QUANTUM ERROR CORRECTION IN QUANTUM FIELD THEORY AND GRAVITY

by

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To my parents, Takahisa, Akiko

My sister, Takako

My sister, Ayako

And my brother, Takaaki

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Ango Sakaguchi, “Discourse on Decadence” (Drakuron, 1946)

Ango Sakaguchi, “Discourse on Decadence, Part II” (Drakuron, 1946)

Sakunosuke Oda, “Boku no dokushohou” (1976)

PREFACE

1. Chapter 2 briefly reviews the basic ideas of classical and quantum information theory.

2. Chapter 3 reviews completely positive maps and their duals in von Neumann algebras.

3. Chapter 4 reviews the basics of quantum error correction (QEC). In particular, operator algebra QEC is introduced. The sections about the passive QEC and active QEC are based on the appendices from my paper [1].

4. Chapter 5 is based on my two papers [1] and [2]. We studied the QEC structure in real-space renormalization group theory.

5. Chapter 6 is based on my two papers [3] and [4]. We studied the information measures in the presence of charges and proposed generalized measures which capture the entanglement due to the pairs of charges. Moreover, we proposed the multiparameter, multistate generalization of quantum relative entropy.

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ABSTRACT

Holographic duality as a rigorous approach to quantum gravity claims that a quantum gravitational system is exactly equal to a quantum theory without gravity in lower space-time dimensions living on the boundary of the quantum gravitational system. The duality maps key questions about the emergence of spacetime to questions on the non-gravitational boundary system that are accessible to us theoretically and experimentally. Recently, various aspects of quantum information theory on the boundary theory have been found to be dual to the geometric aspects of the bulk theory.

In this thesis, we study the exact and approximate quantum error corrections (QEC) in a general quantum system (von Neumann algebras) focused on QFT and gravity [1]. Moreover, we study entanglement theory in the presence of conserved charges in QFT [3] and the multiparameter multistate generalization of quantum relative entropy [4].

1. INTRODUCTION

In 1997, Maldacena conjectured that quantum gravity in the Anti-de Sitter space is equivalent to a conformal field theory. This conjecture is known as the AdS/CFT correspondence[1] [5]. The statement of AdS/CFT is that any theory of quantum gravity in asymptotically $d+1$ dimensional anti-de Sitter space [2] is equivalent to a conformal field theory in d-dimension without gravity. There are two requirements for this duality to hold; i) A sparse spectrum of low-dimension operators, ii) A large number of local degrees of freedom. In other words, a strongly coupled CFT with a large number of degrees of freedom is dual to a weakly coupled gravitational theory. In this sense, AdS/CFT can be understood as a weak/strong duality.

In this thesis, our discussion of AdS/CFT is primarily concerned with the bulk/boundary dictionary. According to the extrapolate dictionary, to every local field ϕ in the bulk there corresponds an operator $\mathcal{O}$ in the boundary theory [6],

$$\lim_{r\to\infty} r^{\Delta}\phi(t,r,x) = \mathcal{O}(t,x) \tag{1.1}$$

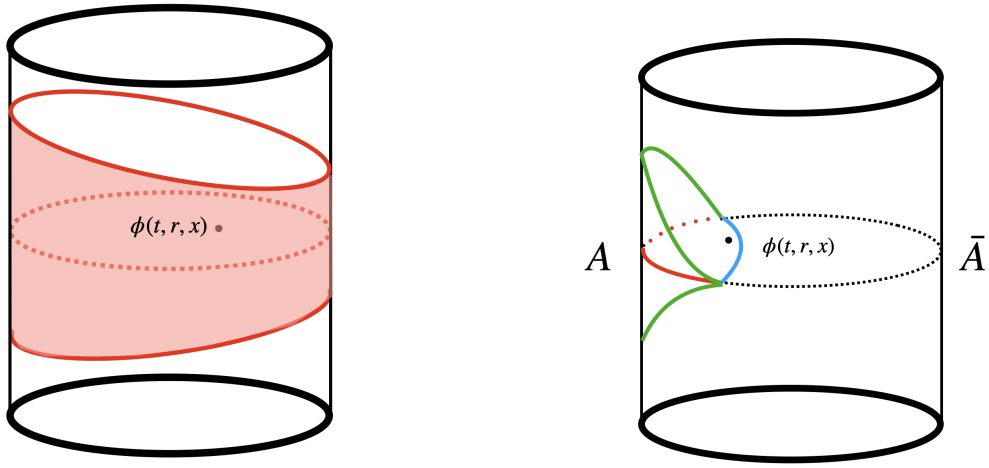
where Δ is a conformal weight, and $\phi(t,r,x)$ is the local bulk operator at (r,x) on a Cauchy slice at time t. Here, r is the radial coordinate of the spacetime. $\mathcal{O}(t,x)$ is the corresponding CFT primary operator. We want to know how to reconstruct the local operators deep in the bulk in a general asymptotically AdS geometry. This type of problem is called the bulk reconstruction. In the strict $N\to\infty$ limit, we can construct a smearing function $K(t',x'|t,r,x)$ such that

$$\phi(t,r,x) = \int dx' K(t',x'|t,r,x)\mathcal{O}(t',x'). \tag{1.2}$$

It is, in a sense, the inverse of the bulk to boundary propagator[7]–[9].

In the $N\to\infty$ limit, there are two ways to reconstruct the operators deep in the bulk: 1) the global reconstruction and 2) the causal wedge reconstruction. One of the procedures, the so-called global reconstruction, represents the bulk operator $\phi(t,r,x)$ as a non-local

[1]↑We use holography and AdS/CFT interchangeably.
[2]↑AdS(Anti-de Sitter) space is the maximally symmetric solution of Einstein equation with a negative cosmological constant.



(a) (b)

Figure 1.1. The cylinder is a Penrose diagram of AdS_3. The position of the bulk operator is denoted as a black dot. (a) The red-dotted horizontal circle denotes the boundary Cauchy slice. The boundary support is colored pink. (b)The boundary subregion on the constant time slice is denoted as a red line. The domain of dependence of A is within the green lines. The causal wedge is the bulk domain of dependence whose boundary is the blue and red line.

CFT operator supported on a full Cauchy slice of the boundary, fig.1.1a[10]. This procedure has the non-local CFT operators supported on the boundary region that is space-like to the position of the bulk operator. On the contrary, in the causal wedge reconstruction, we reconstruct a bulk operator with the CFT operators supported only on the boundary subregion, fig.1.1b[10].

The reconstruction maps above obscure the bulk locality. Consider the bulk operator $\phi(\mathbf{x})$ deep in the bulk, and a boundary operator $\mathcal{O}(\mathbf{x}')$, where we have absorbed all the coordinates into the single variable. $\phi(\mathbf{x})$ and any boundary operators $\mathcal{O}(\mathbf{x}')$ are space-like separated. Then, bulk locality requires them to commute. Hence, the boundary representation $\phi(\mathbf{x}) \to \mathcal{O}_{\phi(\mathbf{x})}$ should commute with all the boundary operators. This seems to contradict the following axiom of the boundary quantum field theory [11]: there exist no non-trivial bounded operators which commute with all local operators. In other words, the commutator cannot hold as an operator statement, i.e., $[\mathcal{O}_{\phi(\mathbf{x})}, \mathcal{O}(\mathbf{x}')] \neq 0$. Quantum error

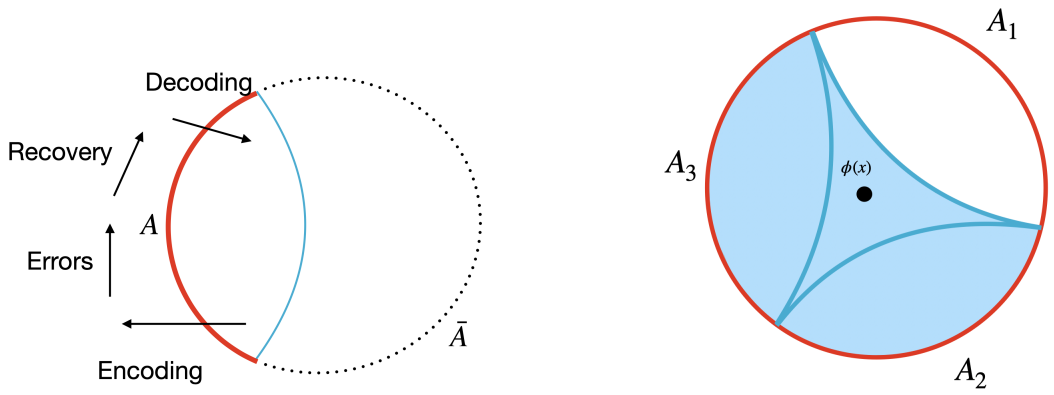
correction (QEC) offers a resolution to this seeming paradox by stating that the bulk locality only requires the operators to commute projected into a "code" subspace.

Quantum error correction (QEC) is one of the essential frameworks and the pillar of quantum information theory. For example, in quantum communication, signals traveling through a fiber are disturbed by the interactions between the fiber and the environment. This results in the deterioration of the signal. In quantum computations, avoiding or correcting errors is essential for fast and high-precision simulations. To model a quantum error correction code, we need 1) an encoding map 2) an error map, 3) a recovery map, and 4) a decoding map. In the Heisenberg picture, the procedure above is often referred to as the *operator algebra quantum error correction* (OAQEC)[12]–[14]. Consider a set of quantum states that one wants to simulate and a set of errors. These states are called *logical states* which span a logical space. The encoding map encodes the logical states into a code subspace which is a vector subspace where all the states are correctable. In OAQEC, one starts with the *logical operators* that act on the logical space. These operators are encoded into a code/correctable subalgebra. Note that in a general QEC, the set of correctable operators need not generate an algebra. In chapter 4 and 5, we work on the case where the set of correctable operators form an algebra.

In AdS/CFT [15]–[17], the error map is, for instance, the erasure of a boundary subregion. The bulk-to-boundary dictionary is the encoding map. The logical operators are bulk operators. Hence, the logical space is the bulk Hilbert space. The logical operators encoded in the boundary theory turn out to be the boundary operators acting on the code subspace in the boundary Hilbert space. Any states in the code subspace or operators acting on the code subspace are correctable against the erasures, fig.1.2a.

As an example, consider the AdS_3 spacetime and partition it into three regions as depicted in fig.1.2b. Consider the error that is the erasure of region A_1. Since the correctability only holds in the code subspace, the radial locality is the statement within the code subspace[15]–[17];

$$P_{code}[\mathcal{O}_{\phi(\mathbf{x})}, \mathcal{O}_1]P_{code} = P_{code}[\mathcal{O}_{23}, \mathcal{O}_1]P_{code} = 0 \tag{1.3}$$



(a) (b)

Figure 1.2. (a)A schematic picture of quantum error correction in holography in the Schrödinger picture. In the case of the Heisenberg picture, the order of "Errors" and "Recovery" switches. (b) A_1, A_2, and A_3 are the boundary subregions. The causal wedge of each boundary subregion lives between the blue curve and its boundary. The operator $\phi(\mathbf{x})$ lies near the center of the bulk.

where $\mathcal{O}_{23}$ is the boundary operators supported on A_2 and A_3, whereas $\mathcal{O}_1$ is supported only on A_1. P_{code} is the projection on the code subspace.

Up to here, we have used the causal wedge reconstruction map to map the bulk operators to the boundary ones. However, there is a more general reconstruction the so-called *entanglement wedge reconstruction*[18] was discovered. It reconstructs bulk regions deeper than the causal wedge.

The entanglement wedge is the bulk domain of dependence of the region between a given boundary subregion and a codimension two bulk surfaces homologous to the boundary region. The codimension two bulk surfaces are called the *Ryu-Takayanagi (RT) surface*. They are determined by extremizing the boundary entanglement entropy, RT formula [19]. HRT formula[20] is its covariant generalization to the time-dependent case;

$$S_A = \frac{Area[\chi_A]}{4G_N} \tag{1.4}$$

where G_N is a gravitational constant. $Area[\,\cdot\,]$ is an area functional of codimension two bulk surfaces χ_A. Since the entanglement wedge always contains the causal wedge [21], [22], we can recover the bulk operators in the entanglement wedge, but not in the causal wedge.

Next, we want to understand the recovery map in the holographic QEC code. As briefly mentioned above, there are four maps in quantum error correction; 1) an encoding map 2) an error map, 3) a recovery map, and 4) a decoding map. In AdS/CFT, encoding or decoding is considered as a map from bulk to boundary, or vice versa. In general, an error map on a set of density matrices $\{\rho\}$ is correctable if there exists a recovery map satisfying the equation $\mathcal{R}^* \circ \Phi^*(\rho) = \rho$ for all the density matrices. When the recovery is approximate, i.e. $\mathcal{R}^*_U \circ \Phi^*(\rho) \approx \rho$, the best recovery map $\mathcal{R}^*_U$ constructed is called the universal recovery map. Constructions and discussions of the universal recovery map on the entanglement wedge are discussed in [23], [24].

As seen above, the applications of quantum error corrections to AdS/CFT are inevitable. However, the conventional theory of quantum error correction is mostly constructed for algebra types that are not the same as the algebras of quantum field theory associated with local regions of spacetime. In general, the local algebras of quantum fields in spacetime are known to be a type-III von Neumann algebra which differs from a finite-dimensional von Neumann algebra. For instance, type-III algebras do not admit a trace, or any density matrices [11], [25]. One needs to reformulate a theory of QEC and recovery map without assuming the existence of trace or density matrices. This motivates us to study quantum error correction in an arbitrary von Neumann algebra (a general quantum system).

2. "INFORMATION IS PHYSICAL"

"Information is physical" [26] is the title of a paper published by Landauer in 1961. The paper claims that "Information" is always encoded in a physical system and there are fundamental energy costs to processing information. For example, the erasure of information encoded in a system surrounded by a heat bath at temperature T always requires an energy cost of $k_B T \ln 2$[1] per single unit of "information", which is known as *Landauer's principle*[2].

In general, a quantity is said to be physical if we can measure and manipulate it in a lab. Hence, any operation (e.g. transfer, processing, or computation) on "information" encoded in a physical system should be constrained by the physical laws of nature. Landauer's principle is the manifestation that information is physical. This led to solving Maxwell's demon problem[28], [29], and the generalization of the second law of thermodynamics including the physical effect due to the information erasure[30].

So far, we have naively used the word "information". If information is physical, we need to define it sharply. In 1948, Shannon quantified the amount of "information" and formulated the theory of information processing, and communication, in the presence of noise[31]. At the time, in physics, there was already a well-founded theory of quantum mechanics, made mathematically rigorous by John von Neumann in the 1920s. However, Shannon was only concerned with information in classical systems. Hence, we call Shannon's theory the classical information theory. It took a few more decades until the essence of information theory was generalized to the quantum realm. In 1984, the first quantum cryptographic protocol, *BB84*, was developed by Charles Bennet and Gilles Brassard [32]. The proposal of a quantum teleportation protocol by Jozsa, William K. Wootters, Charles H. Bennett, Gilles Brassard, Claude Crépeau, and Asher Peres in 1993[33] was another seminal work that laid the foundation of the quantum information theory.

In this section, we briefly review the definition of classical and quantum entropy, distinguishability measures, and channels. For classical information, two measures that play key roles in classical information theory are the Shannon entropy and the Kullback-Leibler divergence. Their quantum analogs are called the entanglement entropy and the quantum

[1]↑k_B is the Boltzmann constant.
[2]↑For the beautiful and concise discussions, see [27].

relative entropy. We also discuss a one-parameter family of entropic measures that generalize entropy, *Rényi measures*. Information processing is described by "channels". However, we postpone its detailed properties and discussions to chapter 3.

2.1 Classical information: Shannon entropy and classical channels

We start this section with the definition of Shannon entropy, as the average "information" or "surprise" associated with a random variable. Intuitively, one can think of the entropy of a random variable as the amount of information we need to know the value of the random variable. Equivalently, it is the amount of surprise one gets after learning the value of the random variable. Consider a probability $p(x)$ of an event x. Then, the amount of "information" or the *surprisal* is defined as

$$\log(\frac{1}{p(x)}) = -\log(p(x)). \tag{2.1}$$

The base of the logarithm defines the unit of information content. In binary systems, it is customary to use base two logarithms and call the unit of information a *bit*. When $p(x)$ is close to 1, the surprisal is close to 0, and there is almost no information in learning the value of the random variable. That is, there is no surprise. On the other hand, if the probability $p(x)$ is close to zero, the surprisal becomes very large. In other words, one has a huge surprise after observing an event that is expected to happen rarely.

Consider, for example, a coin. It has two events {head, tail}, and the associated probability distribution is $\{p_{\text{head}} = 1/2, p_{\text{tail}} = 1/2\}$. The information of each event is equally $log2$, or 1 bit. On the contrary, if the associated probability is uneven, i.e. $\{p_{\text{head}} = 1/4, p_{\text{tail}} = 3/4\}$, the surprisal of the event "head" is larger than that of the event "tail". It means that one gets surprised more after observing the head.

Shannon entropy is an expectation value of the surprisal.

Definition 2.1.1 (Shannon entropy)**.** *Given a set of discrete n events* $\{x_i\}_{i=1}^n$ [3] *and a discrete probability distribution*[4] $\{p\} = \{p(x_i)\}_{i=1}^n$*, Shannon entropy is given by*

$$H(\{x_i\}_{i=1}^n) \equiv \mathbb{E}_{\{p\}}[-\log p(\{x_i\}_{i=1}^n)] = -\sum_{i=1}^n p(x_i)\log p(x_i) \tag{2.2}$$

where $\mathbb{E}_{\{p\}}$ *is an expectation value associated with the probability distribution.*

Next, to make the notion of information processing sharp, we define *channels*. Consider two probability distributions of two random variables $\{p(x_i)\}_{i=1}^n$, $\{p(y_i)\}_{i=1}^n$. The information of each probability distribution is measured by Shannon's entropy. Information processing is the transition from one probability distribution $\{p(x_i)\}_{i=1}^n$ to the other $\{p(y_i)\}_{i=1}^n$. Hence, the channel in information theory is modeled by the map between the probability distributions. In classical information theory, conditional probabilities provide us with the definition of a classical channel. For the probability distributions, $\{p(x_i)\}_{i=1}^n$, $\{p(y_i)\}_{i=1}^n$, the conditional probability is defined by

$$p(y_j|x_i) := \frac{p(y_j, x_i)}{p(x_i)} \tag{2.3}$$

where $p(y_j, x_i)$ is a joint probability. A family of conditional probabilities $\{p(y_j|x_i)\}_{i=1}^n$ of the initial events $\{x_i\}_{i=1}^n$ and the final events $\{y_i\}_{i=1}^n$ is called a *conditional probability distribution*. We define a classical channel between $\{p(x_i)\}_{i=1}^n$ and $\{p(y_i)\}_{i=1}^n$ [5] by the conditional probability distribution $\{p(y_j|x_i)\}_{i=1}^n$.

Definition 2.1.2 (Classical channel)**.** *Given two classical probability distributions* $\{p(x_i)\}_{i=1}^n$ *and* $\{p(y_i)\}_{i=1}^n$*. The classical channel is the conditional probability distribution* $\{p(y_j|x_i)\}_{i=1}^n$ *where* $\{p(y_i)\}_{i=1}^n$ *is obtained by*

$$p(y_j) = \sum_{i=1}^n p(y_j|x_i)p(x_i) \tag{2.4}$$

[3]↑To be precise, x is a random variable on a sample space with n events. x_i are the realizations of the random variable x. We are using the events and the realizations of the random variable interchangeably.
[4]↑In this definition, we have a discrete probability distribution. However, one can, of course, have a version of the continuous probability distribution.
[5]↑Note that one can have a continuous version of the above. In stochastic theory, the multiple steps for some time intervals are considered. In this case, the conditional probability is often called *propagator* which is familiar from classical field theories.

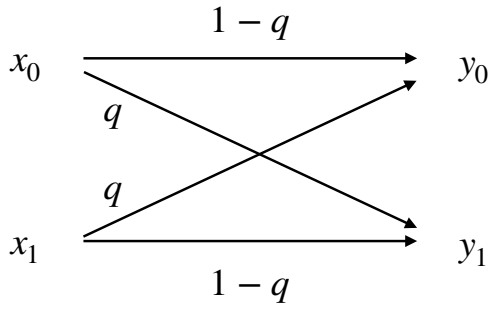


Figure 2.1. The probability distribution $\{p(x_{\rm i})\}_{\rm i=1}^{n}$ is mapped to $\{p(y_{\rm i})\}_{\rm i=1}^{n}$ by the binary symmetric channel $\{p(y_0|x_0) = 1-q,\ p(y_1|x_0) = q,\ p(y_0|x_1) = q,\ p(y_1|x_1) = 1-q\}$.

for all $\rm j = 1, \cdots, n$.

For example, let us consider a *binary symmetric channel* of a single bit, fig.2.1. Suppose the initial probability distribution is given by $\{p(x_0) = p, p(x_1) = 1-p\}$ where $0 \leq p \leq 1$ is the probability of the bit in the state of 0. Suppose the classical channel is $\{p(y_0|x_0) = 1-q,\ p(y_1|x_0) = q,\ p(y_0|x_1) = q,\ p(y_1|x_1) = 1-q\}$. Under the classical channel, the final probability distribution as the output is

$$\{p(y_0) = (1-q)p + q(1-p),\ p(y_1) = qp + (1-q)(1-p)\ \}. \tag{2.5}$$

2.2 Quantum information: von Neumann entropy and quantum channels

In quantum mechanics, the non-commutative analog of a probability distribution is a density matrix ρ on a Hilbert space $\mathcal{K}$ defined by

$$\rho = \sum_{\rm i} p_{\rm i} \left|{\rm i}\right\rangle \left\langle{\rm i}\right| \tag{2.6}$$

where $\left|{\rm i}\right\rangle \left\langle{\rm i}\right|$ are projections on $\mathcal{K}$. In analogy to the classical case, the projections $\left|{\rm i}\right\rangle \left\langle{\rm i}\right|$ correspond to the events where $\{p_{\rm i}\}$ is its probability distribution. A density matrix is

a positive operator because its eigenvalues are positive real numbers[6]. Moreover, for the eigenbasis $\{|\mathrm{i}\rangle\} \in \mathcal{K}$,

$$\mathrm{tr}(\rho) = \sum_{\mathrm{i}} \langle \mathrm{i}| \rho |\mathrm{i}\rangle < \infty. \tag{2.7}$$

Such an operator is called a *trace-class* operator[7]. Roughly speaking, it is *normalizable* with a trace. ρ is called a *pure state* if it cannot be written by a convex composition of any other states. Since the pure states are a one-dimensional projection, they satisfy $\rho^2 = \rho$. This is sometimes called a *purity* of the density matrix. Otherwise, a density matrix is a *mixed* state, and $\rho^2 \leq \rho$.

In quantum information theory, the notion of subsystem plays a significant role in understanding the correlation structures in quantum systems. To define the subsystem, we use partial trace. Consider a density matrix ρ_{AB} of a quantum system AB. We partition the system into A and B. The density matrices of subsystem A and B are defined by $\rho_A = \mathrm{tr}_B(\rho_{AB})$ and $\rho_B = \mathrm{tr}_A(\rho_{AB})$. Conversely, one can compose systems by a *tensor product*. For example, consider two density matrices ρ_A and ρ_B. One can compose them, and get, for instance, $\rho_{AB} = \rho_A \otimes \rho_B$.

One can calculate the quantum information of density matrices similar to Shannon entropy in the classical case. In the quantum case, von Neumann entropy generalizes Shannon entropy.

Definition 2.2.1 (von Neumann entropy)**.** *Given a density matrix ρ, von Neumann entropy is defined as*

$$S(\rho) = -\mathrm{tr}(\rho \log \rho). \tag{2.8}$$

It satisfies the following properties;

1. Additivity:

$$S(\rho \otimes \sigma) = S(\rho) + S(\sigma); \tag{2.9}$$

2. Subadditivity (SA):

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}) \tag{2.10}$$

[6]↑A operator is positive if $\langle \Psi|\rho|\Psi\rangle \geq 0$ for $\forall\, |\Psi\rangle \in \mathcal{K}$.
[7]↑It is also called L^1 bounded operator since L^1-norm is the trace norm. Note that trace norm does not depend on the basis.

3. Concavity: For a set of positive numbers p_{i} such that $\sum_{\mathrm{i}} p_{\mathrm{i}} = 1$,

$$S(\sum_{\mathrm{i}} p_{\mathrm{i}}\rho_{\mathrm{i}}) \geq \sum_{\mathrm{i}} p_{\mathrm{i}} S(\rho_{\mathrm{i}}). \tag{2.11}$$

In addition, its upper-bound is provided by

$$S(\sum_{\mathrm{i}} p_{\mathrm{i}}\rho_{\mathrm{i}}) \leq \sum_{\mathrm{i}} p_{\mathrm{i}} S(\rho_{\mathrm{i}}) - \sum_{\mathrm{i}} p_{\mathrm{i}} \log p_{\mathrm{i}} = \sum_{\mathrm{i}} p_{\mathrm{i}} S(\rho_{\mathrm{i}}) + H(\{p\}). \tag{2.12}$$

In the quantum case, the basic idea of a "channel" is a map between non-commutative probability distributions. A quantum channel is a map between density matrices. *Quantum channel* represents quantum information processing. To define quantum channels, we need to introduce two concepts, i) completely positive maps, and ii) trace-preserving maps. We briefly discuss them here since we will study them in detail in chapter 3.

First, a density matrix is a positive operator. A linear map that sends a positive operator to another one is called a positive map. Thus, quantum channels should be at least positive maps. Consider a density matrix ρ_{SR} of a system S and a d_R-dimensional reference system R. A positive map Φ_{sch}[8] on the system is a completely positive map when

$$(\Phi_{sch} \otimes \mathbb{I}_R)(\rho_{SR}) \tag{2.13}$$

is positive for all d_R. The transpose map is an example of a positive map, but not a completely positive map.

Second, a quantum channel Φ_{Sch} is *trace-preserving* if

$$\mathrm{tr}(\Phi_{sch}(\rho)) = \mathrm{tr}(\rho). \tag{2.14}$$

This is a physical requirement that total probability is preserved under the quantum channel. We summarize the above discussion into the following definition.

[8]↑In this section, a quantum channel on a density matrix is denoted as Φ_{sch} where the subscript indicates the Schrödinger picture. Similarly, we denote a quantum channel on an observable by Φ_H which is in the Heisenberg picture.

Definition 2.2.2 (Quantum channel)**.** *Consider* $\{\rho_A\}$ *the set of density matrices of system* A *and* $\{\rho_B\}$ *the set of density matrices of system* B*, then a quantum channel* Φ_{sch} *is a completely positive and trace-preserving map from* $\rho_A \to \rho_B$*.*

The simplest quantum channel is a unitary transformation $U\rho U^\dagger$ for a density matrix ρ. For instance, the time evolution of a closed quantum system is a unitary flow generated by the Hamiltonian. A general quantum channel models non-unitary and irreversible dynamics as well. Hence, it is suitable to describe the interactions between a system and, for instance, an external system.

To make this statement explicit, let us introduce and construct so-called *Kraus representation*[9] in a simple example below. For a quantum channel and a density matrix, in general, the Kraus representation is defined by

$$\Phi_{sch}(\rho) \equiv \sum_k M_k \rho M_k^\dagger. \tag{2.15}$$

M_k are non-unitary operators called *Kraus operators* and satisfy $\sum_k M_k^\dagger M_k = \mathbb{I}$.

Consider a system S and, for instance, a reservoir R. Given a total Hamiltonian $H_{tot} = H_S \otimes \mathbb{I} + \mathbb{I} \otimes H_R + H_{SR}$ where H_S and H_R are the Hamiltonian of the system and the reservoir. H_{SR} is the interaction Hamiltonian. Let $\rho_S(0)$ be the initial density matrix of the system, and $|r_0\rangle \langle r_0|$ is a projection consisting of the lowest eigenvalue of H_R. Then, the density matrix of the system after time t under the time evolution defined by the unitary $U_{tot} = \mathrm{e}^{\mathrm{i}tH_{tot}}$ is obtained by

$$\begin{aligned} \rho_S(t) &= \mathrm{tr}_R \left(U_{tot} \rho_S(0) \otimes |r_0\rangle \langle r_0| U_{tot}^\dagger \right) \\ &= \sum_k \langle r_k| U_{tot} |r_0\rangle \rho_S(0) \langle r_0| U_{tot}^\dagger |r_k\rangle \\ &= \sum_k M_k \rho_S(0) M_k^\dagger \end{aligned} \tag{2.16}$$

where we put $M_k = \langle r_k| U_{tot} |r_0\rangle$.

[9]↑For its full definition and discussion, see theorem 3.2.2.

For the simplest case, let $|\Psi\rangle\langle\Psi|$ be an initial pure density matrix of a single qubit system where $|\Psi\rangle = p|0\rangle + (1-p)|1\rangle$ for $0 \leq p \leq 1$. Consider a CNOT gate as a dynamics on a two-qubit system, i.e.,

$$U_{tot} = |00\rangle_{SR}\langle 00| + |01\rangle_{SR}\langle 10| + |10\rangle_{SR}\langle 11| + |11\rangle_{SR}\langle 10| \tag{2.17}$$

where $|\mathrm{ij}\rangle_{SR} = |\mathrm{i}\rangle_S \otimes |\mathrm{j}\rangle_R$ $(\mathrm{i},\mathrm{j} = 0,1)$ are the $sigma_z$ basis of the system qubits and resevoir qubits. The Kraus operators in a basis $\{|0\rangle_R, |1\rangle\}$ are

$$M_0 = \langle 0|U|0\rangle = |0\rangle\langle 0|, \; M_1 = \langle 1|U|1\rangle. \tag{2.18}$$

Thus, we get

$$\sum_k M_k |\Psi\rangle\langle\Psi| M_k^\dagger = p|0\rangle\langle 0| + (1-p)|1\rangle\langle 1|. \tag{2.19}$$

The non-unitary process decoheres the system and is irreversible.

2.3 Classical and quantum correlations

Quantum information can be encoded into, for instance, a set of two-level quantum systems. This is often called a *qubit* system. When one encodes the information to a set of d-level quantum systems, we call it a *qudit* system. The correlation structures of these systems are governed by how they interact. In quantum information processing, we actively manipulate the correlation structures to achieve communication or computation protocols. This is how quantum information technologies are built on quantum systems.

In a bipartite system, there are four types of correlation structures that are important to us; i) product states (no correlations), ii) separable states (classical correlations), iii) entangled states (quantum correlations), and iv) classical-quantum states. When the subsystems do not have any correlation or are decorrelated, the density matrix of AB is a *product state*,

$$\rho_{AB} = \rho_A \otimes \rho_B. \tag{2.20}$$

The density matrix is a *separable state* if it is written by

$$\rho_{AB} = \sum_{\mathrm{i}} p_{\mathrm{i}} \rho_{A\mathrm{i}} \otimes \rho_{B\mathrm{i}} \tag{2.21}$$

for a probability distribution $\{p_{\mathrm{i}}\}$. If density matrices cannot be written by any of the two forms, they are called *entangled states*. When the system has both classical and quantum correlations, the density matrices are written by

$$\rho_{AB} = \sum_{\mathrm{i}} p_{\mathrm{i}} \rho_{AB\mathrm{i}} \tag{2.22}$$

for a probability distribution $\{p_{\mathrm{i}}\}$.

To gain intuition as why each form of the density matrices has the corresponding correlations, we now discuss the measures of correlation. One might think that von Neumann entropy is enough. However, as we will see, it is a measure of correlations only for pure bipartite systems.

Consider ρ_{AB} to be a pure state. In general, von Neumann entropy of ρ_A measures total correlations between subsystem A and B. It can capture all the correlations between A and B with neither over-counting nor miss-counting, fig.2.2, because we have $S(\rho_A) = S(\rho_B)$ from the subadditivity (2.10). However, when ρ_{AB} is a mixed state, it cannot capture the correlations properly. For example, even if one has a product state $\rho_{AB} = \rho_A \otimes \rho_B$, the von Neumann entropy $S(\rho_A)$ is not equal to zero despite the fact that there is no correlation between A and B. Hence, von Neumann entropy is not a good measure of correlations of mixed states.

Instead, we define mutual information by

$$I(A:B) := S(\rho_A) + S(\rho_B) - S(\rho_{AB}). \tag{2.23}$$

It is a measure of correlations of both pure and mixed states. For the product states, it always vanishes. Thus, there is no correlation. For the separable states, it equals Shannon entropy. Thus, only the classical correlations exist. For the entangled states, it is calculated by the

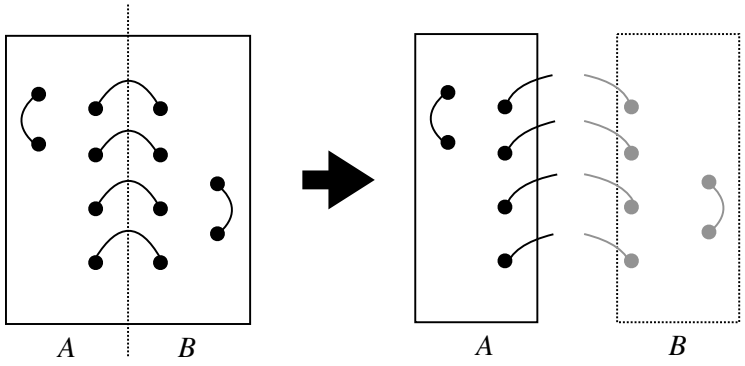


Figure 2.2. A system is partitioned into subsystem A and B. The dots represent, for instance, local lattice sites. The black lines represent the quantum correlations between the lattice sites. Intuitively, what contributes to the von Neumann entropy is the number of free legs.

combination of von Neumann entropy following its definition. In the next section, we will see that there is a much more fundamental information measure than mutual information, which is called *relative entropy*.

2.3.1 Relative entropy

Among the measures, an information measure that is central to the thesis is *relative entropy*. This is a distinguishability measure. It compares two probability distributions and quantifies how distinguishable they are. It generates other information measures, for instance, mutual information. Moreover, it provides operational interpretation for various information and correlation measures[10].

In [35], the relative entropy for the classical probability distributions was proposed by Kullback and Leibler. It was originally proposed for the continuous probability distribution. Here, we give the definition with the discrete probability distributions.

[10]↑For the summary of the role of relative entropy in information theory and physics, see [34].

Definition 2.3.1 (Classical relative entropy/Kullback-Leibler (KL) divergence)**.** *For two probability distributions,* $\{p(x_{\rm i})\}_{\rm i=1}^{n}$*,* $\{q(x_{\rm i})\}_{\rm i=1}^{n}$*, the Kullback-Leibler (KL) divergence is defined by*

$$D_{KL}(\{q\}|\{p\}) = \sum_{\rm i=1}^{n} q(x_{\rm i}) \log\left(\frac{q(x_{\rm i})}{p(x_{\rm i})}\right) \tag{2.24}$$

which satisfies the following properties

1. *If* $p(x) = 0,\ q(x) = 0$*, then*

$$\log(q(x)/p(x)) = 0 \tag{2.25}$$

2. *If* $p(x) \neq 0,\ q(x) = 0$*, then*

$$\log(q(x)/p(x)) = \infty \tag{2.26}$$

In the quantum case, the relative entropy is defined between two density matrices.

Definition 2.3.2 (Quantum relative entropy)**.** *For two density matrices,* ρ*,* σ*, on a Hilbert space* $\mathcal{K}$*,*

$$S(\rho\|\sigma) = \begin{cases} \mathrm{tr}\left(\rho(\log\rho - \log\sigma)\right) & \text{if } supp(\rho) \subseteq supp(\sigma) \\ +\infty & \text{otherwise} \end{cases} \tag{2.27}$$

Let us mention the properties of the quantum relative entropy and its relevance to other information measures. First, the relative entropy is a function from density matrices to the positive real number $\mathbb{R}_+$. That is, $S(\rho\|\sigma) \geq 0$. Note that $S(\rho\|\sigma) = 0$ if and only if $\rho = \sigma$. The remarkable property is the monotonicity under a quantum channel. Consider a quantum channel Φ acting on two density matrices, ρ, σ. Then,

$$S(\rho\|\sigma) \geq S(\Phi(\rho)\|\Phi(\sigma)). \tag{2.28}$$

This implies that the density matrices become less distinguishable after the dynamics modeled by the quantum channel.

It is a quantum generalization of the second law in thermodynamics. Consider σ to be a density matrix that follows the Gibbs distribution and ρ to be a density matrix out of equilibrium. The relative entropy $S(\rho\|\sigma)$ measures how far is ρ from equilibrium. Suppose

a quantum channel is modeled to be a thermalization process. Then, the monotonicity provides the second law[11].

An information measure that satisfies the monotonicity under a local quantum channel is sometimes called a *correlation measure*. A notable correlation measure is a mutual information $I(A:B)$ as discussed above.

Under a local quantum channel Φ on ρ_A, the correlation between subsystem A and B only decreases. This can be seen from the monotonicity of relative entropy, i.e.,

$$I(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B) \geq S((\Phi \otimes \mathrm{i}d)(\rho_{AB})\|\Phi(\rho_A) \otimes \rho_B) = I_\Phi(A:B) \tag{2.29}$$

where $I_\Phi(A:B)$ is the mutual information of the density matrix $(\Phi \otimes \mathrm{i}d)(\rho_{AB})$.

There are two important inequalities obtained from the relative entropy,

1. Subadditivity (SA):

$$S(\rho_A) + S(\rho_B) \geq S(\rho_{AB}) \tag{2.30}$$

2. Strong subadditivity (SSA):

$$S(\rho_{ABC}\|\rho_A \otimes \rho_{BC}) \geq S(\rho_{AB}\|\rho_A \otimes \rho_B) \tag{2.31}$$

Subadditivity is obtained from the positivity of relative entropy or mutual information between ρ_{AB} and $\rho_A \otimes \rho_B$, i.e., $I(A:B) = S(\rho_{AB}\|\rho_A \otimes \rho_B) = S(\rho_A) + S(\rho_B) - S(\rho_{AB}) \geq 0$. The SSA follows simply from the monotonicity under the partial trace tr_C on C as a quantum channel.

The strong subadditivity is saturated on a special class of quantum states called *Markov state*. A tripartite density matrix ρ_{ABC} is a Markov state if it saturates the strong subadditivity. This is a special example of quantum error correction where the error erases subsystem C. Consider a partial trace on C, tr_C, as a quantum channel. Then, the SSA holds from the monotonicity of relative entropy (2.28). When the SSA saturates, it is known

[11]↑For the simplest example, take $\sigma = \mathbb{I}/d$ to be a maximally mixed state. Then, the relative entropy is $S(\rho\|\sigma) = \log d - S(\rho)$. Under the thermalization process, von Neumann entropy increases. Thus, $S(\rho\|\sigma) \geq S(\Phi(\rho)\|\Phi(\sigma)) = \log d - S(\Phi(\rho))$

that there always exists a map from B to BC that can reverse the erasure of C[36]. In other words, one can construct the *recovery map* that recovers the initial state. In this case, the recovery map recovers ρ_{ABC} from ρ_{AB} and $\rho_A \otimes \rho_{BC}$ from $\rho_A \otimes \rho_B$. In general, density matrices that saturate the SSA are always correctable under the erasure error of A or C. Such density matrices are *Markov states*, and they always admit an expansion of the form,

$$\rho_{ABC} = \oplus p_{\mathrm{i}} \rho_{\mathrm{i}}^{AB_1} \otimes \rho_{\mathrm{i}}^{B_2C} \tag{2.32}$$

where $p_{\mathrm{i}} \in \mathbb{R}_+$, $\sum_{\mathrm{i}} p_{\mathrm{i}} = 1$ and $B_1B_2 = B$. ρ_{ABC} is the density matrix on the Hilbert space $\mathcal{K}_{ABC}$ which saturate the SSA. In this case, the Hilbert space splits into two $\mathcal{K}_{ABC} = \oplus_{\mathrm{i}} \mathcal{K}_{AB_1\mathrm{i}} \otimes \mathcal{K}_{B_2C\mathrm{i}}$, and $\rho_{\mathrm{i}}^{AB_1}$ and $\rho_{\mathrm{i}}^{B_2C}$ are supported on $\mathcal{K}_{AB_1\mathrm{i}}$ and $\mathcal{K}_{B_2C\mathrm{i}}$, respectively.

Its physical intuition can easily be grasped by considering another information measure called *conditional mutual information*(CMI) defined by

$$\begin{aligned} I(A:C|B) &:= I(A:BC) - I(A:B) \\ &= S(\rho_{ABC} \| \rho_A \otimes \rho_{BC}) - S(\rho_{AB} \| \rho_A \otimes \rho_B) \\ &= S(AB) + S(BC) - S(B) + S(ABC). \end{aligned} \tag{2.33}$$

It measures the correlation between subsystem A and C conditioned on B. The saturation of the SSA implies that $I(A:C|B) = 0$. Physically, this implies that all correlations between A and C go through B. By staring at (2.32), one can notice that there is no quantum correlation between A and C but the classical correlation provided by the classical mixtures $\{p_{\mathrm{i}}\}$. One should note that this is a special class of density matrices in the class of separable states $\sum_{\mathrm{i}} p_{\mathrm{i}} \rho_{\mathrm{i}} \otimes \sigma_{\mathrm{i}}$.

2.3.2 Rényi measures

We have studied two main quantum information measures, von Neumann entropy, and quantum relative entropy. However, there are various other kinds of information measures, such as max-entropy[37], min-entropy, and collision entropy[38]. Each of them has its own

interpretation. Do they satisfy any relations? Is there a way to interpolate among these information measures? The answer is provided by the construction of Rényi entropy [39].

In [39], Rényi entropy was constructed for a classical probability theory. For a classical probability distribution $\{p\} = \{p(x_{\rm i})\}_{\rm i=1}^{n}$, and a parameter $\theta \in (0,1) \cup (1,\infty)$,

$$H_\theta(\{p\}) = \frac{1}{1-\theta} \log_2 \Big(\sum_{\rm i=1}^{n} p^\theta(x_{\rm i}) \Big). \tag{2.34}$$

This can be generalized to the quantum case. For a density matrix ρ, and a parameter $\theta \in (0,1) \cup (1,\infty)$,

$$S_\theta(\rho) = \frac{1}{1-\theta} \ln \mathrm{tr}(\rho^\theta). \tag{2.35}$$

In the limit $\theta \to 1$, the classical Rényi entropy reduces to Shannon entropy, and the quantum Rényi entropy reduces to von Neuamnn entropy. The beauty of quantum Rényi entropy is once we know $\mathrm{tr}(\rho^\theta)$ for any θ, we get the whole spectrum of ρ.

Again, in our work, the relative entropy is the central information measure. There are two one-parameter generalized measures of quantum relative entropy. One is called *the Petz divergence*[40], and the other is called *the sandwiched Rényi divergence*[41]. For density matrices, ρ, and σ, and a parameter $\theta \in (0,1) \cup (1,\infty)$,

$$\begin{aligned} \text{Petz divergence}: \ D_\theta(\rho\|\sigma) &= \frac{-1}{1-\theta} \log \mathrm{tr}(\rho^\theta \sigma^{1-\theta}) \\ \text{Sandwiched Rényi divergence}: \ S_\theta(\rho\|\sigma) &= \frac{-1}{1-\theta} \log \mathrm{tr} \left((\sigma^{\frac{1-\theta}{2\theta}} \rho \sigma^{\frac{1-\theta}{2\theta}})^\theta \right). \end{aligned} \tag{2.36}$$

The Petz divergence satisfies the monotonicity for $\theta \in (0,1) \cup (1,2)$ while the sandwiched Rényi divergence satisfies it for $\theta \in (1/2,1) \cup (1,\infty)$, fig.2.3a .

θ-r-Rényi divergence was proposed to interpolate the above two one-parameter divergences[42],

$$S_{\theta,r}(\rho\|\sigma) = \frac{-1}{1-\theta} \log tr \Big[\Big(\sigma^{\frac{1-\theta}{2r}} \rho^{\frac{\theta}{r}} \sigma^{\frac{1-\theta}{2r}} \Big)^r \Big]. \tag{2.37}$$

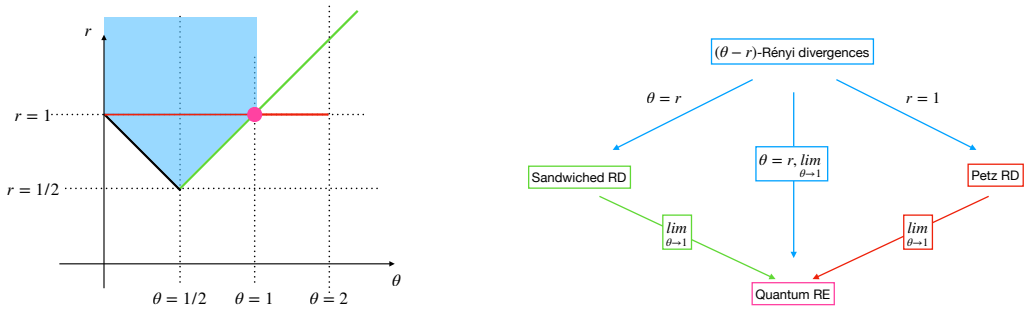


Figure 2.3. (a) Monotonicity of Rényi divergences holds inside the colored regions of the (θ, r) parameter space. The red line represents the Petz divergence. The green line represents the sandwiched Rényi divergence. The black line represents the so-called *reverse sandwiched Rényi divergence*[42]. The pink dot is where quantum relative entropy is placed at. (b) It shows the relations among (θ, r)-Rény divergence, sandwiched Rényi divergence, Petz Rényi divergence, and quantum relative entropy.

When $\theta = r$, it reduces to sandwiched Rényi entropy. When $r = 1$, it reduces to Petz divergence, fig.2.3a. When, $r = 1-\theta$, one gets *reverse Sandwiched Rényi divergence* defined by

$$\hat{S}_\theta(\rho\|\sigma) = \frac{1}{\alpha - 1} \log \mathrm{tr}\left(\rho^{\frac{\theta}{2(1-\theta)}} \sigma \rho^{\frac{\theta}{2(1-\theta)}}\right)^{1-\theta}. \tag{2.38}$$

It has the following symmetry relation;

$$(\theta - 1)\hat{S}_\theta(\rho\|\sigma) = (-\theta) S_{\theta,r}(\rho\|\sigma). \tag{2.39}$$

In chapter 6, we further generalized it to multiparameter multistate Rényi relative entropy.

3. COMPLETELY POSITIVE MAPS AND THEIR DUALS IN VON NEUMANN ALGEBRAS

Our discussion on the information theory in the previous chapter was based on the Schrödinger picture where classical probability distributions and density matrices are dynamical. In this thesis, instead, we work in the Heisenberg picture where operators are dynamical, and the probability distributions and density matrices are stationary. We model the dynamics of observables in the Heisenberg picture by linear maps on C^*-algebras and von Neumann algebras. Thus, the main purpose of this chapter is to provide a brief review of these linear maps and their "dual" linear maps which will be defined later. We first answer the following questions: i) what are those operators? ii) do they form an algebra? iii) if so, what are the types and properties of the algebra?

The outcome of a measurement is an event with a particular probability associated with it. In other words, we measure expectation values and the fluctuations of observables. We model *observables* by Hermitian, or self-adjoint operators a on a Hilbert space $\mathcal{K}$[1]. Consider the spectral decomposition of a Hermitian operator a on $\mathcal{K}$, i.e.,

$$a = \sum_{\mathrm{i}} \lambda_{\mathrm{i}} \left|\mathrm{i}\right\rangle \left\langle\mathrm{i}\right| \tag{3.1}$$

where $\left|\mathrm{i}\right\rangle \in \mathcal{K}$ and $\lambda_{\mathrm{i}} \in \mathbb{R}$ are eigenvalues of a. The expectation value of the projection $\left|\mathrm{i}\right\rangle \left\langle\mathrm{i}\right|$ in a state $\left|\eta\right\rangle$

$$\left\langle\eta\right|(\left|\mathrm{i}\right\rangle \left\langle\mathrm{i}\right|)\left|\eta\right\rangle = \left|\left\langle\mathrm{i}|\eta\right\rangle\right|^2 =: p_{\mathrm{i}} \tag{3.2}$$

gives the probability of the physical state to be $\left|\mathrm{i}\right\rangle \left\langle\mathrm{i}\right|$[2]. Then, the expectation value of a is given by

$$\left\langle\eta|a|\eta\right\rangle = \sum_{\mathrm{i}} p_{\mathrm{i}} \lambda_{\mathrm{i}}. \tag{3.3}$$

[1]↑Self-adjoint is defined for both bounded and unbounded operators. Hermitian operators are bounded self-adjoint operators.

[2]↑This is sometimes called *Born rule*.

One can construct a $*$-algebra $\mathcal{A}$ including a set of the Hermitian operators to algebraically model physics[3]. However, in the infinite-dimensional case, the $*$-algebra is not enough for the purpose. For any sequence $\{a_n\}$ in the $*$-algebra, we should include in the algebra an operator a that satisfies the limit with respect to the so-called *weak operator topology* (WOT),

$$\lim_{n\to\infty} \langle \eta_1 | a_n | \eta_2 \rangle = \langle \eta_1 | a | \eta_2 \rangle \tag{3.4}$$

for $\forall |\eta_1\rangle, |\eta_2\rangle \in \mathcal{K}$. Intuitively, it implies that every matrix element of a_n should converge to that of a in large n. If the closure[4] of the $*$-subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ contains an identity operator $\mathbb{I}$, $\mathcal{A}$ is called a *von Neumann algebra*.

3.1 von Neumann algebras

In this section, we will provide the definition of C^*-algebras and vN algebras. von Neumann (vN) algebras are a special class of C^*-algebras.

Definition 3.1.1 (C^*-algebra)**.** *A C^*-algebra on $\mathcal{K}$ is a $*$-subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ such that*

1. *it is closed under norm topology, i.e., for any sequence of $a_n \in \mathcal{A}$, there exists a limit $a \in \mathcal{A}$,*

$$\lim_{n\to\infty} \|a_n - a\| = 0; \tag{3.5}$$

2. *C^*-property: for $a \in \mathcal{A}$*

$$\|x^\dagger x\| = \|x^\dagger\| \|x\| = \|x\|^2 \tag{3.6}$$

If C^*-algebra contains an identity $\mathbb{I}$, it is called a *unital C^*-algebra*. In this thesis, we always assume C^*-algebra to be unital.

Definition 3.1.2 (von Neumann algebras(vN algebras))**.** *A von Neumann algebra on $\mathcal{K}$ is a $*$-subalgebra $\mathcal{A}$ of $B(\mathcal{K})$ such that*

[3]↑Since mathematicians denote the dagger action by $*$, it is called $*$-algebra. As a physicist, should we call it †-algebra?

[4]↑Including all the limits with respect to convergence is called *closure*. One can choose how operators converge. As we see below, operators in C^*-algebras converge in so-called the *norm topology*.

1. *it is closed under weak operator topology, i.e., for any sequence of* $a_n \in \mathcal{A}$*, there exists a limit* $a \in \mathcal{A}$*,*

$$\lim_{n\to\infty} \langle \eta_1 | a_n | \eta_2 \rangle = \langle \eta_1 | a | \eta_2 \rangle \tag{3.7}$$

for $\forall\, |\eta_1\rangle\,, |\eta_2\rangle \in \mathcal{K}$*.*

2. $\mathbb{I} \in \mathcal{A}$

There is a powerful theorem that characterizes $*$-algebras of $B(\mathcal{K})$ that are von Neumann algebras. It is called the *double commutant theorem* [43]. For a $*$-algebra, the commutant $\mathcal{A}$ is the set of all operators in $B(\mathcal{K})$ that commute with all the operators in $\mathcal{A}$, i.e.,

$$\mathcal{A}' = \{a' \in B(\mathcal{K}) | [a, a'] = 0, \forall a \in \mathcal{A}\}. \tag{3.8}$$

The double commutant $\mathcal{A}''$ is the commutant of $\mathcal{A}'$. In general, $\mathcal{A} \subset \mathcal{A}''$. We do not state the theorem here. Instead, we adopt it as the definition of vN algebras.

Definition 3.1.3 (von Neumann algebras(vN algebras))**.** *A* $*$*-subalgebra* $\mathcal{A}$ *of* $B(\mathcal{K})$ *is a vN algebra if*

$$\mathcal{A} = \mathcal{A}''. \tag{3.9}$$

The center $Z(\mathcal{A})$ of a vN algebra is defined by

$$Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'. \tag{3.10}$$

If the center of vN algebra $\mathcal{A}$ contains only a scalar multiple of an identity operator, i.e., $Z(\mathcal{A}) = \{\mathbb{C}\mathbb{I}\}$, $\mathcal{A}$ is called a *factor*.

3.2 Positive maps, n-positive maps, and completely positive maps

Before we go into the details of representations of vN algebras and Tomita-Takesaki theory, we discuss positive linear maps, n-positive maps, and completely positive maps. In addition, we study the representations of completely positive linear maps.

3.2.1 Definitions and properties

Consider a $d \times d$ matrix algebra $\mathcal{A}$ on a Hilbert space $\mathcal{K}$. We study linear maps that map from an algebra $\mathcal{A}$ to i) complex numbers $\mathbb{C}$, ii) a vector space, and iii) another algebra $\mathcal{B}$ or itself $\mathcal{A}$. Especially, for iii), we sometimes call such a linear map as a *superoperator*. A trace is an example of a linear map from an operator to a complex number, $\mathrm{tr} : a \in \mathcal{A} \mapsto \alpha \in \mathbb{C}$. The linear maps from an algebra $\mathcal{A}$ to a vector space $\mathcal{K}$ can represent operators as vectors, $a \mapsto |a\rangle$. The linear maps from one algebra to another algebra or to itself make the story "dynamical". For example, complex conjugation on the operators $a \mapsto a^*$[5] for $a \in \mathcal{A}$ is a linear map. The dagger operation is also a linear map $a \mapsto a^\dagger$. For a unitary U, a unitary transformation $U^\dagger a U$ is also a very basic linear map. Among various kinds of linear maps, we study positive linear maps, n-positive maps, and completely positive maps.

The positivity of a linear map is characterized by a positive operator in an algebra because positive linear maps map a positive operator to another one or itself.

Definition 3.2.1 (Positive operators)**.** *An operator a on a Hilbert space $\mathcal{K}$ is positive*[6] *if*

$$\langle \Psi | a | \Psi \rangle \geq 0 \tag{3.11}$$

for all $|\Psi\rangle \in \mathcal{K}$.

An example of a positive operator is a density matrix ρ. When an operator is positive, we denote it by $a \geq 0$. Or, we will write it by a_+, and a set of all positive operators of $\mathcal{A}$ by $\mathcal{A}_+$. In operator algebras, a linear map from an algebra to complex numbers $\mathbb{C}$ is often called a *linear functional*. With the notion of positive operators, positive linear functionals are defined by as follows.

Definition 3.2.2 (Positive linear functional)**.** *Let $\mathcal{A}$ be a C^*-algebra*[7]*. A linear functional $\omega : \mathcal{A} \to \mathbb{C}$ is said to be positive linear functional if $\omega(a_+) \geq 0$ for $a_+ \in \mathcal{A}_+$.*

[5]↑We denote the complex conjugate on operators as a^* for $a \in \mathcal{A}$, and that on complex numbers $c \in \mathbb{C}$ as a bar $\bar{c}$.
[6]↑"Positive" in this thesis means the semi-definite positive.
[7]↑or a vN algebra

For a self-adjoint operator $a = a^\dagger$, $\omega(a)$ is a real number. In addition, it preserves the † action, $\omega(a^\dagger) = \overline{\omega(a)}$. If a positive functional satisfies the normalization condition, i.e.,

$$\|\omega\| \equiv \sup_{a\in\mathcal{A},\|a\|\leq 1} |\omega(a)| = 1, \tag{3.12}$$

we call it a *state* which is used to evaluate an expectation value $\omega(a)$ of an operator $a \in \mathcal{A}$. We will return to the state in section 3.3.2. For the simplest example, consider an algebra of $d \times d$ complex matrices $M_d(\mathbb{C})$. We can easily see that the normalized trace, $\mathrm{tr}((\cdot)\mathbb{I}/d)$, is a state on the algebra $M_d(\mathbb{C})$. The positivity of the trace can be directly checked. For $a = \sum_{ij} \alpha_{ij} |i\rangle \langle j|$ where $|i\rangle \langle j|$ are operator basis in $M_d(\mathbb{C})$ and $\alpha_{ij} \in \mathbb{C}$,

$$\mathrm{tr}((aa^\dagger)\mathbb{I}/d) = \frac{1}{d}\sum_{i,j} |\alpha_{ij}|^2 \geq 0. \tag{3.13}$$

Thus, the trace, $\mathrm{tr}(\mathbb{I}/d(\cdot))$, is a state. In a qudit, the normalized trace is understood as a maximally mixed state. As one can easily check, $\mathrm{tr}(\rho(\cdot))$ for a density matrix ρ is also a state. The notion of a state as a positive functional is the key ingredient to study the representations of vN algebras and C^*-algebras.

As a generalization of positive linear functionals, we define a positive linear map from one algebra to another one.

Definition 3.2.3 (Positive linear map)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras*[8]*. A linear map $\pi : \mathcal{A} \to \mathcal{B}$ is said to be positive if $\pi(a_+) \geq 0$ for $\forall a_+ \in \mathcal{A}_+$.*

Let us discuss three important examples of positive linear operators, i)$*$-homomorphisms, ii)conjugating operators, iii)transpose. Let $\mathcal{A}$ and $\mathcal{B}$ are C^*-algebras[9].

$*$-homomorphisms

A $*$-homomorphism $\pi : \mathcal{A} \to \mathcal{B}$ is a structure-preserving map. It is used to describe representations of C^*-algebras and vN algebras in section 3.3, and defined in subsection 3.3.1. It is a positive map because

[8]↑or vN algebras
[9]↑or vN algebras

$$\pi(aa^\dagger) = \pi(a)\pi(a^\dagger) = \pi(a)\pi(a)^\dagger \geq 0. \tag{3.14}$$

A $*$-homomorphism and its variants, such as $*$-isomorphism, and $*$-automorphism, will frequently appear in this thesis. For example, for a representation of an algebra, $*$-isomorphism represents every single operator. The central theorem3.5.1 of Tomita-Takesaki theory claims that, given a von Neumann algebra with a choice of a state, there always exists a canonical $*$-automorphism group called the *modular automorphism group*[10].

Conjugations

The second example is a conjugation $\pi(b) = a^\dagger ba$, $a, b \in \mathcal{A}$. This is also a positive linear map because

$$\pi(b^\dagger b) = a^\dagger b^\dagger ba = (ba)^\dagger ba \geq 0. \tag{3.15}$$

Transpose

The last example is the transpose, i.e., $\pi(a) = a^T$.

$$\pi(a^\dagger a) = (a^\dagger a)^T = a^T(a^\dagger)^T = a^T(a^T)^\dagger \geq 0. \tag{3.16}$$

Transpose as a positive map in quantum information theory provides numerous tools to study entanglement theory, such as entanglement witness [11].

It is important to see that a direct sum of positive maps is again another positive map. Let $\mathcal{A}_\mathrm{i}$ and $\mathcal{B}_\mathrm{i}$ be C^* algebras[12], and $\pi_\mathrm{i} : \mathcal{A}_\mathrm{i} \to \mathcal{B}_\mathrm{i}$ for $(\mathrm{i} = 1, 2)$ be positive maps. The direct sum of the positive maps defined by

$$(\pi_1 \oplus \pi_2)(a_1 \oplus a_2) \equiv \pi_1(a_1) \oplus \pi_2(a_2) \tag{3.17}$$

is a positive map because the direct sum of positive operators is positive, i.e.,

$$(\pi_1 \oplus \pi_2)((a_1 \oplus a_2)^\dagger(a_1 \oplus a_2)) = \pi_1(a_1^\dagger a_1) \oplus \pi_2(a_2^\dagger a_2) \geq 0. \tag{3.18}$$

[10]↑see section 3.5 for the details
[11]↑Unfortunately, we do not discuss entanglement witness in this thesis. For interested readers, see [44].
[12]↑or vN algebras

Now, is the tensor product of positive maps positive again? The answer is no. For example, consider a tensor product $\boldsymbol{\pi}_1 \otimes \boldsymbol{\pi}_2$ of an identity map, $\boldsymbol{\pi}_1 = \mathrm{i}d$, and a transpose, $\boldsymbol{\pi}_2(\cdot) = (\cdot)^T$, on the following positive matrix,

$$\begin{aligned}\begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} &= \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\ &+ \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \otimes \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned} \tag{3.19}$$

Then,

$$(\boldsymbol{\pi}_1 \otimes \boldsymbol{\pi}_2) \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}. \tag{3.20}$$

The resultant matrix is not positive anymore. Hence, the tensor product of positive maps is not necessarily positive. There is a special class of positive maps called *completely positive maps* where the tensor-product of two completely positive maps is again completely positive. This is an essential mathematical object to describe the dynamical evolution in physics.

To define a completely positive map, we provide the notion of *amplification*. Given a C^*-algebra [13], an *amplification* of the algebra by $M_n(\mathbb{C})$ is the set of $n \times n$ matrix $M_n(\mathcal{A})$ whose entries are the operators in the algebra $\mathcal{A}$, i.e.,

$$M_n(\mathcal{A}) \equiv M_n(\mathbb{C}) \otimes \mathcal{A} = \left\{ \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \mid a_{\mathrm{ij}} \in \mathcal{A},\ \mathrm{i,j} = (1, \cdots, n) \right\}. \tag{3.21}$$

The idea of amplification corresponds to the extension of a physical system. Suppose one has an algebra of a single qubit system, which is just a collection of 2×2 complex matrices,

[13] ↑or a vN algebra

$M_2(\mathbb{C})$. One can extend the algebra to the $1+k$-qubit system by tensor-producing the algebra of another k-qubit system, $M_{2^k}(\mathbb{C}) \otimes M_2(\mathbb{C})$. The notion of complete positivity guarantees that the local action of a linear map Φ on the operators of an extended system is positive, i.e., suppose $\Phi : M_2(\mathbb{C}) \to M_2(\mathbb{C})$ be a completely positive

$$M_{2^k}(\mathbb{C}) \otimes M_2(\mathbb{C}) \to \mathbb{I}_{2^k} \otimes \Phi(M_{2^k}(\mathbb{C}) \otimes M_2(\mathbb{C})) \geq 0 \tag{3.22}$$

for all k. Now, we define the completely positive maps.

Definition 3.2.4 (n-positive map, completely positive map)**.** *Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras*[14]*, and $\Phi : \mathcal{A} \to \mathcal{B}$ a linear map. Consider the matrix $M_n(\mathcal{A})$ and $M_n(\mathcal{B})$, the amplification of $\mathcal{A}$ and $\mathcal{B}$, and $\Phi_{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ be a linear map which acts on each matrix elements $a_{\mathrm{ij}} \in \mathcal{A}$ $(\mathrm{i}, \mathrm{j} = 1, \ldots, n)$ of $M_n(\mathcal{A})$, i.e.,*

$$\Phi_{(n)} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} \Phi(a_{11}) & \cdots & \Phi(a_{1n}) \\ \vdots & \ddots & \vdots \\ \Phi(a_{n1}) & \cdots & \Phi(a_{nn}) \end{pmatrix}. \tag{3.23}$$

A linear map Φ is a n-positive map if Φ_n is a positive map on $M_n(\mathcal{A})$. It is a completely positive map if Φ_n is a n-positive map for all n.

Let us work with some examples of a completely positive map.

$*$-homomorphisms

First, one can easily see that $*$-homomorphism is a completely positive map. Let $\mathcal{A}$ and $\mathcal{B}$ be C^*-algebras[15], and $\boldsymbol{\pi} : \mathcal{A} \to \mathcal{B}$ is $*$-homomorphism. Then, $\boldsymbol{\pi}_{(n)} : M_n(\mathcal{A}) \to M_n(\mathcal{B})$ is

[14] ↑or vN algebras
[15] ↑or vN algebras

again a $*$-homomorphism because, for a positive matrix in $M_n(\mathcal{A})$, $\boldsymbol{\pi}_{(n)}$ is positive for all n as follows;

$$\begin{aligned}\boldsymbol{\pi}_{(n)}\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}^{\dagger}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right] \\ = \begin{pmatrix} \boldsymbol{\pi}(a_{11}) & \cdots & \boldsymbol{\pi}(a_{1n}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\pi}(a_{n1}) & \cdots & \boldsymbol{\pi}(a_{nn}) \end{pmatrix}^{\dagger}\begin{pmatrix} \boldsymbol{\pi}(a_{11}) & \cdots & \boldsymbol{\pi}(a_{1n}) \\ \vdots & \ddots & \vdots \\ \boldsymbol{\pi}(a_{n1}) & \cdots & \boldsymbol{\pi}(a_{nn}) \end{pmatrix}.\end{aligned} \tag{3.24}$$

Thus, $*$-homomorphisms are completely positive maps.

Conjugations

Next, we see that the conjugation,

$$\Phi(a) = x^{\dagger}ax \tag{3.25}$$

for $a, x \in \mathcal{A}$, is a completely positive map. The action of $\Phi_{(n)}$ on $M_n(\mathcal{A})$ can be achieved by the conjugation of the diagonal matrix whose diagonal entries are x. That is,

$$\begin{aligned}\Phi_{(n)}\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right] = \begin{pmatrix} x^{\dagger}a_{11}x & \cdots & x^{\dagger}a_{1n}x \\ \vdots & \ddots & \vdots \\ x^{\dagger}a_{n1}x & \cdots & x^{\dagger}a_{nn}x \end{pmatrix} \\ = \begin{pmatrix} x^{\dagger} & 0 & \cdots & 0 \\ 0 & x^{\dagger} & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x^{\dagger} \end{pmatrix}\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{pmatrix}\end{aligned} \tag{3.26}$$

For a positive matrix in $M_n(\mathcal{A})$, there should exists a matrix $[f_{\rm ij}]$ such that

$$\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} = \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}^{\dagger} \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}. \tag{3.27}$$

Then,

$$\begin{aligned} &\Phi_{(n)}\left[\begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix}\right] \\ &\quad = \left[\begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{pmatrix}\right]^{\dagger} \begin{pmatrix} f_{11} & \cdots & f_{1n} \\ \vdots & \ddots & \vdots \\ f_{n1} & \cdots & f_{nn} \end{pmatrix}\begin{pmatrix} x & 0 & \cdots & 0 \\ 0 & x & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & x \end{pmatrix}. \end{aligned} \tag{3.28}$$

Thus, the conjugation is also a completely positive map. Similarly, consider a conjugation by a linear operator V that maps from a Hilbert space $\mathcal{H}$ to another Hilbert space $\mathcal{K}$. Then, the map $\Phi(a) = V^{\dagger}aV$ from $B(\mathcal{K}) \ni a$ to $B(\mathcal{H})$ is completely positive. We will frequently use it for representations of completely positive maps below.

Positive linear functionals

At last, we observe that positive linear functionals are completely positive maps as well. This is the key property in the studies of representations of C^*-algebras and von Neumann algebras. By definition, if $[a_{\rm ij}] \in M_n(\mathcal{A})$ is a positive operator, then

$$\langle \eta | [a_{\rm ij}] \eta \rangle \geq 0 \tag{3.29}$$

for all $|\eta\rangle \in \mathbb{C}^n$. For $|\eta\rangle = \sum_{\mathrm{i}} \eta_{\mathrm{i}} |\mathrm{i}\rangle$ where $|\mathrm{i}\rangle$ are basis of $\mathbb{C}^n$ and $\eta_{\mathrm{i}} \in \mathbb{C}$, the above condition can be equivalently written as[16]

$$\begin{aligned} &\begin{pmatrix} \sum_{\mathrm{ij}} \eta_{\mathrm{i}}^* \eta_{\mathrm{j}} a_{\mathrm{ij}} & 0 & \cdots & 0 \\ 0 & 0 & & \vdots \\ \vdots & \vdots & & \vdots \\ 0 & \cdots & & 0 \end{pmatrix} \\ &= \begin{pmatrix} \eta_1^* & \cdots & \eta_n^* \\ 0 & \cdots & 0 \\ \vdots & \ddots & \vdots \\ 0 & \cdots & 0 \end{pmatrix} \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & \ddots & \vdots \\ a_{n1} & \cdots & a_{nn} \end{pmatrix} \begin{pmatrix} \eta_1 & 0 & \cdots & 0 \\ \vdots & & \ddots & \vdots \\ \eta_n & 0 & \cdots & 0 \end{pmatrix} \geq 0. \end{aligned} \tag{3.30}$$

This implies that $\sum_{\mathrm{ij}} \eta_{\mathrm{i}}^* \eta_{\mathrm{j}} a_{\mathrm{ij}} \geq 0$. Let ω be a positive linear functional on $\mathcal{A}$, and $\omega_{(n)}$ as a linear map from $M_n(\mathcal{A})$ to $M_n(\mathbb{C})$. Then, it is easy to see that $\sum_{\mathrm{ij}} \eta_{\mathrm{i}}^* \eta_{\mathrm{j}} a_{\mathrm{ij}} \geq 0$ is replaced by $\sum_{\mathrm{ij}} \eta_{\mathrm{i}}^* \eta_{\mathrm{j}} \omega(a_{\mathrm{ij}}) \geq 0$. Since ω is linear, $\omega(\sum_{\mathrm{ij}} \eta_{\mathrm{i}}^* \eta_{\mathrm{j}} a_{\mathrm{ij}}) \geq 0$. Therefore, a positive linear functional ω is completely positive.

We can check that the composition of CP maps gives another completely positive map. Given a composition $\Phi_2 \circ \Phi_1$ of two CP maps $\Phi_1 : \mathcal{A}_1 \to \mathcal{A}_2$, $\Phi_2 : \mathcal{A}_2 \to \mathcal{B}$. Let $\Phi_{1(n)} : M_n(\mathcal{A}_1) \to M_n(\mathcal{A}_2)$, $\Phi_{2(n)} : M_n(\mathcal{A}_2) \to M_n(\mathcal{B})$, and $(\Phi_2 \circ \Phi_1)_{(n)} : M_n(\mathcal{A}_1) \to M_n(\mathcal{B})$. By definition, $(\Phi_2 \circ \Phi_1)_{(n)} = \Phi_{2(n)} \circ \Phi_{1(n)}$. Since the composition of positive maps is positive, and $\Phi_{1(n)}$ and $\Phi_{2(n)}$ are positive on each domain for all n, the composition $\Phi_2 \circ \Phi_1$ of the CP maps is completely positive.

The above definition of completely positive maps is cumbersome to work with because one needs to check the positivity of $\Phi_{(n)}$ for all n. There is a powerful yet simple theorem to characterize completely positive maps, known as *Choi's theorem*.

Theorem 3.2.1 (Choi's theorem on completely positive maps[45]). *Let $\Phi : M_n(\mathbb{C}) \to M_n(\mathbb{C})$ be a linear map, and $|\mathrm{i}\rangle \langle \mathrm{j}|$ be an operator basis of $M_n(\mathbb{C})$. The followings are equivalent:*

1. Completely positive: Φ is completely positive;

[16] ↑Since $[a_{\mathrm{ij}}] = [f_{\mathrm{ij}}]^\dagger [f_{\mathrm{ij}}]$ for $[f_{\mathrm{ij}}] \in M_n(\mathcal{A})$, $([\eta_{\mathrm{i}}][f_{\mathrm{ij}}])^\dagger [\eta_{\mathrm{i}}][f_{\mathrm{ij}}] \geq 0$. Here, we denoted the matrix with η_{i} in its first column as $[\eta_{\mathrm{i}}]$.

2. n-positive: $\Phi_{(n)} \equiv \mathrm{i}d_n \otimes \Phi : M_n(\mathbb{C}) \otimes M_n(\mathbb{C}) \to M_n(\mathbb{C}) \otimes M_n(\mathbb{C})$ *is a positive map;*

3. Positivity of Choi's matrix: Choi's matrix,

$$\sigma_\Phi = \Phi_n(|\mathrm{e}\rangle\langle\mathrm{e}|) = \mathrm{i}d_n \otimes \Phi(\sum_{\mathrm{ij}} |\mathrm{i}\rangle\langle\mathrm{j}| \otimes |\mathrm{i}\rangle\langle\mathrm{j}|) = \sum_{\mathrm{ij}} |\mathrm{i}\rangle\langle\mathrm{j}| \otimes \Phi(|\mathrm{i}\rangle\langle\mathrm{j}|) \tag{3.31}$$

where $|\mathrm{e}\rangle = \sum_{\mathrm{i}} |\mathrm{i}\rangle \otimes |\mathrm{i}\rangle$*, is positive.*

Proof. $1 \to 2$: 1 implies 2 by the definition.

$2 \to 3$: One can check that

$$|\mathrm{e}\rangle\langle\mathrm{e}| \equiv \sum_{\mathrm{ij}} |\mathrm{i}\rangle\langle\mathrm{j}| \otimes |\mathrm{i}\rangle\langle\mathrm{j}| \tag{3.32}$$

is a positive operator because $|\mathrm{e}\rangle\langle\mathrm{e}| = 1/n(|\mathrm{e}\rangle\langle\mathrm{e}|)^\dagger |\mathrm{e}\rangle\langle\mathrm{e}|$ where $n = dim(\mathbb{C}^n) > 0$. Since Φ is n-positive, or $\mathrm{i}d_n \otimes \Phi$ is positive, Choi's matrix σ_Φ is positive.

$3 \to 1$: First, from (3.31),

$$\Phi(|\mathrm{i}\rangle\langle\mathrm{j}|) = (\langle\mathrm{i}| \otimes \mathbb{I})\sigma_\Phi(|\mathrm{j}\rangle \otimes \mathbb{I}). \tag{3.33}$$

Since σ_Φ is positive, there is a eigenvalue decomposition

$$\begin{aligned} \sigma_\Phi &= \sum_k \lambda_k |\phi_k\rangle\langle\phi_k| \\ |\phi_k\rangle &= \sum_{\mathrm{i}\alpha} \varphi_{\mathrm{i}\alpha}^{(k)} |\mathrm{i}\rangle \otimes |\alpha\rangle \in \mathbb{C}^n \otimes \mathbb{C}^n \end{aligned} \tag{3.34}$$

where, for all k, $\lambda_k \geq 0$ are the eigenvalues, and $|\phi_k\rangle$ are the eigenvectors of $\mathbb{C}^n \otimes \mathbb{C}^n$. Define a linear map $M_k : \mathbb{C}^n \to \mathbb{C}^n$ to be

$$M_k^\dagger \equiv \sum_{\mathrm{i}\alpha} \varphi_{\mathrm{i}\alpha}^{(k)} |\alpha\rangle\langle\mathrm{i}| \tag{3.35}$$

so that

$$|\phi_k\rangle = \mathbb{I} \otimes M_k^\dagger |\mathrm{e}\rangle . \tag{3.36}$$

Then, Choi's matrix becomes

$$\sigma_\Phi = \sum_k \lambda_k (\mathbb{I} \otimes M_k^\dagger) \left|\mathrm{e}\right\rangle \left\langle \mathrm{e}\right| (\mathbb{I} \otimes M_k). \tag{3.37}$$

From (3.33),

$$\Phi(\left|\mathrm{i}\right\rangle \left\langle \mathrm{j}\right|) = \sum_k (\lambda_k^{1/2} M_k)^\dagger \left|\mathrm{i}\right\rangle \left\langle \mathrm{j}\right| (\lambda_k^{1/2} M_k). \tag{3.38}$$

As discussed in (3.25), each term $(\lambda_k^{1/2} M_k)^\dagger \left|\mathrm{i}\right\rangle \left\langle \mathrm{j}\right| (\lambda_k^{1/2} M_k)$ is completely positive. Moreover, the convex compositions of completely positive maps are completely positive. Therefore, Φ is completely positive. □

Choi's theorem says that there is a canonical choice of an operator, whose positivity implies that the map Φ is CP. The equivalence $(1 \leftrightarrow 3)$ between a completely positive map and a positive operator is known as *Choi-Jamiołkowski isomorphism.*

Since $\left|\mathrm{i}\right\rangle \left\langle \mathrm{j}\right|$ is an operator basis of $M_n(\mathbb{C})$, (3.38) can be written as

$$\Phi(a) = \sum_k M_k^\dagger a M_k \tag{3.39}$$

for $a \in \mathcal{A}$ where we absorbed λ_k into M_K, i.e. $\lambda_k^{1/2} M_k \to M_k$. This is the *Kraus representation* and is proved as a result of Choi's theorem.

Theorem 3.2.2 (Kraus representation)**.** *Let $\mathcal{A}$ be a $C*$-algebra on a Hilbert space $\mathcal{K}$. A map $\Phi : \mathcal{A} \to B(\mathcal{K})$ is a completely positive and $0 \leq \mathrm{tr}(\Phi(b_+)) \leq 1$ for any normalized positive operator, i.e., $tr(b_+) = 1$ for $b_+ \in \mathcal{A}$, if and only if*

$$\begin{aligned} \Phi(a) &= \sum_k M_k^\dagger a M_k \\ \sum_k M_k^\dagger M_k &\leq \mathbb{I}_{\mathcal{H}} \end{aligned} \tag{3.40}$$

for all $a \in \mathcal{A}$. For an unital CP map, i.e., $\Phi(\mathbb{I}) = \mathbb{I}$, $\sum_k M_k^\dagger M_k = \mathbb{I}$.

The Kraus representation of a CP map is non-unique. Let us introduce an auxiliary Hilbert space $\mathcal{K}_R$ with an orthonormal basis $\{|k\rangle\}$. A CP map can be written as

$$\begin{aligned} \Phi(a) &= \sum_k M_k^\dagger a M_k = V^\dagger(\mathbb{I}_R \otimes a)V \\ V &= \sum_k |k\rangle \otimes M_k. \end{aligned} \tag{3.41}$$

Let us define $\tilde{V} = \sum_k U_R |k\rangle \otimes M_k$. We can see that

$$\Phi(a) = V^\dagger(\mathbb{I}_R \otimes a)V = \tilde{V}^\dagger(\mathbb{I}_R \otimes a)\tilde{V}. \tag{3.42}$$

Since $M_k = (\langle k| \otimes \mathbb{I})V$, for $\tilde{V}$, we get another Kraus operator

$$\tilde{M}_k = (\langle k| \otimes \mathbb{I})\tilde{V} = \sum_r U_{R,kr} M_r. \tag{3.43}$$

In physical applications, an unital CP map represents dynamics in an open quantum system. For example, consider a CP map on a matrix algebra $M_n(\mathbb{C})_S$ of a system. The CP map can be expressed with Kraus operators defined by a unitary U on the algebra of a system and a reservoir system, $M_n(\mathbb{C})_R \otimes M_n(\mathbb{C})_S$, as follows.

$$\Phi(a) = \mathrm{tr}_R(U^\dagger(\mathbb{I}_R \otimes a)U) = \sum_{kl} \langle l|U^\dagger|k\rangle\, a\, \langle k|U|l\rangle = \sum_{kl} N_{kl}^\dagger a N_{kl}. \tag{3.44}$$

where $N_{kl} = \langle k|U|l\rangle$[17]. Again, non-uniqueness comes from the freedom of local unitary dynamics in the reservoir. That is, for a unitary U_R on a reservoir,

$$\tilde{N}_{kl} = \langle k|(U_R \otimes \mathbb{I}_S)U|l\rangle = \sum_m \langle k|(U_R \otimes \mathbb{I}_S)\,|m\rangle\, \langle m|\, U|l\rangle = U_{R,km} N_{ml}. \tag{3.45}$$

Physically, the non-uniqueness of the Kraus representation describes that different physical global processes on a system and a reservoir can provide the same process on the system[18].

[17]↑We can put the indices k, l together as $s = \{k, l\}$ without loss of generality so that $\Phi(a) = \sum_s N_s^\dagger a N_s$.
[18]↑For the simple example, see chapter 8 in [46].

Because CP maps describe open quantum dynamics, they can model errors in the quantum error correction conditions.

The representations of CP maps and the above discussions are summarized and generalized into so-called *Stinespring representation*.

Theorem 3.2.3 (Stinespring representation)**.** *Let $\mathcal{A}$ be a C^*-algebra*[19] *and $\Phi : \mathcal{A} \to B(\mathcal{K})$ a completely positive map. Then, there exists a Hilbert space $\hat{\mathcal{H}}$, an unital $*$-representation $\boldsymbol{\pi} : \mathcal{A} \to B(\hat{\mathcal{H}})$ and a linear map $V : \mathcal{K} \to \hat{\mathcal{H}}$ such that*

$$\Phi(a) = V^\dagger \boldsymbol{\pi}(a) V \tag{3.46}$$

for every $a \in \mathcal{A}$. In particular, $\|\phi\| = \|V\|^2 = \|V^\dagger V\| = \Phi(\mathbb{I})$.

If Φ is unital, i.e., $\Phi(\mathbb{I}_{\mathcal{A}}) = \mathbb{I}_{B(\mathcal{H})}$, then V is an isometry and $VV^\dagger = P$ is a projection onto $P\hat{\mathcal{H}} \subset \hat{\mathcal{H}}$.

We provide the proof for an unital CP map. We consider representations of $\mathcal{A} \otimes B(\mathcal{K})$. Choose two vectors $|\phi\rangle$ and $|\psi\rangle$ in $\mathcal{K}$. Given a CP map, we can define a new inner product[20]:

$$\langle a_1, \phi | a_2, \psi \rangle_\Phi \equiv \langle \Phi(a_2^\dagger a_1)\phi | \psi \rangle = \langle \phi | \Phi(a_1^\dagger a_2) | \psi \rangle \ . \tag{3.48}$$

If there are $a \in \mathcal{A}$ such that $\Phi(a^\dagger a) = 0$ then the resulting vector $|a, \phi\rangle$ has zero norm. We quotient by such zero norm vectors to obtain the Hilbert space $\hat{\mathcal{H}}$. When Φ is faithful $\hat{\mathcal{H}} = \mathcal{H}_A \otimes \mathcal{K}$ and the representation $\boldsymbol{\pi}(a) = a \otimes \mathbb{I}_{A'\mathcal{K}}$. The isometry $V : \mathcal{K} \to \mathcal{H}_A \otimes \mathcal{K}$ acts as

$$\begin{aligned} V\,|\phi\rangle &= |\mathrm{e}, \phi\rangle \\ \boldsymbol{\pi}(a_1)\,|a_2, \phi\rangle &= |a_1 a_2, \phi\rangle \ . \end{aligned} \tag{3.49}$$

[19] ↑or a vN algebra

[20] ↑The standard inner product is the special case when the CP map is $\Phi(a) = \mathrm{tr}(a)$. It leads to the Hilbert space $\mathcal{H}_A \otimes \mathcal{K}$:

$$\langle a_1, \phi | a_2, \psi \rangle = \mathrm{tr}(a_1^\dagger a_2)\,\langle \phi | \psi \rangle = \langle a_1 | a_2 \rangle\,\langle \phi | \psi \rangle \ . \tag{3.47}$$

From the inner product in (3.48) it follows that $V^\dagger$ acts as

$$V^\dagger |a, \phi\rangle = \Phi(a) |\phi\rangle . \tag{3.50}$$

As a result, the CP map factors as

$$\Phi(a) = V^\dagger \pi(a) V \ . \tag{3.51}$$

Note that the projection $P = VV^\dagger$ satisfies

$$\begin{aligned} P |a, \phi\rangle &= (\Phi(a) \otimes \mathbb{I}) |1, \phi\rangle \\ P(a \otimes \mathbb{I})P &= \Phi(a) \otimes |\mathrm{e}\rangle \langle \mathrm{e}| \ . \end{aligned} \tag{3.52}$$

Note that Stinespring representation admits freedom in the choice of $\hat{\mathcal{H}}$. When $\{\pi(a)V\mathcal{K}\}$ is dense in $\mathcal{H}$, the Stinespring representation is called the *minimal Stinespring representation*. Among different minimal Stinespring representations, for instance, $(\pi_1, V_1, \mathcal{H}_1)$ and $(\pi_2, V_2, \mathcal{H}_2)$, there always exists a unitary that relates them, i.e.,

$$U : \mathcal{H}_1 \to \mathcal{H}_2, \ UV_1 = V_2, \ U\pi_1 U^\dagger = \pi_2. \tag{3.53}$$

For non minimal Stinespring representations $(\pi, V, \mathcal{H})$, $\pi(a)V\mathcal{K}$ are only dense in a subspace $\tilde{\mathcal{H}}$ of $\mathcal{H} = \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}^\perp$ where $\tilde{\mathcal{H}}^\perp$ is the orthogonal subspace of $\tilde{\mathcal{H}}$. In this thesis, we only consider minimal Stinespring representations.

Completely positive maps $\Phi : \mathcal{A} \to \mathcal{B}$ satisfy so-called the *Schwartz inequality*,

$$\Phi(a_1)\Phi(a_2) \leq \Phi(a_1 a_2) \tag{3.54}$$

for $a_1, a_2 \in \mathcal{A}$.

For a given CP map $\Phi : \mathcal{A} \to \mathcal{A}$ of a C^*-algebra[21], we study two subalgebras; i) invariant algebra $\mathcal{A}_I$, and ii) multiplicative domain $\mathcal{A}_M$. In general, $\mathcal{A}_I \subset \mathcal{A}_M \subset \mathcal{A}$, and they are defined by

$$\begin{aligned} \mathcal{A}_I &\equiv \{c | \Phi(c) = c,\ \forall c \in \mathcal{A}\} \\ \mathcal{A}_M &\equiv \{m | \Phi(m_1 m_2^\dagger) = \Phi(m_1)\Phi(m_2)^\dagger, \Phi(m_1^\dagger m_2) = \Phi(m_1)^\dagger \Phi(m_2),\ \forall m_1, m_2 \in \mathcal{A}\} \end{aligned} \tag{3.55}$$

The action of Φ restricted to the invariant algebra $\mathcal{A}_I$ and the multiplicative domain $\mathcal{A}_M$ saturates the Schwartz inequality by its definition. The invariant subalgebra $\mathcal{A}_I$ satisfies the so-called *bimodule property*, i.e., for $c_1, c_2 \in \mathcal{A}_I$, and $a \in \mathcal{A}$

$$\Phi(c_1 a c_2) = c_1 \Phi(a) c_2. \tag{3.56}$$

The multiplicative domain of Φ also satisfies the bimodule property: for all $m \in \mathcal{A}^M$ and all $a \in \mathcal{A}$ we have:

$$\begin{aligned} \Phi(m^\dagger a) &= \Phi(m^\dagger)\Phi(a) \\ \Phi(a^\dagger m) &= \Phi(a^\dagger)\Phi(m)\ . \end{aligned} \tag{3.57}$$

To prove this, we use the fact that $\Phi^{(2)} = \Phi \otimes \mathrm{id}_2$ is also a CP map that satisfies the Schwartz inequality. Consider the operator $X \in \mathcal{A} \otimes M_2$ (M_2 is the algebra of complex 2×2 matrices)

$$X = \begin{pmatrix} 0 & m^\dagger \\ m & a \end{pmatrix} \tag{3.58}$$

for some $a \in \mathcal{A}$ and $c \in \mathcal{A}^M$. The Schwarz inequality gives

$$\begin{aligned} \begin{pmatrix} \Phi(m^\dagger m) & \Phi(m^\dagger a) \\ \Phi(a^\dagger m) & \Phi(mm^\dagger + a^\dagger a) \end{pmatrix} = \Phi^{(2)}(X^\dagger X) \quad &\geq \Phi^{(2)}(X^\dagger)\Phi^{(2)}(X) \\ &= \begin{pmatrix} \Phi(m^\dagger)\Phi(m) & \Phi(m^\dagger)\Phi(a) \\ \Phi(a^\dagger)\Phi(m) & \Phi(m)\Phi(m^\dagger) + \Phi(a^\dagger)\Phi(a) \end{pmatrix} \end{aligned} \tag{3.59}$$

[21] ↑or a vN algebra

This implies that

$$\begin{pmatrix} 0 & \Phi(m^\dagger a) - \Phi(m^\dagger)\Phi(a) \\ \Phi(a^\dagger m) - \Phi(a^\dagger)\Phi(m) & \Phi(a^\dagger a) - \Phi(a^\dagger)\Phi(a) \end{pmatrix} \geq 0 \tag{3.60}$$

which is possible if and only if its off-diagonal terms are exactly zero which proves (3.57).

3.2.2 Conditional expectations

In this section, we focus on a special class of unital CP maps, *conditional expectations*.

Definition 3.2.5 (Conditional expectation)**.** *Let $\mathcal{A}_I$ be $*$-subalgebra of another $*$-subalgebra $\mathcal{A} \supset \mathcal{A}_I$ on $B(\mathcal{K})$. A completely positive map $\mathcal{E} : \mathcal{A} \to \mathcal{A}_I$ is said to be a conditional expectation when*

1. *Unital: $\mathcal{E}(\mathbb{I}) = \mathbb{I}$;*
2. *Bimodule property: $\mathcal{E}(c_1 a c_2) = c_1 \mathcal{E}(a) c_2$ for $c_1, c_2 \in \mathcal{A}_I$ and $a \in \mathcal{A}$.*

Consider a $d \times d$ complex matrix algebra $\mathcal{A} \equiv M_d(\mathbb{C}) \otimes M_d(\mathbb{C})$ and its subalgebra $\mathcal{A}_I \equiv M_d(\mathbb{C}) \otimes \mathbb{I}$. A simple example of a conditional expectation $\mathcal{E} : \mathcal{A} \to \mathcal{A}_I$ is

$$\mathcal{E}(x_1 \otimes x_2) = x_1 \otimes \mathbb{I}\ \mathrm{tr}(x_2/d) \tag{3.61}$$

for $x_1 \otimes x_2 \in \mathcal{A}$ and $x_1 \otimes \mathbb{I} \in \mathcal{A}_I$ where $x_1, x_2 \in M_d(\mathbb{C})$. It is obviously unital. We can check its bimodule property;

$$\mathcal{E}\Big((x_1 \otimes \mathbb{I})(x_2 \otimes x_3)(x_4 \otimes \mathbb{I})\Big) = x_1 x_2 x_4 \otimes \mathbb{I}\,\mathrm{tr}(x_3/d) = (x_1 \otimes \mathbb{I})\mathcal{E}(x_2 \otimes x_3)(x_4 \otimes \mathbb{I}). \tag{3.62}$$

A conditional expectation is called ω-preserving conditional expectation $\mathcal{E}_\omega : \mathcal{A} \to \mathcal{A}_I$ if it preserves the density matrix ω or a state $\mathrm{tr}(\omega(\cdot))$, i.e., for all $a \in \mathcal{A}$,

$$tr(\omega \mathcal{E}_\omega(a)) = tr(\omega a). \tag{3.63}$$

The conditional expectation in (3.61) preserves the maximally mixed density matrix $\mathbb{I}/d$. Below, we discuss ω-preserving conditional expectation $\mathcal{E}_\omega$ in matrix algebras.

We start with a CP map $\iota_\omega : \mathcal{A}_1 \to \mathcal{A}_1 \otimes \mathcal{A}_2$ given by

$$\iota_\omega(a) = a \otimes \omega, \tag{3.64}$$

where ω is a positive operator with eigenvectors $\{|k\rangle\}$ and eigenvalues λ_k^2. The Stinespring dilation of this map factorizes as a representation on $\mathcal{K}_1 \otimes \mathcal{K}_3$ and the isometry $W : \mathcal{K}_1 \otimes \mathcal{K}_2 \to \mathcal{K}_1 \otimes \mathcal{K}_3$:

$$\begin{aligned} \iota_\omega(a) &= W^\dagger (a \otimes \mathbb{I}_3) W \\ W &= \sum_k \lambda_k (\mathbb{I}_1 \otimes |k\rangle_3 \langle k|_2) \\ \mathbb{I}_3 &= \sum_k |k\rangle_3 \langle k|_3 \ . \end{aligned} \tag{3.65}$$

The Kraus operators are $V_k = \lambda_k(\mathbb{I}_1 \otimes \langle k|_2)$. The dual map $\iota_\omega^* : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ is

$$\iota_\omega^*(a_1 \otimes a_2) = \sum_k V_k (a_1 \otimes a_2) V_k^\dagger = a_1 \operatorname{tr}(\omega a_2), \tag{3.66}$$

with the Stinespring dilation

$$\begin{aligned} \iota_\omega^*(a_1 \otimes a_2) &= W^\dagger (a_1 \otimes a_2 \otimes \mathbb{I}_3) W \\ W &= \sum_k \lambda_k (\mathbb{I}_1 \otimes |kk\rangle_{23}) \end{aligned} \tag{3.67}$$

The map ι_ω is unital when $\omega = \mathbb{I}_2$. In this case, it is an embedding of $\mathcal{A}_1$ in $\mathcal{A}_1 \otimes \mathcal{A}_2$:

$$\iota_1(a_1 a_2) = \iota_1(a_1)\iota_1(a_2) \ . \tag{3.68}$$

The dual ι_1^* is a quantum channel (trace-preserving CP map) $\mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1$ that is partial trace over $\mathcal{A}_2$:

$$\begin{aligned} &\mathrm{tr}(\rho_{12}\, \iota_1(a)) = \mathrm{tr}(\iota_1^*(\rho_{12})a) \\ &\iota_1^*(\rho_{12}) = (\mathbb{I}_1 \otimes \langle \mathrm{e}|_{23})\rho_{12}(\mathbb{I}_1 \otimes |\mathrm{e}\rangle_{23}) = \rho_1 \ . \end{aligned} \tag{3.69}$$

The map ι_ω is a quantum channel (CP and trace-preserving) when ω is a density matrix: $\mathrm{tr}(\omega) = 1$. This channel prepares a density matrix ω on $\mathcal{K}_2$. The composition of two CP maps is also a CP map. For instance, the composite map $\iota_\omega^* \circ \iota_\omega(a_1) = a_1 \,\mathrm{tr}(\omega)$ multiplies operators by a positive constant, whereas $\iota_\omega \circ \iota_\omega^*(a_1 \otimes a_2) = (a_1 \otimes \omega)\,\mathrm{tr}(\omega a_2)$. An important composite map for us is

$$\begin{aligned} &\mathcal{E}_\omega \equiv \iota_1 \circ \iota_\omega^* : \mathcal{A}_1 \otimes \mathcal{A}_2 \to \mathcal{A}_1 \otimes \mathbb{I}_2 \\ &\mathcal{E}_\omega(a_1 \otimes a_2) = (a_1 \otimes \mathbb{I}_2)\,\mathrm{tr}(\omega a_2) \ . \end{aligned} \tag{3.70}$$

It has the property that when ω is a density matrix it leaves the subalgebra $\mathcal{A}_1 \otimes \mathbb{I}_2$ invariant

$$\mathcal{E}_\omega(a_1 \otimes \mathbb{I}_2) = a_1 \otimes \mathbb{I}_2 \ . \tag{3.71}$$

It is the simplest example of a ω-preserving conditional expectation [3].

The conditional expectations in (3.70) are labeled by density matrices ω on $\mathcal{A}_2$. In fact, these are the only conditional expectations from $\mathcal{A}_1 \otimes \mathcal{A}_2$ to $\mathcal{A}_1 \otimes \mathbb{I}_2$. To see this, we use the bimodule property:

$$\mathcal{E}(a_1 \otimes a_2) = (a_1 \otimes \mathbb{I})\mathcal{E}(\mathbb{I} \otimes a_2) = \mathcal{E}((\mathbb{I} \otimes a_2)(a_1 \otimes \mathbb{I})) = \mathcal{E}(\mathbb{I} \otimes a_2)(a_1 \otimes \mathbb{I}) \ . \tag{3.72}$$

Therefore, $\mathcal{E}(\mathbb{I} \otimes a_2)$ commutes with all $a_1 \otimes \mathbb{I}$ and has to take the form

$$\mathcal{E}_\epsilon(a_1 \otimes a_2) = (a_1 \otimes \mathbb{I})\,\epsilon(a_2), \tag{3.73}$$

where $\epsilon(a_2)$ is an unital CP map from $\mathcal{A}_2 \to \mathbb{C}$ which is in one-to-one correspondence with density matrices on $\mathcal{A}_2$:[22]

$$\mathcal{E}_\omega(a_1 \otimes a_2) = (a_1 \otimes \mathbb{I})\ \mathrm{tr}(\omega a_2)\ . \tag{3.74}$$

The conditional expectation $\mathcal{E}_\omega$ preserves all states of the form $\psi \otimes \omega$. Moreover, given a product state $\psi \otimes \omega$ the conditional expectation $\mathcal{E}_\omega$ that preserves it is unique. However, for a generic ω_{12} there does not exist a conditional expectation that preserves it.

To gain more intuition about conditional expectations $\mathcal{E} : \mathcal{A} \to \mathcal{A}_I$ in finite-dimensional matrix algebras consider their Kraus representation $\mathcal{E}(a) = \sum_r V_r^\dagger a V_r$. The Hilbert space $\mathcal{K}$ decomposes as $\mathcal{K} = \oplus_q \mathcal{K}_1^q \otimes \mathcal{K}_2^q$ such that

$$\begin{aligned} c &= \oplus_q c^q \otimes \mathbb{I}_2^q \qquad \forall c \in \mathcal{A}_I \\ V_r &= \oplus_q \mathbb{I}_1^q \otimes V_r^q \qquad \forall r\ . \end{aligned} \tag{3.75}$$

A conditional expectation $\mathcal{E}$ projects every operator in $\mathcal{A}$ to its invariant subalgebra $\mathcal{A}_I$. Denote the projection to the subspace $\mathcal{K}_1^q \otimes \mathcal{K}_2^q$ by P^q. Since $P^q \in \mathcal{A}^C$ from the bi-module property (3.2.5) we have [47]

$$\begin{aligned} \mathcal{E}(a) &= \mathcal{E}\left(\sum_{q'q} P^{q'} a P^q\right) = \sum_q P^q \mathcal{E}(a) P^q = \sum_q \mathcal{E}^q(a) \\ \mathcal{E}^q(a) &= \mathcal{E}\left(P^q a P^q\right), \end{aligned} \tag{3.76}$$

where we have used $P^{q'} c P^q = \delta_{q'q} c^q$ for all $c \in \mathcal{A}_I$. As a result, every conditional expectation $\mathcal{E} : \mathcal{A} \to \mathcal{A}_I$ decomposes as a sum of conditional expectations $\mathcal{E}^q : B(\mathcal{K}_1^q \otimes \mathcal{K}_2^q) \to B(\mathcal{K}_1^q) \otimes \mathbb{I}_2^q$. However, we already showed that the conditional expectations $\mathcal{E}^q$ are labelled by density matrices ω_2^q:

$$\mathcal{E}_\omega^q(a_1^q \otimes a_2^q) = \mathrm{tr}_2\left((\mathbb{I}_1^q \otimes \omega_2^q)(a_1^q \otimes a_2^q)\right)\ . \tag{3.77}$$

[22] $\uparrow \epsilon(a_2)$ is a continuous linear functional on $\mathcal{A}_2$ which by Riesz representation theorem can be associated with a unique vector $|\epsilon\rangle \in \mathcal{K}_2$ such that $\epsilon(a_2) = \langle \epsilon | a_2 \rangle$.

As a result, the conditional expectations from $\mathcal{A} \to \mathcal{A}_I$ are in one-to-one correspondence with unnormalized states $\omega = \oplus_q \mathbb{I}_1^q \otimes \omega_2^q$ on the commutant $(\mathcal{A}_I)'$:

$$\mathcal{E}_\omega(a) = \mathrm{tr}_2(\omega a) \otimes \mathbb{I}_2 = \oplus_q \mathrm{tr}_2 \left((\mathbb{I}_1^q \otimes \omega_2^q) P^q a P^q \right) \otimes \mathbb{I}_2^q \ . \tag{3.78}$$

This conditional expectation preserves every state of the form $\psi = \oplus_q p_q \psi_1^q \otimes \omega_2^q$:

$$\mathrm{tr}(\psi \mathcal{E}_\omega(a)) = \sum_q \mathrm{tr}(\psi \mathcal{E}_\omega^q(a)) = \sum_q p_q \, \mathrm{tr} \left((\psi_1^q \otimes \omega_2^q) a \right) = \mathrm{tr}(\psi a) \ . \tag{3.79}$$

If a state does not have the form we postulated for ψ there exists no conditional expectation that preserves it. The restriction of the state ψ to the subalgebra $\mathcal{A}_I$ is

$$\psi_0 = \oplus_q p_q \psi_1^q \otimes \mathbb{I}_2^q \ . \tag{3.80}$$

The discussion above was restricted to finite-dimensional matrix algebras. In theorem 4.4.3, we show that the necessary and sufficient condition for the existence of a ψ-preserving conditional expectation is

$$\psi^{1/2} c \psi^{-1/2} = \psi_0^{1/2} c \psi_0^{-1/2} \ . \tag{3.81}$$

This condition holds trivially for ω and ω_0 in the example above.

3.3 States and representations of von Neumann algebras

In this section, we discuss how to represent a von Neumann algebra on a Hilbert space. For a representation of a von Neumann algebra $\mathcal{A}$, we input a pair $(\mathcal{A}, \omega)$ of a von Neumann algebra $\mathcal{A}$ and a state ω (3.12), and obtain three objects; i) a linear map $\pi_\omega : \mathcal{A} \to \pi_\omega(\mathcal{A})$, ii) a representation space $\mathcal{H}_\omega$, and iii) "vacuum" state $|\Omega_\omega\rangle$. For any $*$-algebra, *GNS construction* is a canonical way to construct the representation of $\mathcal{A}$. Before we discuss the GNS construction, let us briefly study the linear map π_ω and the representation space $\mathcal{H}_\omega$[23].

[23]↑Since the discussion is generally true for any choice of states, we omit the subscript of the linear map and the representation space in the next section. It will revive when we study the GNS construction.

3.3.1 Basics of representations

To represent C^*-algebras and vN algebras $\mathcal{A}$, we want to at least preserve its algebraic structure in $\boldsymbol{\pi}(\mathcal{A})$. Hence, the linear map $\boldsymbol{\pi}$ should satisfy the following properties. For a linear map $\boldsymbol{\pi} : a \in \mathcal{A} \mapsto \boldsymbol{\pi}(a) \in \boldsymbol{\pi}(\mathcal{A})$,

1. $\boldsymbol{\pi}(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 \boldsymbol{\pi}(a_1) + \alpha_2 \boldsymbol{\pi}(a_2)$

2. $\boldsymbol{\pi}(a_1 a_2) = \boldsymbol{\pi}(a_1)\boldsymbol{\pi}(a_2)$

3. $\boldsymbol{\pi}(a_1^\dagger) = \boldsymbol{\pi}(a_1)^\dagger$

where $a_1, a_2 \in \mathcal{A}$ and $\alpha_1, \alpha_2 \in \mathbb{C}$. Such a map is a $*$-homomorphism in which we found a completely positive map in (3.2).

In addition to the above, we introduce the kernel

$$\ker\boldsymbol{\pi} := \{a | \boldsymbol{\pi}(a) = 0,\ a \in \mathcal{A}\}. \tag{3.82}$$

If $*$-homomorphism $\boldsymbol{\pi}$ has a non-trivial kernel, all the operators in the kernel of $\boldsymbol{\pi}$ are eliminated. We want the representation to be one-to-one. This holds if and only if $\ker\boldsymbol{\pi} = \{0\}$. In such a case, $\boldsymbol{\pi}$ is known as a *faithful representation*.

Cyclic representation is another one that is defined by the linear map $\boldsymbol{\pi}$, the Hilbert space $\mathcal{H}$, and, in addition, a *cyclic vector* $|\Omega\rangle$. A vector $|\Omega\rangle$ is cyclic for $\boldsymbol{\pi}(\mathcal{A})$ if $\{\boldsymbol{\pi}(\mathcal{A})\,|\Omega\rangle\}$ is dense in $\mathcal{H}$[24]. Roughly speaking, it means that one can find a vector from the set $\{\boldsymbol{\pi}(\mathcal{A})\,|\Omega\rangle\}$ that can arbitrarily well approximate any vector in the Hilbert space $\mathcal{H}$ [25].

Definition 3.3.1 (Cyclic representation)**.** *A cyclic representation of a C^*-algebra*[26] *is defined by the triple $(\boldsymbol{\pi}, \mathcal{H}, |\Omega\rangle)$ where $(\mathcal{H}, \boldsymbol{\pi})$ is a representation of $\mathcal{A}$ and $|\Omega\rangle$ is a vector in $\mathcal{H}$ which is cyclic for $\boldsymbol{\pi}(\mathcal{A})$.*

[24]↑$\{\boldsymbol{\pi}(\mathcal{A})\,|\Omega\rangle\}$ is dense in $\mathcal{H}$ if the closure of $\{\boldsymbol{\pi}(\mathcal{A})\,|\Omega\rangle\}$ with respect to the L^2-norm of $\mathcal{H}$ is equivalent to $\mathcal{H}$.
[25]↑In the context of quantum field theory, the cyclic vector provides a crucial property in the application to local QFT known as *Reeh-Schlieder theorem*.
[26]↑or a vN algebra

3.3.2 Gelfand-Naimark-Segal(GNS) construction

We first give the definition of a *state*.

Definition 3.3.2 (State). *A state ω on a C^* algebra*[27] *$\mathcal{A}$ is an unital positive linear functional on $\mathcal{A}$, i.e.,*

1. *Linear: for $a_1, a_2 \in \mathcal{A}$, and $\alpha_1, \alpha_2 \in \mathbb{C}$,*

$$\omega(\alpha_1 a_1 + \alpha_2 a_2) = \alpha_1 \omega(a_1) + \alpha_2 \omega(a_2) \tag{3.83}$$

2. *positivity: for $a \in \mathcal{A}$,*

$$\omega(a^\dagger a) \geq 0 \tag{3.84}$$

3. *normalization:*

$$\|\omega\| \equiv \sup_{a \in \mathcal{A}, \|a\| \leq 1} |\omega(a)| = 1 \tag{3.85}$$

or, equivalently, for $\mathbb{I} \in \mathcal{A}$,

$$\omega(\mathbb{I}) = 1. \tag{3.86}$$

Theorem 3.3.1 (GNS representation). *For any state ω on a C^*-algebra*[28] *$\mathcal{A}$, there exists a Hilbert space $\mathcal{H}_\omega$, a representation $\boldsymbol{\pi}_\omega$ of $\mathcal{A}$ on $\mathcal{H}_\omega$, and a unit vector state $|\Omega_\omega\rangle$, i.e., $\langle\Omega_\omega|\Omega_\omega\rangle = 1$, that satisfy the following conditions;*

1. *for any $a \in \mathcal{A}$,*

$$\omega(a) = \langle\Omega_\omega|\boldsymbol{\pi}_\omega(a)\Omega_\omega\rangle\,; \tag{3.87}$$

2. *$|\Omega_\omega\rangle$ is a cyclic vector for $\boldsymbol{\pi}_\omega(a) \in B(\mathcal{H}_\omega)$, i.e.,*

$$\boldsymbol{\pi}_\omega(\mathcal{A})\,|\Omega_\omega\rangle \equiv \{\boldsymbol{\pi}_\omega(a)\,|\Omega_\omega\rangle \mid a \in \mathcal{A}\} \tag{3.88}$$

is dense in a Hilbert space $\mathcal{H}_\omega$.

[27] ↑or a vN algebra
[28] ↑or a vN algebra

Any triple $(\mathcal{H}_\omega, \pi_\omega, |\Omega\rangle)$ *is unique up to unitary. That is, consider another triple* $(\tilde{\mathcal{H}}_\omega, \tilde{\pi}_\omega, |\tilde{\Omega}_\omega\rangle)$ *that satisfies the above two conditions. Then, there exists a unitary* U *such that, for* $a \in \mathcal{A}$*,*

$$U\pi_\omega(a) = \tilde{\pi}_\omega(a)U,\ U\,|\Omega_\omega\rangle = |\tilde{\Omega}_\omega\rangle\,. \tag{3.89}$$

We will not give the full proof of theorem 3.3.1 because the GNS representation is the special case of minimal Stinspring representation (3.46). Recall that the state is a CP map. Hence, by applying theorem 3.2.3, for $a \in \mathcal{A}$ and $\omega : \mathcal{A} \to \mathbb{C}$, there exists $V : \mathbb{C} \to \mathcal{H}_\omega$[29] and $\pi_\omega : \mathcal{A} \to B(\mathcal{H}_\omega)$ such that

$$\omega(a) = V^\dagger \pi_\omega(a) V = \langle \Omega_\omega | \pi_\omega(a) \Omega_\omega \rangle \tag{3.90}$$

where $V = |\Omega_\omega\rangle$. Cyclicity of the vector $|\Omega_\omega\rangle$ comes from the fact that the GNS representation is the minimal Stinespring representation. That is, $\{\pi_\omega(\mathcal{A})V\} = \{\pi_\omega(\mathcal{A})\,|\Omega_\omega\rangle\}$ is dense in $\mathcal{H}_\omega$.

If the kernel of the state is trivial, it is called a *faithful state*.

Definition 3.3.3 (faithful state)**.** *A state* ω *on a* C^**-algebra or vN algebra* $\mathcal{A}$ *is faithful if* $\omega(a_+) > 0$ *for all nonzero* $a_+ \in \mathcal{A}_+$*.*

In GNS representation, ω is faithful if and only if the vector $|\Omega_\omega\rangle$ is separating for A. Thus, if we construct the GNS representation with a faithful state, the GNS vacuum is cyclic and separating.

In vN algebras, there is a special class of states called *normal states*.

Theorem 3.3.2 (normal state)**.** *Let* ω *be a state on a von Neumann algebra acting on a Hilbert space* $\mathcal{H}$*. The following conditions are equivalent:*

1. ω *is normal;*

2. ω *is* σ*-weakly continuous;*

[29]↑Below theorem 3.2.3 in section 3.2, the domain of V was denoted as $\mathcal{K}$. In GNS representation, obviously, $\mathcal{K} = \mathbb{C}$.

3. *there exists a density matrix* ω*, i.e., a positive trace-class operator* ω *on* $\mathcal{H}$ *with* $\mathrm{tr}(\rho_\omega) = 1$*, such that*

$$\omega(a) = \mathrm{tr}(\rho_\omega a) \tag{3.91}$$

As we will see in section 3.5, the GNS representation of a von Neumann algebra with respect to a faithful normal state provides us deep and beautiful theory known as *Tomita-Takesaki theory* or *modular theory*.

3.3.3 Examples: finite dimensional von Neumann algebras

In this section, we observe the GNS representation of a $d \times d$ matrix algebra over complex number $\mathbb{C}$ on a Hilbert space $\mathcal{K}$. We denote $\mathcal{A}$ as the $*$-subalgebra of $d \times d$ complex matrix algebra $B(\mathcal{K})$ on Hilbert space $\mathcal{K}$. Physically, $\mathcal{A}$ corresponds to the algebra of a d-level quantum system.

Maximally mixed state

For the representation of $\mathcal{A}$, consider a maximally mixed state $\mathrm{tr}(\mathbb{I}/d(\cdot))$. Then, the operators $a \in \mathcal{A}$ are represented by the vectors $|a\rangle \in \mathcal{H}_{\mathrm{tr}}$

$$a \mapsto |a\rangle \equiv \boldsymbol{\pi}_{\mathrm{tr}}(a)\,|\mathrm{e}\rangle = a \otimes \mathbb{I}\,|\mathrm{e}\rangle \tag{3.92}$$

where $|\mathrm{e}\rangle = \sum_{\mathrm{i}} \frac{1}{\sqrt{d}}\,|\mathrm{ii}\rangle$. Similarly, operators in the commutant $a' \in \mathcal{A}'$ are represented by the vectors $|a'\rangle_{\mathrm{tr}} \in \mathcal{H}_{\mathrm{tr}}$

$$a \mapsto |a'\rangle \equiv \boldsymbol{\pi}_{\mathrm{tr}}(a')\,|\mathrm{e}\rangle = \mathbb{I} \otimes a'\,|\mathrm{e}\rangle\,. \tag{3.93}$$

One can easily check that

$$\frac{1}{d}\,\mathrm{tr}(a) = \frac{1}{d}\,\langle \mathrm{e}|a \otimes \mathbb{I}|\mathrm{e}\rangle = \frac{1}{d}\sum_{\mathrm{i}} \langle \mathrm{i}|a|\mathrm{i}\rangle\,. \tag{3.94}$$

Moreover, it is simple to see that $\mathcal{H}_{\mathrm{tr}}$ is equipped with the inner product

$$\langle a|b\rangle_{\mathrm{tr}} = \frac{1}{d}\,\langle \mathrm{e}|a^\dagger b \otimes \mathbb{I}|\mathrm{e}\rangle = \frac{1}{d}\,\mathrm{tr}(a^\dagger b). \tag{3.95}$$

Density matrix

For the representation of $\mathcal{A}$, consider a state $\mathrm{tr}(\omega(\cdot))$ with a density matrix ω[30]. Then, the operators $a \in \mathcal{A}$ are represented by the vectors $|a\rangle_\omega \in \mathcal{H}_\omega$

$$a \mapsto |a\rangle \equiv \boldsymbol{\pi}_\omega(a)\,|\Omega_\omega\rangle = a \otimes \mathbb{I}\,|\omega^{1/2}\rangle \tag{3.96}$$

where $|\Omega_\omega\rangle = |\omega^{1/2}\rangle = \omega^{1/2} \otimes \mathbb{I}\,|\mathrm{e}\rangle$[31]. Similarly, operators in the commutant $a' \in \mathcal{A}'$ are represented by the vectors $|a'\rangle_\omega \in \mathcal{H}_{\mathrm{tr}}$

$$a \mapsto |a'\rangle \equiv \boldsymbol{\pi}_\omega(a')\,|\Omega_\omega\rangle = \mathbb{I} \otimes a'\,|\omega^{1/2}\rangle\,. \tag{3.97}$$

One can easily check that

$$\mathrm{tr}(\omega a) = \langle \omega^{1/2}|a \otimes \mathbb{I}|\omega^{1/2}\rangle\,. \tag{3.98}$$

Then, it is simple to see that $\mathcal{H}_\omega$ is equipped with the inner product

$$\langle a|b\rangle_\omega = \langle \omega^{1/2}|a^\dagger b \otimes \mathbb{I}|\omega^{1/2}\rangle = \mathrm{tr}(\omega a^\dagger b). \tag{3.99}$$

3.4 Representing superoperators as the operators on a GNS Hilbert space

Consisder von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with faithful normal states ω_A on $\mathcal{A}$ and ω_B on $\mathcal{B}$. Let $\mathcal{H}_A \equiv \mathcal{H}_{\omega_A}$ and $\mathcal{H}_B \equiv \mathcal{H}_{\omega_B}$ are the GNS Hilbert spaces. Suppose $\Phi : \mathcal{A} \to \mathcal{B}$ is a CP map. In this section, we construct the *GNS operator* $F : \mathcal{H}_A \to \mathcal{H}_B$ by the Stinespring representation of Φ. This provides the map from a superoperator to an operator on the GNS Hilbert space, i.e.,

$$\Phi : \mathcal{A} \to \mathcal{B} \mapsto F : \mathcal{H}_A \to \mathcal{H}_B. \tag{3.100}$$

This is similar to (3.92) and (3.96) where operators are represented as vectors in the GNS Hilbert space.

[30]↑From now to below, we represent both a state ω and its corresponding density matrix by ω.
[31]↑$|\Omega_\omega\rangle$ can be considered as a purified state of the density matrix ω. Suppose the density matrix is written in its eigenbasis $\omega = \sum_{\mathrm{i}} \lambda_{\mathrm{i}}\,|\mathrm{i}\rangle\,\langle \mathrm{i}|$. With the Schmidt basis, we have $|\Omega_\omega\rangle = \sum_{\mathrm{i}} \lambda_{\mathrm{i}}^{1/2}\,|\mathrm{ii}\rangle$

By the GNS representations, one has

$$\begin{aligned} \omega_A(a) &= \langle \Omega_A | \boldsymbol{\pi}_A(a) \Omega_A \rangle \\ \omega_B(b) &= \langle \Omega_B | \boldsymbol{\pi}_B(b) \Omega_B \rangle \end{aligned} \tag{3.101}$$

for $a \in \mathcal{A}$, $b \in \mathcal{B}$, $|\Omega_A\rangle \equiv |\Omega_{\omega_A}\rangle \in \mathcal{H}_A$ and $|\Omega_B\rangle \equiv |\Omega_{\omega_B}\rangle \in \mathcal{H}_B$ are the cyclic and separating vectors, and $\boldsymbol{\pi}_A$ and $\boldsymbol{\pi}_B$ are the representations of each algebra. We use the Stinespring representation to represent $\boldsymbol{\pi}_B \circ \Phi : \mathcal{A} \to \boldsymbol{\pi}_B(\mathcal{B})$ by

$$\boldsymbol{\pi}_B \circ \Phi(a) = F \boldsymbol{\pi}_A(a) F^\dagger \tag{3.102}$$

where $F : \mathcal{H}_A \to \mathcal{H}_B$. In the following discussion, we use the following notation interchangeably;

$$|a\rangle_A = a\,|\Omega_A\rangle = \boldsymbol{\pi}_A(a)\,|\Omega_A\rangle\,. \tag{3.103}$$

First, we show that F is a contraction, i.e., $\|F\| \leq 1$ with respect to the operator norm. In this notation, we have $F^\dagger F \leq \mathbb{I}_A$ from the Schwartz inequality in (3.54). Then,

$$\begin{aligned} \|\Phi(a)\,|\Omega_B\rangle\,\|^2 &= \|FaF^\dagger\,|\Omega_B\rangle\,\|^2 \\ &= |\,\langle FaF^\dagger\Omega_B | FaF^\dagger\Omega_B\rangle\,|^2 \\ &= |\,\langle \Omega_B | Fa^\dagger F^\dagger FaF^\dagger\Omega_B\rangle\,|^2 \\ &\leq |\,\langle \Omega_B | Fa^\dagger aF^\dagger\Omega_B\rangle\,|^2 \\ &= \|aF^\dagger\,|\Omega_B\rangle\,\|^2 \end{aligned} \tag{3.104}$$

for all $a \in \mathcal{A}$. Thus, by the definition of the operator norm[32],

$$\|F\| = \sup_{a\in\mathcal{A}} \|FaF^\dagger\,|\Omega_B\rangle\,\| \leq \sup_{a\in\mathcal{A}} \|aF^\dagger\,|\Omega_B\rangle\,\| \leq 1. \tag{3.105}$$

[32]↑The operator norm makes sense only when $\{aF^\dagger\,|\Omega_B\rangle\}$ is dense in $\mathcal{H}_A$. Here, $\{aF^\dagger\,|\Omega_B\rangle\}$ is always dense in $\mathcal{H}_A$ since we are assuming the minimal Stinespring representation.

When a CP map is unital, i.e., $\Phi(\mathbb{I}_A) = \mathbb{I}_B$, then, $FF^\dagger$ and $F^\dagger F$ act trivially on the cyclic and separating states $|\Omega_A\rangle$ and $|\Omega_B\rangle$, i.e.,

$$FF^\dagger |\Omega_A\rangle = |\Omega_A\rangle\,,\ F^\dagger F |\Omega_B\rangle = |\Omega_B\rangle\,. \tag{3.106}$$

When a CP map satisfies

$$\omega_A = \omega_B \circ \Phi, \tag{3.107}$$

we define its GNS operator by

$$\Phi(a) |\Omega_B\rangle = F_\omega a F_\omega^\dagger |\Omega_B\rangle\,. \tag{3.108}$$

We can simply show that

$$\begin{aligned} \Phi(a) |\Omega_B\rangle &= F_\omega a |\Omega_A\rangle\,, \\ F_\omega^\dagger |\Omega_B\rangle &= |\Omega_A\rangle\,. \end{aligned} \tag{3.109}$$

From $\omega_A = \omega_B \circ \Phi$ (3.107),

$$\langle F_\omega^\dagger \Omega_B | a F_\omega^\dagger \Omega_B\rangle = \langle \Omega_A | a \Omega_A\rangle \tag{3.110}$$

for $\forall a \in \mathcal{A}$. This implies that

$$F_\omega^\dagger |\Omega_B\rangle = |\Omega_A\rangle \tag{3.111}$$

resulting in

$$\Phi(a) |\Omega_B\rangle = F_\omega a F_\omega^\dagger |\Omega_B\rangle = F_\omega a |\Omega_A\rangle\,. \tag{3.112}$$

If a CP map Φ_ω is unital in addition to the condition (3.107),

$$F_\omega |\Omega_A\rangle = |\Omega_B\rangle\,. \tag{3.113}$$

Table 3.1. : Linear maps of the operator algebra (superoperators) correspond to operators in the GNS Hilbert space. Above is a list of some important superoperators and their corresponding operators. In matrix algebras, this correspondence is one-to-one.

Superoperator		**GNS Operator**
(anti-)linear $\mathcal{T}$		(anti-)linear T
linear CP	unital Φ	$(FF^\dagger - 1)\left\|\Omega_\omega\right\rangle = 0$
	ω-preserving Φ	$F_\omega : (F_\omega^\dagger - 1)\left\|\Omega_\omega\right\rangle = 0$
	unital ω-preserving Φ	$F_\omega \left\|\Omega_\omega\right\rangle = \left\|\Omega_\omega\right\rangle$ and $F_\omega^\dagger \left\|\Omega_\omega\right\rangle = \left\|\Omega_\omega\right\rangle$
	conditional expectation $\mathcal{E}$	projection $E^2 = E$
	isometric embedding ι (faithful representation)	isometry W $W^\dagger W = 1$
	ρ-dual Φ'_ω	co-isometry $F_\omega^\dagger$
	Petz dual Φ_ω^P	$J_B F_\omega^\dagger J_A$
linear non-CP	relative modular operator $\mathcal{D}_{\psi\|\omega}$	$\Delta_{\psi\|\omega} = \psi \otimes \omega^{-1}$
anti-linear non-CP	Tomita map $\mathcal{S}_\omega$	Tomita operator S_ω
	modular conjugation $\mathcal{J}_\omega$	modular conjugation J_ω

Consider the special case where the vN algebras $\mathcal{A} \subset \mathcal{B}$ are represented on the same GNS Hilbert space $\mathcal{H}_\omega$ with respect to a faithful normal state ω. We say that Φ_ω is a *state-preserving* or *ω-preserving* if

$$\omega \circ \Phi = \omega. \tag{3.114}$$

Similar to the above, we have

$$\begin{aligned} \Phi(a)\left|\Omega_\omega\right\rangle &= F_\omega a \left|\Omega_\omega\right\rangle \\ F_\omega^\dagger \left|\Omega_\omega\right\rangle &= \left|\Omega_\omega\right\rangle . \end{aligned} \tag{3.115}$$

3.4.1 Examples

In this section, we study the GNS operators of an isometric embedding and a conditional expectation. In table 3.1, we list the various GNS operators including the ones appearing in the later sections. We will refer to the table in the later sections whenever needed.

Isometric embedding

An isometric embedding of vN algebras from $\mathcal{A}$ to $\mathcal{B}$ is a one-to-one linear map, but not necessarily onto. Consider its GNS operator W defined by

$$\iota(a)\,|\Omega_B\rangle = W\pi_A(a)W^\dagger\,|\Omega_B\rangle\,. \tag{3.116}$$

where $|\Omega_B\rangle \in \mathcal{H}_B$ is a cyclic and separating vector in $\mathcal{H}_B$ associated to a state ω_B. We assume $\pi_A : \mathcal{A} \to B(\mathcal{K}_A)$ is a $*$-isomorphism from $\mathcal{A}$ to $B(\mathcal{K}_A)$ without loss of generality. Then, $W : \mathcal{K}_A \to \mathcal{H}_B$ is an isometry, i.e., $W^\dagger W = \mathbb{I}_A \in B(\mathcal{K}_A)$ and $WW^\dagger$ is a projection on $\mathcal{H}_B$.

ω-preserving Conditional expectations

Consider a ω-preserving conditional expectation $\mathcal{E}_\omega : \mathcal{B} \to \mathcal{A}$ from vN algebra $\mathcal{B}$ to vN algebra $\mathcal{A}$ such that $\omega \circ \mathcal{E}_\omega = \omega$. Let E_ω be its GNS operator on $\mathcal{H}_\omega$ defined by

$$\mathcal{E}_\omega(b)\,|\Omega_\omega\rangle = E_\omega b\,|\Omega_\omega\rangle \tag{3.117}$$

where we have used $E_\omega^\dagger\,|\Omega_\omega\rangle = |\Omega_\omega\rangle$. From the bimodule property in definition 3.2.5, for any $a_1, a_2 \in \mathcal{A}$,

$$\begin{aligned}\langle a_1|E_\omega a_2\rangle_\omega &= \langle\Omega_\omega|a_1^\dagger\mathcal{E}_\omega(a_2)\Omega_\omega\rangle = \langle\Omega_\omega|\mathcal{E}_\omega(a_1^\dagger\mathcal{E}_\omega(a_2))\Omega_\omega\rangle \\ &= \langle\Omega_\omega|\mathcal{E}_\omega(a_1^\dagger)\mathcal{E}_\omega(a_2)\Omega_\omega\rangle = \langle a_1|E_\omega^\dagger E_\omega a_2\rangle_\omega\,.\end{aligned} \tag{3.118}$$

Hence, $E_\omega = E_\omega^\dagger E_\omega$. Together with $E_\omega^2 = E_\omega$, we get $E_\omega = E_\omega^\dagger$. Therefore, E_ω is a projection on $\mathcal{H}_\omega$.

3.5 Tomita-Takesaki theory/Modular theory

Tomita-Takesaki theory or modular theory was initiated by Tomita in unpublished notes in 1967. It was reformulated by Takesaki in [48] in 1970. The main theme of the modular theory is the algebraic relation between a vN algebra and its commutant. The relation is manifested by the antilinear S known as the *Tomita operator*. It was proved that S is an

unbounded, but closable, and invertible operator[33]. This led to the polar decomposition $S = J\Delta^{1/2}$ where the anitunitary J is called the *modular conjugation* and the positive definite operator Δ is called the *modular operator*. A similar mathematical structure was found by Haag, Hugenholtz, and Winnik in the algebraic formulation of a thermal equilibrium system[49]. Their work formulated *Kubo-Martin-Schwinger* condition using a vN algebra. This work showed the deep and close relation between vN algebras and quantum statistical theory.

In an algebraic formulation of quantum field theory, the relation between a vN algebra associated with a local region of spacetime and its commutant was found to be geometric for a certain setup, which is known as *Haag's duality*. It is proven for a free scalar field on causal diamonds by Araki in [50], a free field in Rindler wedges by Bisognano and Wichmann [37], [51], and conformal field theory by Brunetti, Guido, and Longo in [52]. In the work by Bisognano and Wichmann, they proved the action of the unitary flow defined by the modular operator called *modular flow* $\Delta^{is}(\cdot)\Delta^{-is}$ for $\forall s \in \mathbb{R}$ corresponds to the boost transformation of the Rindler wedges in flat spacetime. Moreover, the geometric action of the modular conjugation was found to be *charge-reflection-time (CRT) conjugation*. Recently, the development of vN algebras associated with a local region of spacetime in the presence of a black hole and a local region of de Sitter spacetime is attracting the great attention[53]–[57].

In the next sections, we provide a minimal review of modular theory[34] with the following setup: For a von Neumann algebra $\mathcal{A}$ with a faithful normal state ω, we have the GNS representation $(\pi_\omega, \mathcal{H}_\omega, |\Omega_\omega\rangle)$,

$$\pi_\omega(a)\,|\Omega_\omega\rangle \in \mathcal{H}_\omega, \forall a \in \mathcal{A}, \tag{3.119}$$

where π_ω is a faithful representation on the GNS Hilbert space $\mathcal{H}_\omega$, and $|\Omega_\omega\rangle$ is a cyclic separating state. We use the notations below interchangeably;

$$\pi_\omega(a)\,|\Omega_\omega\rangle = a\,|\Omega_\omega\rangle = |a\rangle_\omega\,. \tag{3.120}$$

[33]↑We will describe what it means in the following section.
[34]↑For the details, see [43], [58]–[61].

Especially, for a finite-dimensional case, we write the GNS vacuum by $|\Omega_\omega\rangle = |\omega^{1/2}\rangle$.

3.5.1 Tomita operator, modular conjugation, modular operator

We start with defining the anti-linear operator called the *Tomita operator* S_ω associated with the GNS representation of a vN algebra $\mathcal{A}$ with respect to a faithful normal state ω. It is defined by

$$S_\omega a\,|\Omega_\omega\rangle = a^\dagger\,|\Omega_\omega\rangle \tag{3.121}$$

for $a \in \mathcal{A}$. One should note that S_ω depends on the state ω.

The Tomita operator is an unbounded operator[35].Fortunately, it is a closable operator[36] From (3.121),

$$S_\omega^2 = \mathbb{I}. \tag{3.122}$$

where $\mathbb{I} \in \mathcal{A}$. Moreover,

$$S_\omega\,|\Omega_\omega\rangle = |\Omega_\omega\rangle \tag{3.123}$$

since $\mathbb{I}$ is self-adjoint.

One can define the Tomita operator S'_ω of the commutant $\mathcal{A}'$ in the same way. One can show that

$$S'_\omega = S_\omega^\dagger \tag{3.124}$$

where $S_\omega^\dagger$ is the adjoint of S_ω[37]. For $a \in \mathcal{A}$ and $a' \in \mathcal{A}'$[38],

$$\langle S'_\omega a'|a\rangle_\omega = \langle a'^\dagger|a\rangle_\omega = \langle a^\dagger|a'\rangle_\omega = \langle S_\omega a|a'\rangle_\omega = \overline{\langle a|S_\omega^\dagger a'\rangle}_\omega = \langle S_\omega^\dagger a'|a\rangle_\omega\,. \tag{3.126}$$

[35]↑Unbounded operators are not bounded operators. A momentum operator in quantum mechanics is an example of an unbounded operator.

[36]↑An operator is closable if, for any sequence $\{|a_n\rangle_\omega\}$ in a domain of S_ω, $D(S_\omega) = \{a\,|\Omega_\omega\rangle\,, a \in \mathcal{A}\}$, there exists a vector $|a\rangle_\omega \in D(S_\omega)$ and $S_\omega\,|a\rangle_\omega$ in the sense of norm convergence. Intuitively, there is a vector on which the action of an unbounded operator is well-defined.

[37]↑$S_\omega^\dagger$ is well-defined. This is because $D(S_\omega)$ is dense in $\mathcal{H}_\omega$. It is so since $|\Omega\rangle$ is a cyclic state for $\mathcal{A}$.

[38]↑We used the definition of the adjoint of an antilinear operator in the fourth inequality, i.e., for vectors $|\psi\rangle\,, |\phi\rangle$, and an antilinear linear operator T,

$$\langle\psi|T\phi\rangle = \overline{\langle T^\dagger\psi|\phi\rangle}. \tag{3.125}$$

It implies $S'_\omega = S^\dagger_\omega$ because $|\Omega\rangle$ is cyclic and separating.

From (3.122), the Tomita operator is invertible. It has a unique polar decomposition

$$S_\omega = J_\omega \Delta_\omega^{1/2} \tag{3.127}$$

where J_ω is antiunitary, and Δ_ω is self-adjoint and positive definite. They are called *modular conjugation* and *modular operator*, respectively. Here, we summarize the properties of three central operators in modular theory,

1. Tomita operator: $S_\omega = J_\omega \Delta_\omega^{1/2},\ S'_\omega = S^\dagger_\omega = J_\omega \Delta_\omega^{-1/2} = \Delta_\omega^{1/2} J_\omega$

2. modular operator: $\Delta_\omega = S^\dagger_\omega S_\omega,\ \Delta'_\omega = \Delta_\omega^{-1} = S_\omega S^\dagger_\omega$

3. modular conjugation: $J_\omega = J'_\omega = J^\dagger_\omega,\ J^2_\omega = \mathbb{I},\ \Delta_\omega^{-1} = J_\omega \Delta_\omega J_\omega$

In addition,

$$\Delta_\omega |\Omega_\omega\rangle = |\Omega_\omega\rangle . \tag{3.128}$$

For any function f,

$$f(\Delta_\omega) |\Omega_\omega\rangle = f(1) |\Omega_\omega\rangle . \tag{3.129}$$

We will frequently see the function $f(\Delta_\omega) = \Delta_\omega^{\mathrm{i}s}$ for $\forall s \in \mathbb{R}$.

Now, we state the principal result of Tomita-Takesaki theory.

Theorem 3.5.1 (Tomita's fundamental theorem)**.** *Let $\mathcal{A}$ be a von Neumann algebra with a cyclic and separating vector $|\Omega_\omega\rangle$, and let Δ_ω be the associated modular operator and J_ω the associated modular conjugation. It follows that*

$$J_\omega \mathcal{A} J_\omega = \mathcal{A}' \tag{3.130}$$

and

$$\Delta_\omega^{\mathrm{i}s} \mathcal{A} \Delta_\omega^{-\mathrm{i}s} = \mathcal{A} \tag{3.131}$$

for $\forall s \in \mathbb{R}$.

Here, we briefly mention the superoperator corresponding to the Tomita operator, modular conjugation, and modular operator using finite-dimensional algebra. In the GNS representation (3.96), one has

$$S_\omega(a \otimes \mathbb{I})\,|\omega^{1/2}\rangle = a^\dagger \otimes \mathbb{I}\,|\omega^{1/2}\rangle\,, \tag{3.132}$$

The superoperator $\mathcal{S}_\omega$ is a antilinear map, i.e., $\mathcal{S}_\omega(a) = a^\dagger$. For the modular conjugation J_ω,

$$J_\omega(a \otimes \mathbb{I})J_\omega = \mathbb{I} \otimes a^T, \tag{3.133}$$

the superoperator $\mathcal{J}_\omega$ is an antiunitary map, i.e.,

$$\mathcal{J}_\omega(a) = a^T \in \mathcal{A}'. \tag{3.134}$$

where a^T is the transpose of the operator a. For the modular operator,

$$\Delta_\omega(a \otimes \mathbb{I})\Delta_\omega^{-1} = \omega a \omega^{-1} \otimes \mathbb{I}_{\mathcal{K}}. \tag{3.135}$$

Motivated by the above, the superopertor $\mathcal{D}_\omega$ of the modular operator Δ_ω is

$$\mathcal{D}_\omega(a) = \omega a \omega^{-1}. \tag{3.136}$$

Similarly, (3.131) can be written as

$$\mathcal{D}_\omega^{\mathrm{i}s}(a) = \omega^{\mathrm{i}s} a \omega^{-\mathrm{i}s} \tag{3.137}$$

for $s \in \mathbb{R}$. The action of J_ω and Δ_ω to the center $Z(\mathcal{A})$ of the algebra $\mathcal{A}$ is trivial to understand.

Proposition 3.5.1. *If $a \in Z(\mathcal{A}) = \mathcal{A} \cap \mathcal{A}'$, then*

$$\mathcal{D}_\omega^{\mathrm{i}s}(x) = \Delta_\omega^{\mathrm{i}s} a \Delta_\omega^{-\mathrm{i}s} = a,\ J_\omega a J_\omega = a^\dagger \in Z(\mathcal{A}) \tag{3.138}$$

for all $s \in \mathbb{R}$.

Because $\Delta_\omega > 0$ is a positive definite operator, $\mathcal{D}_\omega(\cdot) = \Delta_\omega^{is}(\cdot)\Delta_\omega^{-is}$ defines a unitary flow in physics. It forms a one-parameter group called the *modular automorphism group*, which is our next topic.

3.5.2 Modular automorphism group and Kubo-Martin-Schwinger (KMS) boundary condition

Definition 3.5.1 (Modular automorphism group)**.** *Let $\mathcal{A}$ be a von Neumann algebra, ω a faithful normal state on $\mathcal{A}$, $(\mathcal{H}_\omega, |\Omega_\omega\rangle)$ the pair of the corresponding GNS Hilbert space and the cyclic and separating vector, Δ_ω the modular operator associated with the pair $(\mathcal{H}_\omega, |\Omega_\omega\rangle)$. Then, $*$-automorphisms of $\mathcal{A}$, i.e.*

$$\Delta_\omega^{\mathrm{i}s}\mathcal{A}\Delta_\omega^{-\mathrm{i}s} = \mathcal{A}, \tag{3.139}$$

forms the one-parameter group $s \mapsto \mathcal{D}_\omega^{\mathrm{i}s}(\cdot) = \Delta_\omega^{\mathrm{i}s}(\cdot)\Delta_\omega^{-\mathrm{i}s}$. The group is called the modular automorphism group of the pair $(\mathcal{H}_\omega, |\Omega_\omega\rangle)$.

The state ω is invariant under $\mathcal{D}_\omega^{\mathrm{i}s}$ by definition 3.5.1, i.e.,

$$\omega(\mathcal{D}_\omega^{\mathrm{i}s}(a)) = \langle\Omega_\omega|\Delta_\omega^{\mathrm{i}s}a\Delta_\omega^{-\mathrm{i}s}\Omega_\omega\rangle = \langle\Omega_\omega|\Omega_\omega\rangle = \omega(a) \tag{3.140}$$

for $a \in \mathcal{A}$. We define the subalgebra $\mathcal{A}_{\mathcal{D}_\omega}$ that is invariant under the modular automorphism group by

$$\mathcal{A}_{\mathcal{D}_\omega} \equiv \{a|\ \mathcal{D}_\omega^{\mathrm{i}s}(a) = a,\ \forall a \in \mathcal{A}, \forall s \in \mathbb{R}\}. \tag{3.141}$$

One can find that it matches with so-called *centralizer*, $\mathcal{A}_\omega$.

Definition 3.5.2 (Centralizer of a faithful normal state)**.** *For a faithful normal state ω of a von Neumann algebra $\mathcal{A}$, a centralizer $\mathcal{A}_\omega$ is a subalgebra of $\mathcal{A}$ which is a set of operators $c \in \mathcal{A}$ such that*

$$\omega(ac) = \omega(ca) \tag{3.142}$$

for $\forall a \in \mathcal{A}$. That is,

$$\mathcal{A}_\omega \equiv \{c|\ \omega(ac) = \omega(ca), \forall a \in \mathcal{A}\} \subset \mathcal{A} \tag{3.143}$$

We introduce the Kubo-Martin-Schwinger (KMS) condition.

Theorem 3.5.2 (KMS condition/Modular condition)**.** *Consider a von Neumann algebra $\mathcal{A}$ and a state ω. Let $\{\mathcal{D}_\omega^{\mathrm{i}s}\}$ for $\forall s \in \mathbb{R}$ be a modular automorphism group of $\mathcal{A}$ and ω. Then, the state ω is said to satisfy the Kubo-Martin-Schwinger(KMS) condition with respect to $\{\mathcal{D}_\omega^{\mathrm{i}s}\}$ if for any $a_1, a_2 \in \mathcal{A}$ there exists a complex function $F_{a_1,a_2}(z)$ which is analytic on the strip $\mathcal{S} = \{z \in \mathbb{C} | -1 < Imz < 0\}$ and continuous on the closure of this strip $\overline{\mathcal{S}} = \{z \in \mathbb{C} | -1 \leq Imz \leq 0\}$ such that*

$$F_{a_1,a_2}(s) = \omega(a_1 \mathcal{D}_\omega^{\mathrm{i}s}(a_2)) = \langle \Omega | a_1 \Delta_\omega^{\mathrm{i}s} a_2 \Omega \rangle \,,\ F_{a_1,a_2}(s - \mathrm{i}) = \omega(\mathcal{D}_\omega^{\mathrm{i}s}(a_2) a_1) = \langle \Omega | a_2 \Delta_\omega^{-\mathrm{i}s} a_1 \Omega \rangle \tag{3.144}$$

for $\forall s \in \mathbb{R}$. Especially, when $s = -\mathrm{i}$,

$$\omega(a_1 \mathcal{D}_\omega^{-\mathrm{i}s}(a_2)) = \omega(a_2 a_1). \tag{3.145}$$

The restriction of ω to the centralizer $\mathcal{A}_\omega$ leads ω to behave as a tracial state. The modular operator Δ_ω captures the non-tracial character of a faithful normal state ω. A state ω is tracial when

$$\omega(a_1 a_2) = \omega(a_2 a_1) \tag{3.146}$$

holds for all $a_1, a_2 \in \mathcal{A}$. For a von Neumann algebra $\mathcal{A}$, suppose ω is a tracial state. Then, consider

$$\omega(a_2^\dagger a_1^\dagger a_1 a_2) = \langle a_1 a_2 | a_1 a_2 \rangle_\omega \tag{3.147}$$

where $a_1, a_2 \in \mathcal{A}$. One can show that

$$\langle a_1 a_2 \Omega_\omega | a_1 a_2 \Omega_\omega \rangle = \langle a_2 \Delta_\omega a_2^\dagger | a_1^\dagger a_1 \rangle_\omega \,. \tag{3.148}$$

Since ω is tracial, one also has

$$\langle a_1 a_2 | a_1 a_2 \rangle_\omega = \langle a_2 a_2^\dagger | a_1^\dagger a_1 \rangle_\omega \,. \tag{3.149}$$

Because $|\Omega\rangle$ is cyclic and separating, one obtains $\Delta_\omega = \mathbb{I}$ for a tracial state ω. Together with the properties above, it implies $S_\omega = S'_\omega = J_\omega$. Therefore, the modular operator captures the non-tracial character of a state ω.

It becomes clearer in a finite-dimensional case[39]. Suppose $\mathcal{A} \subset \mathcal{B}(\mathcal{K})$ is an algebra of $d \times d$ complex matrices. A state $\omega(\cdot)$ can be written using the density matrix ω[40], $\mathrm{tr}(\omega(\cdot))$. From (3.96), the algebra $\mathcal{A}$ is represented by

$$a \mapsto |a\rangle = a \otimes \mathbb{I}_\mathcal{K} |\omega^{1/2}\rangle \in \mathcal{H}_\omega \equiv \mathcal{K} \otimes \mathcal{K} \tag{3.150}$$

where $|\omega^{1/2}\rangle = \omega^{1/2} \otimes \mathbb{I} |\mathrm{e}\rangle$, and $|\mathrm{e}\rangle = \sum_\mathrm{i} |\mathrm{ii}\rangle$. The modular operator Δ_ω can be expressed by

$$\Delta_\omega = \omega \otimes \omega^{-1}. \tag{3.151}$$

If the state is tracial, $\omega = \mathbb{I}_\mathcal{K}/d$. Equivalently, the state is literally a normalized trace, i.e., $\omega(\cdot) = \mathrm{tr}(\mathbb{I}_\mathcal{K}/d(\cdot))$. Then, $\Delta_\omega \propto \mathbb{I}_\mathcal{K} \otimes \mathbb{I}_\mathcal{K} = \mathbb{I}_\mathcal{H}$. This applies to the modular automorphism group as well. If the state is tracial, then, the modular automorphism group is trivial, $\mathcal{D}^{is}_\omega = \mathrm{i}d$. In this case, the centralizer matches with the whole algebra, i.e., $\mathcal{A}_\omega = \mathcal{A}$.

Before we move on to define the positive cones associated with vN algebras, we provide the important statements about automorphism groups of a given von Neumann algebra, and the uniqueness among the modular automorphism groups of a state without the proofs.

Theorem 3.5.3. *Consider ω is a faithful normal state of a von Neumann algebra $\mathcal{A}$. Suppose there is an one-parameter group $\{\alpha_t\}$ of $*$-automorphisms of $\mathcal{A}$. If it satisfies the KMS condition relative to ω, then it is the modular automorphism group relative to the state ω. That is,*

$$\{\alpha_s\} = \{\mathcal{D}^{is}_\omega\} \tag{3.152}$$

Moreover, the modular automorphism group of a state is unique.

Theorem 3.5.4. *To each faithful normal state of von Neumann algebra $\mathcal{A}$, there exists a unique modular automorphism group.*

[39]↑We will not provide the proof of the following finite dimensional expression. For the proof, see [58]
[40]↑ω represents both a state and its corresponding density matrix.

In short, if one has a one-parameter automorphism group on a given von Neumann algebra, which satisfies KMS and modular condition with respect to a faithful normal state, it is a unique modular automorphism group.

3.5.3 Positive cones associated with a von Neumann algebra

In this section, we study a particular set of vectors in Hilbert space $\mathcal{H}_\omega$. It is called *positive cones* or *convex cones* defined as follows. For a vN algebra $\mathcal{A}$ and a faithful normal state ω, let $\mathcal{A}_+$ be a set of positive operators. A positive cone or convex cone $\mathcal{C}$ is a subspace of the GNS Hilbert space $\mathcal{H}_\omega$ defined by the set of vectors[41]

$$\mathcal{P}_\omega = \{|a_+\rangle_\omega = a_+ |\Omega_\omega\rangle \,,\ \forall a_+ \in \mathcal{A}_+\} \tag{3.153}$$

where $|\Omega_\omega\rangle \in \mathcal{H}_\omega$ is a cyclic separating vector. For the positive cone $\mathcal{P}$, the *dual positive cone* is defined by

$$\mathcal{P}^o_\omega = \{|\eta\rangle_\omega \in \mathcal{H}_\omega|\ \ \langle a_+|\eta\rangle_\omega \geq 0,\ \forall\, |a_+\rangle_\omega \in \mathcal{P}\}. \tag{3.154}$$

If $\mathcal{P}^o_\omega = \mathcal{P}_\omega$, $\mathcal{P}_\omega$ is called *self-dual cone*.

In the theory of vN algebras and modular theory, there is a one-parameter generalization of the positive cones. We start with a unique self-dual positive cone called the *natural positive cone*.

Definition 3.5.3 (Natural positive cone)**.** *The natural positive cone $\mathcal{P}^\natural_\omega$ associated with a von Neumann algebra $\mathcal{A}$ on $\mathcal{H}_\omega$ with a cyclic and separating vector $|\Omega_\omega\rangle$ is defined as the closure of the set*

$$\mathcal{P}^\natural_\omega \equiv \{aJ_\omega a\,|\Omega_\omega\rangle\,|a \in \mathcal{A}\}. \tag{3.155}$$

The elements of the natural positive cone can be written as

$$\mathcal{P}^\natural_\omega \equiv \{\Delta_\omega^{1/4} a a^\dagger\,|\Omega_\omega\rangle\,|a \in \mathcal{A}\}. \tag{3.156}$$

[41]↑To be more precise, the positive cone is a closure of $\{a_+\,|\Omega_\omega\rangle\,,\ \forall a_+ \in \mathcal{A}_+\}$.

It is invariant under $\Delta_\omega^{\mathrm{is}}$ and J_ω, i.e., $\Delta_\omega^{\mathrm{is}}\mathcal{P}_\omega^\natural = \mathcal{P}_\omega^\natural$ and $J_\omega\mathcal{P}_\omega^\natural = \mathcal{P}^\natural$. In [62], [63], one-parameter family of positive cones is constructed; for $\beta \in [0, 1/2]$

$$\mathcal{P}_\omega^{\beta/2} = \{\Delta_\omega^{\beta/2} a \left|\Omega\right\rangle,\ 0 \leq \beta \leq 1,\ a \in \mathcal{A}\}. \tag{3.157}$$

Its dual positive cone is similarly defined by

$$\mathcal{P}_\omega^{1/2-\beta/2} = \{\Delta_\omega^{1/2-\beta/2} a \left|\Omega\right\rangle,\ 0 \leq \beta \leq 1,\ a \in \mathcal{A}\}. \tag{3.158}$$

3.5.4 Relative modular operator

We studied a von Neumann algebra with a single faithful normal state and its associated GNS representation and modular theory. If one has two faithful normal states for a von Neumann algebra, we get two GNS representations and modular theories. In this section, we review the relations between GNS representations and modular theories with two different states.

Consider a von Neumann algebra $\mathcal{A}$ and two faithful normal states ψ and ω. For a pair $(\mathcal{A}, \psi)$, one has a cyclic separating vector $\left|\Omega\right\rangle_\psi$ and the GNS Hilbert space $\mathcal{H}_\psi$. Similarly, for a pair $(\mathcal{A}, \omega)$, one has a cyclic separating vector $\left|\Omega_\omega\right\rangle$ and the GNS Hilbert space $\mathcal{H}_\omega$. For each GNS Hilbert space, one can define the Tomita operator as discussed above, i.e., $S_\psi = J_\psi\Delta_\psi^{1/2}$ and $S_\omega = J_\omega\Delta_\omega^{1/2}$.

In [64], [65], Araki defined the *relative Tomita operator* and the *relative modular operator*. The relative Tomita operator $S_{\psi|\omega}$ is defined by

$$S_{\psi|\omega} a \left|\Omega_\omega\right\rangle = a^\dagger \left|\Omega_\psi\right\rangle. \tag{3.159}$$

Then, the relative modular operator $\Delta_{\psi|\omega}$ is defined by

$$\Delta_{\psi|\omega} = S_{\psi|\omega}^\dagger S_{\psi|\omega}. \tag{3.160}$$

In finite-dimensional case, the relative modular operator can be written using density matrices as

$$\Delta_{\psi|\omega} = \psi \otimes \omega^{-1}. \tag{3.161}$$

Thus, its superoperator is

$$\mathcal{D}_{\psi|\omega}(\cdot) = \psi(\cdot)\omega^{-1}. \tag{3.162}$$

This defines the relative modular flow

$$\mathcal{D}^{\mathrm{i}s}_{\psi|\omega}(\cdot) = \psi^{\mathrm{i}s}(\cdot)\omega^{-\mathrm{i}s} \tag{3.163}$$

for $\forall s \in \mathbb{R}$.

3.6 Duality and positivity

A simple dual linear map of a given CP map is the adjoint map with respect to the inner product of a Hilbert space. Consider the GNS representations of von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with a maximally mixed state $\mathrm{tr}(\mathbb{I}/d(\cdot))$,

$$|a\rangle_{\mathrm{tr}} = a \otimes \mathbb{I}\,|\mathrm{e}\rangle\,,\ |b\rangle_{\mathrm{tr}} = b \otimes \mathbb{I}\,|\mathrm{e}\rangle \tag{3.164}$$

for $a \in \mathcal{A}$ and $b \in \mathcal{B}$. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Then, the *trace-dual map* Φ^*_{tr} of Φ is defined with respect to the trace-inner product, i.e.,

$$\langle b|\Phi(a)\rangle_{\mathrm{tr}} = 1/d \cdot \mathrm{tr}(b^\dagger \Phi(a)) = 1/d \cdot \mathrm{tr}(\Phi^*_{\mathrm{tr}}(b^\dagger)a) = \langle \Phi^*_{\mathrm{tr}}(b^\dagger)|a\rangle_{\mathrm{tr}}\,. \tag{3.165}$$

The trace dual is familiar to physicists as the Heisenberg-Schrödinger duality. Consider a unitary time-evolution U on a density matrix ρ. Then,

$$\langle U\rho U^\dagger|a\rangle_{\mathrm{tr}} = \langle \rho|U^\dagger a U\rangle_{\mathrm{tr}}\,. \tag{3.166}$$

Since $U\rho U^\dagger$ defines the forward time-evolution of the density matrix ρ, its trace dual $U^\dagger a U$ on an operator defines the backward time-evolution. For open quantum dynamics, the

dynamics are described by a completely positive trace-preserving map $\Phi_{\rm Sch}$ on a set of density matrices in the Schrödinger picture. With its Kraus representation $\Phi_{\rm Sch}(\rho) = \sum_k M_k \rho M_k^\dagger$, the dynamics $\Phi_{\rm H}$ in the Heisenberg picture is described by

$$\langle \Phi_{\rm Sch}(\rho)|a\rangle_{\rm tr} = {\rm tr}(\sum_k M_k \rho M_k^\dagger a) = {\rm tr}(\rho \sum_k M_k^\dagger a M_k) = \langle \rho|\Phi_{\rm H}(a)\rangle_{\rm tr}\,. \tag{3.167}$$

A trace daul map can be a recovery map that reverses an error map[42]. Consider a unitary time-evolution $U^\dagger(\cdot)U$ as an error map. Obviously, $U(\cdot)U^\dagger$ is a recovery map that reverses the unitary time-evolution, i.e.,

$$\langle U^\dagger a_1 U|U^\dagger a_2 U\rangle_{\rm tr} = \langle UU^\dagger a_1 UU^\dagger|a_2\rangle_{\rm tr} = \langle a_1|a_2\rangle_{\rm tr}\,. \tag{3.168}$$

If the error map is an isometry, we have

$$\langle V^\dagger a_1 V|V^\dagger a_2 V\rangle_{\rm tr} = \langle P a_1 P|a_2\rangle_{\rm tr}\,. \tag{3.169}$$

The information in the subspace orthogonal to the support of P is erased and cannot be recovered.

In general, a dual map of a given CP map Φ with respect to a non-tracial state is not necessarily CP. In [66], Accardi and Cecchini provided a way to construct the dual CP maps which we call them the *state-dual* and the *Petz dual* map. In the next section, we review them.

3.6.1 State-dual, Petz dual and their positivity

Consider von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ with faithful normal states ω_A on $\mathcal{A}$ and ω_B on $\mathcal{B}$. Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a linear map. Then, its dual linear map with respect to the inner product of the Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ is

$$\langle b|\Phi(a)\rangle_{\omega_B} = \langle \Phi_\omega^*(b)|a\rangle_{\omega_A} \tag{3.170}$$

[42]↑One should note that this way of using dual is distinct from Heisenberg-Schrödinger duality because the Heisenberg-Schrödinger duality relates the state space and algebras. $\Phi : \mathcal{A} \to \mathcal{B}$. $\Phi^* : S(\mathcal{B}) \to S(\mathcal{A})$.

Unfortunately, the dual operator Φ^*_ω is not completely positive. This is due to the non-tracialness of a state ω. To see this, first, let us start with the trace dual. The trace dual map of Φ is defined by

$$\langle b|\Phi(a)\rangle_{\mathrm{tr}_B} = 1/d \cdot \mathrm{tr}(b^\dagger \Phi(a)) = 1/d \cdot \mathrm{tr}(\Phi^*_{\mathrm{tr}}(b^\dagger)a) \tag{3.171}$$

for $b \in \mathcal{B}$, $a \in \mathcal{A}$. One can see that the trace-dual map Φ^*_{tr} is completely positive[43]. For faithful normal non-tracial states ω_B on $\mathcal{B}$ and ω_A on $\mathcal{A}$, since

$$\langle b|\Phi(a)\rangle_{\omega_B} = \langle \Phi^*_\omega(b)|a\rangle_{\omega_A} = \mathrm{tr}(\omega_A \Phi^*_\omega(b^\dagger)a) \tag{3.172}$$

for all $a \in \mathcal{A}$, we get

$$\Phi^*_\omega(b^\dagger) = \omega_A^{-1}\Phi^*_{\mathrm{tr}}(\omega_B b^\dagger). \tag{3.173}$$

Apparently, Φ^*_ω is not necessarily a completely positive map.

In [66], Accardi and Cecchini proved and provided the procedure to construct the dual linear map of a completely positive map which is completely positive.

Proposition 3.6.1 (*State-dual map*, Proposition 3.1 in [66])**.**

Let $\mathcal{A}$ and ω_A, and $\mathcal{B}$ and ω_B are von Neumann algebras and states. Consider their GNS Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$ with the corresponding cyclic and separating vectors $|\Omega_A\rangle$ and $|\Omega_B\rangle$ where $\omega_A(a) = \langle\Omega_A|a\Omega_A\rangle$ for $a \in \mathcal{A}$ and $\omega_B(b) = \langle\Omega_B|b\Omega_B\rangle$ for $b \in \mathcal{B}$. For a completely positive map $\Phi : \mathcal{A} \to \mathcal{B}$ such that

$$\omega_B \circ \Phi = \omega_A, \tag{3.174}$$

there exists a state dual map, or ω-dual map, which is a unique completely positive map $\Phi'_\omega : \mathcal{B}' \to \mathcal{A}'$, defined by

$$\langle b'|\Phi(a)\rangle_{\omega_B} = \langle \Phi'_\omega(b')|a\rangle_{\omega_A}. \tag{3.175}$$

If Φ is unital, i.e., $\Phi(\mathbb{I}_B) = \mathbb{I}_A$, then $\Phi'_\omega(\mathbb{I}_A) = \mathbb{I}_B$ and Φ'_ω is faithful. Here, $\mathbb{I}_A \in \mathcal{A}$ and $\mathbb{I}_B \in \mathcal{B}$ are the identity of the algebras.

[43]↑Let $\Phi(a) = \sum_k M_k^\dagger a M_k$ be the Kraus representation of Φ. The trace dual map has the Kraus representation, $\Phi^*_{\mathrm{tr}}(b) = \sum_k M_k b M_k^\dagger$.

In short, for a completely positive map $\Phi : \mathcal{A} \to \mathcal{B}$, we obtain the dual CP map $\Phi'_\omega : \mathcal{B}' \to \mathcal{A}'$ from the above theorem. Is there a way to construct a dual CP map that maps from $\mathcal{B}$ to $\mathcal{A}$? The answer to the question is yes. Such a question naturally arises in the QEC picture. Let a CP map $\Phi : \mathcal{A} \to \mathcal{B}$ be an error map. A recovery map should be a map from $\mathcal{B} \to \mathcal{A}$ where the range of the recovery map does not have to be the whole algebra $\mathcal{A}$.

Consider a completely map $\Phi' = \mathcal{J}_B \circ \Phi \circ J_A : \mathcal{A}' \to \mathcal{B}'$ where $\mathcal{A}'$ and $\mathcal{B}'$ are the commutant of $\mathcal{A}$ and $\mathcal{B}$, and $\mathcal{J}_A$ and $\mathcal{J}_B$ are the modular conjugations associated with ω_A and ω_B. By theorem 3.6.1, we obtain the dual CP map, which we call the *Petz dual map*, $\Phi^P_\omega : \mathcal{B} \to \mathcal{A}$ of Φ' defined by

$$\langle b|\Phi'(a')\rangle_{\omega_B} = \langle \Phi^P_\omega(b)|a'\rangle_{\omega_A} . \tag{3.176}$$

The left-hand side of (3.176) becomes

$$\langle b|\Phi'(a')\rangle_{\omega_B} = \langle b|\mathcal{J}_B \circ \Phi \circ \mathcal{J}_A(a')\rangle_{\omega_B} = \overline{\langle \mathcal{J}_B(b)|\Phi'(a')\rangle}_{\omega_B} = \langle \mathcal{J}_A \circ \Phi'_\omega \circ \mathcal{J}_B(b)|a'\rangle_{\omega_B} . \tag{3.177}$$

Hence, from (3.176),

$$\Phi^P_\omega = \mathcal{J}_A \circ \Phi'_\omega \circ \mathcal{J}_B. \tag{3.178}$$

Therefore, we conclude that, for a given completely positive map $\Phi : \mathcal{A} \to \mathcal{B}$, we can construct a dual CP map $\Phi^P_\omega : \mathcal{B} \to \mathcal{A}$, see fig.3.1.

3.6.2 GNS operator of the Petz dual map

In the GNS Hilbert space, we define the GNS operators of CP map $\Phi : \mathcal{A} \to \mathcal{B}$ and the Petz dual map $\Phi^P_\omega : \mathcal{B} \to \mathcal{A}$ by

$$\Phi(a)\,|\Omega_B\rangle = FaF^\dagger\,|\Omega_B\rangle\,,\ \Phi^P_\omega(b)\,|\Omega_A\rangle = F^P_\omega a F^{P\dagger}_\omega\,|\Omega_B\rangle \tag{3.179}$$

for $a \in \mathcal{A}$. Here, we study the properties of the GNS operator F^P_ω. We assume Φ is unital and CP, and satisfies the condition

$$\omega_A = \omega_B \circ \Phi. \tag{3.180}$$

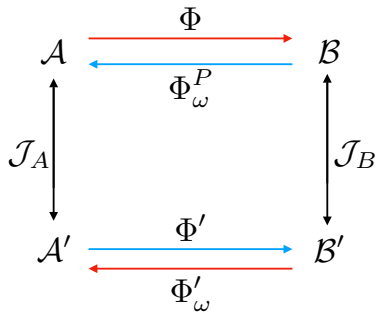


Figure 3.1. The state-dual map of a CP map $\Phi : \mathcal{A} \to \mathcal{B}$ is $\Phi'_\omega : \mathcal{B}' \to \mathcal{A}'$. The maps Φ and Φ'_ω are represented by the red arrows above. The state-dual map of a CP map $\Phi' : \mathcal{A}' \to \mathcal{B}'$ is $\Phi^P_\omega : \mathcal{B} \to \mathcal{A}$. The maps Φ' and Φ^P_ω are represented by the blue arrows above. Φ^P_ω is a Petz dual map of Φ. From the above figure, the Petz dual map can be obtained from the state-dual map Φ'_ω, i.e. $\Phi^P_\omega = \mathcal{J}_A \circ \Phi'_\omega \circ \mathcal{J}_B$.

Thus, we have

$$F^\dagger_\omega |\Omega_B\rangle = |\Omega_A\rangle\,,\ |\Omega_B\rangle = F_\omega |\Omega_A\rangle \tag{3.181}$$

From theorem 3.6.1, Φ^P_ω is again unital and faithful. This implies $F^P_\omega F^{P\dagger}_\omega |\Omega\rangle = |\Omega\rangle$. From (3.176) and (3.179),

$$\langle J_A F^\dagger_\omega J_B b J_B F_\omega J_A |a'\rangle_{\omega_B} = \langle F^P_\omega b F^{P\dagger}_\omega |a'\rangle_{\omega_A}\,. \tag{3.182}$$

for $b \in \mathcal{B}$ and $a' \in \mathcal{A}$. Since $|\Omega_A\rangle$ and $|\Omega_B\rangle$ are cyclic and separating,

$$F^P_\omega b F^{P\dagger}_\omega = J_A F^\dagger_\omega J_B b J_B F_\omega J_A \tag{3.183}$$

for $b \in \mathcal{B}$. Φ^P_ω satisfies $\omega_A \circ \Phi^P_\omega = \omega_B$ because

$$\omega_A \circ \Phi^P_\omega(b) = \langle \Omega_A | F^P_\omega b F^{P\dagger}_\omega \Omega_A\rangle = \langle J_B F J_A \Omega_A | b J_B F J_A \Omega_A\rangle = \langle \Omega_B | b \Omega_B\rangle = \omega_B(b) \tag{3.184}$$

for $\forall b \in \mathcal{B}$. Since Φ^P_ω is unital and satisfies the condition $\omega_A \circ \Phi^P_\omega = \omega_B$, we have

$$F^P_\omega |\Omega_B\rangle = |\Omega_A\rangle\,,\ |\Omega_B\rangle = F^{P\dagger}_\omega |\Omega_A\rangle\,. \tag{3.185}$$

Then, (3.182) reduces to

$$\langle b\Omega_B | J_B F J_A a' \Omega_A \rangle = \langle b\Omega_B | F_\omega^{P\dagger} a' \Omega_A \rangle \tag{3.186}$$

for $\forall a' \in \mathcal{A}'$ and $\forall b \in \mathcal{B}$. Then,

$$F_\omega^{P\dagger} = J_B F_\omega J_A, \ F_\omega^P = J_A F_\omega^\dagger J_B \quad \text{or} \quad J_B F_\omega^{P\dagger} = F_\omega J_A \tag{3.187}$$

because $|\Omega_A\rangle$ and $|\Omega_B\rangle$ are cyclic and separating. Moreover, for $\forall a \in \mathcal{A}$,

$$S_B F a \, |\Omega_A\rangle = S_B \Phi(a) \, |\Omega_B\rangle = \Phi(a^\dagger) \, |\Omega_B\rangle = F S_A a \, |\Omega_A\rangle \, . \tag{3.188}$$

This implies that

$$S_B F = F S_A. \tag{3.189}$$

From (3.187) and (3.188),

$$\Delta_B^{1/2} F = F_\omega^P \Delta_A^{1/2}. \tag{3.190}$$

One should note that we do not have $J_B F = F J_A$ and $\Delta_B^{1/2} F = F \Delta_A^{1/2}$. These are the stronger conditions which we call *Takesaki's condition*. We will study them in section 4.4.1. Below, we summarize the properties of the GNS operator F_ω^P of the Petz dual map.

Corollary 3.6.0.1 (GNS operator of Petz dual map)**.** *For von Neumann algebras $\mathcal{A}$, $\mathcal{B}$, and the faithful normal states ω_A and ω_B, consider an unital CP map Φ satisfying*

$$\omega_A = \omega_B \circ \Phi \tag{3.191}$$

for $b \in \mathcal{B}$. Let its GNS operator $F : \mathcal{H}_A \to \mathcal{H}_B$ is defined by, for $a \in \mathcal{A}$,

$$\Phi(a) \, |\Omega_B\rangle = F_\omega a F_\omega^\dagger \, |\Omega_B\rangle \tag{3.192}$$

where $|\Omega_B\rangle$ is a cyclic and separating vector. It satisfies

$$F_\omega^\dagger |\Omega_B\rangle = |\Omega_A\rangle\,,\ |\Omega_B\rangle = F_\omega |\Omega_A\rangle \tag{3.193}$$

where $|\Omega_A\rangle$ is also a cyclic and separating vector.

The Petz dual map is an unital CP map satisfying

$$\omega_A \circ \Phi_\omega^P = \omega_B. \tag{3.194}$$

The GNS operator $F_\omega^P : \mathcal{H}_B \to \mathcal{H}_A$ of the Petz dual map is defined by

$$\Phi_\omega^P(b) |\Omega_A\rangle = F_\omega^P a F_\omega^{P\dagger} |\Omega_B\rangle \tag{3.195}$$

and satisfies

$$F_\omega^P |\Omega_B\rangle = |\Omega_A\rangle\,,\ |\Omega_B\rangle = F_\omega^{P\dagger} |\Omega_A\rangle\,. \tag{3.196}$$

Furthermore, the GNS operator $F_\omega^P : \mathcal{H}_B \to \mathcal{H}_A$ satisfies

$$J_B F_\omega^{P\dagger} = F_\omega J_A,\ S_B F_\omega = F_\omega S_A. \tag{3.197}$$

As a result,

$$\Delta_B^{1/2} F_\omega = F_\omega^{P\dagger} \Delta_A^{1/2}. \tag{3.198}$$

4. OPERATOR ALGEBRA QUANTUM ERROR CORRECTION (OAQEC)

The main purpose of this section is to study the QEC conditions in the language of von Neumann algebras and modular theory. In section 4.1, we review QEC in the Schrödinger picture. Our first step to the construction of QEC in a general quantum system (vN algebra) is the transition from the Schrödinger picture to the Heisenberg picture. For this purpose, in section 4.2, we discuss the relation between the Schrödinger and the Heisenberg picture of QEC. QEC described by operator algebras is called the *operator algebra quantum error correction* (OAQEC). At the end of this section, we provide the definition of OAQEC and its exact QEC conditions. In QEC, there are two types; i) passive QEC, and ii) active QEC. One has the passive QEC if there is no need for a recovery process even after an error occurs. This is possible only when there is a clever way to encode a logical information so that the information can be unaffected by the error in an encoded system. In the active QEC, one needs to provide a recovery process to undo an error. In section 4.3 and 4.4, we study passive and active QEC in OAQEC, respectively.

4.1 Quantum error correction in the Schrödinger picture

Definition 4.1.1 (Exact quantum error correction in Schrödinger picture)**.**

Consider a set $S(\mathcal{K}_{log})$ of density matrices ρ_{log} on Hilbert space $\mathcal{K}_{log}$ of a logical system. Given an error map Φ^*_{log} as a completely positive and trace-preserving (CPTP) map on $\rho_{log} \in S(\mathcal{K}_{log})$. Consider an isometry $V : \mathcal{K}_{log} \to \mathcal{K}$, i.e., $VV^\dagger = P_C$ is a projection, as a choice of encoding from the logical system to a physical system. This code can correct an error map Φ^* on $S(\mathcal{K})$ if there exists a recovery map $\mathcal{R}^*_V$ such that

$$\mathcal{R}^*_V \circ \Phi^*(\rho_C) = \rho_C \tag{4.1}$$

for $\forall \rho_C \in S(\mathcal{K}_C)$ where

$$S(\mathcal{K}_C) \equiv \{\rho_C | P_C \rho_C P_C = \rho_C, \forall \rho_C \in S(\mathcal{K}_C)\} \subseteq S(\mathcal{K}). \tag{4.2}$$

Equivalently, for $\Phi^*_{log}(\cdot) \equiv \Phi^*(V(\cdot)V^\dagger)$ *and* $\mathcal{R}^*_{log}(\cdot) = V^\dagger \mathcal{R}^*_V(\cdot)V$,

$$\mathcal{R}^*_{log} \circ \Phi^*_{log}(\rho_{log}) = \rho_{log} \tag{4.3}$$

for $\forall \rho_{log} \in S(\mathcal{K}_{log})$.

If an error map Φ is correctable, $\mathcal{K}_C$ is called a *code subspace*. If the recovery map $\mathcal{R}^*_V$ can correct an error map Φ on all the density matrices in $S(\mathcal{K})$, i.e. $S(\mathcal{K}) = S(\mathcal{K}_C)$, then, one can exercise the error correction without the need for encoding, i.e. $P_C = \mathbb{I}$.

Passive QEC has a trivial recovery map. This is possible when an encoding map embeds logical information into a physical system so that the information is untouched. One example is called the *subsystem QEC code* [67]–[69]. In chapter 5, we observe that the QEC code constructed through the real-space renormalization group theory, in particular, continuous multiscale entanglement renormalization ansatz (cMERA), is passive. On the contrary, *active QEC* requires the construction of a non-trivial recovery map. We will come back to the construction of passive QEC code in section 4.3 and active QEC code in section 4.4, but in the Heisenberg picture or OAQEC.

Now, we will provide two of the well-known error-correcting conditions below. These statements are equivalent to the condition of the existence of a recovery map stated in the definition 4.1.1.

Theorem 4.1.1 (Exact QEC conditions in the Schrödinger picture)**.**

Following the notation and setup in definition 4.1.1, the error map Φ^* *is correctable*

1. [*Knill-Laflamme condition*] *[70] if there exists* $\lambda_{kl} \in \mathbb{C}$ *such that*

$$P_C M_k^\dagger M_l P_C = \lambda_{kl} P_C \tag{4.4}$$

where M_k *are Kraus operators of the error map* Φ^**, i.e.* $\Phi^*(\cdot) = \sum_k M_k(\cdot)M_k^\dagger$ *such that* $\sum_k M_k^\dagger M_k = \mathbb{I}$.

2. [Saturation of monotonicity of relative entropy] [71], [72] if

$$S(\rho_C \| \sigma_C) = S(\Phi_V^*(\rho_C) \| \Phi_V^*(\sigma_C)) \tag{4.5}$$

for density matrices $\rho_C, \sigma_C \in S(\mathcal{K}_C)$.

4.2 Operator algebra QEC: from the Schrödinger picture to the Heisenberg picture

The transition from the Schrödinger picture to the Heisenberg picture is done by the trace dual as discussed in section 3.6. Consider an algebra $\mathcal{A} \subseteq B(\mathcal{K})$ of $d \times d$ complex matrices on a Hilbert space $\mathcal{K}$, and a set of density matrices $\rho_A \in S(\mathcal{A})$ on $\mathcal{K}$. In the Schrödinger picture, a unitary U can represent a time evolution of density matrices $U\rho U^\dagger$. The Heisenberg picture is obtained by

$$\mathrm{tr}(aU\rho_A U^\dagger) = \mathrm{tr}(U^\dagger a U \rho_A) \tag{4.6}$$

for $a \in \mathcal{A}$. The trace dual $\Phi : \mathcal{A} \to \mathcal{A}$ of a quantum channel (CPTP) $\Phi^* : S(\mathcal{A}) \to S(\mathcal{A})$is defined by

$$tr(a\Phi^*(\rho_A)) = tr(\Phi(a)\rho_A) \tag{4.7}$$

for $a \in \mathcal{A}$ and $\rho \in S(\mathcal{A})$. For an error map (CPTP) $\Phi^* : S(\mathcal{A}) \to S(\mathcal{A})$ and a recovery map (CPTP) $\mathcal{R}_V^* : S(\mathcal{A}) \to S(\mathcal{A})$ with a choice of encoding V, the transition of QEC from the Schrödinger picture to the Heisenberg picture is achieved by

$$\mathrm{tr}(a\mathcal{R}_V^* \circ \Phi^*(\rho_A)) = \mathrm{tr}(\Phi \circ \mathcal{R}_V(a)\rho_A). \tag{4.8}$$

for $a \in \mathcal{A}$ and $\rho_A \in S(\mathcal{A})$. Here, the trace dual error map $\Phi : \mathcal{A} \to \mathcal{A}$ and the trace dual recovery map $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ are unital CP maps on $\mathcal{A}$. They are unital because, for any CPTP map Φ^*,

$$\mathrm{tr}(\rho) = \mathrm{tr}(\Phi^*(\rho)) = \mathrm{tr}(\Phi(\mathbb{I})\rho) \tag{4.9}$$

for all ρ in $S(\mathcal{K})$.

In general, we define an error map on algebras, $\Phi : \mathcal{A} \to \mathcal{B}$. Then, the recovery map should be constructed as $\mathcal{R}_V : \mathcal{B} \to \mathcal{A}$. To make our story simpler, we mainly work on a physical system rather than starting from a logical system unless we explicitly need them. Then, the definition of OAQEC for the general quantum system (vN algebra) is given by the following.

Definition 4.2.1 (OAQEC)**.** *Given von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$ and an unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$ as an error map. The error Φ is correctable if there exists $\mathcal{R}$ such that*

$$\Phi \circ \mathcal{R}|_C = \mathrm{id}_C \tag{4.10}$$

restricted to the subalgebra $\mathcal{B}_C \subset \mathcal{B}$ where $\mathcal{R} : \mathcal{B} \to \mathcal{A}$. id_C is an identity map on $\mathcal{B}_C \subset \mathcal{B}$, i.e., $\mathrm{id}_C(c) = c$ for $\forall c \in \mathcal{B}_C$. $\mathcal{B}_C$ is a code/correctable subalgebra of $\mathcal{B}$.

We first derive the so-called *recovery equation* which is the equivalent statement to (4.10). Consider a faithful normal state ω_B and its associated GNS vacuum $|\Omega_B\rangle$ in the GNS Hilbert space $\mathcal{H}_B$. Let $c_1, c_2 \in \mathcal{B}_C$ be the operators of the correctable algebra that satisfies (4.10). Let us denote the code subspace as $\mathcal{H}_C$ that is spanned $\{|c\rangle_B = c\,|\Omega_B\rangle\,,\ \forall c \in \mathcal{B}_C\}$. Here, $|\Omega_B\rangle$ is the GNS vacuum associated with the state ω_B on $\mathcal{B}$. From (4.10),

$$\Phi \circ \mathcal{R}(c_1)\,|c_2\rangle_B = c_1\,|c_2\rangle_B\,. \tag{4.11}$$

Suppose the Kraus representation of Φ is written as

$$\Phi(\cdot) = \sum_k M_k^\dagger(\cdot)M_k \tag{4.12}$$

where M_k are the Kraus operators. Then, by (4.11) and (4.12),

$$\begin{aligned} &\sum_k M_k^\dagger \mathcal{R}(c_1) M_k\,|c_2\rangle_B = \sum_k M_k^\dagger M_k c_1\,|c_2\rangle_B \\ &=> \sum_k M_k^\dagger(\mathcal{R}(c_1)M_k - M_k c_1)\,|c_2\rangle_B = 0 \end{aligned} \tag{4.13}$$

for all $c_1, c_2 \in \mathcal{B}_C$. This implies that

$$\mathcal{R}(c)M_k = M_k c \tag{4.14}$$

for all k and $c \in \mathcal{B}_C$[1]. We call (4.14) the *recovery equation*. This is equivalent to (4.10) in the definition. We will use this equation frequently to prove the first QEC condition listed below. It is also useful to identify the nature of error and recovery maps, see section 4.4.

Theorem 4.2.1 (Exact OAQEC conditions)**.** *Following the notation and setup in definition 4.2.1, the unital completely positive map* Φ *is correctable*

1. [*commutator*]

   if and only if

   $$[c, M_k^\dagger M_l] = 0 \tag{4.15}$$

   for all k, l *and for* $c \in \mathcal{B}_C \subset \mathcal{B}$. M_k *are the Kraus operators of* Φ, *i.e.* $\Phi(\cdot) = \sum_k M_k^\dagger(\cdot)M_k$.

2. [*Saturation of monotonicity of relative entropy*]

$$S(\rho_C \| \sigma_C) = S(\Phi^*(\rho_C) \| \Phi^*(\sigma_C)) \tag{4.16}$$

In section 4.3 and 4.4, we study *condition 1*. This approach is the summary of the known results on the exact OAQEC[73]. *condition 2* will be discussed in chapter 5.

4.3 Passive QEC

One of the easiest ways to protect against errors is to find an encoding of the algebra $\mathcal{B}_{log}$ into the algebra that is immune to errors so that we do not need to correct it at all. We achieve this if we choose the encoding $\iota : \mathcal{B}_{log} \to \mathcal{B}_I$ from $\mathcal{B}_{log}$ to the subalgebra $\mathcal{B}_I$ that is invariant under the action of the error map, i.e., $\Phi(b) = b$ for all $b \in \mathcal{B}_I$. We call such an algebra the *noiseless algebra*. The noiseless algebra can be characterized by the commutation relation between Kraus operators of a given CP map.

[1]↑Obviously, this holds only within the code subspace $\mathcal{H}_C$

Theorem 4.3.1 (Noiseless algebra)**.** *Every operator $c \in \mathcal{B}_I \subset \mathcal{B}$ in a noiseless algebra $\mathcal{B}_I$ of an unital CP map $\Phi : \mathcal{B} \to \mathcal{B}$ with the Kraus representation, $\Phi(\cdot) = \sum_k M_k^\dagger(\cdot)M_k$, satisfies*

$$[b, M_k] = [b, M_k^\dagger] = 0 \tag{4.17}$$

for all k and for all $c \in \mathcal{B}_I$.

Proof. For $b_1, b_2 \in \mathcal{B}_I$, we have

$$\Phi(b_1) \left|b_2\right\rangle_B = b_1 \left|b_2\right\rangle_B \tag{4.18}$$

where $\left|b_{\mathrm{i}}\right\rangle_B = b_{\mathrm{i}} \left|\Omega_B\right\rangle$ ($\mathrm{i} = 1, 2$) are the states in the GNS Hilbert state $\mathcal{H}_B$. Using the Kraus representation of $\Phi(\cdot) = \sum_k M_k^\dagger(\cdot)M_k$,

$$\sum_k M_k^\dagger b_1 M_k \left|b_2\right\rangle_B = M_k^\dagger M_k b_1 \left|b_2\right\rangle_B . \tag{4.19}$$

Then,

$$\sum_k M_k^\dagger (b_1 M_k - M_k b_1) \left|b_2\right\rangle_B = 0 \tag{4.20}$$

for all $b_1, b_2 \in \mathcal{B}_I$. Hence,

$$b_1 M_k = M_k b_1 \tag{4.21}$$

for all $b_1 \in \mathcal{B}_I$. The commutator between $\mathcal{B}_I$ and $M_k^\dagger$ follows from the same logic.

□

In the passive QEC, the invariant subalgebra $\mathcal{B}_I$ is the code subalgebra of a given CP map Φ. In this case, we can construct the conditional expectation from Φ. For an unital CP map Φ that preserves some faithful state ω, i.e., $\omega = \omega \circ \omega$,

$$\mathcal{E}_\omega(b) = \lim_{n\to\infty} \frac{1}{n}(b + \Phi(b) + \Phi^2(b) + \cdots + \Phi^{n-1}(b)) \tag{4.22}$$

is a conditional expectation that projects $\mathcal{B}$ down to the invariant subalgebra $\mathcal{B}_I$ of Φ. To see this, consider the Stinespring representation $\Phi(b) = F\boldsymbol{\pi}(b)F^\dagger$. Since Φ is unital, F is coisometry. Because $\boldsymbol{\pi}$ and F are norm non-increasing resulting in $\|\Phi(b)\| \leq \|b\|$, we have

$$\begin{aligned}\|\Phi(\mathcal{E}_\omega(b)) - \mathcal{E}_\omega(b)\| &= \|\lim_{n\to\infty}\frac{1}{n}(\Phi^n(b) - b)\| \\ &\leq \lim_{n\to\infty}\frac{1}{n}(\|\Phi(b)\| + \|b\|) \leq \lim_{n\to\infty}\frac{2}{n}\|a\| = 0.\end{aligned} \tag{4.23}$$

We find that the range of $\mathcal{E}_\omega$ is $\mathcal{B}_I$. This map is evidently CP and leaves every operator in $\mathcal{B}_I$ invariant. Therefore, it is a ω-preserving conditional expectation.

4.4 Active QEC

As opposed to the passive QEC, the active QEC requires one to construct a non-trivial recovery map to undo a given error map. In this section, first, we characterize the correctable algebra by *condition 1.* in theorem 4.2.1. Its proof is basically similar to that of theorem 4.3.1. Second, we explore the nature of an error map and recovery. Third, in section 4.4.1, we discuss *Takesaki's condition* which allows us to construct a unique recovery map. At the end of this section, we study *condition 2.* known as the *sufficiency condition*.

Theorem 4.4.1 (Correctable algebra)**.** *Every operator $c \in \mathcal{B}_C \subset \mathcal{B}$ in a correctable algebra $\mathcal{B}_C$ of a pair $(\Phi, \mathcal{R})$ satisfies*

$$[c, M_k^\dagger M_l] = 0 \tag{4.24}$$

for all k, l and for all $c \in \mathcal{B}_C$.

Proof. For $c_1, c_2 \in \mathcal{B}_C$, from

$$\Phi \circ \mathcal{R}(c_1)\left|a_2\right\rangle_B = c_1\left|c_2\right\rangle_B, \tag{4.25}$$

we have the recovery equation

$$\mathcal{R}(c_1)M_k = M_k c_1 \tag{4.26}$$

as discussed in (4.14). Multiply the equation from both sides by $M_l^\dagger$, we have

$$M_l^\dagger \mathcal{R}(c_1)M_k = M_l^\dagger M_k c_1. \tag{4.27}$$

Repeat the above with $c_1^\dagger$ and take $\dagger$,

$$M_l^\dagger \mathcal{R}(c_1) = c_1 M_l^\dagger. \tag{4.28}$$

Again, multiply the equation from both sides by M_k,

$$M_l^\dagger \mathcal{R}(c_1) M_k = c_1 M_l^\dagger M_k. \tag{4.29}$$

Therefore, by (4.27) and (4.29), we have

$$M_l^\dagger M_k c_1 = c_1 M_l^\dagger M_k. \tag{4.30}$$

□

Now, let us discuss the nature of a given error map and the corresponding recovery map. We observe the following three things;

1. A recovery map $\mathcal{R}$ is unique and unital if the kernel of a given error map is trivial.

2. The range of a recovery map, $\mathcal{A}_C \equiv \mathcal{R}(\mathcal{B}_C)$, is the multiplicative domain of Φ.

3. The unique unital recovery map $\mathcal{R}$ is a faithful representation on $\mathcal{B}_C$.

First, for example, if a given unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$ has a non-trivial kernel, there are two recovery maps $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ and $\tilde{\mathcal{R}} = \mathcal{R} + \chi : \mathcal{B} \to \mathcal{A}$ where the range of χ is in the kernel of Φ, i.e., $\Phi(\chi(c)) = 0$ for $c \in \mathcal{B}_C$. In the QEC, any operators in the kernel is deleted and never be able to recover. Hence, we can remove the kernel from our story. Let P be the projection on $\mathcal{A}$ such that it removes the kernel. We redefine the error map using the projection; $\Phi_P : P\mathcal{A}P \to \mathcal{B}$. This allows us to write the following two recovery equation; for $c_1, c_2 \in \mathcal{B}_C$,

$$\begin{aligned} \mathcal{R}(c_1) P M_k \left|c_2\right\rangle_B &= P M_k c_1 \left|c_2\right\rangle_B \\ \tilde{\mathcal{R}}(c_1) P M_k \left|c_2\right\rangle_B &= P M_k c_1 \left|c_2\right\rangle_B . \end{aligned} \tag{4.31}$$

This implies that the recovery map is unique within the code subspace $\mathcal{H}_C = \{PM_k |c\rangle_B , \ \forall k, \ \forall c \in \mathcal{B}_C\}$, i.e.,

$$\mathcal{R}(c_1)PM_k = \tilde{\mathcal{R}}(c_1)PM_k \tag{4.32}$$

for all k, and all $c_1 \in \mathcal{B}_C$.

For the unital CP map Φ with a trivial kernel, Φ acts faithfully on $\mathcal{A}$. Hence, the operator acted by Φ resulting in the identity operator should uniquely be the identity operator. Because $\Phi \circ \mathcal{R}|_C = \mathrm{i}d_C$ by the definition of correctability, $\mathcal{R}$ should be unital.

Now, let us explicitly consider the range of the recovery map, i.e., $\mathcal{A}_C = \mathcal{R}(\mathcal{B}_C)$. We show that $\mathcal{A}_C$ is the multiplicative domain of an unital CP map Φ with a trivial kernel, i.e.,

$$\Phi(\mathcal{R}(c_1)\mathcal{R}(c_2)) = \Phi(\mathcal{R}(c_1))\Phi(\mathcal{R}(c_2)) \tag{4.33}$$

for $c_1, c_2 \in \mathcal{B}_C$, and hence $\mathcal{R}(c_1), \mathcal{R}(c_2) \in \mathcal{A}_C$. For the left hand side, for all $c_1, c_2, c_3 \in \mathcal{B}_C$,

$$\begin{aligned}
\Phi(\mathcal{R}(c_1)\mathcal{R}(c_2)) |c_3\rangle_B &= \sum_k M_k^\dagger \mathcal{R}(c_1)\mathcal{R}(c_2) M_k |c_3\rangle_B \\
&= \sum_k M_k^\dagger \mathcal{R}(c_1) M_k c_2 |c_3\rangle_B \\
&= \Phi(\mathcal{R}(c_1)) \sum_l M_l^\dagger M_l c_2 |c_3\rangle_B \\
&= \Phi(\mathcal{R}(c_1))\Phi(\tilde{\mathcal{R}}(c_2)) |c_3\rangle_B
\end{aligned} \tag{4.34}$$

where $\tilde{\mathcal{R}}$ is another recovery map. In the second and fourth equality, we used the recovery equation (4.14). Since we are assuming that the unital CP map Φ has a trivial kernel, the recovery map is unique. Hence, $\mathcal{R} = \tilde{\mathcal{R}}$. Therefore, $\mathcal{A}_C \equiv \mathcal{R}(\mathcal{B}_C)$ is the multiplicative domain of Φ.

We have seen that the recovery map $\mathcal{R}$ of the error map (unital CP) with the trivial kernel is unique and unital. To show that $\mathcal{R}$ is the faithful representation of $\mathcal{B}_C$ into $\mathcal{A}_C$, we need to show that it is injective, or one-to-one, i.e. $\mathcal{R}(c_1c_2) = \mathcal{R}(c_1)\mathcal{R}(c_2)$ for $c_1, c_2 \in \mathcal{B}_C$.

First, one should notice that $\mathcal{B}_C$ is the multiplicative domain of $\Phi \circ \mathcal{R}$. Together with the fact that $\mathcal{A}_C$ is the multiplicative domain of Φ, one can show that

$$\Phi \circ \mathcal{R}(c_1 c_2) \left|c_3\right\rangle_B = \Phi(\mathcal{R}(c_1)\mathcal{R}(c_2)) \left|c_3\right\rangle_B \tag{4.35}$$

for all $c_{\mathrm{i}} \in \mathcal{B}_C$ $(\mathrm{i} = 1, 2, 3)$. Thus, we get

$$\mathcal{R}(c_1 c_2) M_k = \mathcal{R}(c_1)\mathcal{R}(c_2) M_k \tag{4.36}$$

for all k and all $c_1, c_2 \in \mathcal{B}_C$. Therefore, $\mathcal{R} : \mathcal{B}_C \to \mathcal{A}_C$ is the faithful representation on the Hilbert space spanned by $\{M_k \left|c\right\rangle_B, \ \forall k, \ \forall c \in \mathcal{B}_C\}$. We summarize the discussion into the following theorem.

Theorem 4.4.2. *Consider an unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$ with the trivial kernel. If there exists a recovery map $\mathcal{R}$ satisfying (4.10) with $\mathcal{B}_C$ being the correctable algebra, it is a unique faithful representation $\mathcal{R} : \mathcal{B}_C \to \mathcal{A}_C \equiv \mathcal{R}(\mathcal{B}_C)$.*

4.4.1 Takesaki's condition

Takesaki's condition will be used to prove that the Petz dual map of a given error map is the unique recovery map if there exists one in the next section. The condition claims a necessary and sufficient condition of the existence of a state-preserving conditional expectation. In particular, for an unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$ between vN algebras and faithful states ω_B and $\omega_A = \omega_B \circ \Phi$, we show that $\mathcal{R} \circ \Phi$ is the conditional expectation that preserves ω_A. Then, we discuss how it is related to Takesaki's condition. We provide Takesaki's condition in the case of matrix algebras and an arbitrary vN algebra.

First, we show that $\mathcal{R} \circ \Phi$ is a conditional expectation. For the unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$ with a trivial kernel, the recovery map $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ is a faithful representation that satisfies

$$\Phi \circ \mathcal{R}(c) \left|\Omega_B\right\rangle = c \left|\Omega_B\right\rangle \tag{4.37}$$

for all $c \in \mathcal{B}_C$. By definition, for $\mathcal{A}_C \equiv \mathcal{R}(\mathcal{B}_C)$,

$$\mathcal{R} \circ \Phi(\mathcal{R}(c)) \left|\Omega_A\right\rangle = \mathcal{R}(c) \left|\Omega\right\rangle \tag{4.38}$$

for all $\mathcal{R}(c) \in \mathcal{A}_C$. Since both Φ and $\mathcal{R}$ are unital and CP, $\mathcal{R} \circ \Phi$ is an unital and CP map. To show that it is a conditional expectation from $\mathcal{A} \to \mathcal{A}_C$, we need to check its bimodule property[2], i.e.,

$$\mathcal{R} \circ \Phi(\mathcal{R}(c_1) a \mathcal{R}(c_2)) = \mathcal{R}(c_1) \mathcal{R} \circ \Phi(a) \mathcal{R}(c_2) \tag{4.39}$$

for $\mathcal{R}(c_1), \mathcal{R}(c_2) \in \mathcal{A}_C$ and $a \in \mathcal{A}$. Using the recovery equation (4.14),

$$\mathcal{R} \circ \Phi(\mathcal{R}(c_1) a \mathcal{R}(c_2)) \left|\Omega_A\right\rangle = \mathcal{R}\Big[\sum_k M_k^\dagger \mathcal{R}(c_1) a \mathcal{R}(c_2) M_k\Big] \left|\Omega_A\right\rangle = \mathcal{R}\Big[c_1 \Phi(a) c_2\Big] \left|\Omega_A\right\rangle . \tag{4.40}$$

Since $\mathcal{R}$ is a faithful representation, its GNS operator is defined by

$$\mathcal{R}(c) \left|\Omega_A\right\rangle = W c W^\dagger \left|\Omega_A\right\rangle \tag{4.41}$$

is a partial isometry, i.e., $WW^\dagger = P_{A_C}$ is a projection on $\mathcal{A}$ to $\mathcal{A}_C$, and $W^\dagger W = P_{B_C}$ is a projection on $\mathcal{B}$ to $\mathcal{B}_C$. Thus,

$$W c_1 \Phi(a) c_2 W^\dagger \left|\Omega_A\right\rangle = W c_1 W^\dagger W \Phi(a) W^\dagger W c_2 W^\dagger \left|\Omega_A\right\rangle = \mathcal{R}(c_1) \mathcal{R} \circ \Phi(a) \mathcal{R}(c_1) \left|\Omega_A\right\rangle \tag{4.42}$$

Therefore, $\mathcal{R} \circ \Phi$ is a conditional expectation. In addition, it preserves ω_A, i.e.,

$$\omega_A \circ \mathcal{R} \circ \Phi = \omega_A. \tag{4.43}$$

The simplest example is when Φ is just an encoding $\iota : \mathcal{A} \to \mathcal{B}$. Suppose we want to simulate a quantum system $\mathcal{B}$ using the algebra of physical operators $\mathcal{A}$. We encode $\mathcal{B}$ as a subalgebra of $\mathcal{A}$ using the isometric embedding map $\iota : \mathcal{B} \to \mathcal{A}$. We also have a decoding map $\alpha : \mathcal{A} \to \mathcal{B}$ such that $\alpha \circ \iota : \mathcal{B} \to \mathcal{B}$ is the identity map. The composite map $\iota \circ \alpha : \mathcal{A} \to \iota(\mathcal{A})$ is a CP map that preserves every operator in $\iota(\mathcal{B})$. The set of states ω_A that

[2]↑For the detail of a conditional expectation, see section 3.2.2 in chapter 3

are invariant under this map are decodable physical states. Assume that ω_A is a decodable faithful state (full-rank density matrix) and $\omega_B = \omega_A \circ \iota$ is its restriction to $\mathcal{B}$. They can be represented as cyclic and separating vectors $|\Omega_B\rangle$ and $|\Omega_A\rangle$ in the GNS Hilbert spaces $\mathcal{H}_{\omega_B}$ and $\mathcal{H}_{\omega_A}$. We denote these Hilbert spaces by $\mathcal{H}_B$ and $\mathcal{H}_A$, respectively. The encoding map as a superoperator is represented by an isometry $W : \mathcal{H}_B \to \mathcal{H}_A$:

$$\iota(b)\,|\Omega_A\rangle = Wb\,|\Omega_B\rangle \ . \tag{4.44}$$

Since we assumed that ρ_A is decodable, this state is preserved under the conditional expectation $\iota \circ \alpha : \mathcal{A} \to \iota(\mathcal{B})$. We will see in theorem 4.4.3 that this is equivalent to the Takesaki condition: $J_A W = W J_B$, where J_A and J_B are the modular conjugation operators corresponding to ω_A and ω_B. This implies that our decoding map corresponds to the GNS operator

$$\alpha(a)\,|\Omega_B\rangle = J_B W^\dagger J_A a\,|\Omega_A\rangle = W^\dagger a\,|\Omega_A\rangle \ . \tag{4.45}$$

In other words, a state is decodable if it satisfies the Takesaki condition, in which case α is the Petz dual of ι.

In matrix algebras, there always exists a trace-preserving conditional expectation $\mathcal{E}_\mathrm{e} : \mathcal{A} \to \mathcal{A}^C$ if $\mathcal{A}^C$ contains the identity operator. To show this, we start with the orthogonal projection P_e in the Hilbert space $\mathcal{H}_\mathrm{e}$ that projects down to $\mathcal{H}_C$ that is the span of $\mathcal{A}^C\,|\mathrm{e}\rangle$. We show that the superoperator that is associated with it is a trace-preserving conditional expectation. Since $P_\mathrm{e} c\,|\mathrm{e}\rangle = c\,|\mathrm{e}\rangle$ the superoperator $\mathcal{E}_\mathrm{e}$ satisfies $\mathcal{E}_\mathrm{e}(c) = c$ for all $c \in \mathcal{A}^C$. Furthermore, we have

$$\langle \mathrm{e}|\mathcal{E}_\mathrm{e}(a)|\mathrm{e}\rangle = \langle \mathrm{e}|P_\mathrm{e} a|\mathrm{e}\rangle = \langle P_\mathrm{e}\mathrm{e}|a|\mathrm{e}\rangle = \langle \mathrm{e}|a|\mathrm{e}\rangle\,, \tag{4.46}$$

therefore $\mathcal{E}_\mathrm{e}$ is trace-preserving. We only need to prove it is CP.

To show that $\mathcal{E}_\mathrm{e}(a_+)$ is positive we need to show the matrix element

$$\langle a_2|\mathcal{E}_\mathrm{e}(a_+)|a_2\rangle = \langle a_2|P_\mathrm{e} a_+ a_2\rangle = \langle P_\mathrm{e} a_2|a_+ a_2\rangle \tag{4.47}$$

is positive. It is clear that if $|a\rangle \in (P_{\rm e})_\perp$ this matrix element is zero, therefore we only need to consider $\langle c|\mathcal{E}(a_+)|c\rangle$ for $c \in \mathcal{A}^C$. The inner product in the Hilbert space $\mathcal{H}_{\rm e}$ has the special property that

$$\langle a_1|a_2a_1\rangle = \mathrm{tr}(a_1^\dagger a_2 a_1) = \mathrm{tr}(a_1 a_1^\dagger a_2) = \langle a_1^\dagger a_1|a_2\rangle \tag{4.48}$$

where we have used the cyclicity of trace. Therefore,

$$\langle c|\mathcal{E}_{\rm e}(a_+)|c\rangle = \langle c^\dagger c|Pa_+\rangle = \langle Pc^\dagger c|a_+\rangle = \langle c^\dagger c|a_+\rangle = \langle c|a_+|c\rangle \geq 0\ . \tag{4.49}$$

Therefore, $\mathcal{E}_{\rm e}$ is a positive map. Similarly, the map $\mathcal{E}_{\rm e} \otimes \mathrm{id}_n$ corresponds to $P_{\rm e} \otimes \mathcal{I}_n$ in the Hilbert space $\mathcal{H}_{\rm e} \otimes \mathcal{K}_n$ which is also positive by the same argument, therefore $\mathcal{E}_{\rm e}$ is CP. The superoperator $\mathcal{E}_{\rm e}$ is the unique trace-preserving conditional expectation from $\mathcal{A} \to \mathcal{A}^C$.[3] We can explicitly write down the ω-preserving conditional expectation in terms of the trace-preserving one:

$$\mathcal{E}_\omega(a) = \omega_C^{-1/2}\mathcal{E}_{\rm e}(\omega^{1/2}a\omega^{1/2})\omega_C^{-1/2} \tag{4.50}$$

where ω_C is the restriction of ω to the subalgebra $\mathcal{A}^C$. These maps are the same as the ω-preserving conditional expectations we constructed in section 3.2.2.

We now prove that the Takesaki condition is the necessary and sufficient condition for a state for the existence of a ω-preserving conditional expectation. The argument trivially generalizes to infinite dimensions [74].

Theorem 4.4.3 (Takesaki's condition: matrix algebras)**.** *The following statements are equivalent:*

1. *There exists a ω-preserving conditional expectation $\mathcal{E}_\omega : \mathcal{A} \to \mathcal{A}^C$.*

2. *For all $c \in \mathcal{A}^C$ we have $\omega^{1/2}c\omega^{-1/2} \in \mathcal{A}^C$.*

[3]↑The bi-module property follows from

$$\mathrm{tr}(c_1\mathcal{E}_{\rm e}(c_2a)) = \langle c_1^\dagger|P_{\rm e}c_2a\rangle = \langle P_{\rm e}c_1^\dagger|c_2a\rangle = \langle c_2^\dagger P_{\rm e}c_1^\dagger|a\rangle = \langle P_{\rm e}c_2^\dagger c_1^\dagger|a\rangle = \mathrm{tr}(c_1c_2\mathcal{E}_{\rm e}(a))\ .$$

3. For all $c \in \mathcal{A}^C$ we have $\omega^{1/2} c \omega^{-1/2} = \omega_C^{1/2} c \omega_C^{-1/2}$.

Here, ω_C is the restriction of ω to $\mathcal{A}^C$.

Proof:

(2 $\rightarrow$ 1): Repeating the argument above for the projection P_ω in the GNS Hilbert space to the subspace $\mathcal{H}_C$ spanned by $\mathcal{A}^C |\omega^{1/2}\rangle$ reveals why there might not exist a ω-preserving conditional expectation for an arbitrary ω. By the same argument, the projection P_ω corresponds to a superoperator $\mathcal{E}_\omega : \mathcal{A} \rightarrow \mathcal{A}^C$ that preserves ω and satisfies $\mathcal{E}_\omega(c) = c$. However, in general, it will not be CP because there is no analog of the property (4.48) in the GNS Hilbert space $\mathcal{H}_\omega$. Instead, we have

$$\langle a_1 | a_2 a_1 \rangle_\omega = \mathrm{tr}(a_1 \omega a_1^\dagger a_2) = \mathrm{tr}(\omega(\omega^{-1} a_1 \omega) a_1^\dagger a_2) = \langle a_1 \mathcal{D}_\omega(a_1^\dagger) | a_2 \rangle_\omega \tag{4.51}$$

where $\mathcal{D}_\omega(a) = \omega a \omega^{-1}$ is the modular superoperator we introduced in section (3.4). If $\mathcal{D}_\omega(c) \in \mathcal{A}^C$ we can repeat the argument above to show

$$\begin{aligned} \langle c | \mathcal{E}_\omega(a_+) c \rangle_\omega &= \langle c \mathcal{D}_\omega(c^\dagger) | P_\omega a_+ \rangle_\omega = \langle P_\omega c \mathcal{D}_\omega(c^\dagger) | a_+ \rangle_\omega \\ &= \langle c \mathcal{D}_\omega(c^\dagger) | a_+ \rangle_\omega = \langle c | a_+ c \rangle_\omega \geq 0 \ . \end{aligned} \tag{4.52}$$

Therefore, if $\mathcal{D}_\omega(c) \in \mathcal{A}^C$ the superoperator $\mathcal{E}_\omega(c)$ is CP and hence it is the unique ω-preserving conditional expectation from $\mathcal{A}$ to $\mathcal{A}^C$. If $\mathcal{D}_\omega^{1/2}(c) \in \mathcal{A}^C$ so is $\mathcal{D}_\omega(c) \in \mathcal{A}^C$, therefore the condition in (2) is sufficient for (1).

(1$\rightarrow$ 2): Assume that $\mathcal{E}_\omega$ exists and P_ω is its corresponding projection operator in $\mathcal{H}_\omega$. Consider the Tomita superoperator $\mathcal{S}(a) = a^\dagger$. Since $\mathcal{E}_\omega$ is a positive map we have $\mathcal{E}_\omega(a^\dagger) = \mathcal{E}_\omega(a)^\dagger$ which implies $\mathcal{E}_\omega(\mathcal{S}(a)) = \mathcal{S}(\mathcal{E}_\omega(a))$. In the GNS Hilbert space, this implies $[P_\omega, S_\omega] = 0$. Since P_ω is self-adjoint when $\mathcal{E}_\omega$ is ω-preserving we also have $[P_\omega, S_\omega^\dagger] = 0$. Therefore, we find $[P_\omega, \Delta_\omega] = 0$, where $\Delta_\omega = S_\omega^\dagger S_\omega$ is the modular operator of ω. Since both operators are positive we have $[P_\omega, \Delta_\omega^{1/2}] = 0$, and using the superoperator representation we obtain $\mathcal{E}(\mathcal{D}_\omega^{1/2}(a)) = \mathcal{D}_\omega^{1/2}(\mathcal{E}(a))$. For any $c \in \mathcal{A}^C$:

$$\mathcal{E}_\omega(\mathcal{D}_\omega^{1/2}(c)) = \mathcal{D}_\omega^{1/2}(\mathcal{E}_\omega(c)) = \mathcal{D}_\omega^{1/2}(c) \ . \tag{4.53}$$

Therefore, $\mathcal{D}_\omega(c) = \omega^{1/2} c \omega^{-1/2} \in \mathcal{A}^C$.

(1→ 3): We saw that (1) implies the commutation relation $[P_\omega, \Delta_\omega] = 0$. Define the state ω_C on the subalgebra $\mathcal{A}^C$ as the restriction $\mathrm{tr}(\omega_C c) = \mathrm{tr}(\omega c)$.[4] Consider its GNS Hilbert space $\mathcal{H}_C$ spanned by $c\,|\omega_C^{1/2}\rangle$ and the linear map $W : \mathcal{H}_C \to \mathcal{H}_A$:

$$W c\,|\omega_C^{1/2}\rangle = c\,|\omega^{1/2}\rangle \ . \tag{4.54}$$

It follows from the definition of ω_C that this linear map is an isometry and $W\mathcal{A}^C W^\dagger$ is an isometric embedding of $\mathcal{A}^C$ in $\mathcal{A}$. Acting with the modular operator we find

$$S_\omega W c\,|\omega_C^{1/2}\rangle = c^\dagger\,|\omega^{1/2}\rangle = W S_C c\,|\omega_C^{1/2}\rangle \ . \tag{4.55}$$

In other words, $S_\omega W = W S_C$ and as a result we have $W^\dagger \Delta_\omega W = \Delta_C$ and $P_\omega \Delta_\omega P_\omega = W \Delta_C W^\dagger$. When $[\Delta_\omega, P_\omega] = 0$ we can take the square root of this equation to find

$$P_\omega \Delta_\omega^{1/2} = W \Delta_C^{1/2} W^\dagger \tag{4.56}$$

or equivalently[5]

$$\Delta_\omega^{1/2} W = W \Delta_C^{1/2} \ . \tag{4.57}$$

This together with $S_\omega W = W S_C$ gives the form of the Takesaki condition $J_\omega W = W J_C$. Then, the constraint that $\mathcal{D}_\omega(c) \in \mathcal{A}^C$ becomes

$$\mathcal{D}_\omega^{1/2}(c)\,|\omega^{1/2}\rangle = P_\omega \Delta_\omega^{1/2} c\,|\omega^{1/2}\rangle = W \Delta_C^{1/2} c\,|\omega_C^{1/2}\rangle = W \mathcal{D}_C^{1/2}(c)\,|\omega_C^{1/2}\rangle = \mathcal{D}_C^{1/2}(c)\,|\omega^{1/2}\rangle \tag{4.58}$$

As a result, we have

$$\omega^{1/2} c \omega^{-1/2} = \mathcal{D}_\omega^{1/2}(c) = \mathcal{D}_C^{1/2}(c) = \omega_C^{1/2} c \omega_C^{-1/2} \tag{4.59}$$

[4]↑Note that $\omega_C = \mathcal{E}_{\mathrm{e}}(\omega)$ because $\mathrm{tr}(c\omega_C) = \mathrm{tr}(c\omega) = \mathrm{tr}(\mathcal{E}_{\mathrm{e}}(c\omega)) = \mathrm{tr}(c\mathcal{E}_{\mathrm{e}}(\omega))$.
[5]↑We act with $W^\dagger$ on the left and take the Hermitian conjugate.

which is the condition in Takesaki's theorem.

(3→ 1): Consider a subalgebra $\mathcal{A}^C \subseteq \mathcal{A}$ and the isometric embedding map $\iota : \mathcal{A}^C \to \mathcal{A}$. The Petz dual $\iota_\omega^P : \mathcal{A} \to \mathcal{A}^C$ is unital and CP. It follows from the definition of the alternate inner product that the Petz dual satisfies

$$\langle \iota_\omega^P(a) | \Delta_C^{1/2} c \rangle_{\omega_C} = \langle a | \Delta_\omega^{1/2} c \rangle_\omega \ . \tag{4.60}$$

We now show that when (3) is satisfied this Petz dual map is a ω-preserving conditional expectation. All we need to show is that $\iota_\omega^P(c) = c$:

$$\begin{aligned} \langle \iota_\omega^P(c_1) | \Delta_C^{1/2} c_2 \rangle_{\omega_C} &= \langle c_1 | \Delta_\omega^{1/2} c_2 \rangle_\omega = \langle c_1 | \mathcal{D}_\omega^{1/2}(c_2) \rangle_\omega \\ &= \langle c_1 | \mathcal{D}_C^{1/2}(c_2) \rangle_\omega = \langle c_1 | \Delta_C^{1/2} c_2 \rangle_{\omega_C} \end{aligned} \tag{4.61}$$

where in the second line we have used (3) and $c_1^\dagger \mathcal{D}_C^{1/2}(c_2) \in \mathcal{A}^C$. Since the isometric embedding is trivial in this case the composite map $\mathcal{E}_\omega^P = \iota \circ \iota_\omega^P : \mathcal{A} \to \mathcal{A}^C$ is a ω-preserving generalized conditional expectation that becomes a conditional expectation (3) is satisfied. □

All the steps of the arguments above can be repeated for an arbitrary von Neumann algebra with $\omega^{1/2}$ replaced with $\Delta_\omega^{1/2}$. The proof did not rely on the existence of a density matrix or a trace, and trivially generalizes to an arbitrary von Neumann algebra and its GNS Hilbert space representation:

Theorem 4.4.4 (Takesaki's condition: von Neumann algebras)**.** *Let $\mathcal{A}^C \subset \mathcal{A}$ be an inclusion of von Neumann algebras. Let ω_A be a faithful state of $\mathcal{A}$ and ω_C be its restriction to $\mathcal{A}^C$. Let $|\omega_A^{1/2}\rangle$ and $|\omega_C^{1/2}\rangle$ be the cyclic and separating vectors in $\mathcal{H}_A$ and $\mathcal{H}_C$. Define the isometry $W : \mathcal{H}_C \to \mathcal{H}_A$ as $Wc|\omega_C^{1/2}\rangle = c|\omega_A^{1/2}\rangle$ for all $c \in \mathcal{A}^C$. The following statements are equivalent:*

1. *There exists a ω_A-preserving conditional expectation $\mathcal{E}_\omega : \mathcal{A} \to \mathcal{A}^C$*
2. *The modular conjugations J_A and J_C corresponding to $|\omega_A^{1/2}\rangle$ and $|\omega_C^{1/2}\rangle$ satisfy $J_A W = W J_C$.*

3. $\Delta_A^{1/2} W = W \Delta_C^{1/2}$.

4.4.2 Sufficiency

Our next question is given a ω-preserving conditional expectation what other states are also invariant under it? To characterize all "sufficient" states of a ω-preserving conditional expectation $\mathcal{E}_\omega$ we show that it preserves another state ψ if and only if the *sufficiency condition*

$$\psi^{1/2} \psi_C^{-1/2} = \omega^{1/2} \omega_C^{-1/2} \tag{4.62}$$

is satisfied [71], [72]. If we are given a ω-preserving conditional expectation $\mathcal{E}_\omega$ the map

$$\mathcal{E}_\omega^\psi(a) = \psi_C^{-1/2} \omega_C^{1/2} \mathcal{E}_\omega \left(\omega^{-1/2} \psi^{1/2} a \psi^{1/2} \omega^{-1/2} \right) \omega_C^{1/2} \psi_C^{-1/2} \tag{4.63}$$

is a ψ-preserving CP map from $\mathcal{A} \to \mathcal{A}^C$. If it preserves every operator in $c \in \mathcal{A}^C$ it becomes an ψ-preserving conditional expectation. It is clear that if the sufficiency condition in (5.35) holds it becomes an ψ-preserving conditional expectation $\mathcal{E}_\psi = \mathcal{E}_\omega$. Therefore, $\mathcal{E}_\omega$ also preserves ψ. We now prove the converse: the conditional expectation $\mathcal{E}_\omega$ preserves ψ only if the condition (5.34) holds. We basically repeat the proof of Takesaki's theorem for the relative Tomita operator $S_{\psi|\omega} a \left|\omega^{1/2}\right\rangle = a^\dagger \left|\psi^{1/2}\right\rangle$. The norm of this operator is the relative modular operator $\Delta_{\psi|\omega} : \mathcal{H}_\omega \to \mathcal{H}_\omega$. The superoperator corresponding to it is $\mathcal{D}_{\psi|\omega}(a) = \psi a \omega^{-1}$. We repeat the argument for the Takesaki theorem with the relative modular map $\mathcal{D}_{\psi|\omega}(a) = \psi a \omega^{-1}$ to find $[P_\omega, \Delta_{\psi|\omega}^{1/2}] = 0$. This implies

$$\mathcal{E}_\omega(\mathcal{D}_{\psi|\omega}^{1/2}(c)) = \mathcal{D}_{\psi|\omega}^{1/2}(\mathcal{E}_\omega(c)) = D_{\psi|\omega}^{1/2}(c) \in \mathcal{A}^C \tag{4.64}$$

We define the isometries

$$\begin{aligned} W_\omega c \left|\omega_C^{1/2}\right\rangle &= c \left|\omega^{1/2}\right\rangle \\ W_\psi c \left|\psi_C^{1/2}\right\rangle &= c \left|\psi^{1/2}\right\rangle \end{aligned} \tag{4.65}$$

so that

$$S_{\psi|\omega}W_\omega = W_\psi S_{\psi_C|\omega_C}$$
$$W_\omega^\dagger \Delta_{\psi|\omega} W_\omega = \Delta_{\psi_C|\omega_C} \ . \tag{4.66}$$

Since $[P_\omega, \Delta_{\psi|\omega}^{1/2}] = 0$ we have

$$P_\omega \Delta_{\psi|\omega}^{1/2} = W_\omega \Delta_{\psi_C|\omega_C}^{1/2} W_\omega^\dagger \ . \tag{4.67}$$

As a result,

$$\begin{aligned}&\mathcal{D}_{\psi|\omega}^{1/2}(c)\,|\omega^{1/2}\rangle = P_\omega \Delta_{\psi|\omega}^{1/2} c\,|\omega^{1/2}\rangle = W_\omega \Delta_{\psi_C|\omega_C}^{1/2} c\,|\omega_C^{1/2}\rangle \\ &= W_\omega \mathcal{D}_{\psi_C|\omega_C}^{1/2}(c)\,|\omega_C^{1/2}\rangle = \mathcal{D}_{\psi_C|\omega_C}^{1/2}(c)\,|\omega^{1/2}\rangle \ .\end{aligned} \tag{4.68}$$

We obtain that

$$\psi^{1/2} c \omega^{-1/2} = \mathcal{D}_{\psi|\omega}^{1/2}(c) = \mathcal{D}_{\psi_C|\omega_C}^{1/2}(c) = \psi_C^{1/2} c \omega_C^{-1/2} \ . \tag{4.69}$$

In other words,

$$\psi_C^{-1/2}\psi^{1/2} c \omega^{1/2}\omega_C^{-1/2} = c = \omega_C^{-1/2}\omega^{1/2} c \omega^{-1/2}\omega_C^{1/2} \tag{4.70}$$

which holds if and only if the sufficiency condition in (5.34) is satisfied.

The sufficiency condition can be expressed as

$$\Delta_{\psi|\omega}^{1/2} = W_\omega \Delta_{\psi_C|\omega_C}^{1/2} W_\omega^\dagger \ . \tag{4.71}$$

Using the integral representation of X^α for $\alpha \in (0,1)$

$$X^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty ds\, s^\alpha \left(\frac{1}{s} - \frac{1}{s+X}\right) \tag{4.72}$$

we find

$$\int_0^\infty ds\, s^{1/2} \left(\frac{1}{s + \Delta_{\psi|\omega}} - W_\omega \frac{1}{s + \Delta_{\psi_C|\omega_C}} W_\omega^\dagger \right) = 0 \ . \tag{4.73}$$

From the monotonicity of the relative modular operator [58], [75] we know that the operator in the integrand above is positive, therefore it has to be zero:

$$\frac{1}{s + \Delta_{\psi|\omega}} = W_\omega \frac{1}{s + \Delta_{\psi_C|\omega_C}} W_\omega^\dagger \tag{4.74}$$

which implies

$$\Delta^\alpha_{\psi|\omega} = W_\omega \Delta^\alpha_{\psi_C|\omega_C} W_\omega^\dagger \ . \tag{4.75}$$

Furthermore, for any continuous function f we have

$$W_\omega f(\Delta_C) \, |\omega_C^{1/2}\rangle = f(\Delta) \, |\omega^{1/2}\rangle \ . \tag{4.76}$$

In particular, choosing $f(x) = x^{\mathrm{i}t}$ for $t \in \mathbb{R}$ we find that $\omega_C^{\mathrm{i}t} \psi_C^{-\mathrm{i}t} = \omega^{\mathrm{i}t} \psi^{-\mathrm{i}t}$. This condition implies that the relative entropy for any pair of sufficient states ω and ψ:

$$S(\psi \| \omega) = S(\psi_C \| \omega_C) \ . \tag{4.77}$$

Intuitively, this says that a coarse-graining (conditional expectation) preserves a set of states $\{\omega_k\}$ (sufficient states) if and only if the distinguishability (relative entropy) of any pair of them remains the same.

5. QUANTUM ERROR CORRECTION AND RENORMALIZATION GROUP(RG)

We have reviewed and constructed the basic tools, such as completely positive maps, von Neumann algebras, modular theory, and the operator algebra quantum error correction (OAQEC). In this chapter, we study the QEC code in quantum field theory, especially, the relevance to real-space renormalization group theory.

Holographic duality conjectures that the $d+1$-dim quantum field theory without gravity is exactly equal to the d-dim quantum field theory without gravity. Although we have some examples [76], [77], we still do not know why the duality holds. In this work, we study holography through the perspective of renormalization group theory, and the theory of quantum error correction.

From the viewpoint of renormalization group theory, the holographic duality has the correspondence between the ultraviolet behavior of the boundary theory and the infrared behavior of the bulk theory, and vice-versa. This is known as *UV-IR connection*[78]. The key is that the radial direction is the extra degree of freedom in the bulk emerging from the RG flow of the coupling constants of the boundary theory.

The other approach is the quantum error correction framework. As already discussed in the introduction, holographic duality is realized as quantum error correction by either isometrically or non-isometrically encoding the bulk theory into the boundary theory[15], [79], [80].

Then, what is the relation between the RG theory and the QEC? Does the RG theory exhibit the QEC structure? In section 5.2 and 5.1, we show that the real-space RG theory in free quantum field theory does exhibit the QEC structure. Previously, a similar question was explored by [81] which shows how real-space renormalization group theory of a lattice system, which is known as multiscale entanglement renormalization ansatz (MERA), exhibits the QEC structure in a lattice system. In section 5.2, we extend this work using a continuous version of MERA (cMERA).

Although we observe that MERA and cMERA provide a good test of whether RG embraces the structure of the QEC code, they are not enough to explain the holographic duality.

The things lacking in those models are large internal degrees of freedom and strong interactions. For example, in AdS_3/CFT_2, one needs to take large N and large gap to realize the two-dim CFT to be holographically dual to semiclassical quantum field theory in AdS_3. In the language of QEC, the difference is observed by the property called the *complimentary recovery*. We see that the theories of general quantum systems without the limit of large N and a large gap, for instance, MERA and cMERA, do not satisfy the complimentary recovery. On the contrary, the ones with the limits satisfy the complimentary recovery.

In summary, we study the following main statements;

1. [Renormalization group as an approximate error correction]

   In section 5.2, we check the following statement in $1+1$-d massive free boson theory.

   Consider a Hilbert space at the level s, $\mathcal{H}_s$, of a system, and an isometry $W_s : \mathcal{H}_s \to \mathcal{H}_{UV}$. We define a family of code subspaces $\mathcal{C}_s$ at a length scale e^s to be

   $$\{\mathcal{C}_s \equiv P_s\mathcal{H}_{UV} | W_s : \mathcal{H}_s \to \mathcal{H}_{UV}, W_s^\dagger W_s = \mathbb{I}_s,\ \forall s,\ -\infty \leq s \leq 0\} \tag{5.1}$$

   A local UV error on any simply connected region A can be approximately corrected up to a small error $\tilde{\delta}$ if $s \gg \log|A|$ where $|A|$ is the size of the region. That is, in the form of the Knill-Laflamme condition, for $|\psi_{\mathrm{i}}^s\rangle \in \mathcal{H}_s$, and $O_A^{UV} \in B(\mathcal{H}_{UV})$,

   $$\langle\psi_{\mathrm{i}}^s|W_s^\dagger O_A^{UV} W_s|\psi_{\mathrm{j}}^s\rangle = \langle C_{\mathrm{i}}^s|O_A^{UV}|C_{\mathrm{j}}^s\rangle \propto \delta_{\mathrm{ij}} + \tilde{\delta} \tag{5.2}$$

   where $|C_{\mathrm{i}}^s\rangle = W_s|\phi_{\mathrm{i}}^s\rangle \in \mathcal{C}_s$. We verify the above statement using $1+1$-d free massive boson theory on a line.

2. [Complementary recovery]

   In section 5.2.4, we show that MERA and cMERA do not have a *complementary recovery* while the QEC code in the holography admits it in an approximate sense.

3. [OAQEC and Petz recovery map] [1]

[1]↑Although we have already discussed this, we included the results regarding OAQEC and Petz recovery map for the completeness of section 5.1

In chapter 3, we studied the Petz dual map of a given CP map Φ of von Neumann algebras $\mathcal{A}$ to $\mathcal{B}$. In section 5.1, we state the main theorem below which identifies the recovery map as a Petz dual map of a given unital CP map. The key to proving the theorem is what we call *Takesaki's condition*[66], [82].

Theorem 5.0.1 (Petz dual)**.** *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an unital completely positive map between von Neumann algebras. Let ρ_B and $\rho_A = \rho_B \circ \Phi$ be faithful states. Denote by $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$ the cyclic and separating vectors that represent ρ_A and ρ_B in their corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. If there exists a normal faithful representation $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ that satisfies $\Phi \circ \mathcal{R} = id$, it is the Petz dual of the error map*

$$\mathcal{R}(b) = \Phi_\rho^P(b) \equiv \mathcal{J}_A \circ \Phi'_\rho \circ \mathcal{J}_B \ . \tag{5.3}$$

where $\mathcal{J}_A : \mathcal{A}' \to \mathcal{A}$ and $\mathcal{J}_B : \mathcal{B} \to \mathcal{B}'$ are the modular conjugation maps corresponding to $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$, respectively.

5.1 Real-space RG, error correction and Petz map

5.1.1 Introduction

In quantum computing, we use the Hilbert space of a quantum system to encode and process information. The interactions with the environment lead to errors and an important challenge is to protect our information from the errors. One of the main goals of the theory of quantum error correction (QEC) is to identify the subalgebra of correctable operators associated with an error model and construct the recovery map that undoes the errors.[2] In local many-body quantum systems, to every

Let us apply a unitary rotation in $\mathcal{A}_A$. We obtain a new algebra inclusion $U\mathcal{A}_C U^\dagger \subset U\mathcal{A}_A U^\dagger$ and a new error correction code; however, the unitary can obscure locality. In fact, every algebra inclusion is an exact quantum error correction code and, if finite-dimensional, can be trivialized by a choice of unitary on A. Intuitively, this means that there is a hidden

[2]↑For completeness, we have included a review of the theory of operator algebra error correction in chapter 4. See also [67], [83], [84].

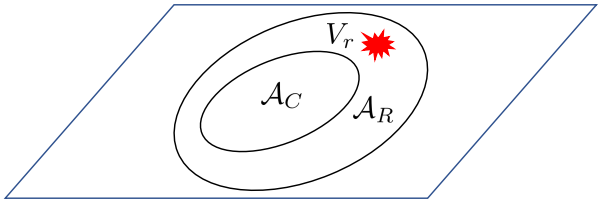


Figure 5.1. The local algebra of region C is a subalgebra of the algebra of a larger region. Any error V_r that acts on the relative commutant $\mathcal{A}_R$ do not disturb the encoded information in $\mathcal{A}_C$.

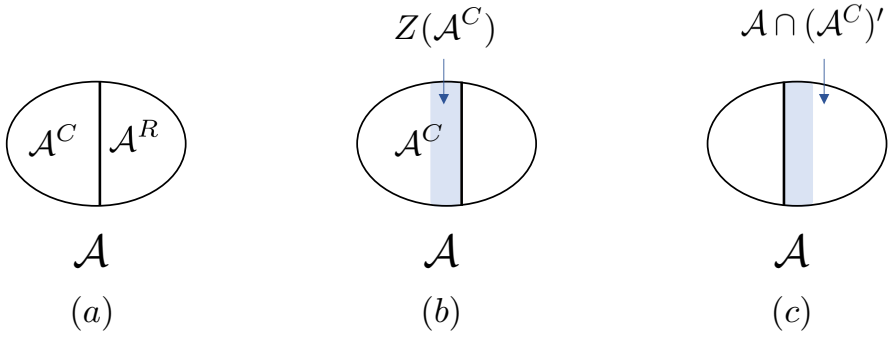


Figure 5.2. (a) If $\mathcal{A}^C$ with trivial center is a subalgebra of a finite dimension algebra $\mathcal{A}$, then we have the tensor product factorization $\mathcal{A} = \mathcal{A}^C \otimes \mathcal{A}^R$. (b) If $\mathcal{A}^C$ has a non-trivial center $Z(\mathcal{A}^C)$, we modify the diagram to represent the center as a blue stripe. (c) The center is part of both $\mathcal{A}^C$ and the relative commutant.

notion of locality in the inclusion of any subalgebra $\mathcal{A}^C \subset \mathcal{A}$.[3] Consider a finite-dimensional matrix algebra with a trivial center (the observable algebra of a qudit). If the subalgebra $\mathcal{A}^C$ also has a trivial center there exists a unitary U in $\mathcal{A}$ such that $U\mathcal{A}U^\dagger = U\mathcal{A}^C U^\dagger \otimes \mathcal{A}^R$ where $\mathcal{A}^R$ is the relative commutant of $U\mathcal{A}^C U^\dagger$ in $U\mathcal{A}U^\dagger$. If $\mathcal{A}^C$ is a subalgebra with a non-trivial center $Z(\mathcal{A}^C)$ then up to the choice of a unitary the algebra $\mathcal{A}$ factors as the direct sum $\oplus_q \mathcal{A}_C^{(q)} \otimes \mathcal{A}_R^{(q)}$ and $\mathcal{A}^C = \oplus_q \mathcal{A}_C^{(q)} \otimes \mathcal{I}_R^{(q)}$. To visualize this structure we use the diagrams in figure 5.2. In this work, we argue that the inclusion of algebras that share the identity

[3]↑With an abuse of notation, we have denoted a general subalgebra that includes the identity operator as $\mathcal{A}^C$ because, in this work, the upper index C in $\mathcal{A}^C$ will stand for "correctable subalgebra".

operator appears naturally in the renormalization group (RG) and holography, however, in these cases the inclusions are not due to any obvious locality principle.

There are two parts to this work. In the first part, in section 5.1.2, we argue that the real-space RG can be modeled as an approximate error correction code that encodes the long-distance operators in the algebra of the short-distance operators. In this picture, the short-distance local perturbations are the errors and the long-distance operators (or a subset of them) are the correctable operators. This is closely related to modeling the holographic map as a quantum error correction code [15], [16], [85].

The connection between the RG and error correction can be seen even in classical systems [2]. The intuition is that exciting a long-range degree of freedom requires acting on a macroscopically large number of short-distance degrees of freedom. The disturbance caused by a local short-distance error cannot alter long-distance modes. Under the RG, local ultraviolet (UV) operators become exponentially weak in the infrared (IR). Deep in the IR, the UV errors are negligible, and in fact, there is no need to actively correct them. Low energy states of a gapped system, do not have excitations at distances much larger than the correlation length. To make our connection concrete, we focus on real-space RG in systems near critical points where the long-range modes of arbitrary wavelength are excited.

As a concrete model of real-space RG that applies to the quantum system near a critical point, in section 5.1.2, we consider the multi-scale renormalization ansatz (MERA) tensor network for lattice models. MERA has found many applications in the study of quantum field theory (QFT) and gravitational theories in AdS/CFT correspondence [86], [87]. To our knowledge, the connection between MERA and error correction codes was first discussed in [81]. This connection was extended to continuous MERA (cMERA) in [2]. The error correction property of MERA is similar to the holographic map modeled as an error correction code with the difference that in a general RG flow, we do not have complementary recovery property.[4] Holography suggests that complementary recovery has to emerge in a special class of theories with a large number of local degrees of freedom (large N) and are strongly

[4]↑See figure 5.4 for complementary recovery in holography. Note that, even in holography, the complementary recovery is an approximate notion. It is known to fail in situations where the code subspace is large [88], [89].

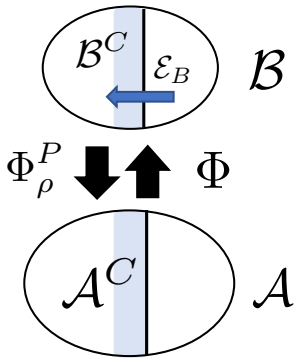


Figure 5.3. We encode the algebra $\mathcal{B}$ in the physical algebra $\mathcal{A}$. If the correctable subalgebra $\mathcal{B}^C \subset \mathcal{B}$ is strictly smaller than B we use a conditional expectation $\mathcal{E}_B$ to project $\mathcal{B}$ down to $\mathcal{B}^C$. Absorbing $\mathcal{E}_B$ in the error map Φ we are back to the case where the whole algebra is correctable.

interacting (large gap). We discuss the role of large N and the large gap in complementary recovery.

Motivated by the connection between the RG and error correction, in the second part of this work in section 5.1.3, we study the operator algebra error correction for an arbitrary von Neumann algebra as a mathematical framework for error correction in continuum quantum field theory (QFT). The error map is modeled by a normal unital completely positive (CP) map $\Phi : \mathcal{A} \to \mathcal{B}$; see figure 5.3. When the whole algebra $\mathcal{B}$ is correctable and the error map has no kernel the recovery map is unique and given by the Petz dual of the error map. It isometrically embeds $\mathcal{B}$ in $\mathcal{A}$. More generally, we consider the setup where only a subalgebra $\mathcal{B}^C$ of the logical operators $\mathcal{B}$ is correctable.[5] Then, the recovery map restricted to the correctable operators is still the Petz dual of the error map. Any unital CP map that projects $\mathcal{B}$ down to $\mathcal{B}^C$ (i.e. any conditional expectation $\mathcal{E}_B : \mathcal{B} \to \mathcal{B}^C$) can be used to redefine the error such that its full image is correctable. Such conditional expectations exist if the inclusion $\mathcal{B}^C \subset \mathcal{B}$ has finite index [90].

For completeness, in the appendices, we have included a self-contained review of the mathematical and information-theoretic background needed for the second part of this work.

[5]↑For instance, in holography, this situation arises when the reconstructable wedge is smaller than the entanglement wedge.

In chapter 4, we review some information theory concepts such as the completely positive (CP) maps and their duals. We also discuss the GNS Hilbert space which has the following two advantages: 1) linear maps on the algebra (superoperators) correspond to linear operators in the GNS Hilbert space. This simplifies the study of error correction. 2) The GNS Hilbert space can be constructed for all quantum systems (von Neumann algebra), including the local algebra of quantum field theory (QFT) that we are ultimately interested in. We show that insisting on the dual of a CP map to remain CP leads to two natural notions of dual maps: 1) the dual map of Accardi and Cecchini that we call the ρ-dual map and 2) Petz dual map. Both of these maps play an important role in error correction. The Petz dual map can be understood as the dual with respect to an alternate inner product that has already found several applications in QFT in the discussion of Rindler positivity [91], [92]. While our discussion applies to any quantum system, to help the readers less familiar with von Neumann algebras we mostly use the more familiar notation of finite quantum systems.

In chapter 4, we review the Heisenberg picture of quantum error correction. We say a subalgebra $\mathcal{B}^C$ is correctable if there exists a recovery map $\mathcal{R} : \mathcal{B}^C \to \mathcal{A}$ such that $\Phi(\mathcal{R}(c)) = c$ for all $c \in \mathcal{B}^C$. We call the constraint $\Phi \circ \mathcal{R} = \text{id}$ the error correction equation. The recovery map is non-unique because any $\mathcal{R} + \mathcal{X}$ satisfies the error correction equation as long as $\Phi(\mathcal{X}(c)) = 0$. In other words, the recovery is non-unique when the kernel of the error map is non-trivial. Another source of non-uniqueness comes from the fact that the error correction equation defines the recovery map from $\mathcal{B}^C$ to $\mathcal{A}$. Any extension of the domain of $\mathcal{R}$ from $\mathcal{B}^C$ to $\mathcal{B}$ can be also called a recovery map. We denote the range of the recovery map by $\mathcal{A}^C \equiv \mathcal{R}(\mathcal{B}^C)$. It is a subalgebra of the physical operators. The recovery map is an isometric embedding of the correctable algebra in $\mathcal{A}$.

Conditional expectations are unital CP maps that project an algebra to a subalgebra that includes the identity. In finite dimension, there is a one-to-one correspondence between conditional expectations $\mathcal{E}_\sigma$ and unnormalized states $\sigma = \oplus_q \mathcal{I}_1^q \otimes \sigma_2^q$ on the relative commutant of $\mathcal{A}^C$ in $\mathcal{A}$.[6] All the density matrices that are preserved under a conditional expectation $\mathcal{E}_\sigma$ take the separable form $\rho = \oplus_q p_q \rho_1^q \otimes \sigma_2^q$. In exact error correction, $\mathcal{R} \circ \Phi$ is a conditional

[6]↑For examples and a more detailed discussion of conditional expectations see chapter 3.2.2.

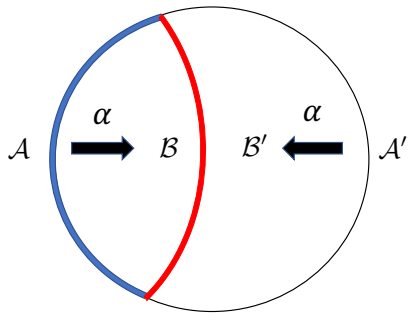


Figure 5.4. The subsystem error correction code in holography satisfies complementary recovery.

expectation and its invariant states are the correctable states. The von Neumann entropy of a correctable state splits into two terms

$$S(\oplus_q p_q \rho_1^q \otimes \sigma_q^2) = H(p) + \sum_q p_q (S(\rho_1^q) + S(\sigma_2^q)) = S(\rho_1) + \sum_q p_q S(\sigma_2^q) \ . \tag{5.4}$$

Note that the second term is a property of the correctable subalgebra and not the correctable state.

In holography, the boundary algebra is our physical algebra, and the bulk is the code algebra. An isometry W encodes the bulk Hilbert space on the boundary. In the Heisenberg picture, the map $\alpha(a) = W^\dagger a W$ maps the boundary operators to the bulk respecting the complementary recovery property: the boundary operators supported on region A go to those in the bulk localized in B and the operator supported on the complementary region A' go to those in B'; see figure 5.4. The bulk operators localized in the region B of the bulk are protected against the erasure of A'. The error map is $\Phi = \alpha \circ \mathrm{tr}_{A'}$ and its Petz dual is the recovery map $\mathcal{R} : \mathcal{B} \to \mathcal{A}$. The complementary recovery implies that the composite map $\mathcal{R} \circ \Phi$ is a conditional expectation.[7] In holography, the second term on the right-hand-side of (5.4) is argued to be similar to the contribution of the area operator to the holographic entanglement [16].

[7]↑A similar observation was made in [85].

5.1.2 Real-space RG as an error correction code

Conventional theory of QEC

We start this section with a quick review of the conventional approach to quantum error correction.In the Schrödinger picture of error correction, consider an encoding isometry $W : \mathcal{K}_B \to \mathcal{H}_A$ from the code Hilbert space $\mathcal{K}_B$ to the physical Hilbert space $\mathcal{H}_A$ and a decoding co-isometry $W^\dagger$. The projection operator $P_C = WW^\dagger$ projects to a subspace of $\mathcal{H}_A$ called the code subspace because it is isomorphic to $\mathcal{K}_B$. Throughout this work, we use the following notation: we denote an irreducible representation of an algebra $\mathcal{B}$ by $\mathcal{K}_B$, and a reducible representation (such as the GNS representation) of $\mathcal{B}$ with $\mathcal{H}_B$. In finite-dimensional matrix algebras, we have $\mathcal{H}_B = \mathcal{K}_B \otimes \mathcal{K}_{B'}$.

A collection of error operators V_r corrupt the physical states and a collection of recovery operators R_r correct the errors; see figure 5.5. In the simple case where the errors V_r are unitary operators, we can undo the error using the correction operators $R_r = V_r^\dagger$. Even when the error is not unitary the correction operator is still made out of the conjugate of the error; see chapter 4. For general errors V_r, the necessary and sufficient condition for the recovery to be possible is the *Knill-Laflamme condition* [8] [70]

$$P_C V_r^\dagger V_s P_C \propto P_C \ . \tag{5.6}$$

When this condition is satisfied the recovery map is $R_r \propto P_C V_r^\dagger$.

[8]↑The physical intuition behind the Knill-Laflamme condition can be seen by defining a set of basis states $\{|C_\mathrm{i}\rangle\}$ in the code subspace $P_C\mathcal{H}_A$. Then,

$$P_C V_r^\dagger V_s P_C = \sum_{\mathrm{ij}} |C_\mathrm{i}\rangle\langle C_\mathrm{i}|V_r^\dagger V_s|C_\mathrm{j}\rangle\langle C_\mathrm{j}| = \sum_{\mathrm{ij}} \langle C_\mathrm{i}|V_r^\dagger V_s|C_\mathrm{j}\rangle |C_\mathrm{i}\rangle\langle C_\mathrm{j}|. \tag{5.5}$$

We satisfy Knill-Laflamme condition if $\langle C_\mathrm{i}|V_r^\dagger V_s|C_\mathrm{j}\rangle = \lambda_{rs}\delta_{\mathrm{ij}}$. This condition implies that the two orthogonal code vectors $|C_\mathrm{i}\rangle$ and $|C_\mathrm{j}\rangle$ remain orthogonal after the action of the error operators. This ensures that the distinguishable states remain distinguishable despite the errors.

For example, consider the 3-qutrit code where the code Hilbert space $\mathcal{K}_B$ is a single qutrit spanned by $|\mathrm{i}\rangle$ with $\mathrm{i} = 0, 1, 2$ that is mapped by an isometry W to the subspace $|\bar{\mathrm{i}}\rangle = W|\mathrm{i}\rangle$:

$$\begin{aligned} |\bar{0}\rangle &= \frac{1}{\sqrt{3}}(|000\rangle + |111\rangle + |222\rangle) \\ |\bar{1}\rangle &= \frac{1}{\sqrt{3}}(|012\rangle + |120\rangle + |201\rangle) \\ |\bar{2}\rangle &= \frac{1}{\sqrt{3}}(|021\rangle + |102\rangle + |210\rangle) \ . \end{aligned} \tag{5.7}$$

An error that occurs on the third qutrit V_3 can be corrected using the $R_3 \propto P_C V_3^\dagger$ because

$$W^\dagger R_3 V_3 W |\mathrm{i}\rangle \propto |\mathrm{i}\rangle \tag{5.8}$$

where we have used (5.6). It is convenient to absorb the encoding isometry W in the definition of the errors and the decoding co-isometry $W^\dagger$ in the definition of the recovery operators

$$\tilde{V}_r = V_r W, \qquad \tilde{R}_r = W^\dagger R_r \ . \tag{5.9}$$

See figure 5.5 (a) and (b). There exists a unitary U and a factorization of the Hilbert space $\mathcal{H}_A = \mathcal{K}_A \otimes \mathcal{K}_{A'}$ such that

$$U|\bar{\mathrm{i}}\rangle = |\mathrm{i}\rangle_A |\chi\rangle_{A'} \tag{5.10}$$

for some state $|\chi\rangle_{A'}$. The unitary trivializes the encoding such that the information is encoded in A and the errors act on A'. The error correction is guaranteed by the locality property $[a, V_r] = 0$ for all a acting on A and error V_r acting on A'.

In the Heisenberg picture of error correction, we have the algebra of code operators $\mathcal{B}$ and that of the physical operators $\mathcal{A}$. An error correction code is a collection of four CP maps $(\iota, \mathcal{R}, \Phi, \alpha)$, where $\iota : \mathcal{B} \to \mathcal{A}$ is an isometric embedding of $\mathcal{B}$ in $\mathcal{A}$ and $\alpha : \mathcal{A} \to \mathcal{B}$

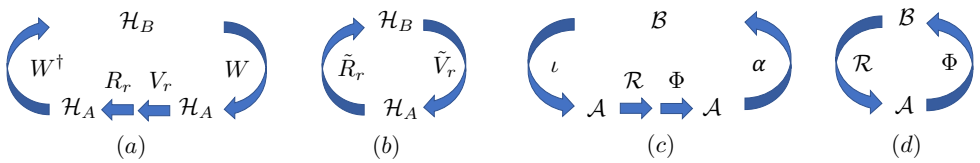


Figure 5.5. (a) Error correction in the Schrödinger picture. The isometry W is the encoding and $W^\dagger$ is the decoding. The errors are V_r and the correction operators are R_r. (b) We can absorb the W and $W^\dagger$ in the definition of the errors and the correction operators. (c) Error correction in the Heisenberg picture. The order of operations is reversed. Both the error map Φ and the recovery map $\mathcal{R}$ are unital completely positive maps. (d) The encoding ι and decoding α can be absorbed in the definition of the error and the recovery maps.

undoes it. The recovery map is $\mathcal{R} : \mathcal{A} \to \mathcal{A}$ and the error map $\Phi : \mathcal{A} \to \mathcal{A}$ is unital. These maps have the Kraus representation

$$\begin{aligned} \alpha(a) &= W^\dagger a W, \qquad \iota(b) = W b W^\dagger \\ \Phi(a) &= \sum_r V_r^\dagger a V_r, \qquad \mathcal{R}(a) = \sum_r R_r^\dagger a R_r \ . \end{aligned} \tag{5.11}$$

We have an error correction if for all the code operators $b \in \mathcal{B}$ we have

$$\alpha \circ \Phi \circ \mathcal{R} \circ \iota(b) = b \ . \tag{5.12}$$

See figure 5.5 (c). The error correction condition above implies the Knill-Laflamme condition in (5.6) as a special case, but it is more general. To simplify the notation, it is often convenient to absorb ι in the definition of the recovery map and α in the definition of the error map. In this way, an error correction code is a doublet $(\mathcal{R}, \Phi)$ where $\Phi : \mathcal{A} \to \mathcal{B}$ is the error and $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ is the recovery map; see figure 5.5 (d):

$$\begin{aligned} \Phi(a) &= \sum_r \tilde{V}_r^\dagger a \tilde{V}_r, \\ \mathcal{R}(b) &= \sum_r \tilde{R}_r^\dagger b \tilde{R}_r. \end{aligned} \tag{5.13}$$

The map $\mathcal{R} \circ \Phi : \mathcal{A} \to \mathcal{A}^C$ projects the physical operators to the subalgebra of correctable operators $\mathcal{A}^C$. These operators are invariant under the action of $\Phi \circ \mathcal{R}$.

A special error channel relevant to the RG flow and holography is erasure. In finite-dimensional matrix algebras, the erasure is the error map that acts as[9][10]

$$\mathcal{E}_{\sigma'}(a \otimes a') = (a \otimes \mathcal{I}') \operatorname{tr}(\sigma' a') \ . \tag{5.14}$$

Any operator $a' \in \mathcal{A}'$ is an error and the necessary and sufficient condition for recovery similar to (5.6) is

$$\forall a' \in \mathcal{A}' : \qquad P_C a' P_C \propto P_C \ . \tag{5.15}$$

This is equivalent to the statement that for any operator b there exists an operator $\mathcal{R}(b)$ acting in subsystem A such that [11]

$$\mathcal{R}(b) W \left|\mathrm{i}\right\rangle = W b \left|\mathrm{i}\right\rangle , \qquad \mathcal{R}(b^\dagger) W \left|\mathrm{i}\right\rangle = W b^\dagger \left|\mathrm{i}\right\rangle \ . \tag{5.16}$$

Since $P_C[\mathcal{R}(b), a']P_C = 0$ any error V_r' supported on A' satisfies

$$\mathcal{R}(b) V_r' W = V_r' W b \ . \tag{5.17}$$

Defining the errors $\tilde{V}_r' = V_r' W$ we have

$$\Phi(\mathcal{R}(b)) = \sum_r (\tilde{V}_r')^\dagger \mathcal{R}(b) \tilde{V}_r' = b \tag{5.18}$$

which is the error correction condition in the Heisenberg picture.

To see how the Heisenberg picture error correction goes beyond the equation (5.6) we consider the subsystem error correction. This is the setup where both the physical Hilbert

[9]↑In the Schrödinger picture, the erasure channel acts on the density matrices according to $\mathcal{E}^*_{\sigma'}(\rho_{AA'}) = \rho_A \otimes \sigma'$.
[10]↑This is the simplest example of a conditional expectation that preserves the states $\rho \otimes \sigma'$.
[11]↑We prove this for an error correction code in a general von Neumann algebra in section 5.1.3. For a proof in finite-dimensional matrix algebras see, for instance, theorem 3.1 of [16]

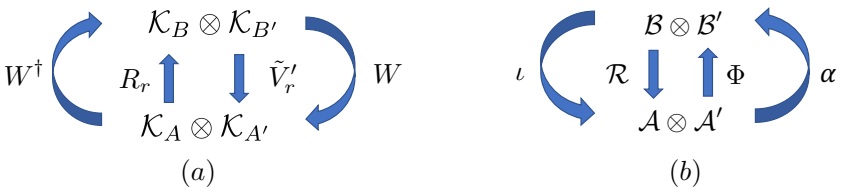


Figure 5.6. The subsystem error correction in (a) the Schrödinger picture (b) the Heisenberg picture.

space and the code Hilbert space admit tensor product forms, respectively $\mathcal{H}_A = \mathcal{K}_A \otimes \mathcal{K}_{A'}$ and $\mathcal{H}_B = \mathcal{K}_B \otimes \mathcal{K}_{B'}$. The goal is to encode the operators b supported on B in the physical Hilbert space such that they are protected against the erasure of A'. In this case, the necessary and sufficient condition generalizes the Knill-Laflamme conditions in (5.6) to

$$W^\dagger a' W \in \mathcal{B}' \ . \tag{5.19}$$

This is to be compared with the condition in (5.15) that can be written as

$$W^\dagger a' W = \lambda \mathcal{I} \ . \tag{5.20}$$

It is a standard result in quantum error correction that (5.19) is equivalent to the existence of a map $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ such that

$$\mathcal{R}(b) W = W b \ . \tag{5.21}$$

We provide proof of this for any von Neumann algebra in section 5.1.3. Since $P_C[\mathcal{R}(b), a']P_C = 0$ for any error $\tilde{V}'_r = V'_r W$ we have

$$\mathcal{R}(b)\tilde{V}'_r = \tilde{V}'_r b \tag{5.22}$$

or equivalently $\Phi(\mathcal{R}(b)) = b$; see figure 5.6.

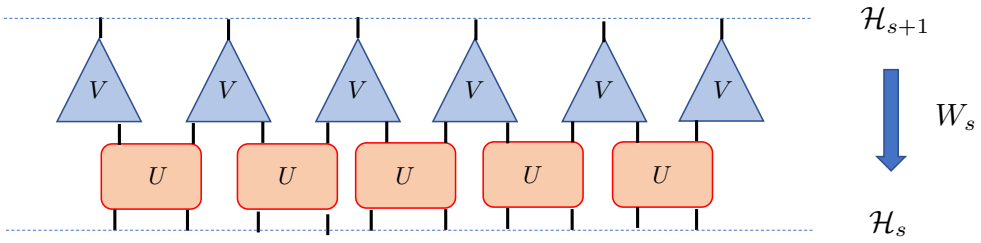


Figure 5.7. A layer of MERA is an isometry $W_s : \mathcal{H}_{s+1} \to \mathcal{H}_s$ that is comprised of two layers: the coarse-graining isometries V and the local disentangling unitaries U.

Entanglement renormalization

As an explicit example of the connection between the real-space renormalization and the quantum error correction codes, we consider a MERA tensor network. A MERA is a sequence of increasingly coarse-grained lattices $\{\mathcal{L}_0, \mathcal{L}_1, \cdots, \mathcal{L}_n\}$ and their corresponding Hilbert spaces $\{\mathcal{H}_0, \mathcal{H}_1, \cdots, \mathcal{H}_n\}$. The Hilbert space $\mathcal{H}_s$ describes the states of the theory at length scale l_s and $l_0 < l_1 < \cdots < l_n$. The states of $\mathcal{H}_0$ are deep in the UV, and the states of $\mathcal{H}_n$ are in the IR. At each site of every lattice $\mathcal{L}_s$ we have a local Hilbert space that we take to be a qudit for simplicity. A sequence of isometries $W_s : \mathcal{H}_{s+1} \to \mathcal{H}_s$ embed $\mathcal{H}_{s+1}$ into the Hilbert space of less coarse-grained states $\mathcal{H}_s$. In the standard MERA, each such isometry is comprised of a layer of local coarse-graining isometries V followed by a layer of disentangling unitaries U; see figure 5.7. The hierarchical structure of correlations in MERA allows for states with long-range correlations. The isometries W_s can be understood as maps that prepare the states $W_1 W_2 \cdots W_n |\Psi_n\rangle$ with long-range correlations. Below, we summarize the argument presented in [81] for the error correction properties of MERA.

In the Heisenberg picture, MERA is a renormalization map for the operators: $\mathcal{A}_s \to \mathcal{A}_{s+1}$ where $\mathcal{A}_s$ is the algebra of observables of the Hilbert space $\mathcal{H}_s$; see figure 5.8:

$$\alpha(a_s) = W_s^\dagger a_s W_s \ . \tag{5.23}$$

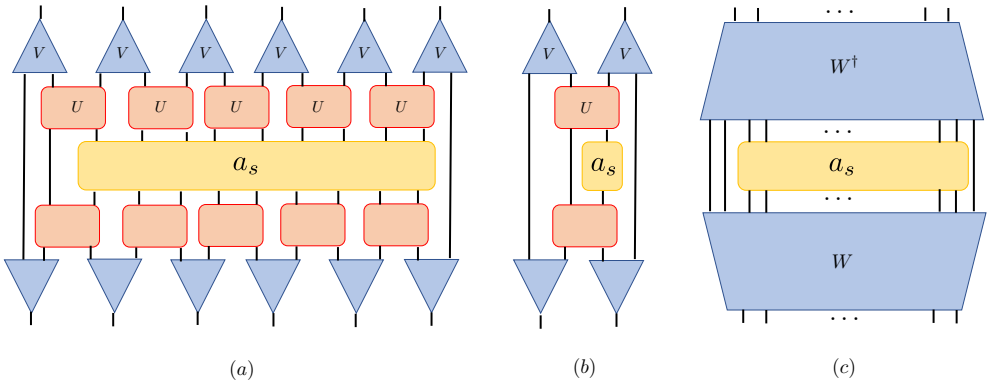


Figure 5.8. (a) One step of RG for a 9-local operator a_s turns it into a 6-local operator acting on $\mathcal{H}_{s+1}$. (b) The support of operators supported on a few sites fluctuates but remains almost constant. (c) The support of k-local operators with $k \gg 1$ shrinks under the RG. For instance, the support of a k-local operator shrinks to at most $\lfloor k/2 \rfloor + 2$. In general, the expectation is that the support of operators shrinks by the coarse-graining factor except for some boundary effects that become important when the operator has support on $O(1)$ number of sites.

The most important property of MERA for us is that it shrinks the support of local operators in the following sense: if a_s is supported on k adjacent sites with $k \gg 1$ on $\mathcal{L}_s$ and the isometries cut down the number of sites by a factor $\gamma > 1$ then the operator $\alpha(a_s)$ is supported on approximately k/γ sites of $\mathcal{L}_{s+1}$ [87]; see figure 5.8. This is not exactly true because of the boundary effects. For instance, for the MERA in figure 5.8, for any k-local operator a the support of $\alpha(a)$ is at most $\lfloor k/2 \rfloor + 2$. For $k = O(1)$ the support of the operator almost remains the same.[12] In higher dimensions, the number of sites in a region scales like the volume of the region, and the number of the sites at the boundary scales like the area of the region therefore it is natural to expect that the volume term in the support a shrinks by γ up to potential area corrections.

A UV operator a_0 supported on region A_0 under the RG flow is mapped to the operator a_s whose support we define to be A_s. After s layers of RG the linear size of A_s is order $\gamma^{-s}|A_0|$. When s becomes comparable to $\log |A_0|$ the support of the operator reaches a few sites. At

[12] ↑It can fluctuate up and down but it can never grow much.

this scale, the second stage of the RG flow starts. As we flow further into the IR, the operator remains local on a few sites, however, its norm falls exponentially fast. This is because, in the Heisenberg picture, the RG flow map is a quantum channel and hence a contraction: its eigenvalues have a norm smaller than one.The operators that are invariant under the RG flow survive deep in the IR forming a subalgebra of exactly correctable operators. These are the eigenoperators with eigenvalue one. All the other operators decay exponentially fast with the exponent set by $h_{min} = -\log|\lambda|$ where λ is the largest eigenvalue of the RG channel with norm less than one [81].[13]

We split the ultra-violet lattice $\mathcal{L}_0$ into a simply connected region A_0 and the complement A_0'. The RG flow respects locality in the sense that operators supported on A_0 are mapped to operators supported on A_s. Therefore, the UV errors a_0' localized on A_0' does not disturb the IR operators a_s in A_s: $[\alpha^s(a_0'), a_s] = 0$. This is a trivial subsystem error correction code. As we flow further into the IR the support A_s shrinks until it reaches a few sites. At this point, the support of the operator no longer shrinks, instead under the RG flow the norm of the operator drops exponentially fast. If there are s layers of coarse-graining between the IR and the UV states a UV operator supported on a region of size A_0 becomes a local operator with a norm that is suppressed by $\mathrm{e}^{-(s-\log|A_0|)}$; for a precise statement see lemma 3 in [81]. Deep in the IR ($s - \log|A_0| \gg 1$) the UV perturbations are vanishingly small. They do not disturb the IR physics; see figure 5.9.

Real-space RG in QFT

In this section, we generalize the connection between MERA and error correction to the RG flow of continuous Poincare invariant QFT. It was shown in [2], that in continuous MERA (cMERA) [93], the RG flow of massive free fields is an approximate quantum error correction code. We comment on the emergence of the complementary recovery in holographic code.

The canonical quantization of a QFT that is a perturbation of massive free fields uses the constant time field operator $\varphi(x)$ and its momentum conjugate $\pi(x)$. For simplicity, we set the mass scale to one. As instructed by cMERA [94], to model the RG flow, we

[13]↑In principle, there can be eigenoperators whose eigenvalues are a phase $\mathrm{e}^{\mathrm{i}\theta}$. If such operators exist, under the RG flow they will show recurrences. We expect a generic RG flow to not have such recurrences.

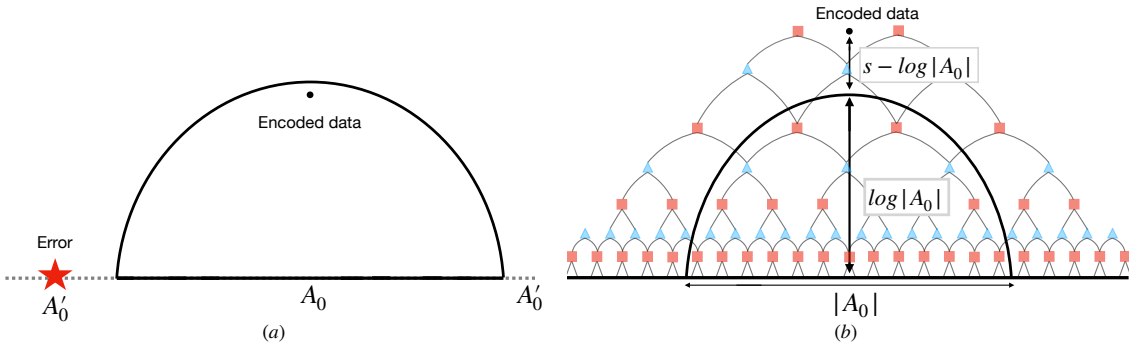


Figure 5.9. (a) Any UV errors a_0' (red star) supported on A_0' do not disturb the IR operators that are originally supported on A_0 before the RG flow. The black dot denoted as the encoded data represents a_s. (b) In the figure, there are s layers between UV and IR where the encoded data is sitting. The size of the support of UV operators shrinks as $\log|A_0|$, though the size drawn in the figure is schematic.

deform the Hamiltonian by adding the irrelevant operator $e^{2s}\partial_i\pi(x)\partial_i\pi(x)$ where the index i runs over spatial directions only and the summation over i is implicit. This term acts as an effective cut-off at the length scale e^s. For $f^\pm(x)$ real test function on the space, we define the annihilation operators $a(f) = \int d^{d-1}x\ (f^-(x) + if^+(x))a(x)$. Under the RG flow this operator renormalizes to $a_s(f_s)$ where a_s is the annihilation operator at scale e^s and the test function f_s is [2]

$$f_s^\pm(x) = (1 - e^{2s}\nabla^2)^{\pm 1/4} f^\pm(x)\ . \tag{5.24}$$

Deep in the UV ($s \to -\infty$) the functions $f^\pm$ are supported on region A. For smooth enough $f^\pm$ as long as $s \ll \log|A|$ the term $e^{2s}\nabla^2 f^\pm$ in (5.24) is smaller than $f^\pm$ and the renormalization of the field is negligible. This is analogous to stage one of the RG flow of the operators in MERA. Here, the support does not change but the cut-off length is growing exponentially fast. The cut-off length is analogous to a single site in MERA (the lattice spacing), therefore the support of f in units of the cut-off length is shrinking exponentially fast.

The support of the operator, in units of the cut-off, shrinks until $e^s \sim |A|$ at which point the operator is supported on a region of cut-off length, and the second stage starts. In the second stage, the second term on the right-hand side of (5.24) is no longer negligible. It was shown in [2] that for large s the projection of the UV coherent operators to the code subspace becomes approximately proportional to the projection to the code subspace:

$$P_C e^{a_s^\dagger(f_s) - a_s(f_s^*)} P_C \simeq P_C \tag{5.25}$$

which is the Knill-Laflamme condition for approximate error correction. More generally, we can directly analyze the spectrum of the RG quantum channel. Deep in the IR, the eigenoperators of the RG quantum channel with the largest eigenvalues are the conformal primaries of the IR fixed point [95], [96]

$$e^{-s\mathcal{D}}(a_h) = e^{-sh} a_h \tag{5.26}$$

where we have defined the superoperator $\mathcal{D}$ that generates the RG flow from the unit length scale to e^s. Here, $h \geq 0$ is the scaling dimension of the eigen-operator. The norm of a non-identity operator decays fast with scale. This implies that any local perturbation in the UV becomes exponentially weak in the IR. The only UV operators that survive the RG flow to the low energies are supported on a macroscopically large number of degrees of freedom.[14] The parameter $h_{min}(s - \log|A|)$ where h_{min} is the dimension of the lightest primary controls how well this error correction code works.

Quantum error correction makes a surprising appearance in quantum gravity and the AdS/CFT duality [15]. The discovery of the Ryu-Takayanagi (RT) formula in holography led to an understanding of the duality at the level of subregion density matrices [97], [98]. It revealed that the map that encodes the bulk operators in the Hilbert space of the boundary theory defines an error correction code. These error correction properties have been used to develop toy models of holography using finite-dimensional quantum systems [99]. It was recently shown that the Petz map gives a reconstruction of the bulk operators in terms of the boundary observables [23]. See [100] for a recent discussion of the Petz map in the reconstruction of operators behind the horizon of a black hole.

At first look, it appears that the approximate error correction in RG is not related to the exact error correction realized in holography because making the error correction above exact requires the conformal dimension of the lightest primary to go to infinity. The holographic QEC code has the complementary recovery property which means that the operators supported on A_0 are mapped to those in A_s and the operators on the complementary region A_s' are encoded in those in the complementary region A_0'.[15] In general, the approximate QEC in RG does not have complementary recovery. This property has to emerge in holographic theories.

The connection with holography becomes clearer when we consider an RG with two groups of primaries: light primaries with conformal dimensions $h_L \ll \Delta$ and heavy primaries

[14]↑In principle, it is plausible that the RG map has invariant local eigenoperators. Such operators would have vanishing conformal dimensions.

[15]↑We will use Latin letters A and A' to refer a region and its complement and $\mathcal{A}_A$ and $\mathcal{A}_{A'}$ to refer to their corresponding algebra of operators. Note that in the presence of conserved charges $\mathcal{A}_{A'} \neq \mathcal{A}_A'$. This happens because the local algebras have non-trivial centers. We assume periodic boundary conditions so that both A' and its complement A can be chosen to be simply connected.

with $h_H \geq \Delta$ for some large parameter Δ. If we choose our code subspace to be the theory at length scale $\mathrm{e}^s l$ with $s = \log|A| + \epsilon$ and l some fixed length scale then any noise $\mathcal{O}_H(A)$ caused by integrals of heavy operators supported on A can be corrected as long as $\epsilon\Delta \gg 1$. As the gap Δ goes to infinity, the error correction becomes exact and we obtain complementary recovery. Note that there is no need for a recovery map as the errors simply do not perturb the code subspace. The commutator between the heavy UV operators on A and any local IR operators $a_{IR}(x)$ vanishes simply because their correlation function vanishes $\langle \mathcal{O}_H(A, l) a_{IR}(\mathrm{e}^s l)\rangle \simeq \mathrm{e}^{-\Delta(s-\log|A|)}$.

In holography, we can correct for the erasure of region A. The error operators include the light operators supported on A in addition to the heavy operators. As opposed to the heavy operators, the light operators on A have non-vanishing correlations with the IR operators. To argue that their effect is correctable in the IR we need a new mechanism specific to holographic theories. Such a mechanism is provided in theories with $N \times N$ matrix degrees of freedom at large N. The light primaries are k trace operators of the form $\mathrm{tr}(X_1)\cdots\mathrm{tr}(X_k)$ with dimension $O(N^0)$. The heavy operators have large dimension $O(N^2)$ that is the size of the gap Δ in holography. It follows from the large N factorization that the commutator of light operators is $1/N$ suppressed.[16] A small commutator is sufficient for the effect of light operators in A to be correctable in the IR.

5.1.3 Error correction in arbitrary von Neumann algebra

The local algebra of quantum field theory is different from the matrix algebras in two important ways: 1) It has no irreducible representations. 2) It does not admit a trace. We need to generalize our discussion of error correction to the GNS Hilbert space to include the local algebra of QFT.[17] In part two of this work, we generalize the formalism of operator algebra error correction to arbitrary von Neumann algebras.

[16]↑We thank Venkatesa Chandrasekaran for insightful conversations about the role of large N in error correction.

[17]↑See chapter 3 for a review of the GNS Hilbert space.

To define the code and the physical GNS Hilbert spaces we need a state ρ_B of $\mathcal{B}$ [18]. After the action of the error map this state becomes $\rho_A = \Phi^*(\rho_B)$.[19] We will choose ρ_B to be full rank (a faithful state). If the error map has a kernel the state ρ_A is no longer faithful. This means that the errors have erased some information permanently and there will not exist any state that is fully correctable. One way to deal with this is to define a projection to the kernel of the error map and use it explicitly in the recovery map. The recovery map will no longer be unital. Another approach is to enlarge the algebra $\mathcal{B}$ by including the degrees of freedom until the extended error map has a trivial kernel. Physically, an error occurs because of the interaction with some environmental degrees of freedom. If there is a kernel for the error map $\Phi : \mathcal{A} \to \mathcal{B}$ it is because the information has left $\mathcal{B}$ and entered the environment. If we add to $\mathcal{B}$ the degrees of freedom of the environment that contain the information that has left $\mathcal{B}$ the extended error map will have a trivial kernel.[20] In the real-space RG in QFT, and in holography, the kernel of the error map is empty. This is because the state ρ_A (the vacuum state of short-distance theory in QFT or the boundary state in holography restricted to a region A) is faithful. In this section, in generalizing our discussion of error correction to an arbitrary von Neumann algebra, we will focus on the case where the kernel of the error map is empty.

To get oriented, let us start with matrix algebras. In finite-dimensional systems, the GNS Hilbert space of a full rank density matrix ρ_A is a double copy Hilbert spaces $\mathcal{H}_{\rho_A} \equiv \mathcal{K}_A \otimes \mathcal{K}_{A'}$ with a distinguished vector $|\rho_A^{1/2}\rangle \in \mathcal{H}_{\rho_A}$ whose density matrix on both A and A' is equal to ρ_A.Such a vector is called cyclic and separating. Given a state ρ_B an arbitrary error map $\Phi : \mathcal{A} \to \mathcal{B}$ is represented in the GNS Hilbert space as a contraction $F : \mathcal{H}_{\rho_A} \to \mathcal{H}_{\rho_B}$ [21]. We assume that the state ρ_A is also full rank therefore the purification of ρ_A is cyclic and separating. There is a one-to-one correspondence between the linear operators in the GNS Hilbert space and the linear superoperators on the algebra; see chapter 4. The operator $F^\dagger$ corresponds to the super-operator $\Phi'_\rho : \mathcal{B}' \to \mathcal{A}'$ that we call the ρ-dual map and the operator

[18]↑A state is a normal positive functional of the algebra. When the algebra has a trace it is a density matrix.
[19]↑In the Schrödinger picture, the error map corresponds to a quantum channel Φ^* that sends the states of $\mathcal{B}$ to those of $\mathcal{A}$.
[20]↑In the extreme case where we include the whole environment in $\mathcal{B}$ the error map is a simple unitary rotation, and completely correctable.
[21]↑A contraction is an operator with $\|F\|_\infty \leq 1$.

$J_A F^\dagger J_B$ corresponds to the Petz dual map $\Phi_\rho^P : \mathcal{B} \to \mathcal{A}$ (see section 3.6.1). Here, J_A and J_B are the modular conjugation operators corresponding to $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$, respectively.

In the special case, F is a co-isometry we call the problem of solving for the recovery map a *reconstruction problem*. Both real-space RG and holography are reconstruction problems. In theorem 5.1.2, we show that any error correction problem where the whole image of the error map is correctable is a reconstruction problem. In reconstruction, the operator F is a co-isometry. In von Neumann algebras, the analog of the Knill-Laflamme condition for exact error correction is the condition $F^\dagger J_B = J_A F^\dagger$ that we refer to as the *Takesaki condition*.[22]

5.1.4 Recovery map in von Neumann algebras

Consider an unital normal CP error map $\Phi : \mathcal{A} \to \mathcal{B}$ between two von Neumann algebras. The Kraus representation $\Phi(a) = \sum_r V_r^\dagger a V_r$ of a CP map generalizes to infinite dimensions[23] . A recovery map is the isometric embedding of the correctable von Neumann subalgebra[24]. The CP map Φ corresponds to a contraction $F : \mathcal{H}_A \to \mathcal{H}_B$:[25]

$$\Phi(a)\,|\rho_B^{1/2}\rangle = Fa\,|\rho_A^{1/2}\rangle \tag{5.27}$$

and if the whole algebra $\mathcal{B}$ is correctable a recovery map corresponds to an isometry $W : \mathcal{H}_B \to \mathcal{H}_A$. Below, we collect all the theorems we need to generalize our discussion of error correction to arbitrary von Neumann algebra.

We start with the definition of the ρ-dual of Φ and its properties.

Theorem 5.1.1 (ρ-dual map: proposition 3.1 [101])**.** *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be a positive map between von Neumann algebras. Let ρ_B and $\rho_A = \rho_B \circ \Phi$ be faithful states of $\mathcal{B}$ and $\mathcal{A}$.*

[22]↑In the remainder of this work, we often denote isometries like $F^\dagger$ with letter W.

[23]↑In matrix algebras, the Kraus operators were maps from $\mathcal{K}_A \to \mathcal{K}_B$ where $\mathcal{K}_A$ and $\mathcal{K}_B$ were the irreducible representations of the algebras $\mathcal{A}$ and $\mathcal{B}$. A general von Neumann algebra does not admit an irreducible representation. As we discuss in chap 3 the generalization of the Kraus representation to an arbitrary von Neumann algebra is in terms of the Kraus operators $V_r : \mathcal{H}_{\rho_B} \to \mathcal{H}_{\rho_A}$.

[24]↑A recovery map satisfies $\mathcal{R}(c)V_r = V_r c, \forall c \in \mathcal{B}^C$. Therefore, $\mathcal{R}(c_1)\mathcal{R}(c_2)V_r\,|\rho_A^{1/2}\rangle = \mathcal{R}(c_1c_2)\,|\rho_A^{1/2}\rangle$. Since we assumed that the kernel of Φ is empty so the union of the range of all V_r cover the whole Hilbert space and we find that a recovery map is multiplicative: $\mathcal{R}(c_1c_2) = \mathcal{R}(c_1)\mathcal{R}(c_2)$. Since it is CP it becomes an isometric embedding.

[25]↑We simplify our notation from $\mathcal{H}_{\rho_A}$ to $\mathcal{H}_A$.

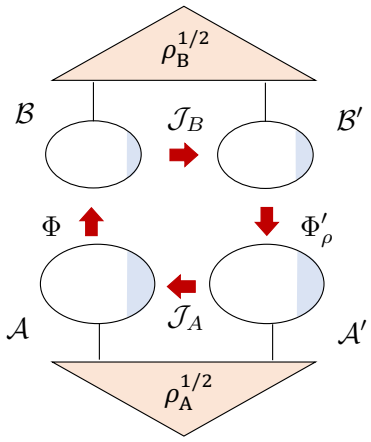


Figure 5.10. The figure shows ρ-dual map of Φ determined by the cyclic and separating vectors $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$ as in theorem 5.1.1. The sequences of $\mathcal{J}_B$, $\mathcal{J}_A$, and Φ'_ρ appears to be a Petz dual map constructed in theorem 5.1.2.

Denote by $|\rho_A^{1/2}\rangle$ *and* $|\rho_B^{1/2}\rangle$ *the cyclic and separating vectors that represent* ρ_A *and* ρ_B *in their corresponding Hilbert spaces* $\mathcal{H}_A$ *and* $\mathcal{H}_B$*. There exists a unique normal positive linear map between the commutants* $\Phi'_\rho : \mathcal{B}' \to \mathcal{A}'$ *defined by*

$$\langle \Phi'_\rho(b')\rho_A^{1/2}|a\rho_A^{1/2}\rangle = \langle b'\rho_B^{1/2}|\Phi(a)\rho_B^{1/2}\rangle\,, \qquad \forall a \in \mathcal{A}, b' \in \mathcal{B}'\ . \tag{5.28}$$

If Φ *is CP so is* Φ'_ρ*, and if* Φ *is unital* Φ'_ρ *is unital and faithful.*

First, consider the case where the whole algebra $\mathcal{B}$ is correctable. This means that there exists a recovery map $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ that isometrically embeds $\mathcal{B}$ in $\mathcal{A}$

$$\mathcal{R}(b)\,|\rho_A^{1/2}\rangle = Wb\,|\rho_B^{1/2}\rangle \tag{5.29}$$

with $W : \mathcal{H}_B \to \mathcal{H}_A$ an isometry. The map $\Phi \circ \mathcal{R} = \mathrm{id}$ and $\mathcal{R} \circ \Phi : \mathcal{A} \to \mathcal{R}(\mathcal{B}^C) \equiv \mathcal{A}^C \subset \mathcal{A}$ is a conditional expectation that preserves the faithful state ρ_A.

Theorem 4.4.4 tells us that the necessary and sufficient condition for the existence of such a conditional expectation is $J_A W = W J_B$ that we call the Takesaki condition. We use

this property in the next theorem to establish that the recovery map is the Petz dual of the error map, see figure 5.10:

Theorem 5.1.2 (Petz dual)**.** *Let $\Phi : \mathcal{A} \to \mathcal{B}$ be an unital completely positive map between von Neumann algebras. Let ρ_B and $\rho_A = \rho_B \circ \Phi$ be faithful states. Denote by $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$ the cyclic and separating vectors that represent ρ_A and ρ_B in their corresponding Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. If there exists a normal faithful representation $\mathcal{R} : \mathcal{B} \to \mathcal{A}$ that satisfies $\Phi \circ \mathcal{R} = id$, it is the Petz dual of the error map*

$$\mathcal{R}(b) = \Phi_\rho^P(b) \equiv \mathcal{J}_A \circ \Phi'_\rho \circ \mathcal{J}_B \ . \tag{5.30}$$

where $\mathcal{J}_A : \mathcal{A}' \to \mathcal{A}$ and $\mathcal{J}_B : \mathcal{B} \to \mathcal{B}'$ are the modular conjugation maps corresponding to $|\rho_A^{1/2}\rangle$ and $|\rho_B^{1/2}\rangle$, respectively.

Proof: The superoperator Φ is unital and CP, therefore it corresponds to a contraction $F : \mathcal{H}_A \to \mathcal{H}_B$. First, we prove that if the whole algebra $\mathcal{B}$ is correctable F is a co-isometry. The image of the recovery map $\mathcal{A}^C \equiv \mathcal{R}(\mathcal{B})$ is a subalgebra of $\mathcal{A}$. The composite map $\mathcal{E} = \mathcal{R} \circ \Phi : \mathcal{A} \to \mathcal{A}^C$ is unital, CP and preserves every operator in $\mathcal{A}^C$, hence it is a conditional expectation. The operator corresponding to this conditional expectation is a projection to the range of W: $WW^\dagger$. Therefore,

$$\mathcal{R} \circ \Phi(a) \, |\rho_A^{1/2}\rangle = WFa \, |\rho_A^{1/2}\rangle = WW^\dagger a \, |\rho_A^{1/2}\rangle \ . \tag{5.31}$$

Since $|\rho_A^{1/2}\rangle$ is cyclic and separating we have $WF = WW^\dagger$ or equivalently $F = W^\dagger$ is a co-isometry. Since this conditional expectation preserves ρ_A we have the Takesaki condition $J_A W = W J_B$.

Now, consider the Petz dual map $\Phi_\rho^P(b)$. We check that it satisfies the recovery equation

$$\Phi \circ \Phi_\rho^P(b) \, |\rho_B^{1/2}\rangle = W^\dagger J_A W J_B b \, |\rho_B^{1/2}\rangle = b \, |\rho_B^{1/2}\rangle \tag{5.32}$$

where we have used the Takesaki condition for ρ_A. Since $|\rho_B^{1/2}\rangle$ is cyclic and separating this implies that $\Phi \circ \Phi_\rho^P(b) = b$ for all $b \in \mathcal{B}$. In the absence of a kernel for the error map this is the unique recovery map from $\mathcal{B} \to \mathcal{A}^C$. $\square$

Next, consider the reconstruction problem where only a proper subalgebra $\mathcal{B}^C \subset \mathcal{B}$ is correctable. The Hilbert space $\mathcal{H}_B$ is a representation of $\mathcal{B}^C$ but the vector $|\rho_B^{1/2}\rangle$ is no longer a cyclic and separating vector for $\mathcal{B}^C$. We can use the theorem below to show that the recovery map is dual to $\Phi(a') = W^\dagger a' W \in (\mathcal{B}^C)'$:

Theorem 5.1.3 (Reconstruction maps: theorem 1 of [85]). *Let $W : \mathcal{H}_B \to \mathcal{H}_A$ be an isometry in between Hilbert spaces that represent von Neumann algebras $\mathcal{B}$ and $\mathcal{A}$, respectively. The following two statements are equivalent:*

1. *For all $a \in \mathcal{A}$ we have $\alpha(a) = W^\dagger a W \in \mathcal{B}$.*

2. *There exists a normal isometric embedding (injective $*$-homomorphism) $\alpha' : \mathcal{B}' \to \mathcal{A}'$ such that $\alpha'(b')W = Wb'$ for all $b' \in \mathcal{B}'$.*

When there exists a vector $W|\rho_B^{1/2}\rangle$ that is cyclic and separating for $\mathcal{A}$, α is faithful and the map α' is the unique ρ-dual and is unital.

The recovery map satisfies the statement (2) therefore it is dual to the map $W^\dagger a' W \in (\mathcal{B}^C)'$ that we call Φ with an abuse of notation. The map Φ acts as $\Phi : \mathcal{A} \to \mathcal{B}$ and $\Phi : \mathcal{A}' \to (\mathcal{B}^C)'$. Since $\mathcal{B}^C$ is smaller than $\mathcal{B}$ we do not have complementary recovery. We cannot combine $\Phi : \mathcal{A} \to \mathcal{B}$ and $\mathcal{R} : \mathcal{B}^C \to \mathcal{A}$ to get a conditional expectation. A simple solution is to look for conditional expectations that project from $\mathcal{B}$ to $\mathcal{B}^C$. As we review in chapter 4, in finite dimensions, there is a one-to-one correspondence between the conditional expectations from $\mathcal{B}$ to $\mathcal{B}^C$ and the states on the relative commutant of $\mathcal{B}^C$ in $\mathcal{B}$. With any conditional expectation $\mathcal{E}_B : \mathcal{B} \to \mathcal{B}^C$ we can redefine the error map to $\Phi \to \mathcal{E}_B \circ \Phi$. We are back to the case where the whole image of the error map is correctable, and the recovery map is the Petz dual of the new error map.

If the inclusion of $\mathcal{B}^C \subset \mathcal{B}$ has finite index there always exists a conditional expectation from $\mathcal{B} \to \mathcal{B}^C$. Any von Neumann subalgebra $\mathcal{B}^C$ is a direct integral of factors: $\mathcal{B}^C = \int_q^\oplus \mathcal{C}^q$.[26]

[26] ↑A factor is a von Neumann algebra with trivial center.

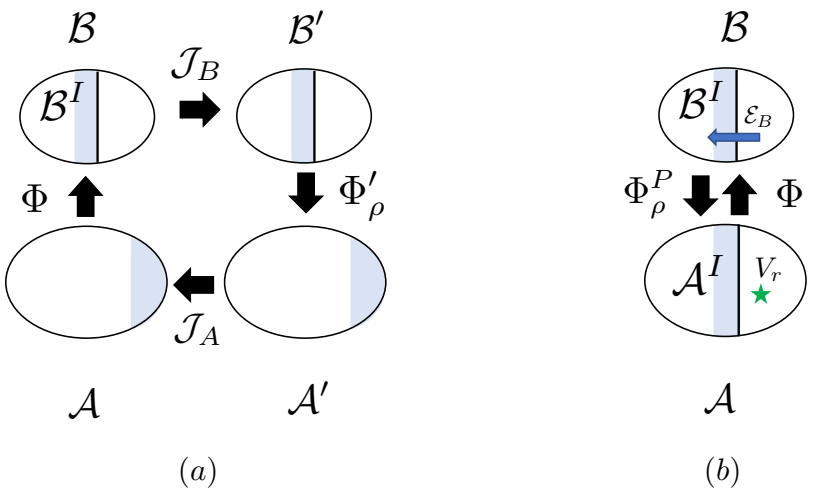


Figure 5.11. (a) Given a correctable state ρ_A we can construct the conditional expectation that projects $\mathcal{B}$ to the invariant subalgebra $\mathcal{B}^I$ of $\Phi \circ \Phi^P_\rho$. (b) The Petz map Φ^P_ρ plays the role of the recovery map sending the operators in $\mathcal{B}^I$ to the subalgebra $\mathcal{A}^I$ that commutes with all errors. This is the von Neumann algebra generalization of the condition $[c, V_r^\dagger V_s] = 0$ for the operators in the correctable subalgebra.

Roughly speaking, the index of a subfactor $[\mathcal{C}^q : \mathcal{B}]$ is a measure of how many times the algebra $\mathcal{C}^q$ fits inside $\mathcal{B}$, and when there exists no conditional expectations from $\mathcal{B}$ to $\mathcal{C}^q$ this index is defined to be infinite. When the index is finite there are conditional expectations $\mathcal{E}^q : \mathcal{B} \to \mathcal{C}^q$ [102]. If all the inclusion of all $\mathcal{C}^q$ in $\mathcal{B}$ have finite indices the direct integral of $\mathcal{E}^q$ is a conditional expectation $\mathcal{E} : \mathcal{B} \to \mathcal{B}^C$.

The correctable subalgebra is the subalgebra of operators that commute with $V_r^\dagger V_s$.[27] We would like to generalize this to arbitrary von Neumann algebras. If there exists no correctable states the correctable subalgebra is empty. Therefore, we consider the case where we have an error map $\Phi : \mathcal{A} \to \mathcal{B}$ and a state ρ_A that is correctable. We follow a strategy similar to the passive error correction in chapter 4. The map $\Phi \circ \Phi^P_\rho : \mathcal{B} \to \mathcal{B}$ is unital and CP. We

[27] ↑See chapter 4.

consider the conditional expectation that projects to its invariant subalgebra that we denote by $\mathcal{B}^I$:

$$\mathcal{E}_B = \lim_{N\to\infty} \frac{1}{N} \sum_{n=1}^{N} (\Phi \circ \Phi_\rho^P)^n \ . \tag{5.33}$$

This is an error correction code for the correctable algebra $\mathcal{B}^I$ with the recovery map Φ_ρ^P because for all $c \in \mathcal{B}^I$ we have $\Phi \circ \Phi_\rho^P(c) = c$. The range of the recovery map is a subalgebra in $\mathcal{A}$ that we denote by $\mathcal{A}^I = \Phi_\rho^P(\mathcal{B}^I)$. The map $\mathcal{E}_A = \Phi_\rho^P \circ \mathcal{E}_B \circ \Phi$ is a conditional expectation from $\mathcal{A}$ down to $\mathcal{A}^I$; see figure 5.11. We can redefine the error map to $\mathcal{E}_B \circ \Phi : \mathcal{A} \to \mathcal{B}^I$. We are back to the standard case above, and the recovery map is once again the Petz dual of the error map.

5.1.5 Sufficiency conditions

To characterize all "sufficient" states of a ρ-preserving conditional expectation $\mathcal{E}_\rho$ we show that it preserves another state ω if and only if the *sufficiency condition*

$$\omega^{1/2}\omega_C^{-1/2} = \rho^{1/2}\rho_C^{-1/2} \tag{5.34}$$

is satisfied [71], [72]. If we are given a ρ-preserving conditional expectation $\mathcal{E}_\rho$ the map

$$\mathcal{E}_\rho^\omega(a) = \omega_C^{-1/2}\rho_C^{1/2}\mathcal{E}_\rho\left(\rho^{-1/2}\omega^{1/2}a\omega^{1/2}\rho^{-1/2}\right)\rho_C^{1/2}\omega_C^{-1/2} \tag{5.35}$$

is a ω-preserving CP map from $\mathcal{A} \to \mathcal{A}^C$. If it preserves every operator in $c \in \mathcal{A}^C$ it becomes an ω-preserving conditional expectation. It is clear that if the sufficiency condition in (5.35) holds it becomes an ω-preserving conditional expectation $\mathcal{E}_\omega = \mathcal{E}_\rho$. Therefore, $\mathcal{E}_\rho$ also preserves ω. We now prove the converse: the conditional expectation $\mathcal{E}_\rho$ preserves ω only if the condition (5.34) holds. We basically repeat the proof of Takesaki's theorem for the relative Tomita operator $S_{\omega|\rho}a\,|\rho^{1/2}\rangle = a^\dagger\,|\omega^{1/2}\rangle$. The norm of this operator is the relative modular operator $\Delta_{\omega|\rho} : \mathcal{H}_\rho \to \mathcal{H}_\rho$. The superoperator corresponding to it is $\mathcal{D}_{\omega|\rho}(a) =$

$\omega a \rho^{-1}$. We repeat the argument for the Takesaki theorem with the relative modular map $\mathcal{D}_{\omega|\rho}(a) = \omega a \rho^{-1}$ to find $[P_\rho, \Delta_{\omega|\rho}^{1/2}] = 0$. This implies

$$\mathcal{E}_\rho(\mathcal{D}_{\omega|\rho}^{1/2}(c)) = \mathcal{D}_{\omega|\rho}^{1/2}(\mathcal{E}_\rho(c)) = D_{\omega|\rho}^{1/2}(c) \in \mathcal{A}^C \tag{5.36}$$

We define the isometries

$$\begin{aligned} W_\rho c \left|\rho_C^{1/2}\right\rangle &= c \left|\rho^{1/2}\right\rangle \\ W_\omega c \left|\omega_C^{1/2}\right\rangle &= c \left|\omega^{1/2}\right\rangle \end{aligned} \tag{5.37}$$

so that

$$\begin{aligned} S_{\omega|\rho} W_\rho &= W_\omega S_{\omega_C|\rho_C} \\ W_\rho^\dagger \Delta_{\omega|\rho} W_\rho &= \Delta_{\omega_C|\rho_C} \ . \end{aligned} \tag{5.38}$$

Since $[P_\rho, \Delta_{\omega|\rho}^{1/2}] = 0$ we have

$$P_\rho \Delta_{\omega|\rho}^{1/2} = W_\rho \Delta_{\omega_C|\rho_C}^{1/2} W_\rho^\dagger \ . \tag{5.39}$$

As a result,

$$\begin{aligned} &\mathcal{D}_{\omega|\rho}^{1/2}(c) \left|\rho^{1/2}\right\rangle = P_\rho \Delta_{\omega|\rho}^{1/2} c \left|\rho^{1/2}\right\rangle = W_\rho \Delta_{\omega_C|\rho_C}^{1/2} c \left|\rho_C^{1/2}\right\rangle \\ &= W_\rho \mathcal{D}_{\omega_C|\rho_C}^{1/2}(c) \left|\rho_C^{1/2}\right\rangle = \mathcal{D}_{\omega_C|\rho_C}^{1/2}(c) \left|\rho^{1/2}\right\rangle \ . \end{aligned} \tag{5.40}$$

We obtain that

$$\omega^{1/2} c \rho^{-1/2} = \mathcal{D}_{\omega|\rho}^{1/2}(c) = \mathcal{D}_{\omega_C|\rho_C}^{1/2}(c) = \omega_C^{1/2} c \rho_C^{-1/2} \ . \tag{5.41}$$

In other words,

$$\omega_C^{-1/2} \omega^{1/2} c \rho^{1/2} \rho_C^{-1/2} = c = \rho_C^{-1/2} \rho^{1/2} c \rho^{-1/2} \rho_C^{1/2} \tag{5.42}$$

which holds if and only if the sufficiency condition in (5.34) is satisfied.

The sufficiency condition can be expressed as

$$\Delta_{\omega|\rho}^{1/2} = W_\rho \Delta_{\omega_C|\rho_C}^{1/2} W_\rho^\dagger \ . \tag{5.43}$$

Using the integral representation of X^α for $\alpha \in (0,1)$

$$X^\alpha = \frac{\sin(\pi\alpha)}{\pi} \int_0^\infty ds\, s^\alpha \left(\frac{1}{s} - \frac{1}{s+X} \right) \tag{5.44}$$

we find

$$\int_0^\infty ds\, s^{1/2} \left(\frac{1}{s+\Delta_{\omega|\rho}} - W_\rho \frac{1}{s+\Delta_{\omega_C|\rho_C}} W_\rho^\dagger \right) = 0 \ . \tag{5.45}$$

From the monotonicity of the relative modular operator [58], [75] we know that the operator in the integrand above is positive, therefore it has to be zero:

$$\frac{1}{s+\Delta_{\omega|\rho}} = W_\rho \frac{1}{s+\Delta_{\omega_C|\rho_C}} W_\rho^\dagger \tag{5.46}$$

which implies

$$\Delta_{\omega|\rho}^{\alpha} = W_\rho \Delta_{\omega_C|\rho_C}^{\alpha} W_\rho^\dagger \ . \tag{5.47}$$

Furthermore, for any continuous function f we have

$$W_\rho f(\Delta_C)\, |\rho_C^{1/2}\rangle = f(\Delta)\, |\rho^{1/2}\rangle \ . \tag{5.48}$$

In particular, choosing $f(x) = x^{\mathrm{i}t}$ for $t \in \mathbb{R}$ we find that $\rho_C^{\mathrm{i}t} \omega_C^{-\mathrm{i}t} = \rho^{\mathrm{i}t} \omega^{-\mathrm{i}t}$. This condition implies that the relative entropy for any pair of sufficient states ρ and ω:

$$S(\omega\|\rho) = S(\omega_C\|\rho_C) \ . \tag{5.49}$$

Intuitively, this says that a coarse-graining (conditional expectation) preserves a set of states $\{\rho_k\}$ (sufficient states) if and only if the distinguishability (relative entropy) of any pair of them remains the same.

5.1.6 Discussion

In summary, we argued that the renormalization group is an approximate error correction code. This is similar to modeling the holographic map as a subsystem error correction code, with the difference that we do not have complementary recovery. We discussed how the complementary recovery emerges in a theory with large N and a large gap.

We studied the operator algebra quantum error correction for an arbitrary von Neumann algebra. If the error map has a kernel some information is irreversibly lost. In real-space RG, the vacuum vector of a QFT is cyclic and separating which implies that the kernel of the RG map is trivial. In von Neumann algebras, the analog of the Knill-Laflamme condition for exact error correction is the Takesaki condition. When recovery is possible, the recovery map is the Petz dual of the error map.

If the kernel of the error map is not empty (we do not have a cyclic and separating vector) the composition of the recovery map and the error $\mathcal{R} \circ \Phi : \mathcal{A} \to \mathcal{A}^C$ is still a CP map that preserves every operator in $\mathcal{A}^C$, but it is no longer unital. In the language of von Neumann algebras, such a map is an operator valued weight: an unbounded unnormalized positive map with dense domain in $\mathcal{A}_+$ (the positive operators of $\mathcal{A}$) that satisfies the bi-module property [28]. There exists a bijection in between the set of operator value weights from $\mathcal{A} \to \mathcal{A}^C$ and those from $(\mathcal{A}^C)'$ to $\mathcal{A}'$ [103]. The study of operator valued weights could shed light on the problem of reconstruction in the absence of a faithful state.

Consider the $\text{AdS}_{d+1}/\text{CFT}_d$ correspondence in $d > 1$ and a simply connected region A. In time-reversal symmetric geometries, the Rangamani-Takayanagi (RT) surface is the co-dimension two surface in the bulk that is anchored on the boundary of A, is homologous to A and has minimal area; see figure 5.12. Denote by B the region in the bulk that is in between the RT surface and A. Consider the map $\mathcal{R}$ that encodes the algebra $\mathcal{B}$ of the bulk

[28] ↑See chapter 4 for a discussion of the bi-module property.

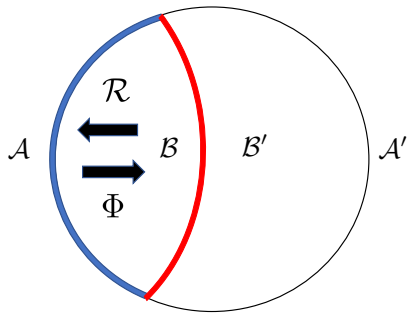


Figure 5.12. A time slice of anti-de Sitter space with $\mathcal{A}$ the algebra of a region A on the boundary and $\mathcal{B}$ the algebra of the bulk region that is in between A and the Ryu-Takayanagi surface of A. The CP map Φ maps the boundary local algebra to the bulk, whereas $\mathcal{R}$ reconstructs the bulk operators on the boundary.

on the boundary (bulk reconstruction map). We choose the error map to be $\Phi = \alpha \circ \mathrm{tr}_{A'}$ where $\alpha(\cdot) = W^\dagger(\cdot)W$ and $W : \mathcal{H}_{bulk} \to \mathcal{H}_{boundary}$ is the encoding isometry. All the bulk operators $b \in \mathcal{B}$ satisfy the error correction condition $\Phi(\mathcal{R}(b)) = b$ and the recovery map $\mathcal{R}$ is an isometric embedding. The holographic map from the boundary algebra to the bulk algebra has no kernel because both of the bulk and boundary vectors are cyclic and separating with respect to their corresponding algebras. We have complementary recovery and the whole bulk algebra $\mathcal{B}$ is reconstructable. The reconstruction map $\mathcal{R}$ is the Petz dual of the holographic map Φ. A similar observation was discussed in a recent paper [85]. Given a ρ-preserving conditional expectation we can define a measure of the information lost under the conditional expectation [3]. This leads to entropic uncertainty relations that play an important role in the derivation of the Ryu-Takayanagi formula in holography [16], [85]. It has been argued that complementary recovery fails in some situations in holography [89]. That brings the holography reconstruction problem closer to the real-space RG.

Finally, we make the following observation: In $\mathrm{AdS}_2/\mathrm{CFT}_1$ the bulk reconstruction map cannot be a conditional expectation, because there exists no conditional expectations from a type I algebra (the boundary theory is $0+1$ dimensional) to a type III von Neumann algebra (the bulk theory is $1+1$ dimensional QFT). We believe that the resolution of this seeming

paradox is that the bulk and boundary relative entropies match only up to $1/N$ corrections. The error correction properties of the holographic map are only approximate. A related observation is that we can define CP maps in between $*$-closed subspaces of observables (operator systems). This generalization can be helpful in moving away from the exact error correction in holography.

5.2 Renormalization group and approximate error correction

5.2.1 Introduction

Renormalization group (RG) flow is a pillar of the twentieth century physics that has allowed us to study the universal dynamics of the emergent long-range effective degrees of freedom. Quantum error correction (QEC) teaches us how to encode quantum information non-locally to protect it against local noise and decoherence. They both involve the physics of states with long-range correlations. In this work, we show that the states deep in the infra-red of an RG flow form an approximate error correction code.

Analogously, we can view the RG as an isometric embedding W of the infrared (IR) states (logical states) into the ultra-violet (UV) states (physical states): $W : \mathcal{H}_{IR} \to \mathcal{H}_{UV}$ [29]. Irrelevant local perturbations are the noise that the encoding protects against: $P\mathcal{O}_{\text{irrel}}P \sim P$. In this work, we explore this connection in three examples: 1) the RG flow of classical Ising model, 2) the real-space RG flow of free massive quantum fields realized as continuous Multi-scale Renormalization Ansatz (cMERA), 3) holographic RG flows as examples of strongly-coupled QFTs.

There are many ways to coarsegrain the observables of a quantum system, and not all of them correspond to isometric embeddings of the IR observables in the RG algebra. However, it is worth mentioning that two commonly used approaches to RG, namely the tensor network renormalization of many-body quantum systems, and exact RG in continuum field theories both lead to isometric flows [104], [105]. In this work, we focus on isometric RG flows.

The idea of a connection between the RG and the QEC is not new. It is known that there is an exact error correction code at the IR end point of an RG flow if there are degenerate

[29] ↑In general, it suffices to take W to be an approximate isometry. In relativistic theories W can be unitary.

ground states that are gapped. Such vacuum subspaces can form either due to spontaneous symmetry breaking or topological order; see [106], [107] and [30]. The vacuum subspace is the code subspace and the noise are local operators.

The spontaneous breaking of a discrete or continuous symmetry group leads to classical error correction codes. For instance, the two-dimensional Ising model at low temperature breaks the $\mathbb{Z}_2$ symmetry spontaneously by forming long-range ordered ferromagnetic states $|00\cdots0\rangle$ and $|1\cdots1\rangle$. This is a classical repetition code that corrects for local bit flips (σ_X Pauli matrix). However, this is not a quantum error correction code because the errors caused by the σ_z Pauli matrix cannot be corrected. In fact, σ_z is precisely the local order parameter that distinguishes different code states.

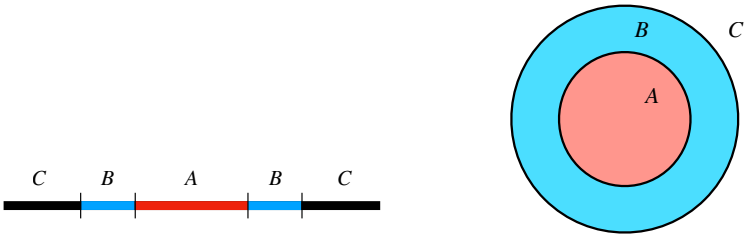
To obtain a QEC code, we need the local density matrices of topologically trivial regions to be the same in all the degenerate vacuum (code states). This implies that no local observable can distinguish the encoded states. In other words, the system has topological order. For example, the four-fold degenerate vacuum subspace of $\mathbb{Z}_2$ Toric code on a torus encodes two logical qubits that are protected against any error localized on a shrinkable subregion. In fact, one can correct for errors localized on disjoint subsystems, e.g. A and C, as long as the combined region, e.g. AC, does not contain non-contractible loops.

In this work, we extend this connection and show that even in the case of a unique ground state the low energy states are approximately protected against the errors on small enough subregions. Moreover, we show that the error correcting code is a local quantum error correction (LQEC) code [108] because the code can correct the errors localized in disjoint subregions A and C which are separated by a region B as shown in Fig. (5.13a) [31]. By definition, a QEC is called local if the recovery map that corrects the errors on A can be chosen to be a local map from B to AB [108], see Fig. (5.13b) for a two-dimensional example.

In section 5.2.2, as our first example, we consider the one-dimensional classical Ising model. To perform RG, we use Kadanoff's spin blocking. The block-spin renormalization of

[30]↑*Entanglement, wormholes and quantum error correction* talk by Brian Swingle, 4th inter. conf. on QEC, qec2017.gatech.edu.

[31]↑A QEC is called local if its stabilizers or gauge generators are supported on a small bounded region of the space.



(a) (b)

Figure 5.13. (a) A one-dimensional system on a line is partitioned into regions A and C which are separated by a small region B. In cMERA, we find that the low energy coherent states are protected against the UV coherent operators localized in subregions A and C.(b) The two-dimensional spatial region is partitioned by ABC. The local erasure tr_A acts on the red-colored region A. The blue-colored region is the spatial domain of a local recovery map.

a classical system leads to approximate classical error correction codes. The quantum analog of spin blocking is the so-called tensor network renormalization [95], [109]. Applied to the Euclidean time evolution operator [109] it yields an isometric RG flow called the Multi-scale Entanglement Ansatz (MERA) [104]. MERA tensor networks are well-suited for describing real-space RG flow [81], [95], topological codes [110], [111], and the low energy states of scale-invariant theories. It was shown in [81] that the MERA states form an approximate QEC. That is to say that the IR information is approximately protected against the erasure of small local regions in the UV lattice [32]. We briefly review this example in section 5.2.3.

In section 5.2.4, we argue that these QEC properties generalize to the RG flow of continuum Poincare-invariant QFTs. In [1], exact QEC in the Heisenberg picture was generalized to arbitrary von Neumann algebras, including the local algebra of QFT. We consider continuous MERA (cMERA) for free massive quantum fields in $1+1$-dimensions. We use the field coherent states to encode quantum information locally and study the RG flow of these code states. We consider the geometry in figure 6.1(a), and show that the encoded low energy states form a local QEC (LQEC) code that is approximately protected against the action of the UV coherent operators in region AC.

To explore the error correction properties of RG in the strongly coupled QFTs, we consider holographic RG flows. In holography, the emergent radial direction of the bulk can be interpreted as the renormalization group scale of the boundary theory. The operators in the IR are deeper in the bulk. They are protected against UV errors that are supported on small regions. In section 5.2.5, we study a 3-dimensional geometry that corresponds to a holographic RG flow from a UV 2-dimensional CFT to an IR one. We observe that not all the encoded logical information is exactly protected. This becomes manifest by considering the so-called holographic Singleton bound that puts an upper bound on the maximum amount of logical information that can be encoded in a bulk subregion [112]. We find that a naive application of the holographic Singleton bound in such a geometry suggests that there are finite volume regions in the bulk where no information can be encoded. We further propose a

[32]↑The authors of [81] showed that MERA as an approximate QEC code satisfies the trade-off bound $kd^{\alpha} \leq cn$ where α is a constant fixed in terms of the size ratio $|AB|/|A|$.

modification to the holographic Singleton bound which resolves the aforementioned paradox. We conclude with a summary and discussion in section 5.2.6.

5.2.2 Classical spin-blocking

In classical physics, the standard real-space RG scheme of a spin system involves splitting the lattice into blocks and coarse-graining the degrees of freedom localized inside each block. The coarse-graining replaces the collection of spins inside the block $\{s_{\mathrm{i}}\}$ with a collective degree of freedom s' [113], [114]. For example, consider the translation-invariant one dimensional classical Ising model with $3N$ sites, a local Hamiltonian

$$H(\{s\}) = -J \sum_{\mathrm{i}=1}^{3N-1} s_{\mathrm{i}} s_{\mathrm{i}+1} + h \sum_{\mathrm{i}}^{3N-1} s_{\mathrm{i}} \tag{5.50}$$

and periodic boundary conditions; see figure 6.2. The configuration space is the set of all sequences $\{s\} = \{s_1 s_2 \cdots s_{3N}\}$ and the thermal state is the probability distribution

$$p(\{s\}) = \mathrm{e}^{-\beta H(\{s\})}/Z \tag{5.51}$$

on this configuration space. A simple coarse-graining scheme is the majority vote scheme

$$s' = \begin{cases} +1 & \text{if } s_1 + s_2 + s_3 \geq 0 \\ -1 & \text{otherwise .} \end{cases} \tag{5.52}$$

It is convenient to represent states as vectors $|s_1 s_2 s_3\rangle$ and think of the coarse-graining map as a matrix T describing a classical information channel acting on probability vectors [33]. We think of the normalized transpose map $T^* \equiv T^T/4$ as an encoding isometry, the s' variable as the logical state and the $\{s\}$ variables as the physical states. The code subspace is spanned by the two states $|T^*(\pm 1)\rangle$ [34]. A simple model of local noise is a one-site bit flip error $\pm 1 \to \mp 1$ with probability p. For instance, its action on the first bit is given by

[33] ↑The matrix elements are $\langle s'|T|s_1 s_2 s_3\rangle$. In our example, the map T is $T = \left(\begin{smallmatrix} 1 & 1 & 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 1 & 1 & 1 \end{smallmatrix}\right)$.

[34] ↑The encoded states are $|T^*(\pm 1)\rangle = \frac{1}{4}(|\pm 1, \pm 1, \pm 1\rangle + |\pm 1, \pm 1, \mp 1\rangle + |\pm 1, \mp 1, \pm 1\rangle + |\mp 1, \pm 1, \pm 1\rangle)$.

the symmetric binary channel $G_1 = \begin{pmatrix} 1-p & p \\ p & 1-p \end{pmatrix}$. In the physical space $|s_1 s_2 s_3\rangle$ this error is represented by $G_1 \otimes \mathcal{I}_{23}$. It propagates to the logical bits $|s'\rangle$ and acts as the matrix

$$T(G_1 \otimes \mathcal{I}_{23})T^* = \begin{pmatrix} 1-\frac{p}{2} & \frac{p}{2} \\ \frac{p}{2} & 1-\frac{p}{2} \end{pmatrix} . \tag{5.53}$$

By translation invariance of the code states, the other two local errors G_2 and G_3 also lead to the same logical error matrix above. The key observation is that the error after coarse-graining (equation 5.53) is the same as the original error but weaker, because the probability of bit-flip is cut in half. It can be viewed as an eigen-operator of the coarse-graining map $\Phi(\cdot) = T(\cdot)T^*$ with eigenvalue $1/2$. In real-space RG, we repeat the coarse-graining map n (large number) times to flow from the short-distances to long-distances. In the case of the above model with $3N$ sites, if we start with the local error map $G_{\rm i}$ on some site $s_{\rm i}$, after the first step the error is

$$\Phi(G_{\rm i}) = \mathcal{T}(G_{\rm i} \otimes \mathcal{I}_{\setminus {\rm i}})\mathcal{T}^* \tag{5.54}$$

where $(G_{\rm i} \otimes \mathcal{I}_{\setminus {\rm i}})$ is the local error in the UV, and $\mathcal{T} = T^{\otimes N}$. After n steps of coarse-graining the errors are exponentially weaker

$$\Phi^n(G_{\rm i}) = \begin{pmatrix} 1-2^{-n}p & 2^{-n}p \\ 2^{-n}p & 1-2^{-n}p \end{pmatrix} . \tag{5.55}$$

As we flow from the very short distances (UV) to very long distances (IR) the local errors are expected to decay exponentially fast $\lim_{n\to\infty} \Phi^n(G_{\rm i}) = \mathcal{I}$.

Next, consider the non-local error $G_1 \otimes G_2 \cdots \otimes G_k$ that corrupts k adjacent sites. After one level of coarse-graining it corrupts $\lfloor (k-2)/3 \rfloor + 2$ sites. After each step of coarse-graining the support of non-local errors shrinks almost by a factor of three, until it becomes local at which point the above analysis applies. This logic extends to arbitrary k-site error model. There are two stages to the renormalization of any error of finite support in the UV Hilbert space. In the first stage, the support of the operator shrinks monotonically. In the

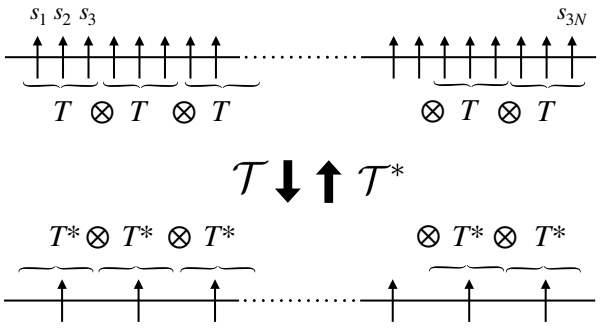


Figure 5.14. This is a single step coarse-graining on a classical Ising model in one dimension with $3N$ sites. Since T coarse-grains three spins into a single spin, $\mathcal{T} = T^{\otimes N}$ coarse-grains $3N$ sites into N sites.

second stage, the error becomes exponentially weaker [81]. Deep in the IR the RG flow is an approximate classical error correction code in the trivial sense that k-local errors are highly unlikely to corrupt the encoded data.

The RG flow map $\Phi(\cdot) = \mathcal{T}(\cdot)\mathcal{T}^*$ is a classical channel and all its eigenvalues have norm less than one [35]. More generally, we could have considered the asymmetric binary channel

$$G = p_1 \begin{pmatrix} 1-p_1 & p_1 \\ p_2 & 1-p_2 \end{pmatrix} = \mathcal{I} + p_1 G_+ + p_2 G_-$$

where both G_+ and G_- are eigenoperators of the RG map with eigenvalue $1/2$ and the identity map is invariant. There are no one-site errors that are left invariant under the RG map. The only fixed point of the RG map corresponds to acting with the noise at every single site. The support of such an operator never shrinks to one-site. The ground states of the one-dimensional nearest neighbor Ising model form an exact repetition code. A simultaneous $\mathbb{Z}_2$ flip on all sites is a logical operation which takes us from one code state to another in the repetition code. In the absence of fixed points, the states after a large but finite number of RG steps are our code-words, the encoding is $(\mathcal{T}^*)^n$ and the largest eigenvalue of the RG map controls how well this approximate error correction code protects the classical information.

[35] ↑A classical channel is a stochastic matrix, or equivalently a conditional probability distribution.

5.2.3 Quantum spin blocking (MERA)

The intuitive discussion above generalizes to the renormalization group flow of quantum systems with local Hamiltonians. In a gapped system, the RG flow becomes trivial at scales above the correlation length. Since we are interested in repeating the RG map many times, we focus on the real-space RG in critical systems. We start with a lattice theory in the IR. The RG map can be viewed as an encoding isometry $W : \mathcal{H}_{IR} \to \mathcal{H}_{UV}$. In MERA, this isometry corresponds to two layers, first a layer of local isometries $V \otimes \cdots \otimes V$, and second a layer of local unitaries $U \otimes \cdots \otimes U$ called the disentanglers. The layer of local isometries is the quantum analog of $\mathcal{T}^*$ map in the example of classical Ising model. In quantum real-space RG, the disentanglers are essential to correctly remove the UV entanglement. By the same logic as in the classical case, we view the RG flow as a quantum channel $\Phi(\mathcal{O}_{UV}) = W^\dagger \mathcal{O}_{UV} W$ acting on UV operators $\mathcal{O}_{UV}$. Repeating the channel many times corresponds to flowing deeper towards the IR. For concreteness consider a MERA where every RG step cuts the number of sites in half.

As discussed in [81], the RG evolution of non-local operators $\mathcal{O}_{UV}(A)$ originally supported on a compact region A follows two stages: first the support of the operator shrinks exponentially fast until after approximately $\log |A|$ steps, and it becomes a single site operator $\tilde{O}$. We decompose the local operator $\tilde{\mathcal{O}}$ as $\tilde{O} = \mathrm{tr}(\tilde{O})\mathbb{I}/D + \sum_{\mathrm{i}} a_{\mathrm{i}} X_{\mathrm{i}}$ with $\{X_{\mathrm{i}}\}$ form an orthonormal basis of local traceless operators. For instance, in the case where every site is a qubit X_{i} corresponds to Pauli matrices.

Then, as we flow further towards the IR, the second stage starts. The RG superoperator leaves the identity operator invariant: $\Phi^s(\mathbb{I}) = \mathbb{I}$, but the norm of all X_{i} operators falls off exponentially fast in s. The reason is simply that Φ is a quantum channel and its eigenvalues have norm less than one. That is, all the eigenoperators $\mathcal{O}_{\mathrm{i}}$ satisfy

$$\Phi(\mathcal{O}_{\mathrm{i}}) = \lambda_{\mathrm{i}} \mathcal{O}_{\mathrm{i}}, \qquad |\lambda_{\mathrm{i}}| \leq 1 \ . \tag{5.56}$$

Denote by λ_1 the second largest eigenvalue of the RG superoperator Φ [36][81]. We refer to $\nu = -\log(Re\lambda_1) > 0$ as the *gap* in the scaling spectrum [37]. After s steps of RG the norm of any traceless operator falls off at least as fast as $2^{-\nu}$. After a total of s RG steps, a noise operator $\mathcal{O}_{UV}(A)$ originally supported on region A has evolved to $\Phi^s(\mathcal{O}_{UV}(A))$ to

$$|\Phi^s(\mathcal{O}_{UV}(A)) - \mathrm{tr}(\tilde{\mathcal{O}})\mathbb{I}/d| \leq 2^{-\nu(s-\log|A|)}|\tilde{\mathcal{O}}|. \tag{5.57}$$

The Knill-Laflamme error correction condition [115] requires that for $\{|\Psi_r\rangle\}$ a basis of the IR states and $\mathcal{O}_{UV}(A)$ an arbitrary UV noise support on compact region A

$$\langle\Psi_r|\Phi^s(\mathcal{O}_{UV}(A))|\Psi_{r'}\rangle = c_{rr'}\delta_{rr'} + Y \tag{5.58}$$

with $|Y| \ll 1$. After s RG steps the equation above is satisfied with

$$|Y| \leq 2^{-\nu(s-\log|A|)}|\tilde{O}| \tag{5.59}$$

which becomes arbitrarily small deep in the IR.

Given a code subspace the distance of a code d is defined to be the minimum support of errors that cannot be corrected. One might guess that the distance of the code we defined above is $|A|$. However, this is incorrect because in MERA there are smaller multi-component regions that contain the information content of the encoded qubit; see figure 5.15. This is because we can erase $\bar{A}$ and smaller regions A_1, A_2 and A_3 inside A and and still recovery our information; see (Fig. (5.15 Right). The authors of [112] called this property uberholography.

In the continuum limit, the RG map of MERA Φ becomes the scaling superoperator, its local eigen-operators are conformal primaries and the eigenvalues are their corresponding

[36]↑The largest eigenvalue is one.

[37]↑We are assuming that there is a gap in the spectrum of the quantum channel; i.e. there exists a smallest a second largest eigenvalue. In finite quantum system, this is obvious, however, for infinite-dimensional local systems it is an assumption that corresponds to ruling out a continuous spectrum of scaling operators above the identity operator.

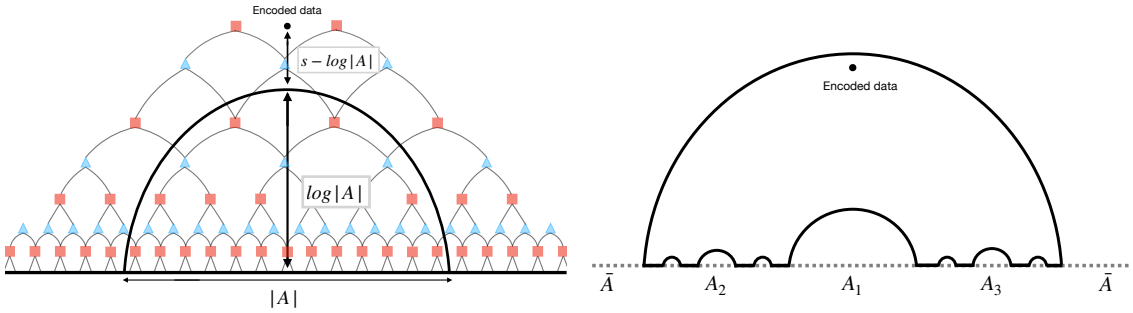


Figure 5.15. (Top) The RG flow of operators localized in A in MERA. The operator shrinks to a point after $s \sim \log|A|$. Deeper in the IR, its norm drops exponentially fast. (Bottom) The encoded data is protected against the erasure of multiple regions: $\bar{A}$, A_1, A_2, etc.

conformal dimensions. If each step of MERA cuts the number of sites down by a factor of γ then

$$\Phi(\mathcal{O}_h) = \gamma^{-h}\mathcal{O}_h \tag{5.60}$$

where h is the conformal dimension of the primary operator $\mathcal{O}_h$. We group the conformal primaries into light $L = \{h : h \leq h_0\}$ and heavy ones $H = \{h : h \geq h_0 + \Delta\}$ and call Δ the gap. Expanding an arbitrary local operator $\tilde{O}$ in terms of local primaries, and one can split it into the light and heavy pieces assuming that all scaling dimensions are positive

$$\begin{aligned} &\tilde{O} = \tilde{O}_L + \tilde{O}_H, \\ &\tilde{O}_L = \sum_{h\in L} a_h \mathcal{O}_h, \qquad \tilde{O}_H = \sum_{h\in H} a_h \mathcal{O}_h \ . \end{aligned} \tag{5.61}$$

Then, after $s \gg 1$ steps of RG flow an arbitrary local operator can be truncated to its light part, i.e. $\tilde{O}_L$:

$$|\Phi^s(\tilde{O}) - \tilde{O}_L| \leq \mathrm{e}^{-s\Delta}|\tilde{O}| \tag{5.62}$$

which can be turned into an approximate QEC statement using the Knill-Laflamme condition in equation 5.58. To see this more explicitly we consider the following example of continuous MERA for a massive free boson in $1+1$-dimensions.

5.2.4 continuous MERA

Consider MERA that goes from deep in the UV to the most IR layer with n local sites and a coarse-graining scheme that, at each step, cuts the number of sites in half. We label the layer deepest in the IR with $s = 0$, and choose the state at $s = 0$ to have zero correlation length and no spatial entanglement $\otimes_x |\Omega(x)\rangle$. At each step of coarse-graining as we flow towards the IR s goes up by one, and the network disentangles modes. Deep in the UV s is a large negative number and we have $2^{-s}n$ sites. In the thermodynamic limit $n \to \infty$ the

MERA network can go on forever and the range of s becomes $(-\infty, 0)$. At each scale we have a ground state wave-function $|\Omega^{(s)}\rangle$.

A generalization of MERA to continuum theories (cMERA) was proposed in [93]. Similar to the discrete case, cMERA is an isometric map that takes the states of a theory with zero correlation length deep in the IR and prepares the low energy states of a QFT (or CFT) in the UV. The IR ground state is taken to be a state $|\Omega^{(0)}\rangle$ with no real-space entanglement. It is more convenient to think of MERA as the isometry from the IR to the UV. The state at energy scale Λe^{-s} is given by

$$|\Omega^{(s)}\rangle = \mathcal{P} e^{i \int_0^s du (K(u)+L(u))} |\Omega^{(0)}\rangle \tag{5.63}$$

where $L(s)$ is the non-relativistic scaling transformation [38], and $K(s)$ is the continuous analog of the layer of entanglers:

$$K(s) = \int d^d k \, \Gamma(|k|/\Lambda) g(s,k) \mathcal{O}_k \ . \tag{5.64}$$

Here, Λ is the cut-off scale and the cut-off function $\Gamma(|k|/\Lambda)$ can be chosen to be sharp or smooth. The operator $\mathcal{O}_k$ is an operator of energy scale k that should be suitably chosen as the generator of the entangling layer. Finally, the function $g(s,k)$ decides the strength of the entangling procedure [116].[39]

For concreteness, consider free massive boson field $\phi(x)$ and its momentum conjugate $\pi(x)$ in one spatial dimension. Following [94] we choose an entangler independent of scale

$$\begin{aligned} K &= \frac{\Lambda}{4} \int dx dy \, e^{-\Lambda|x-y|} \times \\ &\quad (a(x;\Lambda)a(y;\Lambda) - a(x;\Lambda)^\dagger a(y;\Lambda)^\dagger) \end{aligned}$$

[38] ↑It sends $x^i \to \lambda x^i$ keeping time untouched.

[39] ↑In discrete MERA the effective cut-off changes as a function of scale, whereas in cMERA we have kept the cut-off Λ fixed. To compare the two, we consider the rescaled state $|\tilde{\Omega}(u)\rangle = e^{iuL} |\Omega(u)\rangle$.

where

$$a(x;\Lambda) = \sqrt{\frac{\Lambda}{2}}\phi(x) + \frac{\mathrm{i}}{\sqrt{2\Lambda}}\boldsymbol{\pi}(x) \tag{5.65}$$

is the annihilation operator that defines the unentangled state via $a(x;\Lambda)\,|\Omega^{(0)}\rangle = 0$. The cMERA state at scale $\mathrm{e}^{-s}\Lambda$ is

$$|\Omega^{(s)}\rangle = \mathrm{e}^{\mathrm{i}s(L+K)}\,|\Omega^{(0)}\rangle \ . \tag{5.66}$$

This state is the ground state of a massive free boson deformed by a non-relativistic irrelevant term [94][40]:

$$\begin{aligned} H^{(s)}(\Lambda) \ &= \ \int \frac{dx}{2}\Big(\partial_x\phi(x)^2 + \boldsymbol{\pi}(x)^2 \\ &\qquad +\Lambda^2\mathrm{e}^{2s}\phi(x)^2 + \frac{1}{\Lambda^2}(\partial_x\boldsymbol{\pi}(x))^2\Big) \end{aligned} \tag{5.67}$$

We define the annihilation operator that kills $|\Omega^{(s)}\rangle$:

$$\begin{aligned} a_s(k;\Lambda) &= \sqrt{\frac{\alpha_s(k;\Lambda)}{2}}\phi(k) + \frac{\mathrm{i}}{\sqrt{2\alpha_s(k;\Lambda)}}\boldsymbol{\pi}(k) \\ \alpha_s(k;\Lambda) &= \Lambda\sqrt{\frac{k^2+\Lambda^2\mathrm{e}^{2s}}{k^2+\Lambda^2}} \ . \end{aligned} \tag{5.68}$$

The mass term $m(s) = \Lambda\mathrm{e}^s$ runs with scale vanishing in the UV ($s \to -\infty$) and growing in the IR. Since we are interested in the RG flow of a massive theory we fix the mass m and

[40]↑Note that our convention differs from [94] in the sign of s.

vary the cut-off $\Lambda = m\mathrm{e}^{-s}$. To further simplify our discussion we measure all dimensionfull quantities in units of m (we set $m = 1$) [41]:

$$a_s(k) = \sqrt{\frac{\alpha_s(k)}{2}}\phi(k) + \frac{\mathrm{i}}{\sqrt{2\alpha_s(k)}}\pi(k)$$
$$\alpha_s(k) = \sqrt{\frac{k^2+1}{k^2\mathrm{e}^{2s}+1}} \ . \tag{5.71}$$

They satisfy the standard commutation relations

$$[a_s^\dagger(k), a_s(k')] = \delta_{kk'}, \tag{5.72}$$

and the Hamiltonian is

$$H^{(s)} = \int dk\ E_s(k) a_s^\dagger(k) a_s(k) + 1/2$$
$$E_s(k) = \sqrt{k^2+1}\sqrt{1+k^2\mathrm{e}^{2s}} \ . \tag{5.73}$$

The renormalized creation/annihilation operators are related to those of the UV theory

$$a_s^\dagger(k) \pm a_s(k) = \beta_s(k)^{\mp 1}(a^\dagger(k) - a(k))$$
$$\beta_s(k) = \left(1 + k^2\mathrm{e}^{2s}\right)^{1/4} \ . \tag{5.74}$$

Then, the vacuum state $|\Omega^{(s)}\rangle$ is annihilated by the annihilation modes at scale e^{-s}.

[41] ↑If instead of setting $m = 1$ we take the massless limit in (5.68) we find

$$\omega(k;\Lambda) = \frac{\Lambda}{\sqrt{k^2+\Lambda^2}}|k| \ . \tag{5.69}$$

Deep in the IR we have a CFT and the renormalized field operators $\phi^\Lambda(0)$, $\pi^\Lambda(0)$ and $V_p(0) = \mathrm{e}^{\mathrm{i}p\phi(0)}$ are conformal primaries satisfying

$$-\mathrm{i}[L + K, \mathcal{O}_\alpha^\Lambda(0)] = \Delta_\alpha \mathcal{O}_\alpha^\Lambda(0) \tag{5.70}$$

with conformal dimensions $\Delta_\phi = 0$, $\Delta_\pi = 1$ and $\Delta_{V_p} = p^2/2$. The field $\phi(x)$ is not really physical. Its vanishing conformal dimension is a symptom of the infra-red divergences in the two-point function of ϕ.

Coherent operators of the free real scalar fields in 1+1dimensions

Consider the canonical quantization of a free real scalar QFT in finite volume. The vacuum is the tensor product of the vacua corresponding to the annihilation operators

$$a(k) = \sqrt{\frac{\omega(k)}{2}}\phi(k) + \frac{\mathrm{i}\boldsymbol{\pi}(k)}{\sqrt{2\omega(k)}}$$
$$a^\dagger(-k) = \sqrt{\frac{\omega(k)}{2}}\phi(k) - \frac{\mathrm{i}\boldsymbol{\pi}(k)}{\sqrt{2\omega(k)}} \tag{5.75}$$

where we have used the fact that for a real scalar field $\phi^\dagger(k) = \phi(-k)$, and $\boldsymbol{\pi}^\dagger(k) = \boldsymbol{\pi}(-k)$. The commutation relation is $[a(k), a^\dagger(k)] = 2\boldsymbol{\pi}$. Consider the unitary field coherent operator

$$D(f) = \mathrm{e}^{a^\dagger(f) - a(f^*)}, \qquad a(f) = \int dx\, f(x)a(x)\ .$$

We have

$$\begin{aligned}
&a^\dagger(f) - a(f^*) = \int dk \left(f(k)a^\dagger(k) - (f^*)(-k)a(k)\right)\\
&= \int dk \left(f(k)a^\dagger(k) - f(k)^* a(k)\right)\\
&= \int dk \left(\phi(k)\sqrt{2\omega(k)}f_-(-k) - \mathrm{i}\frac{\boldsymbol{\pi}(k)}{\sqrt{2\omega(k)}}f_+(-k)\right)\\
&= \int dk \left(\mathrm{i}\phi(k)f_\phi(-k) - \mathrm{i}\boldsymbol{\pi}(k)f_{\boldsymbol{\pi}}(-k)\right))\\
&= \mathrm{i}\phi(f_\phi) + \mathrm{i}\boldsymbol{\pi}(f_{\boldsymbol{\pi}}),
\end{aligned} \tag{5.76}$$

where we used the Fourier transform of complex conjugate $(f^*)(k) = (f(-k))^*$ and

$$f_\pm(k) = \frac{f(k) \pm f(-k)^*}{2} \tag{5.77}$$
$$f_\phi(k) = -\mathrm{i}\sqrt{2\omega(k)}f_-(k), \qquad f_{\boldsymbol{\pi}}(k) = -\sqrt{\frac{2}{\omega(k)}}f_+(k)\ .$$

Note that it follows from (5.77) and $\omega(k) = \omega(-k)$ that the functions $f_\phi(x)$ and $f_\pi(x)$ are real. Using (5.76) we can write coherent operator as a multi-mode momentum unitary displacement operator

$$\begin{aligned} D(f) &= \prod_k \mathrm{e}^{f(k)a^\dagger(k) - f(k)^* a(k)} \\ &= \prod_k \mathrm{e}^{f(k)a^\dagger(k)} \mathrm{e}^{-f(k)^* a(k)} \mathrm{e}^{-\frac{1}{2} f(k) f(k)^*} \end{aligned} \tag{5.78}$$

where we have used the Baker-Hausdorff-Campbell for $[X, Y] \sim \mathbb{I}$: $\mathrm{e}^{X+Y} = \mathrm{e}^X \mathrm{e}^Y \mathrm{e}^{-\frac{1}{2}[X,Y]}$. This form of the coherent operator allows us to compute the vacuum expectation value

$$\begin{aligned} \langle D(f) \rangle &= \prod_k \langle \mathrm{e}^{-\frac{1}{2} f(-k) f(k)^*} \rangle = \mathrm{e}^{-\frac{1}{2}(f|f)} \\ (f|g) &= \int dx \, f(x)^* g(x) \ . \end{aligned} \tag{5.79}$$

The single-mode displacement operators

$$D(\alpha) = \mathrm{e}^{\alpha a^\dagger - \alpha^* a}$$

satisfy the Weyl algebra

$$D(\alpha) D(\beta) = \mathrm{e}^{\frac{1}{2}(\alpha\beta^* - \alpha^*\beta)} D(\alpha + \beta) \ . \tag{5.80}$$

Therefore, the unitary field coherent operators satisfy

$$\begin{aligned} D(f) D(g) &= \mathrm{e}^{\frac{1}{2}((g|f) - (f|g))} D(f+g) \\ &= \mathrm{e}^{\mathrm{i}((g_-|f_+) - (g_+|f_-))} D(f+g) \\ &= \mathrm{e}^{2\mathrm{i}\mathrm{Im}(g|f)} D(f+g) \end{aligned} \tag{5.81}$$

where we have separated the real and imaginary part of f and g: $f(x) = f_-(x) + \mathrm{i} f_+(x)$ and $g(x) = g_-(x) + \mathrm{i} g_+(x)$.

The quadratic Hamiltonian is

$$H = \int dk\, E(k) a^\dagger(k) a(k) \; + \; 1/2 \tag{5.82}$$

then the energy of the field coherent state is

$$\langle f|H|f\rangle = \int dk\, E(k)|f(k)|^2 \;\; + \, 1/2\,. \tag{5.83}$$

which implies that the coherent states with bounded energy have suppressed $|f(k)|$ at large k.

Encoding a qudit at a point:

We use a coherent operator $D(\mathrm{i}pf_0)$ with a real function f_0 that is an approximation of a delta function at point x_0 with width ϵ, and p running from 0 to $q-1$, i.e.

$$|p, x_0\rangle := D(\mathrm{i}pf_0)\,|\Omega\rangle\,, \tag{5.84}$$

to encode a q-level quantum system at $x = x_0$.

As a concrete example, consider the Gaussian wave-packet

$$f_0(x) = \frac{1}{\sqrt{2\pi}\epsilon} \mathrm{e}^{-\frac{(x-x_0)^2}{2\epsilon^2}} \tag{5.85}$$

that is a regularization of the Dirac delta function [42]:

$$\lim_{\epsilon\to 0}(g|f_0) = g(x = x_0), \qquad (f_0|f_0) = \frac{1}{2\sqrt{\pi}\epsilon}\;. \tag{5.86}$$

These states (5.84) have large energy at small ϵ:

$$\langle p, x_0|H|p, x_0\rangle = \frac{p^2}{2\epsilon^4\sqrt{\pi}} U(-\frac{1}{2}, 0, \epsilon^2) \simeq \frac{p^2}{2\pi\epsilon^4} \tag{5.87}$$

[42]↑In the momentum space we have $f_0(k) = \frac{\mathrm{e}^{\mathrm{i}kx_0 - \frac{\epsilon^2 k^2}{2}}}{\sqrt{2\pi}}$.

where $U(a,b,z)$ is the confluent hypergeometric function. The set of states $|p,x_0\rangle$ are almost orthonormal in the limit of small ϵ because

$$\begin{aligned}\langle p',x_0|p,x_0\rangle &= \langle D(\mathrm{i}(p-p')f_0)\rangle \\ &= \mathrm{e}^{-\frac{1}{2}(p-p')^2(f_0|f_0)} \simeq \delta_{pp'}.\end{aligned}$$

One can construct an algebra of the q-level quantum system encoded at point $x=x_0$ on a code subspace constructed above by considering two types of operators, $D(\mathrm{i}qf_0)$ and $D(g_-)$ for a smooth real function g_-. The operator $D(\mathrm{i}qf_0)$ takes us in between code states

$$\langle p',x_0|D(\mathrm{i}qf_0)|p,x_0\rangle = \delta_{p',p+q} \tag{5.88}$$

and the operator $D(g_-)$ is diagonal in the basis $|p,x_0\rangle$:

$$\langle p',x_0|D(g_-)|p,x_0\rangle = \delta_{pp'}\,\langle D(g_-)\rangle\,\mathrm{e}^{-\mathrm{i}(p+p')(g_-|f_0)}\ .$$

where $(g_-|f_0)\simeq g_-(x=x_0)$ in the $\epsilon\to 0$ limit. If P_0 is the projection to the code subspace spanned by $|p,x_0\rangle$ then the operators P_0, $P_0D(\mathrm{i}qf_0)P_0$ and $P_0D(g_-)P_0$ and their Hermitian conjugates generate the algebra of the q-level system encoded at point $x=x_0$.

Moving a distance ϵ away from $x=x_0$ we can encode a new q-level system because

$$\begin{aligned}&\langle p',x_0|p,x_1\rangle = \langle D(\mathrm{i}pf_1-\mathrm{i}p'f_0)\rangle \\ &= \exp\left(-\frac{1}{2}(f_0|f_0)\left(p^2+(p')^2-2pp'\mathrm{e}^{\frac{-(x_0-x_1)^2}{4\epsilon^2}}\right)\right)\end{aligned}$$

which is vanishing small at small ϵ and $|x_0-x_1|>\epsilon$.

The RG flow of coherent operator

The vacuum state of the massive QFT with the cut-off length scale e^s satisfies

$$a_s(k)\,|\Omega^{(s)}\rangle = 0 \tag{5.89}$$

for the annihilation operators at scale e^s defined in (5.71). The coherent state corresponding to this annihilation operator is

$$\begin{aligned} |f;s\rangle &= D^{(s)}(f)\,|\Omega^{(s)}\rangle \\ D^{(s)}(f) &= \mathrm{e}^{a_s^\dagger(f)-a_s(f^*)} \end{aligned} \tag{5.90}$$

It satisfies the Weyl algebra

$$D^{(s)}(f)D^{(s)}(g) = \mathrm{e}^{2\mathrm{iIm}(f|g)}D^{(s)}(f+g)\ . \tag{5.91}$$

similar to the set of coherent operators $D(f)$ in the UV. Since the theory is Gaussian the renormalization of the coherent operator can be absorbed in the choice of smooth function $D(f) = D^{(s)}(f^s)$ (see section III of the main text):

$$\begin{aligned} f^s_\pm(k) &= \beta_s(k)^{\pm 1} f_\pm(k), \\ f^s_\pm(x) &= B^{\pm 1/4} f_\pm(x), \\ B &:= (1-\mathrm{e}^{2s}\not{\partial}^2)\ . \end{aligned} \tag{5.92}$$

The matrix element of $D(g)$ in the code states correspond to a three-point function of coherent operators that can be computed using the multiplication rule of the algebra in (5.91):

$$\begin{aligned} \mathrm{e}^{A(p',p;s)} \ &:= \ \langle p'f,s|D(g)|pf,s\rangle \\ &= \langle\Omega^{(s)}|D^{(s)}(-p'f)D^{(s)}(g^s)D^{(s)}(pf)|\Omega^{(s)}\rangle \\ &= \mathrm{e}^{2\mathrm{iIm}(-(g^s|p'f)+(pf|g^s-p'f))}\times \\ &\quad \langle\Omega^{(s)}|D(g^s+(p-p')f)|\Omega^{(s)}\rangle\ . \end{aligned} \tag{5.93}$$

In the case where $f = \mathrm{i}f_0$ and $g = g_- + \mathrm{i}g_+$:

$$\begin{aligned} A(p', p; s) &= -\mathrm{i}(p + p')(g_-^s | f_0) \\ &\quad -\frac{1}{2}(g^s + (p - p')f_0 | g^s + (p - p')f_0) \ . \end{aligned}$$

We would like to understand the renormalization of the coherent operators as they flow from the UV to the IR. As an example, consider the test function $f_0(x)$ in (5.85) and the coherent operator $D(f_0 + \mathrm{i}f_0)$. Under the RG flow it goes to $D^{(s)}(f_{0,-}^s + \mathrm{i}f_{0,+}^s)$ with $f_{0,\pm}^s$ real. It follows from (5.92) that

$$f_{0,\pm}^s(x) = (1 - \mathrm{e}^{2s}\partial_x^2)^{\pm 1/4} \frac{\mathrm{e}^{-\frac{(x-x_0)^2}{2\epsilon^2}}}{\sqrt{2\pi}\epsilon} \ . \tag{5.94}$$

Deep in the UV the term $\mathrm{e}^{2s}\partial_x^2$ is small and the renormalization of f_0 is perturbative. The renormalization becomes non-perturbative at the cut-off length scale e^s when $\mathrm{e}^{2s}|\partial_x^2 f_0|$ becomes comparable to $|f_0|$. For the test function f_0 we have

$$\frac{\mathrm{e}^{2s}|\partial_x^2 f_0|}{|f_0|} = \frac{\mathrm{e}^{2s}}{\epsilon^2}\left(\frac{(x - x_0)^2}{\epsilon^2} - 1\right) \ . \tag{5.95}$$

There are two stages to the RG flow of this coherent operator. In the first stage, the cut-off length e^s is much smaller than ϵ, the term $\mathrm{e}^{2s}\partial_x^2$ in (5.94) can be neglected and the renormalization of f_0^s is perturbatively small. The second stage starts when $\mathrm{e}^s \sim \epsilon$. As we flow deeper in the IR $\mathrm{e}^s \gg \epsilon$ the term $\mathrm{e}^{2s}\partial_x^2$ in (5.94) dominates. In stage two, we are in the regime $\epsilon \ll \mathrm{e}^s$ and for points away from $|x - x_0| = \epsilon$ we can use the approximation

$$f_{0,\pm}^s(x) \simeq \mathrm{e}^{\pm s/2}(\partial_x^2)^{\pm 1/4} f_0(x) \ . \tag{5.96}$$

The function $f_{0,+}^s$ ($f_{0,-}^s$) grows (decays) exponentially fast as $\mathrm{e}^{(s - \log\epsilon)/2}$ ($\mathrm{e}^{(-s - \log\epsilon)/2}$) in the IR, respectively; see (Fig. 5.16). Note that s increases as we flow towards the IR.

We can generalize these lessons to the renormalization of any function $g(x)$ that is localized around $x = x_0$ with linear width $\epsilon \ll |A| \ll 1$. Note that, in our units, the unit length

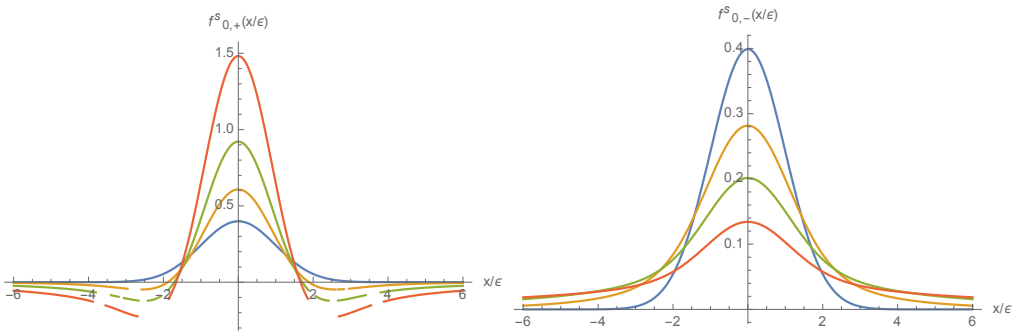


Figure 5.16. The renormalization of the functions $f^s_{0,\pm}(x)$ is insignificant until the cut-off length scale becomes comparable to ϵ (width of $f_0(x)$). As we flow further towards the IR (Top) the function $f^s_{0,+}(x)$ becomes highly peaked (Bottom) the function $f^s_{0,-}(x)$ flattens out. (Blue: $s = -\infty$, Yellow: $s = 1$, Green: $s = 2$, Red: $s = 3$.)

corresponds to the inverse mass at the IR scale, i.e. $m^{-1} = 1$. Intuitively, one expects that near the peak

$$\frac{|\partial_x^2 g(x)|}{|g(x)|}\Big|_{x\simeq x_0} = O(|A|^{-2}) \tag{5.97}$$

or more generally the right-hand-side is some function that is inversely proportional to $|A|$. There are two stages to the RG flow. In the first stage $e^s \ll |A|$, the cut-off grows but the function is frozen. In comparison to MERA, the cut-off can be interpreted as a unit qudit and the operator is supported on $|A|/\epsilon$ number of sites. Therefore, in this stage the support of the UV operator shrinks exponentially fast. Similar to MERA, the second stage starts when the RG scale reaches the size of the unit block $e^s \simeq |A|$. Beyond this point, we find that inside A the function g^s_+ grows exponentially as $e^{(s-\log|A|)/2}$ and g^s_- decays exponentially as $e^{(-s-\log|A|)/2}$. This is reminiscent of the second of the RG flow of operator in MERA. For a general function $g(x)$ the right-hand-side of (5.97) is some more complicated function. The transition scale happens at some $e^s \sim h(|A|)$ for some increasing positive function of $|A|$ and the exponent that controls the exponential growth or decay is $(s - \log h(|A|))$.

Note that the larger $|A|$ is, the later the second stage starts. Because of the above behaviour, for a coherent operator $D(f)$ as an error operator with a function f supported

on a large $|A|$, one needs to flow much deeper in the IR to achieve the error correction code with the same errors.

Error correction condition:

The Knill-Laflamme condition for approximate quantum error operator [117] tells us that we can approximately correct for error caused by the operator $\mathcal{O}$ if and only if this operator is proportional to the projection to the code subspace up to small corrections δ:

$$\langle \Psi_r | \mathcal{O} | \Psi_{r'} \rangle = c\delta_{rr'} + \delta \ . \tag{5.98}$$

We choose the coherent state with cut-off length scale e^s as the states of the code subspace and choose as errors the UV logical operator $D(D(h_-))$ and $D(\mathrm{i}h_+)$ for smooth real functions h_- and h_+ which could be supported on either A, C, or both. One can choose, for instance, $h_- = g_-$ and $h_+ = qf_0$ that we introduced in section (102). To show error correction we need to establish that

$$\begin{aligned} \langle p', x_0; s | D(h_-) | p, x_0; s \rangle &= c(h_-, s)\delta_{pp'} + \delta_1 \\ \langle p', x_0; s | D(\mathrm{i}h_+) | p, x_0; s \rangle &= c(h_+, s)\delta_{p,p'} + \delta_2 \end{aligned} \tag{5.99}$$

for small δ_1 and δ_2. As opposed to the MERA where we were protected against the erasure of a region, in cMERA we have the weaker statement that given a set of local coherent operator we are protected from them if we go deep enough in the IR.

We use the algebra of the free fields to compute these matrix elements:

$$\begin{aligned} \langle p', x_0; s | D(h_-) | p, x_0; s \rangle &\simeq c(h_-, s)\delta_{pp'} \mathrm{e}^{-\mathrm{i}(p-p')(h_-^s | f_0)} \\ \langle p', x_0; s | D(\mathrm{i}h_+) | p, x_0; s \rangle &= \mathrm{e}^{-\frac{1}{2}(h_+^s + (p-p')f_0 | h_+^s + (p-p')f_0)} \\ c(h_\pm, s) &= \langle \Omega^s | D(h_\pm) | \Omega^s \rangle \ . \end{aligned} \tag{5.100}$$

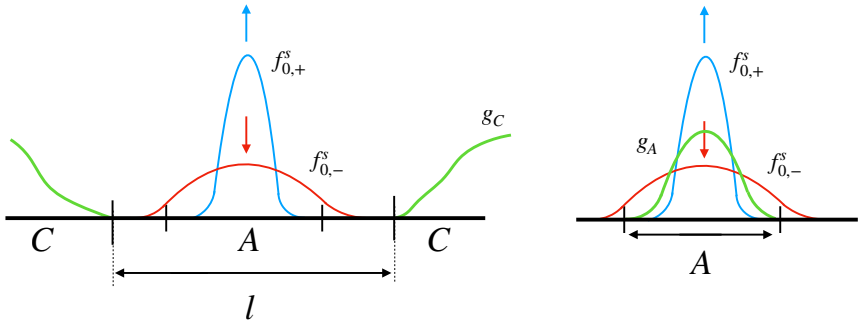


Figure 5.17. Under the RG $f^s_{0,+}$ ($f^s_{0,-}$) localizes (flattens). (Left) Their overlap with g_C supported on C is suppressed if l is large enough. (Right) The overlap of $f^s_{0,+}$ ($f^s_{0,-}$) with g_A supported on A grows (decays), respectively.

We first consider the case where $h_C = h_{-,C} + \mathrm{i}h_{+,C}$ with $h_{\pm,C}$ real functions supported on a region C that is at least $l\epsilon$ away from $x = x_0$; see (Fig. 5.17 Left). We have

$$(h^s_{\pm}|f_0) = (h_{\pm}|f^s_{0,\pm}) \ . \tag{5.101}$$

Since $f^s_{0,-}(x)$ decays exponentially fast with distance away from $x = x_0$ we have

$$(h^s_{\pm,C}|f_0) = O(\mathrm{e}^{-l^2/2}) \ . \tag{5.102}$$

Since all the points in C satisfy $|x - x_0| \gg \epsilon$ it follows that the functions $f_{0,\pm}(x)$ grow at most like $\mathrm{e}^{s/2}/\sqrt{\epsilon}$. For any fixed large s and a large enough l we find that the error caused by any $D(h_C)$ can be corrected because

$$\begin{aligned} &\langle p', x_0; s|D(h_{-,C})|p, x_0; s\rangle \simeq c(h_{-,C}, s)\delta_{pp'} \\ &\langle p', x_0; s|D(\mathrm{i}h_{+,C})|p, x_0; s\rangle \simeq c(h_{+,C}, s)\delta_{pp'} \ . \end{aligned} \tag{5.103}$$

Next, we consider the case where h_A is supported on region A that includes the point $x = x_0$; see (Fig. 5.17). As we argued, under the RG flow, the function h^s_A is unchanged until

$e^s \sim |A|$. After that the second stage of RG starts. Since $e^s \gg |A|$, for all $(x - x_0) \ll |A|$, we can use the approximation in (5.96). Therefore, in the second stage we have

$$(h_{\pm,A}|f^s_{0,\pm}) \simeq e^{\pm 1/2(s-\log|A|)}(h_{\pm,A}|(\ell_x^2)^{\pm 1/4} f_0) \ . \tag{5.104}$$

This combined with (5.101) and (5.100) implies that for $e^s \gg |A|$ we have

$$\begin{aligned} &\langle p', x_0; s|D(h_{-,A})|p, x_0; s\rangle \\ &\simeq c(h_{-,A}, s)\delta_{pp'} e^{-i(p-p')O(e^{-s/2})} \\ &\simeq c(h_{-,A}, s)\delta_{pp'} \ . \end{aligned} \tag{5.105}$$

For the operator $D(ih_{+,A})$ as we saw in equation (5.88) we can distinguish different code states if we tune $h_{+,A} = -(p - p')f_0$ so that the exponent in the second line of (5.100) vanishes. However, this cancellation does not survive under the RG flow because

$$\begin{aligned} &h^s_{+,A} + (p - p')f_0 = (p - p')(f_0 - f^s_{0,+}) \\ &= (p - p')(1 - (e^{2s}\ell_x^2)^{1/4})f_0 \simeq (p' - p)(e^{2s}\ell_x^2)^{1/4} f_0 \end{aligned}$$

where in the last line we use the fact that we are in a regime where $(e^{2s}\ell_x^2)^{1/4}$ dominates over the first term. It is clear from the second line of (5.100) that the norm of the function above controls the size of the matrix element of this coherent operator. This norm can be computed explicitly:

$$(p - p')^2 e^s(\ell_x^{1/2} f_0|\ell_x^{1/2} f_0) = (p - p')^2 \frac{e^s}{2\pi\epsilon^4} \tag{5.106}$$

Since $e^s \gg |A|$ and $|A| \gg 1$ the expression above grows to infinity. Plugging this back into (5.100) we find that deep in the IR

$$\langle p', x_0; s|D(i(p - p')f_0)|p, x_0; s\rangle = e^{-\frac{(p-p')^2}{2}O(e^s)} \tag{5.107}$$

which goes to zero.

In summary, we find that approximate error correction conditions above are satisfied for any fixed set of UV logical operators $D(h)$ with h supported on A, C or both (see figure 6.1) if we go deep enough in the IR. The operators supported on A are correctable when $e^s \gg |A|$ and those on C are correctable when C is far enough from $x = x_0$. In fact, we can consider the operators that are supported on AC and the same argument above implies that deep in the IR the UV operators of AC can be corrected. This is reminiscent of uber-holography and local error correction in [81], [112]. The energy produced by a coherent noise operator $D(h)$ grows as $\int E(k)|h(k)|^2$. If we put a bound M on the energy carried by noise operator, deep in the IR, i.e. $e^s M \ll 1$, we are protected from all such errors. However, we are not protected against erasures because for any fixed IR scale e^s there always exist $D(h)$ supported on A or C with large enough energy that can distinguish the code states.

5.2.5 Holographic RG and error correction

The cMERA discussion was restricted to free fields. To study the QEC structure in the RG flow in strongly coupled QFTs we consider holography. Holography can be viewed as a QEC where the logical algebra of bulk regions are non-locally encoded on the boundary regions such that they are protected against local boundary erasures [118]. Moreover, the logical subalgebra corresponding to some bulk subregion $\mathcal{E}(B)$ can be *reconstructed* from the subalgebra of the boundary subregion B. The bulk region $\mathcal{E}(B)$ is the domain of dependence of a bulk codimension-1 spacelike surface between B and a bulk codimension-2 stationary area surface anchored on B [119]. In case there are more than one possible stationary area surface anchored on B, the surface with the smallest area will determine the bulk region $\mathcal{E}(B)$. The bulk region $\mathcal{E}(B)$ is called the entanglement wedge of B [120] whereas the minimal area stationary area surface is called Hubeny-Rangamani-Takayanagi surface (HRT) surface [121], [122].

Various properties of the holographic QECC were studied in [112] and the authors defined the concept of the distance and the price for a logical algebra of operators in a bulk region. Here, we review some important definitions and theorems from [112].

We start by considering a logical algebra of a single bulk point x. The distance of a logical operator is the size of the smallest boundary region B such that the logical operator cannot be reconstructed from the complement of the boundary region B, which we denote by B^c. This means that the distance d_x of a logical algebra of a point x is the volume of the smallest boundary region B such that x is not in the the entanglement wedge of B^c. That is,

$$d_x = \min_{B:x\notin\mathcal{E}(B^c)} |B| \,, \tag{5.108}$$

where $|B|$ is the volume of the region B. For a logical subalgebra associated to a finite bulk region X, the distance is given by

$$d_X = \min_{x\in X} d_x \,. \tag{5.109}$$

The price p_X of a logical subalgebra corresponding to bulk region X is the volume of the smallest boundary region on which any operator $\phi \in \mathcal{A}(X)$ can be represented. The subregion-subregion duality implies that the price is the volume of the smallest boundary region such that the region X is in the entanglement wedge.

$$p_X = \min_{B:X\in\mathcal{E}(B)} |B| \,. \tag{5.110}$$

By comparing these definitions, one can deduce that $p_X \geq d_X$. This statement is called *no free lunch* in [112].

It is worthwhile to mention that the inequality of the no free lunch can be saturated for a bulk point x if one assumes the notion of *geometric complementarity* [112]. The geometric complementarity states that a bulk point is either in the entanglement wedge of a boundary region B or in the entanglement wedge of the complementary region B^c. That is, if $x \notin \mathcal{E}(B^c)$, then $x \in \mathcal{E}(B)$.

A stronger version of the no free lunch is the holographic strong Singleton bound. It states that the difference of price p_X and distance d_X of the logical algebra associated to a

bulk region X can not be less than the number of logical degrees of freedom k_X in that bulk region [112]. That is,

$$k_X \leq p_X - d_X \,. \tag{5.111}$$

In the following subsection, we consider a simple example of a holographic RG flow in which we find that there exists a finite bulk region for which the price and the distance of the corresponding logical algebra are the same. The holographic Singleton bound then implies that there should not be any logical degrees of freedom in that subregion. We then discuss in Sec. (105) how to modify the definitions of the distance and the price to resolve this problem.

Holographic RG flow

Consider the vacuum AdS_3 with the metric

$$ds^2 = e^{2r/L}\left(-dt^2 + dx^2\right) + dr^2 \,, \tag{5.112}$$

where L is the AdS length scale, and r is the bulk radial coordinate. The boundary is located at $r = \infty$ and $r = -\infty$ is the Poincare horizon. This geometry is dual to the vacuum state of a $(1+1)$-dimensional CFT on a flat spacetime. The AdS length scale and the central charge of the CFT are famously related according to [123]

$$c = 3L/2G_N \,. \tag{5.113}$$

Now suppose we set off an RG flow on the CFT by deforming with a relevant operator. The geometry that is dual to the RG flow on the boundary is given by [124]–[126]

$$ds^2 = e^{2A(r)}\left(-dt^2 + dx^2\right) + dr^2 \,, \tag{5.114}$$

where $A(r)$ is such that $A(r) \sim r/L_{UV}$ near $r = \infty$ whereas $A(r) \sim r/L_{IR}$ near $r = -\infty$. L_{UV} is related to the central charge of the UV theory according to Eq. (5.113), whereas

L_{IR} is related to the central charge of the IR theory to which the UV theory flows [43]. The function $A(r)$ captures the flow of the boundary theory from UV to IR [127], [128].

In this work, we consider a simple example where $A(r)$ is given by a

$$A(r) = \begin{cases} r/L_{UV} & r \geq 0 \\ r/L_{IR} & r \leq 0 \end{cases} . \tag{5.115}$$

This simple model of holographic RG flows has been studied in [129], [130] where the minimal area surfaces corresponding to boundary regions are studied. Here, we review the results from [129].

Consider a single interval on a boundary of size ℓ. For small enough regions, the bulk stationary area surfaces remain near the boundary and do not penetrate to $r < 0$ region (i.e. the IR region.) In particular, this type of stationary area surfaces only exist for $\ell \leq \ell_2 \equiv 2L_{UV}$. For regions of length $\ell \geq \ell_2$, the stationary area surfaces must penetrate to the IR region.

However, it was observed in [129] that even for $\ell < \ell_2$, there can exist stationary area surfaces that go to $r < 0$ region. In fact, such surfaces exist for $\ell \geq \ell_1$, where

$$\ell_1 = 2L_{UV}\sqrt{1-\left(1-\frac{L_{IR}}{L_{UV}}\right)^2} . \tag{5.116}$$

Based on the above discussion, the HRT surfaces for $\ell < \ell_1$ completely stay in the UV region whereas the HRT surfaces for $\ell > \ell_2$ penetrate to the IR region. For intermediate size regions, both types of stationary area surfaces exist but the HRT surface is the one with a smaller area. It was observed in [129] that there exists a critical size of the interval, ℓ_t, below which the surfaces that stay in the UV region has a smaller area and above which the surfaces that reaches the IR region has a smaller area. In Appendix (5.2.7), we numerically calculate the critical size of the interval by comparing the area of the surfaces involved.

[43]↑The holographic c-theorems say that the null energy condition in the bulk implies $L_{IR} < L_{UV}$ [127], [128].

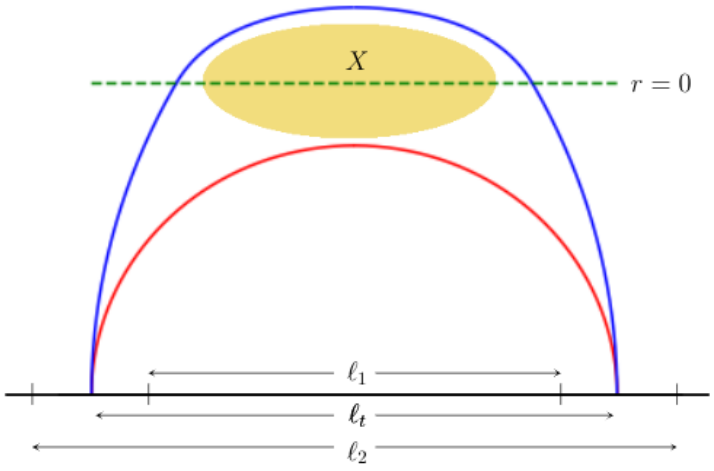


Figure 5.18. Pictorial representation of the phase transition in the HRT surfaces at $\ell = \ell_t$ for $L_{IR} = 0.3$ and $L_{UV} = 1.0$. The critical length scale is in the range $\ell_1 < \ell_t < \ell_2$. The HRT surface for ℓ just bigger/smaller than ℓ_t is shown in blue/red color. The yellow shaded region is an example of a finite size region X for which the distance and price are equal.

The result of this analysis is shown as a plot of ℓ_t versus L_{IR}/L_{UV} in Fig. (5.19 Left) in Appendix (5.2.7).

Due to the 'phase transition' in the HRT surface at $\ell = \ell_t$, there is a jump in the bulk entanglement wedge as well. When the size of the boundary interval is just smaller than ℓ_t, the minimum radial point that is reached by the HRT surface is $r_{UV} > 0$. On the other hand, when the size of the boundary interval is just greater than ℓ_t, the minimum radial point reached by the HRT surface is $r_{IR} < 0$. The jump in the entanglement wedge at $\ell = \ell_t$ can be measured in terms of the proper distance between these minimum radial points. We numerically calculate this proper distance in the Appendix (5.2.7) and find that we can make this proper distance bigger by making the difference between L_{IR} and L_{UV} bigger; see (Fig. 5.19 Right) in Appendix (5.2.7).

The transition in the entanglement wedge at $\ell = \ell_t$ has interesting implications for holographic QECC. Consider a bulk region X which is the intersection of the region $r < r_{UV}$ and the entanglement wedge of a boundary interval of size slightly greater than ℓ_t. The condition $r < r_{UV}$ implies that the region X is not in the entanglement wedge of

any boundary interval of size less than ℓ_t. Hence, according to Eqs. (5.108)-(5.109), the distance of the logical algebra associated to region X is given by ℓ_t. Moreover, according to Eq. (5.110), the price of the region X is also given by ℓ_t[44]. This means that we have found a logical subalgebra associated to a finite size bulk region for which the price and the distance are the same. Comparing this with the holographic strong Singleton bound in Eq. (5.111), we deduce that the number of logical degrees of freedoms in that finite size subregion should be zero or else we get a violation of the holographic strong Singleton bound.

We discuss in the next subsection how to modify the definition of the distance and the price to resolve this apparent paradox.

Price, distance, and the reconstruction wedge

We observed in the previous subsections that a phase transition in the entanglement wedge led us to a violation of the holographic strong Singleton bound. In this subsection, we discuss this violation can be resolved.

The definition of the entanglement wedge in terms of the minimal area surface is only valid at the leading order in $O(1/G_N)$. At subleading order in G_N, we have to take the entanglement entropy of the bulk quantum fields into consideration [131], [132]. More precisely, the entanglement wedge of a boundary region B is the domain of dependence of a bulk region between B and a codimension-2 surface of stationary generalized entropy. The generalized entropy of a region is equal to the area of the boundary of that region (in Planck's units) plus the entropy of the quantum fields in the region. This subleading correction, as is recently emphasized [133]–[135], can be significant when there is a phase transition in the minimal surfaces.

Now suppose there is a state ρ_X of the quantum fields in region X which we introduced in the previous subsection. If this state is pure, then the discussion of the entanglement wedge is unchanged and we end up with a phase transition at $\ell = \ell_t$. However, when the state ρ_X is mixed, then there is no phase transition and the entanglement wedge at $\ell = \ell_t$ is

[44]↑The distance and the price are actually equal to $(\ell_t)^\alpha$ where $\alpha = \log(2)/\log(\sqrt{2}+1)$ [112]. This is the size of the fractal like disconnected intervals such that the entanglement wedge of the disconnected region has the same minimum radial point as the entanglement wedge of a single interval of size ℓ_t. This construction is called *uberholography* in [112].

region between the boundary interval and the red surface shown in (Fig. 5.18). Hence, the entanglement wedge depend on the state ρ_X.

The dependence of state on entanglement wedge has been recently discussed in [133]–[136]. In particular, it was argued in [134] that the bulk region that can be reconstructed given a boundary region B is not the entanglement wedge of B. In fact, this region can be macroscopically smaller than the entanglement wedge, $\mathcal{E}(B)$. The *reconstruction wedge*, $\mathcal{R}(B)$, corresponding to the boundary region B is defined to be the intersection of all the entanglement wedges of B for every state in the code space [134].

In the example that we discussed above, the reconstruction wedge of boundary interval of size $\ell = \ell_t$ is the entanglement wedge when the state ρ_X is mixed. Hence, there is no phase transition in the reconstruction wedge at $\ell = \ell_t$.

With the fact that the reconstruction wedge is smaller than the entanglement wedge, we restate one of the main statements in the main text. To solve the violation of holographic Singleton bound, we propose that the definition of the price for a logical algebra associated to a bulk region X should be modified from Eq. (5.110) to

$$p_X = \min_{B:X\in\mathcal{R}(B)} |B| \,. \tag{5.117}$$

Similarly, we propose that the distance of a subregion X is

$$d_X = \min_{x\in X} d_x \,; \qquad d_x = \min_{B:x\notin\mathcal{R}(B^c)} |B| \,. \tag{5.118}$$

With the new definition, although the distance is still determined by the $\ell = \ell_t$, the price is determined by $\ell = \ell_2$ which is the largest length for which the surface in Eq. (5.119) exists[45].

[45]↑Again, the distance and the price are given by $(\ell_t)^\alpha$ and $(\ell_2)^\alpha$ respectively where $\alpha = \log(2)/\log(\sqrt{2}+1)$ as determined by the uberholography construction.

5.2.6 Discussions

In summary, we studied the connection between RG and approximate error correction codes in three examples: 1) the RG flow of classical Ising model as a classical code 2) continuous MERA for massive free fields as a quantum code. 3) holographic RG flow of a 2-dimensional boundary theory.

In this work, we advertised the picture that the Hilbert space of an effective field theory with the cut-off scale Λ should be viewed as a code subspace of all states that are approximately protected against the short-distance errors localized on a region of linear size A much smaller than the cutoff, $|A| \lesssim 1/\Lambda$. To argue for this point, we used cMERA as a concrete realization of the real-space RG flow of massive free fields. However, there are other approaches to the RG flow. Examples include the continuous Tensor Network Renormalization (cTNR) in [137], the generalization of cMERA using Euclidean path-integrals [116], [138], the RG flow for free $O(N)$ model using Polchinski's exact RG [105]. In some of these approaches the map from the IR physics to the UV is no longer an exact isometry. It is an interesting question to investigate the approximate QECC code appears in these other approaches to the RG flow.

5.2.7 Appendix: Holographic RG flows and phase transition

In this appendix, we present the details of the phase transition in the entanglement wedge that we discussed in section (5.2.5).

Consider a single interval on a boundary of size ℓ. One possible stationary area surface anchored on this interval is the one that does not penetrate to $r < 0$ region (i.e. the IR region). This surface is given by [46]

$$x \;=\; L_{UV}\,\sqrt{\mathrm{e}^{-2r_m/L_{UV}} - \mathrm{e}^{-2r/L_{UV}}}\,, \tag{5.119}$$

[46]↑This equation is only for the half of the surface. The center of the interval is chosen to be at $x = 0$ and the surface is symmetric around $x = 0$.

where r_m is the minimum radial point that is reached by the minimal surface and is related to the size of the interval according to $\ell = 2L_{UV}\mathrm{e}^{-r_m/L_{UV}}$. Note that this type of stationary area surface for which $r_m > 0$ can only exist for $\ell \leq \ell_2$, where $\ell_2 = 2L_{UV}$.

For intervals of size $\ell > \ell_2$, the stationary area surfaces discussed above would penetrate to the IR region. However, it was observed in [129] that even for $\ell \leq \ell_2$, there can exist stationary area surfaces that go to $r < 0$ region. In fact, there can be two such surfaces for a given ℓ and they can be written as $x = x_\pm(r)$ where

$$x_\pm(r) = \begin{cases} L_{IR}\sqrt{K_\pm^2 - \mathrm{e}^{-2r/L_{IR}}}, & r_{m,\pm} \leq r \leq 0 \\ L_{UV}\left(\sqrt{K_\pm^2 - \mathrm{e}^{-2r/L_{UV}}} - K_\pm\right) + \frac{\ell}{2} & r \geq 0\ . \end{cases}$$

The minimum radial point $r_{m,\pm}$ is related to $K_\pm$ according to $K_\pm = \mathrm{e}^{-r_{m,\pm}/L_{IR}}$. The continuity of $x_\pm(r)$ at $r = 0$ implies

$$\ell = 2L_{UV}K_\pm - 2\left(L_{UV} - L_{IR}\right)\sqrt{K_\pm^2 - 1}\,, \tag{5.120}$$

which can be inverted to get

$$K_\pm = \frac{L_{UV}\ell \pm \left(L_{UV} - L_{IR}\right)\sqrt{\ell^2 + 4L_{IR}\left(L_{IR} - 2L_{UV}\right)}}{2L_{UV}^2 - 2\left(L_{UV} - L_{IR}\right)^2}. \tag{5.121}$$

These surfaces can only exist when $K_\pm$ are real-valued which requires $\ell \geq \ell_1$, where

$$\ell_1 = 2L_{UV}\sqrt{1 - \left(1 - \frac{L_{IR}}{L_{UV}}\right)^2}. \tag{5.122}$$

Therefore, there are three possible stationary area surfaces corresponding to a boundary interval of size $\ell_1 \leq \ell \leq \ell_2$. The HRT surface, however, is the one with the smallest area

among these three surfaces. Let us denote the area of the surface in Eq. (5.119) by A_0 and the area of surfaces $x_\pm(r)$ in Eq. (5.120) by $A_\pm$. Then A_0 is given by [121]

$$A_0 \;=\; 2L_{UV}\,\log\left(\frac{\ell}{\delta_{UV}}\right)\,, \tag{5.123}$$

whereas $A_\pm$ are given by [129]

$$\begin{aligned} A_\pm \;&=\; 2L_{UV}\,\log\left(\frac{2L_{UV}K_\pm}{\delta_{UV}}\right) \\ &-\;(L_{UV}-L_{IR})\,\log\left[\frac{K_\pm+\sqrt{K_\pm^2-1}}{K_\pm-\sqrt{K_\pm^2-1}}\right]\,, \end{aligned} \tag{5.124}$$

where δ_{UV} is a UV cutoff of the boundary theory. It was observed in [129] that the area A_- is always greater than A_+ and that there exists a critical size of the interval, ℓ_t, below which $A_0 < A_+$ and above which $A_+ < A_0$[47]. This means that for some critical l_t, when $\ell < \ell_t$ the surface in Eq. (5.119) is the RT surface, whereas when $\ell > \ell_t$ the surface x_+ in Eq. (5.120) is the RT surface. We can calculate the critical size of the interval by numerically solving $A_0 - A_+ = 0$. The result of this analysis is shown as a plot of ℓ_t versus L_{IR}/L_{UV} in Fig. (5.19 Left). As we can see from the figure, ℓ_t increases with L_{IR}/L_{UV} and it approaches ℓ_2 when L_{IR} approaches L_{UV}.

As we discussed in Sec. (5.2.5), there is a jump in the entanglement wedge due to the phase transition in the HRT surfaces. The jump in the entanglement wedge at $\ell = \ell_t$ can be measured in terms of the proper distance between the minimum points reached by the two candidate HRT surfaces. This proper distance is given by

$$\Delta r_m = r_{UV} - r_{IR}\,. \tag{5.125}$$

where $r_{UV} = -L_{UV}\log\left(\ell_t/(2L_{UV})\right)$, and $r_{IR} = -L_{IR}\log K_+(\ell_t)$, and $K_+(\ell_t)$ can be determined using Eq. (5.121). We plot Δr_m as a function of L_{IR}/L_{UV} in Fig. (5.19 Right). As we

[47]↑Even though the areas A_1, A_+, and A_- depend on the UV cutoff, the difference of any two areas is independent of the cutoff. This makes the comparison of the areas meaningful.

can see from the figure, the jump in the HRT surfaces, and hence in entanglement wedges, is more significant when the difference between L_{IR} and L_{UV} is large.

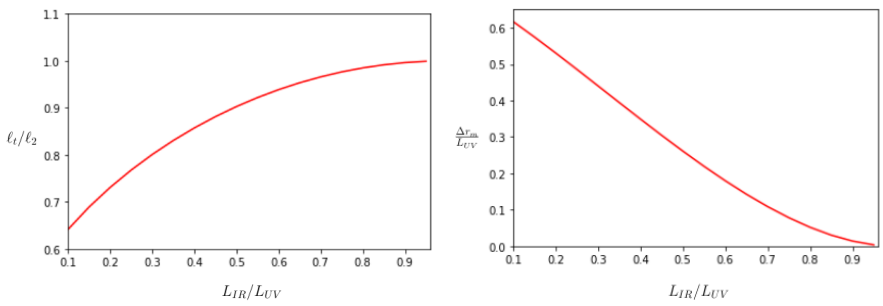


Figure 5.19. Numerical calculation of the (Top) critical length, ℓ_t, and (Bottom) the jump in the entanglement wedge at the phase transition, Δr_m, as a function of L_{IR}/L_{UV}.

6. GENERALIZED INFORMATION MEASURES

In [139], Shannon provided the definition of "information" and *Shannon entorpy* as an average of surprisal as discussed in chapter 2. As a generalization of Shannon entropy, *von Neumann entropy* has been proposed. The von Neumann entropy of a subregion captures the total correlations of a bi-partite pure state in a quantum many-body system. It is also called the *entanglement entropy*. The locality of correlations in quantum many-body systems determines the scaling of entanglement entropy. Consider a quantum many-body system with a finite correlation length. The leading contribution of the entanglement entropy of a subregion scales with the boundary area of the subregion[140], [141]. This is called the *area law*. It is known that the entropy of black holes follows the area law[142], [143]. The leading contribution of the entanglement entropy of holographic conformal field theory is also known to follow the law. In particular, its gravity dual quantity has been proposed and known as Ryu-Takayagi surface[144]. Unfortunately, the von Neumann entropy suffers from ultraviolet divergences. On the contrary, the quantum relative entropy is a UV-finite measure. The quantum relative entropy is known as a distinguishability measure that compares two states. In [64], Araki used the modular theory to define the relative entropy of vN algebra of a general quantum system, for instance, quantum field theory(QFT).

In this chapter, we focus on the family of information measures, called the correlation measures. Correlation measures are the information measures that satisfy the monotonicity, or data-processing inequality, under a local completely positive map. In the QEC, the relative entropy is used to compare the state associated with a physical algebra $\mathcal{B}$ and the ones associated with the correctable algebras $\mathcal{B}_C$, which forms the inclusion of von Neumann algebras $\mathcal{B}_C \subset \mathcal{B}$. In section 6.1, the inclusion is characterized by a group. That is, we consider an inclusion $\mathcal{A} \subset \mathcal{F}$ where $\mathcal{A}$ is the von Neumann subalgebra that is invariant under a group G. Our study proposes the generalized entanglement entropy of a quantum system in the presence of charges due to the symmetry defined by the group G. We show that this can be written by Jaynes entropy[145].

von Neumann entropy and relative entropy are generalized into a one-parameter family of information measures. They are known as Rényi entropies and Rényi relative entropies.

There are two major Rényi relative entropies. One is known as Petz Rényi divergences and the other is sandwiched Rényi divergences. In order to interpolate them, the two-parameter family known as α-z Rényi relative entropy has been proposed[146]. For any family of information measures to be correlation measures, it should satisfy the monotonicity under a CP map. Hence, one of the main interests of quantum information theorists and mathematicians is to specify in which range of parameters the monotonicity holds. These objects actually have physical importance beyond just mathematical generalization. Rényi families were used in quantum thermodynamics of small quantum systems to prove new second laws[147]. In QFT, the sandwiched Rényi divergences were shown to be related to multi-point correlation function[148], and their data-processing inequality was used to prove the novel constraints of unitary and locality[149], [150]. They have also been used to constrain the renormalization group flows of QFTs[151]. In holography, Renyi divergences have gravitational duals in terms of the on-shell action of replicated geometries [152], and their monotonicity has led to new laws of black hole thermodynamics [153].

In section 6.2, we generalize the Rényi families of relative entropy to multistate and multiparameter f-divergences based on the construction of L^p_ω space[154] and Kubo-Ando operator means[155]. Most importantly, we show that it satisfies the monotonicity under a single unital completely positive map. Although we do not have clear physical interpretations of our multistate f-divergence, we make speculation on how it could play a role in quantum state discrimination.

In summary, we show that the following;

1. [Relative entropy of subalgebras and subspaces of an algebra]

   Consider a subspace P and a subalgebra $\mathcal{A}$ of an algebra $\mathcal{F}$. Suppose $\mathcal{E} : \mathcal{F} \to \mathcal{A}$ is a conditional expectation preserving a trace, i.e., $\mathrm{tr}(\mathcal{E}(f)) = \mathrm{tr}(f)$ for $f \in \mathcal{F}$. We propose the relative entropy between $\mathcal{A}$ and $\mathcal{F}$ as, for a state ρ on $\mathcal{F}$,

$$S(\rho\|\mathcal{E}^*(\rho)) = S(\sigma_{max}) - S(\rho) \tag{6.1}$$

where σ_{max} satisfies the Jaynes maximum principle. That is, for a state ρ on $\mathcal{F}$, consider the set $\{\sigma\}$ of all states σ that have the same expectation values as ρ for all operators a in $\mathcal{A}$, i.e., $\mathrm{tr}((\rho-\sigma)a) = 0$. Then, the Jaynes maximum principle states that the Jaynes entropy S_J of a state ρ with respect to a subalgebra $\mathcal{A}$ is the supremum of the von Neumann entropy $S_{vN}(\sigma)$ over the set $\{\sigma\}$;

$$S_J(\rho,\mathcal{A}) = \sup_{\sigma\in\{\sigma\}} \{S_{vN}(\sigma)|\,\mathrm{tr}((\sigma-\rho)a) = 0,\ \forall a\in\mathcal{A}\}. \tag{6.2}$$

σ_{max} provides the supremum. The relative entropy (6.1) between a subspace P and an algebra $\mathcal{F}$ is defined similarly to the above just by replacing $\mathcal{A}$ to P.

2. [A generalized entanglement entropy of a quantum system in the presence of charges in a general quantum system]

   Consider two von Neumann algebras $\mathcal{F}_1$ and $\mathcal{F}_2$ associated with the local disconnected region A_1 and region A_2 on spacetime, and a global symmetry group G [1]. Let us denote the algebras invariant under the group as $\mathcal{A}_1 \subset \mathcal{F}_1$ and $\mathcal{A}_2 \subset \mathcal{F}_2$. Physically, $\mathcal{A}_1$ and $\mathcal{A}_2$ correspond to the set of charge-neutral operators. Note that $\mathcal{F}_{12} = \mathcal{F}_1 \vee \mathcal{F}_2 \cong \mathcal{F}_1 \vee \mathcal{F}_2$ while $\mathcal{A}_{12} \supset \mathcal{A}_1 \vee \mathcal{A}_2 \cong \mathcal{A}_1 \otimes \mathcal{A}_2$. $\mathcal{A}_1 \vee \mathcal{A}_2$ is missing the *bi-local intertwiners* which are operators in $\mathcal{A}_{12}$ that correspond to the creation of a pair of charged particles of opposite charge, one in region A_1 and the other in region A_2. There are two conditional expectations, i) $\mathcal{E} : \mathcal{F}_{12} \to \mathcal{A}_{12}$, ii) $\mathcal{E}_\tau : \mathcal{F}_{12} \to \mathcal{A}_1 \vee \mathcal{A}_2 \cong \mathcal{A}_1 \otimes \mathcal{A}_2$, where their action on each individual local algebra $\mathcal{F}_\mathrm{i}$ is identical, i.e., $\mathcal{E}, \mathcal{E}_\tau : \mathcal{F}_\mathrm{i} \to \mathcal{A}_\mathrm{i}$ for $(\mathrm{i} = 1, 2)$. $\mathcal{E}$ is realized as a Haar average over the group G. $\mathcal{E}_\tau$ washes the local charges away from the local algebras. Then, the generalized entanglement entropy of the disconnected local region of a general quantum system in the presence of charges due to the global symmetry G is defined by

$$S^{\mathcal{A}_{12}}(\rho_{12}\|\mathcal{E}^*_\tau(\rho_1)\otimes\rho_2) = S^{\mathcal{A}_1\otimes\mathcal{A}_2}(\rho_{12}\|\mathcal{E}^*_\tau(\rho_1)\otimes\rho_2) + S^{\mathcal{A}_{12}}(\rho_{12}\|\mathcal{E}^*_\tau(\rho_{12})) \tag{6.3}$$

[1]↑The transformation under a group of global symmetry does not depend on a background spacetime coordinates as oppose to gauge symmetry which provides the local transformations depending on the spacetime coordinates.

The first term is the relative entropy with respect to the charge-neutral operators of A_1 and A_2, and the second term measures the contribution from bi-local intertwiners in $\mathcal{A}_{12}$.

We also provide the explicit upper bound;

$$S^{\mathcal{A}_{12}}(\rho_{12}\|\mathcal{E}^*_\tau(\rho_1)\otimes\rho_2) \leq \log|G| \tag{6.4}$$

where $|G|$ is the cardinality of the group G.

3. [Multi-state quantum f-divergences]

For a von Neumann algebra $\mathcal{A}$ and a state ω on $\mathcal{A}$, consider its GNS representation on the GNS Hilbert space $\mathcal{H}_\omega$ equipped with a Hilbert-Schmidt norm, or L^2-norm $\|\,|a\rangle_\Omega\,\| = \|a\,|\Omega\rangle\,\|$ where $|\Omega\rangle$ be a GNS vacuum. We define a L^p_ω-space equipped with the L^p_ω-norm defined as

$$\begin{aligned}
\|\,|a\rangle_\Omega\,\|_{p,\Omega} &= \sup_{|\Psi^{1/2}\rangle\in\mathcal{H}_\omega} \|\Delta_{\Psi|\Omega}^{\frac{1}{2}-\frac{1}{p}}\,|a\rangle_\Omega\,\| \qquad \forall p\in[2,\infty] \\
\|\,|a\rangle_\Omega\,\|_{p,\Omega} &= \inf_{|\Psi^{1/2}\rangle\in\mathcal{H}_\omega} \|\Delta_{\Psi|\Omega}^{\frac{1}{2}-\frac{1}{p}}\,|a\rangle_\Omega\,\| \qquad \forall p\in[1,2)
\end{aligned} \tag{6.5}$$

where ψ is another state on $\mathcal{A}$ and $\Delta_{\Psi|\Omega}$ is a relative modular operator of $|\Psi\rangle$ and $|\Omega\rangle$.

Suppose we pick n states $\psi_1,\cdots,\psi_n$ on $\mathcal{A}$, and construct the relative modular operator $\Delta_{\Psi_\mathrm{i}|\Omega}$ for $\mathrm{i}=1,\cdots,n$. Note that Ψ_i are the GNS vectors corresponding to ψ_i. With the vector notations $\vec{\Psi}=(\Psi_1,\cdots,\Psi_n)$, $\vec{\theta}=(\theta_1,\cdots,\theta_n)$ and $\vec{f}=(f_1,\cdots,f_{n-1})$, we define the operator

$$\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{\theta}) \equiv \Delta^{\theta_1}_{\Psi_1|\Omega}\sharp_{f_1}\cdots\sharp_{f_{n-1}}\Delta^{\theta_n}_{\Psi_n|\Omega}\,. \tag{6.6}$$

Here, we apply the Kubo-Ando operator means defined below to the relative modular operators.

Definition 6.0.1 (Kubo-Ando mean). *For any operator monotone function f with $f(1) = 1$ and positive operators X and Y the Kubo-Ando mean $\sharp_f$ is defined to be [155], [156]*

$$X\sharp_f Y = X^{1/2} f(X^{-1/2} Y X^{-1/2}) X^{1/2} \tag{6.7}$$

where we are assuming that X is invertible. Note that $X\sharp_f X = X$.

Then, we define the multi-state f-divergence to be

$$S^{\vec{f}}_{\vec{\theta};r}(\vec{\psi}\|\omega) = \frac{-2r}{\prod_{\mathrm{i}=1}^{n}(1-\theta_{\mathrm{i}})} \log \left\| \left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{\theta}) \right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} . \tag{6.8}$$

4. [Monotonicity of multi-state quantum f-divergences under an unital completely positive map]

   For von Neumann algebras $\mathcal{A}$ and $\mathcal{B}$, consider an unital CP map $\Phi : \mathcal{B} \to \mathcal{A}$ such that $\omega_A \circ \Phi = \omega_B$ for states ω_A on $\mathcal{A}$ and ω_B on $\mathcal{B}$. We show the monotonicity of (6.8) under an unital CP map $\Phi : \mathcal{B} \to \mathcal{A}$, i.e., for a set of states $\vec{\psi}_A = (\psi_1, \cdots, \psi_n)$ on $\mathcal{A}$ and a set of states $\vec{\psi}_B = (\psi_1, \cdots, \psi_n)$ on $\mathcal{B}$,

$$S^{\vec{f}}_{\vec{\theta},r}(\vec{\psi_B}\|\omega_B) \leq S^{\vec{f}}_{\vec{\theta},r}(\vec{\psi_A}\|\omega_A) . \tag{6.9}$$

   for $r \geq 1$.

6.1 Generalized entanglement entropy, charges, and intertwiners

6.1.1 Introduction

The study of entanglement in many-body quantum systems has opened new windows to understanding strongly coupled phenomena. Entanglement measures in lattice models have helped identify phases of matter and universal dynamical processes. In Poincare-invariant quantum field theory (QFT), entanglement measures have taught us about universal long-range correlation patterns, and renormalization monotones [157]–[159]. In holographic QFT, entanglement measures play an important role in the emergence of geometry out of quan-

tum states [97]. In this work, we study the entanglement theory in quantum systems with conserved charges.

In the conventional quantum information theory, the Hilbert space of a bipartite system $A_{12} \equiv A_1 \cup A_2$ with A_1 and A_2 non-overlapping is the tensor product of the Hilbert spaces of each: $\mathcal{H}_{12} = \mathcal{H}_1 \otimes \mathcal{H}_2$. There are local algebras of operators on A_1 and A_2 that we denote by $\mathcal{F}_1$ and $\mathcal{F}_2$, respectively. For instance, the algebra of operators of a d-level quantum system (qudit) is the algebra of $d \times d$ complex matrices. The global algebra of the bipartite system A_{12} is $\mathcal{F}_{12} = \mathcal{F}_1 \otimes \mathcal{F}_2$. The local algebra $\mathcal{F}_1$ is a subalgebra of $\mathcal{F}_{12}$, and the reduced state on this subalgebra is given by a partial trace on $\mathcal{F}_2$. The entanglement measure we are interested in captures the amount of information erased by partial trace. Entanglement is a resource that can be distilled in the form of Einstein-Podolosky-Rosen (EPR) pairs and can be used to teleport quantum states. For instance, for a bipartite qudit density matrix $\rho_1 \otimes \rho_2$ the amount of information erased by partial trace on A_2 is $\log d - S_{vN}(\rho_2)$, where $S_{vN}(\rho) = -\mathrm{tr}(\rho \log \rho)$ is the von Neumann entropy. The state $\rho_1 \otimes \mathbb{I}_2/d$ is unique in that it loses no information under partial trace. The distinguishability of an arbitrary state ρ_{12} with respect to the invariant state of partial trace $\rho_1 \otimes \mathbb{I}_2/d$ can be used to quantify the amount of information lost in partial trace of A_2. In quantum information theory, the distinguishability of a state ρ from σ is measured by the relative entropy

$$S(\rho \| \sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) \tag{6.10}$$

which is non-negative and vanishes if and only if $\rho = \sigma$. We choose the relative entropy $S(\rho_{12} \| \rho_1 \otimes \mathbb{I}_2/d) = \log d - S_{vN}(\rho_{12}) + S_{vN}(\rho_1)$ as our measure of the information lost in partial trace.[2]

In systems with symmetries and conserved charges, the degrees of freedom in A_1 and A_2 are not completely independent. Charge conservation requires that any physical process that creates a charge particle in A_1 also creates the opposite charge in A_2. If we superpose states of different charge, there is no information in their relative phase because they cannot be detected in any physical process made out of charge conserving operations. The naive relative

[2]↑It has an operational interpretation in the language of the state merging protocol [160].

entropy for a charged system cannot be used as a resource to distill entangled pairs [161], [162]. In this work, we argue that the measure of entanglement with the correct operational interpretation is the sum of two relative entropies. One term captures the entanglement due to the charge-neutral degrees of freedom. These operators are invariant under the symmetry transformation. The second term captures the contribution of charged operators, and is a measure of the asymmetry of states in the resource theory of symmetry [161], [162]. In section 6.1.2, we motivate a generalized entanglement entropy beyond the case of tensor products, and connect it to the coarse-grained entropy defined by the Jaynes maximum entropy principle [145]. For other definitions of generalized entanglement see [163], [164].

The charge-neutral operators in $\mathcal{F}$ form a sub-algebra that we denote by $\mathcal{A}$; figure 6.1. In the bipartite setup, the algebra of charge-neutral operators localized in A_1 is a subalgebra of all charge-neutral operators of A_{12}: $\mathcal{A}_1 \subset \mathcal{A}_{12}$. However, it is not true that $\mathcal{A}_1$ and $\mathcal{A}_2$ generate all the charge-neutral operators of $\mathcal{A}_{12}$. The operators that spontaneously create a pair of charge particle in A_1 and its anti-charge in A_2 belong to $\mathcal{A}_{12}$, but not to $\mathcal{A}_1 \otimes \mathcal{A}_2$. In section 6.1.5, we call such operators *bi-local intertwiners* due to the role they play in the representation theory of the symmetry group; see figure 6.2. Our goal is to quantify the contribution of the local intertwiners to the entanglement. The key idea is to associate to any state ρ an invariant state $\mathcal{E}^*(\rho)$. The expectation value of all charge-neutral operators $\mathcal{A}$ in $\mathcal{E}^*(\rho)$ and ρ match, however the probability for the spontaneous creation of a charge/anti-charge pair in the invariant state is zero. The relative entropy $S(\rho\|\mathcal{E}^*(\rho))$ measures the distinguishability of the two states. It is a measure of the asymmetry of ρ and captures the information contained in the bi-local intertwiners. In section 6.1.2, we argue that this relative entropy added to the mutual information between region A_1 and A_2 due to the charge-neutral algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ captures the total amount of entanglement between A_1 and A_2. This quantity is also discussed in previous work of [165], [166] and some of the ideas here parallel those of [166].

In section 6.1.5, we review the representation theory of symmetry groups and the superselection sectors. A special role is played by the charge creation/annihilation operators that take charge neutral operators from a superselection sector to another. They are called intertwiners and together with the charge neutral sub-algebra they generate the algebra of

Figure 6.1. a) A charge neutral operator in region A: $a \in \mathcal{A}_A$. b) A charged operator in region A: $b \in \mathcal{F}_A$.

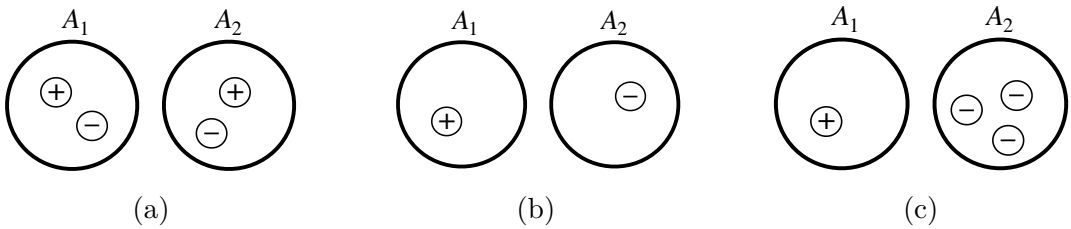


Figure 6.2. (a) Charge neutral operators in region $A_1 \cup A_2$: $a \in \mathcal{A}_1 \otimes \mathcal{A}_2$. (b) A bi-local intertwiner in $A_1 \cup A_2$: $\mathcal{I}_{12} \in \mathcal{A}_{12}$. (c) Local intertwiners, or charged operators in $A_1 \cup A_2$ that belongs to the global algebra $\mathcal{F}_1 \otimes \mathcal{F}_2$.

all charged particles. In section 6.1.5 we provide simple physical examples from qubits to QFT to demonstrate the formalism. A reader who is already familiar with the formalism can skip this section. In section 6.1.13, we make the distinction between global algebras and local algebras. In the global case, we consider the algebra of charge neutral operators as a sub-algebra of all charged operators $\mathcal{A} \subset \mathcal{F}$. In the local case, we consider the tensor product of charge neutral operators in non-overlapping regions A_1 and A_2 as a sub-algebra of charge-neutral operators of $A_1 \cup A_2$: $\mathcal{A}_1 \otimes \mathcal{A}_2 \subset \mathcal{A}_{12}$.

The study of entanglement in QFT is subtle due to absence of a tensor product $\mathcal{H}_A \otimes \mathcal{H}_{A'}$ that reflects itself as ultra-violet divergence in the entanglement entropy [58], [165], [167]. Modular theory is a mathematical framework that is well-suited for the study of entanglement in any quantum system from qubits to QFT. In modular theory, instead of tensor products and local density matrices, the algebra of operators localized in a region and locality constraints among them are used to define entanglement measures. In section 6.1.17,

we use modular theory to define both the relative entropies that measure the entanglement between non-touching regions A_1 and A_2 in a QFT with conserved charges. We highlight the difference in the analysis of entanglement between QFTs and lattice models. Finally, we discuss an extension of the QFT algebra that factors out charged excitations and brings the QFT algebra closer to lattice models.

In this work, we focus on global symmetries, however, the formalism can be generalized to many gauge theories [168], [169]. We postpone this to future work.

6.1.2 Generalizations of entanglement

6.1.3 Conditional expectation as generalization of partial trace

Consider the algebra of operators of two qudits $\mathcal{F}_{12} = \mathcal{F}_1 \otimes \mathcal{F}_2$ and the subalgebra of operators localized on the first system $\mathcal{F}_1 \otimes \mathbb{I}_2$. The reduced density matrix on $\mathcal{F}_1$ is given by the partial trace over $\mathcal{F}_2$: $\rho_1 = \mathrm{tr}_2(\rho_{12})$. In the classical case, $\rho_{12} = \sum_{kk'} p_{kk'} |kk'\rangle \langle kk'|$ the reduced density matrix on the first qudit is $\rho_1 = \sum_k q_k |k\rangle \langle k|$ where $q_k = \sum_{k'} p_{kk'}$ are the classical conditional expectations to obtain result k in a measurement on first qudit: $q_k = \mathrm{tr}(|k\rangle \langle k| \otimes \mathbb{I})\rho_{12}$. In a mathematical analogy, one can think of density matrices as non-commutative probabilities and partial trace as non-commutative conditional expectation [170].

To compute how much information was erased during partial trace we have to pull ρ_1 back to the bipartite Hilbert space by a linear map that we denote by $\alpha^*(\rho_1) = \phi_{12}$ with the following properties:

1. It is consistent with ρ_1: $\mathrm{tr}_2(\phi_{12}) = \rho_1$.

2. The state ϕ_{12} is invariant under partial trace and α^*: $\alpha^*(\phi_1) = \phi_{12}$ so that α^* does not add any information.

We call such α^* maps *recovery maps* or state extensions [170], [171]. In the partial trace case, the recovery map with the properties above is $\alpha^*(\rho_1) = \rho_1 \otimes \mathbb{I}_2/d$.[3] It is convenient to think of partial trace and recovery together as one linear map that sends density matrices

[3]↑An example of a map that satisfies the first property but not the second is $\alpha^*(\rho_1) = \rho_1 \otimes \omega_2$ for some ω_2.

on $\mathcal{F}_{12}$ to the density matrices on the subalgebra $\mathcal{F}_1 \otimes \mathbb{I}_2$: $\mathcal{E}^*(\rho_{12}) = \rho_1 \otimes \mathbb{I}_2/d$. The dual of the $\mathcal{E}^*$ is a projection from $\mathcal{F}_{12}$ down to the subalgebra $\mathcal{F}_1 \otimes \mathbb{I}_2$:

$$E(b_1 \otimes b_2) = b_1 \otimes \mathbb{I}_2 \operatorname{tr}\left(\frac{b_2}{d}\right) \ . \tag{6.11}$$

Here, by duality we mean going from the Schrödinger to the Heisenberg picture[4]

$$\operatorname{tr}(\mathcal{E}^*(\rho_{12})(b_1 \otimes b_2)) = \operatorname{tr}(\rho_{12} E(b_1 \otimes b_2)) = \operatorname{tr}(\rho_1 b_1) \operatorname{tr}\left(\frac{b_2}{d}\right) \ . \tag{6.12}$$

The map $\mathcal{E}$ has the property that it squares to itself, i.e. $\mathcal{E}^2 = E$, so that $\mathcal{E}^*(\rho_{12})$ is invariant state of $\mathcal{E}$:

$$\operatorname{tr}(\mathcal{E}^*(\rho_{12}) E(b_1 \otimes b_2)) = \operatorname{tr}\left(\mathcal{E}^*(\rho_{12})(b_1 \otimes b_2)\right) \ . \tag{6.13}$$

The relative entropy of ρ_{12} with respect to the invariant state $\mathcal{E}^*(\rho_{12}) = \rho_1 \otimes \mathbb{I}_2/d$ measures the asymmetry of the state or the amount of information erased in partial trace: $S(\rho_{12} \| \mathcal{E}^*(\rho_{12})) \geq 0$; see figure 6.3.

A simple way to generalize partial trace is to consider a more general dual map $\mathcal{E} : \mathcal{F}_{12} \to \mathcal{F}_1 \otimes \mathbb{I}_2$:

$$\begin{aligned} E(b_1 \otimes b_2) &= b_1 \otimes \mathbb{D}(b_2) \\ \mathbb{D}(b) &= \sum_k b_{kk} \left|k\right\rangle \left\langle k\right| \end{aligned} \tag{6.14}$$

where $\{|k\rangle\}$ is some distinguished basis of the second qudit. In the Schrödinger picture, the state transforms according to

$$\begin{aligned} \mathcal{E}^*(\rho_{12}) &= \sum_k p_k \rho_1^{(k)} \otimes \left|k\right\rangle_2 \left\langle k\right|_2 \\ p_k \rho_1^{(k)} &= \left\langle k\right|_2 \rho_{12} \left|k\right\rangle_2 , \qquad p_k = \operatorname{tr}(\rho_{12}(\mathbb{I} \otimes \left|k\right\rangle_2 \left\langle k\right|_2)) \end{aligned} \tag{6.15}$$

[4]↑An alternative notation used in [166] is to denote $\mathcal{E}^*(\rho)$ by $\rho \circ E$.

that dephases the density matrix and erases the information in the off-diagonal operators $|k\rangle\langle k'|$. Similar to the $\mathcal{E}$ of partial trace we have the property that $\mathcal{E}(\mathbb{I}) = \mathbb{I}$ so that $\mathcal{E}^*(\rho)$ is properly normalized. Furthermore, $\mathcal{E}$ squares to itself which implies that $\mathcal{E}^*(\rho_{12})$ is an invariant state of $\mathcal{E}$.

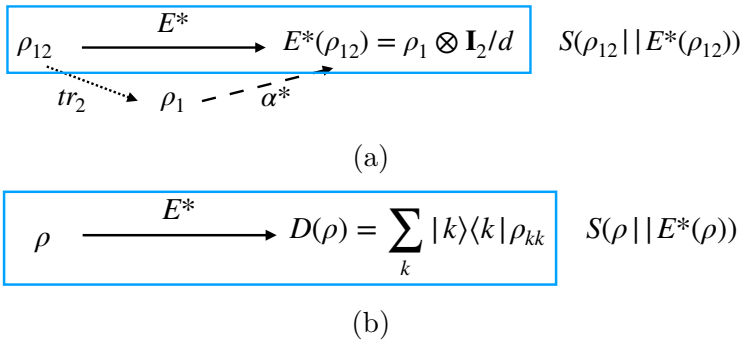


Figure 6.3. Our entanglement measure is the relative entropy of the state ρ with respect to its corresponding invariant state $\mathcal{E}^*(\rho)$: $S(\rho\|\mathcal{E}^*(\rho))$. (a) The example where the map $\mathcal{E}^*$ is a composition of partial trace of system 2 and the recovery map α^* which results in an invariant state $\mathcal{E}^*(\rho_{12})$. (b) The example where the map $\mathcal{E}^*$ decoheres the density matrix ρ in a particular basis $\{|k\rangle\}$.

In systems with conserved charges, the subalgebra of charge-neutral operators corresponds to matrices that are block-diagonal in some basis labelled by charge. For instance, take a qubit and the symmetry transformation σ_z. The Abelian subalgebra $\mathcal{D} \subset \mathcal{F}$ of 2×2 complex matrices diagonal in σ_z basis is the charge neutral algebra. The dephasing map $\mathcal{E}(b) = \mathbb{D}(b)$ projects operators from $\mathcal{F}$ to $\mathcal{D}$. For a general quantum system with symmetry we need to define a linear map $\mathcal{E} : \mathcal{F} \to \mathcal{A}$ with $\mathcal{A} \subset \mathcal{F}$ the subalgebra of charge-neutral operators as a generalization of partial trace. An example of one such map is the Haar average over the group G:

$$E(b) = \frac{1}{|G|} \int dg U_g^\dagger b U_g \ . \tag{6.16}$$

The operator $\mathcal{E}(b)$ is charge-neutral for any charged operator b. In analogy to partial trace, we require this map preserves the identity operator, and it leaves the charge-neutral operators unchanged so that the state $\mathcal{E}^*(\rho)$ defined by $(\mathcal{E}^*(\rho))(b) = \rho(E(b))$ for all $b \in \mathcal{F}$ is invariant under the map $\mathcal{E}$: $\mathcal{E}(\mathcal{E}^*(\rho)) = \mathcal{E}^*(\rho)$. The generalization of partial trace is called the *non-*

commutative conditional expectation (or in short conditional expectation) that is a linear map from $\mathcal{F}$ to an arbitrary subalgebra $\mathcal{A}$ such that $\mathcal{E}(\mathbb{I}) = \mathbb{I}$ and $\mathcal{E}(ab) = aE(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{F}$ [170], [172].[5] Since $\mathcal{E}(a) = a$ all invariant operators are in $\mathcal{A}$ and every operator in $\mathcal{A}$ is invariant. As a result, $\mathcal{E}^*(\rho) \in \mathcal{A}$.

6.1.4 Generalized entanglement entropy and coarse-grained entropy

In conventional quantum information theory, the amount of entanglement between A_1 and A_2 is measured by the distinguishability of the ρ_{12} with respect to the unentangled state $\rho_1 \otimes \rho_2$:

$$S(\rho_{12} \| \rho_1 \otimes \rho_2) = -S_{vN}(\rho_{12}) + S_{vN}(\rho_1) + S_{vN}(\rho_2) \tag{6.17}$$

which is called the mutual information. Consider a multi-partite global state $|\Omega\rangle_{AA'}$ and its reduced states ρ_A and $\rho_{A'}$ on region A and the complementary region A', respectively. The distinguishability of $|\Omega\rangle$ from the tensor product state $\rho_A \otimes \rho_{A'}$ is measured by the relative entropy

$$S(|\Omega\rangle \langle\Omega| \, \| \rho_A \otimes \rho_{A'}) = 2S_{vN}(\rho_A) \ . \tag{6.18}$$

The tensor product state $\rho_A \otimes \rho_{A'}$ has the same expectation values as $|\Omega\rangle$ for all operators in $\mathcal{F}_A \otimes \mathbb{I}$ and $\mathbb{I} \otimes \mathcal{F}_{A'}$, however, all correlations between A and A' are erased. The expectation of all operators $b \otimes b'$ with $b \in \mathcal{F}_A$ and $b' \in \mathcal{F}_{A'}$ factors in the tensor product state $\rho_A \otimes \rho_{A'}$.

To generalize the notion of entanglement to a general subalgebra $\mathcal{A} \subset \mathcal{F}$ we invoke the Jaynes maximum entropy principle. Consider the set of all density matrices σ that have the same expectation values as ρ for operators in $\mathcal{A}$: $\mathrm{tr}((\sigma - \rho)a) = 0$ for all $a \in \mathcal{A}$. According to Jaynes the entropy of a state ρ with respect to a subalgebra $\mathcal{A}$ is the supremum of the von Neumann entropy $S_{vN}(\sigma)$ over the set of all consistent states σ [145]:

$$S_J(\rho, \mathcal{A}) = S_{vN}(\sigma_{max}) \tag{6.19}$$

[5]↑In this paper, the operator b is chosen to belong to the algebra of charged operators, whereas a denotes a charge-neutral operator.

where $\sigma_{\max}$ is consistent with ρ and has the maximum entropy. Hereafter, we suppress the vN index of the von Neumann entropy.

The Jaynes maximum entropy consistent state is precisely the invariant state $\mathcal{E}^*(\rho)$. Given a general conditional expectation $\mathcal{E}$ and a state σ consistent with ρ on $\mathcal{A}$ we have

$$\mathrm{tr}((\mathcal{E}^*(\sigma) - \mathcal{E}^*(\rho))b) = \mathrm{tr}((\sigma - \rho)E(b)) = 0, \tag{6.20}$$

therefore $\mathcal{E}^*(\sigma) = \mathcal{E}^*(\rho)$. At the end of section 6.1.3 we showed that the invariant state is in $\mathcal{A}$, therefore the logarithm of an invariant state is also in $\mathcal{A}$:

$$\begin{aligned} \mathrm{tr}(\sigma \log \mathcal{E}^*(\rho)) &= \mathrm{tr}(\sigma E(\log \mathcal{E}^*(\rho))) = \mathrm{tr}(\mathcal{E}^*(\sigma) \log \mathcal{E}^*(\rho)) \\ &= \mathrm{tr}(\mathcal{E}^*(\rho) \log \mathcal{E}^*(\rho)) = -S(\mathcal{E}^*(\rho)) \ . \end{aligned} \tag{6.21}$$

In the above, we have assumed that the conditional expectation preserves the trace: $\mathrm{tr}(E(b) - b) = 0$ [166].[6] From the definition (6.10) it follows that the relative entropy of any consistent state σ consistent with ρ on A with respect to the invariant state $\mathcal{E}^*(\rho)$ is

$$S(\sigma \| \mathcal{E}^*(\rho)) = -S(\sigma) + S(\mathcal{E}^*(\rho)) \geq 0 \ . \tag{6.22}$$

From the positivity of relative entropy we conclude that the invariant state of a conditional expectation $\mathcal{E}$ is the maximum entropy state appearing in the Jaynes formula:

$$E^*(\rho) = \sigma_{max} \tag{6.23}$$

and the non-degeneracy of relative entropy tells us that this state is unique.[7] Therefore, our proposed measure of the information lost in $\mathcal{E}$ is the entanglement deficit from the maximum value:

$$S(\rho \| \mathcal{E}^*(\rho)) = S(\sigma_{max}) - S(\rho) \ . \tag{6.24}$$

[6]↑We thank Horacio Casini for pointing this out to us.
[7]↑If σ_{max} and σ'_{max} are both maximum entropy then $S(\sigma_{max} \| \sigma'_{max}) = 0$, therefore $\sigma_{max} = \sigma'_{max}$.

As an example, consider the subalgebra of matrices $\mathcal{A} = \mathcal{F}_1 \otimes \mathcal{D}_2$ and the set of all σ that are consistent with ρ on $\mathcal{A}$ and maximize the entropy among them. The consistent states are all σ_{12} that satisfy $\mathrm{tr}((\sigma_{12} - \rho_{12})(a_1 \otimes |k\rangle \langle k|)) = 0$ for all basis vectors $|k\rangle$. The relative entropy of σ_{12} with respect to the invariant state in (6.15) is

$$S(\sigma_{12} \| \sum_k p_k \rho_1^{(k)} \otimes |k\rangle \langle k|) = -S(\sigma_{12}) + H(p) + \sum_k p_k S(\rho_1^{(k)}) \geq 0 \tag{6.25}$$

where $H(p) = -\sum_k p_k \log(p_k)$ is the Shannon entropy of p_k [173]. The maximum entropy state is the invariant state, and the Jaynes entropy is

$$S_J(\rho_{12}, \mathcal{F}_1 \otimes \mathcal{D}_2) = S(\mathcal{E}^*(\rho_{12})) = H(p) + \sum_k p_k S(\rho_1^{(k)}) \ . \tag{6.26}$$

The reduced state on system A_1 is $\rho_1 = \sum_k p_k \rho_1^{(k)}$. The von Neumann entropy of ρ_1 is less than the Jaynes entropy because of the inequality [173]

$$S(\sum_k p_k \rho_1^{(k)}) \leq H(p) + \sum_k p_k S(\rho_1^{(k)}) \ . \tag{6.27}$$

The definition of Jaynes entropy can be generalized beyond subalgebras to any subspace of observables P:

$$S_J(\rho, P) = \sup_{\sigma \in \mathcal{F}^*} \{S_{vN}(\sigma) | \mathrm{tr}((\sigma - \rho)a) = 0, \forall a \in P\} \tag{6.28}$$

where $\mathcal{F}^*$ denotes the set of all states of the global algebra $\mathcal{F}$. This measure is often called the *coarse-grained entropy*. For instance, consider the subspace of observables built out of linear sums of $a_1 \otimes \mathbb{I}$ and $\mathbb{I} \otimes a_2$ and a bipartite density matrix ρ_{12}. The relative entropy $S(\rho_{12} \| \rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2) - S(\rho_{12}) = I(1:2) \geq 0$, where $I(1:2)$ is the mutual information between site one and two. Therefore, the maximum entropy state in the Jaynes formula that reduces to both ρ_1 and ρ_2 is $\rho_1 \otimes \rho_2$ and as a result $S_J(\rho_{12}, P) = S(\rho_1 \otimes \rho_2) = S(\rho_1) + S(\rho_2)$ [171]. Our relative entropy measure

$$S(\rho_{12} \| \sigma_{max}) = S(\rho_{12} \| \rho_1 \otimes \rho_2) = I(1:2) \tag{6.29}$$

equals the mutual information that well captures the amount of correlations between A_1 and A_2. In the absence of a subalgebra and a conditional expectation σ_{max} replaces $\mathcal{E}^*(\rho)$ and we propose $S(\rho\|\sigma_{max})$ as a measure of the information lost under restriction to the subspace of observables P [171]. To find the maximum entropy state consider the Lagrange multipliers $\lambda_{\rm i}$ and the function

$$-\mathrm{tr}(\sigma\log\sigma)+\sum_{\rm i}\lambda_{\rm i}\mathrm{tr}((\rho-\sigma)\mathcal{O}_{\rm i}) \tag{6.30}$$

where $\mathcal{O}_{\rm i}$ is a basis for the subspace of observables P. Setting the variation of the expression above with respect to σ and $\lambda_{\rm i}$ establishes that the maximum entropy state $\log\sigma_{max}=\sum_{\rm i}\mu_{\rm i}\mathcal{O}_{\rm i}\in P$ for some constants $\mu_{\rm i}$. Similar to the case of conditional expectation the maximum entropy state belongs to the subspace P, i.e. $\sigma_{max}\in P$, and the expectation value of every operator that is not in P is zero. As a result

$$S(\rho\|\sigma_{max})=-S(\rho)-\mathrm{tr}(\rho\log\sigma_{max})=S(\sigma_{max})-S(\rho)\ . \tag{6.31}$$

In QFT the von Neumann entropy of a region is divergent[8] and we can only compute the relative entropy of states. This motivates us to replace Jayne's maximum entropy principle with the supremum of $S(\rho\|\sigma)$ over all σ consistent with ρ on P:

$$I_P(\rho)=\sup_{\sigma\in\mathcal{F}^*}\{S(\rho\|\sigma)|\mathrm{tr}((\sigma-\rho)a)=0,\forall a\in P\} \tag{6.32}$$

that is the measure of information entropy produced under the restriction to a subspace of observables P and has the advantage of being well-defined in QFT like in systems with density matrices. We postpone further discussion of the generalized entanglement to future work and in the remainder of this work focus on the case of charge-neutral subalgebras.

In a system with an internal symmetry group G, the symmetry transformation acts on the local algebra of region A as a unitary transformation: $b_{\rm i}\to U_g^\dagger b_{\rm i}U_g$ for all $b_{\rm i}\in\mathcal{F}_{\rm i}$ and U_g

[8]↑It is a property of the algebra and not the states.

some unitary representation of G. The operators in $\mathcal{F}_\mathrm{i}$ that are invariant under the action of the symmetry form a subalgebra of uncharged operators that we denote by $\mathcal{A}_\mathrm{i}$:

$$U_g^\dagger a_\mathrm{i} U_g = a_\mathrm{i}, \forall a_\mathrm{i} \in \mathcal{A}_\mathrm{i} \ . \tag{6.33}$$

On a lattice, there is a unitary operator localized in $\mathcal{F}_\mathrm{i}$ that acts the same way as U_g on $\mathcal{F}_\mathrm{i}$:

$$\tau_g b_\mathrm{i} \tau_g^\dagger = U_g b_\mathrm{i} U_g^\dagger, \qquad \forall b_\mathrm{i} \in \mathcal{F}_\mathrm{i} \tag{6.34}$$

we call this operator the *twist* and it generates another representation of the group that we call the *twist group* G_τ: $\tau_g \tau_h = \tau_{gh}$. The commutator of the twist with the group action is

$$U_g \tau_h U_g^\dagger = \tau_{ghg^{-1}} \ . \tag{6.35}$$

For instance, in a bipartite system with symmetry transformation $U_g = \mathrm{e}^{\mathrm{i}g(Q_1+Q_2)}$ where $Q_1 + Q_2$ is the total charge of A_{12} the twist is $\tau_g = \mathrm{e}^{\mathrm{i}gQ_1}$; see figure 6.2. It belongs to $\mathcal{F}_1$ and acts the same way as U_g on $\mathcal{F}_1$. We postpone the subtleties in defining τ_g in QFT to section 6.1.17. The algebra $\mathcal{A}_{12}$ of charge-neutral operators in A_{12} is larger than the algebra generated by locally charge-neutral operators of A_1 and A_2, namely $\mathcal{A}_1 \otimes \mathcal{A}_2$. This is because there are operators that correspond to the creation of a pair of charged particles of opposite charge one in region A_1 and the other in A_2. We call these operators the bi-local intertwiners $\mathcal{I}_{12}$. We will see in section 6.1.13 that there exists a conditional expectation constructed from the twist group $\mathcal{E}_\tau : \mathcal{A}_{12} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ that washes out the information content of the bi-local intertwiners: $\mathcal{E}_\tau(\mathcal{I}_{12}) = 0$.[9] The amplitude for the invariant state $\mathcal{E}_\tau^*(\rho_{12})$ to spontaneously create an entangled pair of charge/anti-charge particles is zero. The relative entropy $S(\rho_{12} \| E_\tau^*(\rho_{12}))$ measures the amount of correlations due to the bi-local intertwiners. Note that the reduced state on $\mathcal{A}_1 \otimes \mathcal{A}_2$ still contains lots of correlations in between region one and two. It is only the correlations due to intertwiners that are washed out. In the presence of charges, the naive mutual information $S^{\mathcal{F}_{12}}(\rho_{12} \| \rho_1 \otimes \rho_2)$ contains unphysical correlations that cannot be accessed in any charge-conserving process. We would

[9]↑The map $\mathcal{E}_\tau$ is from $\mathcal{F}_{12}$ to $\mathcal{A}_1 \otimes \mathcal{A}_2$. However, we will be mostly concerned with its action on $\mathcal{A}_{12}$.

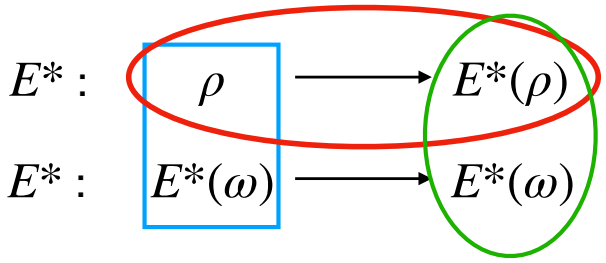


Figure 6.4. A pictorial description of the relative entropy property in equation (6.37) written in terms of states of $\mathcal{F}$. The relative entropy of blue states $S(\rho||\mathcal{E}^*(\omega))$ is the relative entropy the red ellipse $S(\rho\|\mathcal{E}^*(\rho))$ plus the relative entropy of the green ellipse $S(\mathcal{E}^*(\rho)\|\mathcal{E}^*(\omega))$. Note that since both $\mathcal{E}^*(\rho)$ and $\mathcal{E}^*(\omega)$ are invariant under $\mathcal{E}$, the green relative entropy is the same as $S^{\mathcal{A}}(\rho\|\mathcal{E}^*(\omega))$.

like to discard all operators that create charge on A_{12}. First, we restrict the relative entropy to the invariant algebra $\mathcal{A}_{12}$. In general, the relative entropy $S^{\mathcal{A}}(\rho\|\omega)$ is a measure of distinguishability of the two states using only the operators in $\mathcal{A}$. Alternatively, one can think of this relative entropy as

$$S^{\mathcal{A}_{12}}(\rho\|\omega) = S^{\mathcal{F}_{12}}(\mathcal{E}^*(\rho)\|\mathcal{E}^*(\omega)) \tag{6.36}$$

where $\mathcal{E} : \mathcal{F}_{12} \to \mathcal{A}_{12}$. The expression above implies that the distinguishability of invariant states of $\mathcal{E}$ does not change under the restriction to the invariant subalgebra $\mathcal{A}_{12}$ [170]. Second, we replace ρ_1 with $\mathcal{E}^*_\tau(\rho_1)$ to make sure that $\mathcal{E}^*_\tau(\rho_1) \otimes \rho_2$ has no bi-local intertwiners. Therefore, we consider the measure $S^{\mathcal{A}_{12}}(\rho_{12}\|E^*_\tau(\rho_1 \otimes \rho_2))$.

A useful property of relative entropy is that it satisfies the following equality (Theorem 9.3 of [170]); see figure 6.4:

$$S^{\mathcal{F}}(\rho\|\mathcal{E}^*(\omega)) = S^{\mathcal{A}}(\rho\|\mathcal{E}^*(\omega)) + S^{\mathcal{F}}(\rho\|\mathcal{E}^*(\rho)) \tag{6.37}$$

where $\mathcal{E} : \mathcal{F} \to \mathcal{A}$. Applying the identity above to the twist conditional expectation $\mathcal{E}_\tau$ implies that our measure splits into two terms [10]

$$S^{\mathcal{A}_{12}}(\rho_{12}\|E_\tau^*(\rho_1)\otimes\rho_2) = S^{\mathcal{A}_1\otimes\mathcal{A}_2}(\rho_{12}\|E_\tau^*(\rho_1)\otimes\rho_2) + S^{\mathcal{A}_{12}}(\rho_{12}\|E_\tau^*(\rho_{12}))\ . \tag{6.38}$$

The first term is the relative entropy with respect to the charge-neutral operators of A_1 and A_2, and the second term is the contribution due to the bi-local intertwiners. We can use the conditional expectation $\mathcal{E} : \mathcal{F}_{12} \to \mathcal{A}_{12}$ to rewrite both terms in terms of the charged algebras:

$$S^{\mathcal{F}_{12}}(\mathcal{E}^*(\rho_{12})\|\mathcal{E}^*(E_\tau^*(\rho_{12}))) + S^{\mathcal{F}_{12}}(\mathcal{E}^*(E_\tau^*(\rho_{12}))\|\mathcal{E}^*(E_\tau^*(\rho_1)\otimes\rho_2))\ . \tag{6.39}$$

In section 6.1.5, we will see that the conditional expectations $\mathcal{E}$ and $\mathcal{E}_\tau$ are Haar averages over the group and the twist group, respectively. If ρ_{12} is invariant under U_g we get the following simplification

$$S^{\mathcal{F}_{12}}(\rho_{12}\|E_\tau^*(\rho_{12})) + S^{\mathcal{F}_{12}}(E_\tau^*(\rho_{12})\|\mathcal{E}_\tau^*(\rho_1)\otimes\rho_2)\ . \tag{6.40}$$

From the conditional expectation in (6.16) it is clear that our relative entropies have the general form $S(\sum_k p_k\rho_k\|\sum_k q_k\omega_k)$. Relative entropy satisfies the inequality

$$S(\sum_k p_k\rho_k\|\sum_k q_k\omega_k) \le H(p\|q) + \sum_k p_k S(\rho_k\|\omega_k) \tag{6.41}$$

where $H(p\|q)$ is the classical KullbackLeibler divergence of the probability distributions p_k and q_k. To see this, consider the block-diagonal density matrices $\rho = \oplus_k p_k\rho_k$ and $\omega = \oplus_k q_k\omega_k$ the relative entropy

$$S(\rho\|\omega) = H(p\|q) + \sum_k p_k S(\rho_k\|\omega_k)\ . \tag{6.42}$$

[10] ↑This identity was also used in [174] to compute relative entropies in QFT.

With respect to the subalgebra of operators $\mathbb{I} \otimes a$ the density matrix is $\sum_k p_k \rho_k$ and $\sum_k q_k \omega_k$ and since relative entropy is monotonic under restriction to the subalgebra we find that relative entropy satisfies (6.41). In section 6.1.17, we generalize this inequality to QFT and use it to bound the relative entropies in (6.40) from above and below.

6.1.5 Symmetry and intertwiners

6.1.6 Superselection sectors and intertwiners

We start by reviewing some definitions and set the notations for our discussion of quantum systems with symmetries. Consider a quantum system and its Hilbert space $\mathcal{H}$. The set of all bounded linear operators acting on this Hilbert space forms an algebra, $B(\mathcal{H})$, that acts irreducibly on $\mathcal{H}$. We call this algebra the field algebra and denote it by $\mathcal{F}$. All proper subalgebras of $\mathcal{F}$ act reducibly on $\mathcal{H}$. A symmetry is a linear transformation of operators in the algebra $b \to \alpha_g(b) \in \mathcal{F}$ that respects operator multiplication: $\alpha_g(b_1 b_2) = \alpha_g(b_1)\alpha_g(b_2)$ and is invertible.[11] The set of all symmetry transformations of the algebra forms the symmetry group G. By Wigner's theorem, any symmetry is represented by either a unitary or anti-unitary transformation of the Hilbert space, i.e. $|\Psi\rangle \to U_g |\Psi\rangle$, and acts on the algebra as $\alpha_g(b) = U_g^\dagger b U_g$. The set of operators a that commute with U_g form a subalgebra $\mathcal{A} \subset \mathcal{F}$ that we refer to either as the invariant subalgebra, or the subalgebra of charge-neutral operators. On a lattice if the group G is Abelian U_g is itself charge neutral and belongs to $\mathcal{A}$.[12]

If there exist vectors in the Hilbert space such that

$$\langle \Phi | U_g | \Psi \rangle = 0 \tag{6.43}$$

for all $U_g \in G$ we say that $|\Phi\rangle$ and $|\Psi\rangle$ belong to different *selection* sectors.[13] The Hilbert space splits into a direct sum of selection sectors $\mathcal{H} = \oplus_r \mathcal{K}_r \otimes \mathcal{H}_r$ where $\mathcal{K}_r$ is the irreducible representation r of G and $\mathcal{H}_r$ is the Hilbert space corresponding to the charge neutral degrees

[11] ↑In mathematical language, such a transformation is called an automorphism of the algebra. If we relax the invertibility assumption we have an endomorphism of the algebra.
[12] ↑When U_g is not in $\mathcal{A}$ we say the symmetry transformation is an outer automorphism of the algebra $\mathcal{A}$.
[13] ↑If there exists no selection sectors; that is to say the only subspace of $\mathcal{H}$ invariant under the symmetry transformation is the whole $\mathcal{H}$ we say the action of the symmetry is *ergodic*. For instance, the action of modular flow on local algebras of QFT is ergodic.

of freedom. The basis of the Hilbert space is $|r,\mathrm{i}\rangle \otimes |\alpha\rangle$ where $\mathrm{i} = 1, \cdots, d_r$ with d_r the dimension of the irreducible representation r. The group acts as $U = \oplus_r U_g^{(r)} \otimes 1_r$, and by Schur's lemma the invariant operators of each irreducible representation are $\mathbb{I}_r \otimes a$, where $\mathbb{I}_r = \sum_{\mathrm{i}=1}^{d_r} |r,\mathrm{i}\rangle \langle r,\mathrm{i}|$ is the identity operator in the Hilbert space $\mathcal{K}_r$ of representation r. The subalgebra of invariant operators is $\oplus_r \mathbb{I}_r \otimes a$ which has the non-trivial center $\oplus_r \lambda_r \mathbb{I}_r \otimes 1_r$. If the group G is Abelian all its irreducible representations are one-dimensional and we can label them by charge q: $\mathcal{H} = \oplus_q |q\rangle \langle q| \otimes \mathcal{H}_q$ or simply $\mathcal{H} = \oplus_q \mathcal{H}_q$.

Consider the Abelian group $\mathbb{Z}_d$ and its irreducible representations labelled by charge q: $U_g^q = \mathrm{e}^{2\pi \mathrm{i} g q/d}$ with $g = 0, \cdots, d-1$ and $q = 0, \cdots, d-1$. The regular representation of G is the vector space $\mathcal{K}$ of a qudit:

$$U_g = \sum_h |(g+h) \bmod d\rangle \langle h| \tag{6.44}$$

where $g+h$ is the group multiplication and the identity element is zero charge. The irreducible representations are all one-dimensional and correspond to basis where all U_g are diagonal

$$\begin{aligned} U_g &= \sum_q \mathcal{E}^{2\pi \mathrm{i} g q/d} |q\rangle \langle q| \\ |q\rangle &= \sum_g \mathrm{e}^{-2\pi \mathrm{i} g q/d} |g\rangle \ . \end{aligned} \tag{6.45}$$

The *dual group* $\hat{G}$ is the Fourier space generated by

$$\hat{U}_q = \sum_g \mathrm{e}^{-2\pi \mathrm{i} g q/d} |g\rangle \langle g| = \sum_k |(q+k) \bmod d\rangle \langle k| \ . \tag{6.46}$$

The elements of the dual group take us in between irreducible representations and commute with the action of the invariant subalgebra

$$\hat{U}_q |k\rangle \langle k| = |k+q\rangle \langle k+q| \hat{U}_q \ . \tag{6.47}$$

The operators that satisfy the equation above are called the intertwiners, and physically they are charge creation/annihilation operators. Take the infinite Abelian group $G = U(1)$

of rotations around a circle. The irreducible representations are constant momentum modes and the intertwiners are the operators that add momentum $\hat{U}_q = \sum_q |q+k\rangle \langle k|$ and generate the dual group $\hat{G} = \mathbb{Z}$ with the multiplication operation that adds charges $k+q$.

Consider a finite non-Abelian group G represented in its regular representation by a qudit of dimension $|G|$:

$$U_g = \sum_h |gh\rangle \langle h| \tag{6.48}$$

where gh is the group multiplication. The Hilbert space splits into $\mathcal{K} = \oplus_{r,\mathrm{i}} \mathcal{K}_{r,\mathrm{i}}$ where the irreducible representation r with the index i running from zero to the dimension d_r. The irreducible representation r appears d_r times in the decomposition of the regular representation, therefore $\sum_r d_r^2 = |G|$. An operator in $\mathcal{K}_r$ can be written as $\sum_{\mathrm{ij}} b_{\mathrm{ij}} |r,\mathrm{i}\rangle \langle r,\mathrm{j}|$ but by Schur's lemma the invariant operators are proportional to $\mathbb{I}_r$. The intertwiners are linear maps that take us in between different irreducible representations and commute with the action of the invariant operators in the algebra:

$$V_{r,\mathrm{i}} \mathbb{I}_r = |0\rangle \langle 0| V_{r,\mathrm{i}} \ . \tag{6.49}$$

The partial isometry $V_{r,\mathrm{i}} = \frac{1}{\sqrt{d_r}} |0\rangle \langle r,\mathrm{i}|$ satisfies this equation, and is the non-Abelian analog of $|0\rangle \langle q|$. The map ρ_r maps operators from the charged sectors to the vacuum sector:

$$\rho_r(\mathbb{I}_r a) = \sum_{\mathrm{i}} V_{r,\mathrm{i}} \mathbb{I}_r a V_{r,\mathrm{i}}^\dagger \tag{6.50}$$

where $a \in \mathbb{C}$ is a complex number here. In the Abelian case, we constructed a unitary $\hat{U}_k$ by adding $|q+k\rangle \langle q|$ that generates the dual group $\hat{G}$. For an arbitrary charge-neutral operator $\tilde{a} = \sum_q a_q |q\rangle \langle q|$ we have

$$\begin{aligned} \rho_k(\tilde{a}) &= \hat{U}_k^\dagger \tilde{a} \hat{U}_k \\ \hat{U}_k &= \sum_q |q+k\rangle \langle q| \end{aligned} \tag{6.51}$$

which is a generalization of (6.46) to an arbitrary Abelian group. However, in the non-Abelian case, adding a charge r to another charge r' corresponds to the tensor multiplication of two irreducible representations that is not irreducible. The dual $\hat{G}$ to a non-Abelian group G is not a group. The elements of the dual to a non-Abelian group are different representations (not necessarily irreducible), and their multiplication is tensor multiplication but there is no inverse operation. As we will see in the next section, when the representation is infinite dimensional the operators $V_{r,\mathrm{i}}$ can be thought of as isometries that take us between the irreducible representations.

If the symmetry group G is compact there is a normalizable Haar measure dg and we can integrate over the group to project to the zero charge sector $P_0 = |0\rangle\langle 0| \otimes 1$:

$$\frac{1}{|G|}\int_{g\in G} dg\, U_g\,|\Psi\rangle = P_0\,|\Psi\rangle \tag{6.52}$$

where $|G|$ is the volume of the group. The resulting subspace is called the vacuum sector which is spanned by all the invariant states of G. For an Abelian group G the other irreducible representations are found using a Fourier transform with $q \in \hat{G}$ with the group multiplication being the addition of charges:

$$\frac{1}{|G|}\int_{g\in G} dg\, \mathrm{e}^{\frac{2\pi \mathrm{i} gq}{|G|}} U_g\,|\Psi\rangle = P_q\,|\Psi\rangle \ . \tag{6.53}$$

The non-Abelian analog of this projector is

$$P_r = \frac{d_r}{|G|}\int_{g\in G} dg\, \chi_r^*(g)\, U_g \tag{6.54}$$

where $\chi_r(g)$ is the character of the irreducible representation r.

We say two vectors $|\Psi\rangle$ and $|\Phi\rangle$ belong to different *superselection sectors* of algebra $\mathcal{A}$ if $\langle\Psi|a\Phi\rangle = 0$ for all $a \in \mathcal{A}$. For instance, states $|\Psi_q\rangle$ and $|\Phi_{q'}\rangle$ that were in different selection sectors of $\mathcal{F}$, belong to different superselection sectors of the neutral subalgebra $\mathcal{A}$. Given an algebra $\mathcal{F}$ and a compact symmetry group G the linear map $\mathcal{E} : \mathcal{F} \to \mathcal{A}$ that computes

the group average of an operator $b \in \mathcal{F}$ is a conditional expectation to the charge-neutral subalgebra

$$E(b) = \frac{1}{|G|} \int_{g \in G} dg \, U_g^\dagger b U_g \ . \tag{6.55}$$

because it satisfies $\mathcal{E}(ab) = aE(b)$ for all $a \in \mathcal{A}$ and $b \in \mathcal{F}$. This is the conditional expectation that we advocated in section 6.1.2.

We can reconstruct the field algebra $\mathcal{F}$ from the charge-neutral subalgebra algebra $\mathcal{A}_q$ by adding the intertwiners back. In the Abelian case, the intertwiners $\hat{U}_q = \sum_q |q \pm 1\rangle \langle q|$ are unitaries of the dual group. They create or annihilate charges. Enlarging the algebra of charge-neutral operators by added to it $\hat{U}_q$ and taking the closure generates the full algebra of charged operators. In the non-Abelian case, $\hat{G}$ the dual group is mathematically not a group. However, we can still enlarge the charge-neutral algebra by adding the intertwiners to obtain the full algebra $\mathcal{F}$. In representation theory language, enlarging the algebra $\mathcal{A}$ by including intertwiners corresponds to the crossed product of $\mathcal{A}$ by the dual group $\hat{G}$: $\mathcal{A} \rtimes \hat{G}$, see appendix 6.1.23 for the definition of the dual group and crossed product.

In the remainder of this section, we provide several examples of quantum systems with symmetry and highlight the role of the intertwiners. The first four examples have an Abelian symmetry group and the last two have a non-Abelian symmetry. We postpone the discussion of intertwiners for local algebras until the next section.

6.1.7 Example 1: Qudit

Consider the Hilbert space of a qubit $\mathcal{H}_2$ and the algebra of 2×2 complex matrices. Take the symmetry transformation to be the group $\mathbb{Z}_2$ generated by the transformations: $\alpha_1(a) = a$ and $\alpha_g(a) = \sigma_z a \sigma_z$.[14] Here, $U_g = \sigma_z = (-1)^Q$ where $Q = \frac{1}{2}(1 - \sigma_z)$ is the charge operator. The algebra of charge neutral operators $\mathcal{D}_2$ is the algebra of matrices diagonal in the σ_z basis. The Hilbert space splits into two sectors $\mathcal{H}_0 \oplus \mathcal{H}_1$ with $P_q = |q\rangle \langle q|$ projecting to the sector of charge q. The intertwiner $V = |0\rangle \langle 1|$ solves the equation (6.47) and relates the two charged sectors. The dual group is the $\mathbb{Z}_2$ that is generated by $\sigma_x = V + V^\dagger$. If we

[14] ↑We use the notation $\mathbb{Z}_n = \mathbb{Z}/n\mathbb{Z}$.

add the intertwiner (or the generator of the dual group σ_x) to the invariant algebra $\mathcal{D}_2$ we obtain the full algebra of the qubit.

For a qudit the Hilbert space is spanned by $|k\rangle$ with $k = 1, \cdots d$, and we take the symmetry group to be $\mathbb{Z}_d$ generated by the diagonal matrices $\sum_k \mathrm{e}^{2\pi \mathrm{i} g k/d} |k\rangle \langle k|$. The invariant sub-algebras are one-dimensional $\mathcal{A}_k = a |k\rangle \langle k|$ and the projections to the superselection sectors are $P_k = |k\rangle \langle k|$. Each $|k'\rangle \langle k|$ is a unitary intertwiner from $\mathcal{H}_k$ to $\mathcal{H}_{k'}$. The dual group is the Fourier transform $\mathbb{Z}_d$ generated by the unitary $\sum_k |(k+1) \mod d\rangle \langle k|$.

The generalization to infinite dimension is immediate. Take the Hilbert space of a free particle on a circle and the rotation group around the circle: $G = U(1)$. The Hilbert space splits into one dimensional irreducible representations of the rotation group $\mathcal{H} = \oplus_{k\in\mathbb{Z}} |k\rangle \langle k|$ where $|k\rangle$ is a momentum eigenstate. The invariant algebras are $\mathcal{A}_k = a |k\rangle \langle k|$, and the intertwiners are $|k'\rangle \langle k|$. The dual group is $\mathbb{Z}$ generated by the momentum addition/subtraction operator $\sum_k |k \pm 1\rangle \langle k|$. Adding the intertwiners to the invariant algebra gives all operators in the Hilbert space of free quantum particle on a circle.

6.1.8 Example 2: Non-relativistic quantum fields

Consider a non-relativistic bosonic or fermionic field on a circle and assume that the total number of particles is conserved. The particle number operator is $N = \int dx\, a^\dagger(x) a(x)$ and the symmetry transformations are $\mathcal{E}^{\mathrm{i}\alpha N}$. The Fock space is a direct sum of sectors with fixed particle number n: $\mathcal{H} = \oplus_{n\in\mathbb{N}} \mathcal{H}_n$ with vectors in each $\mathcal{H}_n$ represented by totally symmetric (anti-symmetric) wave-functions of n-variable: $\psi^{(n)}_\pm(x_1, \cdots x_n)$. The intertwiners that take us in between sectors are the creation/annihilation operators $a^\dagger_\pm(f)/a_\pm(f)$ that map $\mathcal{H}_n$ to $\mathcal{H}_{n+1}$ and back according to

$$
\begin{aligned}
(a^\dagger_\pm(f)\psi^{(n)})(x_1, \cdots, x_{n+1}) &= \frac{1}{\sqrt{n+1}} \sum_{k=1}^{n+1} (\pm 1)^{k-1} f(x_k) \psi^{(n)}(x_1, \cdots, x_{k-1}, x_{k+1}, \cdots, x_{n+1}) \\
(a_\pm(f)\psi^{(n+1)})(x_1, \cdots, x_n) &= \sqrt{n+1} \int dy \overline{f(y)} \psi^{(n+1)}(y, x_1, \cdots, x_n)
\end{aligned}
\tag{6.56}
$$

and f is a bounded complex function on the circle [59]. There are many intertwiners corresponding to different functions f, however adding one of them to the invariant algebra

suffices to generate the full algebra. We choose $\int dx|f(x)|^2 = 1$ so that the intertwiner is an isometry: $(a(f)a^\dagger(f)\psi^{(n)}) = (a^\dagger(f)a(f)\psi^{(n)}) = \psi^{(n)}$. The full algebra $\mathcal{F}$ is generated by operators $a_\pm(f)$ and $a_\pm^\dagger(f)$ satisfying

$$\begin{aligned} &[a_\pm(f), a_\pm(g)]_\pm = 0, \qquad [a_\pm(f), a_\pm^\dagger(g)]_\pm = \langle f, g\rangle\, \mathbb{I} \\ &[a,b]_- \equiv ab + ba, \qquad [a,b]_+ = ab - ba, \qquad \langle f, g\rangle = \int dx \overline{f(x)} g(x)\ . \end{aligned} \tag{6.57}$$

The dual group is generated by the field operator $\Phi(f) = a(f) + a^\dagger(f)$.

6.1.9 Example 3: Free relativistic fermions

In a general relativistic theory particle number is not conserved. However, in the case of free fermions the transformation $(-1)^Q$ with $Q = \int \mathrm{j}^0(x)$ remains a symmetry, where $\mathrm{j}^0(x) =: \Psi^\dagger(x)\Psi(x) :$ is the charge density operator. The full algebra $\mathcal{F}$ is generated by $\Psi(f) = \int d^2x f(x)\,\Psi(x)$ where f is a function of spacetime that solves the classical equations of motion [165]. The Hilbert space splits into two sectors $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ that correspond to the even and odd number of fermions. The invariant algebra $\mathcal{A}$ is generated by all the operators with an even number of fermions, e.g. $X = \Psi(y)\Psi(z)$ or $Y = \Psi(y)\Psi(z)^\dagger$.[15] The operator $\Psi(f)$ adds a unit of charge and intertwines the two sectors. The unitary $\hat{U}(f) = \Psi(f) + \Psi^\dagger(f)$ with $\int dx\,|f(x)|^2 = 1$ generates the $\mathbb{Z}_2$ dual group: $(1, \hat{U}(f))$. It has the following properties:

$$\begin{aligned} \hat{U}(f)X &= X\hat{U}(f) - f(z)\Psi(y) + f(y)\Psi(z) \\ \hat{U}(f)Y &= Y\hat{U}(f) - f(z)\Psi(y) + f(y)\Psi^\dagger(z)\ . \end{aligned} \tag{6.58}$$

Each choice of f leads to a particular choice of $\mathbb{Z}_2$. If we add any $\hat{U}(f)$ to the algebra of invariant operators all other charged operators $\hat{U}(g)$ are created by closing the algebra, because $\hat{U}(f)^\dagger\hat{U}(g)$ is charge-neutral. Representations with different values of f are unitarily equivalent by the inner automorphism $\hat{U}(f)^\dagger\hat{U}(g)$.

[15] ↑The commutators are $[Q, X] = -2X$ and $[Q, Y] = 0$.

The maps $\rho_f(a) \equiv \hat{U}(f) a \hat{U}(f)^\dagger$ are outer automorphisms of the invariant algebra $a \in \mathcal{A}$:

$$\begin{aligned} &\rho_f(a) \in \mathcal{A} \\ &\rho_f(a_1 a_2) = \rho_f(a_1)\rho_f(a_2) \ . \end{aligned} \tag{6.59}$$

For instance, for the total charge we have

$$\rho_f(Q) = Q + \Psi(f)\Psi^\dagger(f) - \Psi^\dagger(f)\Psi(f) \ . \tag{6.60}$$

The operator $\hat{U}(f)$ has charge one:

$$\rho_f((-1)^Q) = -(-1)^Q \tag{6.61}$$

which implies that an average over the dual group kills the symmetry transformation

$$(-1)^Q + \rho_f((-1)^Q) = 0 \ . \tag{6.62}$$

6.1.10 Example 4: $U(1)$ current algebra

As the next example, consider the algebra of a free compact relativistic boson in two dimensions on a circle. The shift of the scalar field $\phi \to \phi + a$ is a $U(1)$ global symmetry. In the radial quantization frame, we consider the algebra of $W(u) = \mathrm{e}^{\mathrm{i}J(u)}$ with $J(u) = \int \frac{dz}{2\pi \mathrm{i}} J(z)u(z)$ with $u(z)$ a smooth function on the circle. It is generated by the $U(1)$-invariant current $J(z) = (\partial\phi)(z) = \sum_{n\in\mathbb{Z}} z^{-n-1}\mathrm{j}_n$. The scalar field expanded in terms of j_n modes is

$$\phi(z,\bar{z}) = \phi_0 - \mathrm{i}(\mathrm{j}_0 \ln z + \bar{\mathrm{j}}_0 \ln \bar{z}) + \mathrm{i} \sum_{0\neq n\in\mathbb{Z}} \frac{1}{n} \left(\mathrm{j}_n z^{-n} + \bar{\mathrm{j}}_n \bar{z}^{-n} \right) \ . \tag{6.63}$$

The operator ϕ_0 and j_0 are canonical conjugates of each other: $[\phi_0, \mathrm{j}_0] = \mathrm{i}$. The $U(1)$ symmetry group is generated by $U_a = \mathrm{e}^{\mathrm{i}a\mathrm{j}_0}$. The vertex operator $V_k(z,\bar{z}) =: \mathrm{e}^{\mathrm{i}k\phi(z,\bar{z})} :$ acting on the vacuum creates eigenstates of the conjugate momenta $\mathrm{j}_0 |k\rangle = k |k\rangle$ with

$|k\rangle = V_k(0)\,|\Omega\rangle$ and $\langle k|k'\rangle = \delta_{kk'}$. In fact, the vertex operator satisfies $[\mathrm{j}_0, V_k] = kV_k$ which implies that it is a unitary intertwiner.

We can consider $\alpha(z)$ functions on the circle and the unitary vertex operator $V(\alpha) =: \mathrm{e}^{\mathrm{i}\phi(\alpha)} :$ with $\phi(\alpha) = \int dz\alpha(z)\phi(z)$. Under the transformation $\phi \to \phi + 2\pi$ the vertex operator should be invariant therefore the charge $q_\alpha = \int dz\alpha(z)$ is quantized. When $q_\alpha = 0$ the vertex operator $V(\alpha)$ is charge-neutral but when $q_\alpha = \int dz\,\alpha(z) \neq 0$ it is an intertwiner of charge q. The dual group is $\mathbb{Z}$ and is generated by charged vertex operators $V(k\alpha)$ for $k \in \mathbb{Z}$. As in the case of fermions, adding one intertwiner of unit charge adds all of them because $V(\alpha)V^\dagger(\beta)$ with $q_\alpha = q_\beta$ is a charge-neutral operator. The action of the dual group on the invariant algebra at point is

$$\rho_\alpha(J(z)) \equiv V(\alpha)J(z)V^\dagger(\alpha) = J(z) + \alpha(z) \tag{6.64}$$

which does not leave the neutral operators invariant. Instead it shifts it by an element of the center of the algebra [175]. The action of the dual group on the symmetry generator is

$$\rho_\alpha(U_a) = V(\alpha)U_aV^\dagger(\alpha) = \mathrm{e}^{\mathrm{i}aq_\alpha}U_a\ . \tag{6.65}$$

The dual group is not compact, but we can formally define an average over the charged sector as a distribution

$$\sum_{k=-\infty}^{\infty} \rho_{k\alpha}(U_a) = \frac{2\pi}{|q_\alpha|}\delta(a)\ . \tag{6.66}$$

6.1.11 Example 5: Permutation group

The simplest example of a non-Abelian group is the permutation group S_3. Consider three qubits and the symmetry group S_3 that swaps the qubits. The elements of the group are the identity, the two-cycles and the three cycles. The two-cycles are represented by $U_{(12)} = S_{12}$, $U_{(13)} = S_{13}$ and $U_{(23)} = S_{23}$ where S_{ij} is the swap operator of site i and j: $S_{(12)} = \sum_{ab} |ab\rangle\langle ba|$. The three-cycles are $U_{(123)} = \sum_{abc} |abc\rangle\langle bca|$ and $U_{(132)} = \sum_{abc} |abc\rangle\langle cab|$. The invariant algebra $\mathcal{A}$ is the set of 4×4 dimensional matrices $|\alpha_{\mathrm{i}}\rangle\langle\alpha_{\mathrm{j}}|$ where $|\alpha_{\mathrm{i}}\rangle$ are

invariant vectors of S_3: $|\alpha_0\rangle = |000\rangle$, $|\alpha_1\rangle = |111\rangle$, $|\alpha_2\rangle = \frac{1}{3}(|001\rangle + |010\rangle + |100\rangle)$ and $|\alpha_3\rangle = \frac{1}{3}(|011\rangle + |101\rangle + |110\rangle)$.

The Hilbert space has two sectors $\mathcal{H} = (\mathcal{K}_1 \otimes \mathcal{H}_1) \oplus (\mathcal{K}_2 \otimes \mathcal{H}_2)$. The vacuum representation $\mathcal{K}_1$ is the trivial one-dimensional representation, and $\mathcal{H}_1 = \overline{\mathcal{A}\,|000\rangle}$ is the Hilbert space of states invariant under S_3 that is four dimensional and spanned by $|\alpha_{\rm i}\rangle$. The Hilbert space $\mathcal{K}_2$ is the two-dimensional irreducible representation of S_3 corresponding to the Young tableaux $\begin{array}{|c|c|}\hline \ \ & \ \ \\ \hline \ \ \\ \cline{1-1}\end{array}$. The vectors $|v_0\rangle = |100\rangle - |001\rangle$, and $|v_1\rangle = 2\,|010\rangle - (|100\rangle + |001\rangle)$ provide a basis for this representation. It is straightforward to see that the action of S_3 leaves the two-dimensional subspace spanned by these vectors invariant. Acting with the invariant algebra, in particular $|\alpha_3\rangle\langle\alpha_2|$ on these vectors generates two perpendicular vectors $|v_2\rangle = |011\rangle - |110\rangle$ and $|v_3\rangle = 2\,|101\rangle - (|011\rangle + |110\rangle)$, and the new two-dimensional subspace is also preserved under the action of S_3. The sector $\mathcal{K}_2 \otimes \mathcal{H}_2$ is the four dimensional subspace $\overline{\mathcal{A}\,|v_1\rangle}$. There is no totally anti-symmetric representation for qubits.

6.1.12 Example 6: The $O(N)$ model

Consider a real vector field $\Phi(f)$ with N components of form $\varphi^{({\rm j})}(f_{\rm j})$ and f a collection of functions $f_1, \cdots, f_N$. The algebra $\mathcal{F}$ is generated by the Weyl operators $W(f) = {\rm e}^{{\rm i}\sum_{\rm j}\varphi^{({\rm j})}(f_{\rm j})}$. The symmetry group $O(N)$ acts on the vector fields which is equivalent to rotating $f_{\rm i}$: $U_g W(f) U_g^\dagger = W(g.f)$ and $(g.f)_{\rm i} = \sum_{\rm j} g_{\rm ij} f_{\rm j}$. The invariant algebra $\mathcal{A}$ is the algebra of $O(N)$ singlets generated by operators like $\Phi(f)\cdot\Phi(f) = \sum_{\rm i}\varphi^{({\rm i})}(f_{\rm i})\varphi^{({\rm i})}(f_{\rm i})$. The vacuum sector is $\overline{\mathcal{A}\,|\Omega\rangle}$. The other sectors correspond to other irreducible representations of $O(N)$. Take the operator $\Phi(T) = \sum_{{\rm i}_1,\cdots,{\rm i}_k=1}^N T^{{\rm i}_1,\cdots,{\rm i}_k}\varphi^{({\rm i}_1)}(f_1)\cdots\varphi^{({\rm i}_k)}(f_k)$ where the tensor T has symmetries under the permutation of indices that is characterized by a young tableaux $\lambda = (\lambda_1, \cdots, \lambda_s)$ with the total number of boxes $k = \sum_{{\rm j}=1}^s \lambda_{\rm i}$. Such operators acting on the vacuum sector take us to the charged sector with the irreducible representation characterized by the Young tableaux λ and dimension $\dim(\lambda)$. One can find an orthonormal basis of such operators $\Phi(T_{\rm j})$ with ${\rm j} = 1, \cdots, \dim(\lambda)$ [175].

6.1.13 Bi-local intertwiners

Consider a multi-partite quantum system on a lattice with a symmetry $U_g = \mathrm{e}^{igQ}$ and local algebras $\mathcal{F}_A$ associated with each region A (collection of sites on a lattice or a region of space).[16] We say a symmetry of the global algebra $\mathcal{F} = \mathcal{F}_{AA'}$ is internal if it preserves local algebras:

$$U_g a U_g^\dagger \in \mathcal{F}_A \qquad \forall a \in \mathcal{F}_A \ . \tag{6.67}$$

There is a unitary group $\tau_g = \mathrm{e}^{igQ_A}$ localized in A that generates the group action in (6.67) for operators in $\mathcal{F}_A$; see figure 6.5. In section 6.1.2 we called the operator τ_g the twist and its corresponding group the twist group G_τ. When the group is Abelian τ_g is charge-neutral $U_h \tau_g U_h^\dagger = U_g$ and provides a center for the algebra of neutral operators. When the group G is non-Abelian the operator P_r in (6.54) is in the center of the algebra: $Z = \oplus_r \lambda_r P_r$.

Locality implies that $\mathcal{F}_A$ commutes with the algebra of the complementary region $\mathcal{F}_{A'}$. Define the commutant of algebra $\mathcal{F}_A$ to be $\mathcal{F}'_A$: the set of all operators in the global algebra $\mathcal{F}_{AA'}$ that commute with $\mathcal{F}_A$. From locality it follows that $\mathcal{F}_{A'} \subset \mathcal{F}'_A$. We say the region A has the *duality* property if $\mathcal{F}'_A = \mathcal{F}_{A'}$. The full algebra of all charged operators satisfy the duality property, however the algebra of charge-neutral operators $\mathcal{A}$ violate it. For instance, on a lattice the total charge is $Q = Q_A + Q_{A'}$ and $\mathcal{H}_{AA'} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$ the action of the symmetry transformation on $\mathcal{A}_A$ is captured by the twist operator $\tau_g = \mathrm{e}^{igQ_A}$. The local algebra $\mathcal{A}_A$ has a non-trivial center $Z_A = \oplus_r \lambda_r \mathbb{I}_r$ with r irreducible representations of τ_g and λ_r complex numbers. The duality relation for charge-neutral algebras is: $\mathcal{A}'_A = Z_A \otimes \mathcal{A}_{A'}$. Note that here the commutant $\mathcal{A}'_A$ is defined to be the algebra of operators in $\mathcal{A}_{AA'}$ that commute with $\mathcal{A}_A$. On a lattice, the failure of duality is due to a non-trivial center for the algebra of charge-neutral operators. However, in QFT the duality property can fail even though the local charge-neutral algebra has a trivial center. The reason is that the operator P_r defined in (6.54) is not part of the local algebra of region A because it acts singularly on the boundary of A.

[16]↑For the sake of the argument we have assumed G is a Lie group. However, the discussion applies to any group G.

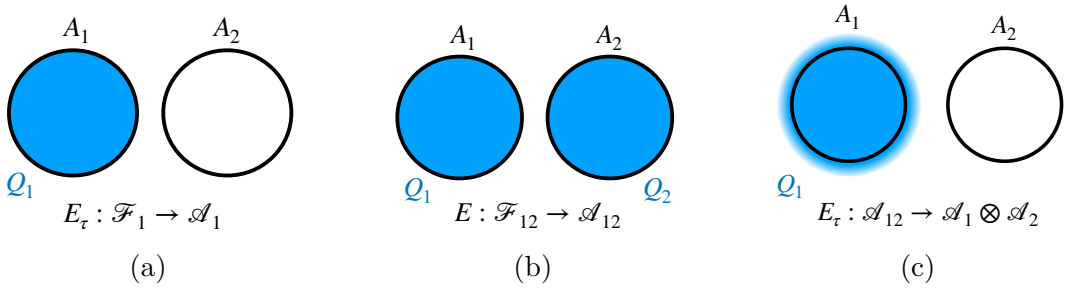


Figure 6.5. Consider the operator $\mathcal{E}^{\mathrm{i}g\int_{x\in B} c(x)\mathrm{j}(x)}$ where $\mathrm{j}(x)$ is the charge density and the region B is the blue region. (a) On a lattice we pick $c(x) = 1$ that is the twist operator $\tau_g = \mathrm{e}^{\mathrm{i}gQ_1}$. It generates the action of the symmetry group on the local algebra of A_1. Averaging over τ_g is a conditional expectation that projects $\mathcal{F}_1$ to $\mathcal{A}_1$. (b) The action of the symmetry on the region A_{12} is given by $\mathcal{E}^{\mathrm{i}g(Q_1+Q_2)}$. Averaging over this unitary projects from $\mathcal{F}_{12}$ to $\mathcal{A}_{12}$ c) In a QFT choosing $c(x) = 1$ in A_1 and $c(x) = 0$ outside of A_1 leads to an operator that has a violent behavior at the boundary of A_1 due to the discontinuity in $c(x)$. If there is a gap between A_1 and A_2 we can choose a $c(x) = 1$ inside A_1 and make it smoothly fall out to zero without entering region A_2. This is the analog of the twist operator in a QFT. Averaging over this twist projects from $\mathcal{A}_{12}$ down to $\mathcal{A}_1 \otimes \mathcal{A}_2$.

More generally, consider the region $A_{12} = A_1 \cup A_2$ with two disconnected pieces A_1 and A_2. On a lattice the algebra of all charged particles is *additive* that is to say $\mathcal{F}_{12} = \mathcal{F}_1 \otimes \mathcal{F}_2$. In QFT, the additivity property holds when A_1 and A_2 are not touching.[17] Both on a lattice or in QFT when we restrict to the subalgebra of locally charge-neutral operators additivity fails: $\mathcal{A}_1 \otimes \mathcal{A}_2 \neq \mathcal{A}_{12}$. Of course, $\mathcal{A}_1 \otimes \mathcal{A}_2$ is a subalgebra of $\mathcal{A}_{12}$ but there exist operators in $\mathcal{A}_{12}$, namely the bi-local intertwiners, that are not generated in $\mathcal{A}_1 \otimes \mathcal{A}_2$. The bi-local intertwiner adds a charge q to region A_1 and the opposite charge $-q$ to the region A_2 so that the total charge $Q_1 + Q_2$ is conserved. The action of the symmetry group on $\mathcal{F}_1$ can be captured by a local transformation $\mathcal{E}^{\mathrm{i}gQ_1}$ on a lattice. In QFT, the operator $\mathcal{E}^{\mathrm{i}gQ_1}$ has a singular behavior at the boundary of A_1. However, as long as there is a gap between region A_1 and A_2 there is a unitary transformation τ_g that matches $\mathcal{E}^{\mathrm{i}gQ_1}$ on A_1 and has a smooth

[17]↑We have assumed that QFT has the split property [169].

tail that leaks outside of A_1 but does not enter A_2; see figure 6.5. In analogy with the lattice systems, we call this operator the twist and the symmetry it generates the twist group:

$$\begin{aligned} b_1 \to \mathrm{e}^{\mathrm{i}gQ} b_1 \mathrm{e}^{-\mathrm{i}gQ} = \tau_g^\dagger b_1 \tau_g \qquad \forall b_1 \in \mathcal{F}_1 \\ [\tau_g, a_2] = 0, \qquad \forall a_2 \in \mathcal{A}_2 \ . \end{aligned} \tag{6.68}$$

In QFT, the local neutral-algebra has a trivial center. When G is compact one has the conditional expectation $\mathcal{E}_\tau : \mathcal{A}_{12} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ that is an average over the twist group:

$$E_\tau(b) = \frac{1}{|G|} \int_{g \in G} dg \ \tau_g^\dagger b \tau_g \ . \tag{6.69}$$

By construction, the conditional expectation above sets any operators charged under Q_1 including bi-local intertwiners to zero. The conditional expectation projects down to the invariant algebra. To go in the opposite direction, we need to enlarge the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ by adding the bi-local intertwiners to obtain $\mathcal{A}_{12}$. Enlarging an algebra $\mathcal{A}$ by the intertwiners of symmetry G is mathematically described by the crossed-product of the algebra with its dual group, $\mathcal{A}_{12} = (\mathcal{A}_1 \otimes \mathcal{A}_2) \rtimes \hat{G}$; see appendix 6.1.23 for details.

In QFT, there is no local Hilbert space $\mathcal{H}_1$, and we only have the global Hilbert space $\mathcal{H}$ and local algebras $\mathcal{A}_1$. In a QFT with charges, analogously, we have the global Hilbert space of type $\oplus_r \mathcal{K}_r \otimes \mathcal{H}_r$. The intertwiner $|r, \mathrm{i}\rangle \langle 0| \otimes 1$ takes us from the global vacuum to the global charged sector $|r, \mathrm{i}\rangle$ but it might not be localized in region A. We come back to this issue in section 6.1.20. Similar to the Abelian case where we added $|q\rangle \langle q+1|$ to get the unitary $\hat{U}_1$, we would like to extend the domain of $|r, \mathrm{i}\rangle \langle 0|$ to an operator that adds charge r to any state. The tensor product of two irreducible representations r and r' is a direct sum of irreducible representations with Clebsch-Gordan coefficients. A charged operator that is localized in A commutes with all $a' \in \mathcal{A}'$ and removes a charge r [176]–[178]:

$$\begin{aligned} V_{r,\mathrm{i}}(|r, \mathrm{i}\rangle \otimes a \, |\Omega\rangle) = a \, |\Omega\rangle \\ V_{r,\mathrm{i}} a \, |\Omega\rangle = |r^*, \mathrm{i}\rangle \otimes a \, |\Omega\rangle \, , \end{aligned} \tag{6.70}$$

where r^* is the conjugate representation of r. The action of $V_{r,\mathrm{i}}$ on a vector $|r', \mathrm{j}\rangle \otimes |\Omega\rangle$ is decided by the Clebsch-Gordan coefficients in the tensor product of representations r and r'. The dual transformation maps the algebra of charge-neutral operators $\mathbb{I}_r \otimes a$ back to the vacuum sector

$$\rho_r(a) = \sum_{\mathrm{i}} V_{r,\mathrm{i}} a V_{r,\mathrm{i}}^\dagger \ . \tag{6.71}$$

The map $\rho_r(a)$ maps the charge-neutral operators $\mathcal{A}$ to itself and since it is the representation of the local algebra it respects the multiplication rule[18]

$$\rho_r(a_1 a_2) = \rho_r(a_1)\rho_r(a_2) \ . \tag{6.72}$$

The condition above together with (6.71) imply that $V_{r,\mathrm{i}}$ should satisfy the algebra

$$\begin{aligned} V_{r,\mathrm{i}}^\dagger V_{r,\mathrm{j}} &= \delta_{\mathrm{ij}} \\ \sum_{\mathrm{i}} V_{r,\mathrm{i}} V_{r,\mathrm{i}}^\dagger &= 1 \ . \end{aligned} \tag{6.73}$$

The algebra above is called the Cuntz algebra [179]. The Cuntz algebra has no finite dimensional representations; however, it is easy to build representations of the Cuntz algebra in infinite dimensions. For instance, take the Hilbert space of a particle on a circle and split it into two sectors defined by projections to the even and odd momenta $P_+ = \sum_k |2k\rangle \langle 2k|$ and $P_- = \sum_k |2k+1\rangle \langle 2k+1|$. The isometries $V_1 = \sum_k |2k+1\rangle \langle k|$ and $V_2 = \sum_k |2k\rangle \langle k|$ satisfy the Cuntz algebra with $\mathrm{i} = 1, 2$.

The particle number is not conserved in relativistic QFT. Acting with $V_{r,\mathrm{i}}^\dagger$ creates one charged particle but applying it again we can have several charged particles. There is a subalgebra of the Cuntz algebra that corresponds to a sector with one charged particle $\sum_{r,\mathrm{i}} a_{\mathrm{ij}} V_{r,\mathrm{i}} V_{r,\mathrm{j}}^\dagger$, where a_{ij} are invariant operators. These operators can be represented by a $d_r \times d_r$ matrix algebra.

[18] ↑Such a map is called an endormophism of the algebra.

The operators $V_{r,\mathrm{i}}$ satisfies the non-Abelian intertwiner equation

$$V_{r,\mathrm{i}}a = \rho_r(a)V_{r,\mathrm{i}}, \forall a \in \mathcal{A} \tag{6.74}$$

and $V^\dagger_{r,\mathrm{i}}$ acting on the vacuum sector creates charged states in representation r: $|r,\mathrm{i}\rangle = \sqrt{d_r}V^\dagger_{r,\mathrm{i}}|\Omega\rangle$. The factor $\sqrt{d_r}$ is needed to make sure $\langle r,\mathrm{i}|r,\mathrm{i}\rangle = 1$. There are also the states in the conjugate representation that are created by $|r^*,\mathrm{i}\rangle = V_{r,\mathrm{i}}|\Omega\rangle$.[19] The conjugate representation is

$$\rho_{r^*}(a) = \frac{1}{d_r}\sum_{\mathrm{i}} V^\dagger_{r,\mathrm{i}}aV_{r,\mathrm{i}} \ . \tag{6.75}$$

In a charged sector the expectation value of a charge neutral operator satisfies

$$\sum_{\mathrm{i}} \langle r,\mathrm{i}|a|r,\mathrm{i}\rangle = \langle\Omega|\rho_r(a)|\Omega\rangle \ . \tag{6.76}$$

If $\rho_r(a) = a$, one cannot distinguish charged sectors. However, if $\rho_r(a) \neq a$, this is no longer true. An example of this is the compact boson example:

$$\rho_\alpha(J(z)) = J(z) + \alpha(z) \ . \tag{6.77}$$

The group transformation $U_g = \oplus_r U^r_g$ acts on the intertwiner according to the equation

$$U^\dagger_g V_{r,\mathrm{i}} = D_r(g)_{\mathrm{ij}}V_{r,\mathrm{j}}U^\dagger_g \tag{6.78}$$

where $D_r(g)_{\mathrm{ij}}$ are the matrix elements of the representation matrix $D_r(g)$ with the orthogonality relations[20]

$$\frac{d_r}{|G|}\sum_g D_r(g)_{\mathrm{i}k}D_{r'}(g)^*_{\mathrm{j}l} = \delta_{rr'}\delta_{\mathrm{ij}}\delta_{kl} \ . \tag{6.79}$$

[19]↑ Note that in this case there is no need for a factor $\sqrt{d_r}$ to normalize the state.

[20]↑In the case of an Abelian group this is $U^\dagger_g V_q = \mathrm{e}^{-\frac{2\pi\mathrm{i}gq}{|G|}}V_qU_g$.

In QFT, in analogy with lattice systems, it is tempting to take the local algebra of A to be all charged operators $|r,\mathrm{i}\rangle\langle r',\mathrm{i}'|\otimes a$, however as we discussed above, in QFT $|r,\mathrm{i}\rangle\langle 0|\otimes 1$ is not localized in A, and the charge neutral algebra has no non-trivial center. Local charges on A are created by the intertwiners $V^\dagger_{r,\mathrm{i}}$, instead of $|r,\mathrm{i}\rangle\langle 0|\otimes 1$. Therefore, we define the local algebra of charge operators to be the algebra generated by charge neutral operators a and the isometries $V^\dagger_{r,\mathrm{i}}$. Consider charged operators $\sum_\mathrm{i} a_\mathrm{i} V_{r,\mathrm{i}}$. Bi-local intertwiners create/annihilate a charge in A_1 and create/annihilate the opposite charge in A_2 so that the net charge is preserved:

$$\mathcal{I}^{(r)}_{12} = \sum_\mathrm{i} (V^{(1)}_{r,\mathrm{i}})^\dagger V^{(2)}_{r,\mathrm{i}} \tag{6.80}$$

with $V^{(1)}_{r,\mathrm{i}}$ and $V^{(2)}_{r,\mathrm{i}}$ supported on A_1 and A_2, respectively. This is a unitary map in the global algebra that is charge-neural. However, from the point of view of algebra $\mathcal{A}_1$ it is an intertwiner.

In the remainder of this section, we provide several examples of bi-local intertwiners in finite quantum systems and QFTs.

6.1.14 Example 1: Qubits

Consider two qubits in $\mathcal{H}_1\otimes\mathcal{H}_2$ and the symmetry group $\mathbb{Z}_2$ corresponding to the action of $(-1)^Q = \sigma_z\otimes\sigma_z$ where the total charge $Q = Q_1\otimes\mathbb{I} + \mathbb{I}\otimes Q_2$ and $Q_\mathrm{i} = \frac{1}{2}(\mathbb{I}-\sigma_z)$ counts the number of excitations "$|1\rangle$". The action of the symmetry group on the local algebra $\mathcal{F}_1$ is captured by the twist group $\tau_1 = (-1)^{Q_1}$ that is localized in A_1. The algebra of global charge-neutral operators $\mathcal{A}_{12}$ is the set of all operators that commute with Q. The charge neutral sub-algebras $\mathcal{A}_1\otimes\mathbb{I}_2$ and $\mathbb{I}_1\otimes\mathcal{A}_2$ commute with Q, however, $\mathcal{A}_{12}$ includes more operators. In particular, the operator that creates a charge on site one and annihilates it on site two commutes with Q:

$$[Q, \sigma^+\otimes\sigma^-] = 0, \qquad \sigma^\pm = \frac{1}{2}(\sigma^{(x)} \mp \mathrm{i}\sigma^y)\ . \tag{6.81}$$

The algebra $\mathcal{A}_{12}$ also includes operators that increase Q by two units, $|00\rangle\langle 11| \in \mathcal{A}_{12}$ and its †. The subspace $\mathcal{H}_1' = \mathcal{A}_{12}|00\rangle$ that is spanned by $|00\rangle$ and $|11\rangle$ is the zero charge sector and the charged sector is $\mathcal{H}_2' = \mathcal{A}_{12}|01\rangle$ which is spanned by $|01\rangle$ and $|10\rangle$.

The subalgebra of $\mathcal{A}_{12}$ invariant under the twist group $(-1)^{Q_1}$ is $\mathcal{A}_1 \otimes \mathcal{A}_2$. Each sector $\mathcal{H}_1'$ and $\mathcal{H}_2'$ further splits into two sectors depending on the eigenvalue of $\sigma_z^{(1)}$. The operator $|11\rangle\langle 00|$ is an internal intertwiner for the twist group that is a unitary in $\mathcal{H}_1'$, and $\sigma^- \otimes \sigma^+ = |10\rangle\langle 01|$ is an internal intertwiner for the twist group in $\mathcal{H}_2'$. Local intertwiners create a pair of charge/anti-charge excitations. The group average over the twist is a conditional expectation $\mathcal{E}_\tau : \mathcal{A}_{12} \to \mathcal{A}_1 \otimes \mathcal{A}_2$ that washes out local intertwiners:

$$E_\tau(b) = \frac{1}{2}\left(b + \sigma_z^{(1)} b \sigma_z^{(1)}\right). \tag{6.82}$$

This can be easily extended to n qubits with the global symmetry $\mathbb{Z}_2$ that is measured by the total charge $(-1)^Q = \otimes_{\mathrm{i}=1}^n \sigma_z^{(\mathrm{i})}$ and the local charge associated with the region A that is the first m qubits $\tau = (-1)^{Q'} = \otimes_{\mathrm{i}=1}^m \sigma_z^{(\mathrm{i})}$. The global Hilbert space splits into two sectors $\mathcal{H} = \mathcal{H}_+ \oplus \mathcal{H}_-$ where $\mathcal{H}_\pm$ is spanned by all $|s_1, \cdots s_n\rangle$ with $s_1 s_2 \cdots s_n = \pm 1$. The twist symmetry $\tau = (-1)^{Q'}$ further splits each sector into two: $s_1 s_2 \cdots s_m = \pm 1$. The operator $|s_1 \cdots s_m, t_{m+1} \cdots t_n\rangle\langle s_1' \cdots s_m', t_{m+1}' \cdots t_n'|$ with $s_1 \cdots s_m = -1 = -s_1' \cdots s_m'$ and $t_{m+1} \cdots t_n = \pm 1 = \mp(t_{m+1}' \cdots t_n')$ is an example of a local intertwiner.

As an example of a region with two non-overlapping pieces consider the local algebras $\mathcal{A}_{12}$ and $\mathcal{A}_1 \otimes \mathcal{A}_2$ where there are a total of three qubits. We first check the duality property. Once we include the centers of local algebra the duality property holds: $\mathcal{A}_{12}' = Z_{12} \otimes \mathcal{A}_3$ and $\mathcal{A}_3' = Z_3 \otimes \mathcal{A}_{12}$. Note that the operator $\tau_{13} = (-1)^{Q_1+Q_3} = \sigma_z^{(1)} \otimes \sigma_z^{(3)}$ is in $\mathcal{A}_3'$ but not in $\mathcal{A}_{12}$. In fact, if we only add τ_{13} to $\mathcal{A}_{12}$ we generate the full $Z_3 \otimes \mathcal{A}_{12}$. The operator τ_{13} is a twist operator similar to the ones in QFT because it acts on $\mathcal{A}_1$ like $(-1)^{Q_1}$, it is supported outside of A_1 but its support does not enter A_2. We learn that another way to express the duality relation for charge-neutral algebras is by enlarging $\mathcal{A}_{12}$ with the twist τ_{13}. In mathematical language, we write the crossed product $\mathcal{A}_3' = \mathcal{A}_{12} \rtimes G_{13}$ where G_{13} is the symmetry group generated by τ_{13}. We could replace τ_{13} with τ_{23} or τ_3 and the result

remains the same. However, for the opposite region we have to enlarge $\mathcal{A}_3$ by τ_{12} to obtain $\mathcal{A}'_{12}$: $\mathcal{A}'_{12} = \mathcal{A}_3 \rtimes G_{12}$. If we have four qubits, then the equations become more symmetric:

$$\begin{aligned} \mathcal{A}'_{12} &= \mathcal{A}_{34} \rtimes G_{12} \\ \mathcal{A}'_{34} &= \mathcal{A}_{12} \rtimes G_{34} \ . \end{aligned} \tag{6.83}$$

A much simpler way to write the duality equation for charge-neutral algebras is $\mathcal{A}'_A = \mathcal{A}_{A'} \rtimes G$ where $G = (-1)^Q$ is the generator of the symmetry in the global algebra.

The interplay between duality and additivity of local algebras plays an important role in the study of quantum systems with symmetries [166]. The action of a symmetry on a local region A_i is captured by the twist group G_{ik} generated by τ_{ik} with A_k some region outside of A_i. On a lattice, one can take the twist to be τ_i. Denote the local intertwiner that creates a charge on A_i and annihilates it in A_j by $\mathcal{I}_\mathrm{ij}$. It generates a group dual to the twist group G_i; or G_{ik} for $k \neq \mathrm{j}$ in QFT. When the algebra is Abelian this duality transformation is a Fourier transform and indeed we find $[\mathcal{I}_\mathrm{ij}, \tau_\mathrm{i}] \neq 0$. In the qubit example, we have $[\mathcal{I}_\mathrm{ij}, \tau_\mathrm{i}] = 2\mathcal{I}_\mathrm{ij}$.

6.1.15 Example 2: Free relativistic fermions

Consider free fermions in $(1+1)$-dimensions. As we discussed in section 6.1.9 the symmetry of the global algebra is $(-1)^N$ where N is the total number of fermions, and the invariant global algebra is all operators with an even number of fermions. The local algebra of a region is generated by $\Psi(f_A)$ with f_A any bounded complex function supported only a region A.[21] The symmetry acts on the local algebra as $\tau = (-1)^{N_A}$ where N_A is the total number of fermions in a region A. This operator is discontinuous at the boundary of A and we can smooth it outside of A. The Hilbert space splits into four sectors corresponding to two charges $(N \mod 2) = 0, 1$ and $(N_A \mod 2) = 0, 1$. The operator $\Psi(f_{A'})\Psi^\dagger(g_A)$ creates a pair of charge/anti-charge particles in A and A'. It is a bi-local intertwiner for $\mathcal{A}_A$.

If we take two regions of space A_1 and A_2 that are non-overlapping and non-touching the complementary region also has two disconnected pieces. This is analogous to the case

[21] ↑We thank Edward Witten for pointing this out to us.

of four qubits we discussed above.[22] In addition to $\mathcal{A}_1 \otimes \mathcal{A}_2$ the algebra of $\mathcal{A}_{12}$ includes the intertwiners from region A_1 to A_2 that are $\Psi^\dagger(f_1)\Psi(f_2)$ with f_i supported in A_i. The twist operator $\tau = (-1)^{Q_1}$ needs to be smoothed out outside of A_1 without leaking inside A_2. We call the smooth twist operator τ_{13} because it is supported on A_{13} and acts like $(-1)^{N_1}$ on $\mathcal{A}_1$. The group average over τ_{13} is a conditional expectation $\mathcal{E}_\tau : \mathcal{A}_{12} \to \mathcal{A}_1 \otimes \mathcal{A}_2$:

$$E_\tau(b) = \frac{1}{2}\left(b + \tau_{13}^{-1} b \tau_{13}\right). \tag{6.84}$$

It kills the local intertwiners: $\mathcal{E}(\Psi(f_1)\Psi^\dagger(f_2)) = 0$. In QFT, there are no local density matrices, instead the local state is a restriction of the global pure state to the local algebra:

$$\omega(a_1 \otimes a_2) = \langle\Omega|a_1 \otimes a_2|\Omega\rangle\,. \tag{6.85}$$

The invariant state is

$$(E_\tau^*(\omega))(a_1 \otimes a_2) = \frac{1}{2}\left(\omega(a_1 \otimes a_2) + \omega(\tau_{13}^{-1}(a_1 \otimes a_2)\tau_{13})\right) \tag{6.86}$$

which can be thought of as the restriction of the global density matrix

$$\frac{1}{2}\left(|\Omega\rangle\langle\Omega| + \tau_{13}|\Omega\rangle\langle\Omega|\tau_{13}^{-1}\right) \tag{6.87}$$

to the local algebra $\mathcal{A}_{12}$.

6.1.16 Example 3: $U(1)$ current algebra

In the free $(1+1)$-dimensional compact boson model, the symmetry group is $\mathcal{E}^{\mathrm{i}\alpha \mathrm{j}_0}$ and the Hilbert space has many sectors $|\alpha\rangle$ with the vertex operators $V(\alpha) =: \mathrm{e}^{\mathrm{i}\Phi(\alpha)} :$ with α some function on the circle intertwining them. If we consider the local algebra generated by $\mathcal{E}^{\mathrm{i}J(f_A)}$ with f_A some smooth function supported only on A then the total charge on A is $\mathrm{j}_0(A) = \frac{1}{2\pi}\oint_A J(z)$ where A is some angle on the unit circle in radial quantization. The

[22] ↑In higher than $(1+1)$-dimensions the complement of A_{12} is connected and the three qubit example is a better analogy.

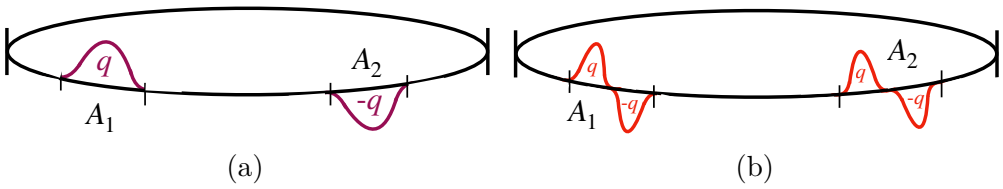


Figure 6.6. (a) The bi-local intertwiners in $A_1 \cup A_2$ conserve the total charge, $\int_{A_1 \cup A_2} dz\, \alpha(z) = q = 0$. (b) The subalgebra $\mathcal{N}_1 \subset \mathcal{A}_1$ does not have any operators that create and annihilate charges inside A_1 (the red excitations). Such an algebra is generated by $J(f)$ with functions localized in A_1.

bi-local intertwiners between two non-touching, non overlapping regions A_1 and A_2 are $V(\alpha)$ with $\int_{A_1} dz\, \alpha(z) = q_A$ and $\int_{A_1 \cup A_2} dz\, \alpha(z) = q = 0$ so that they do not change the global sector; see figure 6.6.

6.1.17 Intertwiners and Modular Theory

In a Poincare-invariant QFT in $(d+1)$-dimensions, the global algebra of spacetime $\mathcal{F}$ is generated by the bounded functions of the field operator $\Phi(f)$ with $\Phi(f) = \int d^{d+1}x f(x)\Phi(x)$ and $f(x)$ a solution to the classical equations of motion that respects the boundary conditions at infinity[23] [165], [169], [178]. This algebra is represented irreducibly on a global Hilbert space $\mathcal{H}$. The local algebra $\mathcal{F}_A \subset \mathcal{F}$ is the subalgebra generated by $\Phi(f)$ where f is only supported in A. The local algebra of QFT does not have an irreducible representation and there is no local Hilbert space [58]. The local algebra $\mathcal{F}_A$ and that of the complementary region $\mathcal{F}_{A'}$ both act on the global Hilbert space. The local states are the restriction of the global state to the local algebra:

$$\omega_A(b) = \langle \Omega | b \Omega \rangle\,, \qquad \forall b \in \mathcal{F}_A\,. \tag{6.88}$$

Since there are no local Hilbert spaces there are no density matrices either. Modular theory is a mathematical formulation that allows us to define information theoretic quantities using

[23]↑Assumptions about the smoothness of the function f are implicit in what is meant by a solution to the classical equations of motion.

only global states and local algebras, with no need for the existence of local density matrices; see [165]. It applies to any quantum system from qubits to QFT. In QFT the algebras $\mathcal{F}_A$ and $\mathcal{F}_{A'}$ are isomorphic and the global vectors of QFT are analogous to the canonical purification of ρ_A a density matrix of A in terms of $|\Omega\rangle$ a pure state of a double copy Hilbert space $\mathcal{H}_A \otimes \mathcal{H}_{A'}$:

$$\begin{aligned} \omega &= \sum_k p_k \, |k\rangle \langle k| \\ |\Omega\rangle &= \sum_k \sqrt{p_k} \, |k\rangle_A \otimes |k\rangle_{A'} \ . \end{aligned} \tag{6.89}$$

If A_1 and A_2 are two non-overlapping and non-touching regions of space, and $\mathcal{F}_1$ and $\mathcal{F}_2$ are their corresponding local algebras in QFT, the additive algebra of the union A_{12} is the algebraic tensor product of local algebras $\mathcal{F}_{12} = \mathcal{F}_1 \otimes \mathcal{F}_2$.[24]. There is no tensor product when the regions A_1 and A_2 touch. The algebra of invariant local operators $\mathcal{A}$ has a trivial center because the twist operator $\mathcal{E}^{igQ_1}$ does not belong to $\mathcal{A}_1$, however, when A_1 and A_2 are not touching the smoothed out twist commutes with both $\mathcal{A}_1$ and $\mathcal{A}_2$; see figure 6.5.

In section 6.1.2, we argued that the correct entanglement measure in the presence of charges is the relative entropy in (6.39):

$$S^{\mathcal{F}_{12}}(\mathcal{E}^*(\omega_{12}) \| \mathcal{E}^*(E_\tau^*(\omega_{12}))) + S^{\mathcal{F}_{12}}(\mathcal{E}^*(\mathcal{E}_\tau^*(\omega_{12})) \| \mathcal{E}^*(E_\tau^*(\omega_1) \otimes \omega_2)) \tag{6.90}$$

with the conditional expectations

$$\begin{aligned} \mathcal{E}^*(\omega_{12}) &= \frac{1}{|G|} \sum_{g \in G} U_g \omega_{12} U_g^\dagger \\ \mathcal{E}_\tau^*(\omega_{12}) &= \frac{1}{|G|} \sum_{g \in G} \tau_g \omega_{12} \tau_g^\dagger \end{aligned} \tag{6.91}$$

[24]↑In infinite dimensions, one has to be careful when tensoring von Neumann algebras since the weak closure of operators depends on the Hilbert space on which it is acting [58]. This is the so-called split property of QFT that we have assumed to hold in any reasonable model.

where $U_g = \mathrm{e}^{\mathrm{i}gQ}$ and τ_g is the smoothed out $\mathcal{E}^{\mathrm{i}gQ_1}$. By a unitary rotation of local states we mean

$$(U\omega U^\dagger)(b) = \omega(U^\dagger b U) \ . \tag{6.92}$$

We have structured this section in the following way: In section 6.1.18, we start by a discussion of the local charged states and a lower and an upper bound on (6.90). In section 6.1.19, we review the Tomita-Takesaki modular theory (see [180] for a more detailed review) and compute the modular operators for charged states $|r, \mathrm{i}\rangle$, and comment on the mirror operators in the presence of charges. Section 6.1.20 discusses the relation between the cocycle operator in modular theory and local charges. Finally, in section 6.1.21 we introduce a canonical enlarging of the algebra of QFT that decouples charged modes across the entangling surface.

6.1.18 Charged states

Consider the global invariant vector $|\Omega\rangle$ and its local state ω on region A. Since $|\Omega\rangle = U_g |\Omega\rangle$, the expectation value of all charged operators of the form $b - E(b)$ vanishes in ω:

$$\omega(b) = \frac{1}{|G|} \sum_{g \in G} \langle \Omega | U_g^\dagger b U_g \Omega \rangle = \langle \Omega | E(b) \Omega \rangle \ . \tag{6.93}$$

All the charged states $|r, \mathrm{i}\rangle$ are perpendicular to the vacuum since they belong to different superselection sectors. We denote by $|r, \mathrm{i}, A\rangle = \sqrt{d_r}(V_{r,\mathrm{i}}^{(A)})^\dagger |\Omega\rangle$ a state with a charge localized in region A. A vector $|\Phi\rangle = \frac{1}{\sqrt{2}}(|\Omega\rangle + |r, \mathrm{i}, A\rangle)$ that superposes the vacuum with a charged state appears mixed to the local charge-neutral subalgebra of A:

$$\langle \Phi | a \Phi \rangle = \frac{1}{2}(\omega(a) + \omega_{r,\mathrm{i}}(a)) \tag{6.94}$$

where $\omega_{r,\mathrm{i}}(a) = \langle r,\mathrm{i}|a|r,\mathrm{i}\rangle$ is the local charged state, which turns out to be independent of i; see (6.98). The same holds for the local state of the vector $|\chi\rangle = \frac{1}{\sqrt{2}}(|r,\mathrm{i},A_1\rangle + |r,\mathrm{j},A_2\rangle)$. With respect to any charge neutral operator $a \in \mathcal{A}_{12}$ the state seems mixed

$$\langle\chi|a\chi\rangle = \frac{1}{2}(\omega_{r,\mathrm{i}}(a) + \omega_{r,\mathrm{j}}(a)) \ . \tag{6.95}$$

This is because

$$\langle r,\mathrm{i},A_1|a|r,\mathrm{j},A_2\rangle = d_r \, \langle\Omega|V^{(1)}_{r,\mathrm{i}} a (V^{(2)}_{r,\mathrm{j}})^\dagger|\Omega\rangle = \frac{d_r}{|G|}\sum_g \langle\Omega|U_g^\dagger V^{(1)}_{r,\mathrm{i}} a (V^{(2)}_{r,\mathrm{j}})^\dagger U_g|\Omega\rangle \ . \tag{6.96}$$

Using the transformation rule of the intertwiner in (6.78) we find

$$\begin{aligned}\langle r,\mathrm{i},A_1|a|r,\mathrm{j},A_2\rangle &= \frac{d_r}{|G|}\sum_{glk} D_r(g)_{\mathrm{i}k} D_r(g)^*_{\mathrm{j}l} \, \langle\Omega|V^{(1)}_{r,k} a (V^{(2)}_{r,l})^\dagger\Omega\rangle \\ &= \delta_{\mathrm{ij}}\sum_k \langle\Omega|V^{(1)}_{r,k} a (V^{(2)}_{r,k})^\dagger|\Omega\rangle = \frac{\delta_{\mathrm{ij}}}{d_r}\sum_k \langle r,k,A_1|a|r,k,A_2\rangle\end{aligned} \tag{6.97}$$

where we have used (6.79). We learn that $\langle r,\mathrm{i},A_1|r,\mathrm{j},A_2\rangle \sim \delta_{\mathrm{ij}}$ and when $\mathrm{i}=\mathrm{j}$ the expectation value of a is independent of j:

$$\omega_{r,\mathrm{j}}(a) = \langle r,\mathrm{j},A|a|r,\mathrm{j},A\rangle = \sum_k \langle\Omega|V_{r,k} a V^\dagger_{r,k}|\Omega\rangle = \omega(\rho_r(a)) \ . \tag{6.98}$$

Therefore, $\omega_r(a) \equiv \omega(\rho_r(a)) = \omega_{r,\mathrm{j}}(a)$ which implies that one cannot distinguish $|r,\mathrm{i}\rangle$ and $|r,\mathrm{j}\rangle$ using charge-neutral operators. For a general vector $|\Psi\rangle = \sum_{r,\mathrm{i}} c_{r,\mathrm{i}}\,|r,\mathrm{i},A\rangle$ we have

$$(\mathcal{E}^*(\psi))(a) = \sum_{r,\mathrm{i}} |c_{r,\mathrm{i}}|^2 \, \langle r,\mathrm{i},A|a|r,\mathrm{i},A\rangle = \sum_r \omega_r(a)\zeta_r = \psi(a), \tag{6.99}$$

where $\zeta_r = \sum_{\mathrm{i}} |c_{r,\mathrm{i}}|^2$.

Now, consider non-touching regions A_1 and A_2 and a global invariant state $U_g\,|\Omega\rangle = |\Omega\rangle$. The local states ω_{12} and $\omega_1 \otimes \omega_2$ both have zero total charge $Q_1 + Q_2 = 0$ and we only need to consider the charge neutral subalgebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ and the bi-local unitary intertwiners $\mathcal{I}_{12,r} = \sum_{\mathrm{i}} (V^{(1)}_{r,\mathrm{i}})^\dagger V^{(2)}_{r,\mathrm{i}}$. The bi-local intertwiner is a unitary operator that creates an entangled pair of

charge/anti-charge particles $\mathcal{I}_{12,r}\left|\Omega\right\rangle = \left|\mathcal{I}_{12,r}\right\rangle$. These states are orthonormal: $\langle\mathcal{I}_r|\mathcal{I}_{r'}\rangle = \delta_{rr'}$ and $\langle\mathcal{I}_{r'}|\mathcal{I}_s|\mathcal{I}_r\rangle = \delta_{r's}\langle\Omega|\mathcal{I}_r\rangle$. They have an overlap with the vacuum state

$$\langle\Omega|\mathcal{I}_{12,r}\rangle = \sum_{\mathrm{i}}\langle\Omega|(V^{(1)}_{r,\mathrm{i}})^\dagger V^{(2)}_{r,\mathrm{i}}|\Omega\rangle = \frac{1}{d_r}\sum_{\mathrm{i}}\langle r,\mathrm{i},A_2|r,\mathrm{i},A_1\rangle \ . \tag{6.100}$$

The vacuum state has a non-zero amplitude to fluctuate to a state with multiple entangled pairs $\langle\Omega|\mathcal{I}_{12,r_1}\cdots\mathcal{I}_{12,r_n}\rangle \neq 0$. If the symmetry group is Abelian $\mathcal{I}_{12,r_1}\cdots\mathcal{I}_{12,r_n} = \mathcal{I}_{12,r_1+\cdots r_n}$.

The average $\mathcal{E}^*_\tau$ projects the algebra $\mathcal{A}_{12}$ to $\mathcal{A}_1\otimes\mathcal{A}_2$ by discarding the bi-local intertwiner $\mathcal{I}_{12}$. The averaged state $\mathcal{E}^*_\tau(\omega_{12})$ has zero amplitude for the creation of an entangled pair of charged particles between region A_1 and A_2. Adding any bi-local intertwiner $\mathcal{I}_{12}$ to $\mathcal{A}_1\otimes\mathcal{A}_2$ immediately enlarges it to $\mathcal{A}_{12}$. If we want to isolate the contribution of any particular $\mathcal{I}_{12} = V^\dagger(x)V(y)$ with $x\in A_1$ and $y\in A_2$ to the relative entropy we need to find a subalgebra of $\mathcal{A}_{12}$ that only includes this particular bi-local intertwiner, and none of the others.

There is a subalgebra of global charged neutral operators that has no bi-local intertwiners in it; that is to say we have discarded $\mathcal{I}_{12}$ for any non-touching A_1 and A_2. This is the algebra of QFT with no charge creation or annihilation operators. For instance, in the example of the $U(1)$ current model, the algebra generated by $J(z)$ without any vertex operators is such a subalgebra. We denote such a subalgebra by $\mathcal{N}$. The restriction of $\mathcal{N}$ to a region A_1 gives a subalgebra $\mathcal{N}_1\subset\mathcal{A}_1$ and a conditional expectation that washes out any bi-local intertwiners within A_1. The subalgebra $\mathcal{N}_1\otimes\mathcal{N}_2\subset A_{12}$ has no bi-local intertwiners within A_1, A_2 or in between A_1 and A_2. Enlarging $\mathcal{N}_1\otimes\mathcal{N}_2$ by adding any $\mathcal{I}_{12}$ gives a subalgebra of $\mathcal{A}_{12}$, rather than immediately generating the whole $\mathcal{A}_{12}$. For instance, in regions A_1 and A_2 we can choose to add a bi-local intertwiner $V^\dagger(x)V(y)$ with $x\in A_1$ and $y\in A_2$; see figure 6.6. The relative entropy

$$S^{(\mathcal{N}_1\otimes\mathcal{N}_2)\rtimes\mathcal{I}_{12}}(\omega_{12}\|E^*_\tau(\omega_{12})) \leq S^{\mathcal{A}_{12}}(\omega_{12}\|E^*_\tau(\omega_{12})) \tag{6.101}$$

measures the contribution of this particular bi-local intertwiner, and we have used the monotonicity of relative entropy to get a lower bound on our entanglement measure due to bi-local intertwiners. The authors of [166] argued that the bi-local intertwiners with the minimal

distance $|x-y|$ in between A_1 and A_2 give the tightest lower bound for the relative entropy $S(\omega_{12}\|E^*_\tau(\omega_{12}))$. In the literature, such bi-local intertwiners are also known as the edge modes.

To find an upper bound on this entanglement measure we use the definition of $\mathcal{E}^*_\tau$ and the inequality in (6.41):

$$S^{\mathcal{F}_{12}}(\omega_{12}\|\mathcal{E}^*_\tau(\omega_{12})) \leq \log|G| \tag{6.102}$$

In section 6.1.19, we demonstrate a generalization of the inequality (6.41) that applies to QFT.

6.1.19 Modular theory in the presence of charges

Consider two global vectors of a QFT, $|\Omega\rangle$ and $|\Psi\rangle$ and a local algebra $\mathcal{F}_A$. The relative Tomita operator is defined using the equation

$$S^{\mathcal{F}_A}_{\Psi|\Omega} b\,|\Omega\rangle = b^\dagger\,|\Psi\rangle\,, \qquad \forall b\in\mathcal{F}_A\ . \tag{6.103}$$

This operator is labelled by the choice of two vectors and an algebra. To simplify the notation, when it is clear from the context we suppress the algebra label. The equation above defines the action of $S^{\mathcal{F}_A}_{\Psi|\Omega}$ and its $\dagger$ everywhere in $\mathcal{H}$ if the action of operators in $\mathcal{F}_A$ and $\mathcal{F}_{A'}$ on $|\Omega\rangle$ is dense in the Hilbert space: $\overline{\mathcal{F}_A\,|\Omega\rangle}=\mathcal{H}$ [58]. Such a vector $|\Omega\rangle$ is called a Reeh-Schlieder vector (cyclic and separating). In a Reeh-Schlieder state, the action of local algebra $\mathcal{F}_A$ on $|\Omega\rangle$ can approximate any excitation in the global Hilbert space, even those supported outside of A.[25] The vector $|\Omega\rangle$ is called Reeh-Schlieder if and only if it is cyclic with respect to both $\mathcal{F}_A$ and $\mathcal{F}_{A'}$. The squared norm of the relative Tomita operator is called the relative modular operator $\Delta_{\Psi|\Omega}=S^\dagger_{\Psi|\Omega}S_{\Psi|\Omega}$ and we define the anti-linear operator $J_{\Psi|\Omega}=S_{\Psi|\Omega}\Delta^{-1/2}_{\Psi|\Omega}$. When both vectors are the same we call $S_\Omega\equiv S_{\Omega|\Omega}$ the Tomita operator

[25]↑In finite quantum systems, the canonical purification of a density matrix ρ is a Reeh-Schlieder vector if and only if all the eigenvectors of ρ are non-zero. That is to say ρ_A is entirely entangled with A'.

and $\Delta_\Omega \equiv \Delta_{\Omega|\Omega}$ the modular operator. The anti-linear operator $J_\Omega = \Delta_\Omega^{1/2} S_\Omega$ is called the modular conjugation of $|\Omega\rangle$ and has the property that

$$b_J \equiv JbJ \in \mathcal{F}_{A'} \qquad \forall b \in \mathcal{F}_A, \tag{6.104}$$

where we have suppressed the Ω index of J. If $|\Omega\rangle$ is Reeh-Schlieder the modular conjugation is an anti-unitary $J = J^{-1} = J^\dagger$ [59]. An important result of the modular theory is that the (relative) modular operator generates a flow called the (relative) modular flow that is an outer automorphism of the algebra $\mathcal{F}_A$. This flow is independent of the second vector (for a proof see [60], [181]); see figure 6.7:

$$\begin{aligned}
&(\Delta_{\Omega|\Psi}^{\mathcal{F}_A})^{\mathrm{i}t} b (\Delta_{\Omega|\Psi}^{\mathcal{F}_A})^{-\mathrm{i}t} = (\Delta_{\Omega}^{\mathcal{F}_A})^{\mathrm{i}t} b (\Delta_{\Omega}^{\mathcal{F}_A})^{-\mathrm{i}t} \in \mathcal{F}_A \qquad \forall b \in \mathcal{F}_A \quad \text{and} \quad \forall t \in \mathbb{R} \\
&(\Delta_{\Omega|\Psi}^{\mathcal{F}_A})^{\mathrm{i}t} b' (\Delta_{\Omega|\Psi}^{\mathcal{F}_A})^{-\mathrm{i}t} = (\Delta_{\Psi}^{\mathcal{F}_A})^{\mathrm{i}t} b' (\Delta_{\Psi}^{\mathcal{F}_A})^{-\mathrm{i}t} \in \mathcal{F}_A \qquad \forall b' \in \mathcal{F}_{A'} \quad \text{and} \quad \forall t \in \mathbb{R}
\end{aligned} \tag{6.105}$$

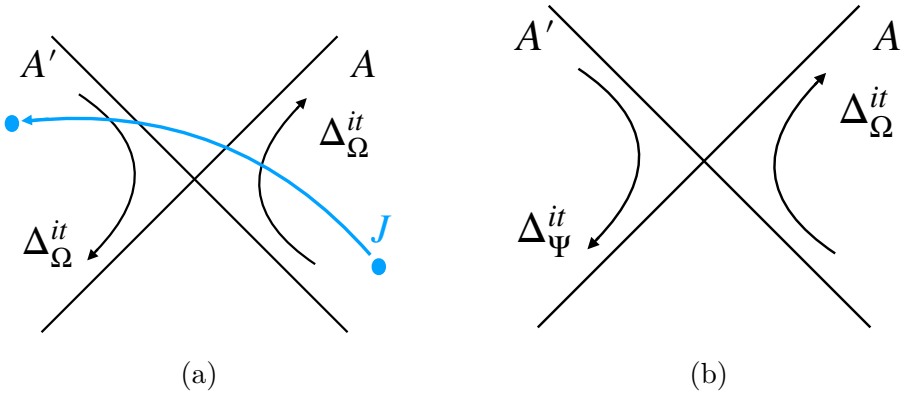


Figure 6.7. If the region A is the Rindler wedge and the state is a QFT in the vacuum, the modular flow is the boost that evolves operators geometrically according to the arrows in (a) [58]. The modular conjugation map J_Ω is the CRT (charge conjugation/reflection/time reversal) that sends operators from A to A' and vice versa. (b) The relative modular flow generated by $\Delta_{\Omega|\Psi}^{it}$ acts as the modular flow of Ω on the operators in A and the modular flow of state Ψ on the operators in A'.

The above relations imply that the operator $\Delta_{\Omega|\Psi}^{it}\Delta_{\Omega}^{-it}$ commutes with all operators in $\mathcal{F}_A$. This operator is called the cocycle. Similarly for the modular conjugation we have

$$J_{\Psi|\Omega}^{\dagger} b J_{\Psi|\Omega} = J_{\Omega} b J_{\Omega} \tag{6.106}$$

which implies that $J_{\Psi|\Omega}J_{\Psi}$ commutes with all $\mathcal{F}_A$. The correlation functions of the operators $b, c \in \mathcal{F}_A$ in the state $|\Omega\rangle$ have the KMS property which can be interpreted as an analytic continuation of the modular flow to complex values of t: $\langle\Omega|b\Delta_{\Omega}c|\Omega\rangle = \langle\Omega|cb|\Omega\rangle$.[26] The set of operators $h \in \mathcal{F}_A$ with the property that $\langle\Omega|[h,b]|\Omega\rangle = 0$ for all $b \in \mathcal{F}_A$ forms a subalgebra of $\mathcal{F}_A$ that we call the centralizer of ω and denote it by $\mathcal{F}_A^{\omega}$ [182], [183]. The KMS property implies that

$$\langle\Omega|b(\Delta - 1)h|\Omega\rangle = 0 \qquad \forall h \in \mathcal{F}_A^{\omega} \ . \tag{6.107}$$

Since $b\,|\Omega\rangle$ is dense in the Hilbert space the vector $h\,|\Omega\rangle$ is an invariant state of the modular operator. The operators in the centralizer have the important property that h and Δ commute [182]

$$\Delta^{z} h \Delta^{-z} = h \qquad \forall z \in \mathbb{C} \ . \tag{6.108}$$

In fact, an operator $h \in \mathcal{F}_A$ that is in the centralizer of Ω commutes with $\Delta_{\Omega|\Psi}$ for any Ψ.[27] Since $h \in \mathcal{F}_A^{\omega}$ are invariant under the modular flow, we sometimes refer to them as the modular zero modes. The modular zero mode satisfies the equation

$$(h^{\dagger} - h_J)\,|\Omega\rangle = 0 \ . \tag{6.109}$$

[26]↑To show this we note that $\langle\Omega|b\Delta c|\Omega\rangle = \langle\Omega|bS^{\dagger}Sc|\Omega\rangle = \langle Sc\Omega|Sb^{\dagger}\Omega\rangle = \langle\Omega|cb|\Omega\rangle$, where we have used the anti-linearity of S.

[27]↑To see this, we first rewrite b as $\lim_{\gamma\to\infty} b_{\gamma}$ in (6.130) that is entire meaning that $b_{\gamma}(z)$ defined in (6.131) is in $\mathcal{F}_1$ for all complex z. Then, from (6.105) it follows that for all h in the centralizer of Ω we have $\Delta_{\Omega}^{z} h \Delta_{\Omega}^{-z} = \Delta_{\Omega|\Psi}^{z} b \Delta_{\Omega|\Psi}^{-z} = h$.

Note that $h_J \in \mathcal{F}_{A'}$ is also in the centralizer of Ω. If the algebra has a center Z, the center is inside the centralizer of all states. The operators in the center $z \in Z$ satisfy $z^\dagger = z_J$ [184]:

$$z^\dagger b \left|\Omega\right\rangle = b z^\dagger \left|\Omega\right\rangle = S(z b^\dagger) \left|\Omega\right\rangle = J\Delta^{1/2} z b^\dagger \left|\Omega\right\rangle = z_J J\Delta^{1/2} b^\dagger \left|\Omega\right\rangle = z_J b \left|\Omega\right\rangle \ . \tag{6.110}$$

The relative Tomita operator for an excited state $h\left|\Omega\right\rangle$ and h an invertible element of the centralizer is

$$\begin{aligned} S_{\Omega|h\Omega} &= \|h\left|\Omega\right\rangle\| S_\Omega (h_J)^{-1} \\ \Delta_{\Omega|h\Omega} &= \|h\left|\Omega\right\rangle\|^2 \Delta_\Omega |h_J|^{-2} \end{aligned} \tag{6.111}$$

where we have used (6.109). The relative entropy of two vectors with respect to an algebra $\mathcal{F}_A$ is given by [185]

$$S^{\mathcal{F}_A}(\Psi\|\Omega) = -\left\langle\Psi\right| \log \Delta^{\mathcal{F}_A}_{\Omega|\Psi} \Psi\right\rangle \ . \tag{6.112}$$

When $\left|\Omega\right\rangle$ and $\left|\Psi\right\rangle$ are the canonical purifications of density matrices σ and ρ in (6.89) the formula above matches the definition:

$$S(\rho\|\sigma) = \mathrm{tr}(\rho \log \rho) - \mathrm{tr}(\rho \log \sigma) \ . \tag{6.113}$$

The elements of the centralizer are the operators that commute with the density matrix. The local state associated with the excited state $h\left|\Omega\right\rangle$ with h in the centralizer is $\rho_h = h\rho h^\dagger / \mathrm{tr}(\rho |h|^2)$ that commutes with ρ the local state of $\left|\Omega\right\rangle$. The relative entropy of these states with respect to the vacuum defined by (6.112) is

$$S(h\Omega\|\Omega) = -2\log\|h\left|\Omega\right\rangle\| + 2\left\langle h\Omega\right| \log|h_J| \left|h\Omega\right\rangle \tag{6.114}$$

where $|h\Omega\rangle$ is the normalized state $h\,|\Omega\rangle$. If v is an isometry in the centralizer of Ω then $v^\dagger\,|\Omega\rangle$ has the same local state as $|\Omega\rangle$:

$$\langle\Omega|vbv^\dagger|\Omega\rangle = \langle\Omega|v^\dagger vb|\Omega\rangle = \langle\Omega|b|\Omega\rangle \ . \tag{6.115}$$

That is why the equation (6.114) implies $S(v^\dagger\Omega\|\Omega) = 0$ for v in the centralizer. Since ρ and ρ_h are simultaneously block diagonalizable their relative entropy can be understood as a classical relative entropy. For instance, take $\rho = \sum_k q_k\,|k\rangle\,\langle k|$ and $h = \sum_k \sqrt{\frac{p_k}{q_k}}\,|k\rangle\,\langle k|$ with p_k a probability distribution that is in the centralizer of ρ. The state $\rho_h = \sum_k p_k\,|k\rangle\,\langle k|$ is simultaneously diagonalized with ρ. The relative entropy above is

$$S(\sum_k p_k\,|k\rangle\,\langle k|\,\|\sum_k q_k\,|k\rangle\,\langle k|) = \sum_k p_k(\log p_k - \log q_k) = H(p\|q) \tag{6.116}$$

which is a special case of (6.42). More generally, for an operator h that in the centralizer of $|\Omega\rangle$ we have

$$\begin{aligned} S_{\Psi|h\Omega} &= \|h\,|\Omega\rangle\,\|S_{\Psi|\Omega}(h_{J_\Omega})^{-1} \\ \Delta_{\Psi|\Omega} &= \|h\,|\Omega\rangle\,\|^2\Delta_{\Psi|\Omega}|h_{J_\Omega}|^{-2} \end{aligned} \tag{6.117}$$

where we have used the fact that $[\Delta_{\Psi|\Omega}, h_{J_\Omega}] = 0$ because $h_{J_\Omega} \in \mathcal{F}_{A'}$ and in the centralizer of Ω. Then, the relative entropy is

$$S(h\Omega\|\Psi) = -2\log\|h\,|\Omega\rangle\,\| + 2\,\langle h\Omega|\log|h_J||h\Omega\rangle - \langle h\Omega|\log\Delta_{\Psi|\Omega}|h\Omega\rangle \ . \tag{6.118}$$

This is a QFT generalization of the equation (6.42). To see this, plug in the equation above the block diagonal density matrices $\rho = \oplus_k q_k\rho_k$, $\psi = \oplus_k p_k\sigma_k$ and the operator $h = \oplus_k\sqrt{\frac{p_k}{q_k}}\mathbb{I}_k$ that is the centralizer of both states:

$$S^{\oplus_k\mathcal{F}_k}(\rho\|\sigma) = H(p\|q) + \sum_k p_k S(\rho_k\|\sigma_k) \ . \tag{6.119}$$

In the presence of an internal symmetry $U_g b U_g^\dagger \in \mathcal{F}_A$ for all $b \in \mathcal{F}_A$. From (6.103) we can solve for the modular operator of $U_g |\Psi\rangle$:

$$\begin{aligned} S^A_{U_g\Psi} &= U_g S^A_\Psi U_g^\dagger \\ \Delta^A_{U_g\Psi} &= U_g \Delta^A_\Psi U_g^\dagger \ . \end{aligned} \tag{6.120}$$

If $|\Omega\rangle$ is the invariant vacuum, i.e. $U_g |\Omega\rangle = |\Omega\rangle$, the modular operator and U_g commute: $\Delta_\Omega U_g = U_g \Delta_\Omega$. As a result, the modular flow Δ_Ω of charge-neutral operators remains charge-neutral if $|\Omega\rangle$ is an invariant vector, and the charge of an operator $V_{r,\mathrm{i}}$ does not change under the modular flow by $\Delta_\Omega^{\mathrm{i}t}$. Now, consider the twist unitary τ_g. On a lattice, the twist operator is in the center of the local charge-neutral algebra. In QFT the twist operator is not in the center of the local algebra, but we still have

$$\begin{aligned} S^{\mathcal{A}_1\otimes\mathcal{A}_2}_{\Psi|\tau_g\Omega} &= S^{\mathcal{A}_1\otimes\mathcal{A}_2}_{\Psi|\Omega}\tau_g^\dagger \\ S^{\mathcal{A}_1\otimes\mathcal{A}_2}(\tau_g\Omega\|\Psi) &= S^{\mathcal{A}_1\otimes\mathcal{A}_2}(\Omega\|\Psi) \ . \end{aligned} \tag{6.121}$$

This is expected because $\tau_g |\Omega\rangle$ has the same local state as $|\Omega\rangle$ with respect to the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$.

The relative Tomita operator of the charged states $|r, \mathrm{i}\rangle = \sqrt{d_r} V^\dagger_{r,\mathrm{i}} |\Omega\rangle$ is

$$S_{(r'\mathrm{i}')|(r\mathrm{i})} b |r, \mathrm{i}, A\rangle = b^\dagger |r', \mathrm{i}', A\rangle \ , \forall b \in \mathcal{F}_A \tag{6.122}$$

which can be solved by setting $S_{(r'\mathrm{i}')|(r\mathrm{i})} = \sqrt{\frac{d_{r'}}{d_r}} V^\dagger_{r,\mathrm{i}} S_\Omega V_{r',\mathrm{i}'}$ in the equation above. Below, we suppress the algebra label in the relative modular operator and relative entropies if the algebra is $\mathcal{F}_A$. Note that S_Ω is the Tomita operator for all charged operators, and we have used $V^\dagger_{r\mathrm{i}} V_{r\mathrm{j}} = \delta_{\mathrm{ij}}$. The relative Tomita operator kills the vectors $b |s, \mathrm{j}, A\rangle$ for $s \neq r$, and on its domain it satisfies

$$\begin{aligned} S_{(r'\mathrm{i}')|(r\mathrm{i})} &= \sqrt{\frac{d_{r'}}{d_r}} V^\dagger_{r,\mathrm{i}} S_\Omega V_{r',\mathrm{i}'} \\ \Delta_{(r'\mathrm{i}')|(r\mathrm{i})} &= \frac{d_{r'}}{d_r} V^\dagger_{r',\mathrm{i}'} S^\dagger_\Omega V_{r,\mathrm{i}} V^\dagger_{r,\mathrm{i}} S_\Omega V_{r',\mathrm{i}'} \ . \end{aligned} \tag{6.123}$$

In particular, we find that $\sum_{\mathrm{i}} \Delta_{(r'\mathrm{i}')|(r\mathrm{i})} = \frac{1}{d_r}\Delta_{(r'\mathrm{i}')|\Omega}$. For an Abelian symmetry, the intertwiner V_q is a unitary operator and

$$\Delta_{q|q'} = V_q^\dagger \Delta_\Omega V_q \ . \tag{6.124}$$

Therefore, the relative entropy states of sectors of charge q and q' is

$$S^{\mathcal{F}_A}(q'\|q) = -\langle q'|V_q^\dagger \log \Delta_\Omega V_q|q'\rangle = -\langle q'-q|\log \Delta_\Omega|q'-q\rangle \ . \tag{6.125}$$

For bi-local intertwiners $\mathcal{I}_{12,r}$ and the algebra $\mathcal{F}_{12}$ we have

$$\begin{aligned}
S^{\mathcal{F}_{12}}_{\Omega|\mathcal{I}_r} &= \mathcal{I}_r S^{\mathcal{F}_{12}}_\Omega \\
S^{\mathcal{F}_{12}}_{\mathcal{I}_r|\Omega} &= S^{\mathcal{F}_{12}}_\Omega \mathcal{I}_r^\dagger \\
S^{\mathcal{F}_{12}}(\mathcal{I}_r\|\Omega) &= -\langle \mathcal{I}_r|\log \Delta_\Omega|\mathcal{I}_r\rangle \\
S^{\mathcal{F}_{12}}(\Omega\|\mathcal{I}_r) &= -\langle \mathcal{I}_r^\dagger|\log \Delta_\Omega|\mathcal{I}_r^\dagger\rangle \ .
\end{aligned} \tag{6.126}$$

where for $\mathcal{F}_1$

$$\begin{aligned}
S^{\mathcal{F}_1}_{\Omega|\mathcal{I}_r} &= \frac{1}{d_r}\sum_{\mathrm{j}} (V^{(1)}_{r,\mathrm{j}})^\dagger S^{\mathcal{F}_1}_\Omega (V^{(2)}_{r,\mathrm{j}})^\dagger, \\
S^{\mathcal{F}_1}_{\mathcal{I}_r|\Omega} &= \sum_{\mathrm{j}} (V^{(2)}_{r,\mathrm{j}}) S^{\mathcal{F}_1}_\Omega (V^{(1)}_{r,\mathrm{j}}) \ .
\end{aligned} \tag{6.127}$$

The relative Tomita equation defines the relative modular operator unambiguously if the vector $|\Omega\rangle$ is Reeh-Schlieder. The Poincare-invariant vacuum of QFT is a Reeh-Schlieder vector for local algebras $\mathcal{F}_A$. In a Reeh-Schlieder vector the excitations inside the region A can approximate an arbitrary excitations outside. We are interested in studying the relative modular operator with respect to the local charge-neutral subalgebras $\mathcal{A}_A$, and below we show that the vacuum vector is Reeh-Schlieder with respect to $\mathcal{A}_A$. That is to say in QFT an arbitrary uncharged operator in $\mathcal{A}$ can be approximated using local uncharged operators $\mathcal{A}_A$: $\overline{\mathcal{A}_A|\Omega\rangle} = \overline{\mathcal{A}|\Omega\rangle}$.

First, let us take a look at the Reeh-Schlieder property for the full algebra $\mathcal{F}$ of QFT. In a Reeh-Schlieder state an arbitrary excitation in $\mathcal{F}_1 \otimes \mathcal{F}_{1'}$ can be approximated using operators in $\mathcal{F}_1$. We want to find $b_m \in \mathcal{F}_1$ such that for some $b' \in \mathcal{F}_{1'}$ we have $b' |\Omega\rangle \simeq b_m |\Omega\rangle$. We call such an operator b_m the *mirror operator* of b'. To construct the mirror operator, we use the following strategy

$$b' |\Omega\rangle = S'_\Omega (b')^\dagger |\Omega\rangle = (\Delta')^{-1/2} J (b')^\dagger |\Omega\rangle = \Delta^{1/2} (b')^\dagger_J \Delta^{-1/2} |\Omega\rangle \tag{6.128}$$

with $(b')^\dagger_J \equiv J(b')^\dagger J \in \mathcal{F}_1$ where we have suppressed the Ω index of Δ_Ω and J_Ω. For a Reeh-Schlieder vector in finite quantum systems, it is straightforward to check that the operator $(\omega^{1/2} b^T \omega^{-1/2} \otimes 1)$ is the mirror of $(1 \otimes b)$ where T is the transpose in the basis picked by the density matrix ω:

$$\sum_k \sqrt{p_k} (\omega^{1/2} b^T \omega^{-1/2} \otimes 1) |kk\rangle = \sum_{kl} \sqrt{p_l} b_{kl} |lk\rangle = \sum_l \sqrt{p_l} (1 \otimes b) |ll\rangle \ . \tag{6.129}$$

where we have used the canonical purification of ω in (6.89). Note that in the example above, the modular conjugation operator J is the anti-linear swap operator in the Schmidt basis of the state: $Jc |kl\rangle = c^* |lk\rangle$ where c is a complex number. In a Reeh-Schlieder state since all $p_k > 0$, $\omega^{-1/2}$ is well-defined. Furthermore, the operator $\Delta^{1/2}(b^T \otimes 1)\Delta^{-1/2} = \omega^{1/2} b^T \omega^{-1/2} \otimes 1 \in \mathcal{F}_1 \otimes 1$. In a QFT, for a general $b \in \mathcal{F}_1$, the modular flow $b(t) \equiv \Delta^{\mathrm{i}t} b \Delta^{-\mathrm{i}t}$ is inside the algebra $\mathcal{F}_1$ for all $t \in \mathbb{R}$, but the operator $\Delta^{1/2} b \Delta^{-1/2}$ need not be in $\mathcal{F}_1$. Luckily, as we demonstrate below, in QFT there are always operators in $\mathcal{F}_1$ that approximate $\Delta^{1/2} b \Delta^{-1/2}$ arbitrarily well.

Consider the operator

$$b_\gamma = \sqrt{\frac{\gamma}{\pi}} \int_{-\infty}^{\infty} dt \, \mathrm{e}^{-\gamma t^2} \Delta^{\mathrm{i}t} b \Delta^{-\mathrm{i}t} \in \mathcal{F}_1 \tag{6.130}$$

In the limit $\gamma \to \infty$ this operator approximates b,[28] and for any γ the modular flow of this operator can be analytically continued to the whole complex plane [182]

$$b_\gamma(z) \equiv \Delta^z b_\gamma \Delta^{-z} = \sqrt{\frac{\gamma}{\pi}} \int_{-\infty}^{\infty} dt\, \mathrm{e}^{-\gamma(t+\mathrm{i}z)^2} \Delta^{\mathrm{i}t} b \Delta^{-\mathrm{i}t} \in \mathcal{F}_1 \ . \tag{6.131}$$

Therefore, we find that mirror operator of b' in the algebra $\mathcal{F}_1$ that satisfies

$$\begin{aligned} b' \left|\Omega\right\rangle &\simeq b_m \left|\Omega\right\rangle \\ b_m &= \lim_{\gamma\to\infty} ((b')^\dagger_J)_\gamma(1/2) \ . \end{aligned} \tag{6.132}$$

If the operator b' is an isometry, the equation

$$\langle\Omega|b' b'^\dagger_m \Omega\rangle = 1 \tag{6.133}$$

implies that the probability for the spontaneous creation of the excitation $b' b'^\dagger_m \left|\Omega\right\rangle$ is almost one. In general, if b' is localized in a small region of $A_{1'}$ its mirror is highly delocalized in A_1. If $[b', \Delta] = 0$ from the mirror equation (6.132) we find that $b_m = (b')^\dagger_J$; see figure 6.8.

Consider the symmetry group G acting on the global algebra $\mathcal{F}$. If $a' \in \mathcal{A}_{1'}$ is a charge-neutral operator from (6.131) it is evident that the mirror operator a_m is also charge-neutral, and is therefore in $\mathcal{A}_1$. This implies that we can generate $\mathcal{A}_{1'} \left|\Omega\right\rangle$ using $\mathcal{A}_1 \left|\Omega\right\rangle$. The only other operators in $\mathcal{A}$ are the bi-local intertwiners between A and A': $I_r = \sum_{\mathrm{i}} V^\dagger_{r,\mathrm{i}} V'_{r,\mathrm{i}}$. Denote the mirror of $V'_{r,\mathrm{i}}$ by $(V_{r,\mathrm{i}})_m$. It has the same charge as $V'_{r,\mathrm{i}}$. Therefore, the operator $\sum_{\mathrm{i}} V^\dagger_{r,\mathrm{i}} (V_{r,\mathrm{i}})_m$ is charge-neutral with respect to the local algebra and therefore belongs to $\mathcal{A}_A$. Moreover, the mirror of all operators in $\mathcal{A}_{1'}$ also belong to $\mathcal{A}_1$, therefore in QFT $\overline{\mathcal{A}_1 \left|\Omega\right\rangle} = \overline{\mathcal{A} \left|\Omega\right\rangle}$.

Since $\overline{A_1 \left|\Omega\right\rangle} = \overline{\mathcal{A}_{12} \left|\Omega\right\rangle}$, it follows that for non-overlapping and non-touching A_1 and A_2 we have $\overline{\mathcal{A}_1 \otimes \mathcal{A}_2 \left|\Omega\right\rangle} = \overline{\mathcal{A}_{12} \left|\Omega\right\rangle} = \overline{\mathcal{A} \left|\Omega\right\rangle}$. All intertwiners between region A_1 and the complement A'_1 can be prepared locally by acting with $\mathcal{A}_1$ which includes the intertwiners

[28]↑Note that in the limit $\gamma \to 0$ the operator b_0 is the modular zero mode, and for finite values of γ this operator sends off-diagonal elements $\left|k\right\rangle \left\langle k'\right| \to \mathrm{e}^{-\frac{(\log p_k - \log p_{k'})^2}{4\gamma}} \left|k\right\rangle \left\langle k'\right|$. It suppresses the off-diagonal terms exponentially with parameter $\frac{1}{\gamma}$. The modular zero mode has the property that its modular flow is trivial: $(b_0)_\gamma = b_0$ for all γ.

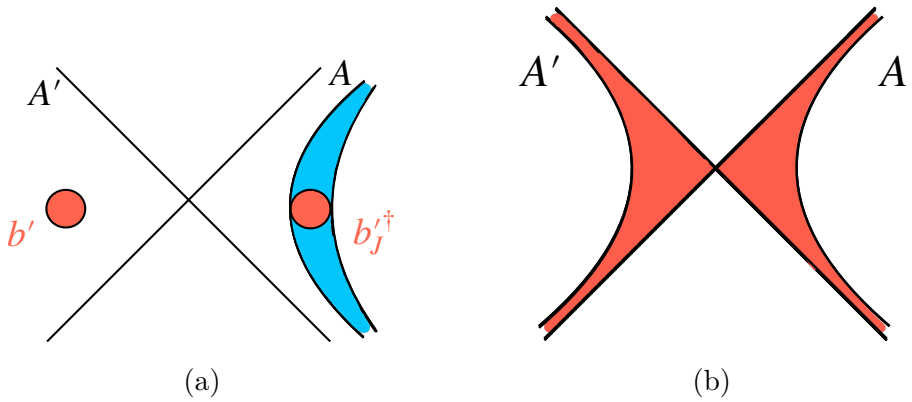


Figure 6.8. Consider the case where region A is the Rindler wedge and we have the vacuum of QFT (a) If b' is localized in the small red circle inside A' the operator $(b'_J)^\dagger$ is also localized in the red circle in A, however the the mirror operator in (6.131) requires boosting that spreads its support in the blue region (b) The operators in A' that are approximately-invariant under modular flow (boost) are localized in a small proper distance from the entangling surface. Their mirror operators are also localized near the entangling surface in A.

between regions A_1 and A_2. As a result, the algebra $\overline{(\mathcal{A}_1 \otimes \mathcal{A}_2)\,|\Omega\rangle} = \overline{\mathcal{A}\,|\Omega\rangle}$, and the Tomita operator for the algebra $\mathcal{A}_1 \otimes \mathcal{A}_2$ is densely defined.

6.1.20 Cocycle and intertwiners

In this section, we show that the intertwiner $V_{r,\mathrm{i}}$ can be understood as an analytic continuation of the unitary cocycle. Consider two vectors $|\Omega\rangle$ and $|\Psi\rangle$ in different superselection sectors of a QFT and the isometry defined by

$$T(a\,|\Psi\rangle) = a\Delta^{1/2}_{\Psi|\Omega}\,|\Omega\rangle \tag{6.134}$$

that maps vectors from the $|\Omega\rangle$ to the $|\Psi\rangle$ sector. This is an intertwiner that takes us from one charged sector to another and commutes with the action of $\mathcal{F}_A$ [58]. When the superselection sectors are due to symmetries, the intertwiner need not be localized in A. We say the intertwiner is localized in A if $Tb'\,|\Omega\rangle = b'T\,|\Omega\rangle$ for all b' charged operators in A'. We

would like to understand when the intertwiner T is localized in A. In the last subsection, we saw that the cocycle operator

$$u_{\Psi|\Omega}(t) = \Delta_{\Psi|\Omega}^{\mathrm{i}t}\Delta_{\Omega}^{-\mathrm{i}t} \tag{6.135}$$

belongs to the algebra. In fact, if both Ω and Ψ are Reeh-Schlieder it is a unitary operator. For real values of t the cocycle is an operator in A and commutes with b' [183], [186]. The isometry in (6.134) can be created by an analytic continuation of the cocycle to imaginary values $\mathrm{Im}(t) = -\mathrm{i}/2$:

$$Tb'\left|\Psi\right\rangle = b'\Delta_{\Psi|\Omega}^{1/2}\Delta_{\Omega}^{-1/2}\left|\Omega\right\rangle = b'u_{\Psi|\Omega}(-\mathrm{i}/2)\left|\Omega\right\rangle \ . \tag{6.136}$$

The isometry in (6.134) commutes with all $b' \in F_{A'}$ if the analytic continuation of the cocycle $u_{\Psi|\Omega}(-\mathrm{i}/2)$ exists and belongs to $\mathcal{F}_A$.

On a lattice the cocycle is $u_{\Psi|\Omega}(t) = \psi^{\mathrm{i}t}\omega^{-\mathrm{i}t} \otimes 1'$ with ψ and ω the reduced density matrices on A of $|\Psi\rangle$ and $|\Omega\rangle$, respectively. The analytic continuation of the cocycle to $\mathrm{Im}(t) = -\mathrm{i}/2$ corresponds to $\psi^{1/2}\omega^{-1/2} \otimes 1$ which is well-defined if all the density matrix ω has no zero eigenvalues. In fact, it suffices to assume that every zero eigenvalue of ω is also a zero eigenvalue of ψ, because if $\omega\left|\xi\right\rangle = \psi\left|\xi\right\rangle = 0$ we can define $\psi^{1/2}\omega^{-1/2}\left|\xi\right\rangle = 0$. In other words, the cocycle has an analytic continuation if there exists a $\lambda > 0$ such that $\omega - \lambda\psi$ is a non-negative operator. This is the necessary condition for the relative entropy $S(\psi\|\omega)$ to be finite. Similarly, in modular theory, the cocycle $u_{\Psi|\Omega}(t)$ can be analytically continued to the $0 \geq \mathrm{Im}(t) \geq -1/2$ if $\omega - \lambda\psi \geq 0$. That is to say there exists a $\lambda > 0$ such that for all $b \in \mathcal{F}_A$:

$$\omega(b^\dagger b) - \lambda\psi(b^\dagger b) \geq 0 \ . \tag{6.137}$$

If $|\Psi\rangle$ and $|\Omega\rangle$ are vectors corresponding to states ψ and ω with $\omega \geq \lambda\psi$ for some positive λ, the cocycle $u_{\psi|\omega}$ has an analytic continuation to the strip $0 \geq \mathrm{Im}(t) \geq -1/2$ that remains

inside the algebra [64]. This implies that there is a map $u(t)$ analytic in the strip and strongly continuous in t with the property that

$$\begin{aligned} u(t) &= u_{\psi|\omega}(t) \\ u(t)b' \left|\Omega\right\rangle &= b'\Delta^{\mathrm{it}}_{\Psi|\Omega} \left|\Omega\right\rangle \ . \end{aligned} \tag{6.138}$$

In particular at $u(-\mathrm{i}/2)$ we have the map

$$u(-\mathrm{i}/2)b' \left|\Psi\right\rangle = b'u(-\mathrm{i}/2) \left|\Psi\right\rangle , \tag{6.139}$$

which is the local intertwiners $V_{r,\mathrm{i}}$ we discussed in the case of charges in section 6.1.13. In our examples of QFT with charges we have $\mathcal{E}^*_\tau(\omega_{12}) > \frac{1}{|G|}\omega_{12}$, therefore the cocycle $u_{\omega_{12}|\mathcal{E}^*_\tau(\omega_{12})}$ takes us from the sector $|\Omega\rangle$ to the sector corresponding to $\mathcal{E}^*_\tau(\omega)$.

6.1.21 Enlarging the QFT algebra

In section 6.1.13, we saw that one main difference between QFT and systems on a lattice is that in QFT the twist operator $\tau_g = \mathrm{e}^{\mathrm{i}gQ_A}$ is not part of the local algebra, and as a result the local invariant algebra has no non-trivial center. It is natural to ask whether one can enlarge the QFT algebra by including τ_g to make QFT more similar to the lattice models. The local algebra of charged operators in QFT has charge neutral operators $\mathbb{I}_r \otimes a \in \mathcal{A}_A$ and charged operators $V^{(A)}_{r,\mathrm{i}}$ supported on A that belong to the dual group $\hat{G}$: $\mathcal{F}_A = \mathcal{A}_A \rtimes_\rho \hat{G}$. If we further enlarge the QFT algebra by adding τ_g that belongs to G to it, we obtain $\mathcal{F}_A \rtimes G$. If the group G is Abelian this is $(\mathcal{A}_A \rtimes \hat{G}) \rtimes G = \mathcal{A}_A \otimes B(L^2(\hat{G}))$. Physically, this corresponds to adding a qudit of dimension $|G|$ to the local algebra of QFT exactly as we do on a lattice [182]. There will be an analogous degree of freedom on the complementary region A' and the global Hilbert space factors as $\mathcal{H} = \oplus_q (\mathcal{K}^A_q \otimes \mathcal{K}^{A'}_q) \otimes \mathcal{H}_q$. The enlarged local algebra is the tensor product of the algebra of charge neutral operators with a qudit of dimension $|G|$ that carries the charge: $\mathcal{A}_A \otimes GL(|G|, \mathbb{C})$ where $GL(|G|, \mathbb{C})$ is the algebra of a qudit. In this enlarged algebra, the charge neutral operators have a non-trivial center: $\oplus_q \lambda_q \left|q\right\rangle \left\langle q\right| \otimes 1$

similar to systems on a lattice. When the group is non-commutative, it is still convenient to consider the algebra represented on the Hilbert space $\oplus_r(\mathcal{K}_r^A \otimes \mathcal{K}_r^{A'}) \otimes \mathcal{H}_r$.

It is desirable to construct a conditional expectation that maps the enlarged algebra $\mathcal{F}_A \rtimes G$ back down to the QFT algebra $\mathcal{F}_A$. In the Abelian case, this is simply an average over the dual group

$$\tilde{E}(|q'\rangle \langle q''| \otimes a) = \frac{1}{|\hat{G}|} \sum_q \hat{U}_q^\dagger (|q'\rangle \langle q''| \otimes a) \hat{U}_q \ . \tag{6.140}$$

The QFT local algebra $\mathcal{F}_A$ is generated by $|0\rangle \langle 0| \otimes a$ and $\hat{U}_q$ which transform under this map to

$$\begin{aligned} \tilde{E}(|0\rangle \langle 0| \otimes a) &= \frac{1}{|\hat{G}|} \sum_q \rho_q(a) \\ \tilde{E}(\hat{U}_q) &= \hat{U}_q \end{aligned} \tag{6.141}$$

where in the second line we have used the fact that the group is Abelian. Under $\tilde{E}$ any new non-identity elements of the twist group $\tau_g = \sum_q \mathrm{e}^{\mathrm{i}gq} |q, A\rangle \langle q, A|$ are washed out

$$\tilde{E}(\tau_g) = \delta_{g0} \mathbb{I} \ . \tag{6.142}$$

In the example of the free boson in section 6.1.10, the dual group is $\mathbb{Z}$ that is not compact but we can still write

$$\rho_\alpha(J(u)) = J(u) + \oint \frac{dz}{2\pi\mathrm{i}} u(z)\alpha(z) \tag{6.143}$$

and the sum over charges in the range $(-q, q)$ vanishes

$$\frac{1}{2q+1} \sum_{k=-q}^{q} \rho_{k\alpha}(J(u)) = J(u) \ . \tag{6.144}$$

as one expects from a conditional expectation.

Our enlarged algebra has a representation in a Hilbert space that factors the charge modes $\oplus_q(\mathcal{K}_q^A \otimes \mathcal{K}_q^{A'}) \otimes \mathcal{H}_q$. We would like to find the vectors in this Hilbert space that

correspond to the states of the QFT algebra. We can extend our QFT states using the conditional expectation $\tilde{E}$ so that the relative entropy of states evaluated in the enlarged algebra remains the same as that of the QFT algebra. For instance, the purification of the state $\mathcal{E}^*(\psi)$ in (6.99) in this enlarged Hilbert space is

$$\begin{aligned} |\Theta_\psi\rangle &= \sum_r \sqrt{\zeta_r}\, |E_r\rangle \otimes |\Psi_r\rangle \\ |E_r\rangle &= \frac{1}{\sqrt{d_r}} \sum_{\mathrm{i}} |r, \mathrm{i}\rangle \otimes |r^*, \mathrm{i}\rangle \end{aligned} \tag{6.145}$$

where r^* is the dual representation of r. The expectation values in this vector are

$$\begin{aligned} &\sum_r \langle\Theta_\psi|(1_r \otimes a)|\Theta_\psi\rangle = \sum_r \zeta_r \omega(\rho_r(a)) \\ &\langle\Theta_\psi|V_{r,\mathrm{i}}|\Theta_\psi\rangle = 0 \end{aligned} \tag{6.146}$$

and

$$\langle\Theta_\psi|U_g^r \otimes 1|\Theta_\psi\rangle = \frac{1}{d_r}\mathrm{tr}(U_g^r) = \delta(g) \tag{6.147}$$

as expected from an invariant state of $\tilde{E}$. This vector is also Reeh-Schlieder with respect to the QFT algebra because the action of $V_{r,\mathrm{i}}$ and $\mathbb{I}_r \otimes a$ take us everywhere in the Hilbert space.

For simplicity, we assume that the symmetry group is Abelian for the remainder of this section. The relative Tomita equation for the vectors $|\Theta_\omega\rangle = |00\rangle \otimes |\Omega\rangle$ and $|\Theta_\psi\rangle = |00\rangle \otimes |\Psi\rangle$ is

$$\begin{aligned} &S^{\mathcal{F}}_{\Theta_\psi|\Theta_\omega}(|q\rangle\langle q'| \otimes a)\,|\Theta_\omega\rangle = (|q'\rangle\langle q| \otimes a^\dagger)\,|\Theta_\psi\rangle \\ &S^{\mathcal{A}}_{\Theta_\psi|\Theta_\omega}(\mathbb{I} \otimes a)\,|\Theta_\omega\rangle = (\mathbb{I} \otimes a^\dagger)\,|\Theta_\psi\rangle \ . \end{aligned} \tag{6.148}$$

The domain of $S^{\mathcal{F}}$ is $|q0\rangle \otimes \mathcal{H}_0$, however, this operator is zero except for the subspace $|00\rangle \otimes \mathcal{H}_0$ that is the domain of $S^{\mathcal{A}}$. On the common domain the two relative modular operators agree. Since the zero vector is not in the domain of $(S^{\mathcal{F}})^\dagger$ the relative modular operators $\Delta^{\mathcal{F}}$ and

$\Delta^{\mathcal{A}}$ are the same map from $|00\rangle \otimes \mathcal{H}_0 \to |00\rangle \otimes \mathcal{H}_0$. In fact, the purification of any state that is invariant under the conditional expectation $\mathcal{E}$ has this property. The distinguishability of invariant states does not change as the restriction map $\mathcal{E}$. Consider two invariant states:

$$\begin{aligned}\omega &= \sum_q p_q \, |q\rangle \langle q| \otimes \omega_q \\ \psi &= \sum_q p'_q \, |q\rangle \langle q| \otimes \psi_q\end{aligned} \tag{6.149}$$

and their corresponding purifications

$$\begin{aligned}|\Theta_\omega\rangle &= \sum_q \sqrt{p_q} \, |q, -q\rangle \otimes |\Omega_q\rangle \\ |\Theta_\psi\rangle &= \sum_q \sqrt{p'_q} \, |q, -q\rangle \otimes |\Psi_q\rangle\end{aligned} \tag{6.150}$$

Their relative modular operator is

$$\Delta^{\mathcal{F}}_{\Theta_\psi|\Theta_\omega} = \Delta^{\mathcal{A}}_{\Theta_\psi|\Theta_\omega} = \sum_{l,m} \frac{p'_l}{p_m} \, |l, -m\rangle \langle l, -m| \otimes \Delta_{\Psi_l|\Omega_m} \tag{6.151}$$

and their relative entropy is

$$S(\Theta_\omega \| \Theta_\psi) = -\langle \Theta_\omega | \log \Delta_{\Theta_\psi|\Theta_\omega} \Theta_\omega \rangle = H(p\|p') + \sum_l p_l S(\Psi_l \| \Omega_l) \tag{6.152}$$

as expected from the equation (6.41).

The algebra of QFT does not admit a tensor factorization when the regions A_1 and A_2 touch, however, as we saw in the presence of a symmetry the extended algebra factors the charged excitations. The local algebra of any quantum field has a symmetry group $\mathbb{R}$ associated with the modular flow. The modular flow is an outer automorphism similar to the twist group. Similar to the case of twist that was not part of the algebra due to the infinities near the entangling surface, the modular Hamiltonian, i.e. $\log \Delta_\Omega$ restricted to A is not part of the algebra because of its discontinuous action at the entangling surface. For instance, in the vacuum QFT and for the Rindler region $|x^1| > |t|$ the modular Hamiltonian is the boost operator $\int_{-\infty}^{\infty} du \, u T_{uu}$ where T_{uu} is the null-null component of the stress tensor. The

half-sided modular Hamiltonian $\int_0^\infty du\, uT_{uu}$ is ill-defined because of its singular behavior at $u = 0$. If we enlarge the local algebra of QFT by the modular group by adding the half-sided modular Hamiltonian to the algebra, every mode that is charged under modular flow factors. The Hilbert space splits into sectors $\mathcal{H}_q$ with projections P_q that project to the subspace with modular frequency q.

The modular group is $\mathbb{R}$ so its dual group is also $\mathbb{R}$ which is non-compact. In the case of vacuum QFT in Rindler space, the centralizer is trivial since there are no local operators that are invariant under boost. This implies that every mode is charged under the modular flow [187]. Enlarging the algebra of QFT by the modular group factors the local algebra of QFT completely: $\mathcal{H} = \oplus_q \mathcal{K}_q^A \otimes \mathcal{K}_q^{A'}$ where q is the modular frequency. The enlarged algebra is type II_∞ and has a trace [188]. Entanglement entropy in the extended Hilbert space is divergent, however the factorization of the Hilbert space resembles the structure of boundary quantum field theory, and the insertion of a resolution of identity that is the center $\oplus_q |q\rangle \langle q|$ in the algebra.[29]

6.1.22 Conclusions

In this work, we generalized the definition of entanglement entropy to the cases with no tensor product structure, and used the new definition to define an entanglement measure that captures the contribution of charges to entanglement in quantum systems with symmetries in equation (6.39). The proposed measure is comprised of two relative entropies. One is the relative entropy with respect to the charge neutral operators and the other is the relative entropy due to the charge creation operators. We used representation theory to introduce the charge creation operators called intertwiners and bi-local intertwiners, and wrote down relative entropy that capture their contributions to entanglement. We set up the formalism to compute these measures in QFT using the Tomita-Takesaki modular theory. We highlighted the differences between QFT and lattice models, and discuss an extension of the algebra of QFT that leads to a factorization of the charged modes.

[29]↑We thanks James Sully for pointing out this connection to us.

6.1.23 Appendix: Group and algebra extensions

Group extension: semi-direct product

Given two groups N and H consider the trivial extension that is the Cartesian product group $N \times H$ where elements of group are (n, h) and the multiplication is $(n_1, h_1).(n_2, h_2) = (n_1 n_2, h_1 h_2)$. If H acts on N by an outer automorphism $\phi_h : n \to hnh^{-1}$ with the composition rule $(\phi_{h_1} \circ \phi_{h_2})(n) = \phi_{h_1 h_2}(n)$ we can consider a subgroup of $G = N \rtimes_\phi H \subset N \times H$ called the semi-direct product and has the multiplication rule $(n_1, h_1).(n_2, h_2) = (n_1 \phi_{h_1}(n_2), h_1 h_2)$. The inverse of (n, h) is $(\phi_{h^{-1}}(n^{-1}), h^{-1})$. All we need for the construction of the semi-direct product is the homomorphism $\phi : H \to \mathrm{Aut}(N)$.

In the semi-direct product extension G, N is a normal subgroup and $H = G/N$ is the quotient group. An important example is the Poincare group that is the semi-direct product of translations and the Lorentz group: $\mathbb{R}^{1,d-1} \rtimes O(1, d-1)$. If N is the center of G the semi-direct product is called a central extension. A trivial example of central extension is the direct product group $N \times H$ where N is Abelian. Non-trivial examples comes from the study of the projective representations of a group. Consider a group H, the Abelian group of complex numbers $\mathbb{C}$ and the map $\phi_h(\alpha) = \alpha c(\alpha, h)$ with $c(\alpha, h)$ a complex number. If $c(\alpha, h)c(\beta, h) = c(\alpha\beta, h)$ and $c(\alpha^*, h) = c(\alpha, h)^*$, this map is an outer automorphism of $\mathbb{C}$, and we can construct $\mathbb{C} \rtimes_\phi H$ with the multiplication rule $(\alpha, h_1).(\beta, h_2) = (\alpha\beta c(\beta, h_1), h_1 h_2)$. We need to further check that $\phi_{h_1}(\alpha)\phi_{h_2}(\alpha) = \phi_{h_1 h_2}(\alpha)$ which imposes $c(\alpha, h_1)c(\alpha, h_2) = c(\alpha, h_1 h_2)$.

Lie algebra extension: semi-direct sum

Consider the groups H and N are Lie groups and their corresponding Lie algebras $\mathfrak{h}$ and $\mathfrak{n}$. The map $\phi : H \to \mathrm{Aut}(N)$ induces a map $\psi : \mathfrak{h} \to \mathrm{Aut}(\mathfrak{n})$ defined by the Lie correspondence

$$\psi_{\hat{h}}(\hat{n}) = \frac{d}{dt}\left(\phi_{\mathrm{e}^{t\hat{h}}}(\mathrm{e}^{t\hat{n}})\right)_{t=0} \tag{6.153}$$

where $\hat{n}$ and $\hat{h}$ are elements of the Lie algebra $\mathfrak{n}$ and $\mathfrak{h}$, respectively. We obtain the notion of a semi-direct sum of Lie algebras with the Lie bracket defined using the equation

$$\Big[(\hat{n}_1, \hat{h}_1), (\hat{n}_2, \hat{h}_2)\Big] = ([\hat{n}_1, \hat{n}_2] + \psi_{\hat{h}_1}(\hat{n}_2) - \psi_{\hat{h}_2}(\hat{n}_1), [\hat{h}_1, \hat{h}_2]) \ . \tag{6.154}$$

There is another method to centrally extend Lie algebras. Every linear map $\chi : \mathfrak{h} \times \mathfrak{h} \to \mathbb{C}$ that is anti-symmetric, i.e. $\chi(\hat{h}_1, \hat{h}_2) = -\chi(\hat{h}_2, \hat{h}_1)$, and satisfies the Jacobi identity leads to an extension defined by the Lie bracket

$$\Big[(\alpha, \hat{h}_1), (\beta, \hat{h}_2)\Big] = (\chi(\hat{h}_1, \hat{h}_2), [\hat{h}_1, \hat{h}_2]) \ . \tag{6.155}$$

A finite-dimensional simple Lie algebra has no non-trivial central extensions. To find examples of non-trivial central extension we have to consider infinite-dimensional Lie algebras. As an example, we work out the central extension of the polynomial loop algebra: Kac-Moody algebra. The loop group is defined to be the algebra of smooth G-valued functions on a circle with group multiplication rule. These are loops C on the group G, $C : S^1 \to G$ with $(C_1 C_2)(\theta) = (C_1)(\theta)(C_2)(\theta)$. A loop Lie algebra is the vector space of smooth functions from S^1 to $\mathfrak{g}$ of G.

Consider the tensor product space $\mathfrak{g} \otimes C^\infty(S^1)$, where $\mathfrak{g}$ is a finite dimensional simple Lie algebra and $C^\infty(S^1)$ is the algebra of smooth functions on S^1. This vector space is a Lie algebra with the bracket defined by

$$[\hat{g}_1 \otimes f_1, \hat{g}_2 \otimes f_2] = [\hat{g}_1, \hat{g}_2] \otimes f_1 f_2 \ , \quad (\hat{g}_1, \hat{g}_2 \in \mathfrak{g}). \tag{6.156}$$

Importantly, this space is not a direct product of the two spaces $\mathfrak{g}$ and $C^\infty(S^1)$ due to the smoothness condition of functions. Instead, it should be thought of as the Lie algebra of smooth $\mathfrak{g}$-valued functions of S^1. The Fourier transform on S^1 gives the basis $\hat{g} \otimes \mathrm{e}^{in\theta}$ where θ is the angle on S^1 and $n \in \mathbb{Z}$. The Lie algebra generated by such generators is the polynomial loop algebra. Another way to think about this algebra is in terms of the algebra of Laurent polynomials $\sum_{n \in \mathbb{Z}} f_n z^n$ with only finitely many non-zero f_n and the standard multiplication

and addition. Then, the algebra of G-valued functions on S^1 is the Lie algebra of formal sums $\sum_{n\in\mathbb{Z}} z^n \otimes \hat{g}_n$ with the Lie bracket

$$[z^n \otimes \hat{g}_1, z^m \otimes \hat{g}_2] = z^{n+m} \otimes [\hat{g}_1, \hat{g}_2] \ . \tag{6.157}$$

The generators of the Lie algebra J_a satisfy

$$[z^n \otimes \hat{J}_a, z^m \otimes \hat{J}_b] = \sum_c C^c_{ab} z^{n+m} \otimes \hat{J}_c \ . \tag{6.158}$$

where C^c_{ab} denotes the structure constants of the Lie algebra $\mathfrak{g}$. The central extension of this algebra is $\mathfrak{g} \otimes C^\infty(S^1) \oplus \mathbb{C}$

$$[(\alpha, z^n \otimes \hat{J}_a), (\beta, z^m \otimes \hat{J}_b)] = (k\, n\, K(\hat{J}_a, \hat{J}_b)\delta_{n+m,0}, \sum_k C^c_{ab} z^{n+m} \otimes \hat{J}_c) \ . \tag{6.159}$$

where $K(J_a, J_b)$ is the Killing form on $\mathfrak{g}$ and k is the central charge. This is an affine Lie algebra.

von Neumann algebra extension: Crossed product

Groups can act on von Neumann algebras and one can extend an algebra $\mathcal{A}$ by a group G that acts on it as outer automorphisms to obtain a larger algebra called the crossed product $\mathcal{A} \rtimes_\phi G$ [61], [189]. If the action of the G on $\mathcal{A}$ is $\phi_g(a) = a_g = u_g a u_g^{-1}$ with $u_g u_h = u_{gh}$ we add u_g to the set of operators in our algebra and consider the algebra of formal sums $\sum_{g\in G} a_g u_g$ with $a_g \in \mathcal{A}$. If $\mathcal{A}$ acts on the Hilbert space $\mathcal{H}$ and $L^2(G)$ is the Hilbert space of square-integrable functions of the group the crossed product algebra acts on $\mathcal{H} \otimes L^2(G)$; that is the space of square-integrable $\mathcal{H}$-valued functions of G. Vectors of this Hilbert space are $|\Psi\rangle = \sum_{g\in G} c_g |\Psi; g\rangle$ and the inner product is

$$\langle \Psi|\Phi\rangle = \sum_{g\in G} c_g^* b_g \langle \Psi; g|\Phi; g\rangle \ . \tag{6.160}$$

The multiplication rules are $u_h |\Psi; g\rangle = |\Psi; hg\rangle$ and $a_g |\Psi; g\rangle = |a\Psi; g\rangle$.

Dual group and non-Abelian Fourier transform

Consider a locally compact Abelian group G. The characters of G are linear maps from G to complex numbers. For instance, for the group $U(1)$ of rotations on a circle the characters are $\chi(U_\theta) = \mathrm{e}^{\mathrm{i}\theta\chi}$ with $\theta \in [0, 2\boldsymbol{\pi})$. The point-wise multiplication $(\chi_1\chi_2)(U_\theta) = \chi_1(U_\theta)\chi_2(U_\theta)$ gives the characters the structure of a group called the *dual group* of G that we denote by $\hat{G}$. The dual group allows us to define a Fourier transform for functions on the group G:

$$\hat{f}(\chi) = \sum_{g\in G} \chi^{-1}(g) f(g) \ . \tag{6.161}$$

If the group is finite the dual Fourier transform is

$$f(g) = \frac{1}{|G|} \sum_{\chi\in\hat{G}} \chi(g)\hat{f}(\chi) \ . \tag{6.162}$$

To generalize Fourier transform to non-Abelian finite groups G we replace the character of group with its irreducible representations $\rho_r(g)$:

$$\hat{f}(\rho_r) = \sum_{g\in G} \rho_r(g) f(g) \ . \tag{6.163}$$

If $\rho_r(g)$ is represented by a $d_r \times d_r$ matrix then $\hat{f}(g)$ is also a matrix of same dimensions. The inverse Fourier transform is

$$f(g) = \frac{1}{|G|} \sum_r d_r \mathrm{tr}\left(\hat{f}(\rho_r)\rho_r(g^{-1})\right) \tag{6.164}$$

where the sum is over irreducible representations ρ_r of group G and we have used the fact that $\frac{1}{|G|}\sum_r d_r \mathrm{tr}\left(\rho_r(g)\right) = \delta_{g\mathbb{I}}$ [190]. The analog of the multiplication of characters in the non-commutative case is the tensor product of irreducible representations which does not form a group, because the tensor product of irreducible representations is not in general irreducible.

6.2 Monotonic multi-state quantum f-divergences

6.2.1 Introduction

Motivation: In classical physics, the state of a system is a probability distribution $p(x)$ over the configuration space X. To distinguish different states one needs to compare probability distributions. The Kullback-Leibler divergence

$$D_{KL}(\{q\}\|\{p\}) = \sum_{x\in X} q(x)\log(q(x)/p(x)) \tag{6.165}$$

is a distinguishability measure that plays a central role in information theory and has an interpretation in terms of the thermodynamic free energy difference of the state $\{q\}$ from the equilibrium distribution $\{p\}$ [191]. It is non-negative, non-degenerate[30] and monotonically non-increasing under the action of a classical channel.[31] The thermodynamic interpretation of relative entropy explains why this measure of distinguishability is not symmetric under the exchange of $\{q\}$ and $\{p\}$. The monotonicity under classical channels is an essential property that any reasonable distinguishability measure should satisfy.[32] We say a quantity satisfies the data processing inequality if it is monotonic under the action of a channel. One can consider symmetric distinguishability measures such as the log-fidelity

$$D_{1/2}(\{q\},\{p\}) = -2\log\sum_{x\in X}\sqrt{q(x)}\sqrt{p(x)} \tag{6.166}$$

or, in general, a one-parameter family $\theta\in(0,1)$ of non-negative, non-degenerate measures

$$D_\theta(\{q\}\|\{p\}) = \frac{1}{(\theta-1)}\log\sum_{x\in X} q(x)^\theta p(x)^{1-\theta} \tag{6.167}$$

[30]↑It is zero if and only if the probability measures are the same.
[31]↑A classical channel a stochastic map $T: X\to Y$ with $\sum_{y\in Y}T(y|x)=1$. In other words, a classical channel is a conditional probability distribution.
[32]↑Intuitively, this is because either the channel is noiseless in which case the distinguishability remains the same, or it is noisy and the distinguishability decreases.

that interpolate between $D_{KL}(\{q\}\|\{p\})$ at $\theta = 1$ and log-fidelity at $\theta = 1/2$ and satisfy the data processing inequality. It is tempting to generalize to a multi-state measure

$$D_{\theta_1,\cdots,\theta_n}(\{p_1\},\cdots,\{p_n\}) = \frac{-1}{(1-\theta_1)\cdots(1-\theta_n)} \log\left(\sum_{x\in X} p_1(x)^{\theta_1}\cdots p_n(x)^{\theta_n}\right)$$
$$\theta_1 + \cdots + \theta_n = 1 \tag{6.168}$$

as a functional that interpolates between $D_{KL}(\{p_{\rm i}\}\|\{p_{\rm j}\})$ and their corresponding log-fidelities for different i and j. Note that the parameters $(\theta_1,\cdots,\theta_n)$ can be thought of as a probability distribution. We are not aware of any arguments in the literature that proves that the measure above satisfies the data processing inequality. In this work, we write down a quantum generalization of the above measure and prove that it satisfies the data processing inequality.

In quantum mechanics, the state of a system is a completely positive (CP) map from the algebra of observables to complex numbers $\omega : \mathcal{A} \to \mathbb{C}$ with $\omega(1) = 1$. If the observable algebra is the algebra of $d \times d$ complex matrices a state is a density matrix (positive operator with unit trace): $\omega > 0$ with $\mathrm{tr}(\omega) = 1$. The quantum relative entropy

$$S(\psi\|\omega) = \mathrm{tr}(\psi\log\psi) - \mathrm{tr}(\psi\log\omega) \tag{6.169}$$

is a measure of distinguishability of the density matrix ψ from ω. It is non-negative, non-degenerate and has an operational interpretation in asymptotic asymmetric hypothesis testing [192]. One can define a symmetric distinguishability measure called log-fidelity:

$$D_{1/2}(\psi\|\omega) = -2\log\mathrm{tr}\sqrt{\omega^{1/2}\psi\omega^{1/2}}\ . \tag{6.170}$$

Since in quantum mechanics the density matrices need not commute there can be many non-commutative versions of the Rényi divergences in (6.167) that interpolate between the

relative entropy $S(\psi\|\omega)$ and log-fidelity. Two important families of measures of this kind are the *Petz Rényi divergences* and the *sandwiched Rényi divergences*, respectively

$$\begin{aligned} D_\theta(\psi\|\omega) &= \frac{1}{\theta-1}\log \mathrm{tr}\left(\psi^\theta \omega^{1-\theta}\right) \\ S_\theta(\psi\|\omega) &= \frac{1}{\theta-1}\log \mathrm{tr}\left(\left(\omega^{\frac{1-\theta}{2\theta}}\psi\omega^{\frac{1-\theta}{2\theta}}\right)^\theta\right) \ . \end{aligned} \tag{6.171}$$

These two families are distinguished because they satisfy the data processing inequality. They have operational interpretations in hypothesis testing [193]. A larger two-parameter family of Rényi divergences called (α, z)-*Rényi relative entropy* interpolates between the two families [146]. In our notation, we call them the (θ, r)-Rényi divergences

$$S_{\theta,r}(\psi\|\omega) = \frac{1}{\theta-1}\log \mathrm{tr}\left[\left(\omega^{\frac{1-\theta}{2r}}\psi^{\frac{\theta}{r}}\omega^{\frac{1-\theta}{2r}}\right)^r\right] \ . \tag{6.172}$$

In fact, they were introduced earlier by [194] as entropic measures in out-of-equilibrium statistical mechanics. They satisfy the data processing inequality in the range of (θ, r) specified by [195].

The generalization of hypothesis testing to a multi-state setup is often called quantum state discrimination. In the asymmetric case, we are given some state and the task is to identify whether the state is ω or any of the alternative hypotheses $\psi_1, \cdots, \psi_k$ by performing measurements on infinite number of copies of ω. The distinguishability measure with a natural operational interpretation in this case is [196]

$$\min_{\psi\in K} S(\psi\|\omega) \qquad K = \{\psi_1, \cdots, \psi_k\} \ . \tag{6.173}$$

Motivated by quantum state discrimination, in this work, we introduce a large family of multi-state quantum Rényi divergences that interpolate between various $S(\psi_{\mathrm{i}}\|\omega)$ and satisfy the data processing inequality. We generalize our measures to multi-state quantum f-divergences.

Method: We employ three main tools to construct the multi-state Rényi divergences and prove their monotonicity. The first tool is the Araki-Masuda non-commutative L^p_ω

spaces [154] that we review in section 6.2.2 and 6.2.3. In particular, we use the Riesz-Thorin theorem to prove that a contraction operator F does not increase the L^p_ω norm of the vectors. The second tool is the monotonicity of the relative modular operator in the Tomita-Takesaki modular theory. A quantum channel Φ^* corresponds to a contraction F in the GNS Hilbert space. The relative modular operator satisfies the inequality

$$F^\dagger \Delta_{\psi|\omega} F \leq \Delta_{\Phi^*(\psi)|\Phi^*(\omega)} \ . \tag{6.174}$$

The third tool is the Kubo-Ando operator mean for positive operators X and Y:

$$X \sharp_f Y = X^{1/2} f(X^{-1/2} Y X^{-1/2}) X^{1/2} \tag{6.175}$$

and an operator monotone function f with $f(1) = 1$. The Kubo-Ando mean has the property that if $X_1 \leq Y_1$ and $X_2 \leq Y_2$ then

$$X_1 \sharp_f X_2 \leq Y_1 \sharp_f Y_2 \ . \tag{6.176}$$

This allows us to construct multi-state operator monotonicity inequalities of the type

$$F^\dagger (\Delta_{\psi_1|\omega} \sharp_f \Delta_{\psi_2|\omega}) F \leq (\Delta_{\Phi^*(\psi_1)|\Phi^*(\omega)} \sharp_f \Delta_{\Phi^*(\psi_2)|\Phi^*(\omega)}) \ . \tag{6.177}$$

The L^p_ω-norm of the vector $\left(\Delta_{\psi_1|\omega} \sharp_f \Delta_{\psi_2|\omega}\right)^{1/2} |\omega^{1/2}\rangle$ is the building block of the class of multi-state Rényi divergences we construct in this work.

Summary of results: In the case of two states in (6.253), we write the (θ, r)-Rényi divergences as the (r, ω)-norm of a vector in the L^p_ω spaces.[33] We generalize them to two-state divergences in (6.259). In theorem 6.2.1 we use the monotonicity of the relative modular operator and the Riesz-Thorin theorem (see appendix 6.2.13) from the complex interpolation theory to prove that these two-state measures satisfy the data processing inequality in the range $r \geq 1$. [34]

[33] ↑A similar expression appears in [194].
[34] ↑The monotonicity of (θ, r)-Rényi divergences was shown using different methods by [197].

Section 6.2.6 generalizes the discussion to multiple states. First, in section 6.2.7, we use the complex interpolation theory to prove a generalization of the Hölder inequality to von Neumann algebras (see theorem 6.2.2). This section follows the arguments by [154], and can be skipped by the readers who are only interested in the multi-state Rényi measures. Then, in section 6.2.8, we use the Kubo-Ando geometric mean to introduce the three-state f-divergence in (6.303) and prove that they are monotonically non-increasing under quantum channels in theorem 6.2.3. This measure depends on an arbitrary operator monotone function f with $f(1) = 1$, the parameters θ_1, θ_2 with $0 \leq \theta_1 + \theta_2 \leq 1$, $r \geq 1/2$ and three states ψ_1, ψ_2 and ω. Specializing to the case $f(x) = x^\alpha$ with $\alpha \in [0, 1]$, in matrix algebras we obtain the three-state Rényi divergences in (6.307).[35] In a special case, this measure reduces to the Rényi measures in [198], [199]:

$$\bar{S}_\theta(\psi\|\omega) = \frac{1}{\theta - 1} \log \mathrm{tr} \left(\omega^{1/2} \left(\omega^{-1/2} \psi \omega^{-1/2} \right)^\theta \omega^{1/2} \right) . \tag{6.178}$$

We write down an n-state f-divergences in equation (6.317), multi-state Rényi divergences in (6.325) and prove that they satisfy the data processing inequality. In matrix algebras, this multi-density matrix measure is (6.326).

In section 6.2.10, we discuss our construction in arbitrary von Neumann algebras, focusing on the case where a trace does not exist. This is important for the applications of this work to infinite dimensional quantum systems such as the algebra of local observables in Poincare-invariant quantum field theory. In section 6.2.11, we conjecture that similar to the Petz divergences and the sandwiched Rényi divergences, the multi-state Rényi divergences in section 6.2.6 have operational interpretations in terms of the optimal error probabilities in various quantum state discrimination setups.

For the marginals of multi-partite systems, one can introduce the so-called swiveled Rényi measures. **Wilde_2015**, [200], [201] In the case all a_S in swiveled measures are non-negative they can be understood as a special case of the multi-state measures introduced in this work.

[35]↑We prove the monotonicity only in the range $r \geq 1$.

6.2.2 Operator L^p spaces

This section reviews the construction of the operator L^p spaces in finite dimensional matrix algebras. The observable algebra of a d-level quantum system is the algebra $\mathcal{A}$ of $d \times d$ complex matrices. The linear map

$$\begin{aligned} \mathcal{A} \ni a \to |a\rangle = (a \otimes \mathcal{I}) \, |\mathrm{e}\rangle \\ |\mathrm{e}\rangle = \sum_k |k, k\rangle \end{aligned} \tag{6.179}$$

represents the algebra on a Hilbert space $\mathcal{H}_\mathrm{e}$ with the inner product

$$\langle a_1 | a_2 \rangle = \mathrm{tr}(a_1^\dagger a_2) \ . \tag{6.180}$$

We use the simplified notation

$$\begin{aligned} a \, |\mathrm{e}\rangle \equiv (a \otimes \mathcal{I}) \, |\mathrm{e}\rangle \\ a' \, |\mathrm{e}\rangle \equiv (\mathcal{I} \otimes a') \, |\mathrm{e}\rangle \end{aligned} \tag{6.181}$$

and refer to the algebra of operators $a' \equiv (\mathcal{I} \otimes a')$ as $\mathcal{A}'$, the commutant of $\mathcal{A}$. The Hilbert space norm of a vector is

$$\| \, |a\rangle \, \| \equiv \|a\|_2 = \mathrm{tr}(a^\dagger a)^{1/2} \tag{6.182}$$

and its ∞-norm (operator norm) is

$$\| \, |a\rangle \, \|_\infty \equiv \|a\|_\infty = \sup_{\| |\chi\rangle \| = \| |\Psi\rangle \| = 1} \langle \chi | a \Psi \rangle \ . \tag{6.183}$$

The advantage of the Hilbert space representation $\mathcal{H}_\mathrm{e}$ is that one can think of superoperators $\Phi : \mathcal{A} \to \mathcal{A}$ as linear operators $F : \mathcal{H}_\mathrm{e} \to \mathcal{H}_\mathrm{e}$:

$$F \, |a\rangle = \Phi(a) \, |\mathrm{e}\rangle \ . \tag{6.184}$$

Linear maps Φ that are completely positive (CP) and unital are specially important in physics. In the Hilbert space, they are represented by operators that are *contractions*, i.e. $\|F\|_\infty \leq 1$.[36] It is clear that F can never increase the 2-norm of vectors

$$\|F\,|a\rangle\,\| \leq \|\,|a\rangle\,\| \ . \tag{6.186}$$

It cannot increase the operator norm either because

$$\begin{aligned} \|F\,|a\rangle\,\|_\infty \quad &= \sup_{\||\Psi\rangle\|=\||\chi\rangle\|=1} |\,\langle\Psi|F(a\otimes 1)|\chi\rangle\,| \\ &\leq \|F\|_\infty \|(a\otimes 1)\|_\infty \leq \|a\|_\infty \ . \end{aligned} \tag{6.187}$$

The 2-norm and the ∞-norm are special cases of the p-norms (Schatten norms) defined by

$$\forall p \in [1,\infty] : \qquad \|a\|_p = \mathrm{tr}(a_+^p)^{1/p} \tag{6.188}$$

where $a = a_+ u$ is the left polar decomposition of a in terms of the positive semi-definite operator a_+ and unitary u. For $p \in (0,1)$, they are quasi-norms because they no longer satisfy the triangle inequality $\|a_1 + a_2\|_p \nleq \|a_1\|_p + \|a_2\|_p$. The Hilbert space norm and the operator norm correspond to $p = 2$ and $p = \infty$, respectively. The map between the operators a and the vectors $|a\rangle$ in matrix algebras is one-to-one.

Definition 6.2.1. *We define the p-norm of a vector in the Hilbert space to be the p-norm of the operator that creates it:*

$$\|\,|a\rangle\,\|_p \equiv \|a\|_p \ . \tag{6.189}$$

[36]↑Consider an unital CP map $\Phi : \mathcal{A} \to \mathcal{B}$. Using the Stinespring dilation theorem, the map decomposes as $\Phi(a) = W^\dagger a W$ where W is an isometry since Φ is unital. The action of the map on the GNS Hilbert space is given by

$$\Phi(a)\,|\Omega_B\rangle = W^\dagger a\,|\Omega_A\rangle \tag{6.185}$$

where W satisfies $W\,|\Omega_B\rangle = |\Omega_A\rangle$. The GNS operator F corresponding to Φ is defined by $\Phi(a)\,|\Omega_B\rangle = Fa\,|\Omega_A\rangle$. Since $\mathcal{A}\,|\Omega_A\rangle$ is dense in $\mathcal{H}_A$, the corresponding GNS operator is a co-isometry $F = W^\dagger$ and a contraction.

Note that since $\|a\|_p = \|uav\|_p$ *for any unitary* u, v *the p-norm of a vector satisfies*

$$\|uu' \left|a\right\rangle\|_p = \|\left|a\right\rangle\|_p \tag{6.190}$$

where $u \in \mathcal{A}$ *and* $u' \in \mathcal{A}'$ *are unitaries.*

Definition 6.2.2. *We define the superoperator norms as*[37]

$$\|\Phi\|_{(p_0 \to p_1)} \equiv \sup_{a \in \mathcal{A}} \frac{\|\Phi(a)\|_{p_1}}{\|a\|_{p_0}} \tag{6.191}$$

and the norm for their corresponding operators as

$$\begin{aligned} F\left|a\right\rangle &= \left|\Phi(a)\right\rangle \\ \|F\|_{(p_0 \to p_1)} &\equiv \|\Phi\|_{(p_0 \to p_1)} \ . \end{aligned} \tag{6.192}$$

A complete normed vector space is called a Banach space. Since the Hilbert space norm is complete with respect to the 2-norm

$$\begin{aligned} \langle a_1 | a_2 \rangle &= \mathrm{tr}(a_1^\dagger a_2) \\ \langle a | a \rangle &= \|a\|_2^2, \end{aligned} \tag{6.193}$$

we sometimes refer to the Hilbert space $\mathcal{H}_\mathrm{e}$ as the L^2 Banach space, or the L^2 space in short. By analogy, we call the algebra $\mathcal{A}$ with the operator norm the L^∞ space.[38] The representation $a \to \left|a\right\rangle$ is then a linear map from $L^\infty \to L^2$. We could also define the linear map $a \to \mathrm{e}_a = \left|a\right\rangle \left\langle \mathrm{e}\right|$ that sends the algebra to a linear space of operators in $B(\mathcal{H}_\mathrm{e})$ that we denote by $\mathcal{A}_*$ and call the *predual* of $\mathcal{A}$. The subspace of operators $\left|a_+\right\rangle \left\langle \mathrm{e}\right|$ is in one-to-one correspondence with the subspace of unnormalized pure density matrices $\left|a_+^{1/2}\right\rangle \left\langle a_+^{1/2}\right|$ of the algebra $\mathcal{A} \otimes \mathcal{A}'$. The predual $\mathcal{A}_*$ equipped with the 1-norm $\mathrm{tr}((\mathrm{e}_a)_+)$ is called the L^1 space. Since the maps $a \to \left|a\right\rangle$ and $a \to \mathrm{e}_a$ are bijections in matrix algebras we can think of the L^1, L^2 and L^∞ spaces as the same space with different norms. As the dimension of algebra

[37] ↑Note that, by definition, $\|T\|_\infty = \|T\|_{(2 \to 2)}$.
[38] ↑Note that the algebra itself is a linear vector space.

goes to infinity an operator with finite 2-norm has finite ∞-norm but not necessarily a finite 1-norm. So we have the hierarchy $L^1 \subseteq L^2 \subseteq L^\infty$.

Our Hilbert space inner product is a map from $L^2 \times L^2 \to \mathbb{C}$ that is anti-linear in the first variable. It could alternatively be interpreted as a map from $L^1 \times L^\infty \to \mathbb{C}$:

$$\langle a|b\rangle = \mathrm{tr}(a^\dagger \mathrm{e}_b) \tag{6.194}$$

where $\mathcal{E}_b \in L^1$. An important property of an inner product is the Cauchy-Schwarz inequality:

$$|\langle a|b\rangle|^2 \leq \langle a|a\rangle \langle b|b\rangle \ . \tag{6.195}$$

The Cauchy-Schwarz inequality is saturated when $|a\rangle$ and $|b\rangle$ are parallel. This allows us to write

$$\| |b\rangle \| = \sup_{\||a\rangle\|=1} |\langle a|b\rangle| \ . \tag{6.196}$$

Similarly, we can use (6.194) to write the operator norm $\|b\|_\infty$ as

$$\|b\|_\infty = \sup_{\mathrm{tr}((\mathrm{e}_a)_+)=1} |\mathrm{tr}(\mathrm{e}_a b)| \ . \tag{6.197}$$

We say the space L^∞ is dual to L^1.

The generalization of the Cauchy-Schwarz inequality to the L^p spaces is called the operator Hölder inequality

$$\forall p \in [1,\infty]: \qquad \|a^\dagger b\|_1 \leq \|a\|_q \|b\|_p \tag{6.198}$$

and $1/p + 1/q = 1$. More generally, if $1/p_0 + 1/p_1 = 1/r$ with $r > 1$ the operator Hölder inequality says

$$\|a^\dagger b\|_r \leq \|a\|_{p_0} \|b\|_{p_1} \ . \tag{6.199}$$

In the range $p_0 \in (0,1)$, the parameter p_1 is negative and we have a reverse Hölder inequality

$$\forall p_0 \in (0,1): \qquad \|a\|_{p_0}\|b\|_{p_1} \leq \|a^\dagger b\|_r \ . \tag{6.200}$$

The reverse Hölder inequality follows from the Hölder inequality and the property $\|a^{-1}\|_{-p} = \|a\|_p^{-1}$ [202]. We will prove the generalization of the operator Hölder inequality in an arbitrary von Neumann algebra in section 6.2.7.

We can realize the p-norm of the vector $|a\rangle \in \mathcal{H}_{\rm e}$ as an inner product between $|a\rangle$ and a vector $|x_0\rangle$ in the Hilbert space $\mathcal{H}_{\rm e}$:

$$\| \, |a\rangle \, \|_p = \mathrm{tr}(a_+^p)^{\frac{1}{p}} = \mathrm{tr}(a_+^p)^{\frac{1}{p}-1} \langle a_+^{p-1}|a_+\rangle = \frac{\langle a_+^{p-1}|a_+\rangle}{\| \, |a_+^{p-1}\rangle \, \|_q} = \langle x_0|a_+\rangle \ . \tag{6.201}$$

The vector $|x_0\rangle \sim |a_+^{p-1}\rangle$ is normalized to have $\| \, |x_0\rangle \, \|_q = 1$. It follows from the Hölder inequality that

$$|\langle b|a_+\rangle| \leq \|b^\dagger a_+\|_1 \leq \|a\|_p\|b\|_q \ . \tag{6.202}$$

We can absorb the unitaries in the polar decomposition of a in b to write

$$|\langle b|a\rangle| \leq \|a\|_p\|b\|_q \ . \tag{6.203}$$

The p-norm is the maximum overlap between $|a\rangle$ and the vectors in the Hilbert space that are normalized to have unit q-norm:

$$\forall p \in [1,\infty] \qquad \|a\|_p = \sup_{\|x\|_q=1} |\langle x|a\rangle| \ . \tag{6.204}$$

Similarly, from the reverse Hölder inequality in (6.200) we have

$$\forall p \in (0,1) \qquad \|a\|_p = \inf_{\|x\|_q=1} |\langle x|a\rangle| \ . \tag{6.205}$$

The equations above generalize (6.196) and (6.197) to arbitrary p. The duality between L^1 and L^∞ is a special case of the duality between L^p and L^q. That is why the parameter q is called the Hölder dual of p.

The vector $|a\rangle$ is a purification of the unnormalized density matrix $aa^\dagger = a_+^2$ of the algebra:

$$\langle a|b|a\rangle = \mathrm{tr}(baa^\dagger) \ . \tag{6.206}$$

All vectors $|a_+u\rangle$ purify the same state a_+^2. To make the purification unique, we define an anti-linear swap map J_e in the basis of $|k\rangle$ in the definition of the vector $|\mathrm{e}\rangle$:

$$J_\mathrm{e}\,|k,k'\rangle = |k',k\rangle \ . \tag{6.207}$$

The map $\mathcal{J}_\mathrm{e}(a) = J_\mathrm{e}aJ_\mathrm{e}$ is an anti-unitary from $\mathcal{A}$ to the commutant algebra $\mathcal{A}'$ that acts as

$$J_\mathrm{e}(a\otimes\mathcal{I})J_\mathrm{e} = (\mathcal{I}\otimes(a^\dagger)^T) \tag{6.208}$$

and the transpose matrix a^T defined in the $\{|k\rangle\}$ basis satisfies the equation

$$(a\otimes\mathcal{I})\,|\mathrm{e}\rangle = (\mathcal{I}\otimes a^T)\,|\mathrm{e}\rangle \ . \tag{6.209}$$

The only purification of the unnormalized density matrix $aa^\dagger$ that is invariant under J_e is

$$|a_+\rangle = (a_+^{1/2}\otimes(a_+^{1/2})^T)\,|\mathrm{e}\rangle \ . \tag{6.210}$$

The set of such vectors is called the natural cone in $\mathcal{H}_\mathrm{e}$ that we denote by $P_\mathrm{e}^\natural$. Vectors in the natural cone are in one-to-one correspondence with the unnormalized density matrices $aa^\dagger = a_+^2$.

To understand the L^p spaces better we define the relative modular operators corresponding to algebra $\mathcal{A}$:

$$\Delta_{\psi|\omega} \equiv \psi \otimes \omega^{-1} \ . \tag{6.211}$$

The vector $|\mathrm{e}\rangle$ reduced to the algebras $\mathcal{A}$ and $\mathcal{A}'$ gives the identity operator as an unnormalized state. We use the notation $\Delta_{\omega|\mathrm{e}} = \omega \otimes \mathcal{I}$. The superoperator on $\mathcal{A}$ that correspond to the relative modular operator is

$$\mathcal{D}_{\psi|\omega}(a) = \psi a \omega^{-1} \in \mathcal{A} \ . \tag{6.212}$$

To every density matrix ω we can associate an operator $|\omega\rangle\langle \mathrm{e}| \in L^1$ with unit 1-norm and a vector in L^p

$$|\omega^{1/p}\rangle = \Delta^{1/p}_{\omega|\mathrm{e}} |\mathrm{e}\rangle = \Delta^{1/p-1/2}_{\omega|\mathrm{e}} |\omega^{1/2}\rangle \ . \tag{6.213}$$

with unit p-norm. We can think of the L^p space as the space of vector $u|\omega^{1/p}\rangle$ for arbitrary ω and unitary u.

We use the Hölder inequality to write the p-norm of a vector as

$$\begin{aligned} \| \, |a\rangle \, \|^2_{2p} = \|aa^\dagger\|_p \quad &= \sup_{\||\psi^{1/2}\rangle\|_q=1} |\, \langle \psi^{1/2}|aa^\dagger\rangle \,| = \sup_{\||\omega^{1/2}\rangle\|=1} |\, \langle \mathrm{e}|\Delta^{1/q}_{\omega|\mathrm{e}}|aa^\dagger\rangle \,| \\ &= \sup_{\||\omega^{1/2}\rangle\|=1} |\, \langle a|\Delta^{1/q}_{\omega|\mathrm{e}}|a\rangle \,| = \sup_{\||\omega^{1/2}\rangle\|=1} \|\Delta^{\frac{1}{2}-\frac{1}{2p}}_{\omega|\mathrm{e}} \, |a\rangle \, \|^2 \ . \end{aligned} \tag{6.214}$$

Above we have used the fact that any vector in the natural cone $|\psi^{1/2}\rangle \in L^q$ can be written as $\Delta^{1/q}_{\omega|\mathrm{e}} |\mathrm{e}\rangle$.[39] After a change of variables from $2p \to p$ we have

$$\| \, |a\rangle \, \|_p = \sup_{\||\omega^{1/2}\rangle\|=1} \|\Delta^{\frac{1}{2}-\frac{1}{p}}_{\omega|\mathrm{e}} \, |a\rangle \, \| \qquad \forall p \in [2, \infty] \ . \tag{6.215}$$

[39] ↑Since $|aa^\dagger\rangle$ is in the natural cone it follows from (6.201) that the vector that saturates the Holder inequality is also in the natural cone. Therefore, in the definition of the q-norm in (6.204) for $|aa^\dagger\rangle$ we can restrict the supremum to the vectors $|\psi^{1/2}\rangle$ in the natural cone.

We remind the reader that the norm of the vectors in the Hilbert space on the right-hand-side of the equations above is the L^2 norm. Similarly, repeating (6.215) for the range $p \in (0,1)$ using (6.205) we obtain

$$\| \, |a\rangle \, \|_p = \inf_{\||\omega^{1/2}\rangle\|=1} \|\Delta_{\omega|\mathrm{e}}^{\frac{1}{2}-\frac{1}{p}} \, |a\rangle \, \| \qquad \forall p \in [1,2) \ . \tag{6.216}$$

The Rényi entropy of a normalized density matrix ω on $\mathcal{A}$ can be written in terms of the $2p$-norm of the vector $|\omega^{1/2}\rangle \in \mathcal{H}_{\mathrm{e}}$:

$$S_p(\omega) \equiv \frac{2p}{1-p} \log \| \, |\omega^{1/2}\rangle \, \|_{2p} = \frac{p}{1-p} \log \|\omega\|_p = \frac{1}{1-p} \log \mathrm{tr}(\omega^p) \ . \tag{6.217}$$

Since p-norms of the vector $u' \, |\omega^{1/2}\rangle$ is independent of the unitary u' the definition above defines the Rényi entropy for the reduced state ω on $\mathcal{A}$ for any vector $|\Omega\rangle \in \mathcal{H}_{\mathrm{e}}$

$$S_p(\omega) = \frac{2p}{1-p} \log \| \, |\Omega\rangle \, \|_{2p} \ . \tag{6.218}$$

The normalized vector $d^{-1/2} \, |\mathrm{e}\rangle$ corresponds to the maximally mixed density matrix and maximizes the Rényi entropy. In the limit $p \to 1$, we obtain the von Neumann entropy:

$$S(\omega) = -2 \lim_{p \to 1} \prime_p \| \, |\omega^{1/2}\rangle \, \|_{2p} \ . \tag{6.219}$$

6.2.3 Operator L^p_ω spaces

The construction of the L^p spaces in the last section used the unnormalized vector $|\mathrm{e}\rangle$. In an infinite dimensional algebra, this vector is not normalizable. The first step in generalizing the discussion of the last section to infinite dimensions is to replace the maximally mixed state with an arbitrary density matrix ω:

$$\omega = \sum_k p_k \, |k\rangle \, \langle k| \tag{6.220}$$

that for simplicity we will assume to be full rank. Not every infinite-dimensional algebra admits density matrices, however as we discuss in section 6.2.10, the construction presented

in this section generalizes to the algebras with no density matrices. We remind the reader that in our notation ω and ψ are the reduced density matrices on $\mathcal{A}$ corresponding to the vectors $|\Omega\rangle$ and $|\Psi\rangle$ in the Hilbert space $\mathcal{H}_{\rm e} = \mathcal{H}_A \otimes \mathcal{H}_{A'}$.

Definition 6.2.3. *We generalize the definition of p-norm in (6.188) to define a (p,ω)-norm:*

$$\|a\|_{p,\omega} \equiv \|a\,|\omega^{1/p}\rangle\,\|_p = \|a\omega^{1/p}\|_p \ . \tag{6.221}$$

Note that the (p,ω)-norm is no longer invariant under $a \to a^\dagger$.[40] Consider the $*$-representation $\boldsymbol{\pi}(a) = a \otimes 1_R$ with some auxiliary system R. The (p,ω)-norm satisfies the equality

$$\|(a\otimes 1_R)\|_{p,\omega_{AR}} = \|a\|_{p,\omega} \tag{6.225}$$

if $\omega_{AR} = \omega_A \otimes \sigma_R$.

We consider the representation map

$$\begin{aligned} a &\to |a\rangle_\omega \equiv a\,|\omega^{1/2}\rangle \\ |\omega^{1/2}\rangle &= \sum_k \sqrt{p_k}\,|k,k\rangle \ . \end{aligned} \tag{6.226}$$

Since ω is full rank this representation is faithful. We call $\mathcal{H}_\omega$ the GNS Hilbert space and sometimes refer to it as the L^2_ω Banach space because the L^2_ω norm is the Hilbert space norm:

$$\|a\,|\omega^{1/2}\rangle\,\|_{2,\omega} = \|a\,|\omega^{1/2}\rangle\,\| \ . \tag{6.227}$$

[40] We can define an alternate $(p,\omega,*)$-norm to be

$$\|a\|_{p,\omega,*} \equiv \|a^\dagger\|_{p,\omega} = \|\omega^{1/p}a\|_p = \|\Delta^{1/p}_{\omega|{\rm e}}\,|a\rangle\,\|_p \ . \tag{6.222}$$

As opposed to the p-norm the (p,ω)-norm is not invariant under $a \to uav$ with u and v unitaries. Instead, we have

$$\|ua\|_{p,\omega} = \|a\|_{p,\omega}, \qquad \|au\|_{p,\omega,*} = \|a\|_{p,\omega,*} \ . \tag{6.223}$$

In other words, for unitaries $u \in \mathcal{A}$ and $u' \in \mathcal{A}'$ we have

$$\|u\,|\Psi\rangle\,\|_{p,\omega} = \|\,|\Psi\rangle\,\|_{p,\omega}, \qquad \|u'\,|\Psi\rangle\,\|_{p,\omega,*} = \|\,|\Psi\rangle\,\|_{p,\omega,*} \ . \tag{6.224}$$

More generally, one can define the Kosaki (p,σ,ω)-norms $\|a\|_{p,\sigma,\omega} = \|\sigma^{1-1/p}a\omega^{1/p}\|_p$ [203].

Since the (∞,ω)-norm is the same as the ∞-norm the algebra itself is the L^∞_ω space. The L^1_ω space is the space of operators $\omega_a = a\,|\omega\rangle\,\langle \mathrm{e}|$ with the L^1 norm. Note that as opposed to the p-norm, for the (p,ω)-norms we have the hierarchy $L^\infty_\omega \subseteq L^2_\omega \subseteq L^1_\omega$ when ω is a normalized density matrix because of the inequalities

$$\begin{aligned} &\|a\|_1 \geq \|a\|_2 \geq \|a\|_\infty \\ &\|a\|_{1,\omega} \leq \|a\|_{2,\omega} \leq \|a\|_{\infty,\omega}\ . \end{aligned} \tag{6.228}$$

The vector $|a\rangle_\omega$ in the Hilbert space $\mathcal{H}_\omega$ corresponds to the state (density martix of $\mathcal{A}$) $\omega_a = a\omega a^\dagger$. However, given a density matrix there are many vectors in $\mathcal{H}_\omega$ that purify it. In the last section, we used the modular conjugation operator J_{e} to fix a canonical vector for each density matrix. To fix a canonical vector we start with the map

$$\psi \to |\psi^{1/2}\rangle = (\psi^{1/2}\omega^{-1/2})\,|\omega^{1/2}\rangle \in \mathcal{H}_\omega\ . \tag{6.229}$$

Any state of the form

$$(\psi^{1/2}u\omega^{-1/2})\,|\omega^{1/2}\rangle \tag{6.230}$$

for unitary u has the same density matrix ψ. To make the correspondence between the density matrices and their purification one-to-one we introduce the modular conjugation operator J_ω that acts as (6.208) in the eigenbasis of ω. From the argument in (6.210) it is clear that the vector $|\psi^{1/2}\rangle$ is the only J_ω invariant vector representative of the density matrix ψ. Therefore, there is a one-to-one correspondence between the density matrices ψ and the vectors

$$|\psi^{1/2}\rangle = \Delta^{1/2}_{\psi|\omega}\,|\omega^{1/2}\rangle \tag{6.231}$$

that are invariant under J_ω. These vectors form the so-called natural cone $P^\natural_\omega$.

We define the (p,ω)-norm of the vectors in the GNS Hilbert space $\mathcal{H}_\omega$ to be

$$\|a\,|\omega^{1/2}\rangle\,\|_{p,\omega} \equiv \|a\|_{p,\omega} \tag{6.232}$$

so that the $(2,\omega)$-norm is the Hilbert space norm of $a\,|\omega^{1/2}\rangle$. Note that $|\omega^{1/2}\rangle$ has unit (p,ω)-norm for all p.[41]

To every density matrix ψ we can canonically associate a unique operator $|\psi\omega^{-1/2}\rangle\,\langle\omega^{1/2}| \in L^1_\omega$ with unit 1-norm and a unique vector in L^p_ω with unit (p,ω)-norm:

$$\psi \to |\psi^{1/p}\omega^{1/2-1/p}\rangle = \Delta^{1/p}_{\psi|\omega}\,|\omega^{1/2}\rangle = \Delta^{1/p-1/2}_{\psi|\omega}\,|\psi^{1/2}\rangle \ . \tag{6.235}$$

As we vary from $p=2$ to $p=\infty$ the vector above interpolates between $|\psi^{1/2}\rangle$ with unit $(2,\omega)$-norm and $|\omega^{1/2}\rangle$ with unit (∞,ψ)-norm. Note that if ψ is not normalized we have

$$\|\Delta^{1/p}_{\psi|\omega}\,|\omega^{1/2}\rangle\,\|_{p,\omega} = \|\psi\|_1^{1/p} \tag{6.236}$$

which is independent of ω. Since ω is invertible and $L^r_\omega \subseteq L^p_\omega$ for any $p \le r$ the vector

$$\Delta^{\theta/p}_{\psi|\omega}\,|\omega^{1/2}\rangle \in L^p_\omega \tag{6.237}$$

for any $\theta \in [0,1]$.[42] In fact, we can extend θ to the complex plane $z=\theta+it$ because

$$\begin{aligned}\Delta^{\mathrm{i}t}_{\psi|\omega}\,|\omega^{1/2}\rangle &= (D\psi:D\omega)_t\,|\omega^{1/2}\rangle \\ (D\psi:D\omega)_t &\equiv \Delta^{\mathrm{i}t}_{\psi|\omega}\Delta_\omega^{-\mathrm{i}t}\end{aligned} \tag{6.238}$$

[41]↑ We can also define the alternate $(p,\omega,*)$-norm of a vector

$$\|a\,|\omega^{1/2}\rangle\,\|_{p,\omega,*} \equiv \|a\|_{p,\omega,*} \equiv \|a^\dagger\,|\omega^{1/2}\rangle\,\|_{p,\omega}\ . \tag{6.233}$$

The $(2,\omega,*)$ is the Hilbert space norm of $a^\dagger\,|\omega^{1/2}\rangle$. The $(p,\omega,*)$-norm of a vector has the advantage that it is independent of unitary rotations $u' \in \mathcal{A}'$:

$$\|u'a\,|\omega^{1/2}\rangle\,\|_{p,\omega,*} \equiv \|u'a\|_{p,\omega,*} = \|a\,|\omega^{1/2}\rangle\,\|_{p,\omega,*}\ . \tag{6.234}$$

Therefore, it only depends on the reduced state on A that is $aa^\dagger$, and not a particular purification choice $u'\,|a\rangle$.

[42]↑Note that in finite dimensions we can take $\theta > 1$ as well. However, in this work, we restrict to the range because it generalizes to infinite dimensions.

and the cocycle $(D\psi : D\omega)_t$ is a partial isometry in the algebra for all real values of t. When ψ is full rank the cocycle is a unitary operator.

As we saw in the last section, the Hölder inequality helps bound the p-norm in terms of simpler norms such as the 2-norm and ∞-norm. In section (6.2.7), we will prove the following Hölder inequality for the (p, ω)-norms

$$\begin{aligned} &\|\Delta_{\psi_0|\omega}^{1/p_0}\Delta_{\psi_1|\omega}^{1/p_1}\,|\omega^{1/2}\rangle\,\|_{r,\omega} \leq \|\psi_0\|_1^{1/p_0}\|\psi_1\|_1^{1/p_1} \\ &\frac{1}{p_0}+\frac{1}{p_1}=\frac{1}{r}\ . \end{aligned} \tag{6.239}$$

Similarly, it is often helpful to relate the $(p, \omega) \to (p, \omega)$ norms of superoperators in (6.191), or equivalently those of their corresponding operators in the GNS Hilbert space in (6.192). This is achieved using an inequality established by the Riesz-Thorin interpolation theorem that we prove in appendix 6.2.13. The theorem says that for $2 \leq p_0, p_1$ and $\theta \in [0, 1]$ and any operator $T : \mathcal{H}_A \to \mathcal{H}_B$ we have

$$\begin{aligned} &\|T\|_{(p_\theta,A)\to(p_\theta,B)} \leq \|T\|^{1-\theta}_{(p_0,A)\to(p_0,B)}\|T\|^{\theta}_{(p_1,A)\to(p_1,B)} \\ &\frac{1}{p_\theta}=\frac{1-\theta}{p_0}+\frac{\theta}{p_1}\ . \end{aligned} \tag{6.240}$$

Lemma 6.2.0.1. *Consider a contraction $F : \mathcal{H}_A \to \mathcal{H}_B$[43] where $\mathcal{H}_A \equiv \mathcal{H}_{\omega_A}$ and $\mathcal{H}_B \equiv \mathcal{H}_{\omega_B}$ are the GNS Hilbert spaces of states ω_A and ω_B, respectively. Then it cannot increase the (p, ω)-norm of a vector for $p \geq 2$, i.e.*

$$\|F\|_{(p,A)\to(p,B)} \leq 1\ . \tag{6.241}$$

Proof. Since $(2, A)$-norm is the Hilbert space norm and (∞, A) norm is the ∞-norm, by the same argument as in (6.187), we have

$$\begin{aligned} &\|F\,|a\rangle_{\omega_A}\,\|_{2,B} \leq \|\,|a\rangle_{\omega_A}\,\|_{2,A} \\ &\|F\,|a\rangle_{\omega_A}\,\|_{\infty,B} \leq \|\,|a\rangle_{\omega_A}\,\|_{\infty,A}\ . \end{aligned} \tag{6.242}$$

[43]↑We remind the reader that a contraction is defined with respect to the infinity norm, and not any other norms we discuss in this work.

$$\begin{array}{ccccccccc} \|T\|_1 \geq \|T\|_2 \geq \|T\|_q \geq \|T\|_\infty = \|T\|_{\infty,\omega_B} \geq \|T\|_{q,\omega_B} \geq \|T\|_{2,\omega_B} \geq \|T\|_{1,\omega_B} \\ \| \\ \|T\|_{2\to 2} \geq \|T\|_{q\to q} \\ \downarrow \\ \|T\|_{(2,\omega_B)\to(2,\omega_A)} \end{array}$$

Figure 6.9. The figure shows the hierarchy of norms for a linear operator $T : \mathcal{H}_B \to \mathcal{H}_A$ with $2 \leq q \leq \infty$. The inequality between $\|T\|_{2\to 2}$ and $\|T\|_{(2,\omega_B)\to(2,\omega_A)}$ saturates when the size of $\mathcal{H}_B$ and $\mathcal{H}_A$ are the same. See appendix 6.2.17 for a comparison of $\|\cdot\|_{2\to 2}$ and $\|\cdot\|_{q\to q}$ norm.

Then, using the Riesz-Thorin inequality in (6.240) completes the proof. This lemma plays a central role in our proof of the data processing inequality. See figure 6.9 for the relation between different norms. □

In (6.215) we used the Hölder inequality to rewrite the p-norm of the vectors as a variational expression in the Hilbert space. In constructing the GNS Hilbert space we replaced $|\mathrm{e}\rangle$ with the state $|\omega^{1/2}\rangle$ and defined the vectors $|a\rangle_\omega = a\,|\omega^{1/2}\rangle$. The definition of the L^p_ω norms in (6.215) generalizes to the GNS Hilbert space:

$$\begin{aligned} \| \, |a\rangle_\omega \, \|^2_{2p,\omega} = \|a\omega^{\frac{1}{2p}}\|^2_{2p} = \|a\omega^{1/p}a^\dagger\|_p \quad &= \sup_{\||\psi^{1/2}\rangle\|=1} \|\Delta^{\frac{1}{2}-\frac{1}{2p}}_{\psi|\mathrm{e}} \, |a\omega^{1/(2p)}\rangle \, \|^2 \\ &= \sup_{\||\psi^{1/2}\rangle\|=1} \|\Delta^{\frac{1}{2}-\frac{1}{2p}}_{\psi|\omega} \, |a\rangle_\omega \, \|^2 \, . \end{aligned} \tag{6.243}$$

After a change of variables from $2p \to p$ we find

$$\begin{aligned} \| \, |a\rangle_\omega \, \|_{p,\omega} &= \sup_{|\psi^{1/2}\rangle\in\mathcal{H}_\omega} \|\Delta^{\frac{1}{2}-\frac{1}{p}}_{\psi|\omega} \, |a\rangle_\omega \, \| \qquad \forall p \in [2,\infty] \\ \| \, |a\rangle_\omega \, \|_{p,\omega} &= \inf_{|\psi^{1/2}\rangle\in\mathcal{H}_\omega} \|\Delta^{\frac{1}{2}-\frac{1}{p}}_{\psi|\omega} \, |a\rangle_\omega \, \| \qquad \forall p \in [1,2) \, . \end{aligned} \tag{6.244}$$

where $|\psi^{1/2}\rangle$ has unit norm. It was observed by [154] that the definition of the (p,ω)-norm above generalize to any von Neumann algebra, even to those that do not admit a trace such as the local algebra of QFT. We will come back to this in section 6.2.10.

6.2.4 Two-state Rényi divergences

Now, we are ready to define the distinguishability measures using the L^p_ω norm of the vectors in the GNS Hilbert space.

Definition 6.2.4. *We define the Petz divergences in terms of the Hilbert space norm of the interpolating vector*

$$D_{1/p}(\psi\|\omega) = \frac{2p}{1-p}\log\|\Delta^{1/(2p)}_{\psi|\omega}\,|\omega^{1/2}\rangle\,\|_{2,\omega} \tag{6.245}$$

and the sandwiched Rényi divergences using the (p,ω)-norm of the vector $|\psi^{1/2}\rangle$ [204], [205]

$$S_p(\psi\|\omega) \equiv \frac{2p}{p-1}\log\|\,|\psi^{1/2}\rangle\,\|_{2p,\omega} = \frac{p}{p-1}\log\|\omega^{-\frac{1}{2q}}\psi\omega^{-\frac{1}{2q}}\|_p \tag{6.246}$$

for $p\in[1/2,\infty]$.[44]

These divergences are the generalizations of the Rényi entropy in (6.217) to the GNS Hilbert space. Their asymmetry has to do with the fact that the reference state ω is used to construct the GNS Hilbert space. These two-state Rényi divergences satisfy the data processing inequality [149], [202], [206]. The $p\to 1$ limit of both quantities gives the relative entropy [207]

$$S(\psi\|\omega) = -2\lim_{p\to 1}\prime_p\|\,|\psi^{1/2}\rangle\,\|_{2p,\omega}\ . \tag{6.247}$$

Since we will be always working in the GNS Hilbert space $\mathcal{H}_\omega$ we simplify our notation by introducing $|\Omega\rangle\equiv|\omega^{1/2}\rangle$. The vector $|\psi^{1/2}\rangle$ is a purification of ψ which is symmetric under J_Ω. It can be written as

$$|\psi^{1/2}\rangle = \Delta^{1/2}_{\psi|\omega}\,|\Omega\rangle\ . \tag{6.248}$$

[44]↑Cases $p=1$ and $p=\infty$ are defined as limits $p\to 1$ and $p\to\infty$.

The definitions in (6.245) and (6.246) are independent of the purification of ψ. To see this, we first define the relative modular operator for an arbitrary vector $|\Psi\rangle$

$$\Delta_{\Psi|\Omega} \equiv \psi_A \otimes \omega_{A'}^{-1} \tag{6.249}$$

so that it remains unchanged for other purifications of ψ:

$$\Delta_{u'\Psi|\Omega} = \Delta_{\Psi|\Omega} \ . \tag{6.250}$$

For an arbitrary vector $|\Psi\rangle \in \mathcal{H}_\omega$ we can write the divergences in (6.171) as

$$\begin{aligned} D_{1/p}(\psi_A\|\omega_A) &\equiv \frac{2p}{1-p} \log \|\Delta_{\Psi|\Omega}^{1/(2p)} |\Omega\rangle \|_{2,\Omega} \\ S_p(\psi_A\|\omega_A) &\equiv \frac{2p}{p-1} \log \|\Delta_{\Psi|\Omega}^{1/2} |\Omega\rangle \|_{2p,\Omega} \ . \end{aligned} \tag{6.251}$$

We also define the (p,Ω)-norm in the GNS Hilbert space $\mathcal{H}_\Omega$ using

$$\begin{aligned} \| |\Psi\rangle \|_{p,\Omega} &= \sup_{\||\chi\rangle\|=1} \|\Delta_{\chi|\Omega}^{1/2-1/p} |\Psi\rangle \| \qquad p \in [2,\infty] \\ \| |\Psi\rangle \|_{p,\Omega} &= \inf_{\||\chi\rangle\|=1} \|\Delta_{\chi|\Omega}^{1/2-1/p} |\Psi\rangle \|, \qquad p \in [1,2) \ . \end{aligned} \tag{6.252}$$

Definition 6.2.5. *To interpolate between the two divergences following [146] we define the (θ, r)-Rényi divergences*

$$S_{\theta,r}(\psi_A\|\omega_A) = \frac{-2r}{1-\theta} \log \|\Delta_{\Psi|\Omega}^{\theta/(2r)} |\Omega\rangle \|_{2r,\Omega} \tag{6.253}$$

for the range $r \in [1/2,\infty]$ and $\theta \in [0,1]$. Even though in matrix algebras one can extend beyond this range we limit our discussion to this limited range because outside of this range, in infinite dimensions, the (θ, r)-Rényi divergences might not be finite. We postpone a study of the extended range to future work.

In matrix algebras, the expression in (6.253) becomes

$$\begin{aligned} S_{\theta,r}(\psi_A\|\omega_A) &= \frac{2r}{\theta-1}\log\|\psi_A^{\theta/(2r)}\omega_A^{(1-\theta)/(2r)}\|_{2r} \\ &= \frac{1}{\theta-1}\log\mathrm{tr}\left[\left(\omega_A^{\frac{1-\theta}{2r}}\psi_A^{\frac{\theta}{r}}\omega_A^{\frac{1-\theta}{2r}}\right)^r\right] \end{aligned} \tag{6.254}$$

where in the first equality we have used

$$(1\otimes\omega_{A'})\,|\Omega\rangle = (\omega_A\otimes 1)\,|\Omega\rangle \ . \tag{6.255}$$

It follows from the definition in (6.253) that the (θ,θ)-Rényi divergences is the θ-sandwiched Rényi divergence and the $(\theta,1)$-Rényi divergences is the θ-Petz divergence. In the remainder of this work, we suppress the subscript A unless there is potential for confusion. Note that the matrix algebra expression enjoys the symmetry

$$(1-\theta)S_{\theta,r}(\psi\|\omega) = \theta S_{1-\theta,r}(\omega\|\psi) \ . \tag{6.256}$$

In the limit $r\to\infty$ we can use the Lie-Trotter formula

$$\lim_{r\to\infty}\left(\mathrm{e}^{a_1/r}\mathrm{e}^{a_2/r}\right)^r = \mathrm{e}^{a_1+a_2} \tag{6.257}$$

for self-adjoint operators a_1, a_2 to write

$$\lim_{r\to\infty} S_{\theta,r}(\psi\|\omega) = \frac{1}{1-\theta}\log\mathrm{tr}\left(\mathrm{e}^{\theta\log\psi+(1-\theta)\log\omega}\right) \ . \tag{6.258}$$

A larger class of two-state f-divergences one can consider if

$$S_r^f(\psi\|\omega) = -2r\log\|f(\Delta_{\Psi|\Omega}^{1/r})^{1/2}\,|\Omega\rangle\,\|_{2r,\Omega} \tag{6.259}$$

where f is an operator monotone function.[45] In the next subsection, we show that these measures satisfy the data processing inequality. They are related to the f-divergences and the

[45]↑A function $f:(0,\infty)\to\mathbb{R}$ is called operator monotone if for positive operators X and Y the inequality $X\le Y$ implies $f(X)\le f(Y)$.

Petz quasi-entropies. [149], [207]–[209] A few examples of the operator monotone functions are

1. $f(x) = x^\alpha$ with $\alpha \in (0,1)$.

2. $f(x) = -x^{-\alpha}$ with $\alpha \in (0,1)$.

3. $f(x) = x \log x$

4. $f(x) = \log x$

For concreteness and the fact that at times we need $f(1) = 1$, we will be mostly concerned with the first case: $f(x) = x^\alpha$. However, we prove the data processing inequality for a general operator monotone function f.

6.2.5 Data processing inequality for (θ, r)-Rényi divergences

Consider a quantum channel Φ^* that sends the density matrices ψ_A and ω_A to $\psi_B = \Phi^*(\psi_A)$ and $\omega_B = \Phi^*(\omega_A)$, respectively. We consider the GNS Hilbert spaces corresponding to ω_A and ω_B and call them $\mathcal{H}_A$ and $\mathcal{H}_B$. We have

$$\langle \Omega_B | b | \Omega_B \rangle = \langle \Omega_A | \Phi(b) | \Omega_A \rangle \ . \tag{6.260}$$

In this subsection, we prove the data processing inequality for the (θ, r)-Rényi divergences in (6.253) and the divergences in (6.259) for $r \geq 1$:

$$\begin{aligned} S_r^f(\Phi^*(\psi_A) \| \Phi^*(\omega_A)) &\leq S_r^f(\psi_A \| \omega_A) \\ S_{\theta,r}(\Phi^*(\psi_A) \| \Phi^*(\omega_A)) &\leq S_{\theta,r}(\psi_A \| \omega_A) \ . \end{aligned} \tag{6.261}$$

In the range we are interested, the monotonicity of the (θ, r)-Rényi divergences was first proved by [197].[46] In the Heisenberg picture, the quantum channel Φ^* is described by an unital CP map $\Phi : \mathcal{B} \to B(\mathcal{H}_A)$ that acts on the algebra. Note that the range of a CP map

[46]↑See theorem 2.1 of [195] for a proof of the data processing inequality in extended range of (θ, r) for matrix algebras.

need not be the full algebra $B(\mathcal{H}_A)$. For simplicity, sometimes we use the notation $\mathcal{A}$ for the operators in $B(\mathcal{H}_A)$.[47]

Theorem 6.2.1. *Let $\Phi : \mathcal{B} \to \mathcal{A}$ be an unital CP map. Let ψ_A, ω_A be states on $\mathcal{A}$ and $\psi_B = \Phi^*(\psi_A)$, $\omega_B = \Phi^*(\omega_A)$ be states on $\mathcal{B}$. For $r \geq 1$, the two-state f-divergences $S_r^f(\psi\|\omega)$ satisfy the data processing inequality*

$$S_r^f(\psi_B\|\omega_B) \leq S_r^f(\psi_A\|\omega_A) \ . \tag{6.262}$$

Proof. Let $|\Omega_A\rangle$ and $|\Omega_B\rangle$ be the vectors corresponding to ω_A and ω_B in their corresponding GNS Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$. Let F be the contraction operator corresponding to Φ in the GNS Hilbert space such that

$$\Phi(b)\,|\Omega_A\rangle = Fb\,|\Omega_B\rangle \ . \tag{6.263}$$

The monotonicity of the relative modular operator is the operator inequality[48]:

$$F^\dagger \Delta_{\Psi_A|\Omega_A} F \leq \Delta_{\Psi_B|\Omega_B} \ . \tag{6.264}$$

Choosing the function $f(x) = -(t+x)^{-1}$ that is operator monotone and operator convex[49] we obtain [210]

$$F^\dagger \frac{1}{t + \Delta_{\Psi_A|\Omega_A}} F \geq \frac{1}{t + \Delta_{\Psi_B|\Omega_B}} \ . \tag{6.265}$$

Any operator monotone function f can be expanded as [211], [212]

$$f(X) = a + bX + \int_0^\infty d\mu(t) \left(\frac{t}{t^2+1} - \frac{1}{t+X} \right) \tag{6.266}$$

[47]↑In general, the range of a CP map is a $*$-closed subspace of observables inside $B(\mathcal{H}_A)$, otherwise known as an operator system.
[48]↑See [210], and [58] for a review of its proof using the Tomita-Takesaki modular theory
[49]↑A function is called operator convex if $f(\theta X + (1-\theta)Y) \leq \theta f(X) + (1-\theta) f(Y)$.

for $a \in \mathbb{R}, b \geq 0$ and $\mu(t)$ a positive measure that satisfies[50]

$$\int_0^\infty \frac{1}{t^2+1} d\mu(t) < \infty \ . \tag{6.268}$$

Therefore, we have the inequality

$$F^\dagger f(\Delta_{\Psi_A|\Omega_A}) F \leq f(\Delta_{\Psi_B|\Omega_B}) \ . \tag{6.269}$$

This implies

$$f(\Delta_{\Psi_B|\Omega_B})^{-1/2} F^\dagger f(\Delta_{\Psi_A|\Omega_A}) F f(\Delta_{\Psi_B|\Omega_B})^{-1/2} \leq 1 \ . \tag{6.270}$$

Define the operator

$$F_f \equiv f(\Delta_{\Psi_A|\Omega_A})^{1/2} F f(\Delta_{\Psi_B|\Omega_B})^{-1/2} \ . \tag{6.271}$$

In appendix 6.2.16, we show that (6.270) implies that F_f is a contraction and satisfies

$$\|F_f\|_{(p,\Omega_B)\to(p,\Omega_A)} \leq 1 \ . \tag{6.272}$$

In the case of function $f(x) = x^\theta$ the integral representation in (6.266) is

$$X^\theta = \frac{\sin(\pi\theta)}{\pi} \int_0^\infty dt\, t^\theta \left(\frac{1}{t} - \frac{1}{t+X} \right) \ . \tag{6.273}$$

which is equivalent to saying that F_θ satisfies:

$$\begin{aligned} &\|F_\theta\|_{(p,\Omega_B)\to(p,\Omega_A)} \leq 1 \\ &F_\theta \equiv \Delta_{\Psi_A|\Omega_A}^{\theta/2} F \Delta_{\Psi_B|\Omega_B}^{-\theta/2} \ . \end{aligned} \tag{6.274}$$

[50]↑When $f(0) := \lim_{t\to 0} f(t) > -\infty$, we can write f as

$$f(X) = f(0) + bX + \int_0^\infty d\mu(t) \left(\frac{1}{t} - \frac{1}{t+X} \right) \tag{6.267}$$

where $\mu(t)$ satisfies $\int_0^\infty \frac{1}{t+t^2} d\mu(t) < \infty$.

This is similar to the argument by [213].

To prove the monotonicity under a contraction we use a proof similar to the one presented by [202]:

$$\begin{aligned} \| f(\Delta_{\Psi_A|\Omega_A})^{1/2} |\Omega_A\rangle \|_{2r,\Omega_A} &= \| F_f f(\Delta_{\Psi_B|\Omega_B})^{1/2} |\Omega_B\rangle \|_{2r,\Omega_A} \\ &\leq \| F_f \|_{(2r,\Omega_B)\to(2r,\Omega_A)} \| f(\Delta_{\Psi_B|\Omega_B})^{1/2} |\Omega_B\rangle \|_{2r,\Omega_B} \\ &\leq \| f(\Delta_{\Psi_B|\Omega_B})^{1/2} |\Omega_B\rangle \|_{2r,\Omega_B} \end{aligned} \tag{6.275}$$

where we have used the definition of the $(2r, \Omega_B) \to (2r, \Omega_A)$ norm for the contraction F_f and the fact that it is less than one. This proves the data processing inequality in the range $r \geq 1$.[51] □

Corollary 6.2.1.1. *Let $\Phi : \mathcal{B} \to \mathcal{A}$ be an unital CP map. Let ψ_A, ω_A be states on $\mathcal{A}$ and $\psi_B = \Phi^*(\psi_A)$, $\omega_B = \Phi^*(\omega_A)$ be states on $\mathcal{B}$. The (θ, r)-Rényi divergences satisfy the data processing inequality*

$$S_{\theta,r}(\psi_B \| \omega_B) \leq S_{\theta,r}(\psi_A \| \omega_A) \tag{6.276}$$

for $r \geq 1$ and $\theta \in [0, 1]$.

In appendix 6.2.14, we show that if $\omega \leq c\psi$ for some constant c the vector

$$\Delta_{\Psi|\Omega}^{-\theta} |\Omega\rangle \in L_\omega^{2r} \tag{6.277}$$

in the extended range $\theta \in [-1, 1]$ and $r \geq 1$. To prove the data processing inequality in (6.275) we used the contraction in (6.274):

$$F_\theta = \Delta_{\Psi_A|\Omega_A}^{\theta/2} F \Delta_{\Psi_B|\Omega_B}^{-\theta/2} . \tag{6.278}$$

[51]↑We restrict to $r \geq 1$ as we proved the Riesz-Thorin theorem for this range in appendix 6.2.13.

The † of this operator is also a contraction

$$F_\theta^\dagger = \Delta_{\Psi_B|\Omega_B}^{-\theta/2} F^\dagger \Delta_{\Psi_A|\Omega_A}^{\theta/2} \ . \tag{6.279}$$

Therefore, we have

$$\|\Delta_{\Psi|\Omega;B}^{-\theta/2} |\Omega_B\rangle \|_{2r,\Omega_B} = \|F_\theta^\dagger \Delta_{\Psi|\Omega;A}^{-\theta/2} |\Omega_A\rangle \|_{2r,\Omega_B} \leq \|\Delta_{\Psi|\Omega;A}^{-\theta/2} |\Omega_A\rangle \|_{2r,\Omega_A} \tag{6.280}$$

which says that the measure

$$S_{\theta,r}(\psi\|\omega) = \frac{-2r \ \mathrm{sign}(\theta)}{1-\theta} \log \|\Delta_{\Psi|\Omega}^{\theta/(2r)} |\Omega\rangle \|_{2r,\Omega} \tag{6.281}$$

satisfies the data processing inequality in the extended range $\theta \in (-1, 1)$. Another way to define a measure with an extended range of monotonicity $\theta \in (-1, 1)$ is

$$\hat{S}_{\theta,r}(\psi\|\omega) \equiv \frac{-2r}{\theta(1-\theta)} \log \|\Delta_{\Psi|\Omega}^{\theta/(2r)} |\Omega\rangle \|_{2r,\Omega} \ . \tag{6.282}$$

Note that this measure no longer vanishes at $\theta \to 0$. For instance, when $r = 1$ it corresponds to a modification of the Petz divergence

$$\frac{-2}{\theta(1-\theta)} \log \|\psi^\theta \omega^{1-\theta}\| \tag{6.283}$$

that interpolates between the relative entropy $S(\omega\|\psi)$ at $\theta \to 0$ and $S(\psi\|\omega)$ at $\theta \to 1$. The measures defined above satisfy the data processing inequality and vanishes for identical states, hence they are non-negative.[52]

In general, when $\theta > 1$ we are not guaranteed that $\Delta_{\Psi|\Omega}^{\theta/(2r)}$ belongs to L_ω^{2r}. It is known that the (θ, r)-Rényi divergences continue to satisfy the data processing inequality in the regime $r \in [1/2, 1)$ and $r \geq \max(\theta, 1-\theta)$ [146]. In this range of parameters, the (θ, r)-Rényi divergences are finite for arbitrary states of infinite systems. However, we will not attempt to prove the data processing inequality in this case. In matrix algebras, one can extend the

[52]↑Consider the CP map that sends all states to the same ω_B. After the channel the measure is zero. Since it has not increased, it was non-negative before applying the channel.

range of the parameters to $\theta \in \mathbb{R}/\{1\}$ and $r > 0$. The full range of parameters for which the (θ, r)-Rényi divergence satisfies the data processing inequality was characterized by [195].

6.2.6 Multi-state measures

We are now ready to generalize the construction of the two-state Rényi divergences to several states. For completeness, we have included a discussion of the Hölder inequality in the first subsection. The reader only interested in the multi-state Rényi divergences can skip this subsection.

6.2.7 Generalized Hölder inequality

Consider the multi-state vector

$$|\Omega_{\vec{\psi}}(\vec{\theta}, \vec{p})\rangle = \Delta_{\Psi_1|\Omega}^{\theta_1/p_1} \cdots \Delta_{\Psi_n|\Omega}^{\theta_n/p_n} |\Omega\rangle \tag{6.284}$$

with $0 \leq \theta_1 + \cdots + \theta_n \leq 1$. We have introduced the compact notation $\vec{\theta} = (\theta_1, \cdots, \theta_n)$, $\vec{p} = (p_1, \cdots, p_n)$ and $\vec{\psi} = (\psi_1, \cdots, \psi_n)$. Note that by the relation (6.250) the vector above only depends on the states ω_1 to ω_n and not their purifications. We define the parameters r_n and $p_{\vec{\theta}}$

$$\begin{aligned} \frac{1}{r_n} &\equiv \frac{1}{p_1} + \cdots \frac{1}{p_n} \\ \frac{1}{p_{\vec{\theta}}} &\equiv \frac{\theta_1}{p_1} + \cdots + \frac{\theta_n}{p_n} \ . \end{aligned} \tag{6.285}$$

We analytically continue the vector in (6.284) to complex variables $z_{\rm i} = \theta_{\rm i} + {\rm i}t$. Since $p_{\vec{\theta}} \geq r_n$ the r_n-norm analytically continued to the complex strip is finite

$$f_{\vec{\psi}|\omega}(\vec{z}, \vec{p}) = \| \, |\Omega_{\vec{\psi}}(\vec{z}, \vec{p})\rangle \, \|_{r_n, \Omega} \ . \tag{6.286}$$

In matrix algebras, the function above is

$$f_{\vec{\psi}|\omega}(\vec{\theta}, \vec{p}) = \| \psi_1^{\theta_1/p_1} \cdots \psi_n^{\theta_n/p_n} \omega^{\frac{1}{r_n} - \frac{1}{p_{\vec{\theta}}}} \|_{r_n} \ . \tag{6.287}$$

Lemma 6.2.1.1 (Hadamard three lines)**.** *Let $f(z)$ be a general function that is bounded and holomorphic in the complex strip $\theta \in [0,1]$ and continuous on its boundaries. Then we have*

$$|f(\theta)| \leq |f(0)|^{1-\theta} \, |f(1)|^{\theta} \ . \tag{6.288}$$

Proof. Define the function $g(z) = f(z) f(0)^{z-1} f(1)^{-z}$ which is also holomorphic and bounded in the strip and continuous on the boundaries of the strip. The function $g(z)$ has value less than or equal to one on the boundaries, therefore by the Phragmén-Lindelöf principle (the maximum modulus principle applied for the holomorphic functions bounded in the strip) it takes its maximum on the boundary. As a result, $|g(z)| \leq 1$ everywhere in the strip. On the real line $z = \theta$ we obtain the inequality

$$|f(\theta)| \leq |f(0)|^{1-\theta} \, |f(1)|^{\theta} \ .$$

This result is sometimes called the Hadamard three-lines theorem. □

Theorem 6.2.2 (Generalized Hölder inequality)**.** *The function in (6.286) is bounded and analytic on the complex domain of $\vec{z}$ with $0 \leq \theta_1 + \cdots + \theta_n \leq 1$. It satisfies the generalized Hölder inequality for the (p, ω)-norms:*[53]

$$\| \Delta^{1/p_1}_{\Psi_1|\Omega} \cdots \Delta^{1/p_n}_{\Psi_n|\Omega} \, |\Omega\rangle \, \|_{r_n,\Omega} \leq \prod_{\mathrm{i}=1}^{n} \| \Delta^{1/p_{\mathrm{i}}}_{\Psi_{\mathrm{i}}|\Omega} \, |\Omega\rangle \, \|_{p_{\mathrm{i}},\Omega} = \prod_{\mathrm{i}=1}^{n} \|\psi_{\mathrm{i}}\|_1^{1/p_{\mathrm{i}}} \ . \tag{6.289}$$

Note that the measure above is independent of the state ω. If the states ψ_{i} are all normalized the right-hand-side is equal to one. In matrix algebras, this is

$$\| \psi_1^{1/p_1} \cdots \psi_n^{1/p_n} \|_{r_n} \leq \prod_{\mathrm{i}=1}^{n} \| \psi_{\mathrm{i}}^{1/p_{\mathrm{i}}} \|_{p_{\mathrm{i}}} \ . \tag{6.290}$$

Defining the operators $a_{\mathrm{i}} \equiv \psi_{\mathrm{i}}^{1/p_{\mathrm{i}}}$ gives the matrix form of the generalized Hölder inequality

$$\| a_1 \cdots a_n \|_{r_n} \leq \| a_1 \|_{p_1} \cdots \| a_n \|_{p_n} \ . \tag{6.291}$$

[53] ↑This was shown in theorem 5 of [154].

Proof. To prove (6.289) we start by showing

$$\begin{aligned}\|\Delta^{1/p_1}_{\Psi_1|\Omega}\cdots\Delta^{1/p_{n-1}}_{\Psi_{n-1}|\Omega}\Delta^{1/p_n}_{\Psi_n|\Omega} \quad & |\Omega\rangle\,\|_{r_n,\Omega} \\ & \leq \|\Delta^{1/p_1}_{\Psi_1|\Omega}\cdots\Delta^{1/p_{n-1}}_{\Psi_{n-1}|\Omega}\,|\Omega\rangle\,\|_{r_{n-1},\Omega}\|\Delta^{1/p_n}_{\Psi_n|\Omega}\,|\Omega\rangle\,\|_{p_n,\Omega}\end{aligned}$$

for arbitrary n. Define

$$\frac{1}{r_{n,\theta}} \equiv \frac{\theta}{r_{n-1}} + \frac{1-\theta}{p_n} \tag{6.292}$$

and the function

$$f_{\vec{\psi}|\omega}(\vec{\theta},\vec{p}) \equiv \|\Delta^{\theta/p_1}_{\Psi_1|\Omega}\cdots\Delta^{\theta/p_{n-1}}_{\Psi_{n-1}|\Omega}\Delta^{(1-\theta)/p_n}_{\Psi_n|\Omega}\,|\Omega\rangle\,\|_{r_{n,\theta},\Omega}\ . \tag{6.293}$$

It can be analytically continued to complex $z = \theta + it$.

Using lemma 6.2.1.1, the function in (6.293) satisfies the inequality

$$f_{\vec{\psi}|\Omega}(\vec{\theta},\vec{p}) \leq \|\Delta^{1/p_1}_{\Psi_1|\Omega}\cdots\Delta^{1/p_{n-1}}_{\Psi_{n-1}|\Omega}\,|\Omega\rangle\,\|^{\theta}_{r_{n-1},\Omega}\|\Delta^{1/p_n}_{\Psi_n|\Omega}\,|\Omega\rangle\,\|^{1-\theta}_{p_n,\Omega}\ . \tag{6.294}$$

Choosing $\theta = 1/2$ and sending $p_{\rm i} \to 2p_{\rm i}$ gives

$$\begin{aligned}\|\Delta^{1/p_1}_{\Psi_1|\Omega}\cdots\Delta^{1/p_n}_{\Psi_n|\Omega} \quad & |\Omega\rangle\,\|_{r_n,\Omega} \\ & \leq \|\Delta^{2/p_1}_{\Psi_1|\Omega}\cdots\Delta^{2/p_{n-1}}_{\Psi_{n-1}|\Omega}\,|\Omega\rangle\,\|^{1/2}_{r_{n-1}/2,\Omega}\|\Delta^{2/p_n}_{\Psi_n|\Omega}\,|\Omega\rangle\,\|^{1/2}_{p_n/2,\Omega} \\ & = \|\Delta^{2/p_1}_{\Psi_1|\Omega}\cdots\Delta^{2/p_{n-1}}_{\Psi_{n-1}|\Omega}\,|\Omega\rangle\,\|^{1/2}_{r_{n-1}/2,\Omega}\|\psi_n\|^{1/p_n}_1\ .\end{aligned} \tag{6.295}$$

Repeating this argument and using

$$\|\Delta^{1/p}_{\Psi|\Omega}\,|\Omega\rangle\,\|_{p,\Omega} = \|\psi\|^{1/p}_1 \tag{6.296}$$

we obtain the generalized Hölder inequality in (6.289). □

6.2.8 Three-state Rényi divergences

In this subsection, we introduce the three-state Rényi divergences and use the monotonicity of the relative modular operator show that they satisfy the data processing inequality.

Definition 6.2.6 (Kubo-Ando mean)**.** *For any operator monotone function f with $f(1) = 1$ and positive operators X and Y the Kubo-Ando mean $\sharp_f$ is defined to be [155], [156]*

$$X\sharp_f Y = X^{1/2} f(X^{-1/2} Y X^{-1/2}) X^{1/2} \tag{6.297}$$

where we are assuming that X is invertible. Note that $X\sharp_f X = X$.

The most important properties of the Kubo-Ando mean for us are the monotonicity relation and the transformer inequality:

1. If $X_A \leq X_B$ and $Y_A \leq Y_B$ then $X_A \sharp_f Y_A \leq X_B \sharp_f Y_B$

2. For any T we have

$$T(X\sharp_f Y)T^\dagger \leq (TXT^\dagger)\sharp_f(TYT^\dagger) \tag{6.298}$$

with equality when T is invertible.

To simplify our equations we introduce the following notation:[54]

$$\begin{aligned} \Delta_{\Psi|\Omega;A} &\equiv \Delta_{\Psi_A|\Omega_A} \\ \Delta^f_{\Psi_1,\Psi_2|\Omega}(\theta_1,\theta_2) &\equiv \Delta^{\theta_1}_{\Psi_1|\Omega}\sharp_f \Delta^{\theta_2}_{\Psi_2|\Omega} \ . \end{aligned} \tag{6.300}$$

[54]↑ In what follows, we could have chosen a more general case

$$\Delta^f_{\Psi_1,\Psi_2|\Omega}(g_1,g_2) \equiv g_1(\Delta_{\Psi_1|\Omega})\sharp_f g_2(\Delta_{\Psi_2|\Omega}) \tag{6.299}$$

where g_1 and g_2 are arbitrary operator monotone functions such that such $g_{\rm i}(x) \geq 0$ for $x \geq 0$, however, to keep the presentation clean we restrict to the operator monotone functions $g_1(x) = x^{\theta_1}$ and $g_2(x) = x^{\theta_2}$ as we did in (6.301). The definition of the multi-state Rényi divergences generalizes in the straightforward way. Our proof of the data processing inequality will apply to this most general case.

Choosing $|\Omega\rangle$ as the reference vector, $|\Psi_1\rangle$ and $|\Psi_2\rangle$ and $\theta \in (0,1)$ we have two monotonicity equations for the relative modular operators. Combining these two inequalities using the Kubo-Ando mean and applying its property in (6.298) we obtain

$$\begin{aligned} F^\dagger \Delta^f_{\Psi_1,\Psi_2|\Omega;A}(\theta_1,\theta_2) F \quad &\leq (F^\dagger \Delta^{\theta_1}_{\Psi_1|\Omega;A} F) \sharp_f (F^\dagger \Delta^{\theta_2}_{\Psi_2|\Omega;A} F) \\ &\leq \Delta^f_{\Psi_1,\Psi_2|\Omega;B}(\theta_1,\theta_2) \ . \end{aligned} \tag{6.301}$$

The first inequality becomes an equality when F is invertible. As before, we can define the contraction

$$F^f_{\theta_1,\theta_2} \equiv \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;A}(\theta_1,\theta_2)\right)^{1/2} F \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;B}(\theta_1,\theta_2)\right)^{-1/2} \ . \tag{6.302}$$

Definition 6.2.7. *For $0 \leq \theta_1, \theta_2 \leq 1$, $r \in [1/2, \infty]$ and f any operator monotone function with $f(1) = 1$, we define the three-state f-divergence as*

$$\begin{aligned} S^f_{\theta_1,\theta_2}(\psi_1,\psi_2\|\omega) \equiv \quad & -2r \log \left\| \left(\Delta^f_{\Psi_1,\Psi_2|\Omega}(\theta_1/r, \theta_2/r)\right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} \\ &= -2r \log \left\| \left(\Delta^{\theta_1/r}_{\Psi_1|\Omega} \sharp_f \Delta^{\theta_2/r}_{\Psi_2|\Omega}\right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} \ . \end{aligned} \tag{6.303}$$

It is clear from (6.250) that the measure is independent of the purifications of ψ_1 and ψ_2.

Theorem 6.2.3. *Let $\Phi : \mathcal{B} \to \mathcal{A}$ be an unital CP map such that for a state ω on $\mathcal{A}$, the corresponding state on $\mathcal{B}$ is $\Phi^*(\omega)$. Let Ω_A and Ω_B denote the states as vectors in the corresponding GNS Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and let $F : \mathcal{H}_B \to \mathcal{H}_A$ be the contraction operator corresponding to the map Φ in the GNS Hilbert space. Given two states ψ_1 and ψ_2 on $\mathcal{A}$, for $r \geq 1$ the three-state f-divergence satisfies the data processing inequality*

$$S^f_{\theta_1,\theta_2;r}(\psi_1,\psi_2\|\omega; B) \leq S^f_{\theta_1,\theta_2;r}(\psi_1,\psi_2\|\omega; A) \ . \tag{6.304}$$

Proof. To prove the data processing inequality for this three-state measure, we use the contraction in (6.302) to write

$$\begin{aligned}
&\left\| \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;A} \quad (\theta_1,\theta_2) \right)^{1/2} |\Omega_A\rangle \right\|_{2r,\Omega_A} \\
&\quad = \left\| F^f_{\theta_1,\theta_2} \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;B}(\theta_1,\theta_2) \right)^{1/2} |\Omega_B\rangle \right\|_{2r,\Omega_A} \\
&\quad \leq \left\| F^f_{\theta_1,\theta_2} \right\|_{(2r,\Omega_B)\to(2r,\Omega_A)} \left\| \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;B}(\theta_1,\theta_2) \right)^{1/2} |\Omega_B\rangle \right\|_{2r,\Omega_B} \\
&\quad \leq \left\| \left(\Delta^f_{\Psi_1,\Psi_2|\Omega;B}(\theta_1,\theta_2) \right)^{1/2} |\Omega_B\rangle \right\|_{2r,\Omega_B} . \qquad (6.305)
\end{aligned}$$

This proves the data processing inequality for $r \geq 1$. □

As a particular example, we choose $f(x) = x^\alpha$ with $\alpha \in (0,1)$ as the operator monotone function. The Kubo-Ando geometric mean is

$$X \sharp_\alpha Y \equiv X^{1/2} \left(X^{-1/2} Y X^{-1/2} \right)^\alpha X^{1/2} \qquad (6.306)$$

which satisfies the properties

1. $(X_1 \otimes X_2) \sharp_\alpha (Y_1 \otimes Y_2) = (X_1 \sharp_\alpha Y_1) \otimes (X_2 \sharp_\alpha Y_2)$

2. If $[X, Y] = 0$ then $X^{\theta_1} \sharp_\alpha Y^{\theta_2} = X^{(1-\alpha)\theta_1} Y^{\alpha\theta_2}$.

Definition 6.2.8. *We define the three-state Rényi divergences as the special case of three-state f-divergences when $f(x) = x^\alpha$ with $\alpha \in (0,1)$*

$$\begin{aligned}
S^\alpha_{\theta_1,\theta_2}(\psi_1,\psi_2\|\omega) &\equiv \frac{-2r}{(1-\theta_1)(1-\theta_2)} \log \left\| \left(\Delta^\alpha_{\Psi_1,\Psi_2|\Omega}(\theta_1,\theta_2;r) \right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} \\
\Delta^\alpha_{\Psi_1,\Psi_2|\Omega}(\theta_1,\theta_2;r) &\equiv \Delta^{\frac{\theta_1}{(1-\alpha)r}}_{\Psi_1|\Omega} \sharp_\alpha \Delta^{\frac{\theta_2}{\alpha r}}_{\Psi_2|\Omega} . \qquad (6.307)
\end{aligned}$$

Note that α in $\Delta^{\alpha}_{\Psi_1,\Psi_2|\Omega}$ is simply an index and not a power. The powers of the relative modular operator are chosen such that when the relative modular operators commute the measure is independent of α. In matrix algebras, this measure is

$$S^{\alpha}_{\theta_1,\theta_2}(\psi_1,\psi_2\|\omega) \equiv \frac{-2r}{(1-\theta_1)(1-\theta_2)} \log \left\| \left(\psi_1^{\frac{\theta_1}{(1-\alpha)r}} \sharp_\alpha \psi_2^{\frac{\theta_2}{\alpha r}} \right)^{1/2} \omega^{\frac{\theta_0}{2r}} \right\|_{2r} \tag{6.308}$$

where $\theta_0+\theta_1+\theta_2=1$.

Special cases: In the $\theta_0 \to 0$, the expression above is independent of ω and we obtain

$$S^{\alpha}_{1-\theta,\theta;r}(\psi_1,\psi_2\|\omega) = \frac{r}{\theta(\theta-1)} \log \left\| \psi_1^{\frac{(1-\theta)}{(1-\alpha)r}} \sharp_\alpha \psi_2^{\frac{\theta}{r\alpha}} \right\|_r \ . \tag{6.309}$$

If we further set $\alpha=\theta$, up to an overall coefficient, it reduces to a generalization of the geometric divergence defined by [198], [199]:

$$S^{\theta}_{1-\theta,\theta;r}(\psi_1,\psi_2\|\omega) = \frac{r}{\theta(\theta-1)} \log \left\| \psi_1^{\frac{1}{r}} \sharp_\theta \psi_2^{\frac{1}{r}} \right\|_r \ . \tag{6.310}$$

In the special cases $\theta_1 \to 0$ (or $\theta_2 \to 0$), the three-state measure in (6.308) reduces to the (θ,r)-Rényi divergence

$$\begin{aligned} S^{\alpha}_{0,\theta;r}(\psi_1,\psi_2\|\omega) &= S_{\theta,r}(\psi_2\|\omega) \\ S^{\alpha}_{\theta,0;r}(\psi_1,\psi_2\|\omega) &= S_{\theta,r}(\psi_1\|\omega) \ . \end{aligned} \tag{6.311}$$

Another special case where we recover the (θ,r)-Rényi divergence is $\psi_1=\psi_2$:

$$\begin{aligned} S_{\theta_1,\theta_2;r}(\psi,\psi\|\omega) &= \frac{-2r}{(\theta_1-1)(\theta_2-1)} \log \|\psi^{(\theta_1+\theta_2)/(2r)} \omega^{\theta_0/(2r)}\|_{2r} \\ &= \frac{\theta_0}{(\theta_1-1)(\theta_2-1)} S_{\theta_1+\theta_2,r}(\psi\|\omega) \ . \end{aligned} \tag{6.312}$$

When $\alpha=1/2$ it is convenient to introduce the notation

$$X \sharp Y = X^{1/2} \left(X^{-1/2} Y X^{-1/2} \right)^{1/2} X^{1/2} \ . \tag{6.313}$$

to write

$$S^{1/2}_{\theta_1,\theta_2;r}(\psi_1,\psi_2\|\omega) = \frac{-2r}{(1-\theta_1)(1-\theta_2)} \log \left\| \left(\psi_1^{\theta_1/r} \sharp \psi_2^{\theta_2/r}\right)^{1/2} \omega^{\theta_0/(2r)} \right\|_{2r} . \tag{6.314}$$

6.2.9 Multi-state Rényi divergences

The generalization to arbitrary number of states is straightforward. We use the vector notation $\vec{\Psi} = (\Psi_1, \cdots, \Psi_n)$, $\vec{\theta} = (\theta_1, \cdots, \theta_n)$ and $\vec{f} = (f_1, \cdots, f_{n-1})$ to define the operator

$$\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{\theta}) \equiv \Delta^{\theta_1}_{\Psi_1|\Omega} \sharp_{f_1} \cdots \sharp_{f_{n-1}} \Delta^{\theta_n}_{\Psi_n|\Omega} . \tag{6.315}$$

We are using the simplified notation[55]

$$X_1 \sharp_{f_1} X_2 \sharp_{f_2} X_3 \equiv X_1 \sharp_{f_1} (X_2 \sharp_{f_2} X_3) . \tag{6.316}$$

Definition 6.2.9. *We define the multi-state f-divergence to be*

$$S^{\vec{f}}_{\vec{\theta};r}(\vec{\psi}\|\omega) = \frac{-2r}{\prod_{\mathrm{i}=1}^n (1-\theta_\mathrm{i})} \log \left\| \left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{\theta})\right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} . \tag{6.317}$$

This is a special case of the more general measure

$$S^{\vec{f},\vec{g}}_{r}(\vec{\psi}\|\omega) = \frac{-1}{N(\vec{g})} \log \left\| \left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{g})\right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega}$$
$$\Delta^{\vec{f}}_{\vec{\Psi}|\Omega}(\vec{g}) \equiv g_1(\Delta_{\Psi_1|\Omega}) \sharp_{f_1} \cdots \sharp_{f_{n-1}} g_n(\Delta_{\Psi_n|\Omega}) \tag{6.318}$$

for operator monotone functions $f_1, \cdots, f_{n-1}$ with $f_\mathrm{i}(1) = 1$ and $g_1, \cdots, g_n$ with g_i satisfying $g_\mathrm{i}(x) \geq 0$ for all $x \geq 0$ and $\frac{-1}{N(\vec{g})}$ is a normalization.

In the remainder of this work, we focus on the measure in (6.317). We will see that when $\theta_1 + \cdots + \theta_n = 1$ this measure is independent of $|\Omega\rangle$.

Theorem 6.2.4. *Let $\Phi : \mathcal{B} \to \mathcal{A}$ be an unital CP map such that for a state ω_A on $\mathcal{A}$, the corresponding state on $\mathcal{B}$ is $\omega_B = \Phi^*(\omega_A)$. Let Ω_A and Ω_B denote the states as vectors in*

[55]↑Multi-variate operator geometric means were discussed by [214].

the corresponding GNS Hilbert spaces $\mathcal{H}_A$ and $\mathcal{H}_B$ and let $F : \mathcal{H}_B \to \mathcal{H}_A$ be the contraction operator corresponding to the map Φ in the GNS Hilbert space. Given a set of states $\vec{\psi}_A = (\psi_1, \cdots, \psi_n)$ on $\mathcal{A}$, for $r \geq 1$ the multi-state f-divergence satisfies the data processing inequality

$$S^{\vec{f}}_{\vec{\theta},r}(\vec{\psi_B}\|\omega_B) \leq S^{\vec{f}}_{\vec{\theta},r}(\vec{\psi_A}\|\omega_A) \ . \tag{6.319}$$

Proof. To prove the data processing inequality, as before, we first construct the inequality

$$F^\dagger \Delta^{\vec{f}}_{\vec{\Psi}|\Omega;A}(\vec{\theta}) F \leq \Delta^{\vec{f}}_{\vec{\Psi}|\Omega;B}(\vec{\theta}) \tag{6.320}$$

by repeatedly using (6.301), from which we get the contraction

$$F^{\vec{f}}_{\vec{\theta}} \equiv \left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;A}(\vec{\theta})\right)^{1/2} F \left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;B}(\vec{\theta})\right)^{-1/2} \ . \tag{6.321}$$

We have

$$\begin{aligned}
\left\|\left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;A}(\vec{\theta})\right)^{1/2}|\Omega_A\rangle\right\|_{2r,\Omega_A} &= \left\|F^{\vec{f}}_{\vec{\theta}}\left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;B}(\vec{\theta})\right)^{1/2}|\Omega_B\rangle\right\|_{2r,\Omega_A} \\
&\leq \|F^{\vec{f}}_{\vec{\theta}}\|_{(2r,\Omega_B)\to(2r,\Omega_A)} \left\|\left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;B}(\vec{\theta})\right)^{1/2}|\Omega_B\rangle\right\|_{2r,\Omega_B} \\
&\leq \left\|\left(\Delta^{\vec{f}}_{\vec{\Psi}|\Omega;B}(\vec{\theta})\right)^{1/2}|\Omega_B\rangle\right\|_{2r,\Omega_B} \ .
\end{aligned} \tag{6.322}$$

This implies that the multi-state f-divergences satisfy the data processing inequality for $r \geq 1$ for any quantum channel Φ^*. □

To be more concrete, we restrict to the geometric mean $\sharp_\alpha$ in (6.306). Consider n operators X_1 to X_n that pairwise commute. Define $\alpha_n = \alpha_0 = 0$ so that

$$\begin{aligned}
&X_1^{\theta_1} \sharp_{\alpha_1} \cdots \sharp_{\alpha_{n-1}} X_n^{\theta_n} = X_1^{\gamma_1\theta_1} \cdots X_n^{\gamma_n\theta_n} \\
&\gamma_{\rm i} = (1-\alpha_{\rm i})(\alpha_1 \cdots \alpha_{{\rm i}-1}) \ .
\end{aligned} \tag{6.323}$$

Note that $\gamma_{\rm i}$ are all positive and add up to one, hence, they are a probability distribution. We define the operator

$$\Delta^{\vec{\alpha}}_{\vec{\Psi}|\Omega}(\vec{\theta};r) \equiv \Delta^{\frac{\theta_1}{r\gamma_1}}_{\Psi_1|\Omega} \sharp_{\alpha_1} \cdots \sharp_{\alpha_{n-1}} \Delta^{\frac{\theta_n}{r\gamma_n}}_{\Psi_n|\Omega} \ . \tag{6.324}$$

The advantage of this definition is that it is independent of $\vec{\alpha}$ when the relative modular operators commute. Then, the multi-state Rényi divergence is

$$S^{\vec{\alpha}}_{\vec{\theta};r}(\vec{\psi}\|\omega) = \frac{-2r}{\prod_{\rm i=1}^{n}(1-\theta_{\rm i})} \log \left\| \left(\Delta^{\vec{\alpha}}_{\vec{\Psi}|\Omega}(\vec{\theta};r)\right)^{1/2} |\Omega\rangle \right\|_{2r,\Omega} \ . \tag{6.325}$$

In matrix algebras, this measure becomes

$$S^{\vec{\alpha}}_{\vec{\theta},r}(\vec{\psi}\|\omega) = \frac{-2r}{\prod_{\rm i=1}^{n}(1-\theta_{\rm i})} \log \left\| \left(\psi_1^{\frac{\theta_1}{r\gamma_1}} \sharp_{\alpha_1} \cdots \sharp_{\alpha_{n-1}} \psi_n^{\frac{\theta_n}{r\gamma_n}} \right)^{1/2} \omega^{\frac{\theta_0}{2r}} \right\|_{2r} \tag{6.326}$$

where $\theta_0 + \theta_1 + \cdots + \theta_n = 1$. We can think of $\theta_{\rm i}$ as a probability distribution associated with states $\psi_{\rm i}$. As before, when $\theta_0 = 0$ the measure above is independent of ω.

Similar to (6.282) we can divide our multi-state Rényi measure by $(1-\theta_0)$ to make it more symmetric among θ_0 and the rest of $\theta_{\rm i}$:

$$\hat{S}^{\vec{\alpha}}_{\vec{\theta},r}(\vec{\psi}\|\omega) \equiv \frac{1}{1-\theta_0} S^{\vec{\alpha}}_{\vec{\theta},r}(\vec{\psi}\|\omega) \ . \tag{6.327}$$

Special cases: In the limit $r \to \infty$, we have the multi-variate Lie-Trotter formula for self-adjoint operators $a_1, \cdots, a_n$ [212], [215]

$$\lim_{r\to\infty} \left({\rm e}^{a_1/r} \cdots {\rm e}^{a_n/r} \right)^r = {\rm e}^{a_1+\cdots+a_n} \ . \tag{6.328}$$

In lemma 3.3 of [216] it was shown that for $\alpha \in [0,1]$ and a_1 and a_2 self-adjoint

$$\lim_{r\to\infty} \left({\rm e}^{a_1/r} \sharp_\alpha {\rm e}^{a_2/r} \right)^r = {\rm e}^{(1-\alpha)a_1+\alpha a_2} \ . \tag{6.329}$$

This was further generalized by [217] to multi-variate geometric means

$$\lim_{r\to\infty}\left(\mathrm{e}^{a_1/r}\sharp_{\alpha_1}\cdots\sharp_{\alpha_{n-1}}\mathrm{e}^{a_{n-1}/r}\right)^r=\mathrm{e}^{\sum_{\mathrm{i}}\gamma_{\mathrm{i}}a_{\mathrm{i}}} \tag{6.330}$$

with γ_{i} given in (6.323). Notice that the right-hand-side of the equation above is invariant under the permutations of a_{i}. Applied to our measure, we find

$$\lim_{r\to\infty}S^{\vec{\alpha}}_{\vec{\theta},r}(\vec{\psi}\|\omega)\quad=\frac{-1}{(1-\theta_1)\cdots(1-\theta_n)}\log\mathrm{tr}\left(\mathrm{e}^{\sum_{\mathrm{i}}\theta_{\mathrm{i}}\log\psi_{\mathrm{i}}+\theta_0\log\omega}\right) \tag{6.331}$$

which is independent of α_{i}. Now, except for an overall $1/(1-\theta_0)$ factor, the reference state ω is no longer distinguished from the rest. We include ω inside $\vec{\psi}$ as ψ_0. We define the vector $\vec{\theta}_\epsilon$ that is $\theta_{\mathrm{j}}=1-\epsilon$ for a particular j, and $\theta_{\mathrm{i}}=\epsilon\beta_{\mathrm{i}}$ for $\mathrm{i}\neq\mathrm{j}$ including $\theta_0=\epsilon\beta_0$. Since $\vec{\theta}_\epsilon$ is a probability distribution the weights β_{i} sum up to one; hence β_{i} is also a probability distribution. In the limit $\epsilon\to0$, all $\theta_{\mathrm{i}}\to0$ except for θ_{j} that goes to one and we find[56]

$$\lim_{\epsilon\to0}S_{\vec{\theta}_\epsilon,\infty}(\vec{\psi})=\sum_{\mathrm{i}=0}^{n}\beta_{\mathrm{i}}\,\mathrm{tr}\left(\psi_{\mathrm{j}}(\log\psi_{\mathrm{j}}-\log\psi_{\mathrm{i}})\right)=\sum_{\mathrm{i}=0}^{n}\beta_{\mathrm{i}}S(\psi_{\mathrm{i}}\|\psi_{\mathrm{j}}) \tag{6.332}$$

which is the weighted average of the relative entropies of ψ_{i} with respect to ψ_{j}.

The same analysis can be repeated at finite r if all the states commute. In this case, we have n probability distributions and our multi-state measure is independent of both r and the vector $\vec{\alpha}$:

$$D_{\vec{\theta}}(\{p_1\},\cdots,\{p_n\})=\frac{-1}{(1-\theta_1)\cdots(1-\theta_n)}\log\left(\sum_{x\in X}p_1(x)^{\theta_1}\cdots p_n(x)^{\theta_n}\right)\ . \tag{6.333}$$

This is the generating functional in (6.168). Taking the same $\epsilon\to0$ limit of $\vec{\theta}_\epsilon$ gives a weighted average of the relative entropies:

$$\lim_{\epsilon\to0}D_{\vec{\theta}_\epsilon}(\vec{p})=\sum_{\mathrm{i}=1}^{n}\beta_{\mathrm{i}}D_{KL}(p_{\mathrm{i}}\|p_{\mathrm{j}})\ . \tag{6.334}$$

[56]↑Since the measure does not depend on $\vec{\alpha}$ we suppress it in the notation.

Consider the the multi-state measure in 6.327. In appendix 6.2.15, we show that in case where we set $\theta_{\mathrm{i}} = \epsilon\beta_{\mathrm{i}}$ and $\theta_0 = 1 - \epsilon$, at finite r, we obtain the same weighted average of relative entropies:

$$\lim_{\epsilon\to 0} \hat{S}^{\vec{\alpha}}_{\vec{\theta}_\epsilon;r}(\vec{\psi}\|\omega) = \sum_{\mathrm{i}=1}^{n} \beta_{\mathrm{i}} S(\psi_{\mathrm{i}}\|\omega) \ . \tag{6.335}$$

6.2.10 Infinite dimensions

In this section, we generalize our discussion of L^p_ω spaces and the multi-state Rényi divergences to an arbitrary von Neumann algebra. . This includes the local algebra of quantum field theory (QFT) that is a type III algebra, meaning that it does not admit a trace.[57] We closely follow the reference by [154].

Any normal CP map $\omega : \mathcal{A} \to \mathbb{C}$ that satisfies $\omega(1) = 1$ is called a state. In infinite dimensions, the vector $|\mathrm{e}\rangle$ or a trace might not exist. However, we can use any normal state ω to define an inner product for the map $a \to a\,|\Omega\rangle$:

$$\langle a_1\Omega|a_2\Omega\rangle = \omega(a_1^\dagger a_2) \ . \tag{6.336}$$

The closure of the set $a\,|\Omega\rangle$ is the GNS Hilbert space $\mathcal{H}_\omega$. For simplicity, we have restricted to the case of faithful normal states.

The Tomita operator $S_\Omega : \mathcal{H}_\omega \to \mathcal{H}_\omega$ is the anti-linear operator defined by

$$S_\Omega a\,|\Omega\rangle = a^\dagger\,|\Omega\rangle \ . \tag{6.337}$$

The closure of S_Ω has a polar decomposition

$$S_\Omega = J_\Omega \Delta_\Omega^{1/2} \tag{6.338}$$

where J_Ω and $\Delta_\Omega = \Delta_{\Omega|\Omega}$ are the generalizations of the modular conjugation and the modular operator to arbitrary von Neumann algebras. The natural cone is the set of vectors that are

[57]↑Formally, a trace is a normal completely positive (CP) map from the algebra to the complex numbers $\mathrm{tr} : \mathcal{A} \to \mathbb{C}$ that satisfies $\forall a_1, a_2 \in \mathcal{A} : \quad \mathrm{tr}(a_1 a_2) = \mathrm{tr}(a_2 a_1)$.

invariant under J_Ω. The vectors in the natural cone are in one-to-one correspondence with the normal states on $\mathcal{A}$. The relative Tomita operator is defined by the equation

$$S_{\Psi|\Omega} a \left|\Omega\right\rangle = a^\dagger \left|\Psi\right\rangle \tag{6.339}$$

with polar decomposition (after closure)

$$S_{\Psi|\Omega} = J_{\Psi|\Omega} \Delta_{\Psi|\Omega}^{1/2}, \tag{6.340}$$

where $\Delta_{\Psi|\Omega}$ is the generalization of the relative modular operator, and $J_{\Psi|\Omega}$ is an anti-unitary operator if both ω and ψ are faithful. When $\left|\Psi\right\rangle$ belongs to the natural cone we have $J_{\Omega|\Psi} = J_\Omega$, otherwise $J_{\Omega|\Psi} J_\Omega$ is a partial isometry in $\mathcal{A}'$; refer to [169].

Motivated by the expression (6.244) we define the (p, Ω)-norm of a vector $\left|\Psi\right\rangle \in \mathcal{H}_\omega$ as

$$\begin{aligned} \| \left|\Psi\right\rangle \|_{p,\Omega} &= \sup_{\left|\chi\right\rangle \in \mathcal{H}_\omega} \| \Delta_{\chi|\Omega}^{1/2-1/p} \left|\Psi\right\rangle \|, \qquad \forall p \in [2, \infty] \\ \| \left|\Psi\right\rangle \|_{p,\Omega} &= \inf_{\left|\chi\right\rangle \in \mathcal{H}_\omega} \| \Delta_{\chi|\Omega}^{1/2-1/p} \left|\Psi\right\rangle \|, \qquad \forall p \in [1, 2) \ . \end{aligned} \tag{6.341}$$

For $p \geq 2$ the (p, Ω)-norm is finite if $\left|\Psi\right\rangle$ is in the intersection of the domains of $\Delta_{\chi|\Psi}^{1/2-1/p}$ for all $\left|\chi\right\rangle \in \mathcal{H}_\omega$. When $\left|\Psi\right\rangle$ is outside of this intersection set we say $\| \left|\Psi\right\rangle \|_{p,\Omega} = \infty$. The closure of the set of all $\left|\Psi\right\rangle$ with finite (p, Ω)-norm is called the L_ω^p space [154]. For $p \in [1, 2)$ the L_ω^p space is defined to be the completion of the Hilbert space $\mathcal{H}_\omega$ with the (p, Ω)-norm. In general, we have $L_\omega^p \subseteq L_\omega^r$ for $r \leq p$ and L_ω^∞ is the algebra itself with its operator norm $\|a\|_\infty$. The L_ω^2 is the GNS Hilbert space $\mathcal{H}_\omega$ and the L_ω^1 is the space of normal linear functionals of $\mathcal{A}$. We can embed the vectors $\left|\Psi\right\rangle \in \mathcal{H}_\omega$ in L_ω^1 using the map

$$\psi(\cdot) = \left\langle\Psi\right| \cdot \left|\Omega\right\rangle \ . \tag{6.342}$$

However, since L_ω^1 is larger than $\mathcal{H}_\omega$ not all states ψ can be expressed this way.

The L^p_ω space is dual to the L^q_ω space when q is the Hölder dual of p:

$$\| \, |\Psi\rangle \, \|_{p,\omega} = \sup_{\||\chi\rangle\|_{q,\omega}=1} | \, \langle\chi|\Psi\rangle \, | \ . \tag{6.343}$$

Given a normal state $\psi \in L^1_\omega$ the vector

$$\Delta^{1/p}_{\Psi|\Omega} \, |\Omega\rangle \in L^p_\omega \tag{6.344}$$

for $p \in [2, \infty)$. For every vector $|\chi\rangle \in L^p_\omega$ there exists a unique $\psi \in L^1_\omega$ such that

$$|\chi\rangle = u \Delta^{1/p}_{\Psi|\Omega} \, |\Omega\rangle \tag{6.345}$$

with some partial isometry $u \in \mathcal{A}$. The vector

$$|\Omega(\theta)\rangle = \Delta^{\theta/2}_{\Psi|\Omega} \, |\Omega\rangle \tag{6.346}$$

is analytic in the complex strip $z = \theta + it$ with $\theta \in [0, 1]$. The reason is that we can write

$$\Delta^{\theta+\mathrm{it}}_{\Psi|\Omega} \, |\Omega\rangle = \Delta^{\theta}_{\Psi|\Omega} (D\Psi : D\Omega)_t \, |\Omega\rangle \tag{6.347}$$

where

$$(D\Psi : D\Omega)_t \equiv \Delta^{\mathrm{it}}_{\Psi|\Omega} \Delta^{-\mathrm{it}}_{\Omega} \in \mathcal{A} \tag{6.348}$$

is the Connes cocycle which is a partial isometry in the algebra for all real values of t.[186]

All the multi-state measures discussed in the previous section and the inequalities they satisfy generalize to arbitrary von Neumann algebras except for (6.258).[58]

[58]↑We do not know how to prove a generalization of (6.256) to arbitrary von Neumann algebras.

6.2.11 Quantum state discrimination

In asymmetric quantum state discrimination, we are given a state ω that we do not know a priori. The task is to perform measurements on this state to decide whether it is ω or any of the alternate hypotheses $K = \{\psi_0, \cdots \psi_k\}$. We would like to know what is the optimal measurement to perform on the state to make the decision and what is the minimum probability of misidentifying the state.

First, consider the case with only one alternate hypothesis ψ. Assume we are given n identical copies of the state prepared in the form $\omega^{\otimes n}$ and we are allowed to use any measurement in the n-copy Hilbert space to identify the state. Denote by β_n the probability that we misidentify the state as ψ with the optimal measurement. Any other measurement strategy to distinguish the two states fails with probability larger than β_n. According to quantum Stein's lemma β_n behaves asymptotically as [192]

$$\lim_{n\to\infty} -\frac{1}{n}\log\beta_n = S(\psi\|\omega) \ . \tag{6.349}$$

This provides an operational interpretation for relative entropy. The asymmetry of the relative entropy is related to the fact that we assumed that in reality the state was ω. Of course, if we were given the state ψ instead the asymptotic error rates are controlled by $S(\omega\|\psi)$. In general, in hypothesis testing we have two types of errors and their corresponding optimal probabilities

1. α_n: the state was ψ and we misidentified it as ω.

2. β_n: the state was ω and we misidentified it as ψ.

There is a trade-off between these two types of errors. Since we do not know whether the state is ω or ψ we should try to adopt a strategy that minimizes a combination of both errors. One might expect that these strategies would fail with minimal probabilities that interpolate between $S(\psi\|\omega)$ and $S(\omega\|\psi)$ as we go from minimizing the type 2 to type 1 errors. This intuition is confirmed in symmetric hypothesis testing when we choose to minimize the

average of the two error probability types. According to the quantum Chernoff bound, the optimal error probability in the symmetric case in the $n \to \infty$ limit is [218]

$$\begin{aligned} E_{\mathrm{e},n} &\leq \mathrm{e}^{-nC(\psi,\omega)} \\ C(\psi,\omega) &= -\log \inf_{\theta\in(0,1)} \mathrm{tr}\left(\psi^{\theta}\omega^{1-\theta}\right) \ . \end{aligned} \tag{6.350}$$

Note that the quantity $C(\psi,\omega)$ is related to a minimization over the Petz divergences in (6.245). The in-between strategies succeed with probabilities that depend on the Petz divergences. For instance, let us restrict to the measurements that leads to type 2 errors smaller than some constant $\mathcal{E}^{-nr}$, i.e. $\beta_n \leq \mathrm{e}^{-nr}$, and denote by $\alpha_{n,r}$ the optimal probability of the type 1 errors among these measurements. In the limit $n \to \infty$ we have [193]

$$\begin{aligned} \alpha_{n,r} &\leq \mathrm{e}^{-nH_r(\psi\|\omega)} \\ H_r(\psi\|\omega) &= \sup_{\theta\in(0,1)} \frac{\theta-1}{\theta}(r - D_\theta(\psi\|\omega)) \ . \end{aligned} \tag{6.351}$$

The quantity $H_r(\psi\|\omega)$ is called the Hoeffding divergence. The inequality above provides an operational interpretation for the Petz divergences $D_\theta(\psi\|\omega)$. It follows from (6.349) that if $r > S(\psi\|\omega)$ the error $\alpha_{n,r}$ tends to one exponentially fast for large n. It was shown by [193] that as $n \to \infty$

$$\begin{aligned} 1-\alpha_{n,r} &\leq \mathrm{e}^{-nH^*_r(\psi\|\omega)} \\ H^*_r(\psi\|\omega) &= \sup_{\theta>1} \frac{\theta-1}{\theta}(r - S_\theta(\psi\|\omega)) \ . \end{aligned} \tag{6.352}$$

The function $H^*_r(\psi\|\omega)$ is often called the converse Hoeffding divergence. It provides an operational interpretation for the sandwiched Rényi divergences.

Now, let us consider the completely asymmetric case where we are given ω but we have several alternate hypotheses $K = \{\psi_1, \cdots, \psi_k\}$. The generalization of the quantum Stein's

lemma in (6.349) to the multi-state setting is called the quantum Sanov's lemma. [219], [220] It says that given ω the optimal probability β_n of mistaking it for other states at large n is

$$\begin{aligned} &\beta_n \leq \mathrm{e}^{-nS(K\|\omega)} \\ &S(K\|\omega) = \min_{\psi_{\mathrm{i}}\in K} S(\psi_{\mathrm{i}}\|\omega) \ . \end{aligned} \tag{6.353}$$

In the symmetric case, given a set of hypothesis K, the multi-state Chernoff bound says that the minimal errors are controlled by the multi-state Chernoff distance [221]

$$\begin{aligned} &E_{\mathrm{e},n} \leq \mathrm{e}^{-n\xi} \\ &\xi = \min_{\mathrm{i}\neq\mathrm{j}} C(\psi_{\mathrm{i}}, \psi_{\mathrm{j}}) \ . \end{aligned} \tag{6.354}$$

However, away from the asymmetric case when we have to minimize various types of errors that generalize the type 1 and type 2 errors to multi-state setting, one expects that the multi-state measures that control the optimal probabilities to interpolate between the relative entropies $S(\psi_{\mathrm{i}}\|\omega)$ and $C(\psi_{\mathrm{i}}, \psi_{\mathrm{j}})$. The optimal error probabilities satisfy a data processing inequality because all distinguishability measures are non-increasing as we restrict the set of allowed measurements. Our multi-state measures interpolate in between these measures as we vary the probability measure $(\theta_0, \theta_1, \cdots, \theta_m)$ and satisfy the data processing inequality. We take this as an evidence to conjecture that the multi-state Rényi divergences in (6.326) have operational interpretations in asymmetric multi-state discrimination where we are given the state ω and the hypotheses are the states $\psi_1, \cdots, \psi_m$. One attempt to make this conjecture more precise is as follows:[59] In the multi-state setting with m alternative hypotheses $\{\psi_1, \cdots, \psi_m\}$ there are m probability errors $\beta_{\mathrm{i},n}$ associated with misidentifying ω with ψ_{i}. Choose a specific j and restrict to the measurements with error probabilities $\beta_{\mathrm{i},n} \leq \mathrm{e}^{-nr_{\mathrm{i}}}$ for $\mathrm{i} \neq \mathrm{j}$ at large number of measurements n. One might expect that the optimal error probability for j is given by an infimum over θ_{i} of some function of r_{i} minus our multi-state measures. However, we do not know what function of r_{i} is relevant or how to fix the value of the α_{i} parameters. In the classical limit, the α_{i} parameters go away making it easier to

[59] ↑We thank Milan Mosonyi for the suggestion.

find the appropriate function of r_{i}, however we will not attempt that here. For more recent developments in quantum state discrimination refer to [222], [223].

6.2.12 Discussion

In this work, we constructed multi-state Rényi divergences and proved that they satisfy the data processing inequality in the range $r \geq 1$ and $\theta_{\mathrm{i}} \in [0, 1]$. Both the Petz and the sandwiched Rényi divergences are monotonic in p; however, we did not explore potential monotonicity of our multi-state Rényi divergences in any of the parameters r or θ. We postpone this question to future work.

Recently, Fawzi and Fawzi used the Kubo-Ando geometric to define new quantum Rényi divergences in terms of a convex optimization program and proved that they satisfy the data processing inequality [224]. It would be interesting to use the non-commutative L^p_ω spaces to rewrite their expressions as (p, ω)-norms and explore their potential multi-state generalizations.

In section 6.2.7 we analytically continued the vector (6.284) to complex θ_{i}. Consider the vectors $|\Omega_{\mathrm{i}}\rangle = u_{\mathrm{i}} |\Omega\rangle$ where $u_{\mathrm{i}} \in \mathcal{A}$ are unitary operators. In that case, the relative modular operator can be written in terms of the modular operator of ω:

$$\Delta_{u\Omega|\Omega} = u \Delta_\Omega u^\dagger \tag{6.355}$$

where Δ_Ω is the modular operator of Ω. Then, our analytically continued vector is

$$|\Omega_{u_1,\cdots u_n}(\vec{z})\rangle = u_1 \Delta_\Omega^{z_1}(u_1^\dagger u_2) \Delta_\Omega^{z_2}(u_2^\dagger u_3) \cdots \Delta_\Omega^{z_n} u_n^\dagger |\Omega\rangle \ . \tag{6.356}$$

If we take all z_{i} to be imaginary we end up with modular evolved operators

$$\begin{aligned} &\| \, |\Omega_{u_1,\cdots,u_n}(\mathrm{i}\vec{t})\rangle \, \| = \|(u_1^\dagger u_2)_{t_1}(u_2^\dagger u_3)_{t_1+t_2} \cdots u_n |\Omega\rangle \, \| \\ &a_t \equiv \Delta_\Omega^{\mathrm{i}t} a \Delta_\Omega^{-\mathrm{i}t} \ . \end{aligned} \tag{6.357}$$

For general values of $t_{\rm i}$ we obtain a $2n$-point modular correlation function that is not modular time-ordered. In fact, since $a \in \mathcal{A}$ belong to L^{∞}_{ω} we can generalize our vector in (6.284) by introducing operators $a_{\rm i} \in \mathcal{A}$ (not necessarily unitaries)

$$\Delta^{z_1}_{\Psi_1|\Omega} a_1 \cdots \Delta^{z_n}_{\Psi_n|\Omega} a_n \left|\Omega\right\rangle \ . \tag{6.358}$$

setting $\left|\Omega_{\rm i}\right\rangle = \left|\Omega\right\rangle$ and all $z_{\rm i} = {\rm i}t_{\rm i}$ imaginary we obtain the out-of-time-ordered modular multi-point correlators. It would be interesting to search for potential connections between these out-of-time-ordered correlators and the notions of modular chaos previously introduced in the literature.[225], [226]

It is important to note that in our definition of the multi-state Rényi divergences in (6.326) we restricted to the range $0 \leq \theta_1 + \cdots \theta_n \leq 1$ to make sure that the resulting vector is in L^{2r}_{ω}. In principle, we can extend beyond this range, for instance, by making some $\theta_{\rm i}$ negative. While the resulting multi-state measure would not always be finite, in an infinite dimensional system that is hyperfinite (approximated by matrix algebras arbitrarily well) one expects that this measure is finite for a large class of states $\psi_1, \cdots, \psi_n$. It would be interesting to explore the data processing inequality in this extended range.[60]

Finally, the analysis with non-commutative L^{p}_{ω} spaces suggests that one might be able to prove an improved data processing inequality using Hirschman's lemma, refer to [213], [227], [228]. We postpone this to future work.

6.2.13 Appendix: Riesz-Thorin theorem

In this appendix, we prove the Riesz-Thorin theorem for the Araki-Masuda (p, ω)-norms [229].

[60]↑Note that our proof only works when all $\theta_{\rm i}$ are positive.

Definition 6.2.10. *Consider the algebras $\mathcal{A}$ and $\mathcal{B}$, faithful states ω_A and ω_B and their corresponding GNS Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For a bounded linear map $T : \mathcal{H}_A \to \mathcal{H}_B$ and $p, q \geq 2$ as in (6.191) and (6.192) we define the $(p, A) \to (q, B)$ norm to be*

$$\|T\|_{(p,A)\to(q,B)} = \sup_{|\chi\rangle\in\mathcal{H}_A} \frac{\|T\,|\chi\rangle\,\|_{(q,\Omega_B)}}{\|\,|\chi\rangle\,\|_{(p,\Omega_A)}} \ . \tag{6.359}$$

Theorem 6.2.5 (Riesz-Thorin)**.** *Consider the algebras $\mathcal{A}$ and $\mathcal{B}$, faithful states ω_A and ω_B and their corresponding GNS Hilbert space $\mathcal{H}_A$ and $\mathcal{H}_B$, respectively. For a bounded linear map $T : \mathcal{H}_A \to \mathcal{H}_B$ and $p_\theta, q_\theta \geq 2$, $\theta \in [0, 1]$ such that*

$$\begin{aligned} \frac{1}{p_\theta} &= \frac{1-\theta}{p_0} + \frac{\theta}{p_1} \\ \frac{1}{q_\theta} &= \frac{1-\theta}{q_0} + \frac{\theta}{q_1}, \end{aligned} \tag{6.360}$$

we have the inequality

$$\|T\|_{(p_\theta,A)\to(q_\theta,B)} \leq \|T\|^{1-\theta}_{(p_0,A)\to(q_0,B)} \|T\|^{\theta}_{(p_1,A)\to(q_1,B)} \ . \tag{6.361}$$

Proof. To prove this inequality, we first use the fact that any $\|\,|\chi\rangle\,\|_{p_\theta,A} = 1$ can be written as $u\Delta^{1/p_\theta}_{\chi|\Omega;A}\,|\omega_A\rangle$ to write the left-hand-side as

$$\|T\|_{(p_\theta,A)\to(q_\theta,B)} = \sup_{|\chi\rangle\in\mathcal{H}_A, u\in\mathcal{A}} \|Tu\Delta^{1/p_\theta}_{\chi|\Omega;A}\,|\Omega_A\rangle\,\|_{q_\theta,\Omega_B} \ . \tag{6.362}$$

We can use the definition of the (q_θ, Ω_B) norm in (6.252) to write the expression above as

$$\|T\|_{(p_\theta,A)\to(q_\theta,B)} = \sup_{u\in\mathcal{A},|\chi\rangle\in\mathcal{H}_A,|\Phi\rangle\in\mathcal{H}_B} \|\Delta^{\frac{1}{2}-\frac{1}{q_\theta}}_{\Phi|\Omega;B} Tu\Delta^{\frac{1}{p_\theta}}_{\chi|\Omega;A}\,|\Omega_A\rangle\,\| \ . \tag{6.363}$$

We define the function

$$f(\theta) = \|\Delta^{\frac{1}{2}-\frac{1}{q_\theta}}_{\Phi|\Omega;B} Tu\Delta^{\frac{1}{p_\theta}}_{\chi|\Omega;A}\,|\Omega_A\rangle\,\| \tag{6.364}$$

and then analytically continue $\theta \to z = \theta + it$ to the complex strip $\theta \in [0,1]$. This function is bounded, holomorphic everywhere inside the strip and is continuous on the boundaries of the strip at $\theta = 1$ and $\theta = 0$. Therefore, by the Phragmén-Lindelöf principle, it takes its maximum value on the boundaries of the strip. Using lemma 6.2.1.1, we find

$$\|\Delta_{\Phi|\Omega;B}^{\frac{1}{2}-\frac{1}{q_\theta}} Tu \Delta_{\chi|\Omega;A}^{\frac{1}{p_\theta}} |\Omega_A\rangle \| \leq \|\Delta_{\Phi|\Omega;B}^{\frac{1}{2}-\frac{1}{q_0}} Tu \Delta_{\chi|\Omega;A}^{\frac{1}{p_0}} |\Omega_A\rangle \|^{(1-\theta)} \|\Delta_{\Phi|\Omega;B}^{\frac{1}{2}-\frac{1}{q_1}} Tu \Delta_{\chi|\Omega;A}^{\frac{1}{p_1}} |\Omega_A\rangle \|^{\theta} \ .$$

Taking the supremum of both sides and using $\sup(fg) \leq \sup(f)\sup(g)$ completes the proof.

□

6.2.14 Appendix: Extended range of θ

Consider the (θ, r)-Rényi divergence. If we choose $\theta \in (-1, 0]$ the measure need not be finite. However, for a dense set of states it is finite. To see this, first assume that there exists a positive constant c such that for all $a_+ \in \mathcal{A}$ we have

$$\omega(a_+) \leq c\psi(a_+) \ . \tag{6.365}$$

In the density matrix setting, it means that the following operator is positive semi-definite

$$c\psi - \omega \geq 0 \ . \tag{6.366}$$

Since the map Φ^* is CP we also have

$$c\Phi^*(\psi) - \Phi^*(\omega) \geq 0 \ . \tag{6.367}$$

For such states we have

$$\langle a\Omega | \Delta_{\Psi|\Omega} a\Omega \rangle = \langle a^\dagger \Psi | a^\dagger \Psi \rangle \geq c^{-1} \langle a^\dagger \Omega | a^\dagger \Omega \rangle = c^{-1} \langle a\Omega | \Delta_\Omega a\Omega \rangle \tag{6.368}$$

which implies the inequality $c\Delta_{\Psi|\Omega} \geq \Delta_\Omega$. For $\theta \in [0,1]$ we obtain[61]

$$c^\theta \geq \Delta_\Omega^{\theta/2} \Delta_{\Psi|\Omega}^{-\theta} \Delta_\Omega^{\theta/2} \ . \tag{6.369}$$

This implies

$$c^\theta \geq \| \Delta_{\Psi|\Omega}^{-\theta/2} \Delta_\Omega^{\theta/2} \|_{\infty,\Omega} \ . \tag{6.370}$$

Therefore, the condition in (6.365) says that the vector

$$\Delta_{\Psi|\Omega}^{\theta/2} \, |\Omega\rangle \in \mathcal{H}_\Omega \tag{6.371}$$

for $\theta \in [-1,1]$. For $r \geq 1$ this vector is in L^{2r}_ω, therefore

$$S_{\theta,r}(\psi\|\omega) = \frac{-2r}{1-\theta} \log \| \Delta_{\Psi|\Omega}^{\theta/2r} \, |\Omega\rangle \, \|_{2r,\Omega} \tag{6.372}$$

is finite. The (θ, r)-Rényi divergences are defined for $r \in [1/2, \infty]$ and $\theta \in [0,1]$ but for $r \geq 1$ we can extend the range of θ to $[-1,1]$.

6.2.15 Appendix: The relative entropy limit

This appendix uses arguments similar to those by [213]. Consider the family of vectors $|\chi_\epsilon\rangle \in L^{2r}_\omega$ such that $|\chi_\epsilon\rangle = |\Omega\rangle + \epsilon \, |\chi_1\rangle + O(\epsilon^2)$. If we normalize the vector $|\chi_\epsilon\rangle$ to $|\bar{\chi}_\epsilon\rangle = |\chi_\epsilon\rangle \, / \| \, |\chi_\epsilon\rangle \, \|_{2,\Omega}$ we obtain

$$\lim_{\epsilon \to 0} \frac{1}{2\epsilon} \| \, |\bar{\chi}_\epsilon\rangle - |\Omega\rangle \, \|^2 = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(1 - \mathrm{Re} \, \langle \bar{\chi}_\epsilon | \Omega \rangle\right) = 0 \ . \tag{6.373}$$

Next, we note that for $r \geq 1$ we have

$$\mathrm{Re} \, \langle \bar{\chi}_\epsilon | \Omega \rangle \leq | \, \langle \bar{\chi}_\epsilon | \Omega \rangle \, | \leq \| \, |\bar{\chi}_\epsilon\rangle \, \|_{2r,\Omega} \| \, |\Omega\rangle \, \|_{s,\Omega} = \| \, |\bar{\chi}_\epsilon\rangle \, \|_{2r,\Omega} \leq \| \, |\bar{\chi}_\epsilon\rangle \, \|^{1/r}_{2,\Omega} = 1 \tag{6.374}$$

[61] ↑See also Lemma 5 of [213].

where in the second inequality we have used the Hölder inequality and the fact that the (s,Ω)-norm of $|\Omega\rangle$ is always one. In the last inequality, we have used the fact that for $r \geq p$

$$\| |\Psi\rangle \|_{r,\Omega}^{r} \leq \| |\Psi\rangle \|_{p,\Omega}^{p} \ . \tag{6.375}$$

This follows from a simple application of the Hadamard three-line theorem to the function $\| |\Psi\rangle \|_{r,\Omega}^{r}$; see lemma 8 and corollary 5 by [229] for more details.

Divide (6.374) by ϵ and take the limit $\epsilon \to 0$. Using (6.373) we obtain

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \left(1 - \| |\bar{\chi}_{\epsilon}\rangle \|_{2r,\Omega}\right) = 0 \ . \tag{6.376}$$

As a result,

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \| |\bar{\chi}_{\epsilon}\rangle \|_{2r,\Omega} = \partial_{\epsilon} \left(\| |\bar{\chi}_{\epsilon}\rangle \|_{2r,\Omega}\right)_{\epsilon \to 0} = 0 \ . \tag{6.377}$$

We are interested in the function

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \| |\chi_{\epsilon}\rangle \|_{2r,\Omega} \ . \tag{6.378}$$

The (p,Ω)-norms are homogeneous therefore

$$\log \| |\chi_{\epsilon}\rangle \|_{2r,\Omega} = \log \| |\bar{\chi}_{\epsilon}\rangle \|_{2r,\Omega} + \log \| |\chi_{\epsilon}\rangle \|_{2,\Omega} \tag{6.379}$$

and

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \| |\chi_{\epsilon}\rangle \|_{2r,\Omega} = \lim_{\epsilon \to 0} \frac{1}{\epsilon} \log \| |\chi_{\epsilon}\rangle \|_{2,\Omega} \ . \tag{6.380}$$

Therefore, we only need to study the $(2,\Omega)$-norm of the vector $|\chi_{\epsilon}\rangle$.

In the three-state Rényi measures our vector of interest is

$$|\chi_{\epsilon}\rangle = \left(\Delta_{\Psi_1|\Omega}^{\frac{\epsilon\beta}{(1-\alpha)r}} \sharp_{\alpha} \Delta_{\Psi_2|\Omega}^{\frac{\epsilon(1-\beta)}{\alpha r}}\right)^{1/2} |\Omega\rangle \ . \tag{6.381}$$

We have

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\log\|\,|\chi_\epsilon\rangle\,\|_{2r,\Omega}=\frac{1}{2}\,\langle\Omega|\partial_\epsilon\left(\Delta_{\Psi_1|\Omega}^{\frac{\epsilon\beta}{(1-\alpha)r}}\sharp_\alpha\Delta_{\Psi_2|\Omega}^{\frac{\epsilon(1-\beta)}{\alpha r}}\right)_{\epsilon\to 0}|\Omega\rangle\ . \tag{6.382}$$

We only need to compute the derivative:

$$\begin{aligned}\partial_\epsilon\left(X^\epsilon\sharp_\alpha Y^\epsilon\right)_{\epsilon\to 0}\qquad&=\log X+\partial_\epsilon\left(X^{-\epsilon/2}Y^\epsilon X^{-\epsilon/2}\right)^\alpha\Big|_{\epsilon=0}\\&=(1-\alpha)\log X+\alpha\log Y\ .\end{aligned} \tag{6.383}$$

Applied to our case in (6.382) we find

$$\lim_{\epsilon\to 0}\frac{1}{\epsilon}\log\|\,|\chi_\epsilon\rangle\,\|_{2r,\Omega}=\frac{-1}{2r}\left(\beta S(\psi_1\|\omega)+(1-\beta)S(\psi_2\|\omega)\right)\ . \tag{6.384}$$

As a result, from eq (6.327) we get

$$\lim_{\epsilon\to 0}\hat{S}^\alpha_{(\epsilon\beta,\epsilon(1-\beta)),r}(\psi_1,\psi_2\|\omega)=\beta S(\psi_1\|\omega)+(1-\beta)S(\psi_2\|\omega)\ . \tag{6.385}$$

To generalize to n states we need to compute

$$\begin{aligned}\partial_\epsilon\left(X_1^\epsilon\sharp_{\alpha_1}\cdots\sharp_{\alpha_{n-1}}X_n^\epsilon\right)_{\epsilon\to 0}\qquad&=(1-\alpha_1)\log X_1+\alpha_1\partial_\epsilon\left(X_2^\epsilon\sharp_{\alpha_2}\cdots\sharp_{\alpha_{n-1}}X_n^\epsilon\right)_{\epsilon\to 0}\\&=\gamma_1\log X_1+\gamma_2\log X_2+\cdots+\gamma_n\log X_n\ .\end{aligned} \tag{6.386}$$

Consider the vector

$$|\chi_\epsilon\rangle=\left(\Delta_{\Psi_1|\Omega}^{\frac{\epsilon\beta_1}{\gamma_1 r}}\sharp_{\alpha_1}\cdots\sharp_{\alpha_{n-1}}\Delta_{\Psi_n|\Omega}^{\frac{\epsilon\beta_n}{\gamma_n r}}\right)^{1/2}|\Omega\rangle\ . \tag{6.387}$$

Then,

$$\lim_{\epsilon\to 0}\hat{S}^{\vec{\alpha}}_{\vec{\theta}_\epsilon,r}(\vec{\psi}\|\omega)=\sum_{\mathrm{i}=1}^n\beta_{\mathrm{i}}S(\psi_{\mathrm{i}}\|\omega)\ . \tag{6.388}$$

6.2.16 Appendix: The $(p \to q)$-norm of contractions

Lemma 6.2.5.1. *Consider a linear operator $F_f : \mathcal{H}_B \to \mathcal{H}_A$ that satisfies $\|F_f^\dagger F_f\|_\infty \leq 1$; see equation (6.270). For $\forall p \in [2, \infty]$*

$$\|F_f\|_{(p,\Omega_B)\to(p,\Omega_A)} \leq 1 \ . \tag{6.389}$$

Proof. First, note that $\|F_f^\dagger F_f\|_\infty \leq 1$ implies that F_f is a contraction, i.e. $\|F_f\|_\infty \leq 1$, because $\|T^\dagger T\|_p = \|T\|_{2p}^2$ for any linear operator $T : \mathcal{H}_B \to \mathcal{H}_A$ and $\forall p \in [1, \infty]$. The proof has two steps: First, we show that for a contraction F_f we have $\|F_f\|_{(p,\Omega_B)\to(p,\Omega_A)} \leq \|F_f\|_\infty$ for $p = 2, \infty$. Then, we use the Riesz-Thorin interpolation theorem to establish (6.389).

For the first step, consider an isometry $V : \mathcal{H}_B \hookrightarrow \mathcal{H}_A$ and a cyclic and separating vector $|\Omega_B\rangle = V^\dagger |\Omega_A\rangle$. For $p = 2$ we have

$$\begin{aligned}
\|F_f\|_{(2,\Omega_B)\to(2,\Omega_A)} :=& \sup_{b\in\mathcal{B}} \frac{\|F_f b |\Omega_B\rangle \|_{2,\Omega_A}}{\|b |\Omega_B\rangle \|_{2,\Omega_B}} \\
=& \sup_{b\in\mathcal{B}} \frac{\|F_f V^\dagger V b V^\dagger |\Omega_A\rangle \|_{2,\Omega_A}}{\|b |\Omega_B\rangle \|_{2,\Omega_B}} \\
\leq& \|F_f V^\dagger\|_\infty \sup_{b\in\mathcal{B}} \frac{\|V b V^\dagger |\Omega_A\rangle \|_2}{\|b |\Omega_B\rangle \|_2} \\
\leq& \|F_f V^\dagger\|_\infty \\
\leq& \|F_f\|_\infty \ .
\end{aligned} \tag{6.390}$$

In the third line, we have used Hölder's inequality and (6.227). By a similar argument, for $p = \infty$, we obtain

$$\begin{aligned}
\|F_f\|_{(\infty,\Omega_B)\to(\infty,\Omega_A)} :=& \sup_{b\in\mathcal{B}} \frac{\|F_f b |\Omega_B\rangle \|_{\infty,\Omega_A}}{\|b |\Omega_B\rangle \|_{\infty,\Omega_B}} \\
=& \sup_{b\in\mathcal{B}} \frac{\|F_f V^\dagger V b V^\dagger |\Omega_A\rangle \|_{\infty,\Omega_A}}{\|b |\Omega_B\rangle \|_{\infty,\Omega_B}} \\
\leq& \|F_f V^\dagger\|_\infty \sup_{b\in\mathcal{B}} \frac{\|V b V^\dagger\|_\infty}{\|b\|_\infty} \\
\leq& \|F_f V^\dagger\|_\infty \\
\leq& \|F_f\|_\infty
\end{aligned} \tag{6.391}$$

where we have used $\|a\,|\Omega_A\rangle\,\|_{\infty,\Omega_A} = \|a\|_\infty$ which follows from equations (6.232) and (6.221). Since $\|F_f\|_\infty \leq 1$, the above inequalities imply that for $p = 2$ or $p = \infty$

$$\|F_f\|_{(p,\Omega_B)\to(p,\Omega_A)} \leq 1 \tag{6.392}$$

In the second step, we use the Riesz-Thorin interpolation theorem,

$$\|F_f\|_{(p_\theta,\Omega_B)\to(p_\theta,\Omega_A)} \leq \|F_f\|^{1-\theta}_{(\infty,\Omega_B)\to\infty,\Omega_A)}\|F_f\|^{\theta}_{(2,\Omega_B)\to 2,\Omega_A)} \tag{6.393}$$

for $\frac{1}{p_\theta} = \frac{1-\theta}{p_0} + \frac{\theta}{p_1}$ with $\theta \in [0,1]$ where we set $p_0 = \infty$ and $p_1 = 2$. From equation (6.392),

$$\|F_f\|_{(p_\theta,\Omega_B)\to p_\theta,\Omega_A)} \leq 1 \tag{6.394}$$

for $\forall p_\theta \in [2,\infty]$. Just by relabeling p_θ to p, we obtain the statement in (6.389). □

6.2.17 Appendix: Comparison of norms

Lemma 6.2.5.2. *For $q \in [2,\infty]$ we have $\|T\|_{q\to q} \leq \|T\|_{2\to 2}$, see figure 6.9.*

Proof.

$$\|T\|_{q\to q} = \sup_a \frac{\|T(a\otimes\mathcal{I})\|_q}{\|(a\otimes\mathcal{I})_q} \leq \sup_a \frac{\|T\|_\infty\|(a\otimes\mathcal{I})\|_q}{\|(a\otimes\mathcal{I})\|_q} = \|T\|_\infty = \|T\|_{2\to 2} \tag{6.395}$$

where we have used the Hölder inequality and the last equality follows from the definition of the $\|\cdot\|_{2\to 2}$ norm. □

7. SUMMARY

In this thesis, we studied the operator algebra quantum error correction(OAQEC) and the generalizations of information measures in a general quantum system (von Neumann algebras).

We formulated the known QEC conditions, such as the Knill-Laflamme condition and the sufficiency condition in the Schrödinger picture in the language of OAQEC for a general von Neumann algebra. Most importantly, we constructed a unique unital recovery map from the Petz dual map of a given error map (unital CP) with a trivial kernel.

We observed that the real-space renormalization theory exhibits the approximate QEC structure. We studied it by applying the continuous multiscale entanglement renormalization ansatz (cMERA) to the $1+1$-dimensional massive free boson fields.

We constructed the generalized entanglement entropy that captures the entanglement due to the presence of charges in a general quantum system using quantum relative entropy. It captures the contributions from the *bi-local intertwiners*. They are charge-neutral operators in the algebra of two local regions in spacetime, but charge operators in each local algebras. We worked out examples in finite-dimensional systems and quantum field theories.

We achieved the generalization of quantum relative entropy into the multistate f-divergence. We showed that the multistate f-divergence satisfies the monotonicity under a single unital completely positive map. Although we do not have its physical interpretation, we speculated its potential application to quantum state discrimination.

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