collisions

Quantum chromodynamics is a theory describing quarks and gluons. In QCD there are three colors for the fundamental quarks and eight for gluons, as the gauge group is SU(3). The $\mathcal{N}=4$ theory seems very different when compared with QCD — for example, the $\mathcal{N}=4$ theory is highly supersymmetric and there are fermions and scalars in the adjoint representation, but no fundamental quarks. Also the theory is a conformal field theory and has the gauge group SU(N).

Nevertheless, there seems to be a certain regime, where these theories behave in a similar fashion. This is the regime when the interactions between particles are strong, but the temperature of the system is at the same time high [1].

At finite temperatures, supersymmetry is broken and the conformal description of QCD thermodynamics seems to be a reasonable approximation at the high temperature regime, see e.g. [69]. It has also been known for a long time that the large N approximation of QCD, i.e. QCD with the gauge group SU(N), gives a quite accurate description of QCD in many cases [70].

Heavy ion collisions probe exactly the high temperature and strongly interacting regime of QCD. Therefore it seems natural to apply duality to study phenomena related to heavy ion collisions. In this chapter, we give an introduction to this subject and discuss the gravity description of heavy ion collision.

[1] But not too high, so that QCD is strongly coupled.

3.1 Heavy ion collisions

An ultrarelativistic heavy ion collision is a process in which two heavy ions, typically gold or lead ions, collide head-on with highly relativistic speeds. A typical heavy ion consists of approximately 200 nucleons. Because of the high velocity, the ions are strongly Lorentz contracted in the longitudinal direction of the motion, and the collision appears more or less as a collision of two disks. In the collision, the disks smash into one another and then pass through each other. In this violent process, some part of the collision energy is transformed into heat and new particles. Heavy ion collision experiments have been carried out recently at RHIC and in future will be carried out at LHC.

At low temperatures and densities, quarks and gluons are tightly bound together as hadrons. In the collision, the pressure and temperature is huge and the protons and neutrons can "melt". Quarks and gluons form a new state of matter called quark-gluon plasma (QGP).

There is compelling evidence that QGP is formed in heavy ion collisions and that it can be described as strongly interacting fluid [71, 72, 73, 74] (see [75] for a review on the subject). After the collision, the plasma starts to rapidly expand and it cools down. When the temperature drops down to approximately $T_c \simeq 175$ MeV, or about 2×10^{12} Kelvin, the plasma *hadronizes*. In the hadronization, the matter goes through a phase transition and quarks and gluons coalesce into hadrons.

3.2 Hydrodynamical description of heavy ion collisions

In the hydrodynamical description of matter one assumes that the system can be modeled in terms of only a few variables, such as energy density $\epsilon(x)$, pressure $p(x)$ and velocity vector $u_\mu(x)$ of the fluid.

In the ideal fluid case, one can write the energy-momentum tensor as

$$T_{\mu\nu} = (\epsilon + p)u_\mu u_\nu + p g_{\mu\nu}. \tag{3.1}$$

In the rest frame of the fluid, it has the form:

$$T^\mu_\nu(x) = \begin{pmatrix} -\epsilon(x) & 0 & 0 & 0 \\ 0 & p(x) & 0 & 0 \\ 0 & 0 & p(x) & 0 \\ 0 & 0 & 0 & p(x) \end{pmatrix}.$$

Further, the energy-momentum tensor is conserved, i.e., it obeys the conservation equation

$$\nabla^{\mu} T_{\mu\nu} = 0. \tag{3.2}$$

However, in reality, the fluid is not ideal [2]. The first order deviation from the ideal fluidity can be expressed as:

$$T_{\mu\nu} = (\epsilon + p) u_{\mu} u_{\nu} + p g_{\mu\nu} - \Pi_{\mu\nu}, \tag{3.3}$$

where

$$\Pi_{\mu\nu} = \eta \left(\nabla_{\mu} u_{\nu} + \nabla_{\nu} u_{\mu} \right) - \left(\frac{2}{3}\eta + \zeta \right) \left(\nabla \cdot u \right) \left(g_{\mu\nu} + u_{\mu} u_{\nu} \right), \tag{3.4}$$

denotes the first order correction to the ideal fluid behavior. The shear viscosity η and the bulk viscosity ζ describe the dissipative forces that affect the flow.

If the fluid is conformal, i.e. $T_{\mu\nu}$ is traceless, then the bulk viscosity is identically zero. In the context of AdS/CFT duality this is usually the case, because the boundary theory is in most cases a conformal field theory [3].

In [76] the following viscosity bound was conjectured:

$$\frac{\eta}{s} \geq \frac{1}{4\pi} \tag{3.5}$$

for all relativistic quantum field theories at finite temperature and zero chemical potential. The conjecture also stated that this inequality is saturated by theories with gravity duals, such as $\mathcal{N} = 4$ Yang-Mills theory. This conjecture has been verified for large class of field theories with gravity duals [77, 78, 79].

Interestingly, all known hydrodynamical systems seem to obey the inequality (3.5). Also, the current data is compatible with the assumption that the ratio η/s for the QGP is a small number, close to the value $1/4\pi$, see [75]. This can be interpreted as a further evidence that perhaps $\mathcal{N} = 4$ matter is reasonable approximation for the collective behavior of quark-gluon plasma at high temperatures and at strong coupling.

[2] Nevertheless, the ideal fluid behavior seem to be a good approximation in the case of quark-gluon plasma.

[3] However, in the case of curved boundary metric, one obtains a conformal anomaly to the energy-momentum tensor. In this case, the energy-momentum tensor can have a non-vanishing trace.

3.2.1 Boost-invariant fluid

In the study of evolution of expanding quark-gluon plasma, one would like to know how do the energy density $\epsilon(x)$ and the pressure $p(x)$ vary as functions of the coordinates x^μ. The problem is in the most general form quite complex. However, in the so called *mid-rapidity region*[4] the problem can be simplified using the following scaling ansatz, introduced first by Bjorken in 1983 [80]:

$$\epsilon(t,x,y,z) \equiv \epsilon(\tau), \;\; p_x(t,x,y,z) \equiv p_x(\tau), \quad p_{\mathrm{T}}(t,x,y,z) \equiv p_{\mathrm{T}}(\tau), \tag{3.6}$$

In here x is the longitudinal direction and τ is the proper time $\tau = \sqrt{t^2 - x^2}$. In general, the longitudinal pressure p_x and the transversal pressure p_{T} can differ from the each other.

Take now new variables $t = \tau \cosh \eta$ and $x = \tau \sinh \eta$. The Minkowski metric now has the form:

$$ds^2 = -d\tau^2 + \tau^2 d\eta^2 + d\vec{x}_T^2. \tag{3.7}$$

Assume that the energy-momentum tensor can depend only on the variable τ and that it has translational and rotational symmetry in the transverse directions and that the system is also symmetric in reflections: $y \to -y$. One finds that the most general traceless (conformal) and conserved energy-momentum tensor respecting these symmetries can be written as:

$$T_{\mu\nu} = \begin{pmatrix} f(\tau) & 0 & 0 & 0 \\ 0 & -\tau^2 f(\tau) - \tau^3 \frac{d}{d\tau} f(\tau) & 0 & 0 \\ 0 & 0 & f(\tau) + \frac{1}{2}\tau \frac{d}{d\tau} f(\tau) & 0 \\ 0 & 0 & 0 & f(\tau) + \frac{1}{2}\tau \frac{d}{d\tau} f(\tau) \end{pmatrix}.$$

Assuming that the weak energy condition holds, i.e., $T_{\mu\nu} t^\mu t^\nu \geq 0$ for all light like vectors t^μ, one obtains further restrictions for the function $f(\tau)$:

$$f(\tau) \geq 0, \qquad f'(\tau) \leq 0, \qquad \tau f'(\tau) \geq -4 f(\tau). \tag{3.8}$$

If one makes the further assumption that the fluid is perfect, one obtains the constraints: $T^\eta_\eta = T^y_y = T^z_z$. Now one can solve the equation for $f(\tau)$ and one ends up with,

$$f(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau} \right)^{4/3}. \tag{3.9}$$

[4]The mid-rapidity region describe the part of the expanding plasma that has small longitudinal velocity after the collision.

One can also study $f(\tau)$, in the presence of the first order correction, due to the shear viscosity. In the co-moving coordinates, τ, η-coordinates, the velocity-vector is:

$$u^\mu = \frac{dx^\mu}{d\tau} = (1,0,0,0). \tag{3.10}$$

Therefore in these coordinates each fluid element is at rest. One can now calculate the first order correction $\Pi_{\mu\nu}$ to the energy-momentum tensor in the boost invariant case, using equation (3.4).

After a short calculation, one finds that:

$$\Pi^\mu_{\ \nu} = \frac{2\eta}{3\tau^2}\begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 2 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \tag{3.11}$$

and therefore

$$T^\mu_{\ \nu} = \begin{pmatrix} -\epsilon(\tau) & 0 & 0 & 0 \\ 0 & p(\tau) - \frac{4\eta}{3\tau^2} & 0 & 0 \\ 0 & 0 & p(\tau) + \frac{2\eta}{3\tau^2} & 0 \\ 0 & 0 & 0 & p(\tau) + \frac{2\eta}{3\tau^2} \end{pmatrix}.$$

The continuity equation now reads:

$$\nabla_\mu T^\mu_{\ 0} = -\epsilon'(\tau) - \frac{4\epsilon(\tau)}{3\tau} + \frac{4\eta(\tau)}{3\tau^2} = 0, \tag{3.12}$$

where the relation $p(\tau) = \epsilon(\tau)/3$ has been applied. Also one knows that in a conformal field theory $\epsilon(\tau) = 3aT^4(\tau)$ and $\eta(\tau) = \eta_0\, T^3(\tau)$. Therefore, one can solve the temperature from the equation (3.12). One obtains that:

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right)^{1/3} - \frac{\eta_0}{6a}\frac{1}{\tau} \tag{3.13}$$

and thus the late time ($\tau \to \infty$) expansion is

$$\epsilon(\tau) = 3aT_0^4 \left(\frac{\tau_0}{\tau}\right)^{4/3} - 2\eta_0 T_0^3 \frac{\tau_0}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^{8/3}}\right) + \ldots. \tag{3.14}$$

When generalizing the analysis to arbitrary dimensions [87], one finds that the corresponding late time expansion is:

$$\epsilon(\tau) = \frac{c_1}{\tau^{2-\nu}} + \frac{c_2}{\tau^2} + \mathcal{O}\left(\frac{1}{\tau^{2+2\nu}}\right) + \ldots, \quad \text{where } \nu = \frac{d-2}{d-1}, \tag{3.15}$$

and c_i are constants.

From the expression (3.15) one notices the uniqueness of the 1+1 dimensional Bjorken flow. In this degenerate case $\nu = 0$, and all terms in the expansion come at the same order. Therefore in 1+1 dimensional Bjorken flow one has:

$$T(\tau) = T_0 \left(\frac{\tau_0}{\tau}\right) \quad \text{and} \quad \epsilon(\tau) = \epsilon_0 \left(\frac{\tau_0}{\tau}\right)^2, \tag{3.16}$$

at all times.

In general, however, any choice of $f(\tau)$ that obey the conditions (3.8), give a proper solution in terms of the conservation equation and the traceless condition. Not all of these solutions are physically interesting or are produced naturally from the dynamics of the gauge theory. Therefore, one can ask the question: does the gauge/gravity duality give any restrictions for the function $f(\tau)$?

3.3 Gravity dual of boost-invariant fluid

In heavy ion collisions one has observed that hot and expanding quark-gluon plasma is produced. There is also evidence that this expanding matter can be modeled using the fluid description of the plasma. An interesting question then is whether one can construct a gravity solution which describes the expanding $\mathcal{N} = 4$ gauge theory fluid using gauge/gravity duality? Even though one does not have a dual gravity description for QCD, one can try to use $\mathcal{N} = 4$ theory to model quark-gluon plasma at temperatures $T > T_c$, as discussed earlier.

One relevant solution would be the dual solution describing a boost-invariant expansion on the boundary. One starts using an ansatz for the metric that respects the symmetries of the problem:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-a(\tau, z)d\tau^2 + \tau^2 b(\tau, z)d\eta^2 + c(\tau, z)d\vec{x}_T^2 + dz^2\right]. \tag{3.17}$$

This simplified ansatz, when inserted to Einstein equations, however, produces complicated nonlinear partial differential equations. To obtain the full exact solution to these equations seems to be an overwhelming challenge. Nevertheless, progress can be made.

Janik and Peschanski assumed in [26] that at late times, i.e. when $\tau \to \infty$, $f(\tau)$ behaves as $f(\tau) \sim \tau^{-s}$ in the leading order. The holographic renormalization techniques, reviewed in section (2.3), give a tool to address the problem of constructing the gravity background corresponding to the field theory setup at hand. In this case, one wants to find a bulk background that produces $f(\tau) \sim \tau^{-s}$ and $g^{(0)}_{\mu\nu} = \text{diag}(-1, \tau^2, 1, 1)$ as the boundary metric. Because the boundary metric is flat, only a coordinate transformation from the Minkowksi metric, the relation between the expectation value of the energy-momentum tensor and the metric is particularly simple:

$$\langle T_{\mu\nu} \rangle = \frac{\mathcal{L}^3}{4\pi G_5} g^{(4)}_{\mu\nu}, \tag{3.18}$$

as discussed in the section (2.3). The conditions for $g^{(0)}_{\mu\nu}$ and $f(\tau)$ provide enough information to construct the bulk metric as a power expansion in z, and to obtain the metric to arbitrary order in z — the solutions parameterized in terms of s only.

However, one can even do better than just solve the equations order by order. This is because one finds that the expansion has schematically the following structure:

$$a(\tau, z) = 1 + \left(\frac{z}{\tau^{s/4}}\right)^4 \left(c_0 + \frac{c_1}{\tau^{\#}} + \dots\right) + \left(\frac{z}{\tau^{s/4}}\right)^6 \left(c_2 + \frac{c_3}{\tau^{\#}} + \dots\right) + \dots, \tag{3.19}$$

where # denote some positive coefficients. The same behavior is found also for the functions $b(\tau, z)$ and $c(\tau, z)$.

One observes that in the scaling limit, $\tau \to \infty$, $z \to \infty$ and $z/\tau^{s/4} =$ constant, one is left with functions that depend only on the combination $v \equiv z/\tau^{s/4}$. This motivates one to use the following scaling ansatz for the metric (3.17):

$$\begin{aligned} a(\tau, z) &= a_0(v) \ + \ \mathcal{O}\left(\frac{1}{\tau^{\#}}\right) \ + \ \dots, \\ b(\tau, z) &= b_0(v) \ + \ \mathcal{O}\left(\frac{1}{\tau^{\#}}\right) \ + \ \dots, \\ c(\tau, z) &= c_0(v) \ + \ \mathcal{O}\left(\frac{1}{\tau^{\#}}\right) \ + \ \dots, \end{aligned} \tag{3.20}$$

and to study the Einstein equations when $v =$ fixed and $\tau \to \infty$. The partial differential equations reduce to ordinary differential equations and one can solve the functions $a_0(v), b_0(v)$ and $c_0(v)$ analytically for arbitrary s [5].

However, the functions $a_0(v)$, $b_0(v)$ and $c_0(v)$ seem to have a singularity at a critical value $v = v_c(s)$. In here, the critical value v_c depends on the

[5] We do not give here the expressions for the the functions $a_0(v), b_0(v)$ and $c_0(v)$ for general s for simplicity. They are written explicitly in [26].

parameter s. To check if the singularity is a true curvature singularity, one must calculate the curvature invariants. One finds [26] that in $\tau \to \infty$ limit,

$$R = -20 \; + \; \mathcal{O}\left(\frac{1}{\tau^{\#}}\right) \; + \; \ldots, \tag{3.21}$$

and

$$R^{\mu\nu\rho\gamma}R_{\mu\nu\rho\gamma} = \frac{1}{(v - v_c)^4} f(v, s) \; + \; \mathcal{O}\left(\frac{1}{\tau^{\#}}\right) \; + \; \ldots, \tag{3.22}$$

where $f(v, s)$ is now a finite function of v and s. Thus there seems to be a true curvature singularity at $v = v_c$.

Janik and Peschanski argued that the physical late time solution should have a non-singular bulk metric, i.e., a metric without naked singularities[6]. Indeed, one finds that there is a specific value of s that gives a finite curvature everywhere — the value $s = 4/3$.

When $s = 4/3$, the function $f(v, 4/3)$ cancels the fourth order pole and gives

$$R^{\mu\nu\rho\gamma}R_{\mu\nu\rho\gamma} = 112, \quad \text{at} \quad v = v_c. \tag{3.23}$$

Therefore, one can argue that the gravity duality predicts perfect fluid flow at late times.

The full late time solution can be now written explicitly:

$$ds^2 = \frac{\mathcal{L}^2}{z^2}\left[-\frac{\left(1 - c\frac{z^4}{\tau^{4/3}}\right)^2}{1 + c\frac{z^4}{\tau^{4/3}}}\, dt^2 + \left(1 + c\frac{z^4}{\tau^{4/3}}\right)\left(\tau^2 d\eta^2 + d\vec{x}_T^2\right) + dz^2 \right], \tag{3.24}$$

where c is an arbitrary integration constant.

The similarity between the anti de Sitter black hole metric in Fefferman-Graham coordinates (2.15) and the metric (3.24) is very interesting. It suggests that the late time metric (3.24) has an horizon at $z_0 = c^{-1/4}\tau^{1/3}$. Applying naively the formula for the static AdS black hole temperature, $T \sim 1/z_0$, one finds that[7]:

$$T(\tau) \sim \frac{1}{z_0} \sim \frac{1}{\tau^{1/3}}. \tag{3.25}$$

As $\epsilon \sim T^4$ and $\epsilon \sim \tau^{-4/3}$, the result (3.25) seems to be consistent.

[6]That is, singularities that are not behind an event horizon.

[7]Although strictly speaking there are two caveats: first, all singularities in the metric do not correspond to true event horizons. Second, in time dependent case, the black hole thermodynamics is a subject that is generally not well understood. Therefore the exact definition for time-dependent temperature for a black hole is unknown.

The argument that the solution corresponding to a physical realistic solution on the boundary, should be non-singular in the bulk, has been applied further in series of papers [81, 82, 83, 84, 85] [8].

The metric (3.24) describes the leading behavior in $\tau \to \infty$ limit. However, there are also subleading corrections to $f(\tau)$. When $s = 4/3$, these corrections come as power series in $\tau^{-2/3}$, as shown in [90]. The first subleading corrections to the solution (3.24) were first calculated in [91]. In [90] it was shown that in order that the singularities in $R^{\mu\nu\rho\gamma}R_{\mu\nu\rho\gamma}$ would cancel also at the order $\mathcal{O}(\tau^{-4/3})$, the ratio η/s must take the value $\eta/s = 1/4\pi$, and in this way reproduced the well known result of $\mathcal{N} = 4$ theory.

The main restriction of the above analysis is that one does not have any understanding of the global properties of the spacetime nor the definitions of temperature and entropy in a time dependent setup. In the papers II-IV of this thesis, the main goal was to study exact solutions, where the global structure of the spacetime is better in control, and try to shed some light on the questions mentioned above. Because of the problems in solving the exact Einstein equations, when using the ansatz (3.17), one needs to simplify the problem, in order to obtain some progress. This can be done, for example, by studying a simplified ansatz in five dimensions or by reducing the dimensions of the bulk spacetime.

In the paper II, we study an ansatz for the five dimensional AdS gravity equations, where the metric has only two unknown functions. In this case, an analytical solution with non-trivial time dependence can be obtained. In the papers III and IV, we study also exact solutions, but in this case for the three dimensional Einstein equations.

In the paper III, we even show that the time dependence can be transformed away, by performing a specific coordinate transformation. In this case, the static form of the metric turns out to be the BTZ black hole metric [48]. As the temperature, mass and entropy of a static BTZ black hole can be easily calculated, using the methods presented in the chapter 2, it is also possible to identify the corresponding quantities in the time dependent coordinate system. This is because one can trace back the coordinate transformation, and identify the correct black hole mass parameter in the time dependent metric.

[8] See also [86, 87, 88, 89] related to the subject of dual description of heavy ion collisions.

Chapter 4

Conclusions

Gauge/gravity duality provides powerful methods for the study of strongly interacting gauge theories. The duality between type IIB string theory on anti de Sitter background and $\mathcal{N} = 4$ supersymmetric Yang-Mills theory in 3+1 dimensions, makes it possible to analyze the non-perturbative regime of the gauge theory, using methods of classical gravity only.

As there is evidence, that at strong coupling and at finite temperature $\mathcal{N} = 4$ theory and QCD behave in similar fashion, one can also try to apply the methods of AdS/CFT to study QCD in this regime.

The gauge/gravity duality framework also gives a chance for a true string description of QCD. The success of some phenomenological AdS/QCD models [62, 63] and the fact that there is a wide range of different realizations of gauge/gravity duality, suggest that maybe also QCD can be formulated in terms of strings on some higher dimensional curved background[1].

The evidence seems to point out [71, 72, 73, 74, 75] that in heavy ion collisions hot and strongly interacting quark-gluon plasma is produced. Therefore, heavy ion collisions provide an ideal context to apply AdS/CFT duality to QCD. Using AdS/CFT, one can study the expansion and dynamics of the plasma in a meaningful way, as is discussed in the chapter 3. The hydrodynamical properties of the expanding plasma, such as the viscosity and the evolution of the energy density, can be obtained consistently.

In the context of applications of AdS/CFT to heavy ion collision, many interesting questions remain to be answered. A better understanding of the dual gravity background corresponding to the 3+1 dimensional Bjorken expansion would be highly desirable. For example, the global properties of this dual gravity background should be better understood. Of course,

[1] This was actually suggested already in 1981 by Polyakov [92].

it would be best if one could discover an exact analytic solution, but this seems unlikely due to the complexity of the problem.

In the introductory part of this thesis, we have reviewed some background and tools needed in the study of strongly interacting gauge theories using gauge/gravity dualities — especially we reviewed some applications of AdS/CFT to QCD and heavy ion collisions. The second part of the thesis consists of four papers, where the gauge/gravity duality framework is applied to QCD in various ways.

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