QUANTUM GROUPS, QUANTUM CATEGORIES AND QUANTUM FIELD THEORY

Jürg Fröhlich
Theoretische Physik
ETH-Hönggerberg
CH 8093 Zürich
Switzerland

Thomas Kerler
Department of Mathematics
Harvard University
Cambridge, MA 02138
USA

4131

Contents

Chapter 1

Introduction and Survey of Results

Our original motivation for undertaking the work presented in this book* has been to clarify the connections between the braid (group) statistics discovered in low-dimensional quantum field theories and the associated unitary representations of the braid groups with representations of the braid groups obtained from the representation theory of quantum groups – such as $U_q(g)$, with deformation parameter $q = q_N := \exp(i\pi/N)$, for some $N = 3, 4, \ldots$. Among quantum field theories with braid statistics there are two-dimensional, chiral conformal field theories and three-dimensional gauge theories with a Chern-Simons term in their action functional. These field theories play an important role in string theory, in the theory of critical phenomena in statistical mechanics, and in a variety of systems of condensed matter physics, such as quantum Hall systems.

An example of a field theory with braid statistics is a chiral sector of the two-dimensional Wess-Zumino-Novikov-Witten model with group $SU(2)$ at level k which is closely related to the representation theory of $\widehat{su}(2)_k$-Kac-Moody algebra, with $k = 1, 2, 3, \ldots$. The braid statistics of chiral vertex operators in this theory can be understood by analyzing the solutions of the Knizhnik-Zamolodchikov equations. Work of Drinfel'd [4] has shown that, in the example of the $SU(2)$-WZNW model, there is a close connection between solutions of the Knizhnik-Zamolodchikov equations and the representation theory

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*This book is based on the Ph.D. thesis of T.K. and on results in [6, 11, 24, 28, 42, 61]

of $U_q(sl_2)$ if the level k is related to the deformation parameter q by the equation
$\exp(i\pi/(k+2))$, and k is *not* a rational number. For an extension of these results to negative rationals see [62]. Unfortunately, the $SU(2)$-WZNW model is a *unitary* quant field theory *only* for the values $k = 1, 2, 3, \cdots$, not covered by the results of Drinfel'd. (goal was to understand the connections between the field theory and the quantum grc for the physically interesting case of positive integer levels. (This motivates much of analysis in Chapters 2 through 7.)

The notion of *symmetry* adequate to describe the structure of superselection sect in quantum field theories with braid statistics turns out to be quite radically differ from the notion of symmetry that is used to describe the structure of superselecti sectors in higher dimensional quantum field theories with permutation (group) statisti (i.e., Fermi-Dirac or Bose-Einstein statistics). While in the latter case compact grou and their representation theory provide the correct notion of symmetry, the situati is less clear for quantum field theories with braid statistics. One conjecture has be that quantum groups, i.e., quasi-triangular (quasi-)Hopf algebras, might provide a usel notion of symmetry (or of "*quantized symmetry*") describing the main structural featur of quantum field theories with braid statistics. It became clear, fairly soon, that t quantum groups which might appear in unitary quantum field theories have a deformati parameter q equal to a root of unity and are therefore *not* semi-simple. This circumstan is the source of a variety of mathematical difficulties which had to be overcome. Wo on these aspects started in 1989, and useful results, eventually leading to the material Chapters 4, 5 and 6, devoted to the representation theory of $U_q(g)$, q a root of unity, a to the so-called vertex-SOS transformation, were obtained in the diploma thesis of T.I see [6]. Our idea was to combine such results with the general theory of braid statisti in low-dimensional quantum field theories, in order to develop an adequate concept "quantized symmetries" in such theories; see Chapter 7, Sects. 7.1 and 7.2.

In the course of our work, we encountered a variety of mathematical subtleties a difficulties which led us to study certain abstract algebraic structures – a class of (r necessarily Tannakian) tensor categories – which we call *quantum categories*. Work

Doplicher and Roberts [29] and of Deligne [56] and lectures at the 1991 Borel seminar in Bern played an important role in guiding us towards the right concepts.

These concepts and the results on quantum categories presented in this volume, see also [61], are of some intrinsic mathematical interest, independent of their origin in problems of quantum field theory. Although problems in theoretical physics triggered our investigations, and in spite of the fact that in Chapters 2, 3 and 7, Sects. 7.1 through 7.4 we often use a language coming from local quantum theory (in the algebraic formulation of Haag and collaborators [17, 18, 19, 20]), all results and proofs in this volume (after Chapter 2) can be understood in a sense of pure mathematics: They can be read without knowledge of local quantum theory going beyond some expressions introduced in Chapters 2 and 3, and they are mathematically rigorous.

In order to dispel possible hesitations and worries among readers, who are pure mathematicians, we now sketch some of the physical background underlying our work, thereby introducing some elements of the language of algebraic quantum theory in a non-technical way. For additional details the reader may glance through Chapter 2.

For quantum field theories on a space-time of dimension four (or higher) the concept of a global gauge group, or symmetry G is, roughly speaking, the following one: The Hilbert space $\mathcal{H}$ of physical states of such a theory carries a (highly reducible) unitary representation of the group G. Among the densely defined operators on $\mathcal{H}$ there are the so-called local field operators which transform covariantly under the adjoint action of the group G. The fixed point algebra, with respect to this group action in the total field algebra, is the algebra of observables. This algebra, denoted by $\mathcal{A}$, is a C^*-algebra obtained as an inductive limit of a net of von Neumann algebras $\mathcal{A}(\mathcal{O})$ of observables localized in bounded open regions $\mathcal{O}$ of space-time. The von Neumann algebras $\mathcal{A}(\mathcal{O})$ are isomorphic to the unique hyperfinite factor of type III_1, in all examples of algebraic field theories that one understands reasonably well. The Hilbert space $\mathcal{H}$ decomposes into a direct sum of orthogonal subspaces, called superselection sectors, carrying inequivalent representations of the observable algebra $\mathcal{A}$. All these representations of $\mathcal{A}$ can be generated by composing a standard representation, the so-called vacuum representation, with *endomorphisms of $\mathcal{A}$. Each superselection sector also carries a representation of the global gauge group G which is equivalent to a mulitple of a distinct irreducible representation of G. As shown by Doplicher, Haag and Roberts (DHR) [19], one can introduce a notion of tensor product, or "composition", of superselection sectors with properties analogous to those of the tensor product of representations of a compact group. The composition of superselection sectors can be defined even if one does not know the global gauge group G of the theory, yet. From the properties of the composition of superselection sectors, in particular from the fusion rules of this composition and from the statistics of superselection sectors, i.e., from certain representations of the permutation groups canonically associated with superselection sectors, one can reconstruct important data of the global gauge group G. In particular, one can find its character table and its 6-j symbols. As proven by Doplicher and Roberts [29], those data are sufficient to reconstruct G. The representation category of G turns out to reproduce all properties of the composition of superselection sectors, and one is able to reconstruct the algebra of local field operators from these data. One says that the group G is dual to the quantum theory described by $\mathcal{A}$ and $\mathcal{H}$.

The results of Doplicher and Roberts can be viewed as the answer to a purely mathematical duality problem (see also [56]): The fusion rules and the 6-j symbols obtained from the composition of superselection sectors are nothing but the structure constants of a symmetric tensor category with C^* structure. The problem is how to reconstruct from such an abstract category a compact group whose representation category is isomorphic to the given tensor category. It is an old result of Tannaka and Kreĭn that it is always possible to reconstruct a compact group from a symmetric tensor category if the category is Tannakian, i.e., if we know the dimensions of the representation spaces and the Clebsch-Gordan matrices, or 3-j symbols, which form the basic morphism spaces. The results of Doplicher and Roberts represent a vast generalization of the Tannaka-Kreĭn results, since the dimensions and Clebsch-Gordan matrices are not known a priori.

Another duality theorem related to the one of Doplicher and Roberts is due to Deligne [56] which requires integrality of certain dimensions but no C^* structure on the symmetric tensor category. (It enables one to reconstruct algebraic groups from certain symmetric tensor categories.) Disregarding some subtleties in the hypotheses of these duality theorems, they teach us that it is equivalent to talk about compact groups or certain symmetric tensor categories.

Quantum field theories in two and three space-time dimensions can also be formulated within the formalism of algebraic quantum theory of DHR, involving an algebra $\mathcal{A}$ of observables and superselection sectors carrying representations of $\mathcal{A}$ which are compositions of a standard representation with *endomorphisms of $\mathcal{A}$. This structure enables us to extract an abstract tensor category described in terms of an algebra of fusion rules and 6-j symbols. Contrary to the categories obtained from quantum field theories in four or more space-time dimensions, the tensor categories associated with quantum field theories in two and three space-time dimensions are, in general, *not* symmetric but only *braided*. Therefore, they *cannot* be representation categories of cocommutative algebras, like group algebras. In many physically interesting examples of field theories, these categories are not even *Tannakian* and, therefore, *cannot* be identified, naïvely, with the representation category of a Hopf algebra or a quantum group; see [61]. The complications coming from these features motivate many of our results in Chapters 6 through 8.

The following models of two- and three-dimensional quantum field theories yield non-Tannakian categories:

(1) *Minimal conformal models [7] and Wess-Zumino-Novikov-Witten models [8] in two space-time dimensions* .

The basic feature of these models is that they exhibit infinite-dimensional symmetries. The example of the $SU(n)$-WZW model can be understood as a Lagrangian field theory with action functional given by

$$S(g) \;=\; \tfrac{k}{16\pi} \;\int_{S^2} tr\,((g^{-1}\partial_\mu g)(g^{-1}\partial^\mu g))\,d^2x$$
$$+\,\tfrac{k}{24\pi}\int_{B^3} tr\,((\bar{g}^{-1}d\bar{g})^{\wedge 3}),$$

where, classically, a field configuration g is a map from the two-sphere S^2 to the group $G = SU(n)$, and $\bar{g}$ is an arbitrary extension of g from $S^2 = \partial B^3$ to the ball B^3; (such an extension always exists, since π_2 of a group is trivial). The second term

in $S(g)$ is the so-called Wess-Zumino term which is defined only $mod\, k\mathbb{Z}$. Classi cally, the theory exhibits a symmetry which is the product of two loop groups, fo right- and left movers, respectively. For $k = 1, 2, 3, \ldots$, the quantum theory associ ated with $S(g)$ has conserved currents generating two commuting $\widehat{su}(n)$-Kac-Mood algebras at level k, whose universal enveloping algebras contain Virasoro algebras (Sugawara construction). From the representation theory of the infinite-dimensiona Lie algebras of symmetry generators in these models, i.e., the representation the ory of Virasoro- or Kac-Moody algebras, one can construct algebras of so-calle *chiral vertex operators* which play the role of Clebsch-Gordan operators of (a semi simple quotient of) the representation category of the Virasoro- or Kac-Moody al gebra. *Local* conformally covariant field operators are then constructed by takin linear combinations of products of two such chiral vertex operators, a holomorphi one (left movers) and an anti-holomorphic one (right movers).

Of interest in relation to the main subject of our work is that the algebras of chira vertex operators, the holomorphic ones, say, appearing in these models provid us with categorial data corresponding to non-Tannakian braided tensor categories (This can be understood by studying the multi-valuedness properties and operato product expansions of chiral vertex operators. A very thorough analysis of th $SU(2)$-WZW model can be found in the papers of Tsuchiya and Kanie and o Kohno quoted in [9]; see also [8, 61].)

Zamolodchikov and others have studied "non-critical perturbations" of minimal con formal models which are integrable field theories [10]. Their results suggest tha there are plenty of massive quantum field theories in two space-time dimensio with fields exhibiting non-abelian braid statistics, as originally described in [11 (A perturbation of minimal conformal models giving rise to massive integrable fie theories is obtained from the $\phi_{(1,3)}$-field; a field with braid statistics is the fie obtained from a chiral factor of the $\phi_{(3,1)}$-field, after the perturbation has bee turned on [12].) To such non-conformal field theories one can also associate certa braided tensor categories. However, the general theory of superselection sectors two-dimensional, massive quantum field theories leads to algebraic structures mo general than braided tensor categories, including ones with non-abelian fusion rule algebras. A general understanding of these structures has not been accomplished, yet.

(2) Three-dimensional Chern-Simons gauge theory, [13, 14, 15] .

Consider a gauge theory in three space-time dimensions with a simply connected, compact gauge group $G \overset{\text{e.g.}}{=} SU(n)$. Let A denote the gauge field (vector potential) with values in $g \equiv Lie(G)$, the Lie algebra of the gauge group G, and let ψ be a matter field, e.g. a two-component spinor field in the fundamental representation of G. There may be further matter fields, such as Higgs fields. The action functional of the theory is given by

$$\begin{aligned} S[A,\bar{\psi},\psi] \overset{\text{e.g.}}{=} \; & g^{-2} \int tr\,(F^2) d\, vol. \\ & - \tfrac{il}{4\pi} \int tr\,(A \wedge dA + \tfrac{2}{3}\, A \wedge A \wedge A) \qquad (1.1) \\ & + \lambda \int \bar{\psi}(\not{D}_A + m)\, \psi\, d\, vol. + \cdots , \end{aligned}$$

where g, λ and m are positive constants, and l is an integer.

This class of gauge theories has been studied in [13, 14, 15]. Although the results in these papers are not mathematically rigorous, the main properties of these theories are believed to be as follows:

The gluon is massive, and there is no confinement of colour. Interactions persisting over arbitrarily large distances are purely topological and are, asymptotically, described by a pure Chern-Simons theory. Thus the statistics of coloured particles in Chern-Simons gauge theory is believed to be the same as the statistics of static colour sources in a pure Chern-Simons theory which is known explicitly [16]. The statistics of coloured asymptotic particles can be studied by analyzing the statistics of fields creating coloured states from the vacuum sector. Such fields are the Mandelstam string operators, $\psi_\alpha(\gamma_x)$, which are defined, heuristically, by

$$\psi_\alpha(\gamma_x) \;=\; \text{“}\sum_\beta N[\psi_\beta(x) P(\exp \int_{\gamma_\alpha} A_\mu(\xi) d\xi^\mu)_{\beta\alpha}]\text{”}, \qquad (1.2)$$

where α and β are group indices; γ_x is a path contained in a space-like surface, starting at x and reaching out to infinity, N is some normal ordering prescription,

and P denotes path ordering. (Similarly, conjugate Mandelstam strings $\bar{\psi}_\alpha(\gamma_x)$ are defined.)

For the field theories described in (1) and (2), one observes that when the group G is $SU(2)$ the combinatorial data of a braided tensor category, an algebra of fusion rules and 6-j symbols (braid- and fusion matrices), can be reconstructed from these field theories which is isomorphic to a braided tensor category that is obtained from the representation theory of the quantum group $U_q(sl_2)$, where

$$q \;=\; e^{\frac{i\pi}{k+2}} , \quad k \;=\; 1, 2, 3, \cdots ,$$

(with $k = l + const.$). These categories are manifestly non-Tannakian. This is the reason why it is not possible to reconstruct field operators transforming covariantly under some representation of $U_q(sl_2)$ on the Hilbert space of physical states of those theories. However, passing to a quotient of the representation category of $U_q(sl_2)$, $q = \exp(i\pi/(k+2))$, described in Chapters 6 and 7, we can construct a semi-simple, non-Tannakian, braided tensor category describing the composition and braid statistics of superselection sectors in these quantum field theories. In this sense, $U_q(sl_2)$ is the "quantized symmetry" dual to the quantum field theories described above. (For precise details see Chapter 7.)

The strategy used to prove this duality is to compare the fusion rules and the 6-j symbols of $U_q(sl_2)$ with the corresponding data of the field theories found, e.g., in [9], and to show that they coincide. More precisely, it is quite easy to show that the representations of the braid groups associated with tensor products of the fundamental representation of $U_q(sl_2)$ coincide with those associated with arbitrary compositions of the "fundamental superselection sector" of the corresponding field theories. One implication of our work is that, in fact, the entire braided tensor categories coincide. This result follows from a much more general uniqueness theorem stating that whenever a braided tensor category with C^* structure is generated by arbitrary tensor products of a selfconjugate object, ρ, whose tensor square decomposes into two irreducible objects, i.e.,

$$\rho \otimes \rho \;=\; 1 \oplus \psi, \qquad (1.3)$$

(where 1 is the neutral object, corresponding to the trivial representation of $U_q(sl_2)$, to the vacuum sector of the field theory, respectively), and a certain invariant associated with ρ, the so-called monodromy of ρ with itself, is non-scalar, then the category is isomorphic to the semi-simple subquotient of the representation category of $U_q(sl_2)$, for $q = \pm e^{\pm \frac{i\pi}{k+2}}$, $k = 1, 2, 3, \ldots$.

The abstract nature of eq. (1.3) suggests that this result applies to a class of local quantum field theories more general than the models described above. This observation and the fact that those models are not rigorously understood in every respect led us to work within the general framework of algebraic field theory. In this framework, ρ and ψ can be interpreted as irreducible *endomorphisms of the observable algebra $\mathcal{A}$, with 1 the identity endomorphisms of $\mathcal{A}$, and eq. (2.3) for a selfconjugate object ρ of a braided tensor category with C^* structure is equivalent to some bounds on a scalar invariant associated with ρ, its statistical dimension, $d(\rho)$; namely (1.3) is equivalent to

$$1 < d(\rho) < 2 . \tag{1.4}$$

The main result of this book is a complete classification of braided tensor categories with C^*-structure that are generated by a not necessarily selfconjugate, irreducible object ρ whose statistical dimension, $d(\rho)$, satisfies (1.4). This is the solution to a very limited generalization of the duality problem for groups. Our method of classification is unlikely to be efficient for much larger values of $d(\rho)$ than those specified in eq. (1.4) – except, perhaps, for certain families of examples connected with more general quantum groups. However, our solution to the problem corresponding to the bounds on $d(\rho)$ in eq. (1.4) might serve as a guide for more general attempts. In particular, our notions of product category and induced category might be useful in a general context.

The constructive part of our classification consists in the description of two families of categories: First, we need to understand the representation theory and tensor-product decompositions of $U_q(sl_2)$, with q a root of unity; (Chapters 4 and 5, and [6]). This will permit us to construct a non-Tannakian, braided tensor category by passing to the semi-simple quotient of the representation category of $U_q(sl_2)$; (vertex-SOS transformation; see

Chapter 6 and [61]). The generating object ρ, of this category can always be multiplied with the generator of a category whose fusion rules are described by the group algebra of a cyclic group $\mathbb{Z}_a$, $a = 2, 3, \ldots$, without changing the statistical dimension. The second task is thus to classify categories whose fusion rules are given by the group algebras of abelian groups.

It turns out that, besides the operation of taking products of categories just alluded to, we also need the notion of induced categories which are, in general, not quotients of representation categories; (Chapter 8, Sect. 8.1).

For a selfconjugate, generating object ρ, with $1 < d(\rho) < 2$, our proof of uniqueness relies on an inductive procedure reminiscent of what is known as cabeling. In order to extend our proof of uniqueness to categories generated by a non-selfconjugate, irreducible object, we have to study the interplay between the group of "invertible objects" in a category and gradings. This will permit us to separate the subcategories corresponding to invertible objects from the entire category and to thereby reduce the classification problem to that of categories with a selfconjugate generator whose statistical dimension satisfies (1.4); (Chapter 8).

As a prerequisite to the classification of braided tensor categories with C^* structure satisfying (1.4), we present a classification of fusion rule algebras which have the same properties as the object algebras of a tensor category; (Chapter 3 and Sect. 7.3). Our classification is limited to fusion rule algebras generated by an irreducible object ρ o statistical dimension $d(\rho)$ satisfying

$$1 \le d(\rho) \le 2 . \qquad (1.5$$

We find that there are many more fusion rule algebras than there are object algebra of braided tensor categories. Our classification relies on results of T.K. in [42].

When $d(\rho) = 2$ we essentially reproduce the fusion rules of the finite subgrou of $SU(2)$ which have been classified and described in terms of certain Coxeter grap by Mac Kay. In the sense that symmetric tensor categories are dual to groups ar braided tensor categories are a natural generalization of symmetric tensor categories, our main result might be viewed as a natural generalization and completion of the Mac Kay correspondence for $d(\rho) = 2$ to the entire range $1 \leq d(\rho) \leq 2$.

One application of our classification theorems to conformal field theory, in particular to minimal conformal models and $SU(2)$-WZW theories, is that we can reproduce the fusion rules, the braid- and the fusion matrices of these models from an algebraically simpler object, a quantum group. This is one way of making "the quantum group structure" of conformal field theories precise. Our uniqueness theorems permit us, moreover, to establish a precise connection between $SU(2)$-WZW theories at level k and $SU(k)$-WZW theories at level 2 which is useful to understand the details of the conformal imbedding of $(\widehat{su}(2)_k \times \widehat{su}(k)_2)$-Kac-Moody algebra into $\widehat{su}(2k)_1$-Kac-Moody algebra. For example, we find that the braided tensor categories constructed from the representation theory of $\widehat{su}(k)_2$-Kac-Moody algebra, with k even, are non-trivially induced by those constructed from $\widehat{su}(2)_k$-Kac-Moody algebra. This result is useful in the context of certain systems in condensed matter physics.

We conclude this introduction with some additional comments on the contents of the various chapters of this book and a summary of our main results, Theorem 3.4.11 and Theorem 8.2.11.

Survey of Contents

In Chapter 2 we explain the appearance of certain braided tensor categories, called C^*-quantum categories, in local quantum theories in two and three space-time dimensions. To this end, we use the formalism of algebraic field theory, which - following the arguments of Section 2.1 and the introduction - is expected to describe two dimensional conformal field theories and three dimensional topological field theories. In Section 2.2 we review the C^*-algebra approach to local quantum theories with braid statistics, in a form developed in [15, 24] generalizing the algebraic field theory of [19] for quantum theories with (para-) permutation statistics. In this framework the objects of the considered C^*-quantum category are a subset of the endomorphisms of the observable algebra $\mathfrak{A}$ and the arrows (or morphisms) are operators in $\mathfrak{A}$ intertwining these endomorphisms. The quantitative description of the structure of these categories in terms of R- and F- matrices is derived in Section 2.3. In Section 2.4 we show how to extract unitary representations of the braid groups equipped with Markov traces from a C^*-quantum category.

The objects of a quantum category together with the operations of taking direct sums and tensor products form a half algebra over the positive integers which we shall call a fusion rule algebra. An axiomatic definition of fusion rule algebras which forgets about their origin from quantum categories is given in Section 3.1. In Section 3.2 we show that notions familiar in C^*-categories can already be defined from the fusion rule algebra itself, namely a unique positive dimension (the statistical or Perron-Frobenius dimension) for rational fusionrules and a universal group of gradings. These concepts are eventually combined in the construction of quotients of fusion rule algebras, so called Perron-Frobenius algebras. In Section 3.3 we demonstrate how non trivially graded invertible objects may be used in order to derive simplified descriptions of fusion rule algebras. In particular, we derive for cyclic grading goups a general presentation of a fusion rule algebra in terms of an accordingly smaller fusion rule algebra, whose invertible objects are all trivially graded. We give several criteria implying that this fusion rule algebra is either $\mathbb{Z}_2$-graded or ungraded. Among the categories that are constructed from $\mathbb{Z}_2$- or ungraded algebras we find those which are generated by a single object ρ of dimension $d(\rho)$ not greater than two (with the exception of two algebras at $d(\rho) = 2$). They are classified in Section 3.4, using the methods developed in the previous section. More precisely, we first determine the fusion rule algebras with a selfconjugate generator of dimension less than or equal to two and we analyze the action of the respective groups of invertible objects. Composing them with $\mathbb{Z}_r$ -algebras and twisting them we obtain the complete list of fusionrules given in Theorem 3.4.11.

In the following three chapters we construct the C^*-quantum categories with A_n-fusionrules from the quantum group $U_q(sl_2)$.

For this purpose, we review in Chapter 4 the general defintion of a quasitriangular Hopf algebra, [3, 5], and the quantum groups $U_q(s\mathcal{L}_n)$, [2]. We introduce anti-cohomomorphic $*$-operations on quasitriangular Hopf algebras and define the finite dimensional examples $U_q^{red.}(s\mathcal{L}_n)$ for q a root of unity.

The representation theory of $U_q(s\ell_2)$ is treated in Chapter 5 following the remarks on invariant forms, commutativity constraints and contragradient representations for general quantum groups made in Section 5.1. In Section 5.2 we give a summary of the irreducible and the unitary representations of $U_q^{red.}(s\ell_2)$, and in Section 5.3 we study their tensor product decompositions. The formula given in Theorem 5.3.1 involves projective representations with vanishing q-dimensions, which naturally form a tensor ideal in the category of representations of $U_q(s\ell_2)$. The subquotient of the abstract representation ring by this ideal is a fusion rule algebra in the sense of Chapter 3, as described in Section 5.4.

In order to obtain a semisimple category we need not only divide out the radical of the objects, i.e., the representation ring, but perform a similar quotient for the entire category including the morphisms, i.e., the intertwiners of representations. This procedure is described in Section 6.1. We give the explicit definition of the structure matrices and verify the polynomial equations for the quotient category in Section 6.2. In Section 6.3 we prove that this category is a C^*-quantum category if $q = exp(\pm\frac{i\pi}{N})$. The connection between balancing (or statistical) phases of a quantum category and the special element of a ribbon-graph Hopf algebra and the relation between Markov traces and quantum traces are explained in Section 6.4.

The first two sections of Chapter 7 are devoted to the mathematical interpretation of the structure matrices found in Chapter 2 and the connection of duality theory for abstract tensor categories and the notion of duality in terms of global gauge symmetries for local quantum theories. We start with a summary of the ingredients entering the definition of an abstract quantum category and show its equivalence to the systems of R- and F- matrices we have used so far. Furthermore, we draw the connection to the theory of inclusions and towers of algebras, see [41, 23], if the category is obtained from a set of quasi-commuting endomorphisms on a hyperfinite von-Neumann algebras, e.g., a local subalgebra of the observable algebra of a local quantum theory. We review the known duality results, [29, 56], for abstract, symmetric categories and the existence o
field operators with global gauge group symmetry entailed by them. For braided, non-Tannakian categories the notion of duality needs to be modified, involving semisimpl
quotients of Tannakian categories arising from non-semisimple quantum groups. In thi
setting, however, the analogous construction of fields which are gauge symmetric with
respect to the dual Hopf algebra does not yield an operator algebra with local brai
relations and a closing operatorproduct expasion. This is explained in Section 7.2.

The goal of Sections 7.3 and 7.4 is to select from the list of fusion rule algebra
given in Theorem 3.4.11 those which are actually realized as the object algebras of
C^*-quantum category and, furthermore, characterize them by the decomposition of th
tensor products $\rho \circ \rho$ and $\rho \circ \bar{\rho}$ of the generator. The precise correspondence betwee
the dimension restriction $1 \leq d(\rho) \leq 2$ and the structure of these fundamental product
is given in Proposition 7.3.1. This result is refined in Proposition 7.3.5, where we sho
that the restriction $1 < d(\rho) < 2$ is equivalent to a two channel decomposition of $\rho \circ$
with one object being invertible so that the projections on the invertible object define
Temperley-Lieb algebra in $End(\rho^{\otimes n})$. In particular, the exclusion of the D_4-type fusio
rule algebras is inferred from the general result in Proposition 7.3.4 asserting that if $\rho \circ$
decomposes completely into M invertible objects, then $M = 2^n$ for some $n \in \mathbb{N}$. I
Section 7.4 we exploit the fact that the natural braid group representation in $End(\rho^{\otimes n}$
factors through a Temperley-Lieb algebra in order to compute the statistical phases for th
C^*-quantum categories with fusionrules given in Theorem 3.4.11.i). We find consistenc
requirements in this computation that allow us to discard the D- and E-type algebras an
certain twisted A-type algebras from the list of admissible object algebras. The remainin
algebras, listed in Proposition 7.4.11 together with their possible statistical phases, can a
be obtained from a direct product of an A_n- algebra and the fusion rule algebra given
the group $\mathbb{Z}_r$, for some $r \in \mathbb{N}$, either by inclusion or by quotienting with some irreducibl
graded fusion rule algebra epimorphism.

The results of Section 7.4 suggest that all relevant quantum categories can
obtained from a product of a category with A_n-fusionrules and a category with $\mathbb{Z}$

fusionrules. Having constructed categories with A_n-fusionrules in Chapters 4,5 and 6 we are left with the characterization of the quantum categories for the $\mathbb{Z}_r$-case. More generally, we classify in Section 7.5 the quantum categories for which all objects are invertible so that the fusionrules are given by a finitely generated, abelian group G. The set of inequivalent quantum categories for a fixed group G carries a natural group structure and we show this group to be canonically isomorphic to the cohomology group $H^4(G, 2; U(1))$, associated to Eilenberg-MacLane spaces. We discuss in some detail the $\mathbb{Z}_2$-obstruction of these categories to be strict , i.e., their non trivial structure if viewed as monoidal categories. In the concluding Proposition 7.5.4 we also give the structure matrices for a convenient choice of morphisms.

It turns out that any fusion rule algebra and any choice of statistical phases for the untwisted cases of Proposition 7.4.11 is realized by a subcategory of a C^*quantum category with A_n-fusionrules and a $\mathbb{Z}_r$-category.

The aim of Chapter 8 is to prove the uniqueness of these categories and to construct the categories with twisted fusionrules. The main tool in this is the notion of induced categories developed in Proposition 8.1.4. We also define an action of the group $H^4(Grad(Obj), 2; U(1))$ on the set of quantum categories with fusion rule algebra Obj, where $Grad(Obj)$ is the corresponding universal grading group. In the second part of Section 8.1 we find conditions that the orbit of a category with respect to this action contains a category, which is induced by a smaller one. The obstructions here are found to be elements of $H^5(Grad(Obj), 2; \mathbb{Z}_2)$, see Lemma 8.1.13.

In Lemma 8.2.4 of Section 8.2 we show that this obstruction is trivial in the case of A-type algebras. Using the uniquenss of induced categories and the uniqueness of A_2-categories given in Proposition 8.2.6 we infer the uniqueness and thereby the classification of the untwisted A-type (not necessarily C^*) quantum categories in Theorems 8.2.8 and 8.2.9 . The respective categories with twisted fusionrules are presented in Theorem 8.2.10 in terms of the untwisted categories they induce. Combining these results with Proposition 7.4.11 we arrive at the classification in Theorem 8.2.11 of C^*-quantum categories with a generator of dimension less than two.

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Chapter 2

Local Quantum Theory with Braid Group Statistics

2.1 Some Aspects of Low-Dimensional, Local Quantum Field Theory

As described in the introduction, it is the purpose of this work to elucidate properties of superselection sectors of local quantum theories with braid (group) statistics. In particular, we are interested in understanding the laws by which two superselection sectors of a local quantum theory with braid statistics can be composed. In more conventional field theoretic jargon, we are interested in understanding the operator algebra and the operator product expansions of analogues of charged fields in theories with braid statistics. This involves, in particular, introducing appropriate *algebras of fusion rules* and attempting to classify them. It involves, furthermore, to characterize and classify the statistics of superselection sectors, or, in other words, the statistics of "charged fields". More precisely, we wish to describe, as completely as possible, those *unitary representations of the braid group*, B_∞, that describe the statistics of superselection sectors in local quantum theories with braid statistics. It is well known [19, 20] that in quantum field theory in four- or higher-dimensional space-time the statistics of superselection sectors, or, equivalently, of

charged fields, is described by unitary representations of the permutation group, S_∞. It is quite a recent result, due to Doplicher and Roberts [29], that the representations of the permutation group S_∞ and the composition laws of the superselection sectors (fusion rules) of a local quantum field theory in four or more dimensions can be derived from the representation theory of some compact group which, in fact, has the interpretation of a global symmetry of the quantum field theory.

It is then natural to ask whether the fusion rules and the representations of B_∞ encountered in local quantum field theories with braid statistics can be derived from the representation theory of a natural algebra which, moreover, can be interpreted as a *generalized global symmetry* ("quantized symmetry") of the quantum field theory? A conjecture proposed frequently, but not really well understood (see, however, [30] for an example that is understood in detail) is that quasi-triangular (quasi-) Hopf algebras, in particular *quantum groups*, could play the role of algebras whose representation theory yields the fusion rules and the braid group representations of local quantum theories with braid statistics and that they can be interpreted as "global symmetries" of such theories [31, 28, 32].

One of our *main goals* in this book is to describe some classes of local quantum theories for which the conjecture just described can actually be proven completely. The quantum groups appearing in our examples are $U_q(s\ell_n)$, and we shall prove that the deformation parameter q must have one of the values $\exp(i\pi/N)$, N a positive integer $(\geq n+1)$ Our results are complete for $U_q(s\ell_2)$. (For some simpler examples, involving quasi-Hop algebras, see also [33].)

Next, we wish to recall some basic facts about braid statistics. In the context o quantum mechanics of point particles in two-dimensional space, braid statistics was discovered in [34, 35, 36]. However, a more precise analysis of braid statistics and a classification of all possible braid statistics requires the principles of local quantum (field) theory Examples of local quantum field theories, more precisely Chern-Simons gauge theories in three space-time dimensions with braid statistics were described in [36, 37, 38] and numerous further articles; see also [13, 14, 15]. It has been recognized in [15] that, apar from permutation statistics, braid statistics is the *most general statistics* of superselectio sectors and charged fields that can appear in local quantum theories in three space-time dimensions; (see also [22] for related, partial results). Historically, braid statistics of fields actually first appeared in quantum field models in two space-time dimensions with topological solitons; (see [11] and refs. given there). It should be emphasized, however, that the theory of statistics of superselection sectors in general local quantum field theories in two space-time dimensions is considerably *more general* than the theory of braid statistics. But, for the chiral sectors of *two-dimensional conformal field theories*, the statistics of superselection sectors and of the corresponding chiral vertex operators is always described by representations of the braid group B_∞, generated by certain Yang-Baxter matrices; see [21, 9, 11, 26, 27, 28, 22]

Inspired by results in [16], it has been argued in [24] that the theory of the statistics of sectors in general three-dimensional, local quantum theory is equivalent to the theory of the statistics of chiral vertex operators in two-dimensional conformal field theory; (i.e., the same braid statistics appear in both classes of theories). We may therefore focus our attention on the analysis of statistics in *three-dimensional* local quantum theory.

Next, we review some characteristic features of local quantum theory in three space-time dimensions.

(a) *Spin in three space-time dimensions.*

According to Wigner, a relativistic particle is described by a unitary, irreducible representation of the quantum mechanical Poincaré group, $\tilde{\mathcal{P}}^{\uparrow}_{+}$, which is the universal covering group of the Poincaré group, $\mathcal{P}^{\uparrow}_{+}$. In three space-time dimensions,

$$\mathcal{P}^{\uparrow}_{+} = SO(2,1) \rtimes \mathbb{R}^3 .$$

The three-dimensional Lorentz group, $SO(2,1)$, is homeomorphic to $\mathbb{R}^2 \times S^1$, its covering group is therefore homeomorphic to $\mathbb{R}^3$. If one imposes the relativistic spectrum condition one concludes that those representations of the quantum mechanical Poincaré group associated with three-dimensional Minkowski space that are relevant for the description of a relativistic particle are characterized by two real parameters, the *mass* $M \geq 0$, and the *"spin"* $s \in \mathbb{R}$. In particular, spin need *not* be an integer or half-integer number.

(b) *Localization properties of one-particle states.*

Let us now consider a local, relativistic quantum theory in three space-time dimensions describing a particle of mass $M > 0$ and spin s. As shown by Buchholz and Fredenhagen [20], one can then in general construct a "string-like field", ψ, with non-vanishing matrix elements between the physical vacuum, Ω, of the theory and one-particle states of mass M and spin s. This result follows from very general principles of local quantum theory; (locality, relativistic spectrum condition, existence of massive, isolated (finitely degenerate) one-particle states). The field ψ is, in general, neither observable nor local. However, as shown in [20], it can always be localized in a space-like cone, C, of arbitrarily small opening angle; (see Sect. 2.2 for precise definitions and results). Physically, C can be interpreted as the location of a fluctuating string of flux attached to a "charged particle". Particles of this kind are encountered in three-dimensional Chern-Simons gauge theories, [13, 14, 37, 38, 15].

It can happen that the field ψ is actually localizable in bounded regions of space-time. (This would be the case in field theories *without* local gauge invariance.) Then a general result, due to Doplicher, Haag and Roberts [19], proves that the spin of particles created by applying ψ to the vacuum Ω is necessarily integer or half-integer, the statistics of ψ is permutation statistics, and the usual spin-statistics connection holds. It follows that if the spin of a particle created by applying some field ψ to the vacuum Ω is neither integer nor half-integer then the field ψ *cannot* be localizable in bounded regions of space-time – but ψ is still localizable in space-like cones. It has also been proven in [15] that if the spin of the particle created by ψ is neither integer nor half-integer then ψ has necessarily *non-trivial braid statistics*, and a fairly non-trivial spin-statistics connection holds. We thus expect that particles with spin $s \notin \frac{1}{2}\mathbb{Z}$ can only be encountered in quantum field theories with a manifest or hidden local gauge invariance.

Another general result of [15] is that, under a certain minimality assumption on the structure of superselection sectors, non-trivial braid statistics can only appear in theories in which the discrete symmetries of space reflections in lines and time reversal are broken.

Thus the only realistic candidates of relativistic quantum field theories in three space-time dimensions describing particles with spin $s \notin \frac{1}{2}\mathbb{Z}$ and with braid statistics, called *anyons* [36], are Chern-Simons gauge theories described in [13, 14, 37, 38, 15], with an action S given e.g. by (1.1), or non-linear $0(3)-\sigma$-models with Hopf terms equivalent to abelian Chern-Simons theories. See also [14, 15] for a heuristic discussion of the properties of these theories.

Since a mathematically rigorous analysis of the quantum field theories just referred to would be difficult and has, in fact, not been carried out, so far, we shall, in this book, follow an *axiomatic approach*. The formalism most convenient for our purposes turns out to be *algebraic quantum field theory*, as originally proposed by Haag and Kastler [17]. Since algebraic quantum field theory does not appear to be terribly well known among theoretical physicists or mathematicians, we shall now give heuristic motivations of some of its main concepts which will then be reviewed more precisely in Sect. 2.2.

The local, gauge-invariant observables of a gauge theory are constructed from real currents, $J^a(x)$, $x \in \mathrm{M}^d$, $a = 1, 2, 3, \ldots$, which commute among each other at space-like separated arguments, from Wilson loop operators, $W(\mathcal{L})$, and Mandelstam string operators, $M(\gamma)$, where $\mathcal{L}$ is an arbitrary smooth, bounded, space-like loop without double points, and γ is an arbitrary smooth, bounded, space-like curve; etc.. In order to obtain densely defined operators on the vacuum sector, $\mathcal{H}_1$, of the theory, one has to smear out these currents, Wilson loops and Mandelstam strings: Let f be a real-valued test function. We define

$$J^a(f) := \int_{\mathrm{M}^d} dx\, J^a(x)\, f(x) .$$

One may expect that $J^a(f)$ defines a selfadjoint operator on the vacuum sector $\mathcal{H}_1$. Moreover, all bounded functions, A, of $J^a(f)$ are localized on the support of f, (in the sense that $\left[A, J^b(y)\right] = 0$ whenever y is space-like separated from the support of f, for all b).

Let $\sum$ be a finite-dimensional parameter space equipped with a smooth measure, $d\sigma$, and let $\{\mathcal{L}(\sigma) : \sigma \in \operatorname{supp} d\sigma \subseteq \sum\}$ be a family of smooth, space-like loops, free of selfintersections, smoothly depending on $\sigma \in \sum$ and contained in a space-time region $\mathcal{O} \subset \mathrm{M}^d$.

Heuristically, we define an operator

$$W_{\mathcal{O}} := \int_{\Sigma} d\sigma\, W(\mathcal{L}(\sigma))$$

(where the integral is interpreted in the weak sense). One can imagine that $W_{\mathcal{O}}$ defines a closed operator on $\mathcal{H}_1$ all of whose bounded functions are localized in $\mathcal{O}$. (Similar ideas apply to the Mandelstam strings $M(\gamma)$.)

We now define *local observable algebras* $\mathfrak{A}(\mathcal{O})$, for $\mathcal{O}$ some bounded space-time region, as the *von Neumann* (weakly closed*) *algebras* [17] *generated by all bounded functions of the operators*

$$\{J^a(f),\ \operatorname{supp} f \subset \mathcal{O}, \quad a = 1, 2, 3, \ldots ; \quad W_{\mathcal{O}} ;\ M_{\mathcal{O}}\} .$$

As explained above, one expects that if $\mathcal{O}_1$ and $\mathcal{O}_2$ are two space-like separated space-time regions then *locality* of the theory implies that

$$[A, B] = 0 \quad \text{for all} \quad A \in \mathfrak{A}(\mathcal{O}_1) , \quad B \in \mathfrak{A}(\mathcal{O}_2) .$$

It is also clear that if $\mathcal{O}_1 \subset \mathcal{O}_2$ then $\mathfrak{A}(\mathcal{O}_1) \subseteq \mathfrak{A}(\mathcal{O}_2)$. The general properties required o the net $\{\mathfrak{A}(\mathcal{O})\}_{\mathcal{O} \subset \mathrm{M}^d}$ of local algebras are discussed in [17, 19] and will be briefly sketche in Sect. 2.2.

Let U_1 denote the unitary representation of $\tilde{\mathcal{P}}_+^\uparrow$ describing the dynamics of th gauge theory on its vacuum sector $\mathcal{H}_1$. Let λ be an element of $\tilde{\mathcal{P}}_+^\uparrow$ projecting onto a element $(\Lambda, a) \in \mathcal{P}_+^\uparrow$, (where Λ is a Lorentz transformation and $a \in \mathbb{R}^d$ is a space-tim translation). Then one expects that, for every observable $A \in \mathfrak{A}(\mathcal{O})$, $U_1(\lambda)\, A U_1(\lambda)^*$ onl depends on (Λ, a) and is contained in the algebra $\mathfrak{A}\left(\mathcal{O}_{(\Lambda,a)}\right)$, where

$$\mathcal{O}_{(\Lambda,a)} = \left\{x \in \mathrm{M}^d : \Lambda^{-1}(x - a) \in \mathcal{O}\right\} .$$

Hence we have a representation, α, of $\mathcal{P}_+^\uparrow$ on the algebra of observables of the theory give by

$$\alpha_{(\Lambda,a)}(A) = U_1(\lambda)\, A U_1(\lambda)^* ,$$

with

$$\alpha_{(\Lambda,a)}(\mathfrak{A}(\mathcal{O})) = \mathfrak{A}\left(\mathcal{O}_{(\Lambda,a)}\right) .$$

We now suppose the theory has some non-trivial conserved charges giving rise to superselection rules. Let $\mathcal{H}_j$ be a Hilbert space of states of "charge j" orthogonal to the vacuum sector $\mathcal{H}_1$; (the charge is here viewed as being "multiplicative"). It is customary to assume that there exist a field $\psi^j(\gamma_x)$ carrying "charge j" with non-vanishing matrix elements between vectors in $\mathcal{H}_j$ and vectors in $\mathcal{H}_1$. Here γ_x is either a point $x \in \mathrm{M}^d$ (charged local fields) or a space-like string starting at a point $x \in \mathrm{M}^d$ and extending to space-like infinity (Mandelstam operators in gauge field theories without colour confinement, such as three-dimensional Chern-Simons gauge theories). Let $\{\gamma(\sigma) : \sigma \in \sum\}$ be a smooth, finite-dimensional family of space-like strings contained in a "space-like cone" $\mathcal{C} \subset \mathrm{M}^d$, and let $d\sigma$ be a smooth measure on $\sum$. Heuristically, one defines

$$\psi^j(\mathcal{C}) := \int\limits_{\sum} d\sigma\, \psi^j(\gamma(\sigma)) .$$

One may imagine that $\psi^j(\mathcal{C})$ defines a *closed* operator on the entire physical Hilbert space of the theory. Then $\psi^j(\mathcal{C})$ has a polar decomposition

$$\psi^j(\mathcal{C}) = U^j_{\mathcal{C}} \left|\psi^j(\mathcal{C})\right| ,$$

where $|\psi^j(\mathcal{C})|$ is a positive, selfadjoint operator of charge 0, hence leaving all superselection sectors invariant, and $U^j_{\mathcal{C}}$ is an operator carrying "charge j" and mapping the orthogonal complement of the null space of $|\psi^j(\mathcal{C})|$ isometrically to (a subspace of) the physical Hilbert space. Heuristically, the operators $U^j_{\mathcal{C}}$ and $|\psi^j(\mathcal{C})|$ commute with all observables localized in regions space-like separated from $\mathcal{C}$. One can now extend $U^j_{\mathcal{C}}$ to an isometric operator $V^j_{\mathcal{C}}$, defined on the entire physical Hilbert space, which carries the same charge as $U^j_{\mathcal{C}}$ and commutes with all observables localized in regions space-like separated from $\tilde{\mathcal{C}}$, for some cone $\tilde{\mathcal{C}}$ containing $\mathcal{C}$.

For every bounded observable A of the theory, the operator

$$\rho^j_{\mathcal{C}}(A) := \left(V^j_{\mathcal{C}}\right)^* A V^j_{\mathcal{C}}$$

is then expected to be again a bounded observable, and if A is localized in a space-time region space-like separated from $\tilde{\mathcal{C}}$ then $\rho^j_{\mathcal{C}}(A) = A$. The map $\rho^j_{\mathcal{C}}$ is therefore called an endomorphism of the observable algebra localized in $\tilde{\mathcal{C}}$.

In the next section, we recall rigorous results, due to Buchholz and Fredenhagen [20], asserting the existence of endomorphisms with the properties of $\rho^j_{\mathcal{C}}$ under very general, physically plausible hypotheses on the theory. The Buchholz-Fredenhagen construction of endomorphisms does *not* involve first constructing operators analogous to $V^j_{\mathcal{C}}$. Rather the existence of such operators – which are bounded versions of charged field operators – is *derived* from the existence of localized endomorphisms. It is one of the major goals of our work to construct operators analogous to the operators $V^j_{\mathcal{C}}$ and discuss their algebraic properties, in particular their *statistics*, for some class of field theories in three space-time dimensions characterized in terms of nets of local observable algebras and families of localized endomorphisms.

From now on, we shall work within the formalism of algebraic field theory [17, 18, 19, 20], motivated by the heuristic considerations sketched above, and our analysis will be mathematically rigorous. We expect that the hypotheses on which our analysis is based can be verified for some two-dimensional conformal field theories [30, 25] and some three-dimensional Chern-Simons gauge theories [38].

It should be mentioned that, in Sects. 2.2-2.4 and in Chapter 6, the reader is expected to be vaguely familiar with one of the references [11, 15, 22].

2.2 Generalities Concerning Algebraic Field Theory

The starting point of the algebraic formulation of local, relativistic quantum theory is a net, $\{\mathfrak{A}(\mathcal{O})\}$, of von Neumann algebras of local observables indexed by bounded, open regions, $\mathcal{O}$, in Minkowski space M^d. If $\mathcal{S}$ is an unbounded space-time region in M^d one defines an algebra of observables localized in $\mathcal{S}$ by setting

$$\mathfrak{A}(\mathcal{S}) = \overline{\bigcup_{\substack{\mathcal{O}\subset\mathcal{S} \\ \mathcal{O}\text{ bounded}}} \mathfrak{A}(\mathcal{O})}^{n} , \tag{2.1}$$

where the closure is taken in the operator norm. We define the algebra $\mathfrak{A}$ of all quasi-local observables as

$$\mathfrak{A} = \mathfrak{A}(\mathcal{S} = \mathrm{M}^d) . \tag{2.2}$$

The algebras $\mathfrak{A}(\mathcal{S})$ and $\mathfrak{A}$ are C^*-algebras. The relative commutant, $\mathfrak{A}^c(\mathcal{S})$, of $\mathfrak{A}(\mathcal{S})$ in $\mathfrak{A}$ is defined by

$$\mathfrak{A}^c(\mathcal{S}) = \{A \in \mathfrak{A} : [A,B] = 0\,, \qquad \text{for all} \qquad B \in \mathfrak{A}(\mathcal{S})\}\,. \tag{2.3}$$

The causal complement, $\mathcal{S}'$, of a region $\mathcal{S} \subset \mathbf{M}^d$ is defined as

$$\mathcal{S}' = \{x \in \mathbf{M}^d : (x-y)^2 < 0, \qquad \text{for all} \qquad y \in \mathcal{S}\}\,. \tag{2.4}$$

Let $\mathcal{C}_0$ be a wedge in $(d-1)$-dimensional space. The causal completion, $\mathcal{C}$, of $\mathcal{C}_o$ is defined by

$$\mathcal{C} = (\mathcal{C}_0')' \tag{2.5}$$

and is called a *simple domain*. If the opening angle of $\mathcal{C}_0$ is less than π $\mathcal{C}$ is called a *space-like cone*.

Locality and relativistic covariance of the theory are expressed in the following two postulates on the structure of the net $\{\mathfrak{A}(\mathcal{O})\}$.

(1) *Locality*:

$$\mathfrak{A}(\mathcal{S}') \subseteq \mathfrak{A}^c(\mathcal{S})\,, \tag{2.6}$$

for any open region $\mathcal{S} \subset \mathbf{M}^d$.

(2) *Poincaré covariance*: There is a representation, α, of the Poincaré group, $\mathcal{P}_+^\uparrow$, as a group of *automorphisms of $\mathfrak{A}$ with the property that

$$\alpha_{(\Lambda,a)}(\mathfrak{A}(\mathcal{S})) = \mathfrak{A}(\mathcal{S}_{(\Lambda,a)})\,, \tag{2.7}$$

where

$$\mathcal{S}_{(\Lambda,a)} = \left\{x \in \mathbf{M}^d : \Lambda^{-1}(x-a) \in \mathcal{S}\right\}\,. \tag{2.8}$$

The properties of a physical system described by $\{\mathfrak{A},\alpha\}$ can be inferred from the *representation theory* of $\{\mathfrak{A},\alpha\}$. We focus our attention on the analysis of physical systems at zero temperature and density. Then it suffices to consider a restricted class of representations of $\{\mathfrak{A},\alpha\}$ which has been described in work of Borchers [18] and Buchholz and Fredenhagen [20]. Buchholz and Fredenhagen start from the assumption that all representations describing a local, relativistic system at zero temperature and density car be reconstructed from what they call massive, single-particle representations [20]. They then *prove* that there exists at least one *vacuum representation*, 1, of $\mathfrak{A}$ on a separabl Hilbert space, $\mathcal{H}_1$, containing a unit ray, Ω, the *physical vacuum*, which is cyclic for and is space-time translation invariant, i.e.,

$$\langle \Omega, 1\,(\alpha_a(A))\,\Omega\rangle = \langle \Omega, 1(A)\Omega\rangle\,, \qquad (2.9$$

for all $A \in \mathfrak{A}$ and all $a \in \mathbf{M}^d$; here $\left\{\alpha_a \equiv \alpha_{(1,a)}\right\}$ is a representation of space-tim translations of $\mathbf{M}^d$. [In the analysis of [20] full Poincaré covariance is not assumed; i is sufficient to require locality and space-time translation covariance. In our analysis space-rotation covariance will be used at some point, but full Poincaré covariance is no needed.] It follows from (2.9) that space-time translations are unitarily implemented o $\mathcal{H}_1$ by a group of operators $U_1(a) = \exp i(a^0 H_1 - \vec{a}\cdot\vec{P}_1)$, $a = (a^0, \vec{a}) \in \mathbf{M}^d$, and it follow from the starting point chosen in [20] that the relativistic spectrum condition,

$$spec\left(H_1, \vec{P}_1\right) \subseteq \overline{V}_+ \qquad (2.10$$

holds.

In the following, we shall assume for simplicity that there is a *unique* vacuum representation, (i.e., there is no vacuum degeneracy). This assumption must be given up i the study of two-dimensional theories with topological solitons [11]. Our analysis can b extended to certain theories with vacuum degeneracy without much difficulty, in particular to a class of two-dimensional theories with solitons. It can also be applied to studyin the *chiral sectors of two-dimensional conformal field theories*; see e.g. [23, 22, 25]. W shall, however, focus our attention on three-dimensional theories, following [15, 24], sinc these have been studied less intensely.

If the vacuum is unique, and under suitable physically plausible hypotheses describe in [20], all representations, p, of $\mathfrak{A}$ encountered in the analysis of relativistic, local system at zero temperature and density have the property that, for an arbitrary space-like co $\mathcal{C} \subset \mathbf{M}^d$, the restriction of p to $\mathfrak{A}^c(\mathcal{C})$ is unitarily equivalent to the restriction of th vacuum representation, 1, to $\mathfrak{A}^c(C)$, i.e.,

$$p\big|_{\mathfrak{A}^c(C)} \simeq 1\big|_{\mathfrak{A}^c(C)} \,. \tag{2.11}$$

A representation of $\mathfrak{A}$ with this property is said to be *localizable in space-like cones* relative to the vacuum representation. In the framework of [20], only representations, p, of $\mathfrak{A}$ satisfying (2.11) are considered which are translation-covariant, i.e., for which there exists a continuous, unitary representation, U_p, of $\mathbb{M}^d$ on the representation space (*superselection sector*) $\mathcal{H}_p$ corresponding to the representation p such that

$$p(\alpha_a(A)) = U_p(a)\, p(A)\, U_p(-a)\,, \tag{2.12}$$

where

$$U_p(a) = \exp i\left(a^0 H_p - \vec{a}\cdot\vec{P}_p\right)\,, \tag{2.13}$$

and

$$spec\left(H_p, \vec{P}_p\right) \subseteq \overline{V}_+\,. \tag{2.14}$$

A fundamental assumption on the choice of the net $\{\mathfrak{A}(\mathcal{O})\}$ of local algebras is *duality*, [19, 20]: One assumes that the algebras $\mathfrak{A}(\mathcal{O})$ are chosen so large that

$$1\,(\mathfrak{A}(S))' = \overline{1(\mathfrak{A}^c(S))}^{w}\,, \tag{2.15}$$

where $\mathfrak{B}'$ denotes the commuting algebra of a subalgebra, $\mathfrak{B}$, of the algebra, $\mathfrak{B}(\mathcal{H}_1)$, of all bounded operators on $\mathcal{H}_1$, and $\overline{\mathfrak{B}}^{w} = (\mathfrak{B}')'$ denotes its weak closure. [Duality (2.15) can be derived from a suitable set of postulates for local, relativistic quantum field theory, [39], and expresses the property that states in $\mathcal{H}_1$ do not carry a non-abelian charge.]

Remark. The analysis presented in this chapter can be applied to the chiral sectors of two-dimensional conformal field theory if Minkowski space is replaced by the circle S^1, a compactified "light-ray", with a distinguished point P, the point at infinity, (corresponding to the auxiliary cone, C_a, introduced below), space-like cones, C, in $\mathbb{M}^d$ are replaced by intervals $I \subset S^1$, and Poincaré covariance is replaced by covariance under $PSL(2,\mathbb{R})$. In this case, the spectrum condition becomes the requirement that the generator, L_0, of rotations of S^1 is a positive operator with discrete spectrum.

Next, we construct an extension of the algebra $\mathfrak{A}$ which will be more convenient for our analysis. First, we note that the vacuum representation 1 of $\mathfrak{A}$ is faithful. In the following, we shall identify $\mathfrak{A}$ with the subalgebra $1(\mathfrak{A})$ of $\mathfrak{B}(\mathcal{H}_1)$. If $\mathfrak{B}$ is a subalgebra of $\mathfrak{A}$ we denote by $\overline{\mathfrak{B}}^{w}$ the weak closure of $1(\mathfrak{B})$ in $\mathfrak{B}(\mathcal{H}_1)$. Let C_a be some auxiliary space-like cone in $\mathbb{M}^d$ of arbitrarily small opening angle, and set

$$C_a + x = \left\{y \in \mathbb{M}^d : y - x \in C_a\right\}.$$

We define an enlarged algebra, $\mathfrak{B}^{C_a}$, containing $\mathfrak{A}$:

$$\mathfrak{B}^{C_a} = \overline{\bigcup_{x\in \mathbb{M}^D} \overline{\mathfrak{A}^c(C_a + x)}^{w}}^{n}\,. \tag{2.16}$$

A fundamental result of Buchholz and Fredenhagen [20] is that every representation p of $\mathfrak{A}$ localizable in space-like cones relative to 1 has a continuous extension to $\mathfrak{B}^{C_a}$. Moreover, given a space-like cone C in the causal complement of $C_a + x$, for some $x \in \mathbb{M}^d$, there exists a *endomorphism, ρ_C^p, of $\mathfrak{B}^{C_a}$ such that

$$\rho_C^p(A) = A, \qquad \text{for all} \qquad A \in \mathfrak{A}^c(C)\,, \tag{2.17}$$

and the representation $1\,(\rho_C^p(\cdot))$ of $\mathfrak{B}^{C_a}$ is unitarily equivalent to the representation p of $\mathfrak{B}^{C_a}$, i.e., there exists a unitary operator V_C from $\mathcal{H}_p$ to $\mathcal{H}_1$ such that

$$\rho_C^p(A) = V_C\, p(A) V_C^* : \tag{2.18}$$

Next, let ρ_C be a *endomorphism of $\mathfrak{B}^{C_a}$ localized in a space-like cone C, in the sense of equation (2.17), and let $\tilde{\rho}_{\tilde{C}}$ be a *endomorphism of $\mathfrak{B}^{C_a}$ localized in a cone $\tilde{C}$, with the property that $\tilde{\rho}_{\tilde{C}}$ is unitarily equivalent to some subrepresentation of ρ_C. Let S be a simple domain in the causal complement of $C_a + x$, for some $x \in \mathbb{M}^d$, with the property that $C \cup \tilde{C}$ is contained in the interior of S. Then there exists a partial isometry $\Gamma^S_{\rho_C,\tilde{\rho}_{\tilde{C}}}$, on $\mathcal{H}_1$, called a *"charge-transport operator"*, such that

$$\rho_C(A)\Gamma^S_{\rho_C,\tilde{\rho}_{\tilde{C}}} = \Gamma^S_{\rho_C,\tilde{\rho}_{\tilde{C}}}\tilde{\rho}_{\tilde{C}}(A)\,, \tag{2.19}$$

for all $A \in \mathfrak{B}^{C_a}$. It follows from (2.17) and *duality*, see (2.15) and [19, 20], that

$$\Gamma^S_{\rho_C,\tilde{\rho}_{\tilde{C}}} \in \overline{\mathfrak{A}(S)}^{w} \subset \mathfrak{B}^{C_a}\,. \tag{2.20}$$

Let p and q be two representations localizable in space-like cones relative to 1, and let ρ^p and ρ^q be two *endomorphisms of $\mathfrak{B}^{C_a}$ localized in space-like cones, C_p and C_q, with the properties that $C_p \subset C'_q, C_p \cup C_q \subset (C_a + x)'$, for some $x \in \mathbb{M}^d$, and p and q are unitarily equivalent to ρ^p and ρ^q, respectively. We define $p \circ q$ to be the unique equivalence class of representations of $\mathfrak{B}^{C_a}$ unitarily equivalent to the representation $\rho^p \circ \rho^q$ of $\mathfrak{B}^{C_a}$ on $\mathcal{H}_1$. It is easy to check that $p \circ q$ is again localizable in space-like cones relative to 1, that it is translation-covariant (see (2.12)) and satisfies the relativistic spectrum condition (see (2.14)), provided p and q are translation-covariant and satisfy the relativistic spectrum condition. It is not hard to see [20] that if C_p and C_q are space-like separated ($C_p \subset C'_q$) then $\rho^p \circ \rho^q = \rho^q \circ \rho^p$. Hence

$$p \circ q = q \circ p . \tag{2.21}$$

Clearly

$$1 \circ p = p \circ 1 = p . \tag{2.22}$$

Fredenhagen [40] has isolated natural physical conditions which imply the following properties of representations of $\mathfrak{A}$ localizable in space-like cones relative to 1; see also [20, 19].

Property P

(P1) Every representation p of $\mathfrak{A}$ which is localizable in space-like cones relative to 1, and which is space-time translation-covariant and satisfies the relativistic spectrum condition can be decomposed into a direct sum of *irreducible*, translation-covariant representations of $\mathfrak{A}$ which satisfy the relativistic spectrum condition and are localizable in space-like cones relative to 1.

(P2) Let p be an equivalence class of irreducible representations of $\mathfrak{A}$ which are translation-covariant, satisfy the relativistic spectrum condition and are localizable in space-like cones relative to 1. Then there exists a unique equivalence class, $\bar{p}$, of conjugate representations of $\mathfrak{A}$ with the same properties as p such that $p \circ \bar{p} = \bar{p} \circ p$ contains the vacuum representation, 1, precisely once.

From now on, Property P is always assumed to hold; see also [23, 24].

Definition 2.2.1 *We denote by $L \equiv L_{\{\mathfrak{A},\alpha\}}$ the complete list of all inequivalent, irreducible, translation-covariant representations of $\mathfrak{A}$ which satisfy the relativistic spectrum condition and are localizable in space-like cones relative to 1.*

It follows from Property P that, for i and j in L, the product representation, $i \circ j$, can be decomposed as follows:

$$i \circ j = \bigoplus_{k \in L} \bigoplus_{\mu=1}^{N_{ij,k}} k^{(\mu)}, \quad \text{with} \quad k^{(\mu)} \simeq k , \tag{2.23}$$

for all $\mu = 1, \ldots, N_{ij,k}$. The multiplicity, $N_{ij,k} = N_{ji,k}$, of k in $i \circ j$ is a non-negative integer and, by Property $(P2)$, can also be defined as the multiplicity of 1 in $\bar{k} \circ i \circ j$. The integers $(N_{ij,k})$ are the *fusion rules* of the theory. By the definition of $i \circ j$, $N_{ij,k}$ can be interpreted as the multiplicity of the representation k of $\mathfrak{A}$ in the representation $i\left(\rho_C^j(\cdot)\right)$ of $\mathfrak{A}$, where ρ_C^j is a *endomorphism of $\mathfrak{B}^{C_a}$ localized in a space-like cone $C \subset (C_a + x)'$, for some x, with the property that j is unitarily equivalent to $1\left(\rho_C^j(\cdot)\right)$. It is not hard t derive from this that, given k, i and j in L, there exists a complex Hilbert space $\mathcal{V}_k\left(\rho_C^j\right)_i$ of operators, V, from the representation space, $\mathcal{H}_k$, of k to the representation space, $\mathcal{H}_i$ of i such that

$$i\left(\rho_C^j(A)\right) V = V\, k(A), \qquad \text{for all} \qquad A \in \mathfrak{A} ; \qquad (2.24$$

the dimension of $\mathcal{V}_k\left(\rho_C^j\right)_i$ is given by $N_{ij,k}$, and the scalar product, $\langle V, W\rangle$, between tw elements V and W of $\mathcal{V}_k\left(\rho_C^j\right)_i$ is defined by

$$V^* W = \langle V, W\rangle 1|_{\mathcal{H}_k} . \qquad (2.25$$

By (2.24), V^*W intertwines the representation k of $\mathfrak{A}$ with itself and hence, by Schur' lemma, must be a multiple of $1|_{\mathcal{H}_k}$, because k is irreducible. Intertwiner spaces $\mathcal{V}_k(\rho^{j_1} \circ \cdots \circ \rho^{j_n})_i$ are defined similarly, for arbitrary $i, j_1, \ldots, j_n$ and k in L.

Remark.

One purpose of Chapters 2 and 7 is to use the intertwiners in $\mathcal{V}_k(\rho_C^j)_i$, i, j, k in L, t construct certain (bounded) operators on the total physical Hilbert space of the theor called *charged fields*, which have non-zero matrix elements between different superselection sectors, are localized in space-like cones and hence can be used to, for example, construct Haag-Ruelle collision states [20]. Quantum groups will appear in the construction of such fields in space-times of dimension $d = 3$ and for some class of theories, including *conformal field theories*, in two space-time dimensions.

The first step in our construction of charged fields is to construct ("horizontal") local sections of orthonormal frames of intertwiners of a bundle, $\mathcal{I}_{ij,k}$, of intertwiners satisfying (2.24), whose base space is a "manifold" of *endomorphisms, ρ^j, of $\mathfrak{B}^{\mathcal{C}_a}$ localized in space-like cones contained in $(\mathcal{C}_a + x)'$, for some x, with the property that $1(\rho^j(\cdot))$ is unitarily equivalent to j, and whose fibres $\mathcal{V}_k(\rho^j)_i$ are isomorphic to $\mathbb{C}^{N_{ij,k}}$. Such local sections of frames are constructed as follows: We choose a *reference morphism*, ρ_0^j, localized in a space-like cone $\mathcal{C}_0 \subset (\mathcal{C}_a + x)'$, for some x, and an orthonormal basis $\left\{V_\alpha^{ik}\left(\rho_0^j\right)\right\}_{\alpha=1}^{N_{ij,k}}$ for the Hilbert space $\mathcal{V}_k\left(\rho_0^j\right)_i$ consisting of partial isometries from $\mathcal{H}_k$ to $\mathcal{H}_i$ satisfying (2.24). Given an arbitrary *endomorphism, ρ^j, of $\mathfrak{B}^{\mathcal{C}_a}$ localized in a space-like cone $\mathcal{C} \subset (\mathcal{C}_a + x)'$, for some x, and unitarily equivalent to ρ_0^j, we choose a unitary charge transport operator $\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}$, see (2.19), which belongs to an algebra $\overline{\mathfrak{B}(\mathcal{S})}^w \subset \mathfrak{B}^{\mathcal{C}_a}$ associated with a simple domain $\mathcal{S} \subset (\mathcal{C}_a + x)'$, containing $\mathcal{C}_0$ and $\mathcal{C}$. A basis for $\mathcal{V}_k(\rho^j)_i$ is then given by $\left\{V_\alpha^{ik}(\rho^j)\right\}_{\alpha=1}^{N_{ij,k}}$, where

$$V_\alpha^{ik}(\rho^j) = i\left(\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}\right) V_\alpha^{ik}\left(\rho_0^j\right) . \tag{2.26}$$

Note that, since $\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j} \in \overline{\mathfrak{A}(\mathcal{S})}^w \subset \mathfrak{B}^{\mathcal{C}_a}$, and i is a representation of $\mathfrak{B}^{\mathcal{C}_a}$, $i\left(\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}\right)$ is a well-defined unitary operator on $\mathcal{H}_i$.

Bundles $\mathcal{I}_{ij_1\ldots j_n,k}$ and local sections of frames of intertwiners in $\mathcal{I}_{ij_1\ldots j_n,k}$ are constructed similarly; see [24].

Remark.

Since, for $j \in L$, ρ^j is an irreducible *endomorphism of $\mathfrak{B}^{\mathcal{C}_a}$, the choice of $\Gamma^{\mathcal{S}}_{\rho^j,\rho_0^j}$ is unique *up to a phase factor*. This phase factor <u>cannot</u> be chosen continuously, even in "small neighbourhoods" of ρ_0^j. These technicalities are of no concern in this book, except in

Theorem 2.3.1, below.

2.3 Statistics and Fusion of Intertwiners; Statistical Dimensions

Let $\mathcal{C}$ be a space-like cone which is the causal completion of a wedge $\mathcal{C}_0$ in $(d-1)$-dimensional space. With $\mathcal{C}$ we associate a unit vector $\vec{e} \in S^{d-2}$ which specifies the asymptotic direction of the central axis of $\mathcal{C}_0$; (for $d = 3$, $\vec{e}$ is the unit vector in $\mathbb{R}^2$ specifying the asymptotic direction of the half-line bisecting $\mathcal{C}_0$). Using polar coordinates, $\vec{e}$ can be described by $d-2$ angles; in particular, for $d = 3$, $\vec{e}$ is described by one angle $\theta \in (-\pi, \pi]$. Our coordinates are chosen such that the unit vector $\vec{e}$ associated with $\mathcal{C}_a$ is given by $(-1, 0, \ldots, 0)$. If ρ is a *endomorphism of $\mathfrak{B}^{\mathcal{C}_a}$ localized in a cone $\mathcal{C}$, the unit vector $\vec{e}$ associated with $\mathcal{C}$ is called the *asymptotic direction*, as (ρ), of ρ. We may choose the reference morphisms ρ_0^j, $j \in L$, such that $as\left(\rho_0^j\right) = (1, 0, \ldots, 0)$. In $d = 3$ dimensions, the asymptotic directions of the morphisms ρ^j inherit the ordering of the angles in $(-\pi, \pi]$.

We say that two *endomorphisms, ρ_1 and ρ_2, of $\mathfrak{B}^{\mathcal{C}_a}$ are *causally independent*, denoted $\rho_1 \times \rho_2$, if they are localized in cones $\mathcal{C}_1$ and $\mathcal{C}_2$ such that $\mathcal{C}_1 \subset \mathcal{C}_2'$.

We now recall a basic result proven in [24].

Theorem 2.3.1 *For p and q in L, let ρ^p and ρ^q be two *endomorphisms of $\mathfrak{B}^{\mathcal{C}_a}$ localized in space-like cones contained in $\mathcal{C}_a'$ and unitarily equivalent to p and q, respectively. Let the intertwiners $\left\{V_\alpha^{ik}(\rho^p)\right\}_{\alpha=1}^{N_{ip,k}}$ and $\left\{V_\beta^{ik}(\rho^q)\right\}_{\beta=1}^{N_{iq,k}}$ be defined as in (2.26). Then there are matrices, called statistics-(or braid-) matrices,*

$$\left(R(j, p, \mathrm{as}\,(\rho^p) q, \mathrm{as}\,(\rho^q), k)_{i\alpha\beta}^{\ell\mu\nu}\right) ,$$

such that

$$V_\alpha^{ji}(\rho^p)\, V_\beta^{ik}(\rho^q) = \sum_{\ell\mu\nu} R(j, p, \mathrm{as}\,(\rho^p), q, \mathrm{as}\,(\rho^q), k)_{i\alpha\beta}^{\ell\mu\nu}\, V_\mu^{j\ell}(\rho^q)\, V_\nu^{\ell k}(\rho^p) , \tag{2.27}$$

provided $\rho^p \times \rho^q$. The statistics matrices are locally independent of the choice of the auxiliary cone C_a. Moreover, the following properties hold.

(a) In $d \geq 4$ space-time dimensions, the matrices

$$R(j, p, \mathrm{as}\,(\rho^p), q, \mathrm{as}\,(\rho^q), k) \equiv R(j,p,q,k) \tag{2.28}$$

are independent of as (ρ^p) and as (ρ^q).

(b) For $d = 3$,

$$R(j, p, \mathrm{as}\,(\rho^p), q, \mathrm{as}\,(\rho^q), k) \equiv R^{\pm}(j,p,q,k)\,, \tag{2.29}$$

for as $(\rho^p) \gtrless$ as (ρ^q). [The matrices $R^{\pm}(j,p,q,k)$ depend on ρ^p and ρ^q only through p and q and the sign of as $(\rho^p) -$ as(ρ^q).]

Remarks.

It is easy to see that

$$\sum_{\ell\mu\nu} R^{\pm}(j,p,q,k)^{\ell\mu\nu}_{i\alpha\beta}\, R^{\mp}(j,q,p,k)^{m\kappa\lambda}_{\ell\mu\nu} = \delta^m_i \delta^\kappa_\alpha \delta^\lambda_\beta \;. \tag{2.30}$$

and that the matrices $R^{\pm}(j,p,q,k)$ satisfy the *Yang-Baxter equations* in SOS-form.

We now assume that the representations $p \in L$ are rotation-covariant. Thus if 0 denotes a space rotation then

$$p(\alpha_0(A)) = U_p(0)\,p(A)\,U_p(0^{-1})\,, \tag{2.31}$$

where U_p is a unitary representation of the universal covering group of $SO(d-1)$ on the representation space $\mathcal{H}_p$ of p. Since p is irreducible and $\alpha_{0_{2\pi}}$ is the identity when $0_{2\pi}$ is a rotation through an angle 2π, it follows that

$$U_p\,(0_{2\pi}) = e^{2\pi i s_p} 1\Big|_{\mathcal{H}_p}\,, \tag{2.32}$$

where the real number s_p is called the *spin* of representation p; (for $d = 3$, s_p can, *a priori*, be an arbitrary real number, while, for $d \geq 4$, $s_p \in \frac{1}{2}\mathbb{Z}$).

Theorem 2.3.2

$$R^+(j,p,q,k)^\ell_i = e^{2\pi i(s_\ell + s_i - s_j - s_k)} R^-(j,p,q,k)^\ell_i\,. \tag{2.33}$$

Remark.

The fact that in $d \geq 4$ space-time dimensions $R^+ = R^-$ and Theorem 2.3.2 imply that $s_p \in \frac{1}{2}\mathbb{Z}$, for all $p \in L$.

All the results reviewed above are proven in [24].

Next, we recall what is called *fusion of intertwiners* [24]. For p, q and r, let ρ^p, ρ^q and ρ^r be three *endomorphisms of $\mathfrak{B}^{C_a}$ unitarily equivalent to p, q and r, respectively, and localized in the interior of a simple domain $S \subset C'_a$. Then there exist $N_{pq,r}$ partial isometries,

$$\Gamma^S_{\rho^p \circ \rho^q, \rho^r}(\mu) \in \overline{\mathfrak{A}(S)}^w \subset \mathfrak{B}^{C_a}\,, \tag{2.34}$$

$\mu = 1, \dots, N_{pq,r}$, such that

$$\rho^p\,(\rho^q(A))\,\Gamma^S_{\rho^p \circ \rho^q, \rho^r}(\mu) = \Gamma^S_{\rho^p \circ \rho^q, \rho^r}(\mu)\,\rho^r(A)\,. \tag{2.35}$$

Let $\sigma_\mu(r;p,q)$ be given by

$$\sigma_\mu(r;p,q) \equiv \left\langle \bar{r}\left(\Gamma^S_{\rho^p \circ \rho^q, \rho^r}(\mu)\right) V^{\bar{r}1}(\rho^r),\, V^{\bar{r}\bar{q}}_\mu(\rho^p)\, V^{\bar{q}1}(\rho^q) \right\rangle . \tag{2.36}$$

Note that $N_{\bar{r}r,1} = 1$, so that there is a unique (up to a phase) isometric intertwiner of the type of $V^{\bar{r}1}(\rho^r)$, for all $r \in L$.

Theorem 2.3.3

(a) There exist matrices $\left(\hat{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta}\right)$ only depending on the representations j, p, q, k, i and r, (but not on the specific choice of ρ^p, ρ^q and ρ^r), such that

$$V^{ji}_\alpha(\rho^p)\, V^{ik}_\beta(\rho^q) =$$

$$\sum_{r\mu\nu} \hat{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta}\, \sigma_\mu(r;p,q)\, j\left(\Gamma^S_{\rho^p \circ \rho^q, \rho^r}(\mu)\right)\, V^{jk}_\nu(\rho^r)\,. \tag{2.37}$$

The matrices $\hat{F}$ can be expressed in terms of the matrices R^+ and R^- as follows

$$\hat{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta} =$$

$$\sum_{\gamma\delta} R^{\pm}(\bar{i},q,\bar{k},1)^{\bar{q}\gamma 1}_{k\beta 1}\, R^{\pm}(j,p,\bar{k},\bar{q})^{r\delta\mu}_{i\alpha\gamma}\, R^{\mp}(j,\bar{k},r,1)^{k\nu 1}_{r\delta 1}\,. \tag{2.38}$$

(b) *There exist matrices $\left(\check{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta}\right)$ only depending on the representations j,p,q,k,i and r, (but not on the specific choice of ρ^p, ρ^q and ρ^r), such that*

$$j\left(\Gamma^{S}_{\rho^p\circ\rho^q,\rho^r}(\alpha)\right) V^{jk}_{\beta}(\rho^r) =$$

$$\sum_{i\mu\nu} \check{F}(j,p,q,k)^{i\mu\nu}_{r\alpha\beta}\overline{\sigma_\alpha(r;p,q)}\, V^{ji}_{\mu}(\rho^p)\, V^{ik}_{\nu}(\rho^q)\,. \tag{2.39}$$

The matrices $\check{F}$ can be expressed in terms of R^+ and R^- by a formula analogous to (2.36); (see Theorem 2.3.4, (1))

(c) *The matrices $\hat{F}$ and $\check{F}$ are related to each other by the following equations*

$$\sum_{i\mu\nu} \check{F}(j,p,q,k)^{i\mu\nu}_{r\alpha\beta}\, \hat{F}(j,p,q,k)^{s\lambda\kappa}_{i\mu\nu} = \delta^{s}_{r}\delta^{\lambda}_{\alpha}\delta^{\kappa}_{\beta}\,, \tag{2.40}$$

and

$$P^{(r,\mu)}(j,p,q,k)^{\ell\lambda\kappa}_{i\alpha\beta} := \sum_{\nu} \hat{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta}\, \check{F}(j,p,q,k)^{\ell\lambda\kappa}_{r\mu\nu} \tag{2.41}$$

are the matrix elements of orthogonal projections, $P^{(r,\mu)}(j,p,q,k)$, with

$$\sum_{r\mu} P^{(r,\mu)}(j,p,q,k) = 1|_{\mathcal{V}_k(\rho^p\circ\rho^q)_j}\,, \qquad \text{(completeness)}. \tag{2.42}$$

Remarks.

(a) The consistency of the two equations (2.38) (+ ↔ −) follows easily from Theorem 2.3.1. Theorem 2.3.3 is proven in [24].

(b) We recall that $\mathcal{V}_k(\rho^p)_j$ is the Hilbert space of intertwiners V from $\mathcal{H}_k$ to $\mathcal{H}_i$ satisfying

$$j(\rho^p(A))\,V = V k(A), \qquad \text{for all} \quad A \in \mathfrak{A},$$

see (2.20). We define $\mathcal{V}_k(\rho^p\circ\rho^q)_j$ to be the Hilbert space spanned by the intertwiners

$$\left\{V^{ji}_{\alpha}(\rho^p)V^{ik}_{\beta}(\rho^q) : i \in L, \quad \alpha = 1,\ldots,N_{pj,i}, \quad \beta = 1,\ldots,N_{qi,k}\right\}.$$

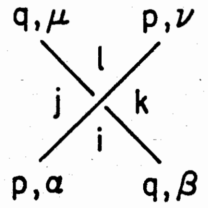
Then the matrices $R^{\pm}(j,p,q,k)$ define *unitary maps* from $\mathcal{V}_k(\rho^p\circ\rho^q)_j$ to $\mathcal{V}_k(\rho^q\circ\rho^p)_j$, provided ρ^p and ρ^q are causally independent $\left(\rho^p \times \rho^q\right)$, the matrices $\hat{F}$ and $\check{F}$ define unitary endomorphisms of $\mathcal{V}_k(\rho^p\circ\rho^q)_j$, and the matrices $P^{(r,\mu)}(j,p,q,k)$ define orthogonal projections on $\mathcal{V}_k(\rho^p\circ\rho^q)_j$.

(c) It is sometimes preferable to use

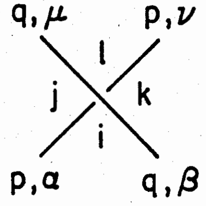
$$\hat{F}(j,p,q,k)^{r,\mu\nu}_{i,\alpha\beta} =$$

$$\sum_{\eta\epsilon\beta} R^{\pm}(1,j,p,i)^{p1\eta}_{j1\alpha}\, R^{\pm}(p,j,q,k)^{r\epsilon r}_{i\eta\beta}\, R^{\mp}(1,r,j,k)^{j1\nu}_{\ell 1r}\, \hat{F}(1,p,q,r)^{r\mu 1}_{p1\epsilon} \tag{2.43}$$

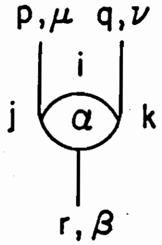
instead of (2.38), in order to compute the $\hat{F}$ matrices from the $R^{\pm}$ matrices. It is useful to express the matrices $R^+, R^-, \hat{F}$ and $\check{F}$ graphically as follows



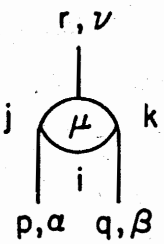
$$\leftrightarrow R^{+}(j,p,q,k)^{\ell\mu\nu}_{i\alpha\beta} \tag{2.44}$$



$$\leftrightarrow R^{-}(j,p,q,k)^{\ell\mu\nu}_{i\alpha\beta} \tag{2.45}$$

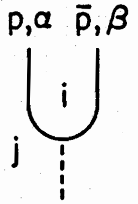


$$\leftrightarrow \hat{F}(j,p,q,k)^{r\mu\nu}_{i\alpha\beta} \tag{2.46}$$



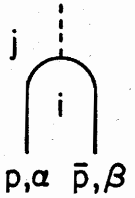
$$\leftrightarrow \check{F}(j,p,q,k)^{i\mu\nu}_{r\alpha\beta}. \tag{2.47}$$

We also introduce the graphical notation



$$\leftrightarrow \hat{F}\binom{j}{pi}_{\alpha\beta} := \hat{F}(j,p,\overline{p},j)^{111}_{i\alpha\beta}, \tag{2.48}$$

and

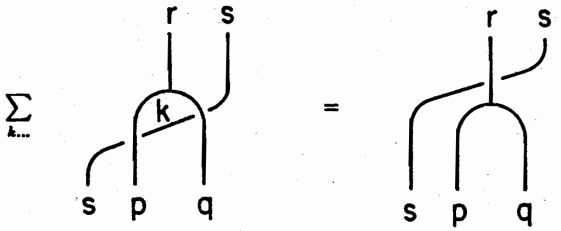


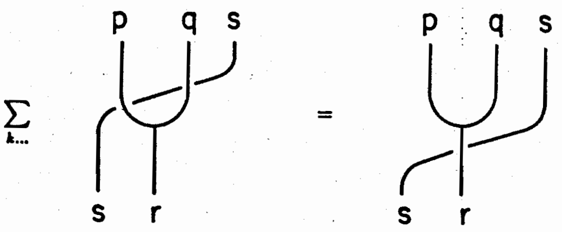
$$\leftrightarrow \check{F}\binom{pi}{j}^{\alpha\beta} := \check{F}(j,p,\overline{p},j)^{i\alpha\beta}_{111}. \tag{2.49}$$

Identities between $R^+, R^-, \hat{F}$ and $\check{F}$ can now conveniently be expressed graphically. It is quite straightforward to prove the following theorem [24].

Theorem 2.3.4 *The matrices $R^\pm, \hat{F}$ and $\check{F}$ satisfy the equations*

(a)

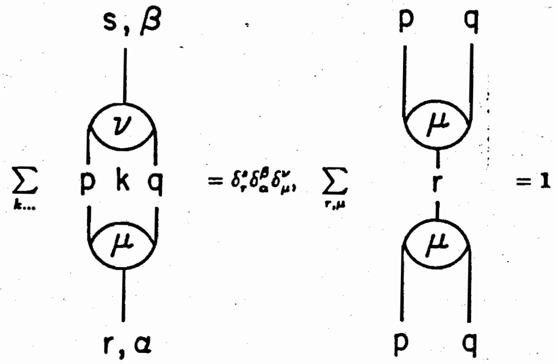




(2.50

*etc.; (*polynomial equations*)*

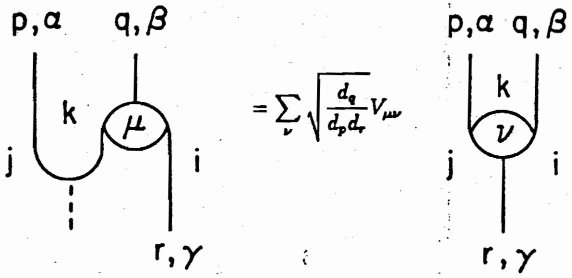
(b)



$$= \delta^s_r\delta^\beta_\alpha\delta^\nu_\mu, \qquad \sum_{r,\mu} \quad = 1$$

(2.51

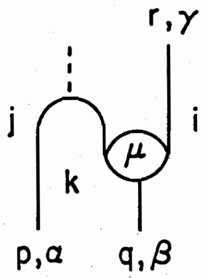
(c) There exist numbers $d_p \geq 1$, for all $p \in L$, and unitary matrices, $V_{\mu\nu} = V_{\mu\nu}\,(p,q,r)$, such that



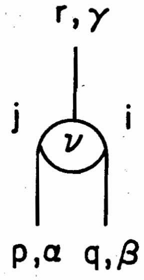
$$= \sum_\nu \sqrt{\frac{d_q}{d_p d_r}} V_{\mu\nu}$$

(2.5

and



$= \sum_{\nu} \sqrt{\frac{d_q}{d_p d_r}} V_{\mu\nu}$



(2.53)

For the proof of Theorem 2.3.4 see [24].

Remarks.

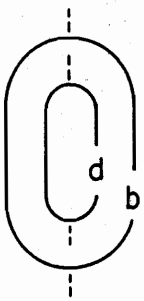
(a) Equation (2.38) for $\hat{F}$ and a similar equation for $\check{F}$ follow from Theorem 2.3.4,(a).

(b) The number $d_p, p \in L$, is called the *statistical dimension* of representation p. If $R^+ = R^-$, i.e., if the theory has permutation group statistics then $d_p \in \mathbb{N}$, [19]. It is shown in [19, 20, 23], that $d_p = d_{\bar{p}}$. From Theorem 2.3.4, (b) and (c), it follows that d_p is the largest eigenvalue of the *fusion rule matrix* N_p, defined by

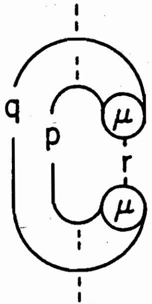
$$(\mathrm{N}_p)_{jk} := N_{jp,k}\,.$$

This can be shown by noticing the identities

$d_p d_q \;=\; d_p d_q$

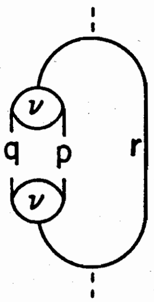


$= \sum_{r,\mu} d_p d_q$

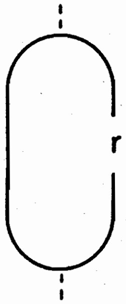


(2.54)

$= \sum_{r,\nu}$



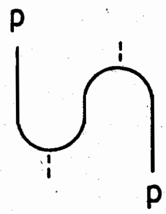
$= \sum_r N^r_{pq} d_r$



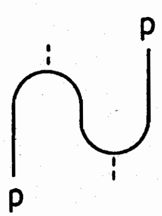
$= \sum_r N^r_{pq} d_r\,.$ (2.55)

Identity (2.54) follows from Theorem 2.3.4,(b) and (c); [22]. Thus $(d_r)_{r\in L}$ is a Perron-Frobenius eigenvector, corresponding to the eigenvalue d_p, of the fusion matrix N_p. Connected to this result is the interpretation of d_p^2 as a *Jones index*, [23, 41]. See Chapter 3.

(c) As a special case of Theorem 2.3.4,(c) we note that



$=$



$= \frac{1}{d_p}$

(2.56)

This identity is important in the construction of invariants of links and of ribbon graphs from the matrices $\left\{R^{\pm}, \hat{F}, \check{F}\right\}$; see [43, 28, 44].

The main result of this section is the insight established in [15, 24, 22] that every local, relativistic quantum theory, in the sense of Sects. 2.1, 2.2 in three or more space-time dimensions [and the chiral sectors of every two dimensional conformal field theory] provides us, in a canonical way, with certain combinatorial data, namely the *fusion rules* $(\mathrm{N}_p)_{p\in L}$, and the *statistics-*(or *braid-*) and *fusion matrices*, $R^{\pm}, \hat{F}$ and $\check{F}$, respectively. In $d \geq 4$ space-time dimensions, we have that $R^+ = R^-$, but in $d = 2, 3$ dimensions R^+ and R^- are, in general, distinct; see Theorem 2.3.2. It is natural to ask, whether these data might be dual to some simpler algebraic object, such as a group or a quantum group. In a remarkable series of papers [29], Doplicher and Roberts have shown that if $R^+ = R^-$, i.e., for *permutation group statistics*, the data $\left\{(\mathrm{N}_p)_{p\in L}, R, \hat{F}, \check{F}\right\}$ are dual to a *compact group*, i.e., L can be viewed as the set of finite-dimensional, irreducible representations of a compact group, $G, N_{pk,j}$ is the multiplicity of representation j of G in the tensor product representation $p \otimes k$, and $R, \hat{F}, \check{F}$ are standard 6-index symbols associated with the representation theory of G.

The point of this work is to show that if $R^+ \neq R^-$, (frequently the case in $d = 2, 3$), then the data $\left\{(\mathrm{N}_p)_{p\in L}, R^\pm, \hat{F}, \check{F}\right\}$ are often dual to some *quantum group*, [1, 2, 3]. We shall discuss in detail one example (see Sect. 6.3.) of a local relativistic quantum theory, encountered in the study of three-dimensional Chern-Simons gauge theory with gauge group $SU(2)$,which leads to quantum $SU(2)$, i.e., $U_q(s\ell_2)$, with q a root of unity. The same example appears in the study of two-dimensional Wess-Zumino-Novikov-Witten models based on $SU(2)$ current algebra and of minimal conformal model [9, 31].

In the next section, we study properties of the representations of the braid groups determined by the statistics matrices $R^\pm$.

2.4 Unitary Representations of the Braid Groups Derived from Local Quantum Theory; Markov Traces

We return to the study of a local quantum theory described by an algebra $\mathfrak{A}$,a *automorphism group, α, and a set, L, of representations localizable in space-like cones. We show how, for $d = 2$ or 3 and assuming that $R^+ \neq R^-$, the quantum theory determines unitary representations of the *braid groups*, B_n, on n strands, for arbitrary n, equipped with a positive Markov trace τ_M. These results are discussed in more detail in [22, 24].

For every $p \in L$ and every $n \in \mathrm{N}$, we define a space $\Omega_p^{(n)}$ of paths of length n, as follows: Every element $\omega \in \Omega_p^{(n)}$ is a sequence of symbols

$$\omega = (\mu_1\alpha_1, \mu_2\alpha_2, \ldots, \mu_n\alpha_n), \qquad \text{with} \qquad \mu_i \in L, \tag{2.57}$$

and $\alpha_i = 1, \ldots, N_{p\mu_{i-1},\mu_i}$, $i = 1, \ldots, n$, with $\mu_0 = 1$. Two neighbors, μ_{i-1} and μ_i, are constrained by the requirement that $N_{p\mu_{i-1},\mu_i} \neq 0$.

We fix a *endomorphism ρ^p of $B^{C_\bullet}$. With each path $\omega \in \Omega_p^{(n)}$, we associate an intertwiner

$$V_\omega = \prod_{i=1}^{n} V_{\alpha_i}^{\mu_{i-1}\mu_i}(\rho^p), \tag{2.58}$$

intertwining the representation $1\left((\rho^p)^n(\cdot)\right)$ of $\mathfrak{A}$ with the representation ω_+ of $\mathfrak{A}$, where $\omega_+ = \mu_n$ is the endpoint of ω. Here, $(\rho^p)^n = \rho^p \circ \ldots \circ \rho^p$ (n-fold composition of ρ^p with itself). The space of these intertwiners carries a natural scalar product $\langle\cdot,\cdot\rangle$, defined as in (2.21), Sect. 2.1. In this scalar product, $\left\{V_\omega : \omega \in \Omega_p^{(n)}, \omega_+ = k\right\}$ is an orthonormal basis for the space, $\mathcal{V}_k\left((\rho^p)^n\right)_1$, of intertwiners between representations $1\left((\rho^p)^n(\cdot)\right)$ of $\mathfrak{A}$ and k, i.e.,

$$\langle V_\omega, V_{\omega'}\rangle = \delta_{\omega,\omega'}\,. \tag{2.59}$$

We define a path algebra [45, 46], $\mathcal{A}\left(\Omega_p^{(n)}\right)$, by setting

$$\mathcal{A}\left(\Omega_p^{(n)}\right) = \bigoplus_{k\in L} \mathfrak{B}\left(\mathcal{V}_k\left((\rho^p)^n\right)_1\right), \tag{2.60}$$

where $\mathfrak{B}\left(\mathcal{H}\right)$ is the algebra of all linear endomorphisms of a Hilbert space $\mathcal{H}$. It is easy to see that [24]

$$\mathcal{A}\left(\Omega_p^{(n)}\right) \simeq 1\left((\rho^p)^n(\mathfrak{A})\right)'\,. \tag{2.61}$$

Next, we define a unitary representation of the braid group B_n on n strands with values in $\mathcal{A}\left(\Omega_p^{(n)}\right)$: Let $\sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1}$ be the usual generators of B_n. We define a unitary representation, $\hat{}$, of B_n on $\mathcal{V}_k\left((\rho^p)^n\right)_1$ by setting

$$\left(\hat{\sigma}_i^{\pm 1} V\right)_\omega = \sum_{\omega'} R_i^\pm(\omega,\omega') V_{\omega'}\,, \tag{2.62}$$

where

$$R_i^\pm(\omega,\omega') = R^\pm\left(\mu_{i-1}, p, p, \mu_{i+1}\right)_{\mu_i\alpha_i\alpha_{i+1}}^{\mu_i'\alpha_i'\alpha_{i+1}'} \tag{2.63}$$

if $\omega = (\mu_\ell, \alpha_\ell)_{\ell=1,\ldots,n}$ and $\omega' = (\mu_\ell', \alpha_\ell')_{\ell=1,\ldots,n}$, where $\mu_\ell' = \mu_\ell$ for $\ell \neq i$, $\alpha_\ell' = \alpha_\ell$, for $\ell \neq i, i+1$. For all other choices of ω', given ω, we set $R_i^\pm(\omega,\omega') = 0$.

Let $b = \prod_{\alpha=1}^{k} \sigma_{k_\alpha}^{\varepsilon_\alpha}$ be an arbitrary element of B_n; $\varepsilon_\alpha = \pm 1$, $k_\alpha \in \{1, \dots, n-1\}$, for $\alpha = 1, \dots k$. We define

$$R_b := \prod_{\alpha=1}^{k} R_{k_\alpha}^{\varepsilon_\alpha}, \tag{2.64}$$

with $R_k^{\pm}$ as in (2.63). The representation $\hat{}$ of B_n on $\mathcal{V}_k((\rho^p)^n)_1$ is then completely determined by setting

$$(\hat{b}V)_\omega := \sum_{\omega'} R_b(\omega, \omega') V_{\omega'}, \quad \omega_+ = \omega'_+ = k. \tag{2.65}$$

It is not hard to show, see [22, 24], that $\hat{}: b \mapsto \hat{b}$ is, in fact, a *unitary* representation of B_n on $\mathcal{V}_k((\rho^p)^n)_1$. This representation admits a unique, positive, normalized *Markov trace*, τ_M^p, constructed as follows [24]: Given $\omega = (\mu_1, \alpha_1, \dots, \mu_n, \alpha_n) \in \Omega_p^{(n)}$, we set $\bar{\omega} = (\mu_1, \dots, \mu_n)$. We define

$$\hat{F}(\omega, \omega') = \prod_{i=1}^{n} \hat{F}\begin{pmatrix} \mu_{i-1} \\ p\mu_i \end{pmatrix}_{\alpha_i \alpha'_i}, \tag{2.66}$$

for $\bar{\omega} = \bar{\omega'}$, and $\hat{F}(\omega, \omega') = 0$, otherwise; the matrices $\hat{F}\binom{j}{p\ell}_{\alpha\beta}$ have been defined in (2.48), Chapter 2.3. The matrix $\check{F}(\omega, \omega')$ is defined similarly; see (2.49), Chapter 2.3. Then τ_M^p is given by

$$\tau_M^p(\hat{b}) := \sum_{\mu_1, \dots, \mu_n} \sum_{\substack{\omega_\ell = (\mu_1, \dots, \mu_n) \\ \text{for } \ell = 1,2,3}} tr\left(\hat{F}(\omega_1, \omega_2)\, R_b(\omega_2, \omega_3) \check{F}(\omega_3, \omega_1)\right).^{*} \tag{2.67}$$

The quantity $\tau_M^p(\hat{\sigma}_1) =: \lambda_p$ is called statistics parameter [23, 22], and one can show [23, 22, 19] that the statistical dimension, d_p, is given by

$$d_p = |\tau_M^p(\hat{\sigma}_1)|. \tag{2.68}$$

The fusion rules $(\mathrm{N}_p)_{p \in L}$ and the values of the Markov traces τ_M^p,

$$\left\{\tau_M^p(\hat{b}) : b \in B_n, p \in L\right\},$$

on B_n, for all $n = 2, 3, 4, \dots$, are *intrinsically* associated with the quantum theory described by $\{\mathfrak{A}, \alpha, L\}$. They do not depend on how the phases and normalizations of the intertwiners $V_\alpha^{ij}(\rho^p)$ are chosen, in contrast to the data $\{R^\pm, \hat{F}, \check{F}\}$.

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[*] Clearly, τ_M^p can be extended to a state on $\mathcal{A}\left(\Omega_p^{(n)}\right)$, for every n.

We know from [43] that a quasi-triangular (quasi-) Hopf algebra $\mathcal{K}$ with universal R-matrix $\mathcal{R}$, and a list, $\mathcal{L}$, of finite-dimensional, irreducible representations of positive q-dimensions of $\mathcal{K}$ also give rise to representations of the braid groups B_n equipped with Markov traces τ_M^p, $p \in \mathcal{L}$, for all $n = 2, 3, 4, \dots$. From the results reviewed in this section we know that only those quasi-triangular (quasi-) Hopf algebras, $\mathcal{K}$, and families, $\mathcal{L}$, of representations of $\mathcal{K}$ can appear in local, relativistic quantum theory for which

(1) the associated representations of B_n are *unitarizable*, for all n; and

(2) the Markov traces τ_M^p, $p \in \mathcal{L}$, are positive.

For $\mathcal{K} = U_q(s\ell_{d+1})$, this restricts the values of q to $q = \exp(i\pi/N)$, $N = d+2, d+3, \dots$. What, as field theorists, we are longing for is a general theorem which completely characterizes those fusion rules and positive Markov traces on B_n, $n = 2, 3, 4, \dots$, which come from quasi-triangular (quasi-) Hopf algebras. We do not know a general result of this type, yet. [In $d \geq 4$ space-time dimensions, the results of Doplicher and Roberts [29] completely settle an analogous problem, with $\mathcal{K}$ the group algebra of a compact group.]

Chapter 3

Superselection Sectors and the Structure of Fusion Rule Algebras

As proposed in [23], it is of interest to investigate the structure of the chain of algebras

$$\mathbf{C}\cdot\mathbf{1} = \rho\left(\mathfrak{B}^{C_\bullet}\right)' \cap \left(\mathfrak{B}^{C_\bullet}\right) \subset \rho\circ\overline{\rho}\left(\mathfrak{B}^{C_\bullet}\right)' \cap \left(\mathfrak{B}^{C_\bullet}\right) \subset \rho\circ\overline{\rho}\circ\rho\left(\mathfrak{B}^{C_\bullet}\right)' \cap \left(\mathfrak{B}^{C_\bullet}\right)\ldots, \quad (3.1)$$

where ρ is an irreducible *-endomorphism an $\overline{\rho}$ a conjugate endomorphism. The point of studying algebra chains obtained by alternating compositions of the form (3.1) is that they admit faithful traces which give rise to conditional expectations and thus to Temperley-Lieb algebras [41] as subalgebras. This structure has been studied in rather much detail. For <u>rational</u> local quantum theories, i.e., theories with a finite number of superselection sectors, one finds that the chain (3.1) eventually leads to a tower in the sense of Jones [41]. The factors in these algebras are distinguished by the inequivalent, irreducible representations occuring in the compositions $\rho\circ\overline{\rho}\circ\rho\circ\overline{\rho}\circ\ldots$, which makes it natural to try to connect the inclusions of the algebras defining the tower to the fusion rules $\{N_{ij,k}\}$ introduced in Sect. 2.2; see also [47, 41]. Assuming that every irreducible representation of $\mathfrak{A}$ is contained in some $\rho^n\circ\overline{\rho}^m$, we shall explain, in some detail, how fusion rules can be recovered from (3.1) and from towers that are in some sense coupled or isomorphic to (3.1).

Since most of the structural information can be obtained from the fusion rules alone, a larger part of this section is devoted to the study of <u>fusion rule algebras</u>, as introduced in [47]. In view of a classification problem solved in Section 7.3, we give a formal treatment of the action of the group of automorphisms in a fusion rule algebra.

On the level of algebra-chains, similar to (3.1), automorphisms give rise to concurrent Temperley-Lieb algebras which, for a special decomposition rule for $\rho\circ\overline{\rho}$, lead to a complete determination of the underlying theory, as we shall see at the end of this section.

3.1 Definition of and General Relations in Fusion Rule Algebras, and their Appearance in Local Quantum Field Theories

A <u>fusion rule algebra</u> (superselection structure, ...) Φ is a positive lattice $\left(|\Phi| = \mathbb{N}^{|L|}\right)$, with a distributive and commutative multiplication

$$\Phi\times\Phi\to\Phi; \qquad a\times b\to a\circ b,$$

an involutive and additive conjugation, $-$,

$$-:\Phi\to\Phi; \qquad a\to\overline{a}$$

with $\overline{a}\circ\overline{b} = \overline{a\circ b}$, a unit $1\in\Phi$ with

$$1\circ a = a \quad \text{and} \quad \overline{1} = 1$$

and an additive evaluation ε

$$\varepsilon \ : \ \Phi\to\mathbb{N} \quad \text{such that}$$
$$\varepsilon(\overline{a}) \ = \ \varepsilon(a), \qquad \varepsilon(1) = 1$$

and $(a,b) := \varepsilon(a\circ\overline{b})$ is the usual euclidian scalar product on $\mathbb{N}^{|L|}$.

It follows, that the scalar product (,) obeys

$$(a\circ x, y) = (x, \overline{a}\circ y), \qquad (3.2$$

so that we have, for the *length* $\|a\| := \sqrt{(a,a)}$ of $a \in \Phi$,

$$\|a\| = \|\overline{a}\|\,, \quad \|a \circ b\| = \|a \circ \overline{b}\|\,, \quad \text{etc.}. \tag{3.3}$$

Minimal elements, ϕ, in Φ, i.e., vectors that cannot be written as the sum of two other nonzero vectors, are characterized by

$$\|\phi\| = 1\,. \tag{3.4}$$

Every vector of Φ can be written uniquely as a sum of the minimal elements $\phi \in \Phi$, and any additive bijection of Φ onto itself corresponds to a permutation in
$L = \{\phi \in \Phi | \phi \text{ minimal}\}$. In particular, we have that $1 \in L$, that the conjugation is an involution, $\overline{\phi}_j \to \phi_{\overline{j}}$, of L, and that

$$\varepsilon\left(\phi_i \overline{\phi}_j\right) = \delta_{ij}\,. \tag{3.5}$$

A fusion rule subalgebra (sub-superselection structure) Φ' is an invariant sublattice of Φ, which contains 1, closes under multiplication and for which (3.4) holds, for all minimal vectors.

Note that a fusion rule algebra is *simple*, in the sense that there do not exist proper ideals, i.e., if Φ_o is a sublattice of Φ spanned by minimal vectors with

$$\overline{\Phi}_o = \Phi_o \qquad \text{and} \qquad \Phi_o \circ \Phi \subset \Phi_o$$

it follows from (3.5) that $1 \in \Phi_o$ and hence $\Phi = \Phi_o$.

The multiplication in Φ is determined by the products of the minimal elements

$$\phi_i \circ \phi_j = \sum_{k \in L} N_{ij,k} \phi_k\,, \tag{3.6}$$

where the structure constants $N_{ij,k} \in \mathbb{N}$ are, what we previously referred to as fusion rules.

In terms of the fusion rules, the definition of the fusion rule algebra is given by:

$$\begin{array}{llcl}
\text{a) commutativity} & N_{ij,k} & = & N_{ji,k} \\
\text{b) associativity} & \sum_k N_{ij,k}\, N_{kr,s} & = & \sum_\ell N_{i\ell,s}\, N_{jr,\ell} \\
\text{c) unit} & N_{i1,j} & = & N_{1i,j} = \delta_{ij} \\
\text{d) involution} & N_{ij,k} & = & N_{\overline{i}\overline{j},\overline{k}} \\
\text{e) evaluation} & N_{i\overline{j},1} & = & \delta_{ij}\,.
\end{array} \tag{3.7}$$

A representation of a fusion rule algebra, π, of Φ on a lattice $A = \mathbb{N}^k$ is an assignment, $a \to \rho(a)$, of elements, a, in Φ to additive mappings of A to itself (i.e., $\rho(a)$ is a nonnegative, integer $k \times k$ matrix), with

$$\rho(1) = \mathbf{1},\ \rho(a)\rho(b) = \rho(a \circ b) \quad \text{and} \quad \rho(\overline{a}) = \rho(a)^t\,. \tag{3.8}$$

The representation we are primarily interested in is given by (right) multiplication of ϕ on $A = \Phi$, so that

$$\rho(\phi_j) = \mathbf{N}_j^t\,, \tag{3.9}$$

where $(\mathbf{N}_j)_{ik} = N_{ij,k}$ are the matrices of fusion rules.

In fact, any lattice A that carries a representation of Φ and has an element ω with

$$(\omega, \rho(a)\omega) = \varepsilon(a) \tag{3.10}$$

can be written as a sum $A = \Phi_r \oplus \Phi_r^\perp$, where $\Phi_r, \Phi_r^\perp$ are Φ-invariant, and Φ_r is equivalent to the right representation. If a representation, ρ, satisfies $\|\rho(\phi_i)\| = \|\mathbf{N}_i\|$, then we call ρ *dimension preserving*. Eqs. (3.8) yield:

$$\mathbf{N}_i^t \mathbf{N}_j^t = \sum_k N_{ij,k} \mathbf{N}_k^t \tag{3.11}$$

$$\mathbf{N}_1 = 1\,, \qquad \mathbf{N}_{\overline{j}} = \mathbf{N}_j^t\ . \tag{3.12}$$

Using (3.10), for $\omega = 1$, and (3.12), (3.7) we see that

$$\mathbf{N}_{\overline{j}} 1 = \mathbf{N}_j^t 1 = \phi_j\,, \quad \text{as well as} \quad \mathbf{N}_{\overline{j}} = C \mathbf{N}_j C\,, \tag{3.13}$$

where $(C)_{ij} = \delta_{i\overline{j}}$.

Moreover, commutativity of $\circ$ implies

$$\left[\mathrm{N}^t_i, \mathrm{N}^t_j\right] = 0\,. \tag{3.14}$$

Suppose we have a lattice Φ_r and nonnegative integer matrices N_j acting on Φ_r, that obey (3.12), (3.13) and (3.14), for a given involution C, then we find that

$$\begin{array}{lrcl} a) & (1, \mathrm{N}_i \mathrm{N}_j\, 1) & = & \delta_{i\bar{j}} \\ b) & (1, \mathrm{N}_i \mathrm{N}_j \mathrm{N}_k\, 1) & = & N_{ij,\bar{k}} \\ c) & (1, \mathrm{N}_i \mathrm{N}_j \mathrm{N}_k \mathrm{N}_\ell\, 1) & = & \sum\limits_{s \in L} N_{ij,s}\, N_{sk,\bar{l}} \end{array} \tag{3.15}$$

By (3.14), these expressions are completely symmetric in the indices i, j, k and ℓ and, by (3.12), are invariant under conjugation $(i, j, k, \ell) \to (\bar{\imath}, \bar{\jmath}, \bar{k}, \bar{\ell})$, so that equations (3.7) are easily verified. Hence any set of matrices obeying (3.12), (3.13) and (3.14) determines a fusion rule algebra.

From the results reviewed in Sections 2.1-2.4 it is clear that every local quantum theory satisfying properties $(P1)$ and $(P2)$ of Section 2.2 defines a fusion rule algebra, Φ. Let

$$\mathcal{B}^{C_a} = \bigvee_{x \in M^n} \overline{\mathfrak{A}\left(C_a + x\right)^{c}}^{w}$$

denote the auxiliary C^* algebra, introduced in Sect. 2.2, containing the observable algebra $\mathfrak{A}$; (C_a is the auxiliary space-like cone). We define Φ to be the fusion rule algebra generated, through arbitrary compositions, by the family L of transportable, irreducible *endomorphisms of $\mathcal{B}^{C_a}$ localizable in space-like cones. Let C be an arbitrary, non-empty space-like cone space-like separated from C_a. We define the von Neumann algebra $\mathfrak{M}$ to be the local algebra

$$\mathfrak{M} \equiv \mathfrak{M}(C) := \overline{\mathfrak{A}(C)}^{w}\,.$$

By Haag duality in the form considered in [20],

$$\mathfrak{M}' = \overline{\mathfrak{A}^c(C)}^{w}\,,$$

on the vacuum sector, $\mathcal{H}_1$, of the theory. Let $U \equiv U(C)$ denote the group of unitary elements in $\mathfrak{M}$, i.e.,

$$U := \left\{V \in \mathfrak{M} : V^* = V^{-1}\right\}.$$

Since every endomorphism in L is transportable, and hence is unitarily equivalent to endomorphism localized in a space-like cone of arbitrarily small opening angle contained the cone C, we can choose a representative which is a *endomorphism of $\mathfrak{M}$ acting trivia on $\mathfrak{M}'$ in *every equivalence class* of unitarily equivalent *endomorphisms in L. By al including arbitrary compositions of such endomorphisms we obtain a subset, $\mathrm{End}_L(C$ of $\mathrm{End}(\mathfrak{M}(C))$ which is closed under composition and hence is a (sub-)semigroup. Tl semigroup $\mathrm{End}_L(C)$ contains the subgroup, $\mathrm{Int}(C)$, of inner *automorphisms of $\mathfrak{M}$ give by

$$\mathrm{Int}(C) := \{\sigma_V : \exists V \in U(C) \quad \text{s.t.} \quad \sigma_V(A) = VAV^*, \forall A \in \mathfrak{M}\}\,.$$

The fusion rule algebra Φ of the local quantum theory under consideration is then give by

$$\Phi \cong \mathrm{End}_L(C) / \mathrm{Int}(C)\,. \qquad (3.16$$

The cone C, although chosen arbitrarily, and the von Neumann algebras $\mathfrak{M} = \mathfrak{M}(C)$ an $\mathfrak{M}'$ can and will be kept fixed throughout this chapter.

3.2 Structure Theory for Fusion Rule Algebras

We review several results on the structure of fusion rule algebras which are based on the theory of non-negative matrices, in particular on connectedness arguments and Perron Frobenius theory. We focus our attention on the description of the fusion rule subalgebra, Φ_ρ*, generated by a distinguished minimal vector* $\phi_\rho \in \Phi$*, and comment on the gradation induced by* Φ_ρ *on* Φ *in terms of "Perron Frobenius algebras" defined over* $\mathbb{R}^+$*. The proofs of the following statements as well as more general aspects of the structure theory will be given elsewhere [42].*

The first observation about fusion rule matrices is that they have non-negative entries, and, since $\mathrm{N}_\rho^t = \mathrm{N}_{\bar\rho}$ is a fusion rule matrix, too, if N_ρ is one, fusion rule matrices are normal, i.e.,

$$\mathrm{N}_\rho \mathrm{N}_\rho^t = \mathrm{N}_\rho^t \mathrm{N}_\rho . \tag{3.17}$$

Note that (3.17) defines a symmetric, non-negative matrix with strictly positive diagonal elements. Hence it can be decomposed into irreducible parts, each of which is primitive. The following lemma holds for arbitrary non-negative matrices. From the superdiagonal block form on every N_ρ-invariant domain, $\Phi_\lambda = \bigoplus_{i\in\mathbb{Z}_{a_\lambda}} \Phi_{(\lambda,i)}$, we see that the period a_λ is identical with the Frobenius imprimitivity index.

Lemma 3.2.1 *Let* N *be a normal* $n \times n$*-matrix, with non-negative (integer) entries and non-zero rows, or columns, and let*

$$\Phi \cong \left(\mathbb{R}^+\right)^n \qquad (\text{or} \cong \mathbb{N}^n)$$

be the cone (positive lattice) on which it is defined, with unit vectors $\phi_1, \ldots, \phi_n$*. Then there is a unique sequence of numbers,* $a_\lambda \in \mathbb{N}$*, with* λ *ranging over some index set* Λ*, and a unique partition of* $\{1, \ldots, n\} : \{1, \ldots, n\} = \dot\bigcup_{\lambda\in\Lambda} \dot\bigcup_{i\in\mathbb{Z}_{a_\lambda}} C_{(\lambda,i)}$*, such that the subcones (sublattices)* $\Phi_{(\lambda,i)} := \left\langle \{\phi_j\}_{j\in C_{(\lambda,i)}} \right\rangle_{\mathbb{R}^+(\mathbb{N})}$*, with*

$$\Phi = \bigoplus_{\lambda\in\Lambda} \bigoplus_{i\in\mathbb{Z}_{a_\lambda}} \Phi_{(\lambda,i)} , \tag{3.18}$$

obey

$$\begin{aligned} &\mathrm{N}\left(\Phi_{(\lambda,i)}\right) \subset \Phi_{(\lambda,i+1)} , \\ \textit{and} \quad & \\ &\mathrm{N}^t\left(\Phi_{(\lambda,i+1)}\right) \subset \Phi_{(\lambda,i)} , \end{aligned} \tag{3.19}$$

and, moreover, $\mathrm{N}^t\mathrm{N}$ *is primitive on each* $\Phi_{(\lambda,i)}$*, i.e.,* $(\mathrm{N}^t\mathrm{N})^m \restriction \Phi_{(\lambda,i)}$ *has strictly positive matrix elements, for some* m*. Furthermore, if there exists an involution,* $\pi \in S_n$*, such that we have*

$$C\mathrm{N}C = \mathrm{N}^t , \tag{3.20}$$

with $C\phi_i := \phi_{\pi(i)}$*, then there is an involution,* $\lambda \to \bar\lambda$*, on* Λ*, with* $a_\lambda = a_{\bar\lambda}$*, and an enumeration of* $\mathbb{Z}_{a_\lambda}$ *such that*

$$C\left(\Phi_{(\lambda,i)}\right) = \Phi_{(\bar\lambda,-i)} \tag{3.21}$$

or

$$C\left(\Phi_{(\lambda,i)}\right) = \Phi_{(\lambda,1-i)} \quad \text{for} \quad \lambda = \bar\lambda . \tag{3.22}$$

From the superdiagonal block form of N_ρ on every N_ρ-invariant domain, $\Phi_\lambda = \bigoplus_{i\in\mathbb{Z}_{a_\lambda}} \Phi_{(\lambda,i)}$, we see that the period a_λ is identical to the Frobenius imprimitivity index . Also we have that the restriction of the matrix N_ρ to a domain Φ_λ is irreducible and by standard Perron Frobenius theory has an eigenvector in Φ_λ (components taken in $\mathbb{R}^+$) with positive eigenvalue, which is unique up to positive scalars. It is called the Perron Frobenius eigenvector of the matrix. Any eigenvector of N_ρ on Φ is thus a convex combination of eigenvectors on components with the same eigenvalue. A more general version of this observation is obtained by induction :

Lemma 3.2.2 *i) Let* $S = \{\mathrm{N}_1, \ldots, \mathrm{N}_k\}$ *be a set of commuting* $n \times n$*- matrices, which closes under transposition, i.e.,* $\mathrm{N}_i^t \in S$ *if* $\mathrm{N}_i \in S$*. Define the set* $PF(S)$ *as*

$$PF(S) = \{\vec d \in (\mathbb{R}^{+,0})^n : \exists (\alpha_i) \in (\mathbb{R}^+)^k \text{ with } \mathrm{N}_i \vec d = \alpha_i \vec d, \forall i\} , \tag{3.23}$$

then there is a partition

$$\{1, \ldots, n\} = \dot\bigcup_{\alpha\in A} \dot\bigcup_{j} C_{(\alpha,j)} \tag{3.24}$$

and Perron Frobenius eigenvectors $\vec{d}_{\alpha,j}$ *with eigenvalues* $\alpha = (\alpha_i)$ *and support in* $C_{(\alpha,j)}$, *i.e.,* $(\vec{d}_{\alpha,j}, \phi_i) \neq 0$ *iff* $i \in C_{(\alpha,j)}$, *such that there is an orthogonal decomposition*

$$PF(S) = \bigoplus_{\alpha \in A} PF(S)_\alpha\,, \tag{3.25}$$

where $PF(S)_\alpha$ *is the convex cone spanned by the set of extremal directions* $\{\vec{d}_{\alpha,j}\}_j$.

ii) Suppose S *is the set of fusion rule matrices of a fusionrule algebra with a finite number of irreducible (or minimal) objects. Then the partition in i) is trivial, i.e. there is a unique vector* $\vec{d} \in (\mathbb{R}^+)^n$, *with*

$$\varepsilon(\vec{d}) = (\phi_1, \vec{d}) = 1 \tag{3.26}$$

and

$$PF(S) = \mathbb{R}^+ \vec{d}. \tag{3.27}$$

The components are

$$\begin{aligned} d_\psi &= (\vec{d}, \phi_\psi) \\ &= \alpha_\psi = \|\mathrm{N}_\psi\|\,, \end{aligned} \tag{3.28}$$

and take values in the set

$$\{\, 2\,cos(\frac{\pi}{N})\,\}_{N=3,4,\ldots} \cup [2, \infty). \tag{3.29}$$

Part i) of Lemma 3.2.2 relates to Lemma 3.2.1 as follows: For $S = \{\mathrm{N}_\rho, \mathrm{N}_\rho^t\}$ to any $\lambda \in \Lambda$ labelling a minimal, invariant sublattice, there corresponds an extremal Perron Frobenius eigenvector labelled by a pair (α, j). This description of smallest common invariant domains in terms of extremal Perron Frobenius eigenvectors of course generalizes to involutive sets, S, of matrices with more than two elements. In the proof of the second assertion in this lemma we make essential use of equation (3.13), which shows immediately that every irreducible object is in the invariant domain containing 1. Using the numbers determined in (3.28) we define a positive function on Φ setting for an arbitrary object $x \in \Phi$, given by $x = \sum_{\psi \in L} x_\psi \phi_\psi$

$$d(x) = \sum_\psi x_\psi d_\psi \tag{3.30}$$

We verify that it satisfies

$$\begin{aligned} d \quad &: \Phi \longrightarrow \mathbb{R}^+\,, \\ d(x+y) \quad &= d(x) + d(y)\,, \\ d(x \circ y) \quad &= d(x) d(y)\,. \end{aligned} \tag{3.31}$$

We call a function with the properties (3.31) a Perron Frobenius dimension. From ii) of Lemma 3.2.2 we conclude that for fusion rule algebras with a finite number of irreducible objects this dimension exists and is unique. Also we have $d(1) = 1$ and $d(x^\vee) = d(x)$. If we consider fusionrules with an infinite set of irreducible objects this dimension is in general not unique as can be seen in the case of ordinary $SU(2)$-fusionrules. For these the numbers, $d(x) = (dim(x))_q$, $q \in \mathbb{R}^+$, provide a one parameter family of Perron Frobenius dimensions.

In the following we define for a subset T of Φ its support in the irreducible objects by

$$\mathrm{supp}(T) := \{\psi \mid (\phi_\psi, s) \neq 0, \text{ for some } s \in T\} \tag{3.32}$$

The result of Lemma 3.2.2, i), can be applied to define a quotient algebra Φ/Φ_o for a fusion rule subalgebra $\Phi_o \subset \Phi$, where two irreducible objects, ψ_1 and ψ_2, are equivalent iff $\psi_1 = x \circ \psi_2$ for some $x \in \Psi_o$. We obtain a partition of Φ by setting $C_{[\psi]} := \mathrm{supp}\,(\phi_\psi \circ \Phi_o$ and $\Phi_\beta := \left\langle \{\phi_\psi\}_{\psi \in C_\beta} \right\rangle_{\mathrm{N}}$, so that $\Phi = \bigoplus_{\beta \in B} \Phi_\beta$ and $\Phi_o \circ \Phi_\beta = \Phi_\beta$, where B is the set of equivalence classes. The fusion rule matrices, N_ψ, of Φ have a unique common Perron Frobeniu eigenvector $\vec{d} \in \mathbb{R}^+ \cdot \Phi$, with $\varepsilon(\vec{d}) = 1$, and the components $\vec{d^\beta} \in \mathbb{R}^+ \Phi_\beta$ of $\vec{d} = \sum_{\beta \in B} \vec{d^\beta}$ spar the cone of common Perron Frobenius eigenvectors of representations in C_o

In order to state the next lemma, we define the positive numbers κ_β and $\overline{N}'_{\psi_1 \psi_2, \beta}$ b setting

$$\kappa_\beta \quad := \quad \left\|\vec{d^\beta}\right\| / \left\|\vec{d^o}\right\| \qquad (3.33$$

and

$$\frac{\kappa_\beta}{\kappa_{[\psi_1]} \kappa_{[\psi_2]}} \overline{N}'_{\psi_1 \psi_2, \beta} \quad := \quad \sum_{\psi_3 \in C_\beta} \frac{d_{\psi_3}}{d_{\psi_1} d_{\psi_2}} N_{\psi_1 \psi_2, \psi_3}\,, \qquad (3.34$$

and the positive vectors

$$\vec{\delta}^{\beta} := \frac{1}{\left\|\vec{d}^{o}\right\| \left\|\vec{d}^{\beta}\right\|} \vec{d}^{\beta} . \tag{3.35}$$

Lemma 3.2.3

i) We have the following equations in $\mathbb{R}^+ \cdot \Phi$*:*

$$\frac{1}{d_\psi} \phi_\psi \circ \vec{\delta}^{o} = \frac{1}{\kappa_{[\psi]}} \vec{\delta}^{[\psi]} \tag{3.36}$$

and

$$\vec{\delta}^{\beta} \circ \vec{\delta}^{o} = \vec{\delta}^{o} . \tag{3.37}$$

ii) The numbers defined in (3.34) do not depend on ψ_1 *and* ψ_2 *explicitly, but only on the classes* $[\psi_1]$ *and* $[\psi_2]$*, so that we may define*

$$\overline{N}_{[\psi_1][\psi_2],[\psi_3]} = \overline{N'}_{\psi_1\psi_2,[\psi_3]} . \tag{3.38}$$

The numbers κ_β *then form the common Perron Frobenius eigenvector,* $\vec{\kappa}$*, of the matrices* $\left(\overline{N}_{[\psi_1][\psi_2],[\psi_3]}\right)$*, and* $\kappa_o = 1$*,* $\kappa_{\beta} = \kappa_{\bar{\beta}}$*, and*

$$\sum_{\gamma \in B} \overline{N}_{\alpha\beta,\gamma} \kappa_\gamma = \kappa_\alpha \kappa_\beta . \tag{3.39}$$

iii) The numbers $\overline{N}_{\alpha\beta,\gamma}$ *are the structure constants of the multiplication table of the* $\vec{\delta}^{\beta}$*'s, i.e.,*

$$\vec{\delta}^{\alpha} \circ \vec{\delta}^{\beta} = \sum_{\gamma \in B} \overline{N}_{\alpha\beta,\gamma} \vec{\delta}^{\gamma} , \tag{3.40}$$

and we have that

$$\overline{N}_{\alpha\bar{\alpha},0} = 1 .$$

Remark. In all statements of Lemma 3.2.3 we understand the multiplication, $\circ$, defined on the $\mathbb{N}$-algebra Φ to be extended to the $\mathbb{R}^+$-algebra $\mathbb{R}^+ \cdot \Phi$. We easily verify, that the structure constants, $\overline{N}_{\alpha\beta,\gamma}$, obey all constraints (3.7), necessary for fusion rule matrices to define a fusion rule algebra, except that they are not necessarily integer-valued. This motivates the following definition.

Definition 3.2.4 *For two fusion rule algebras* $\Phi_o \subset \Phi$*, the* *Perron-Frobenius algebra of* Φ *over* Φ_o *is the cone freely generated by the index set* B*, i.e., all combinations* $\sum_{\alpha \in B} \lambda_\alpha \alpha$*, with* $\lambda_\alpha \geq 0$*, equipped with the conjugation,* $\alpha \rightarrow \bar{\alpha}$*, and the* $\mathbb{R}^+$*-bilinear multiplication,* $\circ$*, defined by the structure constants, i.e., we have that*

$$\alpha \circ \beta := \sum_{\gamma} \overline{N}_{\alpha\beta,\gamma} \gamma .$$

This algebra is denoted by Φ / Φ_o *and is often identified with a subalgebra of* $\mathbb{R}^+ \cdot \Phi$ *through the embedding* $\alpha \mapsto \vec{\delta}^{\alpha}$*. (If we set* $(\alpha, \beta) := \delta_{\alpha\beta} \frac{1}{\|d_o\|^2}$ *this embedding is even seen to be isometric.)*

The use of Perron Frobenius algebras is motivated by the observation that

$$\overline{N}_{\alpha\beta,\gamma} = 0$$

iff there exist $\psi_\alpha \in C_\alpha$ and $\psi_\beta \in C_\beta$ such that

$$\phi_{\psi_\alpha} \circ \phi_{\psi_\beta} \perp \Phi_\gamma \tag{3.41}$$

which is equivalent to

$$\Phi_\alpha \circ \Phi_\beta \perp \Phi_\gamma . \tag{3.42}$$

Thus the algebra Φ / Φ_o tells us which Φ_o-invariant components, Φ_γ, occur in the product of two other components, Φ_α and Φ_β. Definition 3.2.4 can of course also be applied to Perron Frobenius algebras $\Phi_o \subset \Phi$, instead of fusion rule algebras. We can therefore iterate our construction and obtain familiar equations, like $\Phi / \Phi_o \cong (\Phi / \Phi_\infty) / (\Phi_o / \Phi_\infty)$, for $\Phi_\infty \subset \Phi_o \subset \Phi$.

We associate to any pair of sets $T, S \subset \{1, \ldots, n\}$, the composition

$$T \circ S := \bigcup_{i \in T, j \in S} \operatorname{supp}(\phi_i \circ \phi_j) ,$$

so that $T, S \rightarrow T \circ S$ is a commutative and associative operation. Further, we denote by $[T]$, $T \subset [T] \subset \{1, \ldots, n\}$, the set generated by T, more precisely $[T] := \bigcup_{k,\ell \geq 0} T^k \circ \overline{T}^k$. For any set T, the sublattice $\Phi_{[T]} := \mathbb{N}^{[T]} \subset \Phi$ is a fusion rule subalgebra of Φ.

In the simplest cases $T = \{\rho\}$ and $T = \{\rho\} \circ \{\bar{\rho}\}$, these subalgebras are related to the presentation of the fusion rule matrix N_ρ of Lemma 3.2.1 by

$$\begin{aligned} \Phi_{[\rho]} &= \Phi_{\lambda_o} = \bigoplus_{i \in \mathbb{Z}_{\lambda_o}} \Phi_{(\lambda,i)}\,, \\ \text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (3.43) \\ \Phi_{[\rho\circ\bar{\rho}]} &= \Phi_{(\lambda_o,0)}\,, \end{aligned}$$

where λ_o and the enumeration of $\mathbb{Z}_{a_{\lambda_o}}$ are chosen such that $1 \in C_{(\lambda_o,0)}$; hence

$$C\left(\Phi_{(\lambda_o,i)}\right) = \Phi_{(\lambda_o,-i)}.$$

The Perron Frobenius algebra $\Phi/\Phi_{[\rho\circ\bar{\rho}]}$ can be described further by using Lemma 3.2.1.

Proposition 3.2.5 *Suppose that for a representation ρ, the fusion rule matrix N_ρ has imprimitivity indices $a_\lambda \in \mathbb{N}$, $\lambda \in \Lambda$, and define a partition of $\{1,\ldots,n\} = \dot{\cup}_\lambda \dot{\bigcup}_{i\in\mathbb{Z}_{a_\lambda}} C_{(\lambda,i)}$, according to Lemma 3.2.1, with λ_o as above, i.e., $1 \in [\rho\circ\bar{\rho}] = C_{(\lambda_o,o)}$. Then*

i) The Perron Frobenius algebra, $\Phi/\Phi_{[\rho\circ\bar{\rho}]}$, is given by vectors $\vec{\delta}^{(\lambda,i)}$, $\lambda \in \Lambda$, $i \in \mathbb{Z}_{a_\lambda}$, with supp $\left(\vec{\delta}^{(\lambda,i)}\right) = C_{(\lambda,i)}$, *and* $1 := \vec{\delta}^{(\lambda_o,0)}$.

ii) The subalgebra $\Phi_{[\rho]}/\Phi_{[\rho\circ\bar{\rho}]}$ is generated by an automorphism $a := \vec{\delta}^{(\lambda_o,1)}$, with

$$\alpha^{a_{\lambda_o}} = \alpha \circ \bar{\alpha} = 1$$

and is therefore isomorphic to $\mathbb{R}^+([\mathbb{Z}_{a_\lambda}])$. We have that

$$\begin{aligned} & \alpha \circ \vec{\delta}^{(\lambda,i)} = \vec{\delta}^{(\lambda,i+1)}, \quad \forall \lambda \in \Lambda\,, \quad i \in \mathbb{Z}_{a_a}\,, \\ \text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (3.44) \\ & a_\lambda \quad \text{divides} \quad a_{\lambda_o}\,, \quad \forall \lambda \in \Lambda\,. \end{aligned}$$

iii) There are constants $N^{(j)}_{\lambda_1\lambda_2,\lambda_3}$, $\lambda_i \in \Lambda$, depending on j only modulo the greatest common divisor of a_{λ_1}, a_{λ_2} and a_{λ_3}, i.e., $j \in \mathbb{Z}_{(a_{\lambda_1},a_{\lambda_2},a_{\lambda_3})}$, such that

$$\vec{\delta}^{(\lambda_1,k_1)} \circ \vec{\delta}^{(\lambda_2,k_2)} = \sum_{\lambda_3\in\Lambda,k_3\in\mathbb{Z}_{a_{\lambda_3}}} N^{(k_3-k_1-k_2)}_{\lambda_1\lambda_2,\lambda_3}\, \vec{\delta}^{(\lambda_3,k_3)}. \qquad (3.45)$$

iv) The vectors, $\vec{\delta}^\lambda$, $\lambda \in \Lambda$, of the Perron Frobenius algebra $\Phi/\Phi_{[\rho]} \cong \left(\Phi/\Phi_{[\rho\circ\bar{\rho}]}\right)/\Phi_{[\alpha]}$ are given by

$$\vec{\delta}^\lambda = \frac{1}{\sqrt{a_\lambda a_{\lambda_o}}} \sum_{i=0}^{a_\lambda-1} \vec{\delta}^{(\lambda,i)} = \sqrt{\frac{a_\lambda}{a_{\lambda_o}}}\, \vec{\delta}^{\lambda_o} \circ \vec{\delta}^{(\lambda,j)} \qquad (3.46)$$

j arbitrary, and the structure constants are

$$\overline{N}_{\lambda_1\lambda_2,\lambda_3} = \sqrt{\frac{a_{\lambda_1}a_{\lambda_2}a_{\lambda_3}}{a_{\lambda_o}}}\, \frac{1}{(a_{\lambda_1},a_{\lambda_2},a_{\lambda_3})} \sum_{j=1}^{(a_{\lambda_1},a_{\lambda_2},a_{\lambda_3})} N^{(j)}_{\lambda_1\lambda_2,\lambda_3}\,. \qquad (3.47)$$

Roughly speaking, Proposition 3.2.5 shows that the action of $\Phi_{[\rho]}$ on Φ is graded and that the composition law of the invariant domains of $\Phi_{[\rho\circ\bar{\rho}]}$ has a periodicity specified in part iii). In the following, we shall denote the fundamental imprimitivity of N_ρ, a_{λ_o}, characterized in part ii), by a_ρ for any label ρ of the fusion rule algebra and by C^ρ_i, $i \in \mathbb{Z}_{a_\rho}$, the components $C_{(\lambda_o,i)}$. Finally $\Phi^\rho_i := \Phi_{[C^\rho_i]} \equiv \Phi_{(\lambda_o,i)}$.

We collect the consequences of Proposition 3.2.5 that are relevant for the later considerations in the next corollary.

Corollary 3.2.6 *For any label ρ of a fusion rule algebra, there is an integer, a_ρ, the imprimitivity of ρ, and a partition*

$$[\rho] = \dot{\bigcup_{i\in\mathbb{Z}_{a_\rho}}} C^\rho_i$$

of the set $[\rho]$ generated by ρ into a_ρ subsets, C^ρ_i, such that

$$\begin{aligned} 1 \in C^\rho_o &= [\rho\circ\bar{\rho}]; \qquad \overline{C^\rho_i} = C^\rho_{-i} \\ \text{and} \qquad & \qquad\qquad\qquad\qquad\qquad\qquad (3.48) \\ \psi \circ C^\rho_i &= C^\rho_{i+j}\,, \qquad \textit{for all} \quad \psi \in C_j\,. \end{aligned}$$

If ρ is selfconjugate, it follows immediately that $a_\rho = 1$ or $a_\rho = 2$.

For the simplest nontrivial case of $\mathbb{Z}_2$-gradation, we describe the fusion rule algebr more explicitly in terms of fusion rule-matrices. In general the fusion rule matrices o a $\mathbb{Z}_2$-graded algebra defined on $\Phi = \Phi_o \oplus \Phi_1$ ($\Phi = \mathbb{N}^{C_o\cup C_1}$, $\Phi_i = \mathbb{N}^{C_i}$, $i = 1,2$) and th conjugation have the blockform

$$\begin{aligned} C &= C_o \oplus C_1\,, \qquad (C_i \quad \text{involution on} \quad \Phi_i) \\ & \qquad\qquad\qquad\qquad\qquad\qquad (3.49 \\ \mathrm{N}_\psi &= \mathrm{N}^o_\psi \oplus \mathrm{N}^1_\psi\,, \qquad \text{for} \quad \psi \in C_o \end{aligned}$$

and

$$\mathbb{N}_\eta = \begin{pmatrix} 0 & C_o \Lambda^t_\eta C_1 \\ \Lambda_\eta & 0 \end{pmatrix}, \quad \text{for} \quad \eta \in C_1 . \tag{3.50}$$

It is possible to give criteria which determine when matrices of this type define a fusion rule algebra, in the sense of Section 3.1, namely that they obey equations (3.12) - (3.14).

Lemma 3.2.7 *Let $\{N^o_\psi\}_\psi$ be the fusion rule matrices of a fusion rule algebra $\Phi_o = \mathbb{N}^{C_o}$ with conjugation C_o. Suppose, further, there is a representation, π, of Φ_o on a lattice $\Phi_1 = \mathbb{N}^{C_1}$ with conjugation C_1, so that $C_1 \pi(\phi) C_1 = \pi(\bar{\phi}) = \pi(\phi)^t$. Then the matrices $\mathbb{N}^o_\psi$, $\mathbb{N}^1_\psi$ and Λ_η, where $\mathbb{N}^1_\psi := \pi(\phi_\psi)$, $\psi \in C_o$, and*

$$\Lambda_\eta : \Phi_o \to \Phi_1 \quad \textit{is determined by} \quad \Lambda_\eta \phi_{\bar{\psi}} = \mathbb{N}^1_\psi \phi_{\bar{\eta}} ,$$

define a fusion rule algebra $\Phi = \Phi_o \oplus \Phi_1$ with fusion rule matrices given by (3.49) and (3.50) iff

$$\Lambda_\eta \Lambda^t_{\bar{\gamma}} = \Lambda_\gamma \Lambda^t_{\bar{\eta}} . \tag{3.51}$$

Note that $C_1 \Lambda_\eta C_o = \Lambda_{\bar{\eta}}$. So we have the equations

$$\begin{aligned} C_1 \Lambda_\eta \Lambda^t_\eta C_1 &= \Lambda_{\bar{\eta}} \Lambda^t_{\bar{\eta}} , \\ C_o \Lambda^t_\eta \Lambda_\eta C_o &= \Lambda^t_{\bar{\eta}} \Lambda_{\bar{\eta}} \end{aligned} \tag{3.52}$$

where Λ_η is the block matrix of $\mathbb{N}_\eta$, for $\eta \in C_1$.

3.3 Grading Reduction with Automorphisms and Normality Constraints in Fusion Rule Algebras

In this Chapter we show how any simply generated fusion rule algebra, with nontrivially graded automorphisms, can be obtained from a smaller fusion rule algebra with $\mathbb{Z}_b$-grading. We state the most general presentation of a fusion rule algebra, $\Phi_{[\rho]}$, in terms of an algebra, in which all automorphisms lie in the trivially graded component. For this purpose, we introduce two constructions that yield new fusion rule algebras, $\tau_\delta\left(\Phi_{[\rho]}\right)$ and $\mathbb{Z}_n * \Phi_{[\rho]}$, from a given one, $\Phi_{[\rho]}$. We also discuss the crossed product, $\mathbb{Z}_n \times \Phi$, for arbitrary fusion rule algebras, and its use in the classification problem, for $\Phi = \Phi_{[\rho]}$.

The restrictions, Λ_i, of a fusion rule matrix $\mathbb{N}_\rho$ to the components C^ρ_i obey constraints that are due to the normality of $\mathbb{N}_\rho$. We use them to specify classes of Λ_o such that any fusion rule algebra, $\Phi_{[\rho]}$, compatible with one of these Λ_o has an automorphism in C_2 and can thus be obtained from a fusion rule algebra generated by a selfconjugate element $\rho = \bar{\rho}$.

Throughout this section, we assume that the fusion rule algebra Φ, with label set C, is $\mathbb{Z}_a$-graded (e.g., as in Corollary 3.2.6, for $C = [\rho]$). Thus we have a partition $C = \dot{\bigcup}_{i \in \mathbb{Z}_a} C_i$ and a corresponding lattice decomposition, $\Phi = \bigoplus_{i \in \mathbb{Z}_a} \Phi_i$.

To any fusion rule algebra, Φ, we can associate the set of <u>invertible objects</u>

$$\mathrm{Out}(\Phi) := \{\phi \in \Phi \mid \phi \circ \bar{\phi} = 1\} . \tag{3.53}$$

It is immediate that $\mathrm{Out}(\Phi)$ only consists of minimal vectors, and thus can be regarded as a subset of C. Moreover, it defines a discrete, abelian group with multiplication $\circ$ and inversion $\phi^{-1} = \bar{\phi}$. Equivalently, $\mathrm{Out}(\Phi)$ is characterized as the subgroup of permutations, $\pi \in S_{|C|}$ of C such that Π, given by $\Pi_{ij} = \delta_{i\pi(j)}$, commutes with all fusion rule matrices and hence $\Pi = \mathbb{N}^t_{\pi(1)}$. Referring to the fusion rule algebras (3.16) that emerge from the superselection rules generated by transportable *-endomorphisms of a local quantum field theory, the group $\mathrm{Out}(\Phi)$ (and, in particular, the notation) has a natural interpretation. If $\mathrm{Aut}(C)$ is the subgroup of $\mathrm{End}_L(C)$ consisting of *-automorphisms of $\mathfrak{M}$ acting trivially on $\mathfrak{M}'$ then equation (3.16) yields the isomorphism

$$\mathrm{Out}(\Phi) \cong \mathrm{Aut}(\mathcal{C})/\mathrm{Int}(\mathcal{C}) \tag{3.54}$$

Automorphisms (or invertible objects) can be detected fairly easily from the vector, $\vec{d}$, of statistical dimensions or from a common Perron-Frobenius eigenvector, $\vec{d'}$, for finite fusion rule algebras, Φ, by

$$\mathrm{Out}(\Phi) = \left\{ i \in \Phi \mid d'_i = \min_j d'_j \right\} = \{ i \in \Phi \mid d_i = 1 \} \,. \tag{3.55}$$

Since $d_i \geq 1, \forall \phi_i \in \Phi$, the total number of irreducible representations in $\phi_i \circ \phi_j$, $\sum_k N_{ij,k}$, is bounded above by $\sum_k N_{ij,k} d_k = d_i \cdot d_j$. Thus $d_\sigma = d_{\bar{\sigma}} = 1$ implies that $\sigma \circ \bar{\sigma}$ is irreducible, i.e. $\sigma \circ \bar{\sigma} = 1$. Hence σ is a *automorphism, and N_σ is a permutation. (Note that, in general, if a matrix, N_σ, with non-negative, integer entries and non-zero rows and columns admits a positive eigenvector with eigenvalue 1 then N_σ is automatically a bijection.) If the components $\mathcal{C}_o$ and $\mathcal{C}_1$ of a fusion rule matrix N_ρ are finite then we find from the unique Perron-Frobenius vector $\vec{d} = \left(\vec{d^o}, \vec{d^1}\right) \in \mathcal{C}_o \oplus \mathcal{C}_1$ of $\Lambda = \mathrm{N}^t_\rho \mid \mathcal{C}_o : \mathcal{C}_o \to \mathcal{C}_1$, (i.e., $\Lambda \vec{d^o} = d_\rho \vec{d^1}$; $\Lambda^t \vec{d^1} = d_\rho \vec{d^o}$) the automorphisms in $\mathcal{C}_o$ and $\mathcal{C}_1$ by (3.55). A similar result holds for $\mathcal{C}^k = \mathcal{C}_o \oplus \ldots \oplus \mathcal{C}_k$. Since, for $\sigma \in \mathrm{Out}(\Phi)$, $\rho \circ \sigma$ is irreducible , the vertices associated with automorphism in the graph to which N_ρ is the incidence matrix have exactly one incoming and one outgoing edge, (i.e., one undirected edge for $\rho = \bar{\rho}$), joining σ to sites ρ' for which $d_{\rho'} = d_\rho$. For general undirected graphs we only have the *"minimum principle"*, i.e., that the edge degree of sites on which the Perron-Frobenius vector admits its absolute minimum is strictly less than d_ρ, and is equal to d_ρ only if all vertices have edge degree d_ρ and the Perron Frobenius vector is constant. Hence we expect that, for $d_\rho > 2$, non-trivial constraints on the set of admissible fusion rule matrices can be found by considering the position of automorphisms in the fusion rule algebra.

Clearly the restriction of the grading map, $\Phi \to \mathbb{Z}_a : \Phi_i \mapsto i$, to $\mathrm{Out}(\Phi)$ is a group homomorphism, and its kernel is given by the subgroup $\mathrm{Out}\,(\Phi_o) \subset \mathrm{Out}(\Phi)$, where $\Phi_o \subset \Phi$ is the fusion rule subalgebra with trivial grading. Hence the grading gives rise to the embedding

$$D(\Phi) := \mathrm{Out}(\Phi)/\mathrm{Out}\,(\Phi_o) \hookrightarrow \mathbb{Z}_a \,. \tag{3.56}$$

It follows from (3.56) that there are integers r and a' with $a = r \cdot a'$, such that $D(\Phi) \cong \mathbb{Z}_r$ and a' is the smallest integer such that $\Phi_{a'} \cap \mathrm{Out}(\Phi) \neq 0$.

These aspects of grading fit into a general context: Let us consider a general fusion rule algebra Φ. Clearly, Φ contains a natural fusion rule subalgebra on which all gradings are trivial, namely the subalgebra $\Phi_{\mathcal{C}_0}$, where

$$\mathcal{C}_0 = \left[\bigcup_{i \in \mathcal{C}} \mathrm{supp}\, \left(\phi_i \circ \bar{\phi}_i \right) \right] .$$

Our notations are those introduced in Section 3.2. It is not hard to see that the Perron-Frobenius algebra over $\Phi_{\mathcal{C}_0}$, i.e.,

$$\mathrm{Grad}\,(\Phi) := \Phi \,/\, \Phi_{\mathcal{C}_0}$$

is, in fact, an abelian group, or, in other words, that $\overline{N}_{\alpha \circ \bar{\alpha}, \gamma} = \delta_{\gamma,1}$, for arbitrary α and γ in $\mathrm{Grad}(\Phi)$. This observation shows that an arbitrary grading on Φ is described by a character of the group $\mathrm{Grad}(\Phi)$. More precisely, if

$$\Theta : \Phi \to G \,,$$

with G an abelian group, is a grading of Φ, i.e.,

$$\Theta(i)\,\Theta(j) \;=\; \Theta(k) \quad \text{if } N_{ij,k} \neq 0 \,,$$

and

$$\mathrm{grad} : \Phi \to \mathrm{Grad}(\Phi)$$

is the canonical projection from Φ onto the quotient space $\mathrm{Grad}(\Phi)$ then there exist a homomorphism of abelian groups,

$$\overline{\Theta} : \mathrm{Grad}(\Phi) \to G \,,$$

such that the diagram

$$\begin{tikzcd} \Phi \arrow[rr, "\mathrm{grad}"] \arrow[dr, "\Theta"'] & & \mathrm{Grad}(\Phi) \arrow[dl, "\overline{\Theta}"] \\ & G & \end{tikzcd}$$

commutes.

We therefore call the map "grad" and its image, Grad(Φ), the universal grading of Φ. One sees without difficulty that

$$\mathrm{grad}(\bar{\psi}) = \mathrm{grad}(\psi)^{-1}, \qquad \text{for all } \psi \in \Phi,$$

where $\psi \mapsto \bar{\psi}$ is the conjugation on Φ.

In general, the restriction of the map grad: $\Phi \to \mathrm{Grad}(\Phi)$ to the group of invertible elements, Out(Φ), contained in Φ is a group homomorphism. Its kernel consists of all invertible elements of Φ_0, i.e.,

$$\ker(\mathrm{grad} \restriction \mathrm{Out}(\Phi)) = \mathrm{Out}(\Phi_0) .$$

From this remark we conclude that

$$D(\Phi) := \mathrm{grad}(\mathrm{Out}(\Phi)) \cong \mathrm{Out}(\Phi) / \mathrm{Out}(\Phi_0) ,$$

and $D(\Phi)$ is a subgroup of Grad(Φ).

If $\mathrm{Grad}(\Phi) \cong \mathbb{Z}_a$ the map "grad" gives rise to the embedding (3.56).

For any $\sigma \in \Phi_{a'} \cap \mathrm{Out}(\Phi)$, we have the decomposition

$$\mathrm{Out}(\Phi) = \bigoplus_{j=0}^{r-1} \sigma^j \circ \mathrm{Out}(\Phi_o) ,$$

and a bijection

$$\mathcal{C}_i \to \mathcal{C}_{i+a'} : \phi \mapsto \sigma \circ \phi .$$

These facts imply that the multiplication law on the set $\mathcal{C}_o \cup \mathcal{C}_1 \ldots \cup \mathcal{C}_{a'-1}$, together with a specific automorphism $\sigma \in \mathcal{C}_{a'}$, already determine the entire fusion rule algebra. In fact, it is true that one can construct a fusion rule algebra Φ' which is $\mathbb{Z}_{a'}$-graded, with $D(\Phi') = 1$, and from which Φ can be reconstructed. The two operations on the class of fusion rule algebras that are necessary for this description are defined next.

Definition 3.3.1 *Let $\Phi \cong \mathbb{N}^{\mathcal{C}}$ be a fusion rule algebra with multiplication $\circ$ and conjugation $\bar{-}$. Further, let Φ be $\mathbb{Z}_a$-graded, with $\Phi = \bigoplus_{i \in \mathbb{Z}_a} \Phi_i$.*

i) For any $b \in \mathbb{N}$, we define the fusion rule algebra $\mathbb{Z}_b * \Phi$ as follows: The underlying lattice is $\mathbb{N}^{(\mathbb{Z}_b \times \mathcal{C})}$ and is spanned by the minimal vectors (k, ϕ_i), $k \in \mathbb{Z}_b$, $i \in \mathcal{C}$. The conjugation is denoted by $\phi \to \phi^c$ and is given by

$$(k,\phi)^c := \begin{cases} (-k-1, \bar{\phi}), & \text{for } \phi \in \Phi_i, \quad i \neq 0, \\ (-k, \bar{\phi}), & \text{for } \phi \in \Phi_o . \end{cases}$$

The multiplication is denoted by $\times$ and, for $k_j \in \mathbb{Z}_{a_j}$, $\phi_j \in \Phi_{n_j}$, $n_j = 0, \ldots, a-1$, $j = 1, \ldots, m$, is given by

$$(k_1, \phi_1) \times (k_2, \phi_2) \times \ldots \times (k_m, \phi_m) := (r + k_1 + \ldots + k_m, \phi_1 \circ \ldots \circ \phi_m) ,$$

where $r \in \mathbb{N}$ is given by the condition

$$ar \leq n_1 + \ldots + n_m < a(r+1) .$$

ii) For any $\delta \in \mathrm{Out}(\Phi_o)$, we define the fusion rule algebra $\tau_\delta(\Phi)$ as follows:

The lattice of $\tau_\delta(\Phi)$ is the same as for Φ. The conjugation, $\phi \to \bar{\phi}'$, is expressed in terms of the conjugation of Φ by

$$\bar{\phi}' := \begin{cases} \bar{\delta} \circ \bar{\phi}, & \text{for } \phi \in \Phi_i, \quad i \neq 0, \\ \bar{\phi}, & \text{for } \phi \in \Phi_o . \end{cases}$$

The multiplication is denoted by $\circ'$ and, for $\phi_j \in \Phi$ and $r \in \mathbb{N}$ as in part i), is defined by

$$\phi_1 \circ' \ldots \circ' \phi_m := \delta^r \circ \phi_1 \circ \ldots \circ \phi_m .$$

It is straightforward to show that the multiplications and conjugations introduced above define fusion rule algebras, in the sense of Section 3.1. Since the trivially graded automorphisms are not affected by these constructions, we naturally have that

$$\mathrm{Out}(\mathcal{C}_o) \cong \mathrm{Out}((\tau_\delta(\Phi))_o) \cong \mathrm{Out}((\mathbb{Z}_b * \Phi)_o) .$$

However, the situation for Out(Φ) is different: $\mathrm{Out}(\mathbb{Z}_b * \Phi)$ contains the subgroup $\cong \mathbb{Z}_b$ generated by (1,1), so that

$$\mathrm{Out}(\mathbb{Z}_b * \Phi) / \mathbb{Z}_b \cong \mathrm{Out}(\Phi), \quad \text{and} \quad D(\mathbb{Z}_b * \Phi) \cong \mathbb{Z}_{ab} .$$

We can find a grading preserving isomorphism of $\mathrm{Out}\,(\tau_\delta(\Phi))$ onto $\mathrm{Out}(\Phi)$ if $\delta = \alpha^b$, for some $\alpha \in \mathrm{Out}\,(\Phi_o)$. For other choices of δ this is in general not possible.

Furthermore, it follows immediately from Definition 3.3.1 that, for any $\alpha \in \mathrm{Out}(\Phi)$, the map

$$\mathbb{Z}_b * \tau_\alpha(\Phi) \rightarrow \tau_{\alpha^b}(\mathbb{Z}_b * \Phi) : (k, \phi) \rightarrow \left(k, \bar{\alpha}^b \circ \phi\right) \tag{3.57}$$

provides a fusion rule algebra isomorphism. Also, we have that

$$\tau_{\delta_1}\left(\tau_{\delta_2}(\Phi)\right) \cong \tau_{\delta_1 \circ \delta_2}(\Phi)\,, \tag{3.58}$$

by natural identification, and an isomorphism

$$\mathbb{Z}_{b_2} * (\mathbb{Z}_{b_1} * \Phi) \rightarrow \mathbb{Z}_{b_2 \cdot b_1} * \Phi : (k_2, (k_1, \phi)) \rightarrow (k_1 + b_1 \cdot k_2, \phi)\,, \tag{3.59}$$

where k_1 is chosen in $\{0, \ldots, b_1 - 1\}$. We are now in a position to state the presentation of all $\mathbb{Z}_a$-graded fusion rule algebras in terms of algebras, Φ, with $D(\Phi) = 1$.

Proposition 3.3.2 *Suppose Φ is $\mathbb{Z}_a$-graded algebra, with label set $C = \bigcup_{j=0}^{a-1} C_j$, multiplication $\circ$ and conjugation $^-$, such that*

$$D(\Phi) = \mathbb{Z}_r\,,$$

where $r > 1$ is an integer dividing a, and $a'' := a/r$.

Then there exists a $\mathbb{Z}_{a''}$-graded fusion rule algebra Φ'', with corresponding constituents $\left(C'' = \bigcup_{j=0}^{a''-1} C''{}_j, \circ'', {}^{-''}\right)$, an automorphism $\delta \in \mathrm{Out}\,(\Phi''_o)$ and a fusion rule algebra isomorphism β,

$$\beta : \tau_\delta(\mathbb{Z}_r * \Phi'') \quad \overset{\cong}{\rightarrow} \quad \Phi\,, \tag{3.60}$$

such that

- *i) β maps $\left(0, \Phi''_j\right)$ bijectively to Φ_j, for $j = 0, \ldots, a' - 1$, and $\beta(1,1) \in \mathrm{Out}(\Phi) \cap \Phi''$. In particular, β is grading preserving.*
- *ii) $D(\Phi'') = 1$, i.e., we have*

$$\mathrm{Out}(\Phi'') = \mathrm{Out}\,(\Phi''_o) = \mathrm{Out}\,(\Phi_o)\,.$$

By part i) of Proposition 3.3.2 the lattice isomorphism obeys

$$\beta''(\phi \circ'' \Psi) = \beta''(\phi) \circ \beta''(\Psi)\,, \tag{3.61}$$

for $\phi \in C_i$, $\Psi \in C_j$, $i, j \geq 0$, provided the condition

$$i + j < a'' \tag{3.62}$$

holds. This shows that the restriction $\beta'' : \Phi''_o \rightarrow \Phi_o$ is a fusion rule algebra isomorphism. Also, for $a'' > 2$, the restriction of the fusion rule matrix of some element $\rho \in C_1$ to C_o remains unchanged. More precisely, for $\Lambda^\rho_o := \mathbb{N}^t_\rho \restriction \Phi_o \rightarrow \Phi_1$, we have that

$$(\beta'')\Lambda^\rho_o\beta'' = \Lambda^{\rho''}_o\,,$$

with $\rho'' = (\beta'')^{-1}(\rho) \in C''_1$. The proof of Proposition 3.3.2 can be found in [42].

For certain special cases there exist a natural procedure to relate $\mathbb{Z}_a$-graded fusion rule algebras among themselves, with the help of automorphisms. It involves the crossed product, $\Phi_1 \times \Phi_2$ of two fusion rule algebras, Φ_i, $i = 1, 2$, with lattice $\Phi_1 \otimes \Phi_2 = \mathbb{N}^{(C_1 \times C_2)}$, multiplication $(\phi_1 \otimes \phi_2) \circ (\psi_1 \otimes \psi_2) = (\phi_1 \circ \psi_1) \otimes (\phi_2 \circ \psi_2)$, and conjugation $\overline{(\phi_1 \otimes \phi_2)} = \bar{\phi}_1 \otimes \bar{\phi}_2$. By $\mathbb{Z}_n$ we denote the fusion rule algebra with $C = \mathbb{Z}_n$ and $\phi_i \circ \phi_j = \phi_{i+j}$, $\bar{\phi}_i = \phi_{-i}$.

Lemma 3.3.3

- *i) Suppose Φ' is a $\mathbb{Z}_{a'}$-graded fusion rule algebra and $r \in \mathbb{N}$ is prime to a', then $\Phi = \mathbb{Z}_r \times \Phi'$ is $\mathbb{Z}_a$-graded, with $a = a' \cdot r$ and $\Phi_{ia'+jr} = \{\phi_i\} \otimes \Phi'_j$.*
- *ii) Assume that Φ is a $\mathbb{Z}_a$-graded algebra, and $a = r \cdot a'$. Then $\Phi' = \sum_{j=0}^{a'-1} \Phi_{r \cdot j} \subset \Phi$ i a $\mathbb{Z}_{a'}$-graded fusion rule subalgebra. If, in addition, there exists an automorphism c*

$$\alpha \in \Phi_{a'} \cap \mathrm{Out}(\Phi)\,, \quad \text{with} \quad \alpha^r = 1\,,$$

and α' is prime to r, then

$$\mathbb{Z}_r \times \Phi' \rightarrow \Phi : \phi_n \otimes \psi \rightarrow \alpha^n \circ \psi$$

is a fusion rule algebra isomorphism.

Note that if $\Phi = \Phi_{[\rho]}$ is generated by an element $\rho \in C_1$, then also $\Phi' = \Phi_{[\alpha^t \circ \rho]}$ is generated by $\alpha^t \circ \rho \in C'$, where t is determined by the equation $t \cdot a' \equiv -1 \bmod r$. We will be interested mainly in the case where $r = 2$ and a' is odd for which Lemma 3.3.3 shows that it is sufficient to consider even graded fusion rule algebras. This is because any odd graded Φ' will appear, as $\mathbb{Z}_2 \times \Phi'$, in the list of evenly graded fusion rule algebras which, in addition, contain an automorphism $\alpha \notin \mathrm{Out}(\Phi_o)$, with $\alpha^2 = 1$.

Returning to Proposition 3.3.2, we note that if $\Phi = \Phi_{[\rho]}$ is generated by a single element $\rho \in C_1$, then the algebra Φ'' in the presentation (3.60) is generated by the corresponding $\rho'' = (\beta'')^{-1}(\rho) \in C''_1$, i.e., $\Phi'' = \Phi_{[\rho'']}$, if it is nontrivially graded. In the following, we shall characterize a class of fusion rule algebras $\Phi_{[\rho]}$ with generating element ρ, with the property that there is a presentation (3.60) where ρ'' is selfconjugate.

Lemma 3.3.4 *Suppose that* $\Phi = \Phi_{[\rho]}$ *is a* $\mathbb{Z}_a$*-graded fusion rule algebra, with* $a \geq 2$ *and generator* ρ. *Then there is a presentation*

$$\Phi \overset{\beta''}{\cong} \tau_\delta(\mathbb{Z}_r * \Phi''), \quad \delta \in \mathrm{Out}(\Phi_o), \quad r \in \mathbb{N},$$

where the corresponding generator $\rho'' = (\beta'')^{-1}(\rho) \in C''_1$ *is selfconjugate in* Φ'', *if and only if there is an element* $\alpha \in \Phi$, *such that*

$$\alpha \circ \bar{\rho} = \rho. \tag{3.63}$$

If Φ *is ungraded then there exists some* $\alpha \in \Phi$, *with (3.63), if and only if we have a presentation*

$$\mathbb{Z}_2 \times \Phi \overset{\beta''}{\cong} \tau_\delta(\Phi''),$$

where the respective element $\rho'' = (\beta'')^{-1}(\phi_{-1} \otimes \rho)$ *generates* Φ'' *and is selfconjugate.*

In any case α *is an automorphism and* Φ'' *is either* $\mathbb{Z}_2$*-graded or ungraded. Hence* $r = \frac{a}{2}$ *or* $r = a$, *and* $\alpha \in C_2$ *or* $\alpha \in C_o$.

If we introduce the restrictions

$$\Lambda_i = \mathrm{N}_\rho^t \restriction_{C_i} : \Phi_i \to \Phi_{i+1}, \tag{3.64}$$

which we regard as $|C_i| \times |C_{i+1}|$-matrices with non-negative integer matrix elements, then condition (3.63) can be reformulated as follows: There exists a $\phi_\alpha \in \Phi_2$, $\|\phi_\alpha\| = 1$, such that

$$\Lambda_1^t \, \phi_\alpha = \Lambda_o \, 1 = \phi_\rho . \tag{3.65}$$

It follows then that the restrictions $T_i = \mathrm{N}_\alpha^t \restriction_{C_i} : \Phi_i \to \Phi_{i+2}$ are bijections, i.e.,

$$T_i^t \, T_i = T_{i-2} \, T_{i-2}^t = \mathbf{1}_{\Phi_i} , \tag{3.66}$$

and

$$\Lambda_{i+2}^t \, T_{i+1} = T_i \, \Lambda_i^t = \Lambda_{i+1} . \tag{3.67}$$

From the fact that N_ρ is normal we obtain the following constraints on the matrices Λ_i:

$$\Lambda_i^t \, \Lambda_i = \Lambda_{i-1} \, \Lambda_{i-1}^t =: M_i . \tag{3.68}$$

We immediately see that any set of matrices obeying (3.67), with arbitrary bijections T_i, solve the constraint (3.68). Moreover, it follows from (3.68) that

$$\|\Lambda_i\| = \|\mathrm{N}_\rho\| , \tag{3.69}$$

independently of i. The purpose of the next two combinatorial results is to infer equation (3.65) from the knowledge of Λ_o or $M_1 = \Lambda_o \Lambda_o^t$ and from condition (3.68), for $i = 1$ (i.e., $M_1 = \Lambda_1^t \Lambda_1$).

It is standard to define an undirected graph, $\mathcal{G}_\mathrm{A}$, from a symmetric nonnegative integer matrix $\mathrm{A} \in \mathrm{Mat}_n(\mathbb{N})$ by joining two vertices, labelled i and j, by exactly $(\mathrm{A})_{ij}$ edges and attaching $(\mathrm{A})_{jj}$ loops to every vertex j. Conversely, to any undirected graph $\mathcal{G}$, there corresponds a unique symmetric matrix A, the <u>incidence matrix</u>, such that $\mathcal{G} = \mathcal{G}_\mathrm{A}$, and, moreover, for an arbitrary n by m-matrix Λ, $\mathrm{A} := \begin{pmatrix} 0 & \Lambda^t \\ \Lambda & 0 \end{pmatrix}$ defines the respective bicolored, undirected graph. For convenience, we will often use this (equivalent) graph theoretical language throughout the following statements and, later on, in Section 3.4.

The first result only assumes local constraints on M_1, yielding a finite list of possibilities all of which imply (3.65).

Lemma 3.3.5 *Let Λ be any n by m nonnegative integer matrix, and let the n by n symmetric, nonnegative integer matrix M be defined by*

$$M = \Lambda^t \Lambda .$$

Suppose that ϕ_ρ is a unit vector in $\mathbb{N}^n$ such that the vicinity of the vertex, v, corresponding to ϕ_ρ in $\mathcal{G}_M$, i.e., the number of its neighbors, the number of edges joining v with each of its neighbors, and the number of loops at v, is given by on of the following subgraphs,

$$\begin{tikzcd}
& \Phi_1 \arrow[dl, "\Lambda_0^t"'] \arrow[dr, "\Lambda_1"] \arrow[dd, "M_1"] & \\
1 \in \Phi_0 \arrow[rr, "T_0"] \arrow[dr, "\Lambda_0"'] & & \Phi_2 \ni \phi_\alpha \arrow[dl, "\Lambda_1^t"] \\
& \phi_\rho \in \Phi_1 &
\end{tikzcd}$$

Figure 3.1

then there exists a unit vector $\phi_\alpha \in \mathbb{N}^m$ such that

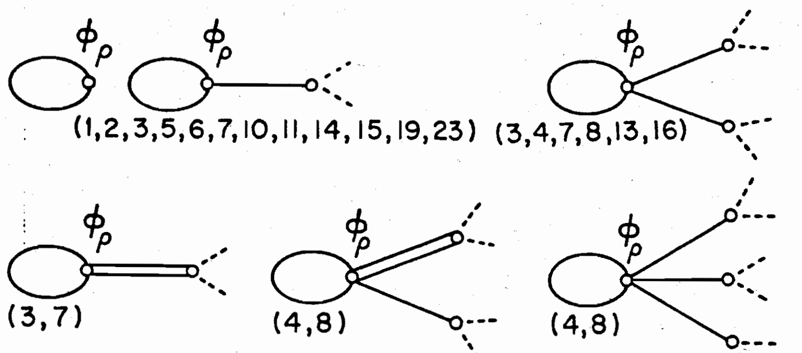
$$\Lambda^t \phi_\alpha = \phi_\rho .$$

Our second result characterizes a class of matrices, Λ, by global constraints with the property that, for two matrices Λ and $\tilde{\Lambda}$ in this class,

$$M := \tilde{\Lambda}^t \tilde{\Lambda} = \Lambda^t \Lambda$$

implies that $\tilde{\Lambda}$ and Λ are equivalent, i.e., the exists a bijection T, with $\Lambda = T\tilde{\Lambda}$.

In the application to fusion rule algebras, we will encounter the case where both matrices, $\tilde{\Lambda} = \Lambda_1$ and $\Lambda = \Lambda_o^t$ defined in (3.64), belong to this class, so that (3.68) implies the existence of $T : \Phi_o \to \Phi_2$, with $T\Lambda_o^t = \Lambda_1$, hence $\Lambda_o = \Lambda_1^t T$. Thus $\phi_\alpha := T1$ is a solution of (3.65), and we can choose $T = T_o$. The situation is summarized in the following commutative diagram:



Lemma 3.3.6

i) Suppose a bicolorable, connected graph $\mathcal{G}$ with incidence matrix Λ has no cycles length two (multiple edge), four or six. Then a component $\mathcal{A}$ of $\mathcal{G}^2$, with inciden matrix $\Lambda^t\Lambda$, has the following properties:

- *a) Except for loops, $\mathcal{A}$ contains only simple edges.*
- *b) If two complete subgraphs of $\mathcal{A}$ have a common edge, they are contained a common, complete subgraph of $\mathcal{A}$.*
- *c) If $\dot{U}(v) \subset \mathcal{A}$ is the subgraph of $\mathcal{A}$ consisting of all next neighbors $v \in \mathcal{A}$, v self excluded, then the number, L_v, of loops at v exceeds the number, E_v, connected components of $\dot{U}(v)$. We put $P_v = L_v - E_v$.*

ii) If $\mathcal{A}$ is a graph with properties a) and b) then $\mathcal{A}$ can be uniquely written as a unio $\mathcal{A} = \bigcup_i Q_i$ of maximal, complete subgraphs Q_i of $\mathcal{A}$ such that every edge of $\mathcal{A}$ contained in exactly one Q_i. Moreover, any two Q_i's can intersect in at most o vertex, and among three distinct Q_i's at least two are disjoint.

iii) For a graph $\mathcal{A}$ satisfying a)-c) we define a bicolored graph $\mathcal{G}_{\mathcal{A}}$ as follows: The vertic of one coloration are identified with the vertices of $\mathcal{A}$. The vertices, p_i, with ea degree greater than one and coloration opposite to those in $\mathcal{A}$ are identified w the Q_i's and joined by simple edges, (p_i, v), to the vertices $v \in Q_i \subset \mathcal{A}$. Additio vertices, p_j^v, of opposite coloration and edge degree one are joined to each $v \in \mathcal{A}$

simple edges, (p_j^v, v), $j = 1, \ldots, P_v$. *It follows that* $\mathcal{G}_{\mathcal{A}}$ *has no cycles of lengths two, four or six and that* $\mathcal{A}$ *is a component of* $\mathcal{G}_{\mathcal{A}}^2$.

iv) $\mathcal{G}_{\mathcal{A}}$ *is unique, i.e., if* $\mathcal{G}$ *is a graph without cycles of lengths two, four or six and* $\mathcal{A}$ *is a component of* $\mathcal{G}$, *then* $\mathcal{G}_{\mathcal{A}} \cong \mathcal{G}$.

For proof of these facts we refer to [42]. From iv) we infer the following Corollary:

Corollary 3.3.7 *If for two bicolored graphs,* $\mathcal{G}$ *and* $\mathcal{G}'$, *without cycles shorter than eight, the components of one coloration of* $\mathcal{G}^2$ *and* $\mathcal{G}'^2$ *are isomorphic then* $\mathcal{G}$ *and* $\mathcal{G}'$ *are isomorphic.*

Although the assumptions in Lemma 3.3.6 are global and very strong, it turns out to be the fitting criteria in the classification problem of Section 3.4., where we impose bounds on the norm of N_ρ, thus by (2.23) also on the norm of Λ_i. In addition we have a prescription of how to construct solutions from M which allows for any easy characterization of a few exponential cases.

3.4 Fusionrules with a Generator of Dimension not Greater than Two

The purpose of this section is to characterize the formal object (half-) algebras of the braided tensor categories to be classified in chapter 8. Not assuming any further structure, this means a classification of fusion rule algebras, in the sense introduced in Chapter 3. In fact we will find fusion rules that do not belong to any braided category. We restrict the classification to fusionrules which are generated by a single, irreducible object, whose Perron Frobenius dimension does not exceed two. Detailed proofs will be given in a separate paper,[42].

The first basic ingredient in the classification of fusion rule algebras is the classification of bicolorable graphs with norm not greater than two. The set of vertices of a bicolorable graph Γ can be divided into two subsets, W and B, such that no two vertices in W and no two vertices in B are joined by an edge. The graph is characterized by a matrix, $\Lambda : \mathbb{N}^W \to \mathbb{N}^B$, whose entries $\lambda_{ij} \in \mathbb{N}$ are the number of edges joining the vertex $i \in W$ with the vertex $j \in B$. The incidence matrix is then

$$N_\Gamma = \begin{pmatrix} 0 & \Lambda^t \\ \Lambda & 0 \end{pmatrix} \qquad (3.70)$$

and the norm of the graph is defined by

$$\|\Gamma\| = \|N_\Gamma\| . \qquad (3.71)$$

The proof of the following theorem can be found in, e.g., [45] and references therein. The graphs referred to here are depicted in Appendix A together with their norms and Perron Frobenius eigenvectors.

Theorem 3.4.1

i) The finite, connected, bicolorable graphs with norm less than two are the following :

$$A_l\,(l \geq 1),\ D_l\,(l \geq 4),\ E_l(l = 6, 7, 8) \qquad (3.72)$$

ii) The finite, connected, bicolorable graphs with norm equal to two are the following :

$$A_l^{(1)}\,(l\geq 2),\ D_l^{(1)}\,(l\geq 4),\ E_l^{(1)}(l=6,7,8). \tag{3.73}$$

Suppose that ρ is a selfconjugate, irreducible object with non-trivial grading in a $\mathbb{Z}_2$- graded fusion rule algebra. Then by equation (3.12) the fusion rule matrix, $\mathbf{N}_\rho$, has to be symmetric and if we use that the grading prescribes an off diagonal block form then we obtain the presentation (3.70) for $\mathbf{N}_\rho$, so that we can associate to it a bicolored graph Γ_ρ. If we assume, further, that the fusion rule algebra is generated by ρ then this graph is connected. Since the Perron Frobenius dimension of ρ is equal to the norm of Γ_ρ we can use Theorem 3.4.1 to establish an apriori list of possible fusion rule matrices labeled by the respective Coxeter graphs if we require d_ρ not to be greater than two. The next lemma is concerned with the question which of these matrices are actually fusion rule matrices of a fusion rule algebra.

Lemma 3.4.2 *Suppose $\Phi=\Phi_{[\rho]}$ is a $\mathbb{Z}_2$-graded fusion rule algebra, with selfconjugate generator, ρ, of dimension*

$$d_\rho < 2\,.$$

Then the fusion rule matrix, $\mathbf{N}_\rho$, of ρ is the incidence matrix of one of the bicolored graphs

$$A_n\,,n\geq 2\,,\quad D_{2n}\,,n\geq 2\,,\quad E_6 \text{ or } E_8\,. \tag{3.74}$$

Furthermore, there is exactly one fusion rule algebra for each of the graphs in (3.74) such that $\mathbf{N}_\rho$ is its incidence matrix. We will thus name these fusion rule algebras by their respective graphs. They have the following properties:

i) The A_n-algebra has trivial conjugation, $C=1$, and $\mathrm{Out}\,(A_n)=\{1,\alpha\}\cong\mathbb{Z}_2$, where α is even-graded, for odd n, and odd-graded, for even n.

If we denote the basis vectors by ρ_j, $j=0,\dots,n-1$, with $\rho_o:=1$, $\rho_1:=\rho$ and

---

$\rho\circ\rho_j=\rho_{j-1}+\rho_{j+1}$, then the structure constants of $\rho_i\circ\rho_j=\sum_k N_{ij,k}\rho_k$ are given

$$N_{ij,k}=\begin{cases}1 \quad \text{if} \quad |i-j|\leq k\leq \min(i+j,2(n-1)-(i+j))\\ \qquad\qquad \text{and} \quad k\equiv i+j \bmod 2\\ 0\,,\quad \text{else}.\end{cases} \qquad (3.7$$

For the statistical dimension we obtain

$$d_\rho = 2\cos\left(\frac{\pi}{n+1}\right) \;=\; (2)_q\,,$$
$$\text{with} \qquad q \;=\; e^{i\frac{\pi}{n+1}}\,.$$

ii) The D_{2n}-algebra has trivial conjugation, for odd n, and, for even n, the represent tions corresponding to the vertices of edge degree one at the short legs in the D_{2} graph are conjugate to each other, while all other representations are selfconjugat For $n>2$, the group of automorphisms of D_{2n} is trivial, and, for $n=2$, we hav that $\mathrm{Out}\,(D_4)\cong\mathbb{Z}_3$. The statistical dimension of the generator of D_{2n} is given by

$$d_\rho = 2\cos\left(\frac{\pi}{4n-2}\right)=(2)_q\,,\qquad \text{with}\quad q=e^{\frac{i\pi}{4n-2}}\,.$$

iii) The E_6- and the E_8-algebra have trivial conjugation; $\mathrm{Out}\,(E_6)=\mathrm{Out}\,((E_6)_o)\cong$ Z and $\mathrm{Out}\,(E_8)=1$. For E_6, the statistical dimension of the generator is given b $d_\rho=2\cos\left(\frac{\pi}{12}\right)=\frac{1}{\sqrt{2}}(\sqrt{3}+1)=(2)_q$ with $q=e^{\frac{i\pi}{12}}$, and, for E_8, we have that $d_\rho=$ $\cos\left(\frac{\pi}{30}\right)=\frac{1}{4}\left[\sqrt{3}(\sqrt{5}+1)+\sqrt{2}\sqrt{5-\sqrt{5}}\right]=(2)_q$, with $q=e^{\frac{i\pi}{30}}$.

From this result and Lemma 3.3.4 we immediately obtain the list of $\mathbb{Z}_2$-graded fusion rul algebras with non-selfconjugate generator and the list of ungraded fusion rule algebras.

Corollary 3.4.3

i) The $\mathbb{Z}_2$-graded fusion rule algebras with non-selfconjugate generator, $\rho\neq\bar{\rho}$, of st tistical dimension $d_\rho<2$ are given by

$$\tau_\alpha\,(A_{2n+1})\,,\quad n\geq 2\,,\qquad \text{and}\quad \tau_\alpha\,(E_6)\,, \qquad (3.76$$

where α is the non-trivial, evenly graded automorphism of A_{2n+1}, E_6, resp. The evenly graded representations thus remain selfconjugate, and the conjugation, restricted to the oddly graded representations, corresponds to the reflection of the Dynkin-diagram.

ii) An ungraded fusion rule algebra with generator, ρ, of statistical dimension $d_\rho < 2$ is given by the fusion rule subalgebra of A_{2n}, for some $n \geq 2$, consisting of the evenly graded representations, so that, in the notation of Lemma 3.4.2, i), the generator is given by $\rho = \rho_{2n-2}$. In particular, we have that $\rho = \bar{\rho}$, and the conjugation is trivial for all of these fusion rule algebras. The fusion rule matrix, $\mathbb{N}_\rho$, of the generator is the incidence matrix of the graph, $\overline{A}_n$. Thus, denoting the fusion rule algebra by this graph, we have that

$$\overline{A}_n \subset A_{2n} . \tag{3.77}$$

The statistical dimension of ρ is given by $d_\rho = 2\cos\left(\frac{\pi}{2n+1}\right) = (2)_q$, with $q = e^{\frac{i\pi}{2n+1}}$. (This also includes the trivial fusion rule algebra $\overline{A}_1 = \{1\}$, which is obtained from $A_2 \cong \mathbb{Z}_2$).

The complete list of $\mathbb{Z}_2$- or ungraded fusion rule algebras with generator, ρ, of statistical dimension $d_\rho < 2$ is thus given by

$$A_n, D_{2n}, E_6, E_8, \overline{A}_n, \tau_\alpha(A_{2n+1}), \tau_\alpha(E_6) . \tag{3.78}$$

A comparison of (3.75) with (6.43) and [7, 8] shows that the fusion rule algebra A_n is realized as the tensor product decomposition rule of $U_q(s\ell_2)$, $q = e^{\frac{i\pi}{n+1}}$, and in the formal operator product expansion of $\widehat{su}(2)_{n-1}$-symmetric WZNW-conformal models.

An independent way of realizing the structure constants of (3.74) as those of a ring over $\mathbb{Z}$ is given as follows:

Consider the sequence of Chebychev polynomials, $\mathcal{P}_k(X) \in \mathbb{Z}[X]$, defined by

$$\begin{aligned} & \mathcal{P}_0(X) && = 1 , \\ & \mathcal{P}_1(X) && = X \\ \textit{and} \quad & X\mathcal{P}_k(X) && = \mathcal{P}_{k-1}(X) + \mathcal{P}_{k+1}(X) . \end{aligned} \tag{3.79}$$

Let $\check{\mathbf{C}}_n$ be the ring of dimension n over $\mathbb{Z}$, given by

$$\check{\mathbf{C}}_n := \mathbb{Z}[X] / \mathcal{P}_n(X) \cdot \mathbb{Z}[X] . \tag{3.80}$$

Then the images of the Chebychev-polynomial in this quotient, $\overline{\mathcal{P}}_k := [\mathcal{P}_k(X)] \in \check{\mathbf{C}}_n$, $k = 0, 1, \ldots, n-1$, for a $\mathbb{Z}$-basis of $\check{\mathbf{C}}_n$, and the multiplication in $\check{\mathbf{C}}_n$ is given by

$$\overline{\mathcal{P}}_i \cdot \overline{\mathcal{P}}_j = \sum_k N_{ij,k} \overline{\mathcal{P}}_k , \tag{3.81}$$

where the $N_{ij,k}$ are precisely the structure constants (3.75) of an A_n-fusion rule algebra.

In order to provide means by which also the D- and E-algebras can be computed, we discuss fusion rule algebra homomorphisms between different algebras, as well as fusion rule algebra automorphisms.

Lemma 3.4.4

i) For the $\mathbb{Z}_2$- or ungraded fusion rule algebras with a generator, ρ, of statistical dimension $d_\rho < 2$, all fusion rule algebra automorphisms are involutive, and there is at most one non-trivial automorphism for every fusion rule algebra. If the fusion rule algebra has a non-trivial conjugation then the automorphism coincides with the conjugation. For the fusion rule algebras with trivial conjugation the automorphisms are given as follows:

a) A_{2n+1}, E_6: The involution γ_n, γ_E, resp., is identical to the conjugation of $\tau_\alpha(A_{2n+1})$, $\tau_\alpha(E_6)$.

b) D_{4n+2}: The involution γ'_{2n+1} exchanges the representations that correspond to the vertices of edge degree one at the short legs of the graph D_{4n+2}.

ii) The non-trivial fusion rule algebra homomorphisms from one of the fusion rule algebras in (3.78) into the algebras A_n or $\overline{A}_n$ are given by

$$A_{2n+1} \quad \xleftrightarrow{\gamma_n} \quad A_{2n+1} , \qquad n \geq 1 , \tag{3.82}$$

$$\mathbb{Z}_2 \cong A_2 \quad \overset{\longleftrightarrow}{\cong} \quad \mathrm{Out}(A_n) \hookrightarrow A_n , \quad n \geq 2 , \tag{3.83}$$

and

$$\overline{A}_n \overset{i}{\hookrightarrow} A_{2n} \overset{\overline{\sigma}_n}{\twoheadrightarrow} \overline{A}_n\,, \quad n \geq 1\,. \tag{3.84}$$

Here i is the inclusion (3.77), and the homomorphism $\overline{\sigma}_n$ is defined by its graph, depicted in (B1) of the Appendix. The composition $\overline{\sigma}_n \circ i$ in (3.83) is the identity on $\overline{A}_n$. Among the fusion rule algebras with generator of statistical dimension 2, $\overline{D}_3$ (to be defined below) is the only one for which there exists a homomorphism to an A-algebra:

$$\overline{D}_3 \hookrightarrow A_5\,. \tag{3.85}$$

The inclusion is defined by noticing that the subalgebra of 0-graded sectors in A_5 is isomorphic to $\overline{D}_3$.

iii) For every $n \geq 2$, there are exactly two fusion rule algebra homomorphisms σ^D_n, $\dot{\sigma}^D_n$, of one of the algebras listed in (3.78) into D_{2n}. They are defined on A_{4n-3} and on $\tau_\alpha(A_{4n-3})$, respectively, and given by the graphs in (B2) of the Appendix. They are related to the automorphisms by the following commutative diagram:

$$\begin{tikzcd}
 & & D_{2n} \arrow[d, leftrightarrow, "\gamma'_n"] \\
A_{4n-3} \arrow[urr, twoheadrightarrow, "\sigma^D_n"] \arrow[rr, twoheadrightarrow, "\sigma^D_n"] \arrow[d, leftrightarrow, "\gamma_{2n-2}"'] & & D_{2n} \\
A_{4n-3} \arrow[urr, "\sigma^D_n"'] & &
\end{tikzcd} \tag{3.86}$$

The map (3.84) can thus be extended to D_4, the image of $\overline{D}_3$ in D_4 being the evenly graded subalgebra isomorphic to $\mathbf{Z}_3$.

iv) The only homomorphisms of one of the fusion rule algebras in (3.78) into E_6 are defined on $A_3 \supset A_2$ and A_{11}. The only possible one on A_3 maps the generator ρ of statistical dimension $d_\rho = (2)_{q_1} = \sqrt{2}$, $q_1 = e^{\frac{i\pi}{4}}$, to the representation corresponding to the endpoint of the shortest leg in the E_6-diagram, with dimension $d_\rho = (4)_{q_2} - (2)_{q_2}$ $q_2 = e^{\frac{i\pi}{12}}$, and the non-trivial automorphism of A_3 to the non-trivial automorphis in E_6, thus providing an inclusion of A_3 into E_6 as a fusion rule subalgebra. Th two possible homomorphisms of A_{11} to E_6 differ from each other by multiplicatio of the automorphism on E_6, described in part i), and one, σ^{E_6}, is given by the grap depicted in (B3) of the Appendix. The following diagram commutes:

$$\begin{tikzcd}
 & & E_6 \arrow[d, leftrightarrow, "\gamma^E"] & A_{11} \arrow[l, twoheadrightarrow, "\sigma^{E_6}"'] \arrow[d, leftrightarrow, "\gamma_{11}"] \\
\mathbb{Z}_2 \cong A_2 \arrow[r, hookrightarrow] & A_3 \arrow[ur, hookrightarrow] \arrow[dr, hookrightarrow] & & \\
 & & E_6 & A_{11} \arrow[l, twoheadrightarrow, "\sigma^{E_6}"']
\end{tikzcd} \tag{3.87$$

Analogous statements hold for the homomorphisms

$$A_2 \hookrightarrow A_3 \hookrightarrow \tau_\alpha(E_6) \overset{\dot{\sigma}^{E_6}}{\twoheadleftarrow} \tau_\alpha(A_{11})\,. \tag{3.8$$

v) On each of the fusion rule algebras $\overline{A}_2$, A_4, D_{16} and A_{29} there exists exactly o homomorphism into E_8, and there is none for all other fusion rule algebras list in (3.78). The homomorphism of $\overline{A}_2$ to E_8 maps the representation of statistic dimension $d_\rho = (2)_{q_1} = \frac{1}{2}(1+\sqrt{5})$, $q_1 = e^{\frac{i\pi}{5}}$, to the representation corresponding the endpoint of the leg of length two in the E_8-diagram, with statistical dimensi $d = (7)_{q_2} - (5)_{q_2} = \frac{1}{2}(1+\sqrt{5})$, $q_2 = e^{\frac{i\pi}{30}}$, and it therefore provides an inclusion, i, $\overline{A}_2$ into E_8 as a fusion rule subalgebra. The homomorphism of A_4 to E_8 is th given by the composition $i \circ \overline{\sigma}_2$, $\overline{\sigma}_n$ being defined in (3.84) and (B1). In (B4) t homomorphism, σ^{DE}, of D_{16} to E_8 is given by its graph. The homomorphism of A to E_8 is the composition $\sigma^{DE} \circ \sigma^D_8$, where σ^D_n is defined in (3.86) and (B2).

With the help of the homomorphisms described in Lemma 3.4.4, it is possible to rederi the explicit fusion rules, e.g., in the form of the structure constants (4.6), of the and E-fusion rule algebras from those of the A-algebra; see (3.75). Except for the triv ones, $A_2 \twoheadrightarrow 1 \hookrightarrow \Phi$ and $\Phi \xleftrightarrow{id} \Phi$, Lemma 3.4.4 describes the entire set of homomorphis among the fusion rule algebras in (3.78). The situation described in Lemma 3.4.4v) can be summarized in the following commutative diagram :

$$\begin{tikzcd}
 & A_{29} \arrow[r, leftrightarrow, "\gamma_{14}"] \arrow[dl, twoheadrightarrow, "\sigma^D_8"'] \arrow[d, twoheadrightarrow, "\sigma^D_8"] & & A_{29} \arrow[dll, twoheadrightarrow, "\sigma^D_8"'] \arrow[ddl, twoheadrightarrow, "\sigma^{DE}\circ\sigma^D_8"] & & \\
D_8 \arrow[r, leftrightarrow, "\gamma'_4"] \arrow[drr, twoheadrightarrow, "\sigma^{DE}"'] & D_8 \arrow[dr, twoheadrightarrow, "\sigma^{DE}"] & & & \bar{A}_2 \arrow[dll, hook, "i"'] & \\
 & & E_8 \arrow[rrr, leftrightarrow, "i\circ\bar{\sigma}_2"'] & & & A_4 \arrow[ul, twoheadrightarrow, "\bar{\sigma}_2"']
\end{tikzcd} \tag{3.89}$$

Next we present the complete list of fusion rule algebras with generators of statistical dimension equal to two. Our presentation is organized in a way similar to the one above, for $d_\rho < 2$, except that the detailed discussion of homomorphisms is replaced by a study of the realizations of these fusion rule algebras by discrete subgroups of $SU(2)$.

Lemma 3.4.5 *Suppose $\Phi = \Phi_{[\rho]}$ is a $\mathbb{Z}_2$-graded fusion rule algebra, with selfconjugate generator, ρ, of dimension*

$$d_\rho = 2 .$$

Then the fusion rule matrix, $\mathbb{N}_\rho$, of ρ is the incidence matrix of one of the following bicolored graphs

$$A_\infty,\ D_\infty,\ D^{(1)}_{\rho+2},\ \rho \geq 2,\ E^{(1)}_6,\ E^{(1)}_7,\ E^{(1)}_8 . \tag{3.90}$$

There exists one fusion rule algebra for each of the graphs in (3.90), such that $\mathbb{N}_\rho$ is its incidence matrix, except for $D^{(1)}_{\rho+2}$, where we have exactly two inequivalent such algebras for each $\rho \geq 2$.

They have the following properties:

i) The A_∞-algebra has trivial conjugation, $C = 1$, and only trivial automorphisms, $\mathrm{Out}\,(A_\infty) = 1$. *We enumerate its basis by ρ_j, $j = 0, 1, \ldots$, such that $\rho_o := 1$, $\rho_1 := \rho$ and $\rho \circ \rho_j = \rho_{j-1} + \rho_{j+1}$. Moreover, $d_{\rho_j} = j + 1$. The structure constants, $N_{ij,k}$, of $\rho_i \circ \rho_j = \sum_k N_{ij,k}\rho_k$ are given by the Clebsch-Gordan rule, i.e.*

$$N_{ij,k} = \begin{cases} 1 & \text{if } |i-j| \leq k \leq i+j \text{ and } k \equiv i+j \bmod 2 \\ 0, & \text{else.} \end{cases} \tag{3.91}$$

The D_∞-algebra has trivial conjugation, $C = 1$, and $\mathrm{Out}\,(D_\infty) = \mathrm{stab}(\rho) = \{1, \alpha\} \cong \mathbb{Z}_2$. *If we set $\omega_o := 1 + \alpha$, $\omega_1 := \rho$ and define basis vectors ω_j, $j \geq 2$, by $\omega_1 \circ \omega_j = \omega_{j-1} + \omega_{j+1}$ then $d_{\omega_j} = 2$, for all j, and*

$$\omega_j \circ \omega_k = \omega_{|j-k|} + \omega_{j+k} . \tag{3.92}$$

ii) The automorphism group of $D^{(1)}_{p+2}$-algebra has order 4, i.e., $\mathrm{Out}\left(D^{(1)}_{p+2}\right) = \{1, \alpha, x, y\}$, *with* $\mathrm{stab}(\rho) = \{1, \alpha\} \cong \mathbb{Z}_2$, *for $p > 2$, and $\alpha \circ x = y$. The two possible fusion rule algebras associated to $D^{(1)}_{p+2}$ are distinguished by their automorphisms, for which we have either* $\mathrm{Out}\left(D^{(1)}_{p+2}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, *with $x^2 = y^2 = 1$ and $C = 1$; or* $\mathrm{Out}\left(D^{(1)}_{p+2}\right) \cong \mathbb{Z}_4$, *so that $xy = 1$, the conjugation is the inversion on* $\mathrm{Out}\left(D^{(1)}_{p+2}\right)$ *and all nonautomorphic representations are selfconjugate. Defining the basis vectors, ω_j, $j = 1, \ldots \ldots, p-1$, as in the case of D_∞, and with $\omega_o := 1 + \alpha$, $\omega_p = x + y$, we have $d_{\omega_j} = 2$ and*

$$\omega_j \circ \omega_k = \omega_{|j-k|} + \omega_{\min(2p-(j+k), j+k)} . \tag{3.93}$$

The automorphisms x and y are evenly graded for even p, and odd-graded for odd p. Thus, for odd p, we have that $\mathrm{Out}\,(\tau_\alpha(\Phi)) \cong \mathbb{Z}_4$, *for* $\mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, *whereas* $\mathrm{Out}\,(\tau_\alpha(\Phi)) \cong \mathrm{Out}(\Phi)$, *for even p.*

iii) For the $E^{(1)}_6$-algebra we have that $\mathrm{Out}\left(E^{(1)}_6\right) = \{1, \alpha, \alpha^{-1}\} \cong \mathbb{Z}_3$, *and there are three representations of dimension two, namely ρ, $\alpha \circ \rho$ and $\alpha^{-1} \circ \rho$, so that the conjugation, given by setting $\bar{\alpha} = \alpha^{-1}$ and $\bar{\rho} = \rho$, exchanges the two legs in the $E^{(1)}_6$ diagram opposite to 1. For the one remaining representation, ψ, of dimension three we have*

$$\psi \circ \psi = 1 + \alpha + \alpha^{-1} + 2\psi . \tag{3.94}$$

Furthermore, a cyclic permutation of the set $\{\rho, \alpha \circ \rho, \alpha^{-1} \circ \rho\}$ provides the isomorphism $\tau_\alpha\left(E_6^{(1)}\right) \cong E_6^{(1)}$. For the $E_7^{(1)}$-algebra, we see that $\mathrm{Out}\left(E_7^{(1)}\right) = \{1, \alpha\} \cong \mathbb{Z}_2$, *where α is evenly graded and N_α is the reflection of the diagram. Moreover, the conjugation on $E_7^{(1)}$ is trivial, and all representations have integer dimension.*

Finally, $E_8^{(1)}$ has trivial conjugation, $\mathrm{Out}\left(E_8^{(1)}\right) = 1$, *and all representations have integer dimension.*

The fusion rule algebras with non-selfconjugate generator, as well as the ungraded fusion rule algebras, are obtained in a similar way as in Corollary 3.4.3.

Corollary 3.4.6

i) The $\mathbb{Z}_2$-graded fusion rule algebras with non-selfconjugate generator, $\rho \neq \bar{\rho}$, of dimension two are given by

$$E_6^{(1)} \quad \tau_\alpha\left(E_7^{(1)}\right) \quad \text{and} \quad \tau_x\left(D_{2p'+2}^{(1)}\right), \quad p' \geq 2\,. \tag{3.95}$$

In the case of $E_6^{(1)}$, the generator ρ is replaced by the representation $\alpha \circ \rho$ (or by $\alpha^{-1} \circ \rho$) which is a generator of $E_6^{(1)}$ with dimension two as well. For $\tau_\alpha\left(E_7^{(1)}\right)$, the conjugation is trivial on the evenly graded representations and reflects the oddly graded ones. In (3.95), both possibilities for $D_{p+2}^{(1)}$ are meant to be included, and we have that $\tau_x\left(D_{2p'+2}^{(1)}\right) \cong \tau_y\left(D_{2p'+2}^{(1)}\right)$.

ii) The ungraded fusion rule algebras with generators of dimension two are given by the evenly graded subalgebras of $\left(D_{2p'+3}^{(1)}\right)$, $p' \geq 1$, so that the generator, ρ', is selfconjugate and given by $\rho' = \rho \circ x = \rho \circ y$. The fusion rule matrix $\mathrm{N}_{\rho'}$ is the incidence matrix of the graph $\overline{D}_{p'+2}$, see (A22). Thus, denoting the fusion rule algebra by this graph, we have that

$$\overline{D}_{p'+2} \subset D_{2p'+3}^{(1)}\,; \tag{3.96}$$

$\overline{D}_{p'+2}$ has trivial conjugation, and $\mathrm{Out}\left(\overline{D}_{p'+2}\right) = \mathrm{stab}(\rho') = \{1, \alpha\} \cong \mathbb{Z}_2$.

The complete list of $\mathbb{Z}_2$- or ungraded fusion rule algebras with a generator of dimension two is given by

$$\begin{gathered} A_\infty\,,\ D_\infty\,,\ D_{p+2}^{(1)}\ (\mathrm{Out} \cong \mathbb{Z}_4 \quad \text{or} \quad \mathbb{Z}_2 \times \mathbb{Z}_2)\,,\ p \geq 2\,,\ \tau_x\left(D_{2p'+2}^{(1)}\right) \\ \overline{D}_{p'+2}\,,\ p' \geq 1\,,\ E_6^{(1)},\ E_7^{(1)},\ \tau_\alpha\left(E_7^{(1)}\right)\,,\ E_8^{(1)}\,. \end{gathered} \tag{3.97}$$

In order to study homomorphisms between those algebras in (3.97) which have a selfconjugate generator, it is useful to find their fusion rule algebra monomorphisms, i.e., inclusions of one of the algebras in (3.97) into another one, and fusion rule algebra endomorphisms which map the generator to an irreducible object . The latter requirement will also be present in our description of general homomorphisms and, further, the object to which the generator is mapped has to have dimension two. One consequence of the following lemma is that object s of dimension two which generate the entire fusion rule algebra can be mapped to the canonical generator by a fusion rule algebra automorphism.

Lemma 3.4.7

i) The fusion rule algebra A_∞ contains no fusion rule subalgebras from (3.97) othe than A_∞, and the only fusion rule algebra endomorphism is the identity.

ii) The endomorphisms of the D_∞-algebra are given by the inclusions $I_k : D_\infty \hookrightarrow D_\infty$ $k = 1, 2, \ldots$, determined by $I_k(\alpha) := \alpha$ and $I_k(\omega_j) := \omega_{k \cdot j}$, in the basis of (3.92) i Lemma 3.4.5 i). All subalgebras of D_∞ from (3.97) are isomorphic to D_∞ and ar given by $[\omega_k] = \mathrm{im}(I_k)$, $k = 1, 2, \ldots$. We have that $I_k \circ I_\ell = I_{k \cdot \ell}$.

iii) There are no fusion rule subalgebras of $E_6^{(1)}$ from (3.97), except $E_6^{(1)}$ itself, an the only non-trivial endomorphism γ^τ, for which the generator is mapped to a irreducible object is identical with the conjugation.

iv) The only fusion rule algebras from (3.97) that can be included into $E_7^{(1)}$ in a nor trivial way are $\overline{D}_3$ and $E_7^{(1)}$ itself. The subalgebra $\overline{D}_3$ is generated by the on evenly graded object of dimension two in $E_7^{(1)}$ and contains, besides the unit an the generator, only the non-trivial automorphism of $E_7^{(1)}$. The fusion rule algeb generated by the second oddly graded object of dimension two is isomorphic to $E_7^{(1}$

The inclusion of $E_7^{(1)}$ into itself is given by the fusion rule algebra automorphism, γ^O, which exchanges the two oddly graded object s of statistical dimension 2 and is the identity on all other object s. The only further $E_7^{(1)}$-endomorphism, $\overline{\sigma}_3^{O1}$, can be described by the unique homomorphism $E_7^{(1)} \twoheadrightarrow D_3^{(1)}$, which maps the generator to the generator, (see below); $\overline{\sigma}_3^{O1}$ is then obtained by composing this homomorphism with the inclusion. Thus we have the following commutative diagram:

$$\begin{tikzcd}
\gamma^0 \arrow[r, hook] & E_7^{(1)} \arrow[rr, "\bar{\sigma}^{01}"] \arrow[drr, two heads] & & E_7^{(1)} & \gamma^0 \arrow[l, hook'] \\
& \bar{D}_3 \arrow[u, hook] \arrow[rr, "\mathrm{id}"] & & \bar{D}_3 \arrow[u, hook] &
\end{tikzcd} \tag{3.98}$$

The endomorphism $\overline{\sigma}_3^{O1}$ is an idempotent on whose image σ^O acts trivially.

v) The only fusion rule algebra from (3.97) which is contained in $E_8^{(1)}$ is $E_8^{(1)}$ itself. The only non-trivial endomorphism is the involutive automorphism, γ^J, which exchanges the two object s with dimension 2 and the two object s with dimension 3 and is the identity on all other object s.

vi) The fusion rule subalgebras from (3.97) of $D_{p+2}^{(1)}$ are given by

$$[\omega_q] \;\cong\; D_{p'+2}^{(1)} \tag{3.99}$$

$$\text{if} \qquad p \;=\; p'(q, 2p)\,, \tag{3.100}$$

and

$$[\omega_q] \;\cong\; \overline{D}_{t+2} \tag{3.101}$$

$$\text{if} \qquad 2p \;=\; (2t+1)(q, 2p)\,, \tag{3.102}$$

where $q = 1, \ldots, p-1$.

Here the structure of the group of automorphisms in $D_{p'+2}^{(1)}$ from (3.99) ($\mathrm{Out}\left(D_{p'+2}^{(1)}\right)$ *either $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$) is the same as the one assumed for $D_{p+2}^{(1)}$. The cases (3.100) and (3.102) are distinguished according to whether $\frac{2p}{(q,2p)}$ is even or odd. The subalgebras of $\overline{D}_{t+2}$ are given by*

$$[\omega_q] \;\cong\; \overline{D}_{t'+2}\,, \tag{3.103}$$

$$\text{where} \quad (2t'+1)(q, 2t+1) \;=\; 2t+1 \tag{3.104}$$

There is exactly one involutive automorphism, $\sigma_{xy}^D \neq 1$, on every $D_{p+2}^{(1)}$ mapping the generator to itself and given by $\sigma_{xy}^D(x) = y$ and $\sigma_{xy}^D(\omega_j) = \omega_j$, and there is none for every $\overline{D}_{t+2}$. For every two-dimensional object , ω_j, in $D_{p+2}^{(1)}$ and $\overline{D}_{t+2}$, there exists precisely one endomorphism for the cases (3.101) and (3.103), and there are two endomorphisms for the case (3.99), differing from each other by σ_{xy}^D, which map the generator ω_1 to ω_j. This exhausts the entire set of endomorphisms.

If a homomorphism, σ, defined on one of the algebras, Φ, from (3.97) does not map the generator ρ to an irreducible object it follows from a comparison of statistical dimensions from (3.55) that $\sigma(\rho)$ is the sum of two automorphisms. Since automorphisms close under multiplication, and since $\sigma(\rho)$ is a generator of the image of σ, it follows that

$$G_\sigma := \mathrm{Out}(\sigma(\Phi)) = \mathrm{supp}(\sigma(\Phi))\,,$$

i.e., σ is a homomorphism $\sigma : \Phi \rightarrow \mathbb{N}[G]$. For all fusion rule algebras with only integer dimensions, in particular, for those listed in (3.97), one homomorphism with these properties is given by $\sigma : \Phi \rightarrow \mathbb{N}_1$, $\phi \rightarrow d_\phi$, (i.e., $G = \{1\}$), and, furthermore, if $\sigma(\Phi)$ is a subalgebra of one of those corresponding to (3.97) we have that $|G| \leq 4$. In the context of *group-duality*, homomorphisms to fusion rule algebras consisting entirely of automorphisms correspond to the *abelian subgroups* of that compact group, whose representation theory reproduces the fusion rules given by Φ. Here, however, we wish to focus our attention on non-abelian subgroups, i.e., we restrict our attention to cases, where $\sigma(\rho)$ is *irreducible* and hence has the same dimension as ρ. For a homomorphism $\sigma : \Phi_1 \rightarrow \Phi_2$ with this property, between fusion rule algebras corresponding to (3.97), $\sigma(\Phi_1)$ is a fusion rule subalgebra of Φ_2 generated by an endomorphism of dimension two. It is therefore isomorphic to some Φ' in (3.97). Thus the homomorphism σ is described by a surjective homomorphism $\sigma' : \Phi_1 \twoheadrightarrow \Phi'$, with $\Phi' = [\sigma'(\rho)]$, and one of the inclusions of fusion rule subalgebras, $i : \Phi' \hookrightarrow \Phi_2$, given in Lemma 3.4.7. Hence $\sigma = i \circ \sigma'$. For a complete discussion of fusion rule algebra homomorphisms it therefore suffices to consider surjective ones, $\sigma : \Phi \twoheadrightarrow [\sigma(\rho)]$.

In the classification of Lemma 3.4.5 we have always fixed a distinct generator, ρ, of statistical dimension two. So we are, in fact, considering pairs (ρ, Φ), where ρ is the *canonical generator*, with $[\rho] = \Phi$. From Lemma 3.4.5 and Corollary 3.4.3 we see that non-isomorphic fusion rule matrices of the selfconjugate generators also lead to non-isomorphic fusion rule algebras (which is seen, e.g., by comparing the number of objects for each dimension). Hence $[\rho'] \cong [\rho]$ implies that there exists a bijection T, $T^t = T^{-1}$, with $T1 = 1$, $T\rho = \rho'$, and $T\mathrm{N}_\rho T^t = \mathrm{N}'_{\rho'}$. By the remark in Section 3.1 following (3.15), the matrices $\mathrm{N}'_{j'} = T\mathrm{N}_j T^t$ define a fusion rule algebra, with conjugation $C' = TCT^t$ and lattice $[\rho']$, which is isomorphic to $[\rho]$, and for which $\mathrm{N}_{\rho'} = \mathrm{N}'_{\rho'}$. Lemma 3.4.5 shows, furthermore, that a given $\mathrm{N}_{\rho'}$ uniquely determines the composition rules, once the group of automorphisms is known. (This is, in fact, only needed in the case of $D^{(1)}_{p+2}$). In particular, this can be used in the case $[\rho'] \cong [\rho]$ to conclude that T extends to a fusion rule algebra isomorphism mapping ρ to ρ'.

In summary, we have that if

$$\rho = \bar{\rho}\,, \quad \rho' = \bar{\rho}'\,, \quad d_\rho = d_{\rho'} \leq 2 \qquad \text{and} \qquad [\rho] \cong [\rho']$$

then

$$(\rho, [\rho]) \cong (\rho', [\rho']) \tag{3.105}$$

holds. A consequence of (3.105) is that, for two selfconjugate generators ρ, ρ', with $d_\rho = d_{\rho'} \leq 2$, of the *same* fusion rule algebra $\Phi = [\rho] = [\rho']$, there exists a fusion rule algebra *automorphism* γ,

$$\gamma : \Phi \to \Phi\,, \qquad \text{with} \qquad \gamma(\rho) = \rho'\,. \tag{3.106}$$

This can also be verified directly from Lemma 3.4.7, where all automorphisms satisfying (3.106) are listed.

For a surjective homomorphism $\sigma : \Phi_1 \to \Phi_2$, between two fusion rule algebras, Φ_1, Φ_2, this means that there always exists an automorphism on Φ_2 mapping $\sigma(\rho_1)$ to ρ_2, so that $\dot{\sigma} := \gamma \circ \sigma$ is a homomorphism $\dot{\sigma} : (\rho_1, \Phi_1) \to (\rho_2, \Phi_2)$, with $\dot{\sigma}(\rho_1) = \rho_2$. It follows that *all* homomorphisms can be obtained from those which map canonical generator t canonical generator, by composing them with an appropriate automorphism, followed by an inclusion. The classification of homomorphisms, $\dot{\sigma}$, with $\dot{\sigma}(\rho_1) = \rho_2$, is given in the next lemma.

Lemma 3.4.8 *All fusion rule algebra homomorphisms between the algebras with selfconjugate generator of statistical dimension two (as listed in Lemma 3.4.5 and Corollary 3.4.6, ii)) which map canonical generators to canonical generators are given by th following ones:*

i) For every algebra Φ among the ones specified above, there is a unique homomorphis $\dot{\sigma}^* : A_\infty \twoheadrightarrow \Phi$, with the required properties. For every $\rho \geq 2$ and $t \geq 1$, there exis unique homomorphisms from D_∞ to $\overline{D}_{t+2}$ and to $D^{(1)}_{\rho+2}$.

ii) There exists exactly one homomorphism between the fusion rule algebras

a) $\overline{D}_{t'+2} \twoheadrightarrow \overline{D}_{t+2}$, iff $t' = t + s + 2ts$, for some $s \geq 1$;

b) $D^{(1)}_{p+2} \twoheadrightarrow \overline{D}_{t+2}$, iff $p = m(2t+1)$, for some $m \geq 1$ and $\mathrm{Out}\left(D^{(1)}_{p+2}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$

c) $D^{(1)}_{p'+2} \twoheadrightarrow D^{(1)}_{p+2}$, iff $p = cp'$, and

$$\textit{either } c \textit{ is } \text{even}, \textit{ and } \mathrm{Out}\left(D^{(1)}_{p'+2}\right) \;\cong\; \mathbb{Z}_2 \times \mathbb{Z}_2\,,$$

$$\textit{or } c \textit{ is } \text{odd}, \textit{ and } \mathrm{Out}\left(D^{(1)}_{p'+2}\right) \;\cong\; \mathrm{Out}\left(D^{(1)}_{p+2}\right)\,,$$

and only in the last case we have to account for a non-trivial fusion rule algebr automorphism which is the identity on the canonical generator.

iii) The only homomorphisms between the E-algebras are one, σ^{OT}, from $E^{(1)}_7$ to $E_6^{(1}$ and one, σ^{JT}, from $E^{(1)}_8$ to $E^{(1)}_6$. There are no homomorphisms from D-algebr to E-algebras, and the only homomorphisms from $E^{(1)}_6$ to a D-algebra are give by a unique homomorphism $\sigma^J_2 : E^{(1)}_6 \to D^{(1)}_4$, for each structure of $D^{(1)}_4$. Th *also yields the entire set of homomorphisms from E-algebras to* $D_4^{(1)}$, *by setting* $\sigma_2^O := \sigma_2^J \circ \sigma^{OT}$ *and* $\sigma_2^J := \sigma_2^{T} \circ \sigma^{JT}$*. There exists a homomorphism* $\sigma_4^O : E_7^{(1)} \to D_6^{(1)}$, *and a homomorphism* $\sigma_3^O : E_7^{(1)} \to D_5^{(1)}$, *for each of the two structures of the D-algebras. If we consider the case* $\mathrm{Out}\left(D_5^{(1)}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ *we obtain, by composing* σ_3^O *with the homomorphism from* $D_5^{(1)}$ *to* $\overline{D}_3$ *given in part ii)b)* $(p = 3, m = t = 1)$, *a homomorphism* $\overline{\sigma}_3^O : E_7^{(1)} \to \overline{D}_3$*. Furthermore, there exist unique homomorphisms* $\sigma_5^J : E_8^{(1)} \to D_7^{(1)}$ *and* $\sigma_3^J : E_8^{(1)} \to D_5^{(1)}$, *for any one of the possible structures of the D-algebras. Extensions,* $\sigma_5^J : E_8^{(1)} \to \overline{D}_4$ *and* $\sigma_3^J : E_8^{(1)} \to \overline{D}_3$, *are found from* σ_5^J *and* σ_3^J *with* $\mathrm{Out}\left(D_5^{(1)}\right) \cong \mathrm{Out}\left(D_7^{(1)}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$, *in the same way as for* $\overline{\sigma}_3^O$.

We give a survey of the fusion rule algebra homomorphisms involving the E-algebras in the commutative diagram on the next page. Here γ_6 is the automorphism of $D_6^{(1)}$ exchanging the two oddly graded objects of dimension two, (compare to (3.99), (3.100), with $q = 3$, $p = p' = 4$, and (3.105)). The unspecified arrow, $D_5^{(1)} \twoheadrightarrow \overline{D}_3$, $D_7^{(1)} \twoheadrightarrow \overline{D}_4$ and $D_6^{(1)} \twoheadrightarrow D_4^{(1)}$, are the homomorphisms given in Lemma 3.4.8 ii), and $D_5^{(1)} \to D_5^{(1)}$ is defined by the adjoining commutative triangle. In this diagram, we always assume $\mathrm{Out}\left(D_{p+2}^{(1)}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$ and omit most arrows from A_∞ to the D-algebras.

A large class of fusion rule algebras with generators of dimension two can be obtained from the tensor-product decomposition rules for a compact group, G, which has a unitary fundamental (in particular faithful) representation ρ of dimension two. By identifying G with $\rho(G)$ we can assume that

$$G \subseteq U(2). \qquad (3.107)$$

For dimension two, the requirement that ρ be irreducible is the same as saying that G is *non-abelian.* The fusion rule algebras we have classified, so far, in Lemma 3.4.5 and Corollary 3.4.6 ii), are all those algebras that have a selfconjugate generator. Therefore, we restrict our attention to those subgroups G of $U(2)$ for which the fundamental representation is selfconjugate. They are given by those compact groups, G, with the property that

$$\text{either} \quad G \subset O(2), \qquad \text{or} \quad G \subset SU(2). \qquad (3.108)$$

The E-Algebra Homomorphisms:

$$\begin{tikzcd}
 & & & & A_\infty \arrow[dlll, two heads, "\gamma_6\circ\sigma^{D_6(1)}"'] \arrow[dll, two heads, "\sigma^{D_6(1)}"] \arrow[dddlll, "\gamma_0^O\sigma^O"] \arrow[dddll, "\neq"] \arrow[dd, two heads, "\sigma^{D_4(1)}"] \arrow[dddr, two heads, "\sigma^T"] \arrow[dddrr, two heads, "\sigma^J\ \neq"] \arrow[dddrrr, two heads, "\gamma^J\circ\sigma^J"] & & & \\
 & D_6^{(1)} \arrow[r, leftrightarrow, "\gamma_6"'] & D_6^{(1)} \arrow[drr, two heads, "\sigma^O"] & & & & & \\
 & & & & D_4^{(1)} & & & \\
 & E_7^{(1)} \arrow[uu, two heads, "\sigma_4^O"] \arrow[r, leftrightarrow, "\gamma^O"] \arrow[dl, "\bar\sigma_3^O"'] \arrow[ddr, two heads, "\bar\sigma_3^O"] \arrow[ddrr, two heads, "\sigma_3^O"] & E_7^{(1)} \arrow[uu, two heads, "\sigma_4^O"'] \arrow[urr, two heads, "\sigma_2^O"] \arrow[rrr, two heads, "\sigma^{OT}"] \arrow[dd, two heads, "\bar\sigma_3^O"] \arrow[ddr, two heads, "\sigma_3^O"] & & & E_6^{(1)} \arrow[ul, two heads, "\sigma_2^T"] \arrow[loop below, "\gamma^T"] & E_8^{(1)} \arrow[ull, two heads, "\sigma_2^J"'] \arrow[l, two heads, "\sigma^{JT}"'] \arrow[r, leftrightarrow, "\gamma^J"] \arrow[ddllll, two heads, "\bar\sigma_3^J"] \arrow[ddlll, two heads, "\sigma_3^J"] \arrow[dd, two heads, "\sigma_5^J"] & E_8^{(1)} \arrow[ddllll, two heads, "\sigma_3^J"] \arrow[ddl, two heads, "\sigma_5^J"] \arrow[dd, two heads, "\bar\sigma_5^J"] \\
E_7^{(1)} \arrow[d, leftrightarrow, "\gamma^O"'] & & & & & & & \\
E_7^{(1)} & & \overline{D}_3 \arrow[ull, hook] \arrow[ll, hook] \arrow[dr, hook] & D_5^{(1)} \arrow[l, two heads] \arrow[d] & & & D_7^{(1)} \arrow[r, two heads] & \overline{D}_4 \\
 & & & D_5^{(1)} & & & &
\end{tikzcd}$$

Since G is assumed to be non-abelian, it cannot be isomorphic to subgroups contained in $O(2) \cap SU(2)$. The compact non-abelian subgroups of $SU(2)$ all contain -1. Thus two different subgroups of $SU(2)$ will yield different subgroups in $SO_3 = SU(2)/\pm 1$. The corresponding SO_3-subgroups have half the order, and, except for the smallest dihedral-group, $\mathcal{D}_2$, which is obtained from $\{\pm 1, \pm i\sigma_j\}_{j=1,2,3} \subset SU(2)$, they are also non-abelian. The non-abelian compact subgroups of $SU(2)$ are thus given by the pre-images of the polyhedral subgroups of SO_3. They are also called binary polyhedral groups.

They are: the dihedral-groups,, $\mathcal{D}_n$, $n = 3, \ldots, \infty$, $(D_\infty \supset U(1))$ of order $4n$ (of order $2n$ as SO_3-subgroups), the tetrahedron-group, $\mathcal{T}$, of order 24 (12 in SO_3), the octahedron-, cube- or hexahedron-group, $\mathcal{O}$, of order 48 (24 in SO_3) and the icosahedron- or dodecahedron-group, $\mathcal{J}$, of order 120 (60 in SO_3). The subgroups of $O(2)$, $\mathcal{R}_n$, $n \geq 3$, have rotations characterized by $\mathcal{R}_n \cap SO_{(2)} \cong \mathbb{Z}_n$, and, for them to be non-abelian, they must contain a reflection. As abstract groups, we have that $\mathcal{R}_n \cong \mathbb{Z}_2 \ltimes \mathbb{Z}_n$, where the adjoint action of $\mathbb{Z}_2$ on $\mathbb{Z}_n$ is just the inversion on $\mathbb{Z}_n$, and $|\mathcal{R}_n| = 2n$. (Let us stress again that the $\mathcal{R}_n$ are not isomorphic to any of the binary dihedral groups, since for the latter we have that $x^2 = 1$ which implies that x is central. This is clearly not true for $\mathcal{R}_n$. Yet, the image of $\mathcal{D}_n$ in SO_3 is isomorphic tor $\mathcal{R}_n$).

For fusion rule algebras, Φ_G, obtained from a compact group, G, there is a natural way to induce a fusion rule algebra homomorphism, σ_π, from a group homomorphism π. If $\pi : G \to H$ is a group homomorphism of compact groups G and H, and $\rho : H \to U(n)$ is an irreducible, unitary representation of H (seen as a group homomorphism with $\rho(H)' \cap U(n) = U(1)\mathbf{1}$), we can define a pull back $\pi * \rho := \rho \circ \pi : G \to U(n)$, which is a unitary representation, irreducible only if $\rho(\pi(G))' \cap U(n) = U(1)\mathbf{1}$. For the action σ_U of $U(n)$ on the space of representations of H, given by inner conjugation, $(\sigma_U \rho)(g) = U\rho(g)U^*$, we have that $\sigma_U \circ \pi^* = \pi^* \circ \sigma_U$. Thus, π^* is a map on equivalence classes of unitary representations, and we have well-defined multiplicities $(\sigma_\pi)_{\gamma\rho}$, of an irreducible representation, γ, of G in the representation $\pi^*\rho$, where ρ is an irreducible representation of H. From $\pi^*(\rho_1 \otimes \rho_2) = \pi^*\rho_1 \otimes \pi^*\rho_2$ we easily derive that the matrix σ_π, consisting of these multiplicities, represents a fusion rule algebra homomorphism, $\sigma_\pi : \Phi_H \to \Phi_G$. Clearly, σ_π is an inclusion of fusion rule algebras whenever π is surjective. If $G \subset H$, and π is the inclusion then it follows from the existence of induced representations, ρ^H, of H, for unitary, continuous representations ρ of G, that $\sigma_\pi : \Phi_H \twoheadrightarrow \Phi_G$ is a surjection. In this case, the matrix elements of σ_π are identical with the branching-rules of $H \downarrow G$. In the following lemma we relate the subgroups of (3.108) to the fusion rule algebras from Lemma 3.4.5 and Corollary 3.4.6 ii), and we explain the possible fusion rule algebra homomorphisms in terms of group homomorphisms.

Lemma 3.4.9

i) The tensor-product decomposition rules of the non-abelian compact groups with a self-conjugate fundamental representation of dimension two are given in the following equations:

$$\Phi_{SU(2)} \cong A_\infty \tag{3.109}$$

$$\Phi_{\mathcal{D}_\infty} \cong \Phi_{O(2)} \cong D_\infty \tag{3.110}$$

$$\begin{aligned} \Phi_{\mathcal{D}_p} \cong D^{(1)}_{p+2}, \textit{ for odd } p \geq 3 \textit{ and } \quad \mathrm{Out}\left(D^{(1)}_{p+2}\right) &\cong \mathbb{Z}_4, \\ \textit{for even } p \geq 2 \textit{ and } \quad \mathrm{Out}\left(D^{(1)}_{p+2}\right) &\cong \mathbb{Z}_2 \times \mathbb{Z}_2 \end{aligned} \tag{3.111}$$

$$\begin{aligned} \Phi_{\mathcal{R}_{2p}} &\cong D^{(1)}_{p+2}, \textit{ for } p \geq 2 \textit{ and } \quad \mathrm{Out}\left(D^{(1)}_{p+2}\right) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \\ \Phi_{\mathcal{R}_{2p+1}} &\cong \overline{D}_{p+2}, \textit{ for } p \geq 1 \end{aligned} \tag{3.112}$$

$$\Phi_{\mathcal{T}} \cong E^{(1)}_6 \qquad (3.113$$

$$\Phi_{\mathcal{O}} \cong E^{(1)}_7 \qquad (3.114$$

$$\Phi_{\mathcal{J}} \cong E^{(1)}_8 \qquad (3.115$$

ii) The automorphisms of the fusion rule algebras in part i) are obtained from th following group-automorphisms:

a) The finite groups, with D-type fusion rule algebras, contain maximal cyclic subgroups, $\mathbb{Z}_{2p} \subset \mathcal{D}_p$ and $\mathbb{Z}_q \subset \mathcal{R}_q$, and reflections, $Q \in \mathcal{D}_p$ and $S \in \mathcal{R}_q$, with $Q^2 = -1$ and $S^2 = 1$, such that $\mathcal{D}_p = \mathbb{Z}_{2p} \dot{\cup} Q \cdot \mathbb{Z}_{2p}$, and $\mathcal{R}_q = \mathbb{Z}_q \dot{\cup} S \cdot \mathbb{Z}_q$. For every $k \neq 1$, with $(k, 2p) = 1$, $(k, q) = 1$, resp., an outer automorphism π_k on $\mathcal{D}_p$, $\mathcal{R}_q$, resp., is defined by taking the k-th power of every element in the cyclic subgroup and mapping the reflection to itself. The derived fusion rule algebra automorphism, σ_{π_k}, obeys the equation $\sigma_{\pi_k}(\omega_1) = \omega_k$. Hence, every automorphism of a D-fusion rule algebra can be written as a product of σ_{π_k} and an automorphism, σ', with $\sigma'(\omega_1) = \omega_1$. $\mathcal{D}_p$ and $\mathcal{R}_{2q}$ admit an outer automorphism, η, which is the identity on the cyclic subgroup and $\eta(Q)Q^{-1}$, rep. $\eta(S)S^{-1}$, is a generating element thereof. σ_η is the only non-trivial automorphism on the D-algebras mapping the canonical generator to itself. (It exchanges the one-dimensional representations, x and y).

b) An outer automorphism on the tetrahedron group, $\mathcal{T}/\{\pm 1\} = \overline{\mathcal{T}} \subset SO_3$, is given by conjugating its elements with the $\frac{\pi}{2}$-rotation, mapping the standard tetrahedron to its dual tetrahedron (the axis of rotation runs through the midpoints of two opposite edges) and so defines (uniquely, up to inner conjugation) the outer automorphism, $\eta_\mathcal{T}$ on $\mathcal{T}$. We have that $\gamma_\mathcal{J} = \sigma_{\eta_\mathcal{T}}$ on $E_6^{(1)}$.

c) From a bicoloration of the centered cube, we obtain a signature representation, $c : \mathcal{O} \to \mathbb{Z}_2$, by assigning $c = 1$ to every element in $\mathcal{O}/\{\pm 1\} = \overline{\mathcal{O}} \subset SO_3$ that matches the bicoloration, and $c = -1$ whenever it matches opposite colorations. If we identify $c \in \mathbb{Z}_2$ with an element of the center of $SU(2)$, then $\eta_\mathcal{O}(g) := c(g)g$ defines an outer automorphism on $\mathcal{O}$, where $\mathrm{Out}(\mathcal{O}) = \mathbb{Z}_2$. We have that $\sigma_{\eta_\mathcal{O}} = \gamma_\mathcal{O}$.

d) The icosahedron-group, $\mathcal{I}/\{\pm 1\} = \overline{\mathcal{I}} \subset SO_3$, admits an outer automorphism which is (contrary to the $\overline{\mathcal{T}}$-case) not given by an SO_3-conjugation. It defines an outer automorphism $\eta_\mathcal{I}$ on $\mathcal{I} \subset SU(2)$, where $\mathrm{Out}(\mathcal{I}) = \mathbb{Z}_2$. We have that $\sigma_{\eta_\mathcal{I}} = \gamma_\mathcal{I}$.

iii) The injections of the fusion rule subalgebras, see Lemma 3.4.7, are obtained from

the following projections onto quotients of the dual groups:

a) $\mathcal{D}_\infty$ has normal subgroups $\mathbb{Z}_k \triangleleft U(1) \triangleleft \mathcal{D}_\infty$, so that, for $j_k : \mathcal{D}_\infty \twoheadrightarrow \mathcal{D}_\infty / \mathbb{Z}_k \cong \mathcal{D}_\infty$, we have that $\sigma_{j_k} = I_k : D_\infty \hookrightarrow D_\infty$.

b) The (binary) octahedron group has normal subgroup $\mathcal{D}_2 \triangleleft \mathcal{O}$ (similarly for the SO_3-subgroups $\overline{\mathcal{D}}_2 \triangleleft \overline{\mathcal{O}}$), with $\mathcal{O}/\mathcal{D}_2 \cong \overline{\mathcal{O}}/\overline{\mathcal{D}}_2 \cong \mathcal{R}_3 \cong S_3$. From the projection of $\mathcal{O}$ onto $\mathcal{R}_3$ we obtain the inclusion $\overline{D}_3 \hookrightarrow E_7^{(1)}$.

c) The normal subgroups of $\mathcal{R}_q$ and $\mathcal{D}_p$, with non-abelian quotients are $\mathbb{Z}_{q'} \triangleleft \mathbb{Z}_q \triangleleft \mathcal{R}_q$, for $q'|q$, and $\mathbb{Z}_\ell \triangleleft \mathbb{Z}_{2p} \triangleleft \mathcal{D}_p$, for $\ell|2p$. We have the following correspondences between group epimorphisms and fusion rule algebra inclusions:

$D_p \twoheadrightarrow \mathcal{D}_p/\mathbb{Z}_{2k} \cong \mathcal{R}_{(\frac{p}{k})}$, with $k|p$, yields

$$\overline{D}_{(\frac{p+3k}{2k})} \ \subset \ D^{(1)}_{p+2} \quad \text{with } \mathrm{Out}\left(D^{(1)}_{p+2}\right) = \mathbb{Z}_4\,, \quad \textit{for odd } p$$

$$\textit{or with } \mathrm{Out}\left(D^{(1)}_{p+2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \quad \textit{for even } p \textit{ and } \tfrac{p}{k} \textit{ odd},$$

$$\textit{and} \quad D^{(1)}_{\frac{p}{2k}+2} \ \subset \ D^{(1)}_{p+2}\,, \quad \text{with } \mathrm{Out}\left(D^{(1)}_{\frac{p}{2k}+2}\right) = \mathrm{Out}\left(D^{(1)}_{p+2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2\,,$$

$$\textit{for even } p \textit{ and even } \tfrac{p}{k}.$$

$\mathcal{D}_p \twoheadrightarrow \mathcal{D}_p/\mathbb{Z}_{2k+1} \cong \mathcal{D}_{(\frac{p}{2k+1})}$, with $(2k+1)|p$, yields

$$D^{(1)}_{\frac{p}{2k+1}+2} \ \subset \ D^{(1)}_{p+2}\,, \quad \text{with } \mathrm{Out}\left(D^{(1)}_{p+2}\right) = \mathrm{Out}\left(D^{(1)}_{\frac{p}{2k+1}+2}\right) = \mathbb{Z}_4\,, \quad \textit{for odd } p\,,$$

$$\textit{and } \mathrm{Out}\left(D^{(1)}_{p+2}\right) = \mathrm{Out}\left(D^{(1)}_{\frac{p}{2k+1}+2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2\,, \quad \textit{for even } p.$$

$\mathcal{R}_q \twoheadrightarrow \mathcal{R}_q/\mathbb{Z}_k \cong \mathcal{R}_{(\frac{q}{k})}$, with $k|q$, yields

$$D^{(1)}_{\frac{q}{2k}+2} \ \subset \ D_{\frac{q}{2}+2}\,, \quad \text{with } \mathrm{Out}\left(D^{(1)}_{\frac{q}{2k}+2}\right) = \mathrm{Out}\left(D^{(1)}_{q+2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

$$\textit{for even } q \textit{ and even } \tfrac{q}{k}\,,$$

$$\overline{D}_{\frac{q+3k}{2k}} \ \subset \ D_{\frac{q}{2}+2}\,, \quad \text{with } \mathrm{Out}\left(D^{(1)}_{\frac{q}{2k}+2}\right) = \mathbb{Z}_2 \times \mathbb{Z}_2$$

for even q and odd $\frac{q}{k}$,

$$\overline{D}_{\frac{q+3k}{2k}} \subset \overline{D}_{\frac{q+3}{2}}\,, \quad \textit{for odd } q \textit{ and odd } \tfrac{q}{k}\,.$$

iv) The surjective fusion rule algebra homomorphisms mapping canonical generator to canonical generator arise from the following group-inclusions:

a) The inclusion $G \subset SU(2)$ yields, for all fusion rule algebras Φ^G of binary polyhedral groups, a homomorphism

$$A_\infty \twoheadrightarrow \Phi^G\,.$$

The inclusions $\mathcal{R}_n \subset O(2)$ and $\mathcal{D}_n \subset \mathcal{D}_\infty$ yield the homomorphisms

$$D_\infty \twoheadrightarrow D^{(1)}_{p+2}\,, \qquad D_\infty \twoheadrightarrow \overline{D}_{t=2}\,,$$

for all possible structures.

b) The non-abelian subgroups of $\mathcal{D}_p$ are $\mathcal{D}_{p'}$ with $p'|p$ and of $\mathcal{R}_n$, $\mathcal{R}_{p'}$, with $n'|n$.

$\mathcal{D}_{p'} \subset \mathcal{D}_p$ yields

$$D^{(1)}_{p+2} \twoheadrightarrow D^{(1)}_{p'+2} \quad \textit{for all } p'|p \textit{ with the respective groups } \mathrm{Out}.$$

$\mathcal{R}_{p'} \subset \mathcal{R}_p$ yields

$$D^{(1)}_{\frac{p}{2}+2} \twoheadrightarrow D^{(1)}_{\frac{p'}{2}+2} \quad \textit{for even } p \textit{ and even } p'\,,$$

$$D^{(1)}_{\frac{p}{2}+2} \twoheadrightarrow \overline{D}_{\frac{p'+3}{2}} \quad \textit{for even } p \textit{ and odd } p'\,,$$

$$\overline{D}_{\frac{p+3}{2}} \twoheadrightarrow \overline{D}_{\frac{p'+3}{2}} \quad \textit{for odd } p \textit{ and odd } p'\,.$$

c) The surjective fusion rule algebra homomorphisms involving E-algebras which are collected in the commutative diagram following Lemma 3.4.8, are realized by inclusions of $SU(2)$-subgroups. These in turn are obtained from the respective embeddings of polyhedra.

From the form of the Perron Frobenius eigenvectors for graphs with norm equal to four it follows that the statistical dimensions, d_ψ, of elements $\psi \in \Phi_{[\rho]}$ of a simply generated fusion rule algebra, whose generator ρ has dimension $d_\rho = 2$, are always integer-valued, i.e., $d_\psi \in \mathbb{N}$. It is therefore possible that a fusion rule algebra from this class can be derived from some *semisimple* Hopf-algebra, $\mathcal{A}$, with a two-dimensional fundamental representation $\rho : \mathcal{A} \to \mathrm{Mat}_2(\mathbb{C})$, with the property that $\bigcap_{n,m} \ker(\rho^{\otimes n} \otimes \bar{\rho}^{\otimes m}) = \{0\}$. In Lemma 3.4.9, the fusion rule algebras with *selfconjugate* generator $\rho = \bar{\rho}$ of dimensio $d_\rho = 2$ have been associated to the non-abelian, compact subgroups, G, of $SU(2)$ and $O(2)$ (i.e. $\mathcal{A} = T[G]$), with $n \geq 2$ and $\mathrm{Out}\left(D^{(1)}_{2n}\right) \cong \mathbb{Z}_4$, for which there do not exist any dual compact groups. Moreover, we managed to relate all fusion rule algebra homomorphisms to group homomorphisms. In particular, all inclusions of one group into another one correspond to fusion rule algebra epimorphisms.

The question remains in which sense this result can be extended to fusion rul algebras with a self-conjugate generator ρ of dimension $d_\rho < 2$. More specifically, w should ask whether there exists a Hopf-subalgebra $\mathcal{A}$ of e.g., $U_q(s\ell_2)$, with $q = e^{\frac{2\pi}{30}}$, suc that the branched tensor product decomposition determined by the representation theor of $\mathcal{A}$ yields E_8-fusion rules? We shall see, however, that such an algebra can not b quasitriangular. We note that the non-abelian, compact subgroups of $U(2)$ reproduce al those fusion rule algebras that are generated by a single element ρ, with $d_\rho = 2$, and ar dual to some compact group. For all these fusion rule algebras, $\rho \otimes \rho$ contains a one dimensional subrepresentation α, namely the one corresponding to the representatio $\alpha(g) := \det(\rho(g))$ of the dual group. Hence the element α of the fusion rule algebra corresponding to this one-dimensional representation of the dual group belongs to Out(Φ We are therefore in the situation of Lemma 3.3.4 and conclude that any fusion rul algebra $\Phi = \Phi_G$ dual to some compact group G with a two-dimensional fundament representation, is of the form

$$\Phi = \tau_\alpha(\mathbb{Z}_n * \Phi')\,,$$

where Φ' is one of the $\mathbb{Z}_2$-graded or ungraded algebras given in (3.97), and n is determine by the cardinality of $\alpha(G) \subset U(1)$. A class of fusion rule algebras for which there is automorphism $\alpha \in \rho \circ \rho$ (and which are therefore not dual to a compact group) consists the algebras Φ for which $D(\Phi) = 1$ and with grading greater than two. For these algebras, the restrictions $\Lambda_0 := \mathrm{N}_\rho \restriction \mathcal{C}_0 \to \mathcal{C}_1$ of the fusion rule matrices N_ρ are determined in the proof of Proposition 7.3.1: They correspond to the graphs $D_4^{(1)}$ and $E_6^{(1)}$ (see (7.48)) and to $A_1^{(1)}$, $A_5^{(1)}$. A detailed description of the corresponding fusion rule algebras appears in the next lemma.

Lemma 3.4.10 *Suppose that Φ is a fusion rule algebra with generator ρ of dimension $d_\rho = 2$, that Φ is $\mathbb{Z}_a$-graded, for some*

$$a \geq 3,$$

and that

$$D(\Phi) = 1.$$

Then Φ is one of the following algebras:

i) For $\Lambda_0 \approx D_4^{(1)}$, the algebra Φ, denoted by $\Phi = D_4^{(1)} \left(A_1^{(1)}\right)^{(a-2)}$, has a basis

$$\{1, \sigma_1, \sigma_2, \sigma_3, \tau_1, \tau_2, \ldots, \tau_{a-1}\},$$

with $\rho = \tau_1$, and the decomposition of Φ, as a lattice,

$$\Phi = \bigoplus_{j \in \mathbb{Z}_a} \Phi_j,$$

has the following presentation:

$$\Phi_0 = \langle 1, \sigma_1, \sigma_2, \sigma_3 \rangle_{\mathrm{N}}\,; \qquad \Phi_j = \mathrm{N}\tau_j\,, \quad j \neq 0.$$

The elements $\{1, \sigma_1, \sigma_2, \sigma_3\} = \mathrm{Out}(\Phi)$ form a group: $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_4$, or $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2$. Their products with other elements of Φ are given by

$$\sigma_i \circ \tau_j = \tau_j\,, \qquad \text{for } i = 1, 2, 3, \quad j = 1, \ldots, a-1.$$

The multiplication table of the τ's is given by

$$\tau_j \circ \tau_k = 2\tau_{j+k}\,, \qquad j \neq -k,$$

$$\text{and} \qquad \tau_j \circ \tau_{-j} = 1 + \sum_{i=1}^{3} \sigma_i\,.$$

The conjugation on Φ is thus given by

$$\overline{\sigma_i} = \sigma_i^{-1}\,, \qquad \overline{\tau_j} = \tau_{-j}\,.$$

ii) For $\Lambda_0 \approx E_6^{(1)}$, one algebra Φ, denoted by $\Phi = E_6^{(1)} \left(A_5^{(1)}\right)^{(a-2)}$, has the following structure: It has a basis $\{1, \alpha, \alpha^{-1}, \psi, \chi_j, \alpha \circ \chi_j, \alpha^{-1} \circ \chi_j\}_{j=1,\ldots,a-1}$, with $\rho = \chi_1$, such that

$$\Phi_0 := \left\langle 1, \alpha, \alpha^{-1}, \psi \right\rangle_{\mathrm{N}}$$

$$\Phi_j := \left\langle \chi_j, \alpha \circ \chi_j, \alpha^{-1} \circ \chi_j \right\rangle_{\mathrm{N}}, \quad j = 1, \ldots, \alpha - 1,$$

form the graded sublattices. The elements $\{1, \alpha, \alpha^{-1}\} = \mathrm{Out}\,(\Phi_0) = \mathrm{Out}(\Phi)$ for a group isomorphic to $\mathbb{Z}_3$, and $\psi = \alpha \circ \psi = \alpha^{-1} \circ \psi$. These relations together with

$$\psi \circ \psi = 1 + \alpha + \alpha^{-1} + 2\psi$$

determine the subalgebra Φ_0. The multiplication of the elements in Φ_j with α is given in the obvious way; ($\mathrm{Out}\,(\Phi_0)$ acts transitively and freely on Φ_j). Moreover,

$$\psi \circ \chi_j = \chi_j + \alpha \circ \chi_j + \alpha^{-1} \circ \chi_j\,.$$

The multiplication table of the χ's is given by

$$\chi_j \circ \chi_k = \alpha \circ \chi_{j+k} + \alpha^{-1} \circ \chi_{j+k}\,, \quad \text{for } j \neq -k,$$

$$\text{and} \quad \chi_j \circ \chi_{-j} = 1 + \psi\,.$$

These relations and associativity determine the entire multiplication table, including products of the form $(\alpha^\varepsilon \circ \chi_j) \circ \left(\alpha^{\varepsilon'} \circ \chi_k\right)$, $\varepsilon, \varepsilon' = -1, 0, 1$. It follows that the conjugation is given by

$$\overline{\alpha^\varepsilon \circ \chi_j} = \alpha^{-\varepsilon} \circ \chi_{-j}\,.$$

The remaining fusion rule algebras with $\Lambda_0 \approx E_6^{(1)}$ and $\mathbb{Z}_a$-grading are then given by $\tau_\alpha \left(E_6^{(1)} \left(A_5^{(1)}\right)^{(a-2)}\right)$ and $\tau_{a-1} \left(E_6^{(1)} \left(A_5^{(1)}\right)^{(a-2)}\right)$.

The direct graphs determining the fusion rule matrix N_ρ for the fusion rule algebras $D_4^{(1)} \left(A_1^{(1)}\right)^{(a-2)}$ and $E_6^{(1)} \left(A_5^{(1)}\right)^{(a-2)}$ are depicted in Figures (A24) and (A25) of the Appendix. So far, we have found all fusion rule algebras Φ with a generator ρ of dimension $d_\rho \leq 2$ and with the property that

$$D(\Phi) = 1\,. \tag{3.116}$$

With the help of Proposition 3.3.2 and identities (3.57) - (3.59) we shall arrive at the following general *classification theorem* for fusion rule algebras not necessarily satisfying condition (3.116).

(The algebras will be distinguished according to whether the statistical dimension d_ρ of their generator ρ satisfies $d_\rho < 2$ or $d_\rho = 2$, and according to numbers a, a'' and r, with $a = ra''$, which are defined by: $\Phi/\Phi_0 \cong \mathbb{Z}_a$ (i.e., Φ is $\mathbb{Z}_a$-graded), $\mathbb{Z}_r \cong D(\Phi)$, and $\mathbb{Z}_{a''} \cong \Phi''/\Phi_0$, where Φ'' is defined through the presentation (3.60), and $D(\Phi'') = 1$. Furthermore, we make use of $\mathrm{Out}\,(\Phi_0)$ to discriminate between different algebras; $\mathrm{Out}(\Phi)$ will be determined.)

Theorem 3.4.11 *Let Φ be a fusion rule algebra generated by an element ρ of dimension d_ρ not exceeding two. Then Φ is one of the algebras described below.*

i) For $d_\rho < 2$, one finds the following list of algebras:

(a) If $a'' = 1$ then $\mathrm{Out}\,(\Phi_0) = \{1\}$, *and*

$$\Phi = \mathbb{Z}_r * \overline{A}_n\,, \quad \textit{for some } n \geq 1\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \qquad (3.117)$$

(b) Let $a'' = 2$. If $\mathrm{Out}\,(\Phi_0) = \{1\}$ *then*

$$\Phi = \mathbb{Z}_r * D_{2n}\,, \quad n \geq 3\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,; \qquad (3.118)$$

or

$$\Phi = \mathbb{Z}_r * E_8\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \qquad (3.119)$$

If $\mathrm{Out}(\Phi) \cong \mathbb{Z}_2 = \{1, \alpha\}$ *then Φ is one of the following algebras:*

$$\textit{For } r \textit{ even:} \quad \mathbb{Z}_r * A_{2n-1}\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.120)$$

$$\tau_\alpha\,(\mathbb{Z}_r * A_{2n-1})\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \qquad (3.121)$$

$$\mathbb{Z}_r * E_6\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.122)$$

$$\tau_\alpha\,(\mathbb{Z}_r * E_6)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,. \qquad (3.123)$$

$$\textit{For } r \textit{ odd:} \quad \mathbb{Z}_r * A_{2n-1}\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r \cong \mathbb{Z}_{2r}\,. \qquad (3.124)$$

$$\mathbb{Z}_r * \tau_\alpha\,(A_{2n-1}) \cong \tau_\alpha\,(\mathbb{Z}_r * A_{2n-1}) \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \qquad (3.125$$

$$\mathbb{Z}_r * E_6\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r \cong \mathbb{Z}_{2r}\,; \qquad (3.126$$

$$\mathbb{Z}_r * \tau_\alpha\,(E_6) \cong \tau_\alpha\,(\mathbb{Z}_r * E_6)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,. \qquad (3.127$$

If $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_3 = \{1, \alpha, \alpha^{-1}\}$ *then Φ is one of the following algebras: For $(3, r) = 1$:*

$$\mathbb{Z}_r * D_4 \cong \tau_\alpha\,(\mathbb{Z}_r * D_4) \cong \mathbb{Z}_r * \tau_\alpha\,(D_4)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_3 \times \mathbb{Z}_r \cong \mathbb{Z}_{3r}\,. \qquad (3.128$$

$$\textit{For } r = 3r': \quad \mathbb{Z}_r * D_4\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_3 \times \mathbb{Z}_r\,; \qquad (3.129$$

$$\tau_\alpha\,(\mathbb{Z}_r * D_4) \cong \tau_{\alpha^{-1}}\,(\mathbb{Z}_r * D_4)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{3r}\,. \qquad (3.130$$

ii) For $d_\rho = 2$, Φ is one of the algebras described in the following list:

(a) If $a'' = 1$ then

$$\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 = \{1, \alpha\}$$

and one finds the following algebras:

$$\textit{For } r \textit{ even:}\ \mathbb{Z}_r * \overline{D_n}\,,\ n \geq 3\,, \quad \textit{with}\mathrm{Out} \quad (\Phi_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.131$$

$$\tau_\alpha\left(\mathbb{Z}_r * \overline{D_n}\right)\,,\ n \geq 3\,, \quad \textit{with} \quad \mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_{2r}\,. \qquad (3.132$$

$$\textit{For } r \textit{ odd:}\ \mathbb{Z}_r * \overline{D_n} \cong \tau_\alpha\left(\mathbb{Z}_r * \overline{D_n}\right)\,,\ n \geq 3\,, \quad \textit{with} \quad \mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_{2r}\,. \qquad (3.133$$

(b) If $a'' = 2$ then Φ is one of the following algebras: For $\mathrm{Out}\,(\Phi_0) \cong 1$,

$$\mathbb{Z}_r * A_\infty\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,; \qquad (3.134$$

$$\mathbb{Z}_r * E_8^{(1)}\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \qquad (3.135$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 = \{1, \alpha\}$, *then*

$$\textit{if } r \textit{ is even:}\ \mathbb{Z}_r * D_\infty\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.136$$

$$\tau_\alpha\,(\mathbb{Z}_r * D_\infty)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \qquad (3.137$$

$$\mathbb{Z}_r * E_7^{(1)}\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.138$$

$$\tau_\alpha\left(\mathbb{Z}_r * E_7^{(1)}\right)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \qquad (3.139$$

With the help of Proposition 3.3.2 and identities (3.57) - (3.59) we shall arrive at the llowing general *classification theorem* for fusion rule algebras not necessarily satisfying ondition (3.116).

(The algebras will be distinguished according to whether the statistical dimension d_ρ f their generator ρ satisfies $d_\rho < 2$ or $d_\rho = 2$, and according to numbers a, a'' and r, ith $a = ra''$, which are defined by: $\Phi/\Phi_0 \cong \mathbb{Z}_a$ (i.e., Φ is $\mathbb{Z}_a$-graded), $\mathbb{Z}_r \cong D(\Phi)$, and $_{a''} \cong \Phi''/\Phi_0$, where Φ'' is defined through the presentation (3.60), and $D(\Phi'') = 1$. Furhermore, we make use of Out(Φ_0) to discriminate between different algebras; Out(Φ) ill be determined.)

heorem 3.4.11 *Let Φ be a fusion rule algebra generated by an element ρ of dimenion d_ρ not exceeding two. Then Φ is one of the algebras described below.*

i) For $d_\rho < 2$, one finds the following list of algebras:

(a) If $a'' = 1$ then Out$(\Phi_0) = \{1\}$, *and*

$$\Phi = \mathbb{Z}_r * \overline{A}_n\,, \quad \textit{for some } n \geq 1\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \tag{3.117}$$

(b) Let $a'' = 2$. If Out$(\Phi_0) = \{1\}$ *then*

$$\Phi \;=\; \mathbb{Z}_r * D_{2n}\,, \quad n \geq 3\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,; \tag{3.118}$$

or

$$\Phi \;=\; \mathbb{Z}_r * E_8\,, \quad \textit{and} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \tag{3.119}$$

If Out$(\Phi) \cong \mathbb{Z}_2 = \{1, \alpha\}$ *then Φ is one of the following algebras:*

$$\textit{For } r \textit{ even:} \quad \mathbb{Z}_r * A_{2n-1}\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \tag{3.120}$$

$$\tau_\alpha(\mathbb{Z}_r * A_{2n-1})\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \tag{3.121}$$

$$\mathbb{Z}_r * E_6\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \tag{3.122}$$

$$\tau_\alpha(\mathbb{Z}_r * E_6)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,. \tag{3.123}$$

$$\textit{For } r \textit{ odd:} \quad \mathbb{Z}_r * A_{2n-1}\,,\ n \geq 2\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r \cong \mathbb{Z}_{2r}\,. \tag{3.124}$$

$$\mathbb{Z}_r * \tau_\alpha(A_{2n-1}) \cong \tau_\alpha(\mathbb{Z}_r * A_{2n-1}) \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \tag{3.125}$$

$$\mathbb{Z}_r * E_6\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r \cong \mathbb{Z}_{2r}\,; \tag{3.126}$$

$$\mathbb{Z}_r * \tau_\alpha(E_6) \cong \tau_\alpha(\mathbb{Z}_r * E_6)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,. \tag{3.127}$$

If Out$(\Phi_0) \cong \mathbb{Z}_3 = \{1, \alpha, \alpha^{-1}\}$ *then Φ is one of the following algebras: For $(3, r) = 1$:*

$$\mathbb{Z}_r * D_4 \cong \tau_\alpha(\mathbb{Z}_r * D_4) \cong \mathbb{Z}_r * \tau_\alpha(D_4)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_3 \times \mathbb{Z}_r \cong \mathbb{Z}_{3r}\,. \tag{3.128}$$

$$\textit{For } r = 3r'\textit{:} \qquad \mathbb{Z}_r * D_4\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_3 \times \mathbb{Z}_r\,; \tag{3.129}$$

$$\tau_\alpha(\mathbb{Z}_r * D_4) \cong \tau_{\alpha^{-1}}(\mathbb{Z}_r * D_4)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{3r}\,. \tag{3.130}$$

ii) For $d_\rho = 2$, Φ is one of the algebras described in the following list:

(a) If $a'' = 1$ then

$$\mathrm{Out}(\Phi_0) \cong \mathbb{Z}_2 = \{1, \alpha\}$$

and one finds the following algebras:

$$\textit{For } r \textit{ even:}\ \mathbb{Z}_r * \overline{D_n}\,,\ n \geq 3\,, \quad \textit{with}\mathrm{Out} \quad (\Phi_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \tag{3.131}$$

$$\tau_\alpha\left(\mathbb{Z}_r * \overline{D_n}\right)\,,\ n \geq 3\,, \quad \textit{with} \quad \mathrm{Out}(\Phi_0) \cong \mathbb{Z}_{2r}\,. \tag{3.132}$$

$$\textit{For } r \textit{ odd:}\ \mathbb{Z}_r * \overline{D_n} \cong \tau_\alpha\left(\mathbb{Z}_r * \overline{D_n}\right)\,,\ n \geq 3\,, \quad \textit{with} \quad \mathrm{Out}(\Phi_0) \cong \mathbb{Z}_{2r}\,. \tag{3.133}$$

(b) If $a'' = 2$ then Φ is one of the following algebras: For Out$(\Phi_0) \cong 1$,

$$\mathbb{Z}_r * A_\infty\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,; \tag{3.134}$$

$$\mathbb{Z}_r * E_8^{(1)}\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r\,. \tag{3.135}$$

For Out$(\Phi_0) \cong \mathbb{Z}_2 = \{1, \alpha\}$, *then*

$$\textit{if } r \textit{ is even:}\ \mathbb{Z}_r * D_\infty\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \tag{3.136}$$

$$\tau_\alpha(\mathbb{Z}_r * D_\infty)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \tag{3.137}$$

$$\mathbb{Z}_r * E_7^{(1)}\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r\,; \tag{3.138}$$

$$\tau_\alpha\left(\mathbb{Z}_r * E_7^{(1)}\right)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \tag{3.139}$$

$$\textit{if } r \textit{ is odd}: \ \mathbb{Z}_r * D_\infty \cong \tau_\alpha(\mathbb{Z}_r * D_\infty)\,, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,; \qquad (3.140)$$

$$\mathbb{Z}_r * E_7^{(1)}, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_r \cong \mathbb{Z}_{2r}\,; \qquad (3.141)$$

$$\tau_\alpha\left(\mathbb{Z}_r * E_7^{(1)}\right) \cong \mathbb{Z}_r * \tau_\alpha\left(E_7^{(1)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r}\,. \qquad (3.142)$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_3 = \{1, \alpha, \alpha^{-1}\}$, *then*

$$\textit{if } (r,3) = 1\!: \mathbb{Z}_r * E_6^{(1)} \cong \tau_\alpha\left(\mathbb{Z}_r * E_6^{(1)}\right) \quad \cong \mathbb{Z}_r \quad *\tau_\alpha\left(E_6^{(1)}\right), \qquad (3.143)$$

$$\text{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_3 \quad \times \mathbb{Z}_r \cong \mathbb{Z}_{3r}\,;$$

$$\textit{if } r = 3r'\!: \ \mathbb{Z}_r * E_6^{(1)}, \quad \textit{with}\mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_3 \times \mathbb{Z}_r\,; \qquad (3.144)$$

$$\tau_\alpha\left(\mathbb{Z}_r * E_6^{(1)}\right) \cong \tau_{\alpha^{-1}}\left(\mathbb{Z}_r * E_6^{(1)}\right), \quad \textit{with}\mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_{3r}\,. \qquad (3.145)$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 \oplus \mathbb{Z}_2 = \{1, \alpha, \xi, \alpha \circ \xi\}$, $\alpha \in \mathrm{stab}(\rho)$, *then*

$$\textit{if } r \textit{ is odd}: \ \mathbb{Z}_r * D_{(p+2)}^{(1)} \cong \tau_\alpha\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,; \qquad (3.146)$$

$$\tau_\xi\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \tau_{\alpha\circ\xi}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{2r} \times \mathbb{Z}_2\,; \qquad (3.147)$$

$$\textit{if } r \textit{ is even}: \ \mathbb{Z}_r * D_{(p+2)}^{(1)}, \quad p \geq 2, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.148)$$

$$\tau_\alpha\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \quad p \geq 2, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,; \qquad (3.149)$$

$$\tau_\xi\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \tau_{\alpha\circ\xi}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,. \qquad (3.150)$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_4 = \{1, \xi, \xi^2, \xi^3\}$, *then*

$$\textit{if } r \textit{ is odd}: \ \mathbb{Z}_r * D_{(p+2)}^{(1)} \cong \tau_{\xi^2}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \quad \mathbb{Z}_{4r}\,; \qquad (3.151)$$

$$\tau_\xi\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \tau_{\xi^3}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \quad \mathbb{Z}_{4r}\,; \qquad (3.152)$$

$$\textit{if } r \equiv 2 \bmod 4: \ \mathbb{Z}_r * D_{(p+2)}^{(1)}, \quad n \geq 2,$$

$$\textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_r \times \mathbb{Z}_4 \quad \cong \mathbb{Z}_{2r} \quad \times \mathbb{Z}_2\,; \qquad (3.153)$$

$$\tau_{\xi^2}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \mathbb{Z}_r * \tau\left(D_{(p+2)}^{(1)}\right), \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_{2r} \quad \times \mathbb{Z}_2\,; \qquad (3.154)$$

$$\tau_\xi\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \tau_{\xi^3}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_{4r} \quad ; \qquad (3.155)$$

$$\textit{if } r = 4r'\!: \ \mathbb{Z}_r * D_{(p+2)}^{(1)}, \quad p \geq 2, \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_r \quad \times \mathbb{Z}_4\,; \qquad (3.156)$$

$$\tau_{\xi^2}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \quad p \geq 2, \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_{2r} \quad \times \mathbb{Z}_2\,; \qquad (3.157)$$

$$\tau_\xi\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right) \cong \tau_{\xi^3}\left(\mathbb{Z}_r * D_{(p+2)}^{(1)}\right), \textit{with} \quad \mathrm{Out}(\Phi) \quad \cong \mathbb{Z}_{4r} \quad . \qquad (3.1$$

(c) If $a'' \geq 3$ then Φ is one of the following fusion rule algebras:

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_3 = \{1, \alpha, \alpha^{-1}\}$, *then*

$$\textit{if } (r,3) = 1: \ \mathbb{Z}_r * E_6^{(1)}\left(A_5^{(1)}\right)^{(a''-2)}, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{3r}\,; \qquad (3.1$$

$$\mathbb{Z}_r * \tau_{\alpha^{\pm 1}}\left(E_6^{(1)}\left(A_5^{(1)}\right)^{(a''-2)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{3r}\,; \qquad (3.1$$

$$\textit{if } r = 3r'\!: \ \mathbb{Z}_r * E_6^{(1)}\left(A_5^{(1)}\right)^{(a''-2)}, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_3 \times \mathbb{Z}_r\,; \qquad (3.1$$

$$\tau_{\alpha^{\pm 1}}\left(\mathbb{Z}_r * E_6^{(1)}\left(A_5^{(1)}\right)^{(a''-2)}\right), \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_{3r}\,. \qquad (3.1$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 = \{1, \alpha, \xi, \alpha \circ \xi\}$, *then*

$$\textit{if } r \textit{ is odd}: \ \mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,; \qquad (3.16$$

$$\textit{if } r \textit{ is even}: \ \mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_2 \times \mathbb{Z}_r\,; \qquad (3.16$$

$$\tau_\sigma\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right), \quad \sigma \neq 1, \quad \textit{with} \quad \mathrm{Out}(\Phi) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,. \qquad (3.16$$

For $\mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_4 = \{1, \xi, \xi^2, \xi^3\}$, *then*

if r is odd:

$$\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)} \quad \cong \quad \tau_{\xi^j}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right),$$

$$j = 0, 1, 2, 3, \quad \textit{with} \quad \mathrm{Out}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right) \cong \mathbb{Z}_{4r} \quad ; \quad (3.16$$

if $r \equiv 2 \bmod 4$:

$$\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)} \quad \cong \quad \tau_{\xi^2}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right),$$

$$\textit{with} \quad \mathrm{Out}\,(\Phi_0) \quad \cong \quad \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,; \qquad (3.1$$

$$\tau_\xi\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right) \quad \cong \quad \tau_{\xi^3}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right),$$

$$\textit{with} \quad \mathrm{Out}\,(\Phi_0) \quad \cong \quad \mathbb{Z}_4\,; \qquad (3.1$$

if $r = 4r'$:

$$\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\,, \quad \textit{with} \quad \mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_4 \times \mathbb{Z}_r\,; \tag{3.169}$$

$$\tau_{\xi^2}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right), \quad \textit{with} \quad \mathrm{Out}\,(\Phi_0) \cong \mathbb{Z}_2 \times \mathbb{Z}_{2r}\,; \tag{3.170}$$

$$\begin{aligned} \tau_{\xi}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right) \quad &\cong \quad \tau_{\xi^3}\left(\mathbb{Z}_r * D_4^{(1)}\left(A_1^{(1)}\right)^{(a''-2)}\right), \\ \textit{with} \quad (\Phi_0) \quad &\cong \quad \mathbb{Z}_{4r}\,. \end{aligned} \tag{3.171}$$

Chapter 4

Hopf Algebras and Quantum Groups at Roots of Unity

We review the basic theory of Hopf algebras, including the Drinfel'd [3] definitions of quasitriangularity, and of the double construction and present, as an example, the algebra $U_q(s\ell_{d+1})$ *first defined by Jimbo [2]. We use results, due to Rosso [48], to define a quotient,* $U_q^{\mathrm{red}}(s\ell_{d+1})$, *of the topologically free algebra,* $U_q(s\ell_{d+1})$, *over* $\mathbb{C}([\log q])$, *which is quasitriangular and specializes* q *to a root of unity. Besides the known Cartan involution, we introduce an antilinear* **-involution and determine its relations with the R-matrix and the coproduct. For* $U_q^{\mathrm{red}}(s\ell_2)$, *the R-matrix is determined, and the center is presented as a* $\mathbb{C}^2$*-variety.*

Quantum groups, as defined in [2], are special types of Hopf algebras, obtained as one-parameter deformations of universal enveloping algebras of classical Lie algebras. We begin our discussion of their general properties with a brief review of quasi-triangular Hopf algebras.

Hopf algebras are associative algebras, carrying a comultiplicative structure, which is given by a homomorphism,

$$\Delta : \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}\,,$$

called comultiplication. The algebra is said to be cocommutative, if $\Delta = \sigma\Delta$, where

$\sigma : \mathcal{K} \otimes \mathcal{K} \to \mathcal{K} \otimes \mathcal{K}$ is the transposition $\sigma(a \otimes b) = b \otimes a$. This is the case for the universal enveloping algebras of classical Lie algebras. In order to describe braid statistics, we perturb cocommutativity by an invertible element $\mathcal{R} \in \mathcal{K} \otimes \mathcal{K}$, called universal R-matrix, satisfying

$$\mathcal{R}\Delta(a) = \sigma\Delta(a)\mathcal{R} \tag{4.1}$$

for all $a \in \mathcal{K}$. For Hopf algebras we require coassociativity

$$(1 \otimes \Delta)\Delta = (\Delta \otimes 1)\Delta\,. \tag{4.2}$$

Since the second comultiplication

$$\Delta' = \sigma\Delta \tag{4.3}$$

is coassociative too, there is a compatibility condition on $\mathcal{R}$:

$$(\mathcal{R} \otimes 1)(\Delta \otimes 1)\mathcal{R} = (1 \otimes \mathcal{R})(1 \otimes \Delta)\mathcal{R}\,. \tag{4.4}$$

In an attempt to describe Knizhnik-Zamolodchikov systems Drinfeld [4] has proposed to perturb coassociativity by an invertible element $\phi \in \mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}$ such that

$$(1 \otimes \Delta)\Delta(a) = \phi(\Delta \otimes 1)\Delta(a)\phi^{-1}\,, \quad \forall a \in \mathcal{K} \tag{4.5}$$

leading to quasi-Hopf algebras. The element ϕ has to satisfy certain relations that are due to pentagon cycles.

The unit element of the coalgebra (counit) is a homomorphism, $E : \mathcal{K} \to \mathbb{C}$, satisfying

$$(E \otimes 1)\Delta(a) = (1 \otimes E)\Delta(a) = a\,. \tag{4.6}$$

The "inverse" on a Hopf algebra is given by an antihomomorphism, $S : \mathcal{K} \to \mathcal{K}$, called the antipode, which is characterized by the property that

$$m_{12}(1 \otimes S)\Delta = m_{12}(S \otimes 1)\Delta = 1 \cdot E\,, \tag{4.7}$$

where $m_{12}(a \otimes b) = ab$.

This enables us to define adjoint representations

$$\begin{aligned} ad^{+}_{\mathcal{K}}(x) &= (L \otimes R)(1 \otimes S)\Delta(x) \\ ad^{-}_{\mathcal{K}}(x) &= (L \otimes R)(1 \otimes S^{-1})\Delta'(x)\,, \end{aligned} \tag{4.8}$$

with L and R being the right and left multiplication on $\mathcal{K}$. For quantum groups the subalgebra on which $ad^{\pm}_{\mathcal{K}}$ acts trivially coincides with the center of $\mathcal{K}$.

We summarize these notions in the following definition.

Definition 4.1 [3] *A quasitriangular Hopf algebra $\mathcal{K}$ is a coassociative Hopf algebra with comultiplication Δ, counit E, antipode S and an invertible universal R-matrix, $\mathcal{R} \in \mathcal{K} \otimes \mathcal{K}$, which intertwines Δ with Δ' and satisfies*

$$\begin{aligned} (1 \otimes \Delta)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{12} \\ (\Delta \otimes 1)\mathcal{R} &= \mathcal{R}_{13}\mathcal{R}_{23}\,.^{*}) \end{aligned} \tag{4.9}$$

From (4.1), (4.7) and (4.9) we can deduce further identities, e.g.

$$(1 \otimes E)\mathcal{R} = (E \otimes 1)\mathcal{R} = 1 \tag{4.10}$$

$$(1 \otimes S^{-1})\mathcal{R} = (S \otimes 1)\mathcal{R} = \mathcal{R}^{-1} \tag{4.11}$$

and the Yang-Baxter-equation

$$\mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12} = \mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23}\,.^{*}) \tag{4.12}$$

As an example we consider the quantum groups $U_q(s\ell_{d+1})$. The dependence on the "deformation"-parameter $q = e^t$ is expressed by the fact that the algebra is an $\mathbb{E}$-algebra, where $\mathbb{E}$ is the ring of meromorphic functions, f, for which $\sinh(t)^m f(t)$ is analytic, for some $m \in \mathbb{N}$. The algebra $U_q(s\ell_{d+1})$ is a topologically free algebra with generators 1, e_i, f_i, h_i, $i = 1, \ldots, d$, meaning that every element can be expressed as a series $\sum\limits_{0 \leq m \leq M,\, n \geq 0} t^n \sinh(t)^{-m} p_{n,m}$, where the $p_{n,m}$ are ordered polynomials in the generators. Further, we impose the following relations on the generators:

$$\begin{aligned} [h_i, e_j] &= a_{ij}e_j \\ [h_i, f_j] &= -a_{ij}f_j \\ [e_i, f_j] &= \delta_{ij}\tfrac{q^{h_i} - q^{-h_i}}{q - q^{-1}}\,, \end{aligned} \tag{4.13}$$

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[*] The subscripts label the positions of $\mathcal{R}$ in $\mathcal{K} \otimes \mathcal{K} \otimes \mathcal{K}$, i.e. $\mathcal{R}_{ij}$ is the image of $\mathcal{R}$ in $\mathcal{K}^{\otimes n}$ under th embedding $a \otimes b \to 1 \otimes \cdots \otimes_i a \otimes 1 \cdots \otimes_j b \otimes 1 \cdots \otimes 1$.

and

$$e_ie_j = e_je_i, \quad f_if_j = f_jf_i, \quad \text{for } |\, i-j \,| \;\geq\; 2\,,$$

$$e_i^2e_{i\pm1} - (q+q^{-1})e_ie_{i\pm1}e_i + e_{i\pm1}e_i^2 \;=\; 0\,,$$

$$\text{and} \quad f_i^2f_{i\pm1} - (q+q^{-1})f_if_{i\pm1}f_i + f_{i\pm1}f_i^2 \;=\; 0\,,$$

where $a_{ii} = 2$, $a_{i\pm1,i} = -1$, and $a_{i,j} = 0$, for $|\, i-j \,|\geq 2$.

Depending on whether we choose the functions in $\mathbb{E}$ to be complex or real, we thus have defined the associative algebras $U_q\,(s\ell_{d+1})_{\mathbb{R}}$ (over $\mathbb{E}_{\mathbb{R}}$) and $U_q\,(s\ell_{d+1})$ (over $\mathbb{E}_{\mathbb{C}}$). Clearly, $U_q\,(s\ell_{d+1})$ is also an $\mathbb{E}_{\mathbb{R}}$-module, and, since $\mathbb{E}_{\mathbb{R}} \subset \mathbb{E}_{\mathbb{C}}$, we have that $U_q\,(s\ell_{d+1})_{\mathbb{R}} \subset U_q\,(s\ell_{d+1})$, as $\mathbb{E}_{\mathbb{R}}$-algebras. Also $U_q\,(s\ell_{d+1})$ can be seen as a $\mathbb{C}$- or $\mathbb{R}$-module, i.e., a $\mathbb{C}$- or $\mathbb{R}$-algebra with additional central generators t and $\frac{1}{\sinh(t)}$.

Other prominent subalgebras are defined as in the classical case: $U_q(\mathfrak{b}^{\pm})$ are the Borel algebras generated by the elements e_i and h_i, resp. f_i and h_i, and $U_q(\mathfrak{n}^{\pm})$ the *sub algebras* generated only by the e_i's, resp. f_i's.

The comultiplication is then the $\mathbb{E}_{\mathbb{C}}$-linear homomorphism $\Delta : \mathcal{K} \to \mathcal{K} \otimes_{\mathbb{E}} \mathcal{K}$, given on the generators by

$$\begin{aligned} \Delta(h_i) &= h_i \otimes 1 + 1 \otimes h_i\,, \\ \Delta(e_i) &= e_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes e_i\,, \\ \Delta(f_i) &= f_i \otimes q^{-\frac{h_i}{2}} + q^{\frac{h_i}{2}} \otimes f_i\,. \end{aligned} \tag{4.14}$$

The $\mathbb{E}_{\mathbb{C}}$-linear counit $E : \mathcal{K} \to \mathbb{E}_{\mathbb{C}}$ is zero on the generators and $E(1) = 1$. By (4.7), the $\mathbb{E}_{\mathbb{C}}$-linear antipode must be given by

$$\begin{aligned} S(e_i) &= -q^{-1}e_i \\ S(f_i) &= -qf_i \\ S(h_i) &= -h_i\,. \end{aligned} \tag{4.15}$$

Note that its square is an inner automorphism, since

$$S^2(a) \;=\; q^{-2\delta}aq^{2\delta}\,, \tag{4.16}$$

$$\text{with} \qquad \delta \;=\; \frac{1}{2}\sum_{\alpha>0} h_\alpha\,.$$

Here the h_α are defined, for every positive root α, as the same combinations of $h_i = h_{\alpha_i}$, α_i primitive, as in the classical $s\ell_{d+1}$-case.

The Hopf algebra defined above is quasitriangular only for generic specializations of $q = e^t$, but not for the entire ring $\mathbb{E}$. We will use computations, already performed in [48], to define a quasitriangular version of a quantum group at a root of unity.

In a quantum double construction of a Hopf algebra $\mathcal{A}$ over a ring $\mathbb{E}$, the space, $\mathcal{A}^*$, of $\mathbb{E}$-linear forms

$$\ell : \mathcal{A} \to \mathbb{E}$$

is considered. It is equipped with a multiplication, by setting

$$\langle \ell \otimes k,\, \Delta(x)\rangle = \langle \ell \cdot k, x\rangle\,, \tag{4.17}$$

so that $\langle 1, .\rangle = E$, an (opposite) comultiplication

$$\langle \Delta(\ell),\, x \otimes y\rangle = \langle \ell, y \cdot x\rangle\,, \tag{4.18}$$

so that $E^*(\ell) = \langle \ell, 1\rangle$, and an antipode by

$$\langle S(\ell), S(X)\rangle = \langle \ell, X\rangle\,, \tag{4.19}$$

for $x, y \in \mathcal{A}$ and $\ell, k \in \mathcal{A}^*$. This obviously defines an associative Hopf algebra over $\mathbb{E}$ which we denote $\mathcal{A}^\circ$. The "double-constructed" algebra, $D(\mathcal{A})$, then consists of the space $\mathcal{A} \otimes_{\mathbb{E}} \mathcal{A}^\circ$, together with an $\mathbb{E}$-linear map

$$m : \mathcal{A}^\circ \otimes_{\mathbb{E}} \mathcal{A} \to \mathcal{A} \otimes_{\mathbb{E}} \mathcal{A}^\circ\,, \tag{4.20}$$

such that $D(\mathcal{A})$, with multiplication

$$\begin{aligned} (x \otimes 1) \cdot (y \otimes 1) \;&:= xy \otimes 1\,, \\ (1 \otimes k) \cdot (1 \otimes \ell) \;&:= 1 \otimes k\ell\,, \\ (x \otimes 1) \cdot (1 \otimes \ell) \;&:= x \otimes \ell\,, \\ \text{and} \qquad (1 \otimes k) \cdot (y \otimes 1) \;&:= m(k \otimes y)\,,\,, \end{aligned}$$

and the resulting extensions of coproduct and antipode, define a Hopf algebra over $\mathbb{E}$. A formula for m has been given in [48], with the property that $D(\mathcal{A})$ is quasitriangular, where $\mathcal{R} \in (\mathcal{A} \otimes 1) \otimes (1 \otimes \mathcal{A}^{\circ}) \subset D(\mathcal{A}) \otimes D(\mathcal{A})$ is precisely the canonical element in $\mathcal{A} \otimes \mathcal{A}^{\circ}$.

If we extend the ring over which $U_q(s\ell_{d+1})$ is defined to meromorphic functions, f, such that $\sinh(n_1 t)^{m_1} \cdot \ldots \cdot \sinh(n_k t)^{m_k} f(t)$ is analytic, for some $n_j, m_j \in \mathbb{N}$, i.e., for generic specialization of t, it is well known, see e.g. [4, 48], that for $\mathcal{A} = U_q(b^+)$, we obtain $D(\mathcal{A}) \cong U_q(s\ell_{d+1}) \otimes U(f)$, where $U(f)$ is a second copy of the Cartan subalgebra, commuting with $U_q(s\ell_{d+1})$. For non-generic specializations of t, the algebra dual to $U_q(b^+)$ will be different from $U_q(b^-)$. However, it is possible to take a quotient of $U_q(b^+)$ such that its dual is a similar quotient of $U_q(b^-)$.

The algebra $U_q(b^+)$ over $\mathbb{E}_{\mathbb{C}}$ has been studied thoroughly in [48]. For the statement of the results, we use the generators $E_i := e_i q^{\frac{h_i}{2}}$, so that

$$\Delta(E_i) = E_i \otimes \mathbf{1} + q^{h_i} \otimes E_i\,; \quad S(E_i) = -q^{-h_i} E_i\,, \tag{4.21}$$

and

$$[h_j, E_i] = a_{ij} E_j\,; \quad ad^+(E_i)^{1-a_{ij}}(E_j) = 0\,, \quad \text{for } i \neq j\,. \tag{4.22}$$

It is then possible to define, for each positive root, $\alpha_{i,j}$, of $s\ell_{d+1}$, with

$$\alpha_{i,j} := \alpha_i + \alpha_{i+1} + \ldots + \alpha_{j-1}$$

for $1 \leq i < j \leq d+1$, an element, E_α, by the recursion

$$E_{\alpha_{i,j}} := ad^+(E_i)\left(E_{\alpha_{i+1,j}}\right)\,, \quad \text{with } E_{\alpha_{i,i+1}} := E_i\,, \tag{4.23}$$

and compute q-analogue commutation relations.

From these it follows that every element of $U_q(b^+)$ can be written as a combination of the expressions

$$E^{m_1}_{\beta(1)} \ldots E^{m_n}_{\beta(n)}\, h_1^{\ell_1} \ldots h_d^{\ell_d} \tag{4.24}$$

where $\beta(1) < \ldots < \beta(n)$, $n = \frac{d(d+1)}{2}$, are all positive roots, with total ordering $\alpha_{ij} < \alpha_{i'j'}$ iff $i < i'$ or $i = i'$ and $j < j'$, and $m_j, \ell_j \in \mathbb{N}$. It is shown in [48] that the monomials (4.24) form indeed a basis of $U_q(b^+)$ over $\mathbb{E}_{\mathbb{C}}$. The subalgebra $U_q(b^-)$ of $U_q(b^+)^{\circ}$ is introduced as follows : It is generated by elements F_i, γ_i, defined by the equations

$$\langle F_i, E_i \rangle = \frac{1}{q^{-1} - q}\,, \quad \text{and} \quad \langle \gamma_i, h_i \rangle = \frac{1}{t}\,, \tag{4.25}$$

and =zero on all other monomials. We immediately obtain the coalgebra relations

$$\Delta(\gamma_i) \;=\; \gamma_i \otimes \mathbf{1} + \mathbf{1} \otimes \gamma_i\,, \quad S(\gamma_i) = -\gamma_i\,, \quad \text{and, with } \tilde{h}_i := -\sum_j \alpha_{ij}\gamma_j\,, \tag{4.26}$$

$$\Delta(F_i) \;=\; \mathbf{1} \otimes F_i + F_i \otimes q^{\tilde{h}_i}\,, \quad S(F_i) = -F_i q^{\tilde{h}_i}\,. \tag{4.27}$$

Furthermore, one finds the algebraic relations

$$[\gamma_i, F_j] = \delta_{ij} F_j\,; \quad ad^-(F_i)^{1-a_{ij}}(F_j) = 0\,, \quad \text{for } i \neq j\,. \tag{4.28}$$

Defining elements F_α in $\overline{U_q(b^-)}$, for every positive root, α, of $s\ell_{d+1}$, by the recursion

$$F_{\alpha_{i,j}} = ad^-(F_i)\left(F_{\alpha_{i+1,j}}\right)\,, \quad \text{for } i < j-1 \quad \text{and } F_{\alpha_{i,i+1}} = F_i\,, \tag{4.29}$$

it is possible to write every element as an $\mathbb{E}_{\mathbb{C}}$-combination of monomials in F_α and γ_i, similar to (4.24). The contraction $\langle .,. \rangle : \overline{U_q(b^-)} \otimes U_q(b^+) \to \mathbb{E}_{\mathbb{C}}$ has been computed in [48] as

$$\begin{aligned}\Big\langle F^{m_1}_{\beta(1)} \ldots F^{m_n}_{\beta(n)}\, \gamma_d^{r_1} \ldots \gamma_d^{r_d}\,, \quad E^{m'_1}_{\beta(1)} \ldots E^{m'_n}_{\beta(n)}\, h_1^{r'_1} \ldots h_d^{r'_d} \Big\rangle &= \\ = \prod_{j=1}^{n} \left(\delta_{m_j m'_j}\, q^{\frac{m_j(m_j-1)}{2}}\, \frac{(-q^{-1})^{\ell(\beta(j))m_j}}{(1-q^{-2})^{m_j}}\, (m_j)_q! \right) \prod_{\ell=1}^{d} \left(\delta_{r_\ell r'_\ell}\,, \frac{r_\ell!}{t^{r_\ell}} \right)&\,,\end{aligned} \tag{4.30}$$

where $\beta(1) < \ldots < \beta(n)$, $n = \frac{d(d+1)}{2}$, are the ordered positive roots, $\ell(\beta(j))$ their lengths, i.e., $\ell(\alpha_{ij}) = j - i$, for $i < j$, and the q-analogue numbers are

$$\begin{aligned}(n)_q &:= \frac{q^n - q^{-n}}{q - q^{-1}} = \frac{\sinh(nt)}{\sinh(t)}\,, \quad \text{for } q = e^t\,, \\ \text{and} \quad (n)_q! &:= (n)_q (n-1)_q \ldots (1)_q\,.\end{aligned} \tag{4.31}$$

To describe specialization to the case where q is a root of unity, we use, for $N \geq 3$ and $(n, N) = 1$, the ring-homomorphism $\frac{\sigma_n}{N} : \mathbb{E}_{\mathbb{C}} \to \mathbb{C}$, which assigns to any $f : t \to f(t)$ in $\mathbb{E}_{\mathbb{C}}$ the value $\frac{\sigma_n}{N}(f) = f\left(i\frac{i\pi}{N}\right)$; $\frac{\sigma_n}{N}$ is well defined since $\frac{\sigma_n}{N}(\sinh(t)) = i \sin\left(\frac{\pi n}{N}\right) \neq 0$.

Then

$$\begin{aligned} \langle .,. \rangle_{sp} \quad &: \quad \overline{U_q(b^+)} \otimes_{\Xi} U_q(b^+) \to \mathbb{C}\,, \\ \langle .,. \rangle_{sp} \quad &= \quad \sigma_{\frac{n}{N}} \circ \langle .,. \rangle\,, \end{aligned} \tag{4.32}$$

defines a contraction of $\overline{U_q(b^-)}$ and $U_q(b^+)$, seen as $\mathbb{C}$-algebras. The nullspaces, $I^+_{\frac{n}{N}} = \left\{ x \in U_q(b^+) \mid \langle k, x \rangle_{sp} = 0,\ \forall k \in \overline{U_q}(b^-) \right\}$, and, similarly, $I^-_{\frac{n}{N}}$, then form $\mathbb{C}$-Hopf ideals, by equations (4.17)-(4.19). So we can define the following $\mathbb{C}$-Hopf algebras:

$$U_q^{\mathrm{red}}\left(b^+\right) = U_q\left(b^+\right) / I^+_{\frac{n}{N}}\,, \quad U_q^{\mathrm{red}}\left(b^-\right) = \overline{U_q(b^-)} / I^-_{\frac{n}{N}}\,, \tag{4.33}$$

which, by the properties of $\langle .,. \rangle_{sp}$, are related as $\mathbb{C}$-algebras as follows.

$$U_q^{\mathrm{red}}\left(b^-\right) = \left(U_q^{\mathrm{red}}\left(b^+\right) \right)^{\circ}\,. \tag{4.34}$$

Using the intrinsic formula for m given in [48] and identifying h_i with $\bar{h}_i$, this formally defines a quasitriangular quantum group, $U_q^{\mathrm{red}}(s\ell_{d+1})$, at a root of unity, $q = e^{i\frac{\pi n}{N}}$.

For a more explicit description we remark that the Borel algebras $U_q^{\mathrm{red}}(b^{\pm})$ are generated by the elements $[E_i]$ and $[h_i]$, resp. $[F_i]$ and $\left[\bar{h}_i\right]$, where $[.] : U_q(b^{\pm}) \to U_q^{\mathrm{red}}(b^{\pm})$ denotes the complex-linear homomorphism onto $U_q^{\mathrm{red}}(b^{\pm})$, and further that t (and $\frac{1}{\sin h(t)}$) can be omitted from the set of generators by setting

$$[f(t)a] = f\left(i\,\frac{\pi n}{N} \right)\, [a]\,. \tag{4.35}$$

From (4.35) we also infer that the generators obey the Hopf algebra relations (4.21)-(4.22) and (4.26)-(4.28), where, e.g., E_i is replaced by $[E_i]$ and the expressions in $q = e^t \in \mathbb{E}_{\mathbb{C}}$ are replaced by the specialized ones in $q = e^{i\frac{\pi n}{N}} \in \mathbb{C}$. In the same way we can obtain the elements $[E_\alpha]$ and $[F_\alpha]$ from the specialized versions of the recursions (4.23) and (4.29) and, further, they obey corresponding commutation relations. Hence every element in $U_q^{\mathrm{red}}(b^{\pm})$ can be written as a linear combination of the respective classes of the monomials in (4.24). Since, by $\sigma_{\frac{n}{N}} \circ \langle a, b \rangle = \langle [a], [b] \rangle_{sp}$, the diagonal form of $\langle .,. \rangle$ in (4.30) with respect to the monomials (4.24) is inherited by $\langle .,. \rangle_{sp}$, we have that the set of monomials in $U_q^{\mathrm{red}}(b^+)$, with

$$\left[E_{\beta(1)}^{m_1} \cdots E_{\beta(n)}^{m_n}\, h_1^{r_1} \cdots h_d^{r_d} \right] \neq 0\,, \tag{4.36}$$

is a basis, by the nondegeneracy of $\langle .,. \rangle_{sp}$, and similarly for $U_q^{\mathrm{red}}(b^-)$. From

$$(m)_q! = 0 \quad \text{iff} \quad m \geq N\,, \quad \text{for} \quad q = e^{i\frac{\pi n}{N}}\,, \tag{4.37}$$

we find, that the expressions in (4.36) are characterized by

$$0 \leq m_i < N\,, \qquad i = 1, \ldots, n\,, \tag{4.38}$$

and the monomials $\left[E_\alpha^N\right]$ vanish.

The formula for the multiplication m (see (4.20)) given in [48] shows that $z_i := h_i - \bar{h}_i$ are central elements, and it yields, after quotienting by the Hopf-ideal generated by the z_i's, the commutator

$$[E_i, F_j] = \delta_{ij}\,(h_i)_q\,. \tag{4.39}$$

We collect these observations, based on computations in [48], in the following proposition.

Proposition 4.2 *In the following statements all equations to which we refer should be understood as specialized, i.e., we have*

$$q = e^{i\frac{\pi n}{N}}\,, \quad \textit{with } N \geq 3 \quad \textit{and } (n, N) = 1\,.$$

*i) The complex, associative algebra, $U_q^{\mathrm{red}}(b^+)$, defined by generators E_i, h_i, **1** and relations (4.22), together with*

$$E_\alpha^N = 0\,, \qquad \textit{for all } \alpha > 0\,, \tag{4.40}$$

where the E_α are defined by (4.23), has a PBW-Basis given by the monomials (4.24), restricted by (4.38). It has a Hopf algebra structure defined by the comultiplication and antipode in (4.21).

ii) The dual algebra $\left(U_q^{\mathrm{red}}(b^+)\right)^{\circ}$, with opposite comultiplication, denoted by $U_q^{\mathrm{red}}(b^-)$, is generated by the elements F_i, $\bar{h}_i$ given in (4.25). It is equally described in terms of relations (4.28) and

$$F_\alpha^N = 0\,, \qquad \textit{for all } \alpha > 0\,, \tag{4.41}$$

and co-relations (4.26) and (4.27), and admits a PBW-Basis analogous to the one of $U_q^{\mathrm{red}}(b^+)$. The contraction $\langle .,. \rangle_{sp} : U_q^{\mathrm{red}}(b^-) \otimes U_q^{\mathrm{red}}(b^+) \to \mathbb{C}$ is given by (4.30) and (4.32).

iii) The algebra, $U_q^{\text{red}}(s\ell_{d+1})$, which is obtained from $U_q^{\text{red}}(b^+) \vee U_q^{\text{red}}(b^-)$ by dividing out the relations (4.39) and $h_i = \tilde{h}_i$, has a PBW-Basis

$$\left\{E_{\beta(1)}^{m_1} \dots E_{\beta(n)}^{m_n}\, h_1^{r_1} \dots h_d^{r_d}\, F_{\beta(1)}^{\ell_1} \dots F_{\beta(n)}^{\ell_n}\right\}, \tag{4.42}$$

with $0 \leq m_i < N$; $0 \leq \ell_i < N$, and is quasitriangular with R-matrix

$$\mathcal{R} = \exp_{\left(\frac{n}{N}\right)}\left((-q)^{\ell(\beta(1))} E_{\beta(1)} \otimes F_{\beta(1)}\right) \cdot \ldots \cdot \exp_{\left(\frac{n}{N}\right)}\left((-q)^{\ell(\beta(n))} E_{\beta(n)} \otimes F_{\beta(n)}\right)\ q^{-t}. \tag{4.43}$$

Here we use the notations

$$\exp_{\left(\frac{n}{N}\right)}(X) := \sum_{m=0}^{N-1} q^{-\frac{m(m-1)}{2}} \left(1 - q^{-2}\right)^m \frac{X^m}{(m)_q!}$$

and

$$t := \sum_{jk} \left(a^{-1}\right)_{jk} h_j \otimes h_k,$$

with the inverse, a^{-1}, of the Cartan matrix a, i.e., $\alpha \otimes \beta \cdot t = (\alpha, \beta)$. The algebra $U_q^{\text{red}}(s\ell_{d+1})$ is identical to $D\left(U_q^{\text{red}}(b^+)\right)$ quotiented by the central subalgebra $U(\mathfrak{h})$ generated by $z_i = h_i - \tilde{h}_i$.

There are, of course, further possibilities of defining a quasitriangular quantum group at a root of unity. For example, if we insisted on having the entire Borel algebra, $U_q(b^+)$, without the relations (4.40), the dual algebra $U_q^- = (U_q(b^+))^\circ$ would contain $U_q^{\text{red}}(b^-)$ as a subalgebra, but, in addition, it would contain elements F_α^N, defined by

$$\left\langle F_\alpha^N,\ E_\alpha^N \right\rangle = 1, \tag{4.44}$$

and =zero on all other monomials. It follows that U_q^- is just the Borel algebra of the quantum group at a root of unity, U_q, defined in [49]. To be precise, we also would have to replace the generators h_i by generators $K_i := q^{h_i}$ and impose the relation $K_i^{2N} = 1$. The algebra $U_q^{\text{red}}(s\ell_{d+1})$, with these modifications in the Cartan generators, is still quasitriangular, but, in addition, it is a *finite-dimensional* subalgebra of U_q. It is possible to show that the $\mathcal{R}$-matrix of $U_q^{\text{red}}(s\ell_{d+1})$ is also an admissible $\mathcal{R}$-matrix of U_q, so that U_q is quasitriangular, although it is not double-constructed. Here we call a quasitriangular

Hopf algebra double-constructed if, for the map

$$\pi'_{\mathcal{R}} : \mathcal{K}^* \to \mathcal{K}, \quad \pi_{\mathcal{R}}(\ell) = (\langle \ell, . \rangle \otimes \mathbf{1})\mathcal{R}$$

and hence

$$\pi_{\mathcal{R}}^t(\ell) = (\mathbf{1} \otimes \langle \ell, . \rangle)\mathcal{R}, \tag{4.45}$$

we have that

$$\mathcal{K} = \operatorname{im} \pi_{\mathcal{R}} \vee \operatorname{im} \pi_{\mathcal{R}}^t.$$

In general, we have for a quasitriangular Hopf algebra

$$\mathcal{R} \in \operatorname{im} \pi_{\mathcal{R}}^t \otimes \operatorname{im} \pi_{\mathcal{R}}, \tag{4.46}$$

so that $\pi_{\mathcal{R}}$ is well defined on $(\operatorname{im} \pi_{\mathcal{R}}^t)^*$. Using equations (4.9) we find that

$$\pi_{\mathcal{R}} : \left(\operatorname{im} \pi_{\mathcal{R}}^t\right)^* \to \operatorname{im} \pi_{\mathcal{R}}$$

is an algebra isomorphism, which is anticohomomorphic. Therefore

$$\left(\operatorname{im} \pi_{\mathcal{R}}^t\right)^\circ \cong \operatorname{im} \pi_{\mathcal{R}}. \qquad (4.47$$

Thus in the case of a double-constructed algebra, $\mathcal{K}$, and by the uniqueness of the multiplication (4.20), (see [3]), we infer that $\mathcal{K}$ is a quotient of $D(\operatorname{im} \pi_{\mathcal{R}})$.

In the following we shall consider only the double-constructed examples $U_q^{\text{red}}(s\ell_{d+1})$ seen either as a $\mathbb{C}$- or $\mathbb{R}$-algebra, and $U_q^{\text{gen}}(s\ell_{d+1})$, which is the quantum group over the extended ring, $\mathbb{E}^{\text{gen}}$ of meromorphic functions, f, such that $\sinh(n_1 t)^{m_1} \dots \sinh(n_k t)^{m_k} f(t$ is analytic for some $n_j, m_j \in \mathbb{N}$. The automorphisms of the Borel algebras can be easil described.

Lemma 4.3

i) For every Hopf automorphism, α, of $U_q^{\text{red}}(b^+)\left(U_q^{\text{gen}}(b^+)\right)$, there are invertible elements, η_i, $i = 1, \dots, d$ in $\mathbb{C}(\mathbb{E}^{\text{gen}})$ and an involution, π, of the A_d-Dynkin diagram i.e., $\pi = id$ or $\pi(j) = d + 1 - j$, such that

$$\alpha(h_j) = h_{\pi(j)} \qquad (4.48$$

and

$$\alpha(E_j) = \eta_j E_{\pi(j)}. \qquad (4.49$$

Moreover, we have that α can be chosen either complex-linear or complex-antilinear for $U_q^{\mathrm{gen}}(b^+)$ with ring $\mathbb{E}^{\mathrm{gen}}$ and specializations $q \in \mathbb{R}$ and for $U_q^{\mathrm{red}}(b^+)$ for real specializations $t \in \mathbb{R}$, so that

$$\alpha_{\Xi}(t) = t \tag{4.50}$$

in both cases. α is complex-linear for non-real specializations and $U_q^{\mathrm{red}}(b^+)$.

Conversely, every map, α, defined on the generators by (4.48), (4.49) and (4.50) extends uniquely to an automorphism on $U_q^{\mathrm{red/gen}}(b^+)$.

ii) Similarly the set of anticohomomorphic automorphisms, $\tilde{\alpha}$, of $U_q(b^+)$ is characterized by

$$\tilde{\alpha}(h_j) = h_{\pi(j)} \tag{4.51}$$

$$\tilde{\alpha}(E_j) = \eta_j E_{\pi(j)} q^{-h_{\pi(j)}} \tag{4.52}$$

$$\text{and} \qquad \tilde{\alpha}_{\Xi}(t) = -t \tag{4.53}$$

Thus anticohomomorphic automorphisms only exist for purely imaginary specializations, i.e., $t \in i\mathbb{R}$ or $|q| = 1$, and for $U_q^{\mathrm{red}}(b^+)$, where they have to be antilinear.

The description of antihomomorphisms can be obtained from the above by compositions with the antipode.

iii) For specialized parameters t, the scalings $E_j \to \eta_j E_j$ can be obtained by conjugating elements of the Cartan torus so that the group of outer automorphisms is isomorphic to $\mathbb{Z}_2$. In particular, every cohomomorphic or anticohomomorphic automorphism maps $U_q(\mathfrak{n}^+)$ to itself and is an involution on f.

Furthermore, the automorphisms specified in i) and ii) have unique extensions to $U_q(s\ell_{d+1})$, given for the generators by

$$\alpha(F_j) = \frac{1}{\eta_j} F_{\pi(j)} \tag{4.54}$$

$$\text{and} \qquad \tilde{\alpha}(F_j) = \frac{1}{\eta_j} q^{h_{\pi(j)}} F_{\pi(j)} . \tag{4.55}$$

These extensions are also cohomomorphic, resp. anticohomomorphic.

iv) If we denote by C the extension of the anticohomomorphic automorphism with $\eta_j = 1$, $\pi = id$, then we have the relations

$$C^2 = 1 \tag{4.56}$$

$$\text{and} \qquad C \otimes C\mathcal{R} = \mathcal{R}^{-1} . \tag{4.57}$$

C will thus be called the <u>conjugation</u> of $U_q(s\ell_{d+1})$.

The symmetry in the sets of generators and relations of $U_q(b^+)$ and $U_q(b^-)$ enables us to define involutions on $U_q(s\ell_{d+1})$, which are important in the study of highest-weight representations. In general for a quasitriangular, double-constructed Hopf algebra, $\mathcal{K}$, we call an $\mathbb{R}$-linear, antihomomorphic involution, θ, on $\mathcal{K}$, a <u>Cartan involution</u>if θ satisfies

$$\begin{aligned} & \theta : \operatorname{im}\pi_{\mathcal{R}} \to \operatorname{im}\pi_{\mathcal{R}}^t \\ \text{thus} \qquad & \theta : \operatorname{im}\pi_{\mathcal{R}}^t \to \operatorname{im}\pi_{\mathcal{R}} \end{aligned} \tag{4.58}$$

and

$$\theta \otimes \theta\mathcal{R} = \sigma\mathcal{R} . \tag{4.59}$$

Similarly a <u>$*$- involution</u> is a $\mathbb{R}$-linear, antihomomorphic involution which also maps $\operatorname{im}\pi_{\mathcal{R}}$ to $\operatorname{im}\pi_{\mathcal{R}}^t$ but instead of (4.59) obeys

$$* \otimes * \mathcal{R} = \sigma\mathcal{R}^{-1} . \tag{4.60}$$

Lemma 4.4

i) Assume, that θ is a Cartan involution and $*$ a $*$-involution on a double-constructed Hopf algebra $\mathcal{K}$. Then we have

$$\theta \otimes \theta \circ \Delta = \Delta \circ \theta , \qquad \theta \circ S = S^{-1} \circ \theta \tag{4.61}$$

$$\text{and} \qquad * \otimes * \circ \Delta = \sigma\Delta \circ * , \qquad * \circ S = S \circ * . \tag{4.62}$$

ii) For the isomorphism $\pi_{\mathcal{R}}$, it follows that

$$\pi_{\mathcal{R}}\theta^t = \theta\pi_{\mathcal{R}}^t \qquad \text{and} \qquad S * \pi_{\mathcal{R}}^t = \pi_{\mathcal{R}} * S^t . \tag{4.63}$$

Thus, if we define nondegenerate, $\mathbb{R}$*-bilinear forms on* $\operatorname{im}\pi^t_{\mathcal{R}}$ *by*

$$(a,b)_\theta := \left\langle \pi^{-1}\theta(a), b\right\rangle \qquad \text{and} \qquad (a,b)_* := \left\langle \pi_{\mathcal{R}}^{-1} * (a), b\right\rangle\,, \tag{4.64}$$

it follows that $(a,b)_\theta$ *is symmetric and obeys*

$$(\Delta(a), b\otimes c)_\theta = (a, cb)_\theta \qquad \text{and} \qquad (S(a),b)_\theta = (a, S(b))_\theta \tag{4.65}$$

and further that

$$(a,b)_* = (S(b),a)_*\,, \tag{4.66}$$

so that

$$\begin{gathered}(\Delta(a), b\otimes c)_* = (a, bc)_*\,; \quad (b\otimes c, \Delta(a))_* = (cb, a)_* \\ \text{and} \qquad (S(a), S(b))_* = (a,b)_*\,.\end{gathered} \tag{4.67}$$

iii) Suppose α *is an automorphism of* $\operatorname{im}\pi^t_{\mathcal{R}}$*, so that for* $\mathcal{J} = \operatorname{im}\pi^t_{\mathcal{R}} \cap \operatorname{im}\pi_{\mathcal{R}}$

$$\alpha(\mathcal{J}) = \mathcal{J} \qquad \text{and} \qquad (\alpha\circ\theta)^2 \restriction_{\mathcal{J}} = id_{\mathcal{J}} \tag{4.68}$$

and

$$(\alpha(a), b)_\theta = (a, \alpha(b))_\theta\,, \quad \text{resp.} \quad (\alpha(a), b)_* = (a, \alpha(b))_*\,. \tag{4.69}$$

Then there exists a unique extension, $\bar{\alpha}$*, to* $\mathcal{K}$*, such that*

$$\theta' = \bar{\alpha}\circ\theta\,, \qquad \text{resp.} \qquad *' = \bar{\alpha}\circ * \tag{4.70}$$

*is a Cartan-(resp. *-) involution. Moreover, given some involutions* θ *and* $*$*, then all other involutions are given by (4.70) for some* α *with (4.68) and (4.69), and the extension,* $\bar{\alpha}$*, is always cohomomorphic, thus a Hopf-automorphism.*

This, together with the characterization of automorphisms of the Borel algebra and the conjugation, C, in Lemma 4.3, put us in a position to find all Cartan- and $*$-involutions of $U_q(s\ell_{d+1})$. They are given as follows:

Lemma 4.5 *i) There exists a Cartan involution,* θ*, on* $U_q(s\ell_{d+1})$ *which is given c*

the generators by

$$\begin{aligned}\theta(E_j) &= q^{h_j}F_j && \text{(4.71}\\ \theta(F_j) &= E_j q^{-h_j} && \text{(4.7}\\ \theta(h_j) &:= h_j && \text{(4.7}\\ \theta_{\Xi}(t) &= t\,. && \text{(4.74}\end{aligned}$$

It can be chosen antilinear only if $\mathbb{E} = \mathbb{E}^{gen}$ *or if* t *is specialized to real values. In a*

other cases, θ *has to be complex-linear.* θ *is determined uniquely by (4.71)-(4.74*

and the sign $\theta(i) = \pm i$*.*

ii) The Hopf automorphisms, α*, of Lemma 4.3 i), which give rise to all other Carta*

involutions by (4.70), are those with

$$\eta_j = \eta_{\pi(j)}\,. \qquad \text{(4.75}$$

iii) The antihomomorphism

$$* := C\circ\theta = \theta\circ C\,, \qquad \text{(4.76}$$

where C *is given in Lemma 4.3 iv), is a *-involution, for all versions of* $U_q(s\ell_{d+1})$

where C *is defined. It is given on the generators*

$$\begin{aligned}*(h_j) &= h_j && \text{(4.77}\\ *(E_j) &= F_j && \text{(4.78}\\ *(F_j) &= E_j && \text{(4.79}\\ *_{\Xi}(t) &= -t\,. && \text{(4.80}\end{aligned}$$

*iv) Equation (4.80) holds for all *-involutions* $*'$*, so that* $*'$ *is defined on a quantu*

group whenever $*$ *is defined. All possible* $*'$ *are given by (4.70) where the automo*

phisms α*, specified in Lemma 4.3 i), are constrained by*

$$*_{\Xi}(\eta_i) = \eta_{\pi(i)}\,. \qquad \text{(4.81}$$

One way to verify formulae (4.59), (4.60) and (4.57) is to directly apply the involutions, resp. their compositions with the antipode, to the expression of the $\mathcal{R}$-matrix. Also we can use the fact that these formulae are equivalent to the symmetry relations of the forms (4.64) and similar constructions. Following this strategy it is useful to know that any bilinear form on $U_q(b^+)$ for which the comultiplication is the transpose of the multiplication (compare (4.67)) is uniquely determined by the scalar products of the generator E_i and h_i. For convenience we give the general forms of the involutions in terms of the original definitions (4.13), using the identification

$$F_j = q^{-\frac{h_j}{2}} f_j . \tag{4.82}$$

They are

$$\begin{aligned} \theta(e_j) &= \eta^{-1}_{\pi(j)} f_{\pi(j)} \\ \theta(f_j) &= \eta_{\pi(j)} e_{\pi(j)} \\ \theta(h_j) &= h_{\pi(j)} ; \end{aligned} \tag{4.83}$$

$$\begin{aligned} *(e_j) &= \eta^{-1}_{\pi(j)} f_{\pi(j)} \\ *(f_j) &= \eta_{\pi(j)} e_{\pi(j)} \\ *(h_j) &= h_{\pi(j)} ; \end{aligned} \tag{4.84}$$

and

$$CE_j = e_j , \qquad CF_j = f_j , \qquad CH_j = h_j . \tag{4.85}$$

As an example, let us have a more detailed look at $U_q^{\mathrm{red}}(s\ell_2)$, for $q = e^{i\pi\frac{n}{N}}$, where we assume $(n, N) = 1$, $N \geq 3$. The relations defining $U_q^{\mathrm{red}}(s\ell_2)$ are

$$\begin{aligned} [h, e] &= 2e \\ [h, f] &= -2f \\ [e, f] &= \frac{q^h - q^{-h}}{q - q^{-1}} \end{aligned} \tag{4.86}$$

and

$$e^N = f^N = 0 .$$

The universal R-matrix is given by

$$\mathcal{R} = \sum_{n=0}^{N-1} q^{-\frac{1}{2}(h\otimes h)} q^{\frac{n(n-1)}{2}} \frac{(1-q^2)^n}{(n)_q!} q^{-\frac{nh}{2}} e^n \otimes q^{\frac{nh}{2}} f^n . \tag{4.87}$$

Here we use the <u>q-numbers</u>, defined by

$$(n)_q := \frac{q^n - q^{-n}}{q - q^{-1}} . \tag{4.88}$$

They arise in the calculation of the commutators

$$\begin{aligned} [e, f^n] &= f^{n-1}(n)_q(h - n + 1)_q \\ [f, e^n] &= e^{n-1}(n)_q(-h - n + 1)_q . \end{aligned} \tag{4.89}$$

For the classification of the irreducible representations of $U_q^{\mathrm{red}}(s\ell_2)$, we next describe the generators of the center:

$$\begin{aligned} Q &= fe + \left(\tfrac{h+1}{2}\right)_q^2 \\ &= ef + \left(\tfrac{h-1}{2}\right)_q^2 \end{aligned} \tag{4.90}$$

and $P = e^{i\pi h}$.

They satisfy the relations

$$\left[\frac{P^{\frac{n}{2}} - P^{-\frac{n}{2}}}{(q-q^{-1})^N}\right]^2 = -\prod_{j=0}^{N-1}\left(Q - \left(j+\frac{1}{2}\right)_q^2\right)$$

or equivalently (4.91)

$$\left[\frac{P^{\frac{n}{2}} + P^{-\frac{n}{2}}}{(q-q^{-1})^N}\right]^2 = -\prod_{j=0}^{N-1}\left(Q - (j)_q^2\right) .$$

Relations (4.91) define a variety $\mathfrak{V}$ in $\mathbb{C}^2$, on which the Casimir values of (Q, P) have to lie. The real part of this variety, $\mathfrak{V}_{\mathrm{real}}$, is the intersection of $\mathfrak{V}$ with

$$\mathbb{R} \times S^1 = \{(Q, P) \mid Q \in \mathbb{R}, \mid P \mid = 1\} ,$$

describing representations, that admit sesquilinear forms. A more detailed description of $\mathfrak{V}$ will be given in Section 5.2.

Chapter 5

Representation Theory of $U_q^{\text{red}}(s\ell_2)$

5.1 Highest Weight Representations of $U_q^{\text{red}}(s\ell_{d+1})$

We show that the irreducible representations of $U_q(s\ell_{d+1})$, for q a root of unity, have a maximal dimension and can be obtained from Verma modules by quotienting by the nullspaces of hermitian and bilinear forms. The contragradient of a representation is defined, and categorical aspects are discussed.

The finite dimensional, irreducible representations of $U_q(s\ell_{d+1})$ and $U_q^{\text{red}}(s\ell_{d+1})$ are representations of highest weight, because the generators h_i of the Cartan subalgebras are bounded operators. In the generic case of $U_q(s\ell_{d+1})$, $q^2 \neq$ root of unity, it is known [50] that the highest weights, characterizing the representations, are (up to irrational shifts $\lambda_i \mapsto \lambda_i + \tau$, where $q^2 = e^{\frac{2\pi i}{\tau}}$) all integral, and the associated representations can be seen as deformations of irreducible representations of the corresponding classical Lie algebras. In the rational case, (i.e., q a root of unity) we see from (4.22) that the subalgebra $U_q(\mathfrak{n}^-)$ is finite dimensional. Therefore, *any* highest weight will lead to a finite dimensional, irreducible representation, the dimension of which is bounded by $\dim U_q(\mathfrak{n}^-)+1$. For $U_q(s\ell_2)$, with $q^2 = e^{2\pi i \frac{n}{N}}$, this bound is equal to N.

A useful tool to determine irreducible representations from their highest weights is the study of real linear forms, $\langle .,. \rangle$ and $(.,.)$, that are invariant with respect to the antiinvolutions $*$ and θ, introduced in (4.23) and (4.24).

The proof of the following lemma uses the direct sum decomposition

$$\begin{aligned} U_q(s\ell_{d+1}) = \\ & \mathbb{C}([h_i]) \oplus \mathbb{C}([h_i]) \cdot U_q(\mathfrak{n}^+) \oplus U_q(\mathfrak{n}^-) \cdot \mathbb{C}([h_i]) \\ & \oplus U_q(\mathfrak{n}^-) \cdot \mathbb{C}([h_i]) \cdot U_q(\mathfrak{n}^+) . \end{aligned} \tag{5.1}$$

and follows from a standard reconstruction argument.

Lemma 5.1.1

a) On any pair of highest weight representations W_{λ^*}, V_λ, (W_{λ^T}, V_λ, respectively) of $U_q(s\ell_{d+1})$, with $\lambda^* = \bar{\lambda} \circ \alpha$ $\left(\lambda^T = \lambda \circ \alpha\right)$, there exist invariant, real linear forms

$$\begin{aligned} \langle .,. \rangle \; : \; W_{\lambda^*} \; \otimes \; V_\lambda \; &\rightarrow \; \mathbb{R}([t]) , \\ (.,.) \; : \; W_{\lambda^T} \; \otimes \; V_\lambda \; &\rightarrow \; \mathbb{R}([t]) , \end{aligned}$$

with the properties

$$\begin{aligned} \langle v, aw \rangle &= \langle *(a)v, w \rangle , \\ (v, aw) &= (\theta(a)v, w) , \end{aligned} \tag{5.2}$$

for all $a \in U_q(s\ell_{d+1})$, and

$$\begin{aligned} \langle f(t)v, g(t)w \rangle &= f(-t)g(t)\langle v, w \rangle , \\ (f(t)v, g(t)w) &= f(t)g(t)(v, w) , \quad \text{for} \quad f, g \in \mathbb{R}([t]) , \end{aligned}$$

which upon specializing to $t \in i\mathbb{R}$ (i.e. $|q| = 1$) become sesquilinear, resp. bilinear forms.

b) The invariant forms $\langle .,. \rangle$ and $(.,.)$ are uniquely determined by $\langle v_{\lambda^*}, v_\lambda \rangle$ $((v_{\lambda^T}, v_\lambda)$ where v_λ are the highest-weight vectors. In particular, if $\langle v_{\lambda^*}, v_\lambda \rangle = 0$ $((v_{\lambda^T}, v_\lambda) = ($ then $\langle .,. \rangle \equiv 0$ $((.,.) \equiv 0)$.

c) If $\mathcal{N}_\lambda$ and $\mathcal{M}_{\lambda^*}$ are defined by

$$\begin{aligned} \mathcal{N}_\lambda &= \{x \mid \langle y, x\rangle = 0, \ \forall y \in W_{\lambda^*}\}, \\ \text{and} \qquad & \\ \mathcal{M}_{\lambda^*} &= \{y \mid \langle y, x\rangle = 0, \ \forall x \in V_\lambda\}, \end{aligned} \tag{5.3}$$

the quotients $V_\lambda/\mathcal{N}_\lambda$ and $W_{\lambda^*}/\mathcal{M}_{\lambda^*}$ are exactly the irreducible representations of highest weights λ and λ^*. The irreducible representations can be obtained from $(.,.)$ and θ in the same way.

In the statements made above, we may as well replace highest weights by lowest weights. By unitary representations we henceforth mean highest weight representations, for which $\langle .,. \rangle$ is positive-definite on $V_\lambda/\mathcal{N}_\lambda$, so that the representation space admits a Hilbert-space structure.

In analogy to the classical case, tensor products of representations are defined by the comultiplication. The trivial representation is the counit, which by (4.6) can also be characterized as the only representation such that $V_\lambda = V_1 \otimes V_\lambda$, for all λ. Furthermore, for any representation ρ on V, we can define a representation, $\rho^\vee$, on the dual space, V^*, ($V^\vee$ as a module) by

$$\rho^\vee = \rho^t \circ S \tag{5.4}$$

called the representation *conjugate* to ρ. We have that $\rho^{\vee\vee} \cong \rho$ and that $\rho^\vee$ is uniquely determined by the requirement that the trivial representation is a subrepresentation of $V \otimes V^\vee$. The latter can be seen by replacing the action of $\mathcal{K}$ on $V \otimes W^\vee$, by the adjoint representation on $\mathrm{Hom}(V, W)$. A trivial subrepresentation of $\mathrm{Hom}(V, W)$ consists of an intertwiner from V to W, so that V and W have to be isomorphic. Finally commutativity, $\rho_i \otimes \rho_j \simeq \rho_j \otimes \rho_i$, is guaranteed by the invertible intertwiner

$$R_{ij} = P_{ij}\rho_i \otimes \rho_j \mathcal{R}, \tag{5.5}$$

where $\mathcal{R}$ is the universal $\mathcal{R}$-matrix in $\mathcal{K} \otimes \mathcal{K}$, and $P_{ij} : V_i \otimes V_j \to V_j \otimes V_i$ is the transposition.

For later applications we want to introduce an antilinear mapping $\chi_\lambda : V_\lambda \to V^\vee_{\lambda^*}$, replacing the Clebsch-Gordan matrix $P_{1,\lambda\lambda^{*\vee}}$, intertwining $V_\lambda \otimes V^\vee_{\lambda^*}$ with V_1:

$$\ell(v) = \langle \chi_\lambda^{-1} \ell, v \rangle = P_{1,\lambda\lambda^{*\vee}}(v \otimes \ell)\,. \tag{5.6}$$

It is related to the antihomomorphisms S and $*$ by

$$\chi_\lambda \rho_\lambda(a) \chi_\lambda^{-1} = \rho_{\lambda^{*\vee}}\left(S^{-1}(a^*)\right) \tag{5.7}$$

and, having (4.11) for the square of the antipode, can be normalized to

$$\chi_{\lambda^{*\vee}} \chi_\lambda = q^{2\delta}\,.$$

5.2 The Irreducible and Unitary Representations of $U_q^{\mathrm{red}}(s\ell_2)$

The irreducible representations of $U_q^{\mathrm{red}}(s\ell_2)$ are classified and given in a highest-weight basis. We use the surjective parameterization by highest weights to discuss the topological structure of the center-variety. We show that representations over non-singular points and with a diagonal Cartan element, k, are completely reducible. We determine the ranges of highest weights for which the irreducible representations are unitarizable with respect to $*$.

In this section we describe the irreducible and unitary representations of $U_q^{\mathrm{red}}(s\ell_2)$, for $q^2 = e^{2\pi i \frac{n}{N}}$ a root of unity. The irreducible representations have been determined in [51] for the algebra without relations (4.22) and generators $e, f, k^2 = q^h$, so that e and f could still be invertible. For $U_q^{\mathrm{red}}(s\ell_2)$, however, we have only highest-weight representations, and any $\lambda \in \mathbb{C}$ appears as a weight. In the next proposition, which summarizes these observations, we will see that integrality of λ is only necessary to obtain representations with dimension less than N (rather than ∞, in the generic case).

Proposition 5.2.1

a) For $U_q^{\mathrm{red}}(s\ell_2)$, with $q^2 = e^{2\pi i \frac{n}{N}}$, any highest weight $\lambda \in \mathbb{C}$ corresponds to an irreducible representation which is given, in the standard basis $\{v_\ell\}_{\ell=0,\dots,p_\lambda-1}$ for highest *weight representations, by*

$$\begin{aligned} hv_\ell &= (\lambda - 2\ell)v_\ell \\ fv_\ell &= v_{\ell+1} \qquad\qquad (5.8) \\ ev_\ell &= (\ell)_q(\lambda + 1 - \ell)_q\, v_{\ell-1}\,, \end{aligned}$$

where the dimension $p_\lambda, 1 \leq p_\lambda \leq N$*, is* N *if* $n\lambda$ *is non-integral and is determined by* $np_\lambda \equiv n(\lambda+1) \bmod N$ *if* $\lambda \in \frac{1}{n}\mathbb{Z}$.

b) The trivial representation is identified with $\lambda = 0$*, and the highest weight,* $\lambda^\vee$*, of the conjugate representation* $\rho_\lambda^\vee = \rho_{\lambda^\vee}$ *is given by*

$$\lambda^\vee = 2(p_\lambda - 1) - \lambda\,. \tag{5.9}$$

A sesquilinear form on V_λ *exists only for* $\lambda \in \mathbb{R}$*. Moreover, there is an algebra automorphism,* T*, with*

$$T(e) = e\,, \quad T(f) = f\,, \quad T(h) = h + 2\frac{N}{n}\,, \tag{5.10}$$

such that there is an invertible mapping $F_\lambda : V_\lambda \to V_{\lambda+2\frac{N}{n}}$*, with*

$$\rho_{\lambda+2\frac{N}{n}}(a)F_\lambda = F_\lambda\rho_\lambda(T(a))\,.$$

To prove a), we only need the commutators (4.89), and the fact that $(x)_q = 0$ whenever $x \in \frac{N}{n}\mathbb{Z}$. The irreducible representations are then obtained in the usual way. ☐

From the automorphism T, defined in (5.10), we can find all irreducible and unitary representations, by only looking at those with $\lambda \in \left[0, 2\frac{N}{n}\right)$. On the center $\mathfrak{z}, T(C) = C$ and $T(P) = e^{2\pi i\frac{N}{n}}P$, so that $T^n \restriction \mathfrak{z} = id$. Hence the representations belonging to λ and $\lambda + 2N$, yield the same values of Casimirs in $\mathfrak{V}$. More precisely, we have the following result:

Proposition 5.2.2

a) Let $\mathfrak{V}$ *be the variety described in (4.91). Then the mapping* $\mathbb{C} \to \mathfrak{V} \subset \mathbb{C}^2$*, assigning to each highest weight the corresponding Casimir values*

$$\lambda \mapsto (P,Q) = \left(e^{i\pi\lambda}, \left(\frac{\lambda+1}{2}\right)_q^2\right) \tag{5.11}$$

is surjective, and can be defined on $\mathbb{C}/2N\mathbb{Z}$.

b) $\mathbb{C}/2N\mathbb{Z} \to \mathfrak{V}$ *identifies exactly* $n(N-1)$ *pairs of points,* $\lambda_+ \sim \lambda_-$*, given by*

$$\begin{aligned} \lambda_\pm + 1 &= \pm a + \frac{N}{n}b \bmod 2N\,, \\ a &= 1\ldots(N-1)\,, \quad b = 0\ldots n-1\,, \end{aligned} \qquad (5.12$$

and is injective for all other values of λ*, so that* V *is an infinitely long tube wit* $n(N-1)$ *singular points.*

c) The subvariety describing representations which admit sesquilinear forms is de scribed by $\mathbb{R}/2N\mathbb{Z} \to \mathfrak{V}_{\text{real}}$*. Thus* $\mathfrak{V}_{\text{real}}$ *can be identified with the lattice edges of*

$$\left\langle \frac{1}{2}\left(\frac{1}{n}, \frac{1}{N}\right), \frac{1}{2}\left(\frac{1}{n}, -\frac{1}{N}\right)\right\rangle_{\mathbb{Z}} \bmod \mathbb{Z} \times \mathbb{Z}$$

on the upper half of the torus $T^2 = \mathbb{R}^2/\mathbb{Z} \times \mathbb{Z}$.

The crucial point of Proposition 5.2.2 is that irreducible representations cannot be dis tinguished completely by their Casimir values. A point in $\mathfrak{V}$ only determines the set o representations that appear as quotients, e.g. in Jordan-Hölder series, of indecomposabl representations. Note that, for the dimensions, we have $p_{\lambda_+} + p_{\lambda_-} = N$, and the successor $(\lambda+2)$, of a highest weight λ is also the lowest weight of an irreducible representation, wit the same values of Casimirs. For non-singular values of Casimirs, the picture become much simpler.

Lemma 5.2.3 *Suppose* W *is a representation space of* $U_q(s\ell_2)$ *on which* h *is diagona and* (P,Q) *has only non-singular eigenvalues in* W*, i.e., all highest weights* λ *occuring i* W *are in* $\left(\mathbb{C}\backslash\frac{1}{n}\mathbb{Z}\right) \dot\cup \left(\frac{N}{n}\mathbb{Z} - 1\right)$*. Then* W *is completely reducible.*

To show this, we restrict our attention to a single Casimir value, (P,Q), such that the set of highest weights is in $\{\lambda + 2kN\}_{k\in\mathbb{Z}}$, for some λ. If $hv = \lambda v$, for some v, then v is a highest-weight vector. Otherwise, we could find some s, $1 \leq s \leq N-1$, with $e^s v$ being a non-zero highest weight vector. Since its weight $(\lambda + 2s)$ is not contained in the above set; this is impossible. With a similar statement for lowest-weight vectors, and since h is diagonal, W decomposes into

$$W = \sum_n{}^{\oplus} W_{\lambda+2nN} \,. \tag{5.13}$$

The invariant subspaces are

$$W_\lambda = \sum_{k=0}^{N-1}{}^{\oplus} \ker(h - (\lambda - 2k)) \,, \tag{5.14}$$

for which we have

$$\ker f \restriction W_\lambda = \ker(h - (\lambda - 2(N-1))) \quad \text{and} \quad \ker e \restriction W_\lambda = \ker(h - \lambda) \,. \tag{5.15}$$

Thus all weight spaces in W_λ have the same dimension, so that, for some basis $\{v_1, \ldots, v_r\}$ of $\ker(h-\lambda)$, we have the direct sum decomposition $W_\lambda = \sum_{\ell=1}^{r}{}^{\oplus} V_\ell$, V_ℓ being the irreducible representation $\langle v_\ell, \ldots, f^{N-1} v_\ell \rangle$. □

Next we state a result on unitarity.

Proposition 5.2.4

a) If the representation on V_λ is unitary, then $\lambda \in \mathbb{R}$, and the representation on $V_{\lambda+\frac{2N}{n}}$ is also unitarizable.

b) For $\lambda = \frac{N+s-n}{n}$, with $s \in [-N, N)$, V_λ is unitarizable iff

$$\begin{gathered} \text{either} \quad s \in [-1,1] \,, \quad \text{or} \quad s = n \,, \quad \text{or} \quad s = (-1)^\ell (n_\ell - n_{\ell+1} r) \,, \\ r = 1, \ldots, p_{\ell+2} \,, \quad \ell = -1, \ldots, f-1 \,, \end{gathered} \tag{5.16}$$

where n_ℓ and p_ℓ are defined by the Euclidian algorithm:
$N = p_1 n + n_1 \,;\, n = p_2 n_1 + n_2, \ldots, n_{k-1} = p_{k+1} n_k + n_{k+1}, \ldots, n_{f-1} = p_{f+1} n_f + 1,$
with $n_k > n_{k+1}$, $N = n_{-1}$, $n = n_0$.

c) There exist unitary representations for all singular points in $\mathfrak{D}$, i.e., all dimensions $p_\lambda = 1, \ldots, N$, only if $n = 1$. In this case V_λ is unitarizable for

$$\lambda \in \{0, 1, \ldots, N-2\} \cup (N-2, N] \quad (\mathrm{mod} 2N) \,.$$

The proof is elementary, although somewhat tedious, and will not be reproduced here, see [6].

In the case of unitarity, we define an orthonormal basis $\{\xi^p_m\}$ with $p = p_\lambda$, and $m = -j, -j+1, \ldots, j$, with $2j+1 = p_\lambda$, which is obtained from (5.2) by setting

$$\xi^p_{j-\ell} = \frac{1}{(\ell)_q! \sqrt{\binom{p}{\ell}_q}} v_\ell \,.$$

The representation then has the form

$$\begin{aligned} \left(h - k_\lambda \frac{N}{n}\right) \xi^p_m &= 2m\xi^p_m \,, \\ e\xi^p_m &= \sqrt{(j-m)_q (j+m+1)_q}\, \xi^p_{m+1} \,, \\ f\xi^p_m &= \sqrt{(j+m)_q (j-m+1)_q}\, \xi^p_{m-1} \,, \end{aligned} \tag{5.17}$$

where we have set $k_\lambda = \frac{n}{N}(\lambda + 1 - p_\lambda) \in \mathbb{Z}$.

5.3 Decomposition of Tensor Product Representations

We present a basic result on the tensor product decomposition of two irreducible, integral highest-weight representations of $U_q^{\mathrm{red}}(s\ell_2)$, for q a root of unity, using non-degenerate bilinear forms. We discuss the structure of the indecomposable representations arising in this procedure and state the fusion rules for irreducible representations with non-zero q-dimensions.

In this section, we investigate the decomposition of a tensor product of irreducible representations into its indecomposable parts. If, for two highest weights λ and μ, $\lambda + \mu \notin \frac{1}{n}\mathbb{Z}$, then, using the Casimir P from (4.90), with $\Delta(P) = P \otimes P$, we deduce from Lemma 5.2.3 complete reducibility, so that

$$V_\lambda \otimes V_\mu = \sum_{n=0}^{\min(p_\lambda, p_\mu)}{}^{\oplus} \; V_{\lambda+\mu-2n} \,. \tag{5.18}$$

In the case where $\lambda, \mu \notin \frac{1}{n}\mathbb{Z}$, but $\lambda + \mu \in \frac{1}{n}\mathbb{Z}$, the decomposition of $V_\lambda \otimes V_\mu$ is similar to the one where the highest weights belong to $\frac{1}{n}\mathbb{Z}$. The interesting case is the one where $\lambda, \mu \in \frac{1}{n}\mathbb{Z}$. We use the basis (5.17), regardless of unitarity, with $k_\lambda \equiv \frac{n}{N}(\lambda + 1 - p_\lambda) = 0$. All other decompositions can be generated from the automorphisms $T_\pm^{\frac{1}{2}}(e) = \pm ie$; $T_\pm^{\frac{1}{2}}(f) = \pm if$; $T_\pm^{\frac{1}{2}}(h) = h + \frac{N}{n}$.

Our main result is that the $\widehat{su}_2$-fusion rules of rational conformal field theory and of SU_2-Chern-Simons gauge theory can be recovered from the representation theory of $U_q^{\text{red}}(s\ell_2)$, in the following algebraic sense.

Theorem 5.3.1 *The tensor product of two irreducible representation-spaces* V_{p_1} *and* V_{p_2}*, with heighest weights* $\lambda_i = 2j_i = p_i - 1$, $1 \le p_i \le N-1$, *and with the action of* $U_q^{\text{red}}(s\ell_2)$ *defined in terms of the comultiplication, has a decomposition into invariant subspaces given by*

$$V_{p_1} \otimes V_{p_2} = \sum_{\substack{i=|p_1-p_2|+1 \\ i \equiv p_1+p_2+1 \bmod 2}}^{\min(p_1+p_2-1, 2N-1-(p_1+p_2))}{}^{\oplus} \; V_i \ominus \sum_{\substack{i=2N+1-(p_1+p_2) \\ i \equiv p_1+p_2+1 \bmod 2}}^{N}{}^{\oplus} \; W_i \,. \tag{5.19}$$

The spaces W_i *are indecomposable subspaces, with* $W_N = V_N$ *and* $\dim W_i = 2N$, *for* $i < N$, *on which* $\left[Q - (\frac{i}{2})_q^2\right]^2$, *but not* $\left[Q - (\frac{i}{2})_q^2\right]$, *vanishes.*

In the proof of Theorem 5.3.1, we make strong use of the fact that the bilinear form $(.,.)$, introduced in Lemma 5.1.1, naturally extends to tensor products, because θ commutes with Δ, (Lemma 4.4), and is non-degenerate. The derivation of the decomposition amounts to an explicit construction of the representation spaces W_i in a natural basis. The first step is the computation of all highest weight vectors and of their squares with respect to $(.,.)$.

Lemma 5.3.2

a) For every i, $i = 2j+1$, *with*

$$\begin{gathered} |\,p_1 - p_2\,| + 1 \le i \le p_1 + p_2 - 1\,, \\ i \equiv p_1 + p_2 + 1 \bmod 2 \end{gathered} \tag{5.20}$$

there exists exactly one vector, ξ_j^i, *of highest weight in* $V_{p_1} \otimes V_{p_2}$, *i.e.,*

$$h\xi_j^i = (i-1)\xi_j^i \,, \qquad \text{and} \qquad e\xi_j^i = 0\,. \tag{5.21}$$

The ξ_j^i *form a basis of* $\ker e$.

b) The squares $\left(\xi_j^i, \xi_j^i\right)$ *vanish iff*

$$2N + 1 - (p_1 + p_2) \le i \le N - 1\,.$$

In order to determine the vectors ξ_j^i, we express them in the basis $\xi_m^{p_1} \otimes \xi_{j-m}^{p_2}$ with coefficients α_m^i:

$$\xi_j^i = \sum_{n=0}^{j_1+j_2-j} \alpha_n^i \, \xi_{j_1-n}^{p_1} \otimes \xi_{j-j_1+n}^{p_2} \,.$$

From $\Delta(e)\xi_j^i = 0$ we find the recursion

$$\begin{aligned} 0 \;&=\; q^{-(j+i)} \alpha_{n+1}^i \sqrt{(n+1)_q (2j_1 - n)_q} \\ &\quad + \alpha_n^i \sqrt{(j_2 + j_1 - j - n)_q (1 + j_2 + j - j_1 + n)_q}\,. \end{aligned} \tag{5.22}$$

Solving this in terms of $\alpha_i \equiv \alpha_o^i q^{j_1(j+1)} \sqrt{\binom{j_1+j_2+j}{2j_1}_q}$, we find for the highest weight vector

$$\xi_j^i = \alpha_i \sum_{n=0}^{j_1+j_2-j} (-1)^n \, q^{-(j_1-n)(j+1)} \sqrt{\frac{(2j_1-n)_q!(j_2+j-j_1+n)_q!}{(n)_q!(j_1+j_2-j-n)_q!}} \tag{5.23}$$

$$\xi_{j_1-n}^{p_1} \otimes \xi_{j-j_1+n}^{p_2}$$

with $j_i = \frac{p_i - 1}{2}$.

This recursion can only be solved for i in the range given in (5.20), so that we have found all vectors of highest weight.

The expression for the square $\left(\xi^i_j, \xi^i_j\right)$ is obtained by use of the q-analogue binomial identity

$$\binom{\alpha+\beta}{n}_q = \sum_{k=0}^{n} q^{\pm(\alpha k-(n-k)\beta)} \binom{\alpha}{k}_q \binom{\beta}{n-k}_q , \tag{5.24}$$

with

$$\binom{\alpha}{n}_q = \frac{(\alpha)_q \dots (\alpha-n+1)_q}{(n)_q \dots (1)_q}, \quad \text{for} \quad \alpha,\beta \in \mathbb{R},\ n \in \mathbb{N}. \tag{5.25}$$

It is given by

$$\begin{aligned} &\left(\xi^i_j, \xi^i_j\right) = \\ &\qquad \alpha_i^2 q^{(j_2(j_2+1)-j_1(j_1+1)-j(j+1))}(j_1-j_2+j)_q!(j_2-j_1+j)_q! \\ &\qquad\qquad \binom{j_1+j_2+j+1}{j_1+j_2-j}_q . \end{aligned} \tag{5.26}$$

To show Lemma 5.3.2 b) it is now sufficient to find the zeros of the q-analogue binomial coefficients.

The non-degeneracy of the bilinear form $(.,.)$ now enables us, to assign to each vector ξ^i_j with $(\xi^i_j, \xi^i_j) = 0$ an indecomposable subspace W_i within which it is contained. In contrast to the classical case, the ξ^i_j are no longer cyclic with respect to W_i. However, a candidate for a cyclic vector of W_i is given in the next lemma.

Lemma 5.3.3

a) *The square, with respect to $(.,.)$, of a vector of highest weight, ξ^i_j, in $V_{p_1} \otimes V_{p_2}$ is zero, iff there exists a vector $\tilde{\xi}^i_j \in V_{p_1} \otimes V_{p_2}$, such that*

$$\text{and} \qquad \begin{aligned} h\tilde{\xi}^i_j &= (i-1)\tilde{\xi}^i_j \\ \xi^i_j &= fe\tilde{\xi}^i_j . \end{aligned} \tag{5.27}$$

b) *ξ^i_j and $\tilde{\xi}^i_j$ can be chosen uniquely, up to a sign, by imposing the normalization conditions*

$$\text{and} \qquad \begin{aligned} \left(\xi^i_j, \tilde{\xi}^i_j\right) &= 1 , \\ \left(\xi^i_j, \xi^i_j\right) &= \left(\tilde{\xi}^i_j, \tilde{\xi}^i_j\right) = 0 . \end{aligned} \tag{5.28}$$

c) *The subspace W_i, generated by $\tilde{\xi}^i_j$, also contains ξ^{2N-i}_{N-1-i} and is the desired component of $V_{p_1} \otimes V_{p_2}$ in Theorem 5.3.1.*

Proof.

a) One easily derives from the invariance of the bilinear form $(.,.)$, that if (5.27) holds for some vectors $\tilde{\xi}^i_j$ and ξ^i_j, the square of ξ^i_j is zero:

$$\left(\xi^i_j, \xi^i_j\right) = \left(\xi^i_j, fe\tilde{\xi}^i_j\right) = \left(e\xi^i_j, e\tilde{\xi}^i_j\right) = 0 .$$

To prove the converse, we can assume, for ξ^i_j with $\left(\xi^i_j, \xi^i_j\right) = 0$, that by Lemma 5.3.2 b) $2N+1-(p_1+p_2) \leq i \leq N-1$.

Since $(.,.)$ is non-degenerate, and since both h and Q are symmetric and commute, there has to be a vector $\tilde{\xi}^i_j$ that belongs to the same generalized eigenspaces of h and Q as ξ^i_j, but has nonvanishing scalar-product with ξ^i_j, i.e.,

$$\begin{aligned} h\tilde{\xi}^i_j &= (i-1)\tilde{\xi}^i_j \\ \left(Q - \left(\frac{i}{2}\right)_q^2\right)^a \tilde{\xi}^i_j &= (fe)^a \tilde{\xi}^i_j , \end{aligned} \tag{5.29}$$

$$\text{for } a \text{ sufficiently large, and} \qquad \left(\xi^i_j, \tilde{\xi}^i_j\right) \neq 0 .$$

In the following line of arguments, we will see, that any such $\tilde{\xi}^i_j$ has the desired property (5.27).

From the relationship of Casimir values with highest weights, as computed in Proposition 5.2.2 b), and from the bounds on the weights in (5.20), we see that the only highest weight vector, having the same Casimir values as $\tilde{\xi}^i_j$ and ξ^i_j, is ξ^{2N-i}_{N-j-1}. Since we have $N+1 \leq 2N-i \leq p_1+p_2-1$, this vector has non-zero square. As $e^s\tilde{\xi}^i_j$ has to be a non-zero highest weight vector, for some $1 \leq s < N$, we immediately conclude from the previous observations, that

$$e^{N-i}\tilde{\xi}^i_j = \alpha\xi^{2N-i}_{N-j-1} , \tag{5.30}$$

for some $\alpha \neq 0$. The case $s = 0$ is excluded, because $\tilde{\xi}^i_j$, having, by (5.29), a non-zero scalar product with ξ^i_j, cannot be proportional to ξ^i_j. Applying $Q - (\frac{i}{2})^2_q$ to the vector $\xi^{2N-i}_{N-j-1} = \frac{1}{\alpha} e^{N-i} \tilde{\xi}^i_j$, we find $e^{N-i} f e \tilde{\xi}^i_j = 0$.

The argument used above now shows that $e^s f e \tilde{\xi}^i_j$ is a non-zero highest-weight vector, iff $s = 0$ and $fe\tilde{\xi}^i_j \neq 0$. Finally we show that $fe\tilde{\xi}^i_j \neq 0$, which, for some suitable rescaling, implies $\xi^i_j = fe\tilde{\xi}^i_j \neq 0$. Assuming the opposite, $e\tilde{\xi}^i_j$ should be a lowest-weight vector which has, by calculations similar to the ones at the beginning of the proof, vanishing square with respect to $(.,.)$. From Lemma 5.1.1 for lowest-weight representations, we conclude that $(.,.)$ vanishes identically on the sub-representation generated by the lowest weight vector $e\tilde{\xi}^i_j$. This contradicts, with (5.28) and

$$\alpha^2 \left(\xi^{2N-i}_{N-j-1}, \xi^{2N-i}_{N-j-1}\right) = \left(e^{N-i}\tilde{\xi}^i_j, e^{N-i}\tilde{\xi}^i_j\right) = 0,$$

the fact, that ξ^{2N-i}_{N-j-1} has non zero square.

b) We suppose that there are two vectors obeying (5.27). Then their difference, δ, has to be a multiple of ξ^i_j. Otherwise, we have from $fe\delta = 0$, that $e\delta$ is a non-zero lowest-weight vector with zero square. By the same reasoning as for $e\tilde{\xi}^i_j$ in part a) this is impossible. The proof of statement b) concerning the uniqueness is now just a matter of scaling and adding.

c) So far, we have constructed a direct sum of cyclic subspaces in $V_{p_1} \otimes V_{p_2}$, generated by vectors ξ^i_j, for $\mid p_1 - p_2 \mid +1 \leq i \leq \min(p_1 + p_2 - 1, 2N - 1 - (p_1 + p_2))$, or $i = N$, and by $\tilde{\xi}^i_j$, for $2N + 1 - (p_1 + p_2) \leq i \leq N - 1$, $(i \equiv p_1 + p_2 + 1 \bmod 2)$.

In both cases it can be verified, that $f^j \xi^i_j$ is in the kernel of e, by using the commutators (4.89). For $i \leq N - 1$, its weight is $-2(j+1)$; but by Lemma 5.3.2 a), there do not exist highest-weight vectors with weights below $\mid p_1 - p_2 \mid$, so that we have

$$f^i \xi^i_j = 0 . \qquad (5.31)$$

Hence if $\left(\xi^i_j, \xi^i_j\right) \neq 0$, i.e., i satisfies the restriction in the first summand of (5.19), ξ^i_j generates an irreducible subspace $V_i = U^{\mathrm{red}}_q(s\ell_2)\xi^i_j$, on which the bilinear form $(.,.)$ is non-degenerate. We therefore have $V_i \cap V_i^{\perp} = 0$ and can complement V_i by $V_i^{\perp}$, i.e., $V_{p_1} \otimes V_{p_2} = V_i \ominus V_i^{\perp}$.

This yields a decomposition

$$V_{p_1} \otimes V_{p_2} = \sum_{\substack{i=|p_1-p_2|+1 \\ i \equiv p_1+p_2+1 \bmod 2}}^{\min(p_1+p_2-1, 2N-1-(p_1+p_2))} {}^{\oplus} \quad V_i \ominus \sum_{\substack{i=2N+1-(p_1+p_2) \\ i \equiv p_1+p_2+1 \bmod 2}}^{N} {}^{\oplus} \quad W_i', \qquad (5.32$$

where the W_i' have the same Casimir values as the subspaces W_i generated by th $\tilde{\xi}^i_j$. In order to prove c), without constructing W_i explicitly, we want to show tha W_i'/W_i does not contain any vectors of highest weight, and therefore has to be zero Suppose $[\xi_r]$ is of highest weight in W_i'/W_i, with weight

$$\lambda_r \in \{-2N + i - 1, -(i+1), i - 1, 2N - i - 1\} .$$

A representative ξ_r in W_i', with the same weight, cannot be of highest weight itself because all highest weight vectors are already contained in W_i, so that again $e^s\xi_r$ i of highest weight for some $1 \leq s \leq N - 1$. The only combinations left are:

$$e^i \xi_r = \xi^i_j \qquad \text{with} \qquad \lambda_r = -(i+1) \qquad (5.33)$$

or

$$e^{N-i} \xi_r = \xi^{2N-i}_{N-j-1} \qquad \text{with} \qquad \lambda_r = i - 1 . \qquad (5.34)$$

In the second case (5.34), we have $e^{N-i}\left(\xi_r - \frac{1}{\alpha}\tilde{\xi}^i_j\right) = 0$, so that, by a similar reasoning, $\xi_r - \frac{1}{\alpha}\tilde{\xi}^i_j$ is of highest weight. We then have $\xi_r = \frac{1}{\alpha}\tilde{\xi}^i_j + \beta\xi^i_j$, which is impossible, since $[\xi_r] \neq 0$.

In order to exclude the second case, we first note that

$$e f^i \tilde{\xi}^i_j = f^i e \tilde{\xi}^i_j = f^{i-1}(fe)\tilde{\xi}^i_j = f^{i-1}\xi^i_j = \gamma \xi^i_{-j} , \qquad (5.35)$$

for some $\gamma \neq 0$. Since $f^i\xi^i_j = 0$, ξ^i_{-j} is the lowest-weight vector of the subrepresentation generated by ξ^i_j. Furthermore we have that $e^{i-1}\xi^i_{-j} = \gamma\xi^i_j$, so that, by (5.33) we have that $e^{i-1}\left(\xi^i_{-j} - \gamma e \xi_r\right) = 0$. Thus $\xi^i_{-j} = \gamma e \xi_r$. By (5.35), this implies tha $e\left(\gamma^2\xi_r - f^i\tilde{\xi}^i_j\right) = 0$, and again, with Lemma 5.3.2 a), $\gamma^2\xi_r = f^i\tilde{\xi}^i_j$. This shows tha $[\xi_r] = 0$, completing the proof. □

We complete our analysis on the decomposition of tensor products with an explicit de scription of the representation spaces W_i, equipped with natural bases determined by ξ and $\tilde{\xi}^i_j$ in Lemma 5.3.3, with normalization (5.28).

The space W_i is spanned by $2N$ vectors

$$\begin{aligned} \xi^i_m,\ \tilde{\xi}^i_m, \qquad m &= j, (j-1) \ldots -j\,; \\ \varphi^{i+}_m,\ \varphi^{i-}_m, \qquad m &= j', (j'-1) \ldots -j'\,; \end{aligned}$$

with $j = \frac{i-1}{2}$ and $j' = \frac{N-i-1}{2}$.

The representation is given by

$$\begin{aligned} h\xi^i_m &= 2m\xi^i_m, \qquad h\tilde{\xi}^i_m = 2m\tilde{\xi}^i_m, \\ h\varphi^{i\pm}_m &= (\pm N + 2m)\varphi^{i\pm}_m\,; \end{aligned} \tag{5.36}$$

$$\begin{aligned} f\tilde{\xi}^i_m &= \sqrt{(j+m)_q(j-m+1)_q}\,\tilde{\xi}^i_{m-1}, \\ f\xi^i_m &= \sqrt{(j+m)_q(j-m+1)_q}\,\xi^i_{m-1}, \qquad m \geq -(j-1)\,; \\ f\varphi^{i\pm}_m &= \sqrt{(j'+m)_q(j'-m+1)_q}\,\varphi^{i\pm}_{m-1}, \qquad m \geq -(j'-1)\,; \end{aligned} \tag{5.37}$$

$$\begin{aligned} e\xi^i_m &= \sqrt{(j+m+1)_q(j-m)_q}\,\xi^i_{m+1}, \\ e\tilde{\xi}^i_m &= \sqrt{(j+m+1)_q(j-m)_q}\,\tilde{\xi}^i_{m+1} \\ &\quad + \frac{1}{\sqrt{(j+m+1)_q(j-m)_q}}\,\xi^i_{m+1}, \qquad m \leq (j-1)\,; \\ e\varphi^{i\pm}_m &= -\sqrt{(j'+m+1)_q(j'-m)_q}\,\varphi^{i\pm}_{m+1}, \qquad m \leq (j'-1)\,; \end{aligned} \tag{5.38}$$

and

$$\begin{aligned} f\xi^i_{-j} &= 0, & f\tilde{\xi}^i_{-j} &= \varphi^{i-}_{j'}, \\ e\xi^i_j &= 0, & e\tilde{\xi}^i_j &= \varphi^{i+}_{-j'}, \\ f\varphi^{i+}_{-j'} &= \xi^i_j, & e\varphi^{i-}_{j'} &= \xi^i_{-j}, \\ e\varphi^{i+}_{j'} &= 0, & f\varphi^{i-}_{-j'} &= 0. \end{aligned} \tag{5.39}$$

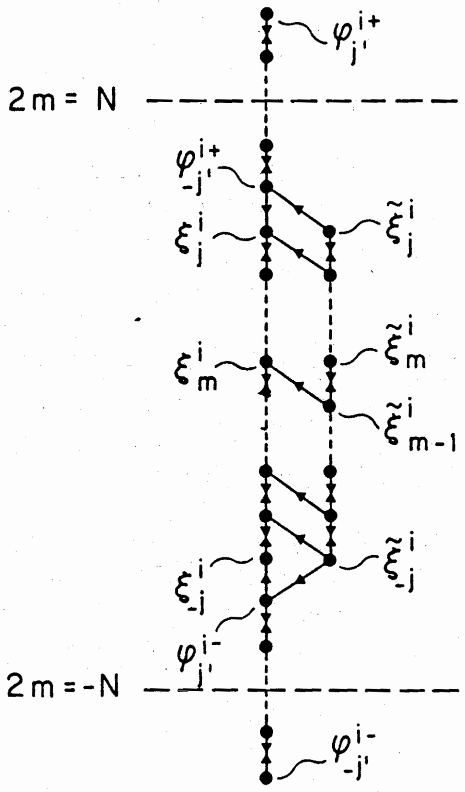


Figure 5.1

The representation W_i is visualized in Figure 5.1. Each dot marks a basis vector, its height in the diagram indicating its weight. The arrows in upward- or downward direction stand for non-zero matrix elements of the step operators e, f, respectively.

For a better understanding, we introduce the Casimir element

$$\begin{aligned} D_i = \left[Q - \left(\frac{i}{2}\right)^2_q\right] &= fe - \left(\frac{h}{2}+j+1\right)_q \left(j - \tfrac{h}{2}\right)_q \\ &= ef - \left(\frac{h}{2}+j\right)_q \left(j+1-\tfrac{h}{2}\right)_q . \end{aligned} \tag{5.40}$$

By construction, we have that $\xi^i_j = fe\tilde{\xi}^i_j = D_i\tilde{\xi}^i_j$. So if we inductively define $\tilde{\xi}^i_m$ by (5.37) the action of f on ξ^i_m is determined by $\xi^i_m = D_i\tilde{\xi}^i_m$. Having $\xi^i_{-j} = ef\tilde{\xi}^i_{-j}$, the $\varphi^{i\mp}_{\pm j'}$ can be consistently defined by equation (5.39). Notice that the ξ^i_m are the basis of an irreducible subrepresentation and that D_i is zero on ξ^i_m and $\varphi^{i\mp}_{\pm j'}$. This is now used for an inductive definition of φ^{i+}_m as in (5.38) and φ^{i-}_m as in (5.37). By comparison with the proof of Lemma 5.3.3, we see that, e.g., $\varphi^{i+}_{j'}$ is proportional to $\tilde{\xi}^{N-i}_{N-1-j}$.

In this basis the form $(.,.)$ has the values

$$\begin{aligned} \left(\xi^i_m, \tilde{\xi}^i_m\right) &= 1, \qquad (\varphi^{i\pm}_m, \varphi^{i\pm}_m) = (-1)^{j'\pm m} \\ \left(\tilde{\xi}^i_m, \tilde{\xi}^i_m\right) &= \sum_{k=1}^{j-m} \frac{1}{(k)_q(i-k)_q} \end{aligned} \tag{5.41}$$

and on all other pairs $(.,.)$ vanishes.

5.4 Fusion Rules, and q-Dimensions: Selecting a List of Physical Representations

In order to show that the tensor product decomposition of $U^{\text{red}}_q(s\ell_2)$ defines a fusion-rule algebra, in the sense of Section 2.5.1, we need to verify associativity, i.e., we have to show that the excluded representations are an ideal under forming tensor products. This is done using a condition introduced by Pasquier and Saleur [52] which characterizes saturated representations of the Borel algebras of $U_q(s\ell_2)$. It is shown that this criterion entails the vanishing of q-dimensions of representations for which it holds. Our criterion an the vanishing of q-dimensions are, in fact, equivalent for indecomposable representation The group-like elements of $U^{\text{red}}_q(s\ell_2)$ are used to define characters which diagonalize th fusion rules, and the so-called S-matrix is expressed in terms of q-numbers. We defin a subset of representations which will be used in our duality theory.

It was already pointed out in [52] that the representation spaces W_i have the propert that $\ker e = \operatorname{im} e^{N-1}$, which we will abbreviate in the following by (E). It is conclude from a simple calculation for $V_2 \otimes W$ and an iteration of tensorproducts, that if (E) hold on some space W, it is also true for $V_p \otimes W$.

Lemma 5.4.1

i) If (E) holds on some module W and $W = A \oplus B$, then (E) holds on A and B.

ii) (E) holds on V_p only if $p = N$.

iii) If (E) holds on W then it also holds on $V_p \otimes W$, $p = 1, \ldots, N$.

Part i) is a trivial consequence of the definition of direct sums of modules, and ii) i immediately checked for the representations given in Proposition 5.2.1. We show ii first for $p = 2$. Let $v = \xi^2_{\frac{1}{2}} \otimes w_+ + \xi^2_{-\frac{1}{2}} \otimes w_-$ be in $\ker e$, with $hw_\pm = (2m \mp 1)w_\pm$, s that $\Delta(h)v = 2mv$. Then $0 = \Delta(e)v = q^{\frac{1}{2}}\xi^2_{\frac{1}{2}} \otimes ew_+ + q^{-(m+\frac{1}{2})}\xi^2_{\frac{1}{2}} \otimes w_- + q^{-\frac{1}{2}}\xi^2_{-\frac{1}{2}} \otimes ew$ implies $ew_- = 0$ and $ew_+ = q^{-(m+1)}w_-$. By hypothesis $w_- = q^{\frac{N-1}{2}}e^{N-1}y$, so that $\delta v :=$ $\Delta(e)^{N-1}\left(\xi^2_{-\frac{1}{2}} \otimes y\right) = \xi^2_{\frac{1}{2}} \otimes w'_+ + \xi^2_{-\frac{1}{2}} \otimes w_-$ for some w'_+, where we use that the $(0,n$ graded summand of $\Delta(e)^n$ is $q^{n-\frac{h}{2}} \otimes e^n$. Hence it is sufficient to show, that $v' \in \operatorname{im} e$, wi $v' = v - \delta v = \xi^2_{\frac{1}{2}} \otimes \left(w_+ - w'_+\right)$, i.e., we can assume $w_- = 0$. In this case we have fro $\Delta(e)v = 0$ that $ew_+ = 0$, thus $w_+ = q^{-\frac{N-1}{2}}e^{N-1}x$ and $w = \Delta(e)^{N-1}\left(\xi_{\frac{1}{2}} \otimes x\right)$. In order show iii) for general p, we use the fact, that $V_p \otimes W$ occurs in $V_2^{\otimes(p-1)} \otimes W$ as a dire summand and apply i).

This statement only depends on the representations of the Borel-algebra generat by e and k. For these, however, the tensor product decomposition is solved by a sim basis transformation, showing immediately the invariance of "saturated" representatio as proposed in iii). This, together with the fact that if a direct sum satisfies (E) then all the summands do, makes the convenience of working with this property evident. All of this can be understood from a more general representation-theoretical point of view in a very natural way [6].

In the decomposition given in (5.19), (E) is true on the right summands and false on the left ones, so that we are led to the definition of the fusion rules

$$N_{p_1 p_2, i} = \begin{cases} 1 \quad \text{if } \mid p_1 - p_2 \mid + 1 \leq i \\ \qquad\qquad\qquad\quad \leq \min(p_1 + p_2 - 1\,, 2N - 1 - (p_1 + p_2)) \\ \qquad\qquad\qquad\quad i \equiv p_1 + p_2 + 1 \bmod 2 \\ 0\,, \quad \text{else}\,. \end{cases} \tag{5.42}$$

The fusion matrix N_j is then defined in the usual way, i.e., $(\mathrm{N}_j)_{ik} := \mathrm{N}_{ij,k}$.

These fusion rules show that the list of algebraic objects producing the combinatorics of the A_{N-1} series, beginning with $\widehat{su}_2$-symmetric models in rational conformal field theory, and continuing with SU_2-Chern-Simons-gauge-theory and towers of algebras arising in local quantum theory, can be completed with the quantum group $U_q(s\ell_2)$, with $q = \exp(i\pi/N)$.

In order to compute the eigenvalues of N_j, we introduce quantum group characters.

Lemma 5.4.2 *If we define the r-th q-dimension of a representation space V, d^r_V, as the character*

$$d^r_V = tr_V\left(q^{rh}\right)\,, \tag{5.43}$$

then

a) for the irreducible representations V_p, with highest weight $\lambda = p - 1 \in \mathbb{Z}$, these characters have the values

$$d^r_p = \frac{(rp)_q}{(r)_q}\,; \tag{5.44}$$

b) $d^1_p = (p)_q$ is positive, for all $p = 1 \ldots N-1$, if and only if $n = 1$, as for unitary representations.

In the next lemma we draw the connection of vanishing q-dimensions and property (E).

Lemma 5.4.3 *If* $\ker \mathrm{e} = \mathrm{im}\mathrm{e}^{N-1}$, *i.e., if (E) holds on some representation space V, and if I intertwines V with itself, then*

$$tr_V\left(q^{rh}I\right) = 0\,, \qquad \text{for} \qquad r = 1, \ldots, N-1\,.$$

Proof. We derive from (E) by induction, that $V_\ell = \ker \mathrm{e}^\ell = \mathrm{im}\mathrm{e}^{N-\ell}$ for all ℓ, with

$$0 = V_o \subset V_1 \ldots \subset V_{N-1} \subset V_N = V\,.$$

Because of $\left(q^{rh}I\right)(V_\ell) \subset V_\ell$, the trace can now be rewritten as a sum over characters on the successive quotients:

$$tr_V\left(q^{rh}I\right) = \sum_{\ell=0}^{N-1} tr_{V_{\ell+1}/V_\ell}\left(q^{rh}I\right)\,. \tag{5.45}$$

Obviously, e maps $V_{\ell+1}$ onto V_ℓ, with $\mathrm{e}^{-1}(V_{\ell-1}) = V_\ell$. We therefore have an isomorphism e^*, with

$$\mathrm{e}^* : V_{\ell+1}/V_\ell \to V_\ell/V_{\ell-1} \tag{5.46}$$

with

$$[h, \mathrm{e}^*] = 2h \qquad \text{and} \qquad [I, \mathrm{e}^*] = 0\,.$$

Hence

$$tr_{V_{\ell+1}/V_\ell}\left(q^{rh}I\right) = q^{-2r} tr_{V_\ell/V_{\ell-1}}\left(q^{rh}I\right)\,,$$

leading to

$$tr_V\left(q^{rh}I\right) = \left(\sum_{\ell=0}^{N-1} q^{-2r\ell}\right) tr_{V_1/V_o}\left(q^{rh}I\right) = 0\,.$$

□

With these tools in our hands, we are now in a position to compute the eigenvalues of the fusion matrix and to show that the fusion rules are well defined, in the sense that we have associativity, i.e., $\mathrm{N}_i\mathrm{N}_j = \mathrm{N}_j\mathrm{N}_i$. As the fusion rules themselves (with the representation-labeling introduced in Section 5.3) do not depend on n, we will restrict our analysis to the cases $n = \pm 1$; ($\mid n \mid > 1$ will just permute eigenvectors and eigenvalues).

Proposition 5.4.4 *For $q = e^{\pm\frac{i\pi}{N}}$ let $N_{ij,k} = (\mathrm{N}_j)_{ik} \in \{0,1\}$, be the multiplicity of V_k in $V_i \otimes V_j$.*

Then the eigenvalues of N_j are exactly d_j^r, $r = 1\ldots(N-1)$, and we have that

$$\|\mathrm{N}_j\| = d_j^1 \,. \tag{5.47}$$

Proof. Taking traces of q^{hr} on both sides of the decomposition, we arrive at the familiar equation

$$d_j^r \cdot d_i^r = \sum N_{ij,k}\, d_k^r \,, \tag{5.48}$$

or in terms of the eigenvectors

$$\begin{aligned} q_r &= \left(d_1^r, \ldots, d_{N-1}^r\right) \\ \mathrm{N}_j q_r &= d_j^r\, q^r \,. \end{aligned} \tag{5.49}$$

In the special case of $q = e^{\pm\frac{i\pi}{N}}$, the vectors q_r, $r = 1, \ldots, N-1$, are linearly independent, and q_1 has positive components. Note that $\mathrm{N}_1 = 1$ and $\mathrm{N}_{N-1}^2 = 1$. For even $j < N-1$, we can infer the ergodicity of N_j from the fact that any unitary representation is contained in a tensor product of V_j. By a Perron-Frobenius argument, we conclude that q_1 is the unique vector with $\mathrm{N}_j q_1 = \|\mathrm{N}_j\| q_1$. Similarly, we find that, for odd values of j, N_j has two ergodic invariant subspaces, one spanned by even-dimensional representations, one by the odd-dimensional ones. $\|\mathrm{N}_j\|$ is now doubly degenerate, with Perron-Frobenius vectors $q_1 \pm q_{N-1}$.

With these results, it is not hard to see that the converse holds, too.

The multiplicity matrices obey $\|\mathrm{N}_j\| = d_j^1$, for $j = 1, \ldots, N-1$, only if $q = e^{\pm\frac{i\pi}{N}}$.

□

Since the matrices N_j are all diagonalized by the same matrix,

$$\psi_{ij} = d_i^j(j)_q = (ij)_q \,, \tag{5.50}$$

they evidently commute. In terms of representation spaces, this can also be inferred from the associativity of the comultiplication (4.2) and the invariance of (E) under tensor products.

The first q-dimensions, $d_j^1 = \|\mathrm{N}_j\|$, can be interpreted as the quantum dimensions
the V_j's. For the fundamental representation V_2, for which N_2 is indecomposable, this
the well known formula

$$\|\mathrm{N}_2\| = 2\cos\frac{\pi}{N} \,. \qquad \text{(5.5}$$

We conclude this section with a summary of those conditions imposed on a quantum-gro
and a list of those representations of the quantum group that appear in applications
local relativistic quantum theory. The rational fusion rules are only reproduced by
subset of representations with $\lambda \in \frac{1}{n}\mathbf{Z}$; (i.e. $P^{2n} = 1$). If we denote the representation
in (5.3) by $[p_\lambda, k_\lambda]$ then we have that

$$\begin{gathered} N_{[p_\lambda^3,k_\lambda^3][p_\lambda^2,k_\lambda^2],[p_\lambda^1,k_\lambda^1]} = 1 \,, \\ \text{for} \quad k_\lambda^1 = k_\lambda^2 + k_\lambda^3 \qquad \text{and} \qquad N_{p_{\lambda 3} p_{\lambda 2}, p_{\lambda 1}} = 1 \end{gathered} \qquad \text{(5.}$$

and zero otherwise. The smallest subset of representations, invariant under fusion,
therefore obtained by setting $k_\lambda = 0$. From Proposition 5.2.1 we see that it contains
trivial representation and closes under conjugation.

By Proposition 5.2.4, these representations are unitarizable only if $n = 1$
$n = N-1$.

Chapter 6

Path Representations of the Braid Groups for Quantum Groups at Roots of Unity

6.1 Quotients of Representation Categories : The Vertex-*SOS* Transformation for Non-Semisimple Quantumgroups

We develop an intertwiner calculus for non-semisimple Hopf algebras in which the notion of irreducibility is replaced by indecomposability, so that Schur's Lemma is not applicable. We use this to generalize the "vertex-SOS-transformation" which is defined as a map from an intertwinerspace, e.g., a space of intertwiners between tensor product representations, to linear maps on quotients of intertwiner spaces. This yields a rigorous procedure to obtain braid group representations of rational local field theories and Boltzmann weights of the restricted RSOS-models from quantum groups at roots of unity. (In this context, we shall speak of a "rational", or "restricted" vertex-SOS-transformation.) The ideal property of the excluded representations is used to show that the resulting SOS-forms of the intertwiners can be written as linear maps on path spaces. A trace formula for the rational vertex-SOS-transformation is given. A more compact presentation of this construction may be found in [61]

From the universal element $\mathcal{R} \in \mathcal{K} \otimes \mathcal{K}$ of a quasitriangular Hopf-algebra $\mathcal{K}$ one can derive representations R^V of the braid groups B_n on an n-fold tensor product

$$V = \sum_{\pi \in S_n}^{\oplus} V_{j_{\pi(1)}} \otimes \ldots \otimes V_{j_{\pi(n)}}$$

of representation spaces V_{j_i} of the algebra $\mathcal{K}$, by setting

$$R^V(\sigma_i) = 1 \otimes \ldots \otimes R_{i\,i+1} \otimes \ldots \otimes 1$$

for the generator σ_i of B_n. Here the matrix

$$R_{i\,i+1} : V_{j_{\pi(i)}} \otimes V_{j_{\pi(i+1)}} \rightarrow V_{j_{\pi(i+1)}} \otimes V_{j_{\pi(i)}}$$

is given by

$$R_{i\,i+1} = P_{i\,i+1} \left(\rho_{j_{\pi(i)}} \otimes \rho_{j_{\pi(i+1)}} \right) \; \mathcal{R} \, ,$$

and commutes with the action of $\mathcal{K}$. If the representations of $\mathcal{K}$ are completely reducible it is well known [53, 43] how to construct representations, R^P, of B_n on the *path space* $\mathcal{P}(i|\{j_i\}|j) = \sum_{\pi \in S_n} {}^{\oplus} \mathcal{P}(i|j_{\pi(1)}, \ldots, j_{\pi(n)}|j)$.

Here the path space $\mathcal{P}(i|j_1, \ldots, j_n|j)$ is defined to be the linear span of paths $\omega = (\mu_1\alpha_1, \mu_2\alpha_2, \ldots, \mu_n\alpha_n)$, with $\mu_n = j$, $\mu_o = i$, and $V_{\mu_k}^{\alpha_k}$ is an irreducible subrepresentation of $V_{\mu_{k-1}} \otimes V_{j_k}$, where $\alpha_k = 1, \ldots, N_{\mu_{k-1} j_k, \mu_k}$ labels the multiplicity.

The construction of R^P (Vertex-SOS-transformation) uses the fact that the compositions of Clebsch-Gordan matrices

$$\begin{aligned} P_{\omega(j_i),j} = (P_{ij_1,\mu_1}(\alpha_1) \otimes \ldots \otimes \mathbf{1}_{j_n}) \ldots \left(P_{\mu_{k-1} j_k, \mu_k}(\alpha_k) \otimes \mathbf{1}_{j_{k+1}} \ldots \otimes \mathbf{1}_{j_n} \right) \\ \ldots \left(P_{\mu_{n-2} j_{n-1}, \mu_{n-1}}(\alpha_{n-1}) \otimes \mathbf{1}_{j_n} \right) \; \left(P_{\mu_{n-1} j_n, j}(\alpha_n) \right) \end{aligned} \tag{6.1}$$

and

$$\begin{aligned} P_{j,\omega(j_i)} = P_{j,\mu_{n-1} j_n}(\alpha_n) \left(P_{\mu_{n-1}, \mu_{n-2} j_{n-1}}(\alpha_{n-1}) \otimes \mathbf{1}_{j_n} \right) \ldots \\ \left(P_{\mu_k, \mu_{k-1} j_k}(\alpha_k) \otimes \ldots \otimes \mathbf{1}_{j_n} \right) \ldots (P_{\mu_1, i j_1}(\alpha_1) \otimes \ldots \otimes \mathbf{1}_{j_n}) \end{aligned} \tag{6.2}$$

are a basis of intertwiners between the spaces V_j and $V_i \otimes V_{j_1} \dots \otimes V_{j_n}$, and can be normalized such that

$$P_{j',\omega'(j_i)} P_{\omega(j_i),j} = \delta_{\omega\omega'} \delta_{jj'} \,. \tag{6.3}$$

The matrix elements of

$$R^P(b) : \mathcal{P}(i|j_1, \dots, j_n|j) \to \mathcal{P}(i|j_{\sigma(1)}, \dots, j_{\sigma(n)}|j) \,,$$

where σ is the image of b under the natural projection of B_n onto S_n, are given by

$$R^V(b) P_{\omega(j_i),j} = \sum_{\omega' \in \mathcal{P}(i|j_{\sigma(1)} \dots j_{\sigma(n)}|j)} R^P(b)_{\omega',\omega} \, P_{\omega'(j_{\sigma(i)}),j} \,. \tag{6.4}$$

Let us note, at this point, that the path spaces carry a multiplicative structure by simple composition

$$\sum_j^{\oplus} \mathcal{P}(i|j_1, \dots, j_\ell|j) \times \mathcal{P}(j|j_{\ell+1}, \dots, j_n|k) \cong \mathcal{P}(i|j_1, \dots, j_n|k) \,, \tag{6.5}$$

giving rise to a *path algebra.*

In the absence of complete reducibility, e.g. when $\mathcal{K} = U_q(s\ell_{d+1})$ with $q = e^{i\pi\frac{n}{N}}$, the Vertex-SOS transformation has to be modified. For this purpose, let us introduce linear spaces of intertwiners between an irreducible representation space V and an arbitrary representation space W.

In order to describe the set of irreducible subrepresentations of W isomorphic to V, we shall make use of their embeddings. Therefore, let us introduce the linear space of intertwiners,

$$\mathrm{Int}(W,V) := \{I : V \to W,\ Ia = aI, \quad \forall a \in \mathcal{K}\} \,. \tag{6.6}$$

By $\mathrm{Int}(V,W)$ we denote the space of intertwiners in reverse direction*. It identifies subrepresentations, $V^c = \ker I$, with the property that $W/V^c \simeq V$. As an example, let us consider $\mathcal{K} = U_q(s\ell_2)$, with $q = e^{\frac{i\pi}{N}}$ and $W = V_{N-1} \otimes V_2 \otimes V_2$ and $V = V_{N-1}$. Since the number of highest- and lowest-weight vectors for a given weight is the same as in the generic case, the dimension of $\mathrm{Int}(W,V)$ is unchanged.

---

*We prefer the more suggestive notation $\mathrm{Int}(W,V)$ to $\mathrm{Hom}_{\mathcal{K}}(V,W)$.

Since intertwiners $P_{ij,k}$ can be defined for all $i,j,k \le N$ obeying the fusio ordinary $SU(2)$, see (5.20), a basis of $\mathrm{Int}(W,V)$ is given by

$$\text{and} \quad \begin{aligned} P^-_{(N-1)22,(N-1)} &= \left(P_{(N-1)2,N-2} \otimes \mathbf{1}_2\right) P_{(N-2)2,(N-1)} \\ P^+_{(N-1)22,(N-1)} &= \left(P_{(N-1)2,N} \otimes \mathbf{1}_2\right) P_{N2,(N-1)} \,. \end{aligned}$$

As in the generic case, we have a natural map from the space

$$\mathrm{Int}(W_2,W_1) = \{R : W_1 \to W_2, \quad Ra = aR, \quad \forall a \in \mathcal{K}\}$$

into $\mathrm{Hom}\,(\mathrm{Int}(W_1,V),\ \mathrm{Int}(W_2,V))$ by left multiplication, denoted by

$$\begin{aligned} \mathcal{P} : \mathrm{Int}(W_2,W_1) &\to \mathrm{Hom}\,(\mathrm{Int}(W_1,V),\ \mathrm{Int}(W_2,V)) \\ R &\to \mathcal{P}(R) \,. \end{aligned}$$

To recover the path structure for the rational case, we have to divide out subsp intertwiners. For this purpose let

$$\mathrm{Int}_o(W_1,W_2) = \{I \in \mathrm{Int}(W_1,W_2) \mid tr(gIJ) = 0 \,, \quad \forall J \in \mathrm{Int}(W_2,W_1)\} \,,$$

where g implements the square of the antipode; e.g., for $U_q(sl_n)$ it is given by If one of the representations $W_i = V$ is an *irreducible* representation with non v q-dimension we see that $\mathrm{Int}_o(W_1,W_2)$ can be given as the subspaces of intert without left or right inverse. More precisely, we have

$\mathrm{Int}_o(W,V) = \{I \in \mathrm{Int}(W,V) \mid JI =$ [illegible] $\forall J \in \mathrm{Int}(V,W)\}$

and

$$\mathrm{Int}_o(V,W) = \{I \in \mathrm{Int}(V,W) \mid IJ = 0,\ \forall J \in \mathrm{Int}(W,V)\}$$

If we assume V only to be indecomposable rather than irreducible, "$JI = 0$" and in (6.9') have to be replaced by "JI and IJ nilpotent, $\forall J$". These sets are line and yield common invariant subspaces of the generic Vertex-SOS-transformat sense, that

$$\mathcal{P}(R) : \mathrm{Int}_o(W_1,V) \to \mathrm{Int}_o(W_2,V) \,,$$

all $R \in \mathrm{Int}(W_2, W_1)$. The complemented irreducible representations in W isomorphic are identified with points in

$$\mathrm{Int}(W, V)/\mathrm{Int}_o(W, V)\,.$$

our example $\mathrm{Int}_o(V_{N-1} \otimes V_2 \otimes V_2, V_{N-1})$ is spanned by $P^+_{(N-1)22,N-1}$. This can be seen m the explicit form of the intertwiner

$$P^+_{(N-1)22,(N-1)}\,\xi^{N-1}_m =$$

$$\begin{array}{lllll} q^{m-\frac{N}{2}}\sqrt{\left(\frac{N-2}{2}+m\right)_q\left(\frac{N}{2}-m\right)_q} & \xi^{N-1}_{m-1} & \otimes & \xi^2_{\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}} & \\ + \quad q^{m-\frac{1}{2}}\left(\frac{N}{2}\right)_q & \xi^{N-1}_m & \otimes & \xi^2_{-\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}} & \\ + \quad q^{m+\frac{1}{2}}\left(\frac{N}{2}+m\right)_q & \xi^{N-1}_m & \otimes & \xi^2_{\frac{1}{2}} \otimes \xi^2_{-\frac{1}{2}} & \\ + \, q^{m+\frac{N}{2}}\sqrt{\left(\frac{N-2}{2}-m\right)_q\left(\frac{N}{2}+m\right)_q} & \xi^{N-1}_{m+1} & \otimes & \xi^2_{-\frac{1}{2}} \otimes \xi^2_{-\frac{1}{2}}\,. & \end{array} \tag{6.11}$$

n this case

$$\begin{aligned} P^+_{(N-1)22,(N-1)}\xi^{N-1}_{\frac{N-2}{2}} &= \Delta^2(f)\left(\xi^{N-1}_{\frac{N-2}{2}} \otimes \xi^2_{\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}}\right), \\ \text{and} \qquad P^+_{(N-1)22,(N-1)}\xi^{N-1}_{-\frac{N-2}{2}} &= \Delta^2(e)\left(\xi^{N-1}_{-\frac{N-2}{2}} \otimes \xi^2_{-\frac{1}{2}} \otimes \xi^2_{-\frac{1}{2}}\right), \end{aligned}$$

here $\Delta^2 \equiv (\Delta \otimes \mathbf{1})\Delta$. A left inverse intertwiner $P^+_{(N-1),(N-1)22}$ to $P^+_{(N-1)22,(N-1)}$ therefore s to be ill defined on $\xi^{N-1}_{\frac{N-2}{2}} \otimes \xi^2_{\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}}$, as

$$\xi^{N-1}_{\frac{N-2}{2}} = f\left(P^+_{(N-1),(N-1)22}\xi^{N-1}_{\frac{N-2}{2}} \otimes \xi^2_{\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}}\right) \in \mathrm{im} f$$

not possible for highest-weight vectors of irreducible representations. A similar result first obtained in [30].

As in the case of the tensor product decomposition, we can find a vector $\tilde{\xi}^{N-1}_{\frac{N-2}{2}} \in$ $_{1} \otimes V_2 \otimes V_2$, given by

$$\begin{aligned} 2\tilde{\xi}^{N-1}_{\frac{N-2}{2}} &= \sqrt{(2)_q}\,\xi^{N-1}_{\frac{N-4}{2}} \otimes \xi^2_{\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}} + q^{\frac{N-1}{2}}\,\xi^{N-1}_{\frac{N-2}{2}} \otimes \xi^2_{-\frac{1}{2}} \otimes \xi^2_{\frac{1}{2}} \\ &\quad + q^{-\frac{N-1}{2}}\,\xi^{N-1}_{\frac{N-2}{2}} \otimes \xi^2_{\frac{1}{2}} \otimes \xi^2_{-\frac{1}{2}}\,, \end{aligned} \tag{6.12}$$

which satisfies (5.25) and (5.26) of Lemma 5.3.3, and therefore yields a subrepresentation W_{N-1} of $V_{N-1} \otimes V_2 \otimes V_2$. In fact, it can be shown that all tensor products decompose into three sets of subrepresentations:

a) irreducible representations with highest weights $\lambda \in \{0, \ldots, N-2\}$

b) irreducible representations of dimension N and weights $\lambda \in N\mathbb{Z} - 1$

c) $2N$-dimensional, indecomposable representations, whose structure differs from the one given for the W_i in Section 5.3 only by shifts, $\alpha \to \alpha + N$, in the weights.

In order to define the rational Vertex-SOS-transformation, we put

$$\mathcal{P}(W, V) = \mathrm{Int}(W, V)/\mathrm{Int}_o(W, V)\,. \tag{6.13}$$

For any linear map $T : \mathrm{Int}(W_1, V) \to \mathrm{Int}(W_2, V)$, that maps

$$\mathrm{Int}_o(W_1, V) \quad \text{into} \quad \mathrm{Int}_o(W_2, V)\,,$$

we have a well defined map $\tilde{T} : \mathcal{P}(W_1, V) \to \mathcal{P}(W_2, V)$, given by the condition, that the diagram

$$\begin{tikzcd} \mathrm{Int}_o(W_1, V) \arrow[r, hook] \arrow[d, "T"] & \mathrm{Int}(W_1, V) \arrow[r, two heads] \arrow[d, "T"] & \mathcal{P}(W_1, V) \arrow[d, "\tilde{T}"] \\ \mathrm{Int}_o(W_2, V) \arrow[r, hook] & \mathrm{Int}(W_2, V) \arrow[r, two heads] & \mathcal{P}(W_2, V) \end{tikzcd}$$

Figure 6.1:

commutes. Stated differently, if $[TI]_2 = 0$ whenever $[I]_1 = 0$, then $\tilde{T}$ is defined by $\tilde{T}[I]_1 = [TI]_2$, from the set of equivalence classes of $\mathrm{Int}(W_1, V)/\mathrm{Int}_o(W_1, V)$ into the quotient space $\mathrm{Int}(W_2, V)/\mathrm{Int}_o(W_2, V)$.

As mentioned in (6.10) this is the case for $T = \mathcal{P}(R)$, for any $R \in \mathrm{Int}(W_2, W_1)$, so that we have the following definition:

Definition 6.1.1 *The rational Vertex-SOS-transformation is the map*

$$\begin{aligned} \mathrm{Int}(W_2,W_1) &\rightarrow \mathrm{Hom}(\mathcal{P}(W_1,V),\,\mathcal{P}(W_2,V)) \\ R &\rightarrow \mathcal{P}^{\mathrm{rat}}(R), \end{aligned} \tag{6.14}$$

where $\mathcal{P}^{\mathrm{rat}}(R)$ *is the extension of* $\mathcal{P}(R)$ *given by* $\mathcal{P}^{\mathrm{rat}}(R) := \widetilde{\mathcal{P}(R)}$.

For $\mathbf{1}_{N-1}\otimes R \in \mathrm{Int}(V_{N-1}\otimes V_2\otimes V_2,\ V_{N-1}\otimes V_2\otimes V_2)$, with $R=\lambda_1\mathbf{P}_1+\lambda_o\mathbf{P}_o$, $\mathbf{P}_i$ being the projections onto the respective subrepresentations of $V_2\otimes V_2$, the ordinary Vertex-SOS-transform is given in the form

$$\mathcal{P}(R)=\begin{pmatrix} \lambda_1 & \frac{i(\lambda_o-\lambda_1)}{\sqrt{(2)_q}} \\ 0 & \lambda_o \end{pmatrix} \tag{6.15}$$

where the invariant subspace of $\mathcal{P}(R)$, spanned by the vector $\binom{1}{0}$, is identified with $\mathrm{Int}_o(V_{N-1}\otimes V_2\otimes V_2,\ V_{N-1})=\left\langle P^{+}_{(N-1)22,N-1}\right\rangle$. Taking quotients for the rational case we arrive at the one dimensional space $\mathcal{P}(V_{N-1}\otimes V_2\otimes V_2,\ V_{N-1})$, on which $\mathcal{P}^{\mathrm{rat}}(R)$ acts as multiplication by λ_o.

Clearly this map factors through the composition of intertwiners

$$\mathrm{Int}(W_3,W_2)\times\mathrm{Int}(W_2,W_1)\rightarrow\mathrm{Int}(W_3,W_1)\,. \tag{6.16}$$

Next, we use the results of the tensor product decomposition (Section 5.3) of $U_q(s\ell_2)$ to identify $\mathcal{P}(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$ with the *restricted path space*, $\mathcal{P}_{\mathrm{rest}}(i\mid j_1,\ldots,j_n\mid j)$. The latter space is defined in the same way as in the case of complete reducibility, with the restriction, that $N_{\mu_{k-1}j_n,\mu_k}\neq 0$ and $\mu_k\in L$, for all k. For any restricted path, the intertwiners $P_{\omega(j_i),j}$ and $P_{j,\omega(j_i)}$ given in (6.1) and (6.2) are well defined and can be normalized as in (6.3).

Lemma 6.1.2

a) If for $I\in\mathrm{Int}(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$, $P_{j,\omega(j_i)}I=0$, *for every restricted path* ω, *then* $I\in\mathrm{Int}_o(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$.

b) The images of intertwiners $P_{\omega(j_1),j}$ *in the quotient* $\mathcal{P}(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$ *for a basis in* $\mathcal{P}(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$.

The corresponding statements are true, if we pick a different ordering of the intertwine in (6.1) and (6.2) and, moreover, if we exchange left with right intertwiners.

Proof.

We first show, that if $V_j\subset V_{\mu_{k-1}}\otimes V_{j_k}\otimes\ldots\otimes V_{j_n}$ is complemented, i.e., its injectio has a left inverse $I: V_{\mu_{k-1}}\otimes V_{j_k}\otimes\ldots\otimes V_{j_n}\rightarrow V_j$, then there exists some μ_k, for whic $N_{\mu_{k-1}j_k,\mu_k}\neq 0$, so that $\bar{V}_j^{\mu_k}=\left(P_{\mu_k,\mu_{k-1}j_k}\otimes\mathbf{1}_{j_{k+1}}\ldots\otimes\mathbf{1}_{j_n}\right)(V_j)$ is non-zero and compl mented in $V_{\mu_k}\otimes V_{j_{k+1}}\otimes\ldots\otimes V_{j_n}$. Statement a) then follows by induction.

Suppose that, for any μ_k, $\bar{V}_{j_k}^{\mu_k}$ is either zero or not complemented. This means th $I\cdot\left(P_{\mu_{k-1}j_k,\mu_k}\otimes\mathbf{1}_{j_{k+1}}\ldots\otimes\mathbf{1}_{j_n}\right)\left(\bar{V}_{j_k}^{\mu_k}\right)=0$, for all μ_k. Hence $I\mathbb{P}_k(V_j)=0$, where

$$\mathbb{P}_k=\sum_{\mu_k:N_{\mu_{k-1}j_k,\mu_k}\neq 0} P_{\mu_{k-1}j_k,\mu_k}\,P_{\mu_k,\mu_{k-1}j_k}\otimes\mathbf{1}_{j_{k+1}}\ldots\otimes\mathbf{1}_{j_n}$$

is the projector on the first summand of the decomposition

$$V_{\mu_{k-1}}\otimes V_{j_k}\otimes\ldots\otimes V_{j_n}=$$

$$\left(\sum_{\substack{\mu_{ki}\\ N_{\mu_{k-1}j_k,\mu_k}\neq 0}}^{\oplus} V_{\mu_k}\otimes V_{j_{k+1}}\otimes\ldots\otimes V_{j_n}\right)\oplus\left(\sum_i^{\oplus}W_i\otimes V_{j_{k+1}}\otimes\ldots\otimes V_{j_n}\right). \qquad (6.1$$

Therefore $\tilde{V}_j=(\mathbf{1}-\mathbb{P}_k)(V_j)\neq 0$ can be complemented and is contained in the secon summand of (6.17). However, we know that property (E), introduced in Section 5. extends to tensor products and direct summands. As (E) is satisfied for all W_i, it al has to hold on $\tilde{V}_j$. For $j\leq\frac{N-2}{2}$, this leads to a contradiction. The second statement an immediate consequence of a), since for any $I\in\mathrm{Int}(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)$

$$I-\sum_{\omega}c_{\omega}\,P_{\omega(j_i),j}\in\mathrm{Int}_o(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_j)\,,$$

with $\mathbf{1}_jc_{\omega}=P_{j,\omega(j_i)}I$, (by the normalization chosen in (6.3)). From Lemma 6.1.2, a) find that the rational Vertex-SOS-transformation preserves the multiplicative structu of the path spaces, as explained in the following remarks.

Under the natural composition, the spaces $\mathrm{Int}_o(W,V)$ have the ideal property

$$\begin{aligned}\mathrm{Int}\left(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_k},V_j\right) \;&\times\; \mathrm{Int}_o\left(V_j\otimes V_{j_{k+1}}\otimes\ldots\otimes V_{j_n},V_s\right)\\ &\subset\; \mathrm{Int}_o\left(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_s\right),\end{aligned}\tag{6.18}$$

as well as

$$\begin{aligned}\mathrm{Int}_o\left(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_k},V_j\right) \;&\times\; \mathrm{Int}\left(V_j\otimes V_{j_{k+1}}\otimes\ldots\otimes V_{j_n},V_s\right)\\ &\subset\; \mathrm{Int}_o\left(V_i\otimes V_{j_1}\otimes\ldots\otimes V_{j_n},V_s\right).\end{aligned}\tag{6.19}$$

With the identifications made above, we can view the rational Vertex-SOS-transformation, in the case of $U_q(s\ell_2)$, as the map

$$\begin{aligned}\mathrm{Int}\left(V_{s_1}\otimes\ldots\otimes V_{s_k},V_{j_1}\otimes\ldots\otimes V_{j_\ell}\right) \;&\rightarrow\; \mathrm{Hom}\left(\mathcal{P}_{(i|j_1\ldots j_\ell|j)},\mathcal{P}_{(i|s_1\ldots s_k|j)}\right)\\ R \;&\mapsto\; \mathcal{P}^{\mathrm{rat}}_{ij}(R).\end{aligned}\tag{6.20}$$

By (6.18), (6.19) and (6.10), this map is evidently compatible with the multiplicative structure defined in (6.5), in the sense that for

$$\begin{aligned}&A \;\in\; \mathrm{Int}(V_{s_1}\otimes\ldots\otimes V_{s_k},\; V_{j_1}\otimes\ldots\otimes V_{j_n})\\ \text{and}\quad&\\ &B \;\in\; \mathrm{Int}\left(V_{s_{k+1}}\otimes\ldots\otimes V_{s_{k+\ell}},V_{j_{n+1}}\otimes\ldots\otimes V_{j_{n+m}}\right),\end{aligned}$$

$\mathcal{P}^{\mathrm{rat}}_{ij}(A\otimes B)$ maps $\mathcal{P}(i|j_1,\ldots,j_n|p)\times\mathcal{P}(p|j_{n+1},\ldots,j_{n+m}|j)$ into the product of path spaces $\mathcal{P}(i|s_1,\ldots,s_k|p)\times\mathcal{P}(p|s_{k+1},\ldots,s_{k+\ell}|j)$ by $\mathcal{P}^{\mathrm{rat}}_{ip}(A)\otimes\mathcal{P}^{\mathrm{rat}}_{pj}(B)$, for all $p\in L$.

The kernel of the rational Vertex-SOS-transformation is given by

$$\bigcap_{r,N_{ir,j}\neq 0} \mathcal{K}\left(V_{s_1}\otimes\ldots\otimes V_{s_k},\; V_{j_1}\otimes\ldots\otimes V_{j_\ell}\mid V_r\right),\tag{6.21}$$

where $\mathcal{K}(W_2,W_1\mid V)$ is the subspace of intertwiners in $\mathrm{Int}(W_2,W_1)$, which map all intertwiners $\mathrm{Int}(W_1,V)$ to $\mathrm{Int}_o(W_2,V)$.

A more efficient way of characterizing $\mathcal{K}(W_2,W_1\mid V)$ can be given with the help of Lemma 5.4.3. From the proof of Lemma 6.1.2, one can see that the common kernel $W^o_{(j_i)}:=\bigcap_{j\omega}\ker P_{j,\omega(j_i)}$ is the maximal subspace in $W:=V_{j_1}\otimes\ldots\otimes V_{j_n}$ satisfying (E). We associate to it the projection $\mathbb{P}_o=\mathbf{1}-\sum_{j,\omega}P_{\omega(j_i),j}P_{j,\omega(j_i)}$, and, as $\mathbb{P}_o\in\mathrm{Int}(W,W)$, $W^o_{(j_i)}$ is seen to be a subrepresentation of W.

Lemma 6.1.3 *If $C\in\mathrm{Int}(W,W)$, where W is an n-fold tensor product of irreducible representations with dimensions less than N, and $a\in U_q(s\ell_2)$, then the following trace formula holds*

$$\begin{aligned}&\mathrm{tr}(aC\upharpoonright W)=\\ &\quad\sum_{j=0}^{N-1}\mathrm{tr}(a\upharpoonright V_j)\,\mathrm{tr}\left(\mathcal{P}^{\mathrm{rat}}(C)\upharpoonright\mathcal{P}(W,V_j)\right)\quad+\mathrm{tr}\left(aC_o\upharpoonright W^o_{(j_i)}\right),\end{aligned}\tag{6.22}$$

where $C_o=\mathbf{P}_oC\mathbf{P}_o$.

Proof.

The second term on the r.h.s of (6.22) can be identified with the second term on the r.h.s of $\mathrm{tr}(aC)=\mathrm{tr}((1-\mathbb{P}_o)aC(1-\mathbb{P}_o))+\mathrm{tr}(\mathbb{P}_oaC\mathbb{P}_o)$.

In order to evaluate the first term, we note that

$$\begin{aligned}&\mathrm{tr}\left(P_{\omega(j_i),j}P_{j,\omega(j_i)}aC\;P_{\omega'(j_i),j'}P_{j',\omega'(j_i)}\right)\\ &\quad=\;\delta_{jj'}\mathrm{tr}\left(aP_{j,\omega(j_i)}C\;P_{\omega'(j_i),j'}\;P_{j',\omega'(j_i)}P_{\omega(j_i),j}\right)\\ &\quad=\;\delta_{\omega\omega'}\delta_{jj'}\;c^j_{\omega\omega'}\mathrm{tr}(a\upharpoonright V_j),\\ \text{where}\qquad &\quad\;\; c^j_{\omega\omega'}\mathbf{1}_j:=P_{j,\omega(j_i)}C\;P_{\omega'(j_i),j}\end{aligned}$$

are the matrix elements of $\mathcal{P}^{\mathrm{rat}}(C)$ on $\mathcal{P}(W,V_j)$. □

Next, we choose functions $\{f_p\}_{p=1,\ldots,N-1}$, such that $\mathrm{tr}\left(f_p(q^h)\upharpoonright V_{p'}\right)=\delta_{p,p'}$ and $\mathrm{tr}\left(f_p(q^h)\upharpoonright W\right)=0$, if W has property (E). With the help of Lemma 5.4.3 we see that any function with $f_p(1)=\delta_{p,1}$, $f_p(-1)=-\delta_{p,N-1}$, $f_p(q^r)=\frac{1}{2}(\delta_{p,r+1}-\delta_{p,r-1})$, $r=1,\ldots,N-1$ and $f_p(q^r)=f_p(q^{-r})$, is a candidate. This defines an inner product of $A\in\mathrm{Int}(W_2,W_1)$ with $B\in\mathrm{Int}(W_1,W_2)$, by

$$\begin{aligned}\langle A,B\rangle_p\;&=\;\mathrm{tr}\left(f_p(q^h)AB\upharpoonright W_2\right)\\ &=\;\mathrm{tr}\left(\mathcal{P}^{\mathrm{rat}}(A)\,\mathcal{P}^{\mathrm{rat}}(B)\upharpoonright\mathcal{P}\left(W_2,V_p\right)\right).\end{aligned}\tag{6.23}$$

Since the map $\mathcal{P}^{\mathrm{rat}} : \mathrm{Int}(W_2, W_2) \to \mathrm{End}(\mathcal{P}(W_2, V_p))$ is surjective, we have from (6.23) that

$$\mathcal{K}(W_2, W_1 \mid V_p) = \{A \mid \langle A, B\rangle_p = 0\,, \quad \forall B \in \mathrm{Int}(W_1, W_2)\}\,. \tag{6.24}$$

Let us conclude our discussion of the rational Vertex-SOS-transformation, with some comments on the structural properties that are present in the vertex picture, but not observed in the SOS-picture. First, it is essential to restrict $j \leq \frac{N-2}{2}$, since every subrepresentation isomorphic to $V_{\frac{N-1}{2}}$ is, by Lemma 5.2.3, complemented, i.e., $\mathrm{Int}_o\left(W, V_{\frac{N-1}{2}}\right) = 0$, and since the dimension of the highest weight spaces is larger than in the generic case, $\mathcal{P}\left(W, V_{\frac{N-1}{2}}\right)$ is described by unbounded paths. An explicit example is given by $W = V_{j_1} \otimes V_{j_2}$ and $V = V_j$, with $N_{j_1 j_2, j} = 0$, $j \neq \frac{N-1}{2}$. From the decomposition of tensor products, discussed in Section 5.3, we see that $\mathrm{Int}_o(W, V_j)$ is given by the embedding of V_j into W_j, (mapping $\xi^j_m \to \xi^j_m$), and $\mathrm{Int}_o(V_j, W)$ by D_j, $\left(\text{mapping } \tilde{\xi}^j_m \to \xi^j_m, \text{ rest} \to 0\right)$, where D_j is defined in (5.38).

6.2 Braid Group Representations and Fusion Equations.

With the help of the rational Vertex-SOS-transformation, as defined above, we obta a faithful representation of the braid group on n-strings, B_n, on the space of restrict paths

$$\mathcal{P}_{\mathrm{rest}}\left(i \mid \{j_i\} \mid j\right) = \sum_{\pi \in S_n}{}^{\oplus}\, \mathcal{P}_{\mathrm{rest}}\left(i \mid j_{\pi(1)}, \ldots, j_{\pi(n)} \mid j\right)\,. \qquad (6.2$$

By compatibility with the multiplicative structure of the path algebra, if is sufficient give the generators in $\mathrm{Hom}(\mathcal{P}(k \mid p, q \mid i), \mathcal{P}(k \mid q, p \mid i))$, by

$$\begin{gathered}\left(\mathbf{1}_k \otimes R^{\pm}_{pq}\right)\left(P_{kp,j}(\beta) \otimes \mathbf{1}_q\right) P_{jq,i}(\alpha) = \sum_{\ell\nu\mu} \rho^{\pm}(k, q, p, i)^{j\beta\alpha}_{\ell,\nu\mu} \left(P_{kq,\ell}(\nu) \otimes \mathbf{1}_p\right) P_{\ell p,i}(\mu) \\ \mathrm{mod}\ \mathrm{Int}_o\left(V_k \otimes V_q \otimes V_p, V_i\right)\,.\end{gathered} \qquad (6.2$$

Here we put $R^-_{pq} = \left(R^+_{qp}\right)^{-1}$.

Since by the arguments of Section 5.4, for all $d \in \mathbb{N}$ and q a root of unity, can find a family of indecomposable representations, with nonzero q-dimensions, a fusion rules for $U_q(s\ell_{d+1})$, such that $\mathcal{P}\left(V_i \otimes V_{j_1} \otimes \ldots \otimes V_{j_n}, V_k\right)$ admits a path basis the above sense, we explicitly include the multiplicities in the following formulas. It convenient to use the following graphical notation for products of intertwiners. A tens product $V_{j_1} \otimes \ldots \otimes V_{j_n}$ is represented by n-ordered strings with colours $j_1, \ldots, j_n$, and intertwiner $I : V_{j_1} \otimes \ldots \otimes V_{j_n} \to V_{i_1} \otimes \ldots \otimes V_{i_m}$ by a "deformation" of the strings $j_1, \ldots,$ into $i_1, \ldots, i_m$. Schematically, this is shown in Figure 6.2:

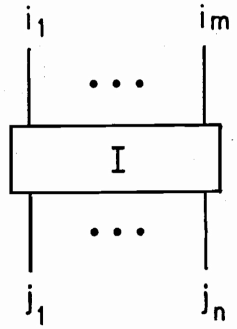
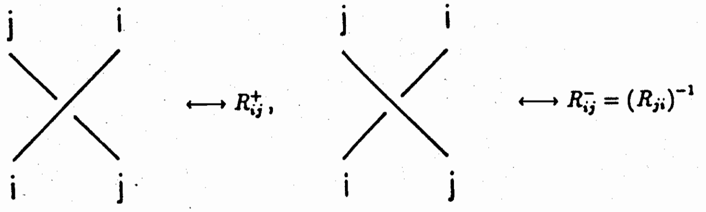


Figure 6.2

The generators of these "deformations" shall consist of the intertwiners $P_{ij,k}(\alpha)$, $P_{k,ij}(\beta)$ and $R^{\pm}_{ij}$,which we represent, graphically, by braids

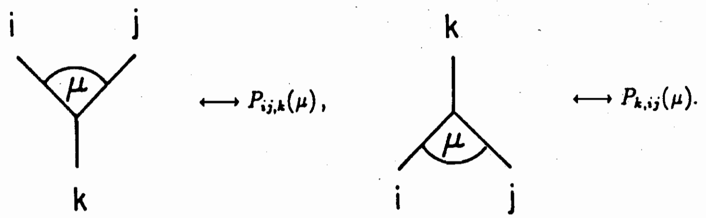


$\longleftrightarrow R^{+}_{ij}$, $\longleftrightarrow R^{-}_{ij} = (R_{ji})^{-1}$

Figure 6.3

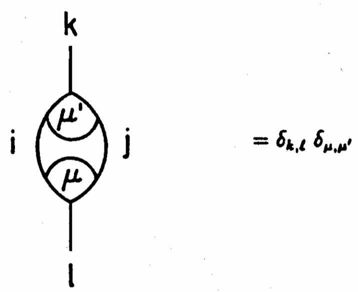
and forks

Figure 6.4



$\longleftrightarrow P_{ij,k}(\mu)$, $\longleftrightarrow P_{k,ij}(\mu)$.

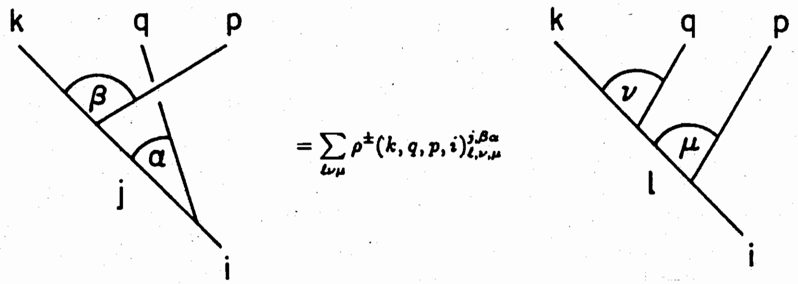
The normalization (6.3) is then represented by



$= \delta_{k,\ell}\, \delta_{\mu,\mu'}$

Figure 6.5

and equation (6.26) by



$= \sum_{\ell\nu\mu} \rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu,\mu}$

Figure 6.6

The proof of Lemma 6.1.2 shows that a choice of basis in $\mathcal{P}(V_i \otimes V_{j_1} \otimes \ldots \otimes V_{j_n}, V_k)$ can be given for any ordering of the Clebsch-Gordan matrices. In fact, a change of basis by reordering can be entirely expressed in terms of the SOS-weights $\left(\rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}\right)$. The following fusion identities mainly rely on the duality relations (4.9) which can be reexpressed in terms of intertwiners by

$$\left(\mathbf{1}_i \otimes R^{\pm}_{\ell j}\right) \left(R^{\pm}_{\ell i} \otimes \mathbf{1}_j\right) \left(\mathbf{1}_\ell \otimes P_{ij,m}(\alpha)\right) = \left(P_{ij,m}(\alpha) \otimes \mathbf{1}_\ell\right) R^{\pm}_{\ell m} \qquad (6.27)$$

and is represented, graphically, as follows

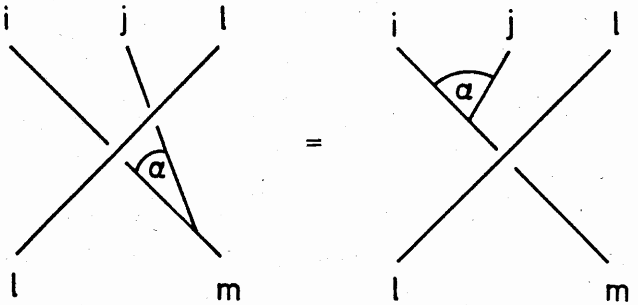


Figure 6.7

Analogous equations hold, for the reflected version of Figure 6.7; see also [43, 44]. Note also, that the labels in (6.27) do not have to correspond to irreducible representations.

Let us assume that we have chosen a basis of intertwiners such that

$$(\mathbf{1}_p \otimes \lambda) P_{p1,p} = (\lambda \otimes \mathbf{1}_p) P_{1p,p} = \lambda(v_o)\, \mathbf{1}_p\,, \tag{6.28}$$

for all $\lambda \in V_1^*$, with v_o independent of p. Then we have

Lemma 6.2.1

a) The images of $\{(\mathbf{1}_i \otimes P_{j\ell,m}(\alpha))\,(P_{im,k}(\beta))\}_{m,\alpha,\beta}$ *in* $\mathcal{P}(V_i \otimes V_j \otimes V_\ell, V_k)$ *form a second basis.*

b) The coefficients, $\hat{\check{\varphi}}$ *, expressing the change of basis*

$$\begin{aligned}&(\mathbf{1}_i \otimes P_{j\ell,m}(\alpha))\, P_{im,k}(\beta) = \\ &\quad \sum_{r\mu\nu} \hat{\varphi}(i,j,\ell,k)^{m,\alpha\beta}_{r,\mu\nu}\, (P_{ij,r}(\mu) \otimes \mathbf{1}_\ell)\, P_{r\ell,k}(\nu) \quad \mathrm{mod}\ \ \mathrm{Int}_o(V_i \otimes V_j \otimes V_\ell, V_k)\end{aligned} \tag{6.29}$$

and

$$\begin{aligned}&(P_{ij,r}(\mu) \otimes \mathbf{1}_\ell)\, P_{r\ell,k}(\nu) = \\ &\sum_{\ell\alpha\beta} \check{\varphi}(i,j,\ell,k)^{r,\mu\nu}_{m,\alpha\beta}\, (\mathbf{1}_i \otimes P_{j\ell,m}(\alpha))\, P_{im,k}(\beta) \quad \mathrm{mod}\ \mathrm{Int}_o(V_i \otimes V_j \otimes V_\ell, V_k)\end{aligned} \tag{6.30}$$

can be computed from ρ-matrices:

$$\begin{aligned}&\hat{\varphi}(i,j,\ell,k)^{m,\alpha\beta}_{r,\mu\nu} = \\ &\quad \sum_{\eta\xi} \rho^{\mp}(1,m,i,k)^{i,1\beta}_{m,1\eta}\, \rho^{\pm}(j,i,\ell,k)^{m,\alpha\eta}_{r,\xi\nu}\, \rho^{\pm}(1,i,j,r)^{j,1\xi}_{i,1\mu}\,,\end{aligned} \qquad (6.31$$

and

$$\begin{aligned}&\check{\varphi}(i,j,\ell,k)^{r,\mu\nu}_{m,\alpha\beta} = \\ &\quad \sum_{\eta\xi} \rho^{\mp}(1,i,m,k)^{m,1\eta}_{i,1\beta}\, \rho^{\pm}(j,\ell,i,k)^{r,\xi\nu}_{m,\alpha\eta}\, \rho^{\pm}(1,j,i,r)^{i,1\mu}_{j,1\xi}\,.\end{aligned} \qquad (6.32$$

The *proof of Lemma 6.2.1* is purely computational. For convenience, we understan the following equations modulo $\mathrm{Int}_o(W,V)$, without further mentioning. Since $\lambda \otimes \mathbf{1}$ $V_1 \otimes W \to W$ is an intertwiner, for any $\lambda \in V_1^*$, we obtain from (6.26), for $k = 1$,

$$R^{\pm}_{pq} P_{pq,i}(\alpha) = \sum_{\mu} \rho^{\pm}(1,q,p,i)^{p,1\alpha}_{q,1\mu}\, P_{qp,i}(\mu)\,. \qquad (6.33$$

Applying $R^{\pm}_{kq} \otimes \mathbf{1}_p$ to (6.26) and making use of (6.27), we have that

$$\begin{aligned}&(\mathbf{1}_q \otimes P_{kp,j}(\beta))\, R^{\pm}_{jq}\, P_{jq,i}(\alpha) = \\ &\quad \sum_{\ell,\nu,\mu} \rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu} \left(R^{\pm}_{kq} P_{kq,\ell}(\nu) \otimes \mathbf{1}_p\right) P_{\ell p,i}(\mu)\,.\end{aligned} \qquad (6.34$$

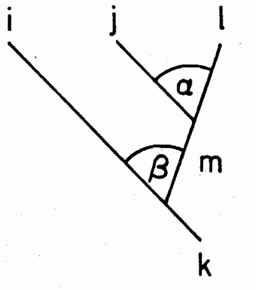
We now use (6.11) on both sides of (6.12) and invert $\rho^{\pm}(1,j,q,i)^{q,1\alpha}_{j,1\eta}$ by using

$$\sum_{\alpha} \rho^{\mp}(1,q,j,i)^{j,1\mu}_{q,1\alpha}\, \rho^{\pm}(1,j,q,i)^{q,1\alpha}_{j,1\eta} = \delta_{\mu\eta}\,. \qquad (6.3$$

This yields the desired expansion of the basis $\{(\mathbf{1}_q \otimes P_{kp,j}(\beta))\, P_{qj,i}(\eta)\}_{j,\beta,\eta}$ in terms the path basis, with coefficients $\hat{\varphi}(q,k,p,i)^{j,\beta\eta}_{\ell,\xi\mu}$ given by (6.31).

The expression for $\check{\varphi}(p,k,q,i)^{j,\eta\alpha}_{\ell,\nu\xi}$ are obtained by applying the product of R-matric $\left(R^{\pm}_{kp} \otimes \mathbf{1}_q\right) \cdot \left(\mathbf{1}_k \otimes R^{\mp}_{qp}\right)$ to (6.26) and proceeding in the same way as above.

Expression (6.29) is expressed, graphically, as follows



$$= \sum_{r\mu\nu} \hat{\varphi}(i,j,\ell,k)^{m,\alpha\beta}_{r,\mu\nu}$$

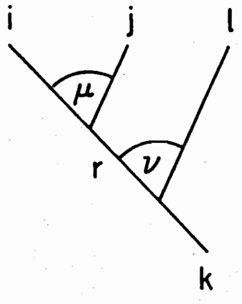
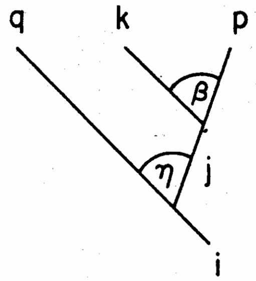
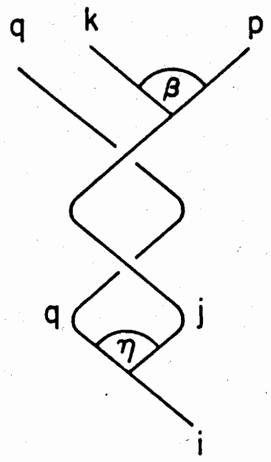


Figure 6.8

To demonstrate the convenience of the graphical notation, we repeat the proof of equation (6.31):

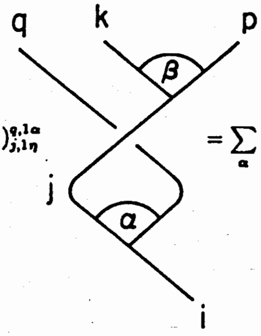


$$=$$

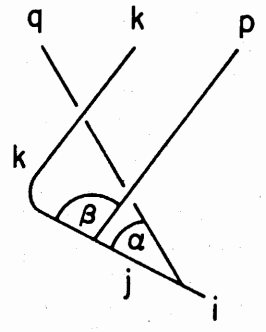


$$=$$

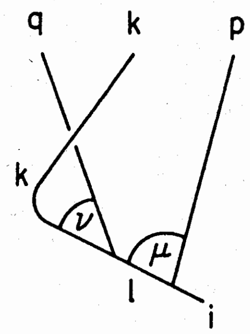
$$= \sum_{\alpha} \rho^{-}(1,j,q,i)^{q,1\alpha}_{j,1\eta}$$



$$= \sum_{\alpha} \rho^{-}(1,j,q,i)^{q,1\alpha}_{j,1\eta}$$



$$= \sum_{\alpha\nu\mu\ell} \rho^{-}(1,j,q,i)^{q,1\alpha}_{j,1\eta}\, \rho^{+}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}$$



$$= \sum_{\alpha\nu\mu\ell\xi} \rho^{-}(1,j,q,i)^{q,1\alpha}_{j,1\eta}\, \rho^{+}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}\, \rho^{+}(1,q,k,\ell)^{k,1\nu}_{q,1\xi}$$

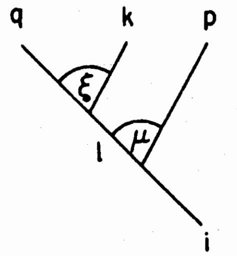
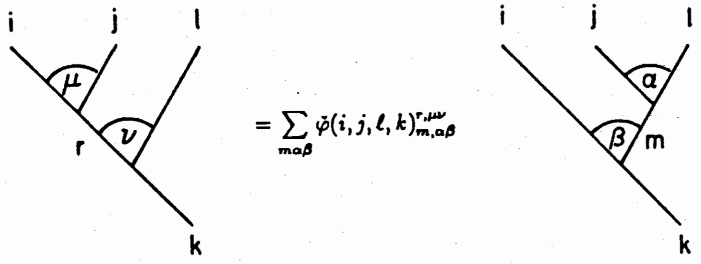


Figure 6.9

□

In the same way we obtain



$$= \sum_{m\alpha\beta} \check{\varphi}(i,j,\ell,k)^{r,\mu\nu}_{m,\alpha\beta}$$

Figure 6.10

Since the $P_{ij,k}(\alpha)$ form a basis of left invertible intertwiners in $\mathrm{Int}(V_i \otimes V_j, V_k)$, we find a dual basis $P_{k,ij}(\beta)$ in $\mathrm{Int}(V_k, V_i \otimes V_j)$, with

$$P_{k,ij}(\beta)\, P_{ij,k}(\alpha) = \delta_{\alpha\beta}\,. \tag{6.36}$$

The path expansions of these intertwiners are evidently given by

$$\begin{aligned}&(\mathbf{1}_\ell \otimes P_{p,rs}(\nu))\,(P_{\ell r,m}(\xi) \otimes \mathbf{1}_s)\, P_{ms,i}(\gamma) = \\ &\sum_\gamma \check{\varphi}(\ell,r,s,i)^{m,\xi\gamma}_{p,\nu\mu}\, P_{\ell p,i}(\mu) \bmod \mathrm{Int}_o\,(V_\ell \otimes V_p, V_i)\,.\end{aligned} \tag{6.37}$$

With these orthogonality relations, we obtain the fusion equations in SOS-form, by expanding both sides of a version of (6.27):

$$R^\pm_{pq} = (P_{q,rs}(\delta) \otimes \mathbf{1}_q)\, \left(\mathbf{1}_r \otimes R^\pm_{ps}\right)\, \left(R^\pm_{pr} \otimes \mathbf{1}_s\right)\, (\mathbf{1}_p \otimes P_{rs,q}(\delta))\,.$$

Together with the reflected version, this yields

$$\begin{aligned}&\delta_{qq'}\,\delta_{\delta\delta'}\,\rho^\pm(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu} = \\ &\sum_{\substack{mm'\eta\\ \ell\ell'\gamma\gamma'}} \hat{\varphi}(j,r,s,i)^{q',\delta'\alpha}_{m,\xi\gamma}\,\rho^\pm(k,r,p,m)^{j,\beta\xi}_{m',\xi'\eta} \\ &\rho^\pm(m',s,p,i)^{m,\eta\gamma}_{\ell,\gamma'\mu}\,\check{\varphi}(k,r,s,\ell)^{m',\xi'\gamma'}_{q,\delta\nu}\end{aligned} \tag{6.38}$$

and

$$\begin{aligned}&\delta_{qq'}\,\delta_{\delta\delta'}\,\rho^\pm(k,p,q,i)^{\ell,\nu\mu}_{j,\beta\alpha} = \\ &\sum_{\substack{mm'\eta\\ \ell\ell'\gamma\gamma'}} \hat{\varphi}(k,r,s,\ell)^{q',\delta'\nu}_{m',\xi'\gamma'}\,\rho^\pm(m',p,s,i)^{\ell,\gamma'\mu}_{m,\eta\gamma} \\ &\rho^\pm(k,p,r,m)^{m',\xi'\eta}_{j,\beta\alpha}\,\check{\varphi}(j,r,s,i)^{m,\xi\gamma'}_{q,\delta\alpha}\,.\end{aligned} \tag{6.39}$$

6.3 Unitarity of Braid Group Representations Obtained from $U_q(s\ell_{d+1})$

If it is possible to generate all representations in L out of a set of fundamental represe tations $\mathcal{F} := \{f_1, \ldots, f_r\}$ by taking tensorproducts and decomposing, then, by equatio (6.38) and (6.39), all SOS-weights can be obtained from the weights $\rho^\pm(k,f_i,f_j,i)^{\ell,}_{j,}$ and $\hat{\check{\varphi}}(k,r,f_i,\ell)^{q,\delta\nu}_{m,\xi\gamma}$. Comparing the complex conjugate of (6.38) to (6.39), we arrive the following expression of unitarity:

Lemma 6.3.1 *For a given choice of basis $\{P_{ij,k}(\alpha)\}$, the representations of B_n on t path sapce $\mathcal{P}(i \mid \{j_1, \ldots, j_n\} \mid j)$, as defined in (6.25), are unitary iff the representatio of B_n are unitary on $\mathcal{P}\,(i \mid \{f_{k_1}, \ldots, f_{k_n}\} \mid j)$, for arbitrary $f_{k_\ell} \in \mathcal{F}$, $\ell = 1, \ldots, n$, and*

$$\overline{\hat{\varphi}(i,j,f_r,\ell)^{q,\delta\nu}_{m,\alpha\beta}} = \check{\varphi}(i,j,f_r,\ell)^{m,\alpha\beta}_{q,\delta\nu}\,. \qquad (6.4$$

As an example we may apply this result to $U_q(s\ell_2)$, where $\mathcal{F} = \{V_2\}$.

Since all the multiplicities are unity, we can set $P_{i,jk} = P^T_{jk,i}$. In this case

$$\check{\varphi}(k,2,p,i)^\ell_m = \hat{\varphi}(k,2,p,i)^m_\ell\,, \qquad (6.4$$

and it is sufficient to check, that the expressions

$$\pi^m(k,2,p,i)^\ell_\ell := \hat{\varphi}(k,2,p,i)^\ell_m\,\check{\varphi}(k,2,p,i)^m_\ell\,, \qquad (6.4$$

which are invariant under scalings of $P_{ij,k}$, are positive.

Their values can be expressed in terms of q-numbers:

$$\pi^{p\pm1}(k,2,p,i)^{k\pm1}_{k\pm1} = \frac{\left(j_p + \frac{1}{2} + j_i - j_k\right)_q \left(j_k - j_p + \frac{1}{2} + j_i\right)_q}{(2j_p + 1)_q\,(2j_k + 1)_q} \qquad (6.4$$

and

$$\pi^{p\mp1}(k,2,p,i)_{k\pm1}^{k\pm1} = 1 - \pi^{p\pm1}(k,2,p,i)_{k\pm1}^{k\pm1} = \frac{\left(j_p + j_k + j_i + \frac{3}{2}\right)_q \left(j_p + \frac{1}{2} + j_k - j_i\right)_q}{(2j_p+1)_q\,(2j_k+1)_q}\,, \tag{6.44}$$

where $2j_k + 1 = k$.

The computation of the braid matrices, for $p = q = 2$, gives

$$\rho^{\pm}(k+2\eta,2,2,k)_{k+\sigma'}^{k+\sigma} = q^{\mp\frac{1}{2}}\,\delta_{\eta\sigma}\,\delta_{\eta\sigma'}\,,$$

for $\eta, \sigma, \sigma' = \pm1$, and

$$\rho^{\pm}(k,2,2,k) = \frac{q^{\pm\frac{1}{2}}}{(k)_q}\begin{bmatrix} -q^{\pm k} & \sqrt{(k+1)_q(k-1)_q} \\ \sqrt{(k+1)_q(k-1)_q} & q^{\mp k} \end{bmatrix}. \tag{6.45}$$

These representations are therefore unitarizable, iff all q-numbers $(n)_q$, with $0 \le n < N$, are positive; or, stated differently, iff $q = e^{\pm i\frac{\pi}{N}}$. We will see in Section 7.3 how this is related to the result obtained in [54] for Hecke-algebras. It is possible to rewrite the expressions (6.43) and (6.44) in the form

$$\pi^{p+1}(k,2,p,i)_{k+\eta}^{k+\eta} = \frac{1}{(p)_q!^2}\,\frac{\lambda_{k(p+1),i}\,\lambda_{2\,p,p+1}}{\lambda_{(k+\eta)p,i}\,\lambda_{k\,2,k+\eta}}\,, \tag{6.46}$$

with

$$\lambda_{k\,p,i} = \frac{(j_k+j_p+j_i+1)_q!\,(j_p+j_k-j_i)_q!\,(j_p+j_i-j_k)_q!\,(2j_p+1)_q}{(j_k-j_p+j_i)_q!\,(2j_k+1)_q} \tag{6.47}$$

where $k = 2j_k + 1$ and $\eta = \pm1$.

This enables us to set $\breve{\varphi}(k,2,p,i)_{p+1}^{k+\eta} = \frac{1}{(p)_q!}$, for all k, p, i and η, in a normalization, where $P_{kp,i}^T\,P_{kp,i} = \lambda_{k\,p,i}$. The recursions given in (6.38), and (6.39) then take the form

$$\rho_o^{\pm}(k,p,q+1,i)_j^{\ell} = \sum_{\sigma=\pm1} \rho_o^{\pm}(k+\eta,p,q,i)_{j+\sigma}^{\ell}\;\rho_o^{\pm}(k,p,2,j+\sigma)_j^{k+\eta} \tag{6.48}$$

and

$$\rho_o^{\pm}(k,q+1,p,i)_{\ell}^{j} = \sum_{\sigma} \rho_o^{\pm}(k,2,p,j+\eta)_{k+\sigma}^{j}\;\rho_o^{\pm}(k+\sigma,q,p,i)_{\ell}^{j+\eta}\,, \tag{6.49}$$

where the subscript "o" refers to our new normalization in which $P_{kp,i}^T\,P_{k\,p,i} = \lambda_{k\,p,i}$, with $\lambda_{kp,i}$ as in (6.47). These expressions and the equations

$$\begin{aligned} \rho_o^{\pm}(k+2\eta,2,2,k)_{k+\sigma'}^{k+\sigma} &= q^{\mp\frac{1}{2}}\delta_{\eta\sigma}\,\delta_{\eta\sigma'}\,, \quad \text{for} \quad \eta = \pm1\,, \\ \rho_o^{\pm}(k,2,2,k) &= \frac{q^{\pm\frac{1}{2}}}{(k)_q}\begin{bmatrix} -q^{\pm k} & (k+1)_q \\ (k-1)_q & q^{\mp k} \end{bmatrix} \end{aligned} \tag{6.50}$$

show that the $\rho_o^{\pm}$-braid matrices can be identified with those found in [9, 55], with the indices in reversed order; (one must compare the recursions given there with (6.49)).

In the following, we show how to construct an inner product on the representation spaces of on the braid group representations derived from $U_q(s\ell_{d+1})$, in order to isolate requirements for the spins in the spectrum of the monodromy matrix and investigate unitarizability of these representations. We conclude this section with a more systematic proof of the above result on unitarizability for $U_q(s\ell_2)$. We start by taking the star-conjugate of (6.26) and insert the transposition

$$P_{pk} : V_p \otimes V_k \rightarrow V_k \otimes V_p : v \otimes w \mapsto w \otimes v\,.$$

Using

$$R_{pq}^{\pm\,*} = P_{qp}\,R_{pq}^{\mp}\,P_{qp}\,, \tag{6.51}$$

see formula (4.24), we find the following equation for $R_{pq}^{\mp}$.

$$\begin{aligned} &\overline{P}_{i,qj}(\alpha)\left(\mathbf{1}_q \otimes \overline{P}_{j,pk}(\beta)\right)\left(R_{pq}^{\mp} \otimes \mathbf{1}_k\right) = \\ &\sum_{\ell,\nu\mu} \overline{\rho^{\pm}(k,q,p,i)_{\ell,\nu\mu}^{j,\beta\alpha}}\;\overline{P}_{i,p\ell}(\mu)\;\left(1_p \otimes \overline{P}_{\ell,qk}(\nu)\right)\,, \quad \text{mod Int}_o(V_i, V_p \otimes V_q \otimes V_k)\,, \end{aligned} \tag{6.52}$$

where we have set

$$\overline{P}_{i,qj}(\alpha) = P_{jq,i}(\alpha)^*\,P_{qj}\,. \tag{6.53}$$

If we represent $\overline{P}_{i,qj}(\alpha)$ by

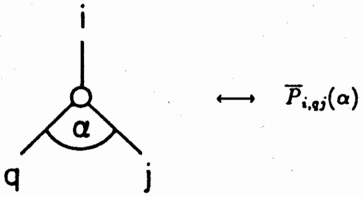
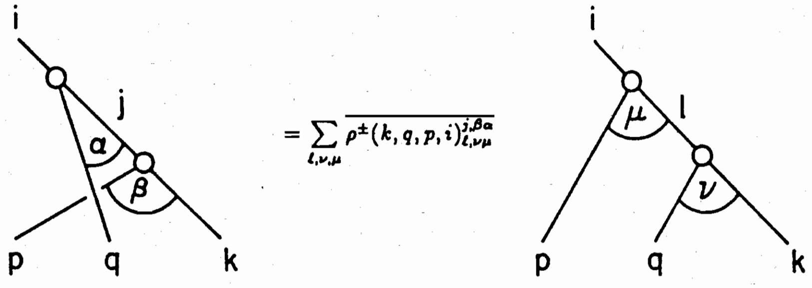


Figure 6.12

then, in addition to Figure 6.6, we obtain the graphical expansion (see(6.45)):



$$= \sum_{\ell,\nu,\mu} \overline{\rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}}$$

Figure 6.13

Applying

$$\left(\mathbf{1}_p \otimes R^{\mp}_{kq}\right)\ \left(R^{\mp}_{kp} \otimes \mathbf{1}_q\right)\ \left(\mathbf{1}_k \otimes R^{\mp}_{qp}\right)$$

to (6.52), and making use of (6.27) and of the Yang-Baxter equation (4.12), (6.52) takes the form

$$\overline{P}_{i,qj}(\alpha)\, R^{\mp}_{jq}\ \left(\overline{P}_{j,pk}(\beta)\, R^{\mp}_{kp} \otimes \mathbf{1}_q\right)\ \left(\mathbf{1}_k \otimes R^{\mp}_{qp}\right) =$$
$$\sum_{\ell,\nu\mu} \overline{\rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}}\ \overline{P}_{i,p\ell}(\mu)\ R_{\ell p}\left(\overline{P}_{\ell,qk}(\nu)\ R_{kq} \otimes \mathbf{1}_p\right)\ , \text{mod } \mathrm{Int}_o\left(V_i, V_k \otimes V_q \otimes V_p\right). \tag{6.54}$$

From the *SOS*-form of (6.54) we find the following factorized relation for the SC

weights:

$$\sum_{\nu'\mu'} \overline{\rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu'\mu'}}\ \left(\mathcal{N}^{\mp}_{kq,\ell}\right)^{\nu'\nu}\ \left(\mathcal{N}^{\mp}_{kp,j}\right)^{\mu'\mu} =$$
$$\sum_{\alpha'\beta'} \left(\mathcal{N}^{\mp}_{jq,i}\right)^{\beta\beta'}\ \left(\mathcal{N}^{\mp}_{jq,i}\right)^{\alpha\alpha'}\ \rho^{\mp}(k,p,q,i)^{\ell,\nu\mu}_{j,\beta'\alpha'}\,.$$

(6.

Here the sesquilinear forms $\mathcal{N}^{\mp}_{kq,\ell}$ on $\mathbb{C}^{N_{kq,\ell}}$, the spaces of multiplicities, are defined b

$$\left(\mathcal{N}^{+}_{kq,\ell}\right)^{\nu'\nu} := \overline{P}_{\ell,qk}(\nu')\, R^{+}_{kq}\, P_{kq,\ell} = P_{kq,\ell}{}^{*}(\nu')\, \mathcal{R} P_{kq,\ell}(\nu)\,,$$

and

$$\left(\mathcal{N}^{-}_{kq,\ell}\right)^{\nu'\nu} := \overline{P}_{k,qk}(\nu')\, R^{-}_{kq}\, P_{kq,\ell}(\nu) = P_{kq,\ell}{}^{*}(\nu')\, \sigma \mathcal{R}^{-1}\, P_{kq,\ell}(\nu)\,,$$

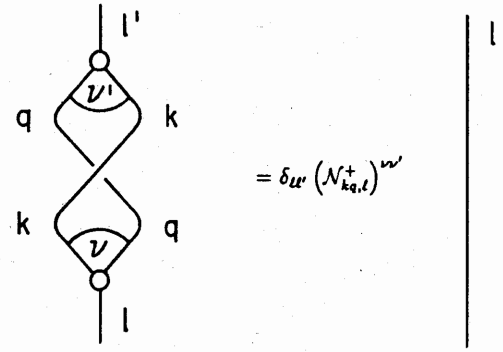
(6.

with

$$\mathcal{N}^{+}_{kq,\ell}{}^{*} = \mathcal{N}^{-}_{kq,\ell}\,.$$

(6.

Using the graphical expression for (6.56)



$$= \delta_{\ell\ell'} \left(\mathcal{N}^{+}_{kq,\ell}\right)^{\nu\nu'}$$

Figure 6.13

and with the help of Figure 6.12, (6.55) can also be derived from the diagram Figure 6.14, by either expanding the first braid from above, according to Figure 6.12, the first braid from below, according to Figure 6.6.

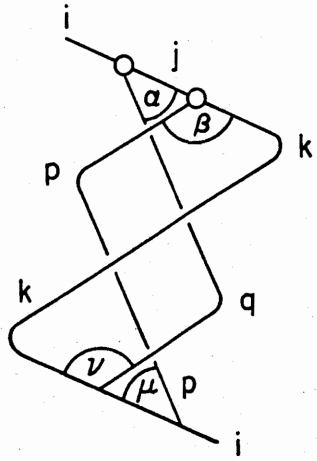


Figure 6.14

The symmetry properties of these forms can be expressed by the monodromy matrix

$$\mathcal{M} = \sigma \mathcal{R}\,\mathcal{R}\,, \tag{6.58}$$

which has matrix elements $\mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'}$, defined by

$$\mathcal{M} P_{kq,\ell}(\nu) = \sum_{\nu'} \mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'}\; P_{kq,\ell}(\nu')\,,\ \text{mod } \mathrm{Int}_o\,(V_k \otimes V_p, V_\ell)\,. \tag{6.59}$$

If we set

$$\langle x, y\rangle^{-}_{kq,\ell} = \sum_{\nu\nu'} \bar{x}_\nu \, \left(\mathcal{N}^{-}_{kq,\ell}\right)^{\nu\nu'} \, y_{\nu'} \tag{6.60}$$

for $x, y \in \mathbb{C}^{N_{kq,\ell}}$, we compute from (6.56) – (6.59) that

$$\overline{\langle x, y\rangle^{-}_{kq,\ell}} = \langle y, \mu(1,k,q,\ell)x\rangle^{-}_{kq,\ell}\,. \tag{6.61}$$

Identifying $\mathcal{P}(i|j_1,\ldots,j_n|k) = \sum^{\oplus}_{\{\mu\}} \mathbb{C}^{N_{ij_1,\mu_1}} \otimes \ldots \otimes \mathbb{C}^{N_{\mu_{n-1}j_n,\mu_n}}$, the inner product $\langle .,.\rangle_-$ defined in (6.60) extends naturally to the pathspace $\mathcal{P}(i|\{j_i\}|k)$, so that by (6.61)

$$\overline{\langle\omega',\omega\rangle_-} = \langle\omega, \mu\omega'\rangle_- \quad \text{for} \quad \omega,\omega' \in \mathcal{P}(i \mid \{j_i\} \mid k)\,, \tag{6.62}$$

where μ acts on $\mathbb{C}^{N_{ij_1,\mu_1}} \otimes \ldots \otimes \mathbb{C}^{N_{\mu-1j_n,\mu_n}}$ by $\mu(1,i,j_1,\mu_1) \otimes \ldots \otimes \mu(1,\mu_{n-1},j_n,\nu_n)$. With the help of $\langle .,.\rangle_-$, (6.55) can be reexpressed by

$$\left\langle R^P(\tau_i)\omega',\; R^P(\tau_i)\omega\right\rangle_- = \langle\omega',\omega\rangle_- \tag{6.63}$$

and

$$\left\langle R^P\left(\tau_i^{-1}\right)\omega',\; \mu R^P\left(\tau_i^{-1}\right)\mu^{-1}\omega\right\rangle_- = \langle\omega',\omega\rangle_-\,. \tag{6.64}$$

Since by the definition of the intertwiners $\{P_{kq,\ell}(\alpha)\}$, the form $\langle .,.\rangle_-$ is nondegenerate, we conclude from (6.64), that μ commutes with $R^P\,(b)$ for all $b \in B_n$, and equations (6.61) and (6.64) simplify to

$$\left\langle R^P(b)\omega',\; R^P(b)\omega\right\rangle_- = \langle\omega',\omega\rangle_- \tag{6.65}$$

and furthermore

$$\langle\mu\omega',\mu\omega\rangle_- = \langle\omega',\omega\rangle_-\,. \tag{6.66}$$

If we assume certain weak indecomposability conditions on R^P, we can deduce from $\mu \in \left(R^P(B_n)\right)'$, that μ is diagonal, i.e.

$$\mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'} = \delta_{\nu\nu'}\, e^{2\pi i\, S^{\nu}_{kq,\ell}} \tag{6.67}$$

and proportional to unity on $\mathcal{P}(i \mid \{j_i\} \mid k)$, which implies

$$S^{\alpha}_{kp,j} + S^{\beta}_{jq,i} = S^{\gamma}_{kq,\ell} + S^{\delta}_{\ell p,i} \qquad \text{mod } 1\,, \tag{6.68}$$

whenever all indices obey fusion rules. A solution of (6.68) has the form

$$S^{\nu}_{kq,j} = s_k + S_q - s_j + m_{kq\bar{j}} \qquad \text{mod } 1\,, \tag{6.69}$$

with $s_1 = 0$, $S_k = S_{\bar{k}}$ and m_{kqj} is totally symmetric,

$$\begin{aligned} m_{kqj} &= -m_{\bar{k}\bar{q}\bar{j}} \\ \text{and} \qquad\qquad & \\ m_{kp\bar{j}} + m_{jq\bar{i}} &= m_{kq\bar{\ell}} + m_{\ell p\bar{i}}\,. \end{aligned} \tag{6.70}$$

For highest weight representations, it follows, by application of χ, see (5.6), to (6.59) that

$$S^{\nu}_{kq,\ell} = S^{\nu}_{\bar{k}\bar{q},\bar{\ell}}$$

and eventually

$$m_{kqj} = 0 \qquad \text{and} \qquad S_k = S_{\bar{k}} \, .$$

Hence μ is given by

$$\mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'} = \delta_{\nu\nu'} \, e^{2\pi i(s_k + S_q - s_\ell)} \, . \tag{6.71}$$

Since $\mu(1,k,q,\ell)$ and $\mu(1,q,k,\ell)$ are equivalent matrices (by conjugation with a ρ-matrix) and, further, are unity if either $k=1$ or $q=1$, we find with (6.71) that

$$S_k = s_k \, . \tag{6.72}$$

The spins of the monodromy-spectrum can be deduced more directly if we assume that $\mathcal{K}$ is a ribbon-graph Hopf-algebra (Section 6.4 or [43, 44]), i.e.

$$\mathcal{M} = v \otimes v \, \Delta\left(v^{-1}\right) \qquad \text{and} \qquad v \text{ central in } \mathcal{K} \, ,$$

so that (6.59) reads

$$\begin{aligned} &\rho_k(v) \otimes \rho_q(v) \, P_{kq,\ell}(v) = \\ &\sum_{\nu'} \mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'} \, P_{kq,\ell}(\nu') \, \rho_\ell(v) \ \bmod \ \mathrm{Int}_o(V_k \otimes V_p, V_\ell) \end{aligned} \tag{6.73}$$

For an indecomposable representation V_p we have

$$\rho_p(v) = e^{2\pi i s_p} \mathbf{1} \qquad \bmod \mathrm{Int}_o(V_p, V_p) \, ,$$

so that again

$$\mu(1,k,q,\ell)^{k,1\nu}_{k,1\nu'} = \delta_{\nu\nu'} \, e^{2\pi i(s_k + S_q - s_\ell)} \tag{6.74}$$

with $s_o = 0$, $s_p = s_{\bar{p}}$, by $E(v) = 1$ and $S(v) = V$.

For $U_q(s\ell_{d+1})$ the spins are determined for highest-weight representations, with highest-weight λ, by the classical Casimir values

$$C_\lambda = \langle \lambda, \lambda \rangle + 2\langle \rho, \lambda \rangle \, , \tag{6.75}$$

so that

$$\rho_\lambda(v) = e^{2\pi i s_\lambda} = q^{C_\lambda} \, . \tag{6.76}$$

The original computation [43] used the fact that $U_q(s\ell_{d+1})$ is a one parameter-deformati of $U(s\ell_{d+1})$ and proceeded by analytically continuing the spectrum of $\mathcal{M}$ in q. A mc explicit way to find these values is given by computing the ribbon-graph-algebra eleme

$$v = L \otimes R\left(\sigma \mathcal{R}^{-1}\right) \left(q^{\delta}\right) , \qquad (6.7$$

with $L \otimes R(a \otimes b)(x) = axb$, from known ((4.87), [48]) formulae. If one applies the e pression (6.77) to highest-weight-vectors v_λ, only the term $q^{\left(\sum_{\alpha>0} h_\alpha + \sum_i h_i h^i\right)} v_\lambda$ will survi yielding the above expression for $\rho_\lambda(v)$. For $U_q(s\ell_2)$ we obtain

$$C_{\lambda_p} = \frac{1}{2}\left(p^2 - 1\right) , \qquad (6.7$$

so that for $q = e^{i\frac{\pi}{N}}$

$$s_p = \frac{p^2 - 1}{4N} \, . \qquad (6.7$$

Continuing our discussion

$$\overline{\langle x, y \rangle^{-}_{kq,\ell}} = e^{2\pi i(s_k + s_q - s_\ell)} \langle y, x \rangle^{-}_{kq,\ell} \, , \quad \text{with} \quad s_i \in \mathbb{R}/\mathbb{Z} \, ,$$

so that for any choice of $s_i \in \mathbb{R}/2\mathbb{Z}$ the form

$$\langle x, y \rangle^{o}_{kq,\ell} = \langle x, y \rangle^{\mp}_{k,q\ell} \, e^{\pm i\pi(s_k + s_q - s_\ell)} \qquad (6.8$$

is symmetric and hence admits an orthogonal basis $\{e_\nu\}$ of $\mathbb{C}^{N_{kq,\ell}}$ with

$$\langle e_\nu, e_\mu \rangle = (-1)^{n^\nu_{kq,\ell}} \, \delta_{\nu,\mu} \, . \qquad (6.8$$

Inserting (6.81) into (6.55) we see by

$$\overline{\rho^{\pm}(k,q,p,i)^{j,\beta\alpha}_{\ell,\nu\mu}} (-1)^{n^\nu_{kq,\ell} + n^\mu_{\ell p,i}} = (-1)^{n^\beta_{kp,j} + n^\alpha_{jq,i}} \, \rho^{\mp}(k,p,q,i)^{\ell,\nu\mu}_{j,\beta\alpha} \qquad (6.8$$

that for this choice of basis R^P represents B_n in some $\mathcal{U}(N,M)$. If we assume unitari the numbers $n^\nu_{kq,\ell} \in \mathbb{Z}_2$ will satisfy constraints similar to the ones imposed on $S^\nu_{kq,\ell}$ (6.68), so that they can also be presented as

$$n^\nu_{ij,k} = n_i + n_j - n_k \qquad \bmod 2$$

and thus correspond just to a redefinition of the spins. We summarize these argume in the following Lemma.

Lemma 6.3.2 *The representations R^P of the braid group, defined by a quasi-triangular ribbon-graph Hopf-algebra with* $*$-involution *are unitarizable iff there exists a choice of spins*

$$s_i \in \mathbb{R}/2\mathbb{Z}$$

such that all the forms $\langle .,. \rangle^{o}_{kq,\ell}$ defined in (6.60) and (6.56) are positive definite.

As an application of Lemma 6.3.2 we shall show unitarizability of R^P for $\mathcal{K} = U_q(s\ell_2)$ with q^2 a *primitive* root of unity.

For $p_1, p_2 = 1, \ldots, N-1$, $q \neq 0$, we define the continuously q-dependent matrices $e(q)$ and $f(q)$ in $\mathrm{Mat}(V_p)$, V_p being the inner product space $V_p = \left\langle \xi^{p}_{-\frac{p-1}{2}}, \ldots, \xi_{\frac{p-1}{2}} \right\rangle$, by the normalized representation (5.17), so that

$$e(q) = e(q^{-1}) \qquad \text{and} \qquad e(q)^* = f(\bar{q}) \,. \tag{6.83}$$

In the domain $D_N = \{t \in \mathbb{C} \mid t \neq 0;\ t^{4j} \neq 1\ j = 1, \ldots, N-1\}$ the map

$$\begin{aligned} D_N &\rightarrow \mathrm{Mat}\,(V_{p_1} \otimes V_{p_2}) \\ t &\rightarrow \mathcal{R}(t) \end{aligned}$$

is by (4.87), with $t = q^{\frac{1}{2}}$, well defined, continuous and obeys by (4.23), (4.24) and (6.83)

$$\mathcal{R}(t)^{-1} = \mathcal{R}\left(t^{-1}\right) \qquad \text{and} \qquad \mathcal{R}(t)^* = \sigma \mathcal{R}(\bar{t}) \,. \tag{6.84}$$

The spins $s_p \in \mathbb{R}/2\mathbb{Z}$ are determined by

$$e^{i\pi s_p} = t^{\frac{p^2-1}{2}} \,. \tag{6.85}$$

From (5.23) we have highest weight vectors ξ^i in $V_{p_1} \otimes V_{p_2}$, with

$$h\xi^i(t) = (i-1)\,\xi^i(t) \qquad \text{and} \qquad P_{p_1p_2}\xi^i(t) = \xi^i(t^{-1}) \,, \tag{6.86}$$

for $i = |p_1 - p_2|, \ldots, p_1 + p_2 - 2, p_1 + p_2$. We now consider the expression

$$\mathcal{N}^{o}_{p_1p_2,i}(t) = \left\langle \xi^i(t),\ \mathcal{R}(t)\xi^i(t) \right\rangle\ t^{-\frac{1}{2}(p_1^2+p_2^2-i^2+1)} \tag{6.87}$$

for which we find by (6.84), (6.85) and (6.86)

$$\overline{\mathcal{N}^{o}_{p_1p_2,i}(t)} = \left\langle \xi^i(t),\ \sigma(\mathcal{R}(\bar{t})\,\mathcal{R}(t))\ \mathcal{R}(t)\,\xi^i(t) \right\rangle \cdot (|\,t\,| \cdot t)^{-\frac{1}{2}(p_1^2+p_2^2-i^2+1)} \,. \tag{6.88}$$

Here $\langle .,. \rangle$ denotes the canonical inner product on the tensor product space $V_{p_1} \otimes V_{p_2}$.

If we restrict the values of $t \in D_N$, by $|t| = 1$, (6.88) implies

$$\mathcal{N}^{o}_{p_1p_2,i}(t) \in \mathbb{R} \,. \tag{6.89}$$

Comparing (6.87) to (6.56) we see that $\mathcal{N}^{o}_{p_1p_2,i}(t)$ is the square of a multiplicity vector with respect to the form defined in (6.80), and is therefore nonzero for

$$\dim \mathcal{P}_t\,(V_{p_1} \otimes V_{p_2}, V_i) = 1 \,. \tag{6.90}$$

If for fixed i, p_1, p_2 (6.90) is true for $t = e^{i\pi\frac{n}{2N}}$, $(n, N) = 1$, then we find from the fusion rules (5.19), that it also holds for $t = e^{i\pi\frac{n'}{2N'}}$, $(n', N') = 1$ with $N' \geqslant N$ or for generic t.

Hence if, for $t = e^{\frac{i\pi}{2N}}$, $N_{p_1p_2,i} = 1$, then we have

$$\mathcal{N}^{o}_{p_1p_2,i}(t) \neq 0 \qquad \text{for} \qquad \arg(t) \leqslant \frac{\pi}{2N} \,. \tag{6.91}$$

From

$$\mathcal{N}^{o}_{p_1p_2,i}(1) = \left\langle \xi^i(1),\ \xi^i(1) \right\rangle > 0 \tag{6.92}$$

we obtain

$$\mathcal{N}^{o}_{p_1p_2,i}(t) > 0 \qquad \text{for} \qquad t = e^{i\frac{\pi}{2N}} \,. \tag{6.93}$$

Combining (6.93) with Lemma 6.3.2 we find the following lemma.

Lemma 6.3.3 *For $U_q(s\ell_2)$, with $t = e^{i\frac{\pi}{2N}}$, the braid matrices*

$$\rho^{\pm}(i, p, q, k)^{j}_{\ell}$$

define unitarizable *representations of the braid group.*

6.4 Markov Traces

The definition of q-dimensions is generalized, using the observation of Drinfel'd [5], that the square of the antipode of a quasi-triangular Hopf algebra is an inner automorphism . We extend the selection criteria already encountered in Section 5.4 to the general case, i.e., we show that the set of indecomposable representations with zero-q-dimensions is an ideal under forming tensor products. This completes the rigorous construction of brai group representations on path spaces from quasi-triangular Hopf algebras in the case where semisimplicity is not assured. We define a Markov trace on these representations and identify spin, statistics parameter and statistical dimension with central elements of a ribbon-graph Hopf algebra, as defined by Reshetikhin and Turaev [44].

The discussion of Markov traces and their role in the vertex-SOS-correspondence requires certain restrictions on the Hopf algebra $\mathcal{K}$ and its representations. The first is the restriction to quasitriangular ribbon-graph Hopf algebras, introduced in [44], that contain a central element v with

$$v^2 = uS(u)\,; \qquad S(v) = v\,, \qquad E(v) = 1 \tag{6.94}$$

and

$$\mathcal{M} = v \otimes v\,\Delta(v^{-1})\,, \tag{6.95}$$

where $u = m(1 \otimes S^{-1})\sigma\mathcal{R}$. Suppose that $\mathcal{K}$ admits a star involution satisfying (4.24). In this case u is unitary, so that we have $p^2 = 1$, for $p = vv^*$. On unitary representations we therefore have, from $\rho_j(p) > 0$, that v is unitary. Its eigenvalue on V_j is thus identified with the phase factor $e^{2\pi i s_j}$ and we have $s_j = s_{\bar{j}}$.

The element $g = uv^{-1}$ satisfies

$$\begin{array}{rcll} & S^2(a) & = & gag^{-1} \qquad \forall a \in \mathcal{K} \\ \text{and} & & & \\ & \Delta(g) & = & g \otimes g \end{array} \tag{6.96}$$

and gives rise to a general definition of the q-dimension, d_p, of an indecomposable representation V_p:

$$d_p := tr(g \restriction V_p)\,. \tag{6.97}$$

In the following we shall consider a set $\mathcal{L}^t$ of indecomposable representations that close under taking tensor products, i.e.,

$$W_\alpha \otimes W_\beta = \sum_\gamma{}^{\oplus}\, W_\gamma \otimes \mathbb{C}^{N_{\alpha\beta,\gamma}}\,, \qquad \text{(6.98}$$

and conjugation, i.e., for each $\alpha \in \mathcal{L}^t$, there is some $\alpha^\vee \in \mathcal{L}^t$, with $\rho_{\alpha^\vee}(a) = \rho^t_\alpha(S(a)$ The fusion rules $\{N_{\alpha\beta,\gamma}\}$ again commute and are symmetric in the first two indices. W have the following result.

Lemma 6.4.1 *For a system $\mathcal{L}^t$ of indecomposable representations of a ribbon-graph Ho algebra, closed under taking tensor products and conjugation, we have*

$$N_{\alpha\beta^\vee,1} = 1 \qquad \text{iff} \qquad \alpha = \beta \quad \text{and} \quad d_\alpha \neq 0\,.$$

This follows from the fact that we have the identifications

$$\text{Int}\,(W_\alpha \otimes W_{\beta^\vee}, 1) \cong \text{Int}\,(W_\beta, W_\alpha) \cong \text{Int}\,(1, W_\alpha \otimes W_{\beta^\vee})$$

given by

$$\begin{array}{rrcl} & \langle \ell \otimes x\,,\ P_{\alpha\beta^\vee,1}(I)\rangle & = & \ell(Ix) \\ \text{and} & & & \\ & P_{1,\alpha\beta^\vee}(I')\,\ell \otimes x & = & \ell(I'gx) \end{array} \qquad \text{(6.9}$$

for $I, I' \in \text{Int}(W_\alpha, W_\beta)$, $x \in W_\beta$, $\ell \in W^*_\alpha$.

The composition is given by

$$\begin{array}{rcl} P_{1,\alpha\beta^\vee}(I)\,P_{\alpha\beta^\vee,1}(I') & = & tr(gII' \restriction W_\alpha) \\ & = & \dfrac{d_\alpha}{(\dim W_\alpha)}\,tr(II' \restriction W_\alpha)\,. \end{array} \qquad \text{(6.10}$$

In the last identity we used the fact that the W_α is indecomposable, i.e.

$$\text{Int}\,(W_\alpha, W_\alpha) = \mathbb{C}\cdot\mathbf{1} \oplus \text{Int}_o(W_\alpha, W_\alpha)\,, \qquad \text{(6.10}$$

where $\text{Int}_o(W_\alpha, W_\alpha)$ only consists of nilpotent mappings. The expression (6.100) is nc zero iff $d_\alpha \neq 0$ and II' is invertible. Since W_β is also indecomposable, by assumption, t latter implies $W_\alpha \cong W_\beta$.

From the commutativity and associativity of the tensor product we infer, that

$$N_{\alpha\beta\gamma,1} = N_{\alpha\beta,\gamma^\vee}\, N_{\gamma^\vee\gamma,1}$$

is completely symmetric in all representation labels. If $\mathcal{L}^t$ is generated from some fundamental set $\mathcal{F}$ with $\mathcal{F}^\vee = \mathcal{F}$ and $\mathcal{F} \cap \mathcal{L}_g = \emptyset$, where $\mathcal{L}_g$ is the set of indecomposable representations with vanishing q-dimension, then $\mathcal{L}_o := \mathcal{L}_g \cap \mathcal{L}^t$ is a maximal conjugation invariant subset that obeys

$$\begin{aligned} \mathcal{L}^t \otimes \mathcal{L}_o &\subset \mathcal{L}_o \\ \text{and} \qquad \mathcal{F} \cap \mathcal{L}_o &= \emptyset . \end{aligned} \tag{6.102}$$

Hence, defining $\mathcal{L} = \mathcal{L}^t \backslash \mathcal{L}_o$, we have the following decomposition laws

$$\begin{aligned} V_i \otimes V_j &= \sum_{k\in\mathcal{L}}^{\oplus} V_k \otimes \mathbb{C}^{N_{ij,k}} \oplus \sum_{\alpha\in\mathcal{L}_o}^{\oplus} W_\alpha \otimes \mathbb{C}^{N_{ij,\alpha}} \\ V_i \otimes W_\alpha &= \sum_{\beta\in\mathcal{L}_o}^{\oplus} W_\beta \otimes \mathbb{C}^{N_{i\alpha,\beta}} \\ W_\alpha \otimes W_\beta &= \sum_{\gamma\in\mathcal{L}_o}^{\oplus} W_\gamma \otimes \mathbb{C}^{N_{\alpha\beta,\gamma}} . \end{aligned} \tag{6.103}$$

Generalizing Lemma 6.1.3 by using (6.101), this allows us to identify the quotient space $\mathcal{P}(V_i \otimes V_{j_1} \otimes \ldots \otimes V_{j_n}, V_k)$ with the path spaces $\mathcal{P}(i|j_1, \ldots, j_n|k)$ constructed from the fusion rules $\{N_{ij_r,k}\}$ for $i, j_r, k \in \mathcal{L}$.

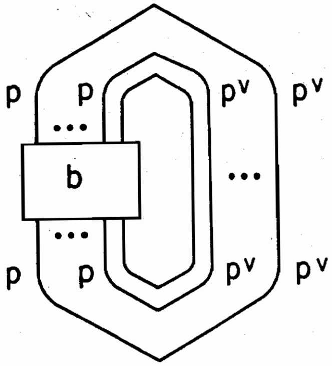


Figure 6.15

Consequently, it is always possible to assign to a pair $(\mathcal{K}, \mathcal{F})$ a path representation of the braid group in a unique way. For any path representation of the braid group B_n with fusion matrices $\{F\}$ (resp. $\{\varphi\}$), we can define Markov trace as in (2.67). If we take the *SOS* expansion of the operator depicted in Figure 6.15 and use (6.99) we obtain the following expression in the vertex picture.

Lemma 6.4.2 *For any pair $(\mathcal{K}, \mathcal{F})$ with $\mathcal{F}^\vee = \mathcal{F}$ and $\mathcal{F} \cap \mathcal{L}_g = \emptyset$, and for the definition of the path representation of B_n given as above, the Markov trace is determined by*

$$\tau_M^p(b) = \frac{1}{d_p^n}\, tr\left(g^{\otimes n} R^P(b) \restriction V_p^{\otimes n}\right) . \tag{6.104}$$

□

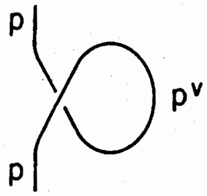
This trace has an obvious generalization to different colorations (i.e., different representations involved) if we restrict b to the appropriate subgroup of B_n.

By Lemma 6.1.3 we have that

$$\tau_M^p(b) = \sum_{k\in\mathcal{L}} \frac{d_k}{d_p^n}\, tr\left(R^p(b) \restriction \mathcal{P}\left(V_p^{\otimes n}, V_k\right)\right) . \tag{6.105}$$

This show that τ_M^p is positive for all n iff $d_p > 0$ for all $p \in \mathcal{L}$.

We can easily compute the statistical parameter of a representation from the diagram



$$= (\mathbf{1}_p \otimes P_{1,pp^\vee})\ (R_{pp} \otimes \mathbf{1}_{p^\vee})\ (\mathbf{1}_p \otimes P_{pp^\vee,1}) \tag{6.106}$$

$$= \lambda_p$$

From $a \otimes \mathbf{1}_{p^\vee} P_{pp^\vee,1} = \mathbf{1}_p \otimes S(a) P_{pp^\vee,1}$, $\forall a \in \mathcal{K}$, $u = m(1 \otimes S^{-1})\sigma\mathcal{R}$ and (6.100), we have

$$\lambda_p = \frac{1}{tr(u \restriction V_p)} = \frac{\rho_p(v^{-1})}{d_p} = \frac{e^{-2\pi i s_p}}{d_p}.$$

The analogue of Theorem 2.4.c) can be shown by inserting the projection $\mathbb{P}^r_{pq} = \sum_\nu P_{pq,r}(\nu)\, P_{r,pq}(\nu)$ into (6.105) and making use of

$$tr\left(\mathbb{P}^r_{pq} \restriction \mathcal{P}(V_p \otimes V_q, V_\ell)\right) = N_{pq,r}\delta_{\ell r}\,.$$

Chapter 7

Duality Theory for Local Quantum Theories, Dimensions and Balancin in Quantum Categories

7.1 General Definitions, Towers of Algebras

In this section we give the complete definition of a quantum category. We show that quantum category can equivalently be described by a system of structure constants, name fusionrules, and R- and F- matrices. We also introduce C^*-structures and discuss th consequences for the existence of balancing phases, positive traces and dimensions. explain the result of Doplicher and Roberts on the duality of compact groups and propos generalized notion of duality. Finally, we show how quantum categories arise in algebr field theory and relate them to the theory a subfactors and towers developed by Jones.

The structural data of local quantum theories, in terms of fusionrules and R- and matrices, which we investigated in chapter 2, and the data obtained from the intertwi calculus for quasitriangular Hopf algebras explained in chapter 6 fullfill the same ty of equations, which were, in our language, interpreted in the graphical Yang-Baxt and Polynomial Equations. In fact, in the construction of charged field operators w permutation statistics and gauge group symmetry, as proposed in [19] it is needed that these two sets of structure matrices are equal. In order to organize our language, it is helpful to observe that fusionrules, R- and F- matrices are precisely the structure constants needed to determine (up to equivalence) a certain type of braided tensor categories. We review the notions entering their definition :

i) We start with a semisimple, abelian, finite, reduced category over $\mathbb{C}$. It consists of a set, $\mathcal{O}bj$, called the objects. To any pair of objects $X, Y \in \mathcal{O}bj$ is associated a vectorspace, denoted $Mor(X,Y)$ or $Int(Y,X)$, over $\mathbb{C}$, called the (space of) morphisms from X to Y. We have distributive, associative composition

$$Mor(Y,Z) \otimes Mor(X,Y) \rightarrow Mor(X,Z)$$

so that, in particular, $End(X) := Mor(X,X)$ is an associative $\mathbb{C}$- algebra with unit. Semisimplicity of the category means that $End(X)$ is semisimple and that the pairing $Mor(X,Y) \otimes Mor(Y,X) \rightarrow End(Y)$ is non-degenerate. In this case the category is abelian iff it has subobjects and direct sums. The subobject requirement is that to any projector $\Pi \in End(X)$ there exists an object U and morphisms $I_U \in Mor(U,X)$ and $P_U \in Mor(X,V)$, such that $P_U I_U = 1$ and $\Pi = I_U P_U$. If we consider also the object V and morphisms P_V and I_V associated to the projector $1 - \Pi$ we obtain what is called a biproduct, $X = U \oplus V$. The axiom of direct sums states that to any pair of objects, U and V, there exists an object X with a biproduct, $X = U \oplus V$. We call a category reduced if equivalent objects are equal, i.e., if for two objects X and Y there are morphisms $f \in Mor(X,Y)$ and $g \in Mor(Y,X)$, with $fg = 1$ and $gf = 1$, then $X = Y$. With these assumptions any object, X, with $dim(End(X)) < \infty$ can be decomposed into a finite direct sum of irreducible objects,

$$X = \bigoplus_{j \in \mathcal{L}} N_{X,j} j ,$$

where $j \in \mathcal{L}$ iff $End(j) = \mathbb{C}$. The category is said to be finite if $dim(End(X)) < \infty$ for all objects $X \in \mathcal{O}bj$ and rational if $|\mathcal{L}| < \infty$. Thus, the objects are naturally identified with $\mathbb{N}^{\mathcal{L}}$.

ii) A tensorproduct on such a category consists of a binary operation, $\circ : \mathcal{O}bj \times \mathcal{O}bj \rightarrow \mathcal{O}bj : (X,Y) \rightarrow X \circ Y$, together with a bilinear product of morphisms

$$\circ : Mor(X,X') \otimes Mor(Y,Y') \rightarrow Mor(X \circ Y, X' \circ Y') : I \otimes J \rightarrow I \circ J .$$

This product shall be compatible with composition, in the sense that

$$(I \circ J)(I' \circ J') = (II') \circ (JJ') ,$$

whenever defined, which makes $\circ$ into a distributive operation on $\mathbb{N}^{\mathcal{L}}$. Thus, the tensorproduct on $\mathcal{O}bj$ is completely determined by the fusionrules :

$$i \circ j = \sum_{k \in \mathcal{L}} N_{ij,k} k .$$

with $i, j \in \mathcal{L}$.

iii) A category is called a tensor category or monoidal category if there is an isomorphism, $\alpha(X,Y,Z) \in Mor(X \circ (Y \circ Z), (X \circ Y) \circ Z)$, which satisfies the pentagonal equation

$$\alpha(W \circ X, Y, Z)\,\alpha(W, X, Y \circ Z) = (\alpha(W,X,Y) \circ 1_Z)\,\alpha(W, X \circ Y, Z)\,(1_W \circ \alpha(X,Y,Z))$$

and the isotropy equation

$$\alpha(X',Y',Z')(I \circ (J \circ K)) = ((I \circ J) \circ K)\,\alpha(X,Y,Z)$$

for all possible objects. This makes $(\mathcal{O}bj, \circ)$ into an associative algebra. Moreover, we may define F- matrices by the commutative diagram of isomorphisms :

$$\begin{tikzcd}
\oplus_{l \in \mathcal{L}} Mor(l, j \circ k) \otimes Mor(t, i \circ l) \arrow[r, "F(i,j,k,t)"] \arrow[d, "\cong"] & \oplus_{l \in \mathcal{L}} Mor(l, i \circ j) \otimes Mor(t, l \circ k) \arrow[d, "\cong"] \\
Mor(t, i \circ (j \circ k)) \arrow[r, "\alpha(i,j,k)"] & Mor(t, (i \circ j) \circ k),
\end{tikzcd} \tag{7.1}$$

with $i, j, k, t \in \mathcal{L}$. Here the vertical arrows are given by the compositions $I \otimes J \rightarrow (1 \circ I)J$, and $I \otimes J \rightarrow (I \circ 1)J$, and the lower horizontal arrow is defined by left multiplication of α. The F-matrices obey an analogous pentagonal equation,

$$\begin{aligned}(\oplus_s F(i,j,k,s) \quad \otimes 1_{N_{sl,t}})\,(\oplus_s 1_{N_{jk,s}} \otimes F(i,s,l,t))\,(\oplus_s F(j,k,l,s) \otimes 1_{N_{is,t}})\\ = (\oplus_s 1_{N_{ij,s}} \otimes F(s,k,l,t))\,T_{12}\,(\oplus_s 1_{N_{kl,s}} \otimes F(i,j,s,t)),\end{aligned} \tag{7.2}$$

and any such system of F-matrices defines a unique associativity constraint α. The category is called strict if $\alpha = 1 \in End(X \circ Y \circ Z)$.

iv) A tensor category is called braided if there exists for any pair of objects $X, Y \in \mathcal{O}bj$ an isomorphism $\varepsilon(X,Y) \in Mor(X \circ Y, Y \circ X)$, which satisfies the hexagonal equations:

$$\alpha(Z,X,Y)\,\varepsilon^{\pm}(X \circ Y, Z)\,\alpha(X,Y,Z) = (\varepsilon(X,Z)^{\pm} \circ 1_Y)\,\alpha(X,Z,Y)\,(1_X \circ \varepsilon^{\pm}(Y,Z)),$$

where $\varepsilon \equiv \varepsilon^+$ and $\varepsilon^-(X,Y) = \varepsilon(Y,X)^{-1}$,and the isotropy equation

$$\varepsilon(X',Y')(I \circ J) = (J \circ I)\varepsilon(X,Y).$$

We define structure matrices,

$$r^{\pm}(i,j,k) : Mor(k, i \circ j) \xrightarrow{\cong} Mor(k, j \circ i)$$

by left multiplication with $\varepsilon^{\pm}(i,j)$. They fullfill the respective hexagonal equation,

$$\begin{aligned}(\oplus_l r^{\pm}(i,k,l) \quad \otimes 1_{N_{lj,t}})\; F(i,k,j,t)\; (\oplus_l r^{\pm}(j,k,l) \otimes 1_{N_{il,t}})\\ = \quad F(k,i,j,t)\; (\oplus_l 1_{N_{ij,l}} \otimes r^{\pm}(l,k,t))\; F(i,j,k,t),\end{aligned} \tag{7.3}$$

and a system of r-matrices obeying (7.3) defines a unique commutativity constraint ε. Frequently, we shall use the R-matrices,

$$R^{\pm}(i,j,k,t) : \bigoplus_{l \in \mathcal{L}} Mor(l, i \circ j) \otimes Mor(t, l \circ k) \longrightarrow \bigoplus_{l \in \mathcal{L}} Mor(l, i \circ k) \otimes Mor(t, l \circ j),$$

defined by

$$R^{\pm}(i,j,k,t) = F(i,k,j,t)(\bigoplus_s r(j,k,s)^{\pm} \otimes 1)F(i,j,k,t)^{-1}. \tag{7.4}$$

A braided tensor category is called symmetric if $\varepsilon^+ = \varepsilon^-$.

v) The category is rigid if to any object $X \in \mathcal{O}bj$ one can associate a conjugate objec $X^\vee$ and morphisms $\vartheta_X \in Mor(1, X^\vee \circ X)$ and $\vartheta_X^\dagger \in Mor(X \circ X^\vee, 1)$, such that

$$\begin{aligned}(\vartheta_X^\dagger \circ 1_X)\,\alpha(X, X^\vee, X)\,(1_X \circ \vartheta_X) &= 1_X\\ \text{and} \qquad (1_{X^\vee} \circ \vartheta_X^\dagger)\,\alpha(X^\vee, X, X^\vee)^{-1}\,(\vartheta_X \circ 1_{X^\vee}) &= 1_{X^\vee}.\end{aligned}$$

(7.

If these objects and morphisms exist then they are unique up to isomorphism starting in $X^\vee$. Also the equations $(X \oplus Y)^\vee = X^\vee \oplus Y^\vee$ and $(X \circ Y)^\vee = Y^\vee \circ X$ hold true in a reduced category. A choice of conjugates yields a transposition

$${}^t : Mor(X,Y) \xrightarrow{\cong} Mor(Y^\vee, X^\vee)$$

and more generally an isomorphism

$$Mor(X, Y \circ Z) \cong Mor(X \circ Z^\vee, Y),$$

which for the symmetric, bilinear form $(X,Y) = dim(Mor(X,Y))$ provides equation (3.2). The conjugation defines an involution on the set of irreducible object $\mathcal{L}$, and we can verify the axioms of a fusion rule algebra given in chapter 3.2 for th algebra $(\mathcal{O}bj, \circ)$.

In the following we shall call an abelian, semisimple, finite, rigid, braided tens category a quantum category. As opposed to symmetric categories the equation

$$(1_{X^\vee} \circ \tau(X))\vartheta_X = \mu(X^\vee, X)\vartheta_X \;,$$

(7.

with $\mu(X,Y) := \varepsilon(Y,X)\varepsilon(X,Y)$, defines set of non-trivial automorphisms $\tau(X)$ $End(X)$.

Lemma 7.1.1 *The automorphisms defined in (7.6) have the following properties:*

a) $\tau(X)$ *is independent of the choice of conjugates* $(X^\vee, \vartheta_X^\dagger, \vartheta_X)$

b) $\tau(Y)I = I\tau(X)$ *for all* $I \in Mor(X,Y)$

c) $\tau(X^\vee) = \tau(X)^t$

d) $\tau(X) \circ \tau(Y) = \mu(X,Y)^2 \tau(X \circ Y)$

Considering equation d) of Lemma 7.1.1, it is reasonable to introduce a notion of a square root of $\tau(X)$. Also we wish to introduce categories with a $*$-structure :

vi) A quantum category is balanced if there exist automorphisms $\sigma(X) \in End(X)$ such that

  a) $\sigma(X)^2 = \tau(X)$

  b) $\sigma(Y)I = I\sigma(X)$ for all $I \in Mor(X,Y)$

  c) $\sigma(X^\vee) = \sigma(X)^t$

  d) $\sigma(X) \circ \sigma(Y) = \mu(X,Y)\sigma(X \circ Y)$

  It is evident that any balancing $\{\sigma(X)\}_X$ can be multiplied by a $\mathbb{Z}_2$-grading of the category, in order to obtain a new balancing structure and that any two balancings differ by a $\mathbb{Z}_2$-grading. From b) we have that a balancing is uniquely determined by the numbers $\sigma(j) \in \mathbb{C}$.

vii) A C^* category is an abelian category if the morphisms form Banach spaces with an antilinear involution * : $Mor(X,Y) \to Mor(Y,X)$ such that $\|IJ\| \leq \|I\|\|J\|$, $\|I^*\| = \|I\|$, $\|I^*I\| = \|I\|^2$ and $(IJ)^* = J^*I^*$. It is clear that any C^*-category is semisimple and that it is, up to *-isomorphism, uniquely determined by the set $\mathcal{L}$ of irreducible objects. A C^*-quantum category is a quantum category with a C^*-structure such that $(I \circ J)^* = I^* \circ J^*$ and α and ε are unitary. The spaces $Mor(k, i \circ j)$ thus admit an inner product and the R- and F- matrices are unitary with repect to this product. Conversely, any unitary set of such structural data uniquely defines a C^*-quantum category.

A peculiar feature of C^*-quantum categories is that they are always balanced.

Lemma 7.1.2 *In a C^*-quantum category let $\lambda_X \in End(X)$ be defined by*

$$(\varepsilon(X^\vee, X)\vartheta_X)^* = \vartheta_X^\dagger(\lambda_X \circ 1) \tag{7.7}$$

We have that λ_X is normal and that its unitary part $\sigma_o \in U(X)$ in a polar decomposition $\lambda_X = \sigma_o(X)^{-1}P_X$, with$P_X > 0$, is a balancing structure of the category.

A final important structural ingredient in the study of C^*-quantum categories are traces. In order for a trace on the endomorphism spaces to factorize with respect to the tensorproduct, we have to use the balancing structure in its definition:

Lemma 7.1.3 *For a balanced quantum category we define a set of linear functionals, $tr_X \in (End(X))^*$, by*

$$tr_X(I) = \vartheta_X^\dagger((I\sigma(X)^{\mp 1}) \circ 1)\varepsilon^\pm(X^\vee, X)\vartheta_X \tag{7.8}$$

It has the following properties:

a) tr_X is independent of the choice of conjugates.

b) $tr_Y(IJ) = tr_X(JI)$ for all $I \in Mor(X,Y)$ and $J \in Mor(Y,X)$.

c) $tr_{(X \circ Y)}(I \circ J) = tr_X(I)tr_Y(J)$ for all $I \in End(X)$, $J \in End(Y)$.

d) $tr_X(I) = tr_{X^\vee}(I^t)$ for all $I \in End(X)$.

e) If we have a C^*-quantum category and tr_X is defined with respect to the canonical balancing $\{\sigma_o(X)\}_X$ given in Lemma 7.1.2 then it is a positive state on the C^*-algebra $End(X)$.

From Lemma 7.1.3 it follows that

$$d(X) := tr_X(1_X) \tag{7.9}$$

is a dimension and, for C^*-quantum categories, it is positive for the balancing $\{\sigma_o(X)\}_X$. Hence, in the latter case it coincides, for rational categories, with the unique Perron Frobenius dimension given in (3.30).

The best known example of a C^*-quantum category is the representation category, $Rep(G)$, of a compact group G. Its obejects are the inequivalent, finite dimensional, unitary representations of G and the morphisms are the intertwiners, $Hom_G(V,W) \equiv Int(W,V) \equiv Mor(V,W)$, between these representations. The conjugation is given by passing to the contragredient representations, and the commutativity constraint is given by the transposition $\varepsilon(V,W)(v \otimes w) = w \otimes v$ of factors in $V \circ W \equiv V \otimes W$. This is a strict, symmetric C^*-quantum category, with $\sigma_o(X) = 1$, for all $X \in \mathcal{O}bj$.

More generally, we can consider the representation category $Rep(\mathcal{K})$ of a quasi-triangular quasi Hopf algebra $\mathcal{K}$. The antipode, the $\mathcal{R}$-matrix and the ϕ-matrix yield the conjugate objects, the commutativity constraint and the associativity constraint, respectively, using formulae (5.4) and (5.5). A balancing structure is implemented for a ribbon-graph Hopf algebra by the special, central element v from (6.94) and(6.95). This category is semisimple - and hence a quantum category - if $\mathcal{K}$ is semisimple. However, in the case of primary interest to us $\mathcal{K}$ is not semisimple and we have to divide out the ideal of intertwiners discussed in Chapter 6.1. Using the trace introduced above we can give a more general and concise definition of the Int_o-spaces, namely

$$Int_o(V,W) := \{I \in Int(V,W) : tr_{(V)}(IJ) = 0,\ \forall J \in Int(W,V)\}.$$

We denote this quotient category by $\overline{Rep}(\mathcal{K})$. Here, the trace $tr_(V)$, defined on $End_{\mathcal{K}}(V) \equiv End((V))$ in Lemma 7.1.3, is related to the canonical trace tr_V on $End_{\mathcal{C}}(V)$ by

$$tr_(V)(I) = tr_V(gI)\,,$$

where g is as in (6.96).

Two quantum categories are <u>equivalent</u> if there exists an invertible, compatible tensor functor between them. On the level of structural data, equivalence is expressed as follows: Suppose we have two quantum categories, one characterized by the set of structural data $\{\mathcal{L}, N_{ij,k},\ F(i,j,k,t),\ R(i,j,k,t)\}$, the other one by the respective set of data $\{\tilde{\mathcal{L}}, \tilde{N}_{ij,k},\ \hat{\varphi}(i,j,k,t),\ \rho(i,j,k,t)\}$. Then the two categories are equivalent iff

a) There is a bijection

$$' : \mathcal{L} \to \tilde{\mathcal{L}} \quad : i \to i' \tag{7.10}$$

such that

$$N_{ij,k} = \tilde{N}_{i'j',k'}$$

b) There is a set of isomorphisms

$$T^k_{ij} : Mor(k, i \circ j) \to \tilde{Mor}(k', i' \circ j')$$

such that

$$\begin{aligned} \hat{\varphi}(i,j,k,t)\,(\oplus_l\, T^l_{jk} \otimes T^t_{il}) &= (\oplus_l\, T^l_{ij} \otimes T^t_{lk})\, F(i,j,k,t) \\ \rho(i,j,k,t)\,(\oplus_l\, T^l_{ij} \otimes T^t_{lk}) &= (\oplus_l\, T^l_{ik} \otimes T^t_{lj})\, R(i,j,k,t) \end{aligned} \qquad (7.1$$

Note that it is sufficient to specialize to $i = 1$, i.e., to the $r(i,j,k)$ -matrices, in t second equation of b). In the case of C^*-categories the isomorphisms T^k_{ij} are assumed be unitary. We next quote the famous result of Doplicher and Roberts on the duality compact groups.

Theorem 7.1.4 *[29] Suppose C is a strict, symmetric C^*-quantum category with $\sigma_o(X)$ 1, for all $X \in \mathcal{O}bj$. Then there exists a unique compact group G such that C is equivale to $Rep(G)$.*

In local quantum field theories in the formulation of [19], as described in Chapter C^*-quantum categories arise in a natural way. The fusion rule algebra was already deriv at the end of Chapter 3.1, using $*$-endomorphisms of the local algebra $\mathfrak{M}$ localized in given spacelike cone. More generally we consider these endomorphisms to be the objec of a category where the tensorproduct is given by the composition of endomorphism The morphisms are the intertwiners

$$Mor(\rho_1, \rho_2) := \{I \in \mathfrak{M} : I\rho_1(A) = \rho_2(A)I,\ \forall A \in \mathfrak{M}\}$$

and the tensorproduct is given by

$$I \circ J := I\sigma'(J) = \sigma(J)I, \qquad \text{for all } I \in Mor(\sigma', \sigma),\ J \in Mor(\rho', \rho).$$

The category is strict and the commutativity constraint is obtained from the cha transport operators. The structural data of this category are disussed in Chapter 2. four and more dimensions, this category is also symmetric and the natural balancin trivial, so that we can apply Theorem 7.1.4. We say that the local quantum theory is d to the group G associated with the category where the commutativity constraint $\varepsilon(i,j)$ is multiplied with a - sign if i and j obey para-Fermi statistics [19].

The main purpose of finding a dual group is to construct field operators with a group symmetry. To this end we use the intertwiners between the representations of the local algebra rather than the intertwiners between the endomorphisms. They are related to each other by (2.18). We define the physical Hilbert space of the theory as

$$\mathcal{H}_{\mathrm{phys.}} := \bigoplus_{j \in L} V_{j'} \otimes \mathcal{H}_j \,, \tag{7.12}$$

where $\mathcal{H}_j$ is the representation space of representation $j \in L$ of $\mathfrak{A}$, and $V_{j'}$, is the representation space of the corresponding representation j' of G. Let $\{e_\lambda\}_{\lambda=1}^{d_{j'}}$ be an orthonormal basis in $V_{j'}$. We define a linear map $P_{k'i'}(\alpha; e_\lambda)$ from $V_{i'}$ to $V_{k'}$ by the equation

$$\langle v,\, P_{k'i'}(\alpha; e_\lambda) w \rangle = \langle P_{i'j',k'}(\alpha) v,\, w \otimes e_\lambda \rangle \ , \tag{7.13}$$

for arbitrary $v \in V_{k'}$ and $w \in V_{i'}$.

If the local quantum theory under consideration is dual to the group G, in the sense of the definition given above, we can introduce charged "field operators", $\psi^\dagger_\lambda(\rho^j)$, by setting

$$\psi^\dagger_\lambda(\rho^j) = \sum_{ik\alpha} P_{k'i'}(\alpha; e_\lambda) \otimes V^{ik}_\alpha(\rho^j)^* \,, \tag{7.14}$$

where the two intertwiners are related to each other by the isomorphisms T^k_{ij}. It is easy to check that these fields obey ordinary Bose- or Fermi local commutation relations: If ρ^j and ρ^k are localized in space-like separated space-like cones then

$$\psi^\dagger_\lambda\left(\rho^j\right)\, \psi^\dagger_\mu\left(\rho^k\right) = \pm \psi^\dagger_\mu\left(\rho^k\right)\, \psi^\dagger_\lambda\left(\rho^j\right) , \tag{7.15}$$

where the minus sign is chosen if j and k obey para-Fermi statistics, and the plus sign is chosen otherwise.

Let π and π' denote the representations of $\mathfrak{A}$ and G, respectively, on $\mathcal{H}_{\mathrm{phys.}}$. Then we have from (2.20) and (7.14) that

$$\pi(A) \psi^\dagger_\lambda\left(\rho^j\right) = \psi^\dagger_\lambda\left(\rho^j\right)\, \pi\left(\rho^j(A)\right) , \tag{7.16}$$

for all $A \in \mathfrak{A}$, and

$$\pi'(g) \psi^\dagger_\lambda\left(\rho^j\right)\, \pi'\left(g^{-1}\right) = \sum_{\lambda'} j'(g)_{\lambda\lambda'} \psi^\dagger_{\lambda'}\left(\rho^j\right) , \tag{7.17}$$

where $\{ j'(g)_{\lambda\lambda'} \}$ are the matrix elements of $j'(g)$ in the basis $\{e_\lambda\}$ of $V_{j'}$.

In low dimensional quantum theories with braid statistics our notion of duality must be modified. We say that a local theory is dual to a quasitriangular Hopf algebra $\mathcal{K}$ iff its category of superselection sectors is equivalent to the quotient category $\overline{Rep}(\mathcal{K})$. Contrary to the case of semisimple groups or Hopf algebras, this causes difficulties in the construction of field operators with an explicit Hopf-algebra symmetry, since $\overline{Rep}(\mathcal{K})$ is in general non-Tannakian for non-semisimple $\mathcal{K}$, i.e., it is not realizable in terms of vectorspaces and linear maps between them. The extent to which analogous field operators obey local braid relations is discussed in Chapter 7.2.

An important consequence of properties $(P1)$ and $(P2)$ of Chapter 2 - in particular of the rigidity assumption - is that the <u>index</u> of an irreducible sector is finite, i.e.,

$$\mathrm{Ind}(\rho) = [\rho(\mathfrak{M}) : \mathfrak{M}] < \infty \,, \tag{7.18}$$

where the index, $[N : M]$, of the embedding of a von Neumann algebra N in M is defined in [41]. It has been shown in [23] that (7.18) is equivalent to $(P1)$ and $(P2)$. Also it is proven in [23] that the dimension given in (7.9) is related to the index by

$$\mathrm{Ind}(\rho) \;=\; d(\rho)^2 \,. \tag{7.19}$$

For an irreducible endomorphism ρ, we have by rigidity isometries $\Gamma_{\rho\circ\bar\rho,1} \in Mor(1, \rho \circ \bar\rho)$ and $\Gamma_{\bar\rho\circ\rho,1} \in Mor(1, \bar\rho \circ \rho)$, with

$$\rho\left(\Gamma^*_{\bar\rho\circ\rho,1}\right) \Gamma_{\rho\circ\bar\rho,1} = \pm d(\rho)^{-1} \quad \text{and} \quad \bar\rho\left(\Gamma^*_{\rho\circ\bar\rho,1}\right) \Gamma_{\bar\rho\circ\rho,1} = \pm d(\rho)^{-1} \,, \tag{7.20}$$

where the sign is always + if $\rho \ncong \bar\rho$ and an invariant with respect to normalization if ρ is selfconjugate. In this case, if the + sign appears we call ρ <u>real</u> and <u>pseudoreal</u> if the - sign appears.

Finally, we present some elements in the categorial description of local theories that are related to the theory of subfactors. Assume that $\mathfrak{N} \subset \mathfrak{M}$ is an inclusion of type III_1

von Neumann algebras (with the same units). Let $L^2(\mathfrak{M})$ be the Hilbertspace obtained from $\mathfrak{M}$ and from a state on $\mathfrak{M}$, and let $J_{\mathfrak{M}}$ denote the modular conjugation with respect to a cyclic, separating vector $\Omega \in L^2(\mathfrak{M})$. We then define the modular extension $\mathfrak{M}_1 \subset \mathfrak{B}(L^2(\mathfrak{M}))$ of $\mathfrak{M}$ over $\mathfrak{N}$ as

$$\mathfrak{M}_1 := J_{\mathfrak{M}} \mathfrak{N}' J_{\mathfrak{M}} \,.$$

It is shown in [23] that the modular extension of $\rho(\mathfrak{M})$ by $\rho \circ \bar{\rho}(\mathfrak{M})$ is isomorphic to $\mathfrak{M}$. The action of $\mathfrak{M}$ on $L^2(\rho(\mathfrak{M}))$ is given by

$$M.\rho(A) := \pm d(\rho)\rho(\Gamma_{\bar{\rho}\circ\rho,1})^* \rho \circ \bar{\rho}(M\rho(A)\Gamma_{\rho\circ\bar{\rho},1})$$

where $A, M \in \mathfrak{M}$ and the sign is as in (7.20). For the projection

$$e_0 := \Gamma_{\rho\circ\bar{\rho},1}\Gamma^*_{\rho\circ\bar{\rho},1} \quad \in \mathfrak{M}\,,$$

we then check that

$$e_0.\rho(A) = \varepsilon_0(A)\,, \qquad \text{for } A \in \mathfrak{M}$$

where

$$\begin{array}{rlcl} \varepsilon_0 : & \rho(\mathfrak{M}) & \longrightarrow & \rho \circ \bar{\rho}(\mathfrak{M}) \\ & \rho(A) & \mapsto & \rho \circ \bar{\rho}(\Gamma^*_{\rho\circ\bar{\rho},1}\rho(A)\Gamma_{\rho\circ\bar{\rho},1}) \end{array}$$

is a conditional expectation

indexconditional expectation, i.e., a positive, linear map $\varepsilon : \mathfrak{M} \to \mathfrak{N}$ between included von Neumann algebras such that

$$\varepsilon(mn) = \varepsilon(m)n\,, \quad \text{if} \quad n \in \mathfrak{N}\,.$$

It is a known fact that this projector together with the extended algebra generates the extension :

$$\langle \rho(\mathfrak{M}), e_0 \rangle = \mathfrak{M}\,. \tag{7.21}$$

Inductively, we thus have a tunnel (tower) of successive modular extensions

$$\ldots \subset \rho \circ \bar{\rho} \circ \rho(\mathfrak{M}) \subset \rho \circ \bar{\rho}(\mathfrak{M}) \subset \rho(\mathfrak{M}) \subset \mathfrak{M}\,. \tag{7.22}$$

Furthermore, from the series of isometries

$$\begin{array}{rcl} \Gamma_{(2n)} & := & (\rho \circ \bar{\rho})^n (\Gamma_{\rho\circ\bar{\rho},1})\,, \\ \Gamma_{(2n+1)} & := & (\rho \circ \bar{\rho})^n \circ \rho (\Gamma_{\bar{\rho}\circ\rho,1})\,. \end{array} \qquad \text{(7.2}$$

we obtain the conditional expectations

$$\varepsilon_n(A) = \rho \circ \bar{\rho}(\Gamma_{(n)}{}^* A \Gamma_{(n)})\,. \qquad \text{(7.2}$$

They correspond to the sequence of projectors

$$e_n := \Gamma_{(n)}\Gamma^*_{(n)}\,, \qquad \text{(7.2}$$

which obey the relations of the so called Temperley-Lieb algebra:

and

$$\begin{array}{rcl} \beta_\rho e_n\, e_{n\pm 1}\, e_n & = & e_n\,, \\ e_n\, e_m & = & e_m\, e_n \quad \text{if} \quad |n-m| \geq 2\,. \end{array} \qquad \text{(7.2}$$

Here

$$\beta_\rho \equiv Ind(\rho)\,.$$

As an alternative to the chain in (7.22) we can consider the sequence of inclusion

$$\ldots \subset M_n \subset M_{n+1} \subset \ldots\,, \qquad \text{(7.2}$$

where

$$\begin{array}{rcll} M_{2n} & := & (\rho \circ \bar{\rho})^n(\mathfrak{M})' \cap \mathfrak{M} & \equiv \; End((\rho \circ \bar{\rho})^n)\,, \\ M_{2n+1} & := & (\rho \circ \bar{\rho})^n \circ \rho(\mathfrak{M})' \cap \mathfrak{M} & \equiv \; End((\rho \circ \bar{\rho})^n \circ \rho)\,. \end{array} \qquad \text{(7.2}$$

The advantage of confining ourselves to the commutants M_n is that they are all type I_k II_1 von Neumann algebras and that they are purely categorial. We also have condition expectations $E^\rho_{n+1} : M_{n+1} \to M_n$ by setting

$$E^\rho_{n+1}(a) := \Gamma^*_{(n)} a \Gamma_{(n)}\,, \quad a \in M_{n+1}\,, \qquad \text{(7.2}$$

However, (7.27) is in general*not* a sequence of modular extensions (tower), i.e., t modular extension of M_n over M_{n-1} is contained in, but *not* equal to M_{n+1}. Still, if t theory or category is rational then the sequence (7.27) becomes a tower for $n > |\mathcal{L}|$.Mc precisely, we have

Lemma 7.1.5

Let $M_1 \subset \ldots \subset M_{n-1} \subset M_n \subset \ldots$ *be the chain of algebras defined in (7.27), and let* $C_{\#}^{(n)}$, $\# = 0, 1$, *be given by* $C_o^{(n)} = \{j \mid j \in (\rho \circ \bar{\rho})^n\}$ *and* $C_1^{(n)} = \{k \mid k \in (\rho \circ \bar{\rho})^n \circ \rho\}$. *Then* $M_{2n+\#}$ *has a decomposition into simple factors*

$$M_{2n+\#} = \bigoplus_{k \in C_{\#}^{(n)}} M_{2n+\#}^k$$

where $M_{2n+\#}^k$ *acts faithfully on* $Mor\big(k, \big((\rho \circ \bar{\rho})^n \circ \rho^{\#}\big)$, *by left multiplication, i.e.,*

$$M_{2n+\#}^k \cong End\Big(Mor\Big(k, ((\rho \circ \bar{\rho})^n \circ \rho^{\#})\Big)\Big).$$

The inclusion matrix, $\Lambda^{(2n+\#)}$, *of* $M_{2n+\#-1} \subset M_{2n+\#}$ *is equal to the restrictions of the fusion rule matrix* $\mathrm{N}_\rho : C_1^{(n-1)} \to C_o^{(n)}$, *for* $\# = 0$, *and* $\mathrm{N}_{\bar{\rho}} : C_o^{(n)} \to C_1^{(n)}$, *for* $\# = 1$.

The sequence $\ldots \subset C_{\#}^{(n)} \subset C_{\#}^{(n+1)} \ldots \subset C_{\#}$ *is strictly monotonously increasing, or* $C_{\#}^{(n)} = C_{\#}$, *where* $C_{\#}$ *are the minimal invariant sets of* $\mathrm{N}_\rho^t \mathrm{N}_\rho$ *given in Chapter 3.2.*

A very important ingredient in the study of inclusions of von Neumann algebras are Markov traces. On the algebra $M_\infty = \bigcup_n M_n$ a Markov trace, τ_M, is characterized by the properties that it is a positive trace and that

$$\tau_M(ae_n) = \beta_\rho^{-1} \tau_M(a) \quad \text{, for all} \quad a \in M_{n+1} \, .$$

It is easily shown that the functional given by the formula

$$\tau_M := d(\rho)^{-(2n+\#)}\, tr_{(\rho\circ\bar{\rho})^n \circ \rho^{\#}}(a) \, , \quad \text{for} \quad a \in M_{2n+\#} \, ,$$

where tr is as in Lemma 7.1.3, is well defined on M_∞ and is a Markov trace. It also satisfies

$$\tau_M(E_n^\rho(a)) = \tau_M(a) \, , \quad \text{for} \quad a \in M_n$$

so that

$$\tau_M(a) = E_1^\rho \ldots E_n^\rho(a) \, , \quad \text{for} \quad a \in M_n \, .$$

This trace is in fact the only possible normalized Markov trace on M_∞.

7.2 Quantum Group Symmetries of Charged Fields[*]

We start from a physical Hilbert space which carries unitary representations of an observable algebra over $\mathbb{M}^3$ *and a Hopf algebra,* $\mathcal{K}$, *and define, in analogy to the case where* $\mathcal{K}$ *is a group algebra, spaces of field operators that transform covariantly (contravariantly) under the adjoint action of* $\mathcal{K}$. *We explain how this notion of symmetry extends to conjugates and compositions of field operators and derive the resulting commutation relations and operator product expansions, in case* $\mathcal{K}$ *is semisimple. We show that commutation relations and operator product expansions hold for non-semisimple algebras* $\mathcal{K}$ *only in a weak sense, i.e., the respective equations have to be contracted with* $\mathcal{K}$*-tensors with non-zero quotients in the intertwiner calculus of Section 6.1. For* $U_q(s\ell_2)$, *we show that if the total order of the monomials does not exceed the level these contractions can be omitted. It would be interesting to see how these subtleties have to be treated in conformal theories [9], where we have a similar construction of primary fields in which the quantum group is replaced by a current algebra.*

In general, there is no procedure to construct a field algebra, $\mathcal{F}(\mathcal{C})$, generated by charged fields, $\psi(\rho^p)$, where ρ^p is a morphism of the observable algebra A localized in a cone $\mathcal{C}$, which has a quasi-triangular Hopf algebra $\mathcal{K}$ as a symmetry algebra and closes under the commutation relations determined by the universal R-matrix of $\mathcal{K}$.

In our context, charged fields with $\mathcal{K}$ symmetry are defined as follows.

The "fieldspace" $\mathcal{F}_1^{cov}(\mathcal{C})$, with elements $\psi(\rho^p)$, ρ^p being localized in $\mathcal{C}$, is a subspace of $\mathfrak{B}(\mathcal{H}_{\mathbf{phys.}})$. The Hilbert space $\mathcal{H}_{\mathbf{phys.}}$ carries unitary representation s, denoted π, of $\mathcal{K}$ and $\mathfrak{A}$, with

$$\mathcal{K} \subset \mathfrak{A}' \, , \tag{7.30}$$

and contains the vacuum sector, $\mathcal{H}_1$, which is determined by

$$\begin{aligned} \mathcal{H}_1 = \{v \in \mathcal{H}_{\mathbf{phys.}} \mid \pi(a)v &= E(a)v \, ; \forall a \in \mathcal{K}\} \\ \text{and} \qquad \Omega &\in \mathcal{H}_1 \, , \end{aligned} \tag{7.31}$$

where Ω denotes the vacuum vector.

The space $\mathcal{F}_1^{\mathrm{cov}}(\mathcal{C})$ is defined as the span of finite dimensional Banach spaces, $F_{\rho^p} \subset \mathfrak{B}(\mathcal{H}_{\mathrm{phys.}})$, that are characterized, for any *endomorphism ρ^p localized in $\mathcal{C}$, by

$$F_{\rho^p} = \{\psi(\rho^p) \in \mathcal{F}_1^{\mathrm{cov}}(\mathcal{C}) \mid \psi(\rho^p)\pi(A) = \pi(\rho^p(A))\ \psi(\rho^p),\ \forall A \in \mathfrak{A}\}\ . \tag{7.32}$$

$\mathcal{K}$ symmetry is expressed by the fact, that $\mathcal{F}_1^{\mathrm{cov}}(\mathcal{C})$ is invariant under the action of $\mathcal{K}$ on $\mathfrak{B}(\mathcal{H}_{\mathrm{phys.}})$given by the adjoint representation $ad_{\mathcal{K}}^-$ defined in (4.8).

It is not hard to see that the finite dimensional spaces F_{ρ^p} are also invariant under $\mathcal{K}$, and if ρ^p and $\tilde{\rho}^p$ are equivalent as representations of $\mathfrak{A}$ on $\mathcal{H}_1$, then F_{ρ^p} and $F_{\tilde{\rho}^p}$ are equivalent as $\mathcal{K}$ modules.

We now *assume* that F_{ρ^p} is *irreducible* as a $\mathcal{K}$ representation, and

$$\pi(\mathfrak{A})\mathcal{F}_1^{\mathrm{cov}}(\mathcal{C})\Omega = \mathcal{H}_{\mathrm{phys.}}\,. \tag{7.33}$$

F_{ρ^p} is identified with an irreducible $\mathcal{K}$-representation V_p by

$$V_p \to F_{\rho^p} : x \to \psi(x, \rho^p)\ ,$$

with

$$ad_{\mathcal{K}}^-(a)(\psi(x,\rho^p)) = \psi(ax, \rho^p)\ . \tag{7.34}$$

For the charge transport operator $\Gamma_{\tilde{\rho}^p,\rho^p} \in \overline{\mathfrak{B}}^{\mathcal{C}_a}$, see (2.19), with

$$\Gamma_{\tilde{\rho}^p,\rho^p}\,\rho^p(A) = \tilde{\rho}^p(A)\,\Gamma_{\tilde{\rho}^p,\rho^p}\ ,$$

we have

$$\pi(\Gamma_{\tilde{\rho}^p,\rho^p})\ \psi(x,\rho^p) = \psi(R(\tilde{\rho}^p,\rho^p)\,x, \tilde{\rho}^p) = r(\tilde{\rho}^p,\rho^p)\ \psi(x,\tilde{\rho}^p)\ , \tag{7.35}$$

where we use that $\Gamma_{\tilde{\rho}^p,\rho^p} \in \mathfrak{B}^{\mathcal{C}_a}$ and hence $R(\tilde{\rho}^p,\rho^p)$ commutes with the action of $\mathcal{K}$.

From (7.31)-(7.35) it follows that $\mathcal{H}_{\mathrm{phys.}}$ is described by

$$\mathcal{H}_{\mathrm{phys.}} = \sum_{p\in L} V_p \otimes \mathcal{H}_p \tag{7.36}$$

and the fields are given by

$$\psi(x,\rho^p) = \sum_{ij\nu}(\mathbf{1}_i \otimes \langle x, .\rangle)\, P_{ip,j}(\nu) \otimes V_\nu^{ij}(\rho^p) \tag{7.37}$$

for some set of intertwiners $P_{ip,j}(\nu) \in \mathrm{Int}\,(V_i \otimes V_p, V_j)$.

From now on we assume that the $\mathcal{K}$-representations associated to different secto are inequivalent and that the intertwiner $P_{ik,j}(\nu)$ are a basis of $\mathcal{P}(V_i \otimes V_k, V_j)$. With th conventions (5.6) and (6.51) we can compute the *-conjugate of (7.37):

$$(\psi(x,\rho^p))^* = \psi^\dagger(\chi x, \rho^p) \qquad (7,3$$

where

$$\psi^\dagger(x,\rho^p) = \sum_{ij\nu}\left(\overline{P}_{i,pj}(\nu)(x \otimes \mathbf{1}_j)\right) \otimes \overline{V_\nu^{ij}}(\rho^p)$$

with

$$\overline{V_\nu^{ij}}(\rho^p) := V_\nu^{ji}(\rho^p)^*\ .$$

The relations of these fields with $\mathcal{K}$ and $\mathfrak{A}$ are given by

$$ad_{\mathcal{K}}^+(a)\,\psi^\dagger(x,\rho^p) = \psi^\dagger(ax,\rho^p) \qquad (7.3$$

and

$$\pi(A)\,\psi^\dagger(x,\rho^p) = \psi^\dagger(x,\rho^p)\ \pi(\rho^p(A))\ . \qquad (7.4$$

The total covariant (contravariant) field-algebra $\mathcal{F}^{\mathrm{cov}}(\mathcal{C})\,(\mathcal{F}^{\mathrm{cont}}(\mathcal{C}))$ is the algebra gene ated by elements in $\mathcal{F}_1^{\mathrm{cov}}(\mathcal{C})$. Note that $\mathcal{F}_1^{\mathrm{cont}}(\mathcal{C}) = \mathcal{F}_1^{\mathrm{cov}}(\mathcal{C})^*$. The transformation laws the monomials in $\mathcal{F}^{\mathrm{cov}}(\mathcal{C})\,(\mathcal{F}^{\mathrm{cont}}(\mathcal{C}))$ are

$$\begin{aligned} &ad_{\mathcal{K}}^-(a)(\psi(x_1,\rho^{p_1})\ldots\psi(x_n,\rho^{p_n})) = \\ &\qquad \left\langle \Delta'^{(n-1)}(a)\, x_1 \otimes \ldots \otimes x_n\,,\ \psi(.,\rho^{p_1})\ldots\psi(.,\rho^{p_n})\right\rangle \\ &ad_{\mathcal{K}}^+(a)\left(\psi^\dagger(x_1,\rho^{p_1})\ldots\psi^\dagger(x_n,\rho^{p_n})\right) = \\ &\qquad \left\langle \Delta^{(n-1)}(a)\, x_1 \otimes \ldots \otimes x_n,\ \psi^\dagger(.,\rho^{p_1})\ldots\psi^\dagger(.,\rho^{p_n})\right\rangle \end{aligned} \qquad (7.4$$

with $x_1 \otimes \ldots \otimes x_n \in V_{p_1} \otimes \ldots \otimes V_{p_n}$.

Let us assume that $\mathcal{K}$ is dual to $\mathfrak{A}$, in the sense explained in Chapter 7.1, so that t identifications of $\mathcal{K}$-representations with superselection sectors coincides with (7.10).

$\mathcal{K}$ is semisimple, this implies commutation relations for the fields, that close in $\mathcal{F}^{\text{cov}}(\mathfrak{A})$ and are given by the universal R-matrix of $\mathcal{K}$:

$$\begin{aligned}
&\psi(x_p,\rho^{\bar{p}})\ \psi(y_q,\rho^{\bar{q}}) = \\
&\quad = \left\langle x_p \otimes y_q,\ R^{\pm}_{\bar{q}\bar{p}}\, \psi(.,\rho^{\bar{q}})\ \psi(.,\rho^{\bar{p}})\right\rangle \qquad (7.42)\\
&\quad = \left\langle P_{pq}\, R^{\pm}_{qp} P_{pq}\, x_p \otimes y_p\,, \psi(.,\rho^{\bar{q}})\ \psi(.,\rho^{\bar{p}})\right\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\psi^{\dagger}(y_q,\rho^{q})\ \psi^{\dagger}(x_p,\rho^{p}) = \\
&\quad = \left\langle R^{\pm}_{qp}\, y_q \otimes x_p\,,\ \psi^{\dagger}(.,\rho^{p})\ \psi^{\dagger}(.,\rho^{q})\right\rangle \qquad (7.43)\\
&\quad = \left\langle y_q \otimes x_p\,,\ P_{\bar{p}\bar{q}}\, R^{\pm}_{\bar{q}\bar{p}} P_{\bar{p}\bar{q}}\, \psi^{\dagger}(.,\rho^{p})\ \psi^{\dagger}(.,\rho^{q})\right\rangle ,
\end{aligned}$$

where ρ^p and ρ^q, resp. $\rho^{\bar{p}}$ and $\rho^{\bar{q}}$, are spacelike separated, P_{pq} is the transposition of tensorfactors, and $x_p \in V_p$, $y_q \in V_q$. Moreover, with the relation (7.11), we have the operator product expansions:

$$\begin{aligned}
&\psi(x_p,\rho^{\bar{p}})\ \psi(y_q,\rho^{\bar{q}}) = \qquad (7.44)\\
&\quad = \sum_{\bar{r},\mu,\nu} \sigma_\mu(\bar{r};\bar{p},\bar{q})\ F(1,\bar{p},\bar{q},\bar{r})^{\bar{r},\mu 1}_{\bar{p},1\nu}\ \Gamma_{\rho^{p}\circ\rho^{q},\rho^{r}}(\mu)\,\langle x_p \otimes y_q\,,\ P_{\bar{p}\bar{q},\bar{r}}(\nu)\ \psi(.,\rho^{\bar{r}})\rangle
\end{aligned}$$

and

$$\begin{aligned}
&\psi^{\dagger}(x_p,\rho^{p})\ \psi^{\dagger}(y_p,\rho^{q}) = \qquad (7.45)\\
&\quad = \sum_{r,\mu,\nu} \overline{\sigma_\mu(r;q,p)}\ \overline{F(1,q,p,r)^{r,\mu 1}_{q,1\nu}}\ \psi^{\dagger}\left(\overline{P}_{r,pq}(\nu)\,(x_p \otimes y_q)\,,\rho^{r}\right)\Gamma_{\rho^{q}\circ\rho^{p},\rho^{r}}(\mu)\,.
\end{aligned}$$

If we turn to the case in which $\mathcal{K}$ is no longer semisimple equations (7.42)-(7.45) no longer hold, since the intertwiner spaces $\text{Int}_o(V_i \otimes V_{\bar{q}} \otimes V_{\bar{p}}, V_k)$ and $\text{Int}_o(V_i, V_q \otimes V_p \otimes V_k)$ are nontrivial. There is, however, a way of understanding commutation relations if we consider the subspace of $\mathfrak{B}(\mathcal{H}_{\text{phys.}})$, spanned by the monomials $\psi(.,\rho^{\bar{p}_1})\ldots\psi(.,\rho^{\bar{p}_n})$ "smeared out" only over a certain subspace of $V_{p_1}\otimes\ldots\otimes V_{p_n}$. To be precise, we define the subspaces $\mathcal{F}^{\text{cov}}_{\text{rest}}(\rho_{\bar{p}_1},\ldots,\rho_{\bar{p}_n})$ as the restriction of $\psi(.,\rho^{\bar{p}_1})\ldots\psi(.,\rho^{\bar{p}_n})$ to

$$\sum_{p\in L}^{\oplus} V_p \otimes \text{Int}(V_{\bar{p}}, V_{\bar{p}_1}\otimes\ldots\otimes V_{\bar{p}_r})\,,$$

seen as a subspace of $V_{p-1}\otimes\ldots\otimes V_{p_n}$, i.e., $\mathcal{F}^{\text{cov}}_{\text{rest}}(\rho^{\bar{p}_1},\ldots,\rho^{\bar{p}_n})$ is the linear span of all fields

$$\langle x, I_{\bar{p},\bar{p}_1\ldots\bar{p}_n}\,\psi(.,\rho^{\bar{p}_1})\ldots\psi(.,\rho^{\bar{p}_n})\rangle\ ,$$

$$\text{with}\qquad x\in V_{\bar{p}}\qquad\text{and}\qquad I_{\bar{p},\bar{p}_1\ldots\bar{p}_n}\in\text{Int}(V_{\bar{p}}, V_{\bar{p}_1}\otimes\ldots\otimes V_{\bar{p}_n})\,.$$

Similarly we define $\mathcal{F}^{\text{cov}}_{\text{rest}}(\rho^{p_1},\ldots,\rho^{p_n})$ by restriction to the subspace

$$\sum_{p\in L}^{\oplus}\text{Int}(V_{p_1}\otimes\ldots\otimes V_{p_n}, V_p)\otimes V_p$$

of $V_{p_1}\otimes\ldots\otimes V_{p_n}$, i.e., the span of all

$$\left\langle I_{p_1\ldots p_n,p}x,\ \psi^{\dagger}(.,\rho^{p_1})\ldots\psi^{\dagger}(.,\rho^{p_n})\right\rangle .$$

Note that the spaces $\mathcal{F}^{\text{cov/cont}}_{\text{rest}}(\rho^{p_1},\ldots,\rho^{p_n})$ are invariant under the adjoint action of $\mathcal{K}$ and coincide, for semisimple $\mathcal{K}$, with the total space of monomials. However, the collection of these subspaces *does not* form an algebra.

From the definition of the vertex-*SOS* transformation and assuming that we have duality in the sense of the equivalence (7.11) , we see that we have to reinterpret the commutation relations (7.42) and (7.43) as being valid only inside of the contractions restricting them to $\mathcal{F}^{\text{cov}}_{\text{rest}}$.

They can be expressed in coordinates if we fix a basis e_{α_i} in V_{p_i} and a dual basis ℓ^{α_i} in $V_{\bar{p}_i}$.

If we denote the matrix elements of $I_{\bar{p},\bar{p}_1\ldots\bar{p}_n}\in\text{Int}(V_{\bar{p}}, V_{\bar{p}_1}\otimes\ldots\otimes V_{\bar{p}_n})$, $I_{p_1\ldots p_n,p}\in\text{Int}(V_{p_1}\otimes\ldots\otimes V_{p_n}, V_p)$ and $R^{\pm}_{qp}$ by

$$\begin{aligned}
\langle e_\beta, I_{\bar{p},\bar{p}_1\ldots\bar{p}_n}\ell^{\alpha_1}\otimes\ldots\otimes\ell^{\alpha_n}\rangle &= (I_{\bar{p},\bar{p}_1\ldots\bar{p}_n})^{\alpha_1\ldots\alpha_n}_{\beta}\\
\langle \ell^{\alpha_1}\otimes\ldots\otimes\ell^{\alpha_n},\ I_{p_1\ldots p_n,p}e_\beta\rangle &= (I_{p_1\ldots p_n,p})^{\alpha_1\ldots\alpha_n}_{\beta}\\
\text{and}\qquad\left\langle \ell^{\gamma}\otimes\ell^{\delta},\ R^{\pm}_{qp}\,e_\beta\otimes e_\alpha\right\rangle &= \left(R^{\pm}_{qp}\right)^{\gamma\delta}_{\beta\alpha}
\end{aligned}$$

and the field components

$$\psi_\alpha(\rho^{\bar{p}}) := \psi(e_\alpha,\rho^{\bar{p}})\ ;\psi^{\dagger}_\alpha(\rho^{p}) := \psi^{\dagger}(e_\alpha,\rho^{p})$$

For this purpose we consider three irreducible, arbitrarily localized *-endomorphisms, ρ_i, $i = 1,2,3$, which are equivalent to causally independent endomorphisms, $\hat{\rho}_i$, with $\mathrm{as}(\hat{\rho}_1) > \mathrm{as}(\hat{\rho}_2) > \mathrm{as}(\hat{\rho}_3)$. Then there exist charge transport operators $\Gamma_{\hat{\rho}_i,\rho_i} \in \mathfrak{B}^{C^*}$ obeying (2.19). We choose the frames, $\left\{V^{\ell k}_{\alpha}(\rho_i)\right\}$, $\left\{V^{\ell k}_{\alpha}(\hat{\rho}_i)\right\}$, of different fibres $\mathcal{V}_{\ell}(\rho_i)_k$, $\mathcal{V}_{\ell}(\hat{\rho}_i)_k \hookrightarrow \mathcal{I}_{\ell\rho_i,k}$, related by (2.26). This yields a relation between the natural frames, $\left\{V^{\ell k_1}_{\alpha_1}(\rho_1)\dots V^{k_{n-1}\rho}_{\alpha_n}(\rho_n)\right\}$, of $\mathcal{V}_{\ell}(\rho_1 \circ \dots \circ \rho_n)_k \hookrightarrow \mathcal{I}_{k\rho_1\dots\rho_n,k}$ gven by the equations

$$V^{ik}_{\alpha}(\hat{\rho}_1)\; V^{k\ell}_{\beta}(\hat{\rho}_2) = i\left(\Gamma_{\hat{\rho}_1,\rho_1}\rho_1\left(\Gamma_{\hat{\rho}_2,\rho_2}\right)\right)\; V^{ik}_{\alpha}(\rho_1)\; V^{k\ell}_{\beta}(\rho_2) \tag{7.59}$$

and

$$\begin{aligned} &V^{ik}_{\alpha}(\hat{\rho}_1)\; V^{k\ell}_{\beta}(\hat{\rho}_2)\; V^{\ell m}_{\gamma}(\hat{\rho}_3) = \\ &i\left(\Gamma_{\hat{\rho}_1,\rho_1}\rho_1\left(\Gamma_{\hat{\rho}_2,\rho_2}\right)\rho_1\circ\rho_2\left(\Gamma_{\hat{\rho}_3,\rho_3}\right)\right)\; V^{ik}_{\alpha}(\rho_1)\; V^{k\ell}_{\beta}(\rho_2)\; V^{\ell m}_{\gamma}(\rho_3)\,,. \end{aligned} \tag{7.60}$$

The statistics operator $\varepsilon^{+}(\rho_1,\rho_2)$ is given by

$$\varepsilon^{+}(\rho_1,\rho_2) = \rho_2\left(\Gamma^{*}_{\hat{\rho}_1,\rho_1}\right)\;\Gamma^{*}_{\hat{\rho}_2,\rho_2}\,\Gamma_{\hat{\rho}_1,\rho_1}\;\rho_1\left(\Gamma_{\hat{\rho}_2,\rho_2}\right). \tag{7.61}$$

Clearly, it is a unitary operator in $\mathfrak{B}^{C^*}$ intertwining $\rho_1\circ\rho_2$ with $\rho_2\circ\rho_1$ and, a priori, it might also depend on $\hat{\rho}_1$ and $\hat{\rho}_2$. The connection to the statistics matrices is obtained if we combine eqs. (7.59), (7.61) with (2.27), using that $\hat{\rho}_1 \times \hat{\rho}_2$, and $\mathrm{as}(\hat{\rho}_1) > \mathrm{as}(\hat{\rho}_2)$. We find that

$$i\left(\varepsilon^{+}(\rho_1,\rho_2)\right)\; V^{ik}_{\alpha}(\rho_1)\; V^{k\ell}_{\beta}(\rho_2) = \sum_{k'\alpha'\beta'} R^{+}\;(i,\hat{\rho}_1,\hat{\rho}_2,\ell)^{k'\alpha'\beta'}_{k\alpha\beta}\; V^{ik'}_{\alpha'}(\rho_2)\; V^{k'\ell}_{\beta'}(\rho_1)\,. \tag{7.62}$$

Since $\varepsilon^{+}(\rho_1,\rho_2) : \mathrm{Int}(\rho_2\circ\rho_1,\rho_1\circ\rho_2) \to \mathrm{Hom}\left(\mathcal{V}_i(\rho_1\circ\rho_2)_k\,,\mathcal{V}_i(\rho_2\circ\rho_1)_k\right)$ is an isomorphism, we infer from the properties of statistics matrices (see Theorem 2.3.1) that $\varepsilon^{+}(\rho_1,\rho_2)$ only depends on the asymptotic direction of $\hat{\rho}_1$ and $\hat{\rho}_2$, and we will write $\varepsilon^{-}(\rho_1,\rho_2)$ if $\mathrm{as}(\rho_1) < \mathrm{as}(\rho_2)$. Obviously we have that

$$\varepsilon^{-}(\rho_1,\rho_2) = \left(\varepsilon^{+}(\rho_2,\rho_1)\right)^{-1}. \tag{7.63}$$

Equations (7.60) and (7.62) also imply the identity

$$\begin{aligned} &i\left(\rho_1\left(\varepsilon^{\pm}(\rho_2,\rho_3)\right)\right)\; V^{ik}_{\alpha}(\rho_1)\; V^{k\ell}_{\beta}(\rho_2)\; V^{\ell m}_{\gamma}(\rho_3) = \\ &\sum_{\ell'\beta'\gamma'} R^{\pm}(k,\rho_2,\rho_3,m)^{\ell'\beta'\gamma'}_{\ell\beta\gamma}\; V^{ik}_{\alpha}(\rho_1)\; V^{k\ell'}_{\beta'}(\rho_3)\; V^{\ell' m}_{\gamma'}(\rho_2)\,. \end{aligned} \tag{7.64}$$

This identity, eq. (7.62), the Yang-Baxter equations for the statistics matrices, and the injectivity of $\mathrm{Int}(\rho_3\circ\rho_2\circ\rho_1,\rho_1\circ\rho_2\circ\rho_3) \hookrightarrow \mathrm{Hom}\left(\mathcal{V}_i(\rho_1\circ\rho_2\circ\rho_3)_m\,,\mathcal{V}_i(\rho_3\circ\rho_2\circ\rho_1)_m\right)$ yield the *Yang-Baxter equations for the statistics operators*, i.e.,

$$\begin{aligned} &\varepsilon^{\pm}(\rho_2,\rho_3)\;\rho_2\left(\varepsilon^{\pm}(\rho_1,\rho_3)\right)\;\varepsilon^{\pm}(\rho_1,\rho_2) = \\ &\rho_3\left(\varepsilon^{\pm}(\rho_1,\rho_2)\right)\;\varepsilon^{\pm}(\rho_1,\rho_3)\,\rho_1\left(\varepsilon^{\pm}(\rho_2,\rho_3)\right). \end{aligned} \tag{7.65}$$

For detailed calculations see [24]. If we specialize (7.62) to $i = 1$, $k = [\rho_1]$, and use the normalization $V^{1[\rho_1]}(\rho_1)\; V^{[\rho_1]\ell}_{\alpha}(\rho_2) = \Gamma_{\rho_1\circ\rho_2,\rho^{\ell}}(\alpha)\, V^{1\ell}\left(\rho^{\ell}\right)$for the isometries, $\Gamma_{\rho_1\circ\rho_2,\rho^{\ell}}(\alpha)$, we obtain the following presentation

$$\varepsilon^{\pm}(\rho_1,\rho_2) = \sum_{\ell\alpha\alpha'} R^{\pm}(1,\rho_1,\rho_2,\ell)^{\rho_2 1\alpha'}_{\rho_1 1\alpha}\;\Gamma_{\rho_2\circ\rho_1,\rho^{\ell}}(\alpha')\,\Gamma^{*}_{\rho_1\circ\rho_2,\rho^{\ell}}(\alpha)\,. \tag{7.66}$$

In the case of interest we have that $\rho_1 = \rho_2 = \rho$, and the summation in (7.66) ranges over $\rho^{\ell} \in \{\sigma,\psi\}$, $\psi \not\cong \sigma$, $\alpha = \alpha' = 1$, so that

$$\varepsilon^{+}(\rho,\rho) = z_{\rho}\left((q_{\rho}+1)\; e_{\sigma}(\rho,\rho) - 1\right), \tag{7.67}$$

where

$$q_{\rho} = -\frac{R(1,\rho,\rho,\sigma)^{\rho 11}_{\rho 11}}{R(1,\rho,\rho,\psi)^{\rho 11}_{\rho 11}} \neq -1\,,\quad z_{\rho} = -R(1,\rho,\rho,\psi)^{\rho 11}_{\rho 11}\,.$$

The consequence of having a two-channel decomposition is that the braid group representation given by the generators $\tau_n := \frac{1}{z_{\rho}}\rho^{n}\left(\varepsilon^{+}(\rho,\rho)\right)$, with $\tau_n\tau_{n+1}\tau_n = \tau_{n+1}\tau_n\tau_{n+1}$, i contained in the set of representations of the Hecke algebra, $H_{q_{\rho},\infty}$, since we also have tha $\tau_n^2 = (q_{\rho}-1)\,\tau_n + q_{\rho}$, i.e., the ideal $I \subset \mathbb{C}[B_{\infty}]$, with $\mathbb{C}[B_{\infty}]/\,I = H_{q_{\rho},\infty}$, is annihilated b our representation of B_{∞}. As remarked in [22], one can then utilize the classification c unitary representations of $H_{q_{\rho}}$, as given in [54], to find the possible values of q_{ℓ}: $q_{\ell} \doteq e^{\pm\frac{2\pi i}{N}}$ $N = 4,5,\dots,\infty$.

For the associated projections $e_n = \frac{\tau_n+1}{q_{\rho}+1} = \rho^{n}\left(e_{\sigma}(\rho,\rho)\right)$, we find the usual Tempe ley-Lieb relations (7.27), with $\beta = d_{\rho}^2$, provided σ is an automorphism and using (7.5 and (7.58), with $\rho = \rho_i$, $i = 1,2,3$, and $\sigma = \sigma' = \sigma''$. In this case, one finds, by inserti (7.67) into the Yang-Baxter equation, the compatibility condition

$$\beta = q_{\rho} + q_{\rho}^{-1} + 2\,. \qquad (7.6$$

decompositions.

We first note that if $\sigma_1 \in \rho \circ \rho$, which means that there exists an isometry $0 \neq \Gamma_{\rho\circ\rho,\sigma_1} \in \mathrm{Int}(\rho \circ \rho, \sigma_1)$, and σ_1 is a localized automorphisms, then we have, for the *-endomorphisms $\bar{\rho} := \sigma_1^{-1} \circ \rho \cong \rho \circ \sigma_1^{-1}$ ("=", for $\sigma_1 \times \rho$), that $\bar{\rho} \circ \rho(A)\Gamma_{\bar{\rho}\circ\rho,1} = \Gamma_{\bar{\rho}\circ\rho,1}A$, for all $A \in \mathfrak{B}^{C_{\bullet}}$, where $\Gamma_{\bar{\rho}\circ\rho,1} := \sigma_1^{-1}(\Gamma_{\rho\circ\rho,\sigma_1})$. Similarly, we find an operator $\Gamma_{\rho\circ\bar{\rho},1}$, with $\rho \circ \bar{\rho}(A)\Gamma_{\rho\circ\bar{\rho},1} = \Gamma_{\rho\circ\bar{\rho},1}A$, $\forall A \in \mathfrak{B}^{C_{\bullet}}$, such that there always exists a conjugate sector. It follows from the result in [23], that $\mathrm{Ind}(\rho) < \infty$, for all cases. Hence $\mathrm{Ind}(\psi) < \infty, \forall \psi \in \Phi$, Φ being the sectors generated by ρ. Moreover, we find from i) that

$$\rho \circ \bar{\rho} = 1 \oplus \psi' \tag{7.50}$$

and from ii) that

$$\rho \circ \bar{\rho} = 1 \oplus \sigma \oplus \psi' \tag{7.51}$$

where $\psi' := \sigma_1^{-1} \circ \psi$, and $\sigma := \sigma_1^{-1} \circ \sigma_2$.

The assertion follows for cases o), iii) and iv) from the basic properties of statistical dimensions, namely: $d_{\rho_1\circ\rho_2} = d_{\rho_1} \cdot d_{\rho_2}$, $d_{\rho_1\oplus\rho_2} = d_{\rho_1} + d_{\rho_2}$ and $d_\sigma = 1$ iff σ is an automorphism. For o), we have $d_\rho = 1$, for iii) $d_\rho^2 = d_1 + d_{\sigma_1} + d_{\sigma_2} + d_{\sigma_3} = 4$, and for iv), $d_\rho^2 = 1 + d_\psi$, $d_\psi d_\rho = d_\rho + d_{\sigma_1} d_\rho + d_{\sigma_2} d_\rho = 3d_\rho$, hence $d_\psi = 3$ and $d_\rho = 2$. The proofs of i) and ii) require some additional knowledge on connections of automorphisms to conditional expectations and Temperley-Lieb algebras.

We first consider two irreducible *-endomorphisms, ρ_1 and ρ_2, which are arbitrarily localized and have finite index, and assume that there exists a localized automorphisms, σ', with

$$\sigma' \in \rho_1 \circ \rho_2 . \tag{7.52}$$

From an isometry, $\Gamma_{\rho_1\circ\rho_2,\sigma'}$, intertwining σ' with a subrepresentation of $\rho_1 \circ \rho_2$, we find an isometry $(\sigma')^{-1}(\Gamma_{\rho_1\circ\rho_2,\sigma'})$ which intertwines the vacuum representation with the *-endomorphism $\left((\sigma')^{-1} \circ \rho_1\right) \circ \rho_2$, so that $\bar{\rho}_2 \cong (\sigma')^{-1} \circ \rho_1$ by property $(P1)$ and $\Gamma_{\rho_1\circ\rho_2,\sigma'}$ is unique up to a phase factor. It follows that $N_{\rho_1\rho_2,\sigma'} = 1$ and $d_{\rho_1} = d_{\rho_2}$. In particular, for some choice of $\bar{\rho}_2$, there exist a localized unitary operator, $\Gamma_{\sigma'\circ\bar{\rho}_2,\rho_1}$, and an isometry,

$\Gamma_{\bar{\rho}_2\circ\rho_2,1}$, such that

$$\Gamma_{\rho_1\circ\rho_2,\sigma'} = \Gamma^*_{\sigma'\circ\bar{\rho}_2,\rho_1}\, \sigma'(\Gamma_{\bar{\rho}_2\circ\rho_2,1}) . \tag{7.53}$$

These properties imply that the projections $e_{\sigma'}(\rho_1, \rho_2) := \Gamma_{\rho_1\circ\rho_2,\sigma'}\Gamma^*_{\rho_1\circ\rho_2,\sigma'}$ are unique and obey

$$e_{\sigma'}(\rho_1, \rho_2) = \Gamma^*_{\sigma'\circ\bar{\rho}_2,\rho_1}\, \sigma'(e_1(\bar{\rho}_2, \rho_2))\, \Gamma_{\sigma'\circ\bar{\rho}_2,\rho_1} , \tag{7.54}$$

where $e_{\rho_2} = e_1(\bar{\rho}_2, \rho_2)$ is just the Temperley-Lieb projection introduced in Section 2.5. Moreover, we have that $e_\sigma(\rho_1, \rho_2) = e_\sigma(\rho_2, \rho_1)$. Let us assume there exist an endomorphism, ρ_3, and an automorphism, σ'', with $\sigma'' \in \rho_2 \circ \rho_3$. We immediately find that $\bar{\rho}_2 \circ \sigma'' \cong \rho_3$, $d_{\rho_1} = d_{\rho_2} = d_{\rho_3}$, that the isometry, $\Gamma_{\rho_2\circ\rho_3,\sigma''}$, is unique up to a phase and can be expressed,similarly to (7.53), by

$$\Gamma_{\rho_2\circ\rho_3,\sigma''} = \rho_2\left(\Gamma^*_{\bar{\rho}_2\circ\sigma'',\rho_3}\right) \Gamma_{\rho_2\circ\bar{\rho}_2,1} , \tag{7.55}$$

where $\Gamma_{\bar{\rho}_2\circ\sigma'',\rho_3}$ is unitary, and $\Gamma_{\rho_2\circ\bar{\rho}_2,1}$ is normalized, relative to $\Gamma_{\bar{\rho}_2\circ\rho_2,1}$, such that (7.20) holds. The identity following from (7.55) for the (unique) projection $e_{\sigma''}(\rho_2, \rho_3) = \Gamma_{\rho_2\circ\rho_3,\sigma''}\Gamma^*_{\rho_2\circ\rho_3,\sigma''}$ is expressed by

$$e_{\sigma''}(\rho_2, \rho_3) = \rho_2\left(\Gamma^*_{\bar{\rho}_2\circ\sigma'',\rho_3}\right)\; e_1(\rho_2, \bar{\rho}_2)\;\; \rho_2(\Gamma_{\bar{\rho}_2\circ\sigma'',\rho_3}) . \tag{7.56}$$

It is straightforward to derive the generalized Temperley-Lieb relation

$$\beta \rho_1(e_{\sigma''}(\rho_2, \rho_3))\;\; e_{\sigma'}(\rho_1, \rho_2)\;\; \rho_1(e_{\sigma''}(\rho_2, \rho_3)) = \rho_1(e_{\sigma''}(\rho_2, \rho_3)) \tag{7.57}$$

from the previous equations and from relation (7.27), i.e.,

$$\beta \bar{\rho}_2(e_1(\rho_2, \bar{\rho}_2))\;\; e_1(\bar{\rho}_2, \rho_2)\;\; \bar{\rho}_2(e_1(\rho_2, \bar{\rho}_2)) = \bar{\rho}_2(e_1(\rho_2, \bar{\rho}_2)) ,$$

and, similarly

$$\beta e_{\sigma'}(\rho_1, \rho_2)\;\; \rho_1(e_{\sigma''}(\rho_2, \rho_3))\;\; e_{\sigma'}(\rho_1, \rho_2) = e_{\sigma'}(\rho_1, \rho_2) . \tag{7.58}$$

From (7.57) and (7.58) we can infer that $\mathrm{Ind}(\rho) \leq 4$, in case i), by using the statistics operator which is fundamental to previous approaches [19] to braid statistics and whose definition requires the explicit use of charge-transport operators and reference-(spectator-)endomorphisms. We therefore briefly rederive its properties from those of the statistics-matrices discussed in Theorem 2.3.1.

constraint (3.68) imposed on $\tilde{\Lambda}^t_\infty := \mathrm{N}^t_\rho \restriction \mathcal{C}^\infty_1 : \mathcal{C}^\infty_1 \to \mathcal{C}^\infty_2$, i.e., by determining the solutions of $\Lambda_\infty \Lambda^t_\infty = \tilde{\Lambda}_\infty \tilde{\Lambda}^t_\infty$. We first note that, since $2 \geq \|\Lambda_\infty\| = \|\Lambda_\infty \Lambda^t_\infty\|^{\frac{1}{2}} = \left\|\tilde{\Lambda}_\infty\right\|$, the graph associated to $\tilde{\Lambda}_\infty$ has to appear in the classification of graphs with norms not larger than two, given in Theorem 3.4.1 . Since any cycle (subgraph isomorphic to $A^{(1)}_n$) has norm equal to two, the only graphs in this set with cycles of length two, four or six, are $A^{(1)}_1$, $A^{(1)}_3$ and $A^{(1)}_5$. All other graphs, in particular those listed in o) - iv) of Proposition 7.3.1, fulfill the prerequisites of Corollary 3.3.7. Thus if there is some Λ_∞, $\|\Lambda_\infty\| \leq 2$, for which there exists a non-isomorphic solution, Λ'_∞, of the normality constraint, Λ'_∞ has to be of type, $A^{(1)}_{2n+1}$, $n = 0, 1, 2$. Since $\Lambda_\infty : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_1$ has no cycles of length two, four or six, we have, by Lemma 3.3.6, that the component $\mathcal{A} := (\Gamma^2_\infty)_c = \left(\Gamma'^2_\infty\right)_c$ of the twice iterate of one coloration obeys a) - c) of Lemma 3.3.6, so, by iv) of the same lemma, $\Gamma_\infty = \mathcal{G}_\mathcal{A}$. For $A^{(1)}_3$, we see that $\mathcal{A}_2 := \left(A^{(1)^2}_3\right)_c$ contains a double edge and is therefore excluded as a candidate for Γ'_∞, but, for $\mathcal{A}_1 := \left(A^{(1)^2}_1\right)_c$ and $\mathcal{A}_3 := \left(A^{(1)^2}_5\right)_c$ statements a) - c) are easily verified. We therefore obtain the only $\Gamma_\infty = (\Lambda_\infty, \mathcal{C}^\infty_o, \mathcal{C}^\infty_1)$ with non-isomorphic Γ'_∞ by going through the construction of $\mathcal{G}_\mathcal{A}$ given in Lemma 3.3.6. We conclude that

$$\begin{array}{lrcl} & \mathcal{G}_{\mathcal{A}_1} & \cong & D^{(1)}_4 , \\ \text{and} & & & \\ & \mathcal{G}_{\mathcal{A}_3} & \cong & E^{(1)}_6 . \end{array} \tag{7.48}$$

It follows from (7.48) and the positions of (*) in (A10) in Appendix A, that, for realizations of $D^{(1)}_4$ in a fusion rule algebra, we have that $\mathrm{Out}\,(\mathcal{C}^\infty_o) = \mathcal{C}^\infty_o = \{1, \sigma_1, \sigma_2, \sigma_3\}$, (thus $\cong \mathbb{Z}_2 \times \mathbb{Z}_2$ or $\mathbb{Z}_4$), and decomposition iii) follows. Similarly, we have for $E^{(1)}_6$, that $\mathrm{Out}\,(\mathcal{C}^\infty_o) = \{1, \sigma_1, \sigma_2, \}$ (hence $\cong \mathbb{Z}_3$ and $\sigma_1 = {\sigma_2}^2$), and therefore $\mathcal{C}^\infty_1 = \{\rho, \sigma_1 \circ \rho, \sigma_2 \circ \rho\}$. If we define $\psi \in \mathcal{C}^\infty_o$ by $\rho \circ \bar{\rho} = 1 \oplus \psi$, (i.e., $\mathcal{C}^\infty_o = \{1, \sigma_1, \sigma_2, \psi\}$), decomposition iv) follows, since ψ is a neighbor of every element in $\mathcal{C}^\infty_1$ in the graph $E^{(1)}_6$. For all other graphs, listed in o) - ii), we have by Corollary 3.3.7, a bijection, $\Sigma : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_2$, such that $\tilde{\Lambda}_\infty \Sigma = \Lambda_\infty$. If we apply Σ to the $\mathcal{C}^\infty_o$-part, d^o, of the Perron-Frobenius eigenvector of N_ρ, $d = (d^o, d^1, \ldots) \in \mathcal{C}^\infty_o \oplus \mathcal{C}^\infty_1 \oplus \ldots$ (if $\mathcal{C}^\infty_o = \mathcal{C}^\infty_i$ put $d^o = d^i$), obtained from the statistical dimensions, d_ρ, which is the Perron-Frobenius eigenvector of Λ_∞ in the finite case, then we find that

$$\Sigma d^o = d^{-1}_\rho \Sigma \Lambda^t_\infty d^1 = d^{-1}_\rho \left(\Lambda_\infty \Sigma^{-1}\right)^t d^1 = d^{-1}_\rho = d^{-1}_\rho \tilde{\Lambda}^t_\infty d^1 = d^2 .$$

By (3.55) the equation $d_{\Sigma(i)} = d_i$ implies that $\Sigma\,(\mathrm{Out}\,(\mathcal{C}^\infty_o)) = \mathrm{Out}\,(\mathcal{C}^\infty_2)$. Hence there i at least one automorphism in $(\mathcal{C}^\infty_2)$, namely $\sigma_1 := \Sigma 1$. More generally, the set

$$\mathrm{stab}(\rho) := \{\sigma \mid \sigma \circ \rho = \rho\}$$

is a subgroup of $\mathrm{Out}\,(\mathcal{C}^\infty_o)$, since $1 = \frac{d_{\sigma \circ \rho}}{d_\rho} = d_\sigma$ and because of grading considerations Hence, using $\mathrm{stab}(\rho) \hookrightarrow \mathrm{Aut}\,(\Gamma_\infty) : \sigma \mapsto \mathrm{N}^t_\sigma \restriction \mathcal{C}^\infty_o$, $\mathrm{stab}(\rho)$ is also a subgroup of the graph automorphisms of Γ_∞ that fix the vertex associated to ρ. It consists of the vertices in of edge degree one that are joined to ρ and is given, in the case where $\|\Gamma_\infty\| \leq 2$, by or $\mathbb{Z}_2 \times \mathbb{Z}_2$ for $D^{(1)}_4$, $\mathbb{Z}_3$ for D_4, $\mathbb{Z}_2$ for D_∞, $D^{(1)}_{n \geq 5}$ and A_3, and is trivial for all other cases.

If $\Sigma : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_2$, as defined above, exists it follows, that $\Sigma(\sigma) \circ \bar{\rho} = \rho$ for any $\mathcal{I} := \Sigma(\mathrm{stab}(\ell)) \subset \{\rho \circ \rho\}$. Also, $\mathcal{I}$ consists of automorphisms and, as $\Delta := \Sigma^{-1}\Sigma'$ is an automorphism of Γ_∞ fixing ρ, for any $\Sigma' : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_2$ with $\tilde{\Lambda}_\infty \Sigma' = \Lambda_\infty$, we have that Δ maps $\mathrm{stab}(\rho)$ to itself. Thus $\mathcal{I}$ is independent of Σ. Conversely, if, for $\alpha \in \mathcal{C}^\infty_2$, $\alpha \circ \bar{\rho} = \rho$ holds we have, from Lemma 3.3.4, that $\alpha \in \{\rho \circ \rho\} \cap \mathrm{Out}\,(\mathcal{C}^\infty_2)$. For any such automorphism, we can define $\Sigma_\alpha := \mathrm{N}^t_\alpha \restriction \mathcal{C}^\infty_o : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_{2.}$, with $\tilde{\Lambda}_\infty \Sigma_\alpha = \Lambda_\infty$, so that $\alpha = \Sigma_\alpha 1 \in \mathcal{I}$. In conclusion

$$\begin{array}{l} \mathcal{I} = \Sigma_\alpha(\mathrm{stab}(\rho)) = \{\alpha \in \mathcal{C}^\infty_2 : \alpha \circ \bar{\rho} = \rho\} = \{\rho \circ \rho\} \cap \mathrm{Out}\,(\mathcal{C}^\infty_2) , \\ \text{and} \\ |\mathcal{I}| = |\mathrm{stab}(\rho)| , \quad \text{for} \quad \mathrm{Out}\,(\mathcal{C}^\infty_2) \neq \emptyset , \quad \text{and} \quad \mathcal{I} = \emptyset , \quad \text{for} \quad \mathrm{Out}\,(\mathcal{C}^\infty_2) = \emptyset . \end{array} \tag{7.49}$$

We now can assign to the remaining inclusions the decompositions o) - ii) by comparing the isomorphic inclusions Γ_∞ and Γ'_∞ and their automorphisms. Since, in these cases, the number, v_ρ, of representations in $\rho \circ \bar{\rho}$ is less than four, we have that $v_\rho = \|\rho \circ \bar{\rho}\|^2 = \|\rho \circ \rho\|^2$ is equal to the number of representations in $\rho \circ \rho$; (here $\|(.)\|$ is the euclidean norm of eq. (3.3)). Since $v_\rho \leq 3$ and $|\mathrm{stab}(\rho)| \geq 1$, the number $\eta_\rho = v_\rho - |\mathrm{stab}(\rho)|$ o non-automorphic representations in $\rho \circ \rho$ obeys $\eta_\rho \leq 2$ and $\eta_\rho = 2$ only for $v_\rho = 3$ anc $|\mathrm{stab}(\rho)| = 1$. However, the only cases with $v_\rho = 3$ are $D^{(1)}_{n \geq 5}$, D_∞ and D_4 for whicl $\mathrm{stab}(\rho) = \mathbb{Z}_2$ or $\mathbb{Z}_3$, so that there is at most one non-automorphic representation in $\rho \circ$ This completes the first part of the proof, showing that $\mathrm{Ind}(\rho) \leq 4$ leads to the decompc sitions listed in o) - iv).

It remains to prove the converse implication, i.e., to derive $\mathrm{Ind}(\rho) \leq 4$ from the give *restriction of the fusion rule matrix to fusion between 0- and 1-graded sectors is given by the incidence matrix of one of the following bicolorable graphs:*

o) A_2;

i) $A_{n\geq 3}\,, D_{n\geq 5}\,, E_{6,7,8}(\mathrm{Ind}(\rho) < 4)$; $\quad A_\infty\,, E^{(1)}_{6,7,8}(\mathrm{Ind}(\rho) = 4)$;

ii) $D_4\,(\psi = \sigma_3\,, \mathrm{Ind}(\rho) < 4)$; $\quad D^{(1)}_{n\geq 5}\,, D_\infty\,(\psi \neq \sigma_3\,, \mathrm{Ind}(\rho) = 4)$;

iii) $D^{(1)}_4$;

iv) $E^{(1)}_6$.

(For the definition of these graphs see Appendix A.)

Proof. We first assume that $\mathrm{Ind}(\rho) \leq 4$ and show that o) - iv) are the only possible inclusion graphs. We consider the superselection structure, Φ, generated by ρ. Since $\mathrm{Ind}\,(\rho_1 \circ \rho_2) = [\rho_1 \circ \rho_2(\mathfrak{M}) : \mathfrak{M}] \leq [\rho_1 \circ \rho_2(\mathfrak{M}) : \rho_2(\mathfrak{M})]\,[\rho_2(\mathfrak{M}) : \mathfrak{M}] = \mathrm{Ind}\,(\rho_1)\,\mathrm{Ind}\,(\rho_2)$, [23], we have that $\mathrm{Ind}\,(\rho_i) < \infty$, for any sector $\rho_i \in \Phi$, and can thus assume properties $(P1)$ and $(P2)$ of Chapter 2 to hold on Φ. It follows that the fusion rule matrix, $\mathbb{N}_\rho$, is well defined on Φ and has only finitely many entries in each column and row. Moreover, we can assign to the sectors $\rho_i \in \Phi$ the statistical dimensions, $d_i = \mathrm{Ind}\,(\rho_i)^{\frac{1}{2}} < \infty$, which form a positive eigenvector of $\mathbb{N}_\rho$, according to (2.54), with eigenvalue $d_\rho = \beta_\rho^{\frac{1}{2}} = \mathrm{Ind}(\rho)^{\frac{1}{2}}$. Further, it follows from $(P1)$ and $(P2)$ that, in sequences $\ldots \mathcal{C}^{(n)}_{\#} \subset \mathcal{C}^{(n+1)}_{\#} \subset \ldots$, $\# = 0, 1$, each of the subsets $\mathcal{C}^{(n)}_{\#}$ of Φ, as defined in Lemma 7.1.5, is finite. We denote by $\mathcal{C}^\infty_{\#} = \bigcup_n \mathcal{C}^{(n)}_{\#} \subset \Phi$ the (possibly infinite) union of these sets, whose elements are the "#-graded sectors". We use the inclusion matrices, $\Lambda^{(m)}$, of the commutant algebras $M_{m-1} \subset M_m$ which are, by Proposition 3.2.1 i), just the restrictions $\Lambda^{(2n)} := \mathbb{N}_\rho \restriction \mathcal{C}^{(n-1)}_o \to \mathcal{C}^{(n)}_1$, $\Lambda^{(2n+1)} := \mathbb{N}^t_\rho \restriction \mathcal{C}^{(n)}_o \to \mathcal{C}^{(n)}_1$, $\mathcal{C}^{(n)}_{\#}$ labelling the factors of $M_{2n+\#}$, in order to define matrices $\Lambda_n : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_1$, with only finitely many non-zero entries, by setting

$$\Lambda_{2n}\,(\Lambda_{2n+1}) := \begin{cases} \Lambda^{(2n)^t}\left(\Lambda^{(2n+1)}\right) & \text{on } \mathcal{C}^{(n)}_o \\ 0\,, & \text{elsewhere}. \end{cases}$$

For these matrices, $\Lambda_{n+1} - \Lambda_n$ has non-negative entries, which are zero at positions where entries of Λ_n are non-zero. Thus, for the graphs, $\Gamma_{2n} = \left(\Lambda_{2n}, \mathcal{C}^{(n)}_o, \mathcal{C}^{(n-1)}_1\right)$, $\Gamma_{2n+1} = \left(\Lambda_{2n+1}, \mathcal{C}^{(n)}_o, \mathcal{C}^{(n)}_1\right)$, we have that Γ_n is a subgraph of Γ_{n+1} (i.e., it is obtained from Γ_{n+1} by amputating only the edges that are joined to vertices in $|\Gamma_{n+1}| \setminus |\Gamma_n|$). These graphs therefore inductively define $\Gamma_\infty = (\Lambda_\infty, \mathcal{C}^\infty_o, \mathcal{C}^\infty_1)$, with $\Lambda_\infty = \mathbb{N}_\rho \restriction \mathcal{C}^\infty_o : \mathcal{C}^\infty_o \to \mathcal{C}^\infty_1$, $\Lambda^t_\infty = \mathbb{N}^t_\rho \restriction \mathcal{C}^\infty_1 : \mathcal{C}^\infty_1 \to \mathcal{C}^\infty_o$. As in iv) of Proposition 3.2.1, we have Perron-Frobenius vectors $(\eta^n, \gamma^n) \in \mathcal{C}^\infty_o \times \mathcal{C}^\infty_1$, with finitely many non-zero components, such that $\Lambda_n \eta^n = \beta_n^{\frac{1}{2}} \gamma^n$ and $\Lambda^t_n \gamma^n = \beta_n^{\frac{1}{2}} \eta^n$, and, for the vectors formed from the statistical dimensions $d^{\#} \in \mathcal{C}^\infty_{\#}$, we have that $\Lambda_\infty d^o = \beta_\rho^{\frac{1}{2}} d^1$ and $\Lambda^t_\infty d^1 = \beta_\rho^{\frac{1}{2}} d^o$. Since $\Lambda_\infty - \Lambda_n$ has non-negative entries, it follows from $0 \leq (d^1, [\Lambda_\infty - \Lambda_n]\,\eta^n) + (\gamma^n, [\Lambda_\infty - \Lambda_n]\,d^o) = \left(\beta_\rho^{\frac{1}{2}} - \beta_n^{\frac{1}{2}}\right) [(d^1, \eta^n) + (d^o, \gamma^n)]$ that $\beta_n \leq \beta_\rho$. Thus, as the β_n are monotone increasing, $\beta_n \to \sup_n \beta_n \leq \beta_\rho$. (In order to show that $\beta_\rho = \sup_n \beta_n$ for the general infinite case, one has to go back to the definition of the index [23], since, for general infinite $\mathbb{N}_\rho$, there corresponds to any eigenvalue $\sqrt{\beta'} \geq \sqrt{\sup_n \beta_n}$ a sequence of numbers, d'_i, which form an eigenvector of $\mathbb{N}_\rho$). For $\mathrm{Ind}(\rho) \leq 4$, it follows that any subgraph $\Gamma_n \subset \Gamma_\infty$ has norm $\|\Gamma_n\| \equiv \|\Lambda_n\| \leq 2$. The finite, bicolorable graphs with norm not larger than two have been classified in Theorem 3.4.1 and are given by A_n, $A^{(1)}_{2n+1}$, D_n, $D^{(1)}_n$, $E_{6,7,8}$, $E^{(1)}_{6,7,8}$, from which we also find the non-bicolorable graphs $\overline{A}_n$, $A^{(1)}_{2n}$, $\overline{D^{(1)}_{2n}}$ and $\overline{\overline{A}}_n$. (These graphs are given in Appendix A.) It follows from $\Lambda_n 1 = \phi_\rho$, that each of the indecomposable graphs, Γ_n, has at least one vertex, 1, with edge degree one, which excludes $A^{(1)}_\ell$ from the above list of bicolorable graphs. It is easily verified that the only infinite series of graphs $\Gamma_n \subsetneqq \Gamma_{n+1} \subsetneqq \ldots$, which can be constructed from the above list are $A_n \subset A_{n+1} \subset \ldots$ and $D_n \subset D_{n+1} \subset \ldots$, where the common vertex, 1, is given by an endpoint of A_n, in the first series, and the endpoint of the short leg of D_n, in the second series. Besides the infinite graphs A_∞ and D_∞, we are left with the finite graphs A_n, D_n, $D^{(1)}_n$, $E_{6,7,8}$, $E^{(1)}_{6,7,8}$, which are listed in o) - iv). In any case, we have that $\beta_n \to \beta_\rho$, since $\beta_n = 4\cos^2\frac{\pi}{n+1}\left(= 4\cos^2\frac{\pi}{2n-1}\right)$ tends to 4, as $A_n \uparrow A_\infty$ $(D_n \uparrow D_\infty)$, and we have that $\beta_n = \beta_\rho$, for $n > \mathrm{diam}\,(\Gamma_\infty)$, when the graph Γ_∞ is finite. The sites in Γ_∞ at which we have automorphisms, i.e., the sites corresponding to the smallest component of the Perron-Frobenius vector of the incidence matrix of Γ_∞, have been indicated by $(*)$ in the graphs of the Appendix.

We are now in a position to derive the decompositions of $\rho \circ \rho$ and $\rho \circ \bar{\rho}$, stated in Proposition 7.3.1, from the list of possible inclusions by considering the normality

7.3 The Index and Fundamental Decompositions

In this section we investigate, for C^*-quantum categories, the connection between the structure of the tensorproducts $\rho \circ \rho$ and $\rho \circ \bar{\rho}$ of an irreducible object ρ and its statistical dimension $d(\rho)$. In particular, we find criteria in terms of these fundamental decompositions which are equivalent to the statements $d(\rho) \leq 2$ and $d(\rho) < 2$. We also prove that if the fundamental decompositions contain only invertible objects then the elements in $\rho \circ \bar{\rho}$ form a group isomorphic to $(\mathbb{Z}_2)^M$. The proofs are given in the formalism of C^*-quantum categories as arising in local quantum theories (see Chapters 2 and 7.1). They can be translated into the language of abstract tensor categories without difficulty.

The classification of fusion rule algebras presented in Section 3.4 was based on the ADE classification of graphs with norms not larger than two and is therefore associated to local quantum theories that are generated by a single localized *-endomorphism, ρ, with $\mathrm{Ind}(\rho) \leq 4$. In general, the computation of the index, $\mathrm{Ind}(\rho) = [\rho(\mathfrak{M}) : \mathfrak{M}]$, is rather difficult, so one is interested in replacing the index by other more computable quantities, which involve the use of locality and braid group statistics.

From the obvious inequalities for statistical dimensions,

$$d_\psi \geq 1\,, \qquad \text{and} \qquad d_\rho^2 = \sum_\psi N_{\rho\circ\rho,\psi}\, d_\psi \geq \#\{\psi : \psi \in \rho \circ \rho\}\,,$$

it is clear that if $\mathrm{Ind}(\rho) \leq 4$, $\rho \circ \rho$, as well as $\rho \circ \bar{\rho}$, cannot contain more than four irreducible subrepresentations. Also, it has been observed in [23] that, for selfconjugate sectors ρ with two-channel decompositions, $\rho \circ \rho = 1 \oplus \psi$, the existence of a unitary braid group representation enforces that $\mathrm{Ind}(\rho) \leq 4$. Below, we extend this result and list five classes of endomorphisms, characterized by the decomposition of $\rho \circ \rho$ and $\rho \circ \bar{\rho}$, for which $\mathrm{Ind}(\rho) \leq 4$ follows. The purpose of Proposition 7.3.1 is to show that it is possible to find constraints on the decompositions of $\rho \circ \rho$ (resp. $\rho \circ \bar{\rho}$) *equivalent* to the index restriction. More precisely, we prove that, for any endomorphism ρ which does not belong to one of the five classes, $\mathrm{Ind}(\rho) > 4$. In the description of these decompositions we not only count the total number of irreducible subrepresentations, but also the number of automorphisms in $\rho \circ \rho$ ($\rho \circ \bar{\rho}$, resp.). We shall see that the representation σ in the decomposition

$\rho \circ \rho \simeq \sigma \oplus \psi$ is found to be an automorphism if and only if the corresponding projection $e_\sigma(\rho,\rho) \in \rho^2(\mathfrak{M})' \cap \mathfrak{M}$ satisfies a Temperley-Lieb relation. The group of automorphisms in $\rho \circ \bar{\rho}$, i.e.,

$$\mathrm{stab}(\rho) := \{\sigma \mid \sigma \circ \rho \cong \rho\}$$

— which is important in cases ii) and iii) of Proposition 7.3.1 — is studied in generality, in the course of the proof. During the proof, we shall have to make a small detour, in order to rederive the braid-statistics formulation in terms of statistics operators from the theory developed in Section 2.2. The possible forms of the fusion rule matrix, N_ρ, restricted to the 0-graded sectors, will be given in terms of graphs, for each case separately, and knowing that the index of ρ is given by the square-norm of these graphs we can find the possible values of $\mathrm{Ind}(\rho)$: $\mathrm{Ind}(\rho) \in \left\{4\cos^2\frac{\pi}{N}\right\}_{N=3,\dots,\infty}$. In the more complicated cases, ii) ($\psi \neq \sigma$) iii) and iv), of Proposition 7.3.1, we shall reach the accumulation point, 4, of this set, and it turns out that, for $\rho = \bar{\rho}$, the fusion rules are dual, in the classical sense of [29], to the dihedral-(ii) and iii)) and the tetrahedron-group (iv)), regarded as discrete subgroups of SU(2).

Proposition 7.3.1 *Suppose that ρ is a localized irreducible *-endomorphism of a loca quantum theory with braid group statistics. Then*

$$\mathrm{Ind}(\rho) \leq 4$$

if and only if the composition of ρ with itself has one of the following decompositions int irreducible endomorphisms:

$$\begin{array}{rlcl}
o) & \rho & & \textit{is an automorphism} \\
i) & \rho \circ \rho & = & \sigma \oplus \psi\,; \\
ii) & \rho \circ \rho & = & \sigma_1 \oplus \sigma_2 \oplus \psi\,, \quad \textit{or, equivalently,} \quad \rho \circ \bar{\rho} = 1 \oplus \sigma \oplus \psi'\,; \\
iii) & \rho \circ \bar{\rho} & = & 1 \oplus \sigma_1 \oplus \sigma_2 \oplus \sigma_3\,; \\
iv) & \rho \circ \bar{\rho} & = & 1 \oplus \psi\,, \quad \text{with} \quad \psi \circ \rho = \rho \oplus \rho \circ \sigma_1 \oplus \rho \circ \sigma_2\,;
\end{array}$$

where σ, σ_i, $i = 1,2,3$ are localized *-automorphisms, i.e., $\sigma_i \circ \bar{\sigma}_i \cong \bar{\sigma}_i \circ \sigma_i \cong \mathrm{id}$, an ψ, ψ' are irreducible localized *-endomorphisms. Under these assumptions ρ generat a $\mathbb{Z}$- or $\mathbb{Z}_n$-graded superselection structure in which all sectors have finite index, and th we obtain in place of (7.42) and (7.43):

$$\begin{aligned}\sum_{\{\alpha_i\}} (I_{\bar{p},\bar{p}_1\ldots\bar{p}_n})_{\beta}^{\alpha_1\ldots\alpha_n}\ \psi_{\alpha_1}(\rho^{\bar{p}_1})\ldots\psi_{\alpha_n}(\rho^{\bar{p}_n})\,v_o \\ = \sum_{\{\alpha_i\}\gamma_h\gamma_{h+1}} (I_{\bar{p},\bar{p}_1\ldots\bar{p}_n})_{\beta}^{\alpha_1\ldots\alpha_n} \left(R^{\pm}_{p_{h+1}p_h}\right)^{\gamma_h\gamma_{h+1}}_{\alpha_{h+1}\alpha_h}\ \psi_{\alpha_1}(\rho^{\bar{p}_1})\ldots \\ \psi_{\gamma_{h+1}}(\rho^{\bar{p}_{h+1}})\ \psi_{\gamma_h}(\rho^{\bar{p}_h})\ldots\psi_{\alpha_n}(\rho^{\bar{p}_n})\ v_o\end{aligned} \tag{7.46}$$

and

$$\begin{aligned}\sum_{\{\alpha_i\}} (I_{p_1\ldots p_n,p})_{\beta}^{\alpha_1\ldots\alpha_n}\ \psi^{\dagger}_{\alpha_1}(\rho^{p_1})\ldots\psi^{\dagger}_{\alpha_n}(\rho^{p_n})\,v_o \\ = \sum_{\{\alpha_i\}} (I_{p_1\ldots p_n,p})_{\beta}^{\alpha_1\ldots\alpha_n}\ \left(R^{\pm}_{p_{h-1}p_h}\right)^{\gamma_h\gamma_{h-1}}_{\alpha_{h-1}\alpha_h}\ \psi^{\dagger}_{\alpha_1}(\rho^{p_1})\ldots \\ \psi^{\dagger}_{\gamma_h}(\rho^{p_h})\ \psi^{\dagger}_{\gamma_{h-1}}(\rho^{p_{h-1}})\ldots\psi^{\dagger}_{\alpha_n}(\rho^{p_n})\ v_o\,.\end{aligned} \tag{7.47}$$

for any β, $I_{\bar{p},\bar{p}_1\ldots\bar{p}_n} \in \mathrm{Int}\,(V_{\bar{p}}, V_{\bar{p}_1} \otimes \ldots \otimes V_{\bar{p}_n})$, $I_{p_1\ldots p_n,p} \in \mathrm{Int}\,(V_{p_1} \otimes \ldots \otimes V_{p_n}, V_p)$, $v_o \in \mathcal{H}_1$ and $\rho^{\bar{p}_{h+1}}$ and $\rho^{\bar{p}_h}$, resp. $\rho^{p_{h-1}}$ and ρ^{p_h}, spacelike separated. The expressions from (7.46) and (7.47) are contained in $\langle e_\beta\rangle \otimes \mathcal{H}_p$, $e_\beta \in V_p$, and vanish if we insert $I_{\bar{p},\bar{p}_1\ldots\bar{p}_n} \in \mathrm{Int}_o(V_{\bar{p}}, V_{\bar{p}_1} \otimes \ldots \otimes V_{\bar{p}_n})$, resp. $I_{p_1\ldots p_n,p} \in \mathrm{Int}_o(V_{p_1} \otimes \ldots \otimes V_{p_n}, V_p)$, so that the "internal states" on which $\mathcal{K}$ acts are actually described by the path spaces $\mathcal{P}(V_{\bar{p}}, V_{\bar{p}_1} \otimes \ldots \otimes V_{\bar{p}_n})$ and $\mathcal{P}(V_{p_1} \otimes \ldots \otimes V_{p_n}, V_p)$.

In the same way we find operator product expansions in constructions generalizing (7.44) and (7.45), relating the restricted monomial spaces by e.g.

$$\mathcal{F}^{\mathrm{cov}}_{\mathrm{rest}}(\rho^{\bar{p}_1},\ldots,\rho^{\bar{p}_n}) \subset \pi(\mathfrak{A}) \sum_{p\in\mathcal{L}} \mathcal{F}^{\mathrm{cov}}_{\mathrm{rest}}(\rho^{\bar{p}_1},\ldots,\rho^{\bar{p}_{h-1}},\rho^{\bar{p}},\rho^{\bar{p}_{h+2}},\ldots,\rho^{\bar{p}_n})\ ,$$

so that, eventually,

$$\mathcal{F}^{\mathrm{cov}}_{\mathrm{rest}}(\rho^{\bar{p}_1},\ldots,\rho^{\bar{p}_n}) \subset \pi(\mathfrak{A}) \sum_{p\in\mathcal{L}} \mathcal{F}_1(\rho^{\bar{p}})\,.$$

The necessity of contracting the fields is in fact not surprising, since we cannot expect to recover the entire R-matrix, R_{ij}, if $V_i \otimes V_j$ contains representations of zero q-dimension, from the information (the braid matrices) given by the statistics of superselection sectors.

We conclude our discussion of the field construction with a remark on $U_q(s\ell_2)$.

If we put $q = e^{\frac{i\pi}{N}}$, we see from the tensor-product decomposition (5.19), that any monomial expression $\psi(.,\rho^{\bar{p}_n})\ldots\psi(.,\rho^{\bar{p}_1})$ can be reproduced from the contracted products, i.e., we have

$$\psi_{\alpha_1}(\rho^{\bar{p}_1})\ldots\psi_{\alpha_n}(\rho^{\bar{p}_n}) \in \mathcal{F}^{\mathrm{cov}}_{\mathrm{rest}}(\rho^{\bar{p}_1},\ldots,\rho^{\bar{p}_n})$$

for all $\{\alpha_i\}$, whenever $\sum_{i=1}^{n} p_i - (n-1) < N$, where the labels p_i are the dimensions of the quantum group representations. Thus, with these bounds on the dimensions, (7.46) and (7.47) hold even when the tensors $I_{\bar{p},\bar{p}_1\ldots\bar{p}_n}$ and $I_{p_1\ldots p_n,p}$ are omitted.

b) stab$(\rho) = \mathbb{Z}_2 \times \overline{G}$, *with* $\overline{G}$ *as in a), and an additional generator* τ *of* $\mathbb{Z}_2$, *so that*

$$q\,(\tau^\varepsilon g) = i^{-\varepsilon}\, q(g)\,, \qquad \textit{for } \varepsilon \in \{0,1\}$$

and $g \in \overline{G}$, $q(g) \in \mathbb{Z}_2$ *as in a). Furthermore,* $A = \pm e^{\frac{i\pi}{4}}$, *and*

$$\theta_\alpha \pm \frac{1}{4} = 4\theta_\rho \bmod 1\,, \tag{7.105}$$

where $+$ *applies for* $\alpha \in \sigma_1 \circ \overline{G}$, *and* $-$ *for* $\alpha \in \tau \circ \sigma_1 \circ \overline{G}$.

c) stab$(\rho) = (\mathbb{Z}_2 \times \mathbb{Z}_2) \times \overline{G}$, *with additional generators* τ *and* b, *so that*

$$q\left(\tau^\varepsilon b^\delta g\right) = (-1)^{\varepsilon\delta}\, i^\delta\, q(g)\,, \tag{7.106}$$

with $\varepsilon, \delta \in \{0,1\}$ *and* $g \in \overline{G}$, $q(g) \in \mathbb{Z}_2$ *as in a). We have* $A = \pm 1$, *and*

$$\begin{aligned} \theta_\alpha &= 4\theta_\rho \bmod 1\,, \quad \textit{for } \alpha \in G'\,, \\ \theta_\alpha + \frac{1}{2} &= 4\theta_\rho \bmod 1\,, \quad \textit{for } \alpha \in b \circ G'\,, \end{aligned} \tag{7.107}$$

where $G' = \{1, \tau\} \times \overline{G}$.

Proof:

i) These are simple consequences of the fact that $\sigma_1 \in \rho \circ \rho$ and $\sigma_1 \circ \overline{\sigma_1} = 1$ implies $\bar{\rho} = \overline{\sigma_1} \circ \rho$.

ii) For some σ_1, let A and q be defined by equations (7.98). We first show (7.99), using the fact that $q(\alpha)$ can be interpreted as the ratio of two particular intertwiners. To see this let R_α, $L_\alpha \in \mathrm{Int}\,(\rho \circ \rho, \sigma_1 \circ \alpha)$ be given by

$$R_\alpha := \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\,(\Gamma_{\rho\circ\alpha,\rho}) \quad \text{and} \quad L_\alpha := \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho}\,. \tag{7.108}$$

First, it follows from

$$\begin{aligned} A\, e^{-2\pi i\theta_\rho}\, q(\alpha)\, R_\alpha &= R_\alpha\, \varepsilon^+(\rho,\rho) = \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\,(\Gamma_{\rho\circ\alpha,\rho})\, \varepsilon^+(\rho,\rho) = \\ &= \Gamma^*_{\rho\circ\rho,\sigma_1}\, \varepsilon^+(\rho,\rho)\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho} \\ &= A\, e^{-2\pi i\theta_\rho}\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho} = A\, e^{-2\pi i\theta_\rho}\, L_\alpha \end{aligned}$$

that

$$R_\alpha = q(\alpha)^{-1}\, L_\alpha\,. \tag{7.109}$$

This implies

$$\begin{aligned} &\Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\,(\Gamma_{\rho\circ\alpha,\rho}\, \Gamma_{\rho\circ\beta,\rho}) = q(\alpha)^{-1}\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho}\, \rho\,(\Gamma_{\rho\circ\beta,\rho}) \\ &\quad= q(\alpha)^{-1}\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\alpha,\rho)\, \alpha\,(\Gamma_{\rho\circ\beta,\rho})\right)\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\rho\left(\varepsilon^-(\beta,\alpha)\right)\, \Gamma_{\rho\circ\beta,\rho}\, \varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, \sigma_1\left(\varepsilon^-(\beta,\alpha)\right)\, R_\beta\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, q(\beta)^{-1}\, \sigma_1\left(\varepsilon^-(\beta,\alpha)\right)\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\beta,\rho)\right)\, \Gamma_{\rho\circ\beta,\rho}\, \rho\left(\varepsilon^+(\alpha,\rho)\right)\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, q(\beta)^{-1}\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\rho\left(\varepsilon^-(\beta,\alpha)\right)\, \varepsilon^+(\beta,\rho)\, \beta\left(\varepsilon^+(\alpha,\rho)\right)\right)\, \Gamma_{\rho\circ\beta,\rho}\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, q(\beta)^{-1}\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\alpha,\rho)\, \alpha\left(\varepsilon^+(\beta,\rho)\right)\, \varepsilon^-(\beta,\alpha)\right)\, \Gamma_{\rho\circ\beta,\rho}\, \Gamma_{\rho\circ\alpha,\rho} \\ &\quad= q(\alpha)^{-1}\, q(\beta)^{-1}\, R^-(\rho,\beta,\alpha,\rho)^\rho_\rho\, \Gamma^*_{\rho\circ\rho,\sigma_1}\, \rho\left(\varepsilon^+(\alpha,\rho)\, \alpha\left(\varepsilon^+(\beta,\rho)\right)\right)\, \Gamma_{\rho\circ\alpha,\rho}\, \Gamma_{\rho\circ\beta,\rho}\,. \end{aligned}$$

For fixed $\alpha, \beta \in$ stab(ρ) there is a unitary, $\Gamma_{\alpha\circ\beta,\alpha\beta}$, with $\rho\left(\Gamma^*_{\alpha\circ\beta,\alpha\beta}\right)\, \Gamma_{\rho\circ\alpha,\rho}\, \Gamma_{\rho\circ\beta,\rho} = \Gamma_{\rho\circ\alpha\beta,\rho}$ and $\rho\left(\Gamma^*_{\alpha\circ\beta,\alpha\beta}\right)\, \varepsilon^+(\alpha,\rho)\, \alpha\,(\varepsilon^+(\beta,\rho)) = \varepsilon^+(\alpha\beta,\rho)\, \Gamma_{\alpha\circ\beta,\alpha\beta}$. Hence multiplying both sides of the above equation with $\sigma_1\left(\Gamma^*_{\alpha\circ\beta,\alpha\beta}\right)$ from the right we obtain

$$R_{\alpha\beta} = q(\alpha)^{-1}\, q(\beta)^{-1}\, f(\beta,\alpha)\, L_{\alpha\beta}\,. \tag{7.110}$$

But by (7.109) this implies (7.99). This, however, implies that f is symmetric as well as skew symmetric, hence $f \in \mathbb{Z}_2 \subset U(1)$ and $f^2 \equiv 1$. Now we use the non-degeneracy of f on stab$(\rho) \times$ stab(ρ) and the normal form of Lemma 7.3.2 to see that all $\nu_j = 2$ in (7.85), i.e., the claim of ii), stab$(\rho) = (\mathbb{Z}_2)^M$, is true. If we specialize

$$f(\alpha,\beta) = q(\alpha)\, q(\beta)\, q(\alpha \circ \beta)^{-1} \tag{7.111}$$

to $\alpha = \beta$ and use $\alpha^2 = 1$, we find $q(\alpha)^2 = f(\alpha,\alpha)$, which together with (7.87) gives (7.100). Finally $q(\alpha)^4 = f(\alpha,\alpha)^2 = 1$, so $q : \text{stab}(\rho) \to \mathbb{Z}_4 \subset U(1)$.

The formulae (7.102), expressing the dependence of A and q on σ_1, follow directly from the defining equations (7.98).

where we use the basic commutation relations and (7.20). If we insert (7.91) we obtain

$$I_\beta^* \, E_\delta \, I_\alpha = \frac{1}{|\mathrm{stab}(\rho)|} \left(\overline{f(\delta,\beta)} \, \Gamma^*_{\rho\circ\delta,\rho} \, \Gamma^*_{\beta\circ\rho,\rho}\right) \Gamma_{\beta\circ\rho,\rho} \left(f(\delta,\alpha) \, \Gamma_{\alpha\circ\rho,\rho} \, \Gamma_{\rho\circ\delta,\rho}\right)$$

$$= \frac{1}{|\mathrm{stab}(\rho)|} f\left(\delta, \alpha \circ \bar{\beta}\right), \tag{7.94}$$

hence

$$E_\delta \, I_\alpha = \frac{1}{|\mathrm{stab}(\rho)|} \sum_{\beta \in \mathrm{stab}(\rho)} f\left(\delta, \bar{\beta}\right) \; I_{\alpha\circ\beta} \,. \tag{7.95}$$

For any character $\sigma \in \hat{G}$ of a finite abelian group G, we know that $\frac{1}{|G|} \sum_{g \in G} \sigma(g) = \delta_{\sigma,1}$. So if $\mathcal{N} \subset \mathrm{stab}(\rho)$ is the degenerate subgroup of a bihomomorphic form f, i.e., $\mathcal{N} = \{\alpha \mid f(\alpha,\beta) = 1 \; \forall \beta \in \mathrm{stab}(\rho)\}$, then this means

$$\frac{1}{|\mathrm{stab}(\rho)|} \sum_\beta f(\alpha,\beta) = \begin{cases} 1 & \text{for } \alpha \in \mathcal{N} \\ 0 & \text{else}. \end{cases} \tag{7.96}$$

With this formula we find from (7.95)

$$\sum_\delta E_\delta \, I_\alpha = \sum_{\gamma \in \mathcal{N}} I_{\alpha\circ\gamma} \,. \tag{7.97}$$

However, by completeness, $\sum_\delta E_\delta = \mathbf{1}$, this implies $\mathcal{N} = \{1\}$, i.e., f is non degenerate. With this knowledge the orthogonality relations $E_\alpha E_\beta = \varepsilon_{\alpha\beta} E_\alpha$ can be easily verified. □

The remarks made in Lemma 7.3.3 will now serve as an important tool to prove the following assertions on the situation where $\rho \circ \rho$ decomposes entirely into invertible elements. Proposition 7.3.4 classifies the possible groups, stab(ρ), to be of the type $(\mathbb{Z}_2)^M$ and gives the general spectra of the statistics operators $\varepsilon(\alpha,\beta)$, $\varepsilon(\rho,\rho)$, in a suitable choice of generators of stab(ρ).

Proposition 7.3.4 *Suppose ρ is an irreducible object of a quantum category and assume that $\rho \circ \rho$ decomposes into invertible elements. Then*

i) $\mathrm{supp}(\rho \circ \bar{\rho}) = \mathrm{stab}(\rho)$, *and for any $\sigma_1 \in \rho \circ \rho$ we have* $\rho \circ \rho = \sum_{\alpha \in \mathrm{stab}(\rho)} \sigma_1 \circ \alpha$.

ii) All elements in stab(ρ) *are selfconjugate, i.e., there is some $M \in \mathbb{N}$, such that*

$$\mathrm{stab}(\rho) \cong (\mathbb{Z}_2)^M \,,$$

so, in particular, $|d_\rho| = 2^{\left(\frac{M}{2}\right)}$.

iii) For any given $\sigma_1 \in \rho \circ \rho$, let $q : \mathrm{stab}(\rho) \to U(1)$ and $A \in U(1)$ be defined by

$$\varepsilon(\rho,\rho) = A \, e^{-2\pi i \theta_\rho} \sum_{\alpha \in \mathrm{stab}(\rho)} q(\alpha) \, e_{\sigma_1 \circ \alpha}(\rho,\rho) \tag{7.98}$$

and $q(1) = 1$. Then we have that the bihomomorphism f defined in Lemma 7.3.3 is a 2-coboundary given by

$$f = \delta q \,. \tag{7.99}$$

We have

$$q(\alpha)^2 = e^{2\pi i \theta_\alpha} \,, \tag{7.100}$$

and further

$$q^4 \equiv 1 \,, \qquad f^2 \equiv 1 \,.$$

The constant A^2 is given by

$$A^2 = e^{2\pi i \left(4\theta_\rho - \theta_{\sigma_1}\right)} \,. \tag{7.101}$$

If σ_1 is replaced by $\sigma_1' = \beta \circ \sigma_1$, $\beta \in \mathrm{stab}(\rho)$, then the quantities A' and q' associated to σ_1' are given by

$$A' = q(\beta) \, A \,, \quad \textit{and} \quad q'(\alpha) = f(\beta,\alpha) \, q(\alpha) \,. \tag{7.102}$$

iv) There is a choice of σ_1 and a system of generators of stab(ρ) *such that the quadratic function $q : \mathrm{stab}(\rho) \to \mathbb{Z}_4$ is as in one of the following cases:*

a) $\mathrm{stab}(\rho) = \overline{G} = (\mathbb{Z}_2 \times \mathbb{Z}_2)^N$ *with generators ξ_i, η_i, $i = 1, \ldots, N$, and*

$$q\left(\prod_{i=1}^N \xi_i^{\varepsilon_i} \, \eta_i^{\delta_i}\right) = (-1)^{\sum_{i=1}^N \varepsilon_i \delta_i} \,, \quad \textit{for } \varepsilon_i, \delta_i \in \{0,1\} \,. \tag{7.103}$$

In this case $A = \pm 1$ according to whether ρ is real or pseudoreal (if selfconjugate) and

$$\theta_\alpha = 4\theta_\rho \bmod 1 \,, \qquad \forall \alpha \in \rho \circ \rho \,. \tag{7.104}$$

the skew symmetry, $f(\alpha,\beta)=\overline{f(\beta,\alpha)}$. The polynomial equations on the one-dimensional intertwiner spaces, Int $(\rho\circ\alpha_1\circ\cdots\circ\alpha_2,\rho)$, are given by

$$\check{F}(\rho,\alpha,\beta,\rho)^{\rho}_{\alpha\circ\beta}\,R^{\pm}(\rho,\alpha\circ\beta,\gamma,\rho)^{\rho}_{\rho}=R^{\pm}(\rho,\alpha,\gamma,\rho)^{\rho}_{\rho}\,R^{\pm}(\rho,\beta,\gamma,\rho)^{\rho}_{\rho}\,\check{F}(\rho,\alpha,\beta,\rho)^{\rho}_{\alpha\circ\beta}$$

etc., which imply that f is homomorphic in every component, after cancelling the F-matrices.

It is clear that we always have a normalization of intertwiners such that

$$\Gamma^{*}_{\alpha\bar{\alpha},1}\,\alpha\left(\Gamma_{\bar{\alpha}\alpha,1}\right)=\alpha\left(\Gamma^{*}_{\bar{\alpha}\alpha,1}\right)\,\Gamma_{\alpha\bar{\alpha},1}=1 \tag{7.88}$$

for all $\alpha\in\mathrm{stab}(\rho)$, with $\alpha\neq\bar{\alpha}$. However, for $\alpha^2=1$ this can *a priori* still vary by a sign (pseudoreality). To exclude this possibility note that in general

$$\varepsilon\left(\alpha,\bar{\alpha}\right)=e^{2\pi i\theta_\alpha}\,\Gamma_{\bar{\alpha}\alpha,1}\,\Gamma^{*}_{\alpha\bar{\alpha},1}\,. \tag{7.89}$$

For $\alpha^2=1$ this specializes to $\varepsilon(\alpha,\alpha)=e^{2\pi i\theta_\alpha}$ and hence $f(\alpha,\alpha)=e^{2\pi i\theta_\alpha}\in\mathbb{Z}_2$. The statistical parameter is then $\lambda_\alpha:=\alpha\left(\Gamma^{*}_{\alpha\alpha,1}\right)\varepsilon(\alpha,\alpha)\,\alpha\left(\Gamma_{\alpha\alpha,1}\right)=e^{2\pi i\theta_\alpha}$ but also $\lambda_\alpha:=e^{2\pi i\theta_\alpha}\Gamma^{*}_{\alpha\alpha,1}\alpha\left(\Gamma_{\alpha\alpha,1}\right)$. With $e^{4\pi i\theta_\alpha}=1$ this implies reality for α and (7.88) holds for any α, i.e., by unitarity we have $\Gamma_{\alpha\circ\bar{\alpha},1}=\alpha\left(\Gamma_{\bar{\alpha}\circ\alpha,1}\right)$. For the unitary intertwiners, $\Gamma_{\rho\circ\alpha,\rho}$ we have an F-matrix identity

$$\Gamma_{\rho\circ\alpha,\rho}\,\Gamma_{\rho\circ\bar{\alpha},\rho}=\varphi_\alpha\,\rho\left(\Gamma_{\alpha\circ\bar{\alpha},1}\right)$$

for some φ_α. We obtain

$$\Gamma_{\rho\circ\bar{\alpha},\rho}=\varphi_\alpha\,\Gamma^{*}_{\rho\circ\alpha,\rho}\,\rho\left(\Gamma_{\alpha\circ\bar{\alpha},1}\right)=\varphi_\alpha\,\Gamma^{*}_{\rho\circ\alpha,\rho}\,\rho\circ\alpha\left(\Gamma_{\bar{\alpha}\circ\alpha,1}\right)=\varphi_\alpha\,\rho\left(\Gamma_{\bar{\alpha}\circ\alpha,1}\right)\Gamma^{*}_{\rho\circ\alpha,\rho}\,,$$

which yields, after multiplication of $\Gamma_{\rho\circ\alpha,\rho}$ from the right, $\varphi_\alpha=\varphi_{\bar{\alpha}}$. Hence

$$\begin{aligned}\rho\left(\varepsilon\left(\alpha,\bar{\alpha}\right)\right)\,\Gamma_{\rho\circ\alpha,\rho}\,\Gamma_{\rho\circ\bar{\alpha},\rho}&=\varphi_\alpha\,e^{2\pi i\theta_\alpha}\,\rho\left(\Gamma_{\bar{\alpha}\circ\alpha,1}\right)\\&=e^{2\pi i\theta_\alpha}\,\Gamma_{\rho\circ\bar{\alpha},\rho}\,\Gamma_{\rho\circ\alpha,\rho}\\\text{and therefore}\quad f(\alpha,\alpha)&=e^{2\pi i\theta_\alpha}\,,\quad\text{for general }\alpha\,.\end{aligned}$$

iii) First let us record a simple consequence of the equation

$$\check{\varphi}(i,j,k,\ell)^{s\mu\nu}_{r\alpha\beta}=\sum_{\beta'\alpha'}R^{\pm}(1,i,r,\ell)^{\ell 1\beta'}_{\ell 1\beta}\,R^{\mp}(j,k,i,\ell)^{s\alpha'\nu}_{r\alpha\beta'}\,R^{\mp}(1,j,i,s)^{s1\mu}_{s1\alpha'} \tag{7.90}$$

(compare (2.38)) for the matrices, $\check{\varphi}$, defined by

$$\rho^{i}\left(\Gamma_{\rho^{j}\circ\rho^{k},\rho^{r}}(\alpha)\right)\,\Gamma_{\rho^{i}\circ\rho^{r},\rho^{\ell}}(\beta)=\sum_{s\mu\nu}\check{\varphi}(i,j,k,\ell)^{s\mu\nu}_{r\alpha\beta}\,\Gamma_{\rho^{i}\circ\rho^{j},\rho^{s}}(\mu)\,\Gamma_{\rho^{s}\circ\rho^{k},\rho^{\ell}}(\nu)\,.$$

If (7.90) is specialized to $j=\ell=s=r=\rho$ and $k=\alpha$, $i=\beta\in\mathrm{stab}(\rho)$, then we find

$$\check{\varphi}(\beta,\rho,\alpha,\rho)^{\rho}_{\rho}=R^{\pm}(\rho,\alpha,\beta,\rho)^{\rho}_{\rho}$$

and therefore

$$\beta\left(\Gamma_{\rho\circ\alpha,\rho}\right)\,\Gamma_{\beta\circ\rho,\rho}=f(\alpha,\beta)\,\Gamma_{\beta\circ\rho,\rho}\,\Gamma_{\rho\circ\alpha,\rho}\,. \tag{7.91}$$

We introduce an orthonormal basis, $\{I_\alpha\}_{\alpha\in\mathrm{stab}(\rho)}$, on the $|\mathrm{stab}(\rho)|$-dimensional Intertwiner space Int $(\rho\circ\bar{\rho}\circ\rho,\rho)$ by

$$I_\alpha:=\Gamma^{*}_{\alpha\circ\rho,\rho}\,\alpha\left(\Gamma_{\rho\circ\bar{\rho},1}\right)\,\Gamma_{\alpha\circ\rho,\rho} \tag{7.92}$$

and consider the action of the complete set of orthogonal projectors, $\{E_\delta\}_{\delta\in\mathrm{stab}(\rho)}$ given by

$$E_\delta=\rho\left(\Gamma_{\bar{\rho}\circ\rho,\delta}\,\Gamma^{*}_{\bar{\rho}\circ\rho,\delta}\right)=\rho\left(\bar{\rho}\left(\Gamma^{*}_{\rho\circ\delta,\rho}\right)\,\Gamma_{\bar{\rho}\circ\rho,1}\,\Gamma^{*}_{\bar{\rho}\circ\rho,1}\,\bar{\rho}\left(\Gamma_{\rho\circ\delta,\rho}\right)\right)\,, \tag{7.93}$$

on Int $(\rho\circ\bar{\rho}\circ\rho,\rho)$, with respect to the basis (7.92). A matrix element of E_δ is given by

$$\begin{aligned}I^{*}_{\beta}\,E_\delta\,I_\alpha&=\Gamma^{*}_{\beta\circ\rho,\rho}\,\beta\left(\Gamma^{*}_{\rho\circ\bar{\rho},1}\right)\,\Gamma_{\beta\circ\rho,\rho}\,\rho\left(\bar{\rho}\left(\Gamma^{*}_{\rho\circ\delta,\rho}\right)\,\Gamma_{\bar{\rho}\circ\rho,1}\,\Gamma^{*}_{\bar{\rho}\circ\rho,1}\,\bar{\rho}\left(\Gamma_{\rho\circ\delta,\rho}\right)\right)\,\Gamma^{*}_{\alpha\circ\rho,\rho}\\&\quad\alpha\left(\Gamma_{\rho\circ\bar{\rho},1}\right)\,\Gamma_{\alpha\circ\rho,\rho}\\&=\Gamma^{*}_{\beta\circ\rho,\rho}\,\beta\left(\Gamma_{\rho\circ\bar{\rho},1}\right)\,\beta\circ\rho\left(\bar{\rho}\left(\Gamma^{*}_{\rho\circ\delta,\rho}\right)\Gamma_{\bar{\rho}\circ\rho,1}\right)\,\Gamma_{\beta\circ\rho,\rho}\,\Gamma^{*}_{\alpha\circ\rho,\rho}\,\alpha\circ\rho\left(\Gamma^{*}_{\bar{\rho}\circ\rho,1}\right)\\&\quad\bar{\rho}\left(\Gamma_{\rho\circ\delta,\rho}\right)\,\alpha\left(\Gamma_{\rho\circ\bar{\rho},1}\right)\,\Gamma_{\alpha\circ\rho,\rho}\\&=\Gamma^{*}_{\beta\circ\rho,\rho}\,\beta\left(\Gamma^{*}_{\rho\circ\delta,\rho}\right)\,\beta\left(\Gamma^{*}_{\rho\circ\bar{\rho},1}\,\rho\left(\Gamma_{\bar{\rho}\circ\rho,1}\right)\right)\,\Gamma_{\beta\circ\rho,\rho}\,\Gamma^{*}_{\alpha\circ\rho,\rho}\,\alpha\left(\rho\left(\Gamma^{*}_{\bar{\rho}\circ\rho,1}\right)\,\Gamma_{\rho\circ\bar{\rho},1}\right)\\&\quad\alpha\left(\Gamma_{\rho\circ\delta,\rho}\right)\,\Gamma_{\alpha\circ\rho,\rho}\\&=\frac{1}{|\mathrm{stab}(\rho)|}\,\Gamma^{*}_{\beta\circ\rho,\rho}\,\beta\left(\Gamma^{*}_{\rho\circ\delta,\rho}\right)\,\Gamma_{\beta\circ\rho,\rho}\,\Gamma^{*}_{\alpha\circ\rho,\rho}\,\alpha\left(\Gamma_{\rho\circ\delta,\rho}\right)\,\Gamma_{\alpha\circ\rho,\rho}\end{aligned}$$

For the proof of these assertions the following presentation of skew-bihomomorphic forms on abelian groups will be useful.

Lemma 7.3.2 *Let G be a finite abelian group and*

$$f : G \times G \to U(1)$$

a nondegenerate bihomomorphic form.

i) If f has trivial diagonal, i.e., $f(\alpha,\alpha) = 1$, $\forall \alpha \in G$, then

$$G \cong (\mathbb{Z}_{\nu_1} \times \mathbb{Z}_{\nu_1}) \underset{\perp}{\times} (\mathbb{Z}_{\nu_2} \times \mathbb{Z}_{\nu_2}) \underset{\perp}{\times} \cdots \underset{\perp}{\times} (\mathbb{Z}_{\nu_k} \times \mathbb{Z}_{\nu_k}) \qquad (7.84)$$

where the orders divide each other as $\nu_1 \mid \nu_2 \mid \ldots \mid \nu_k$ and "$\underset{\perp}{\times}$" means orthogonal with respect to f. On each factor $G_j = \mathbb{Z}_{\nu_j} \times \mathbb{Z}_{\nu_j}$ with generators ξ and η, f is determined by

$$f(\xi,\eta) = e^{\frac{2\pi i}{\nu_j}} \,. \qquad (7.85)$$

ii) If f is only skew symmetric, i.e., $f(\alpha,\beta) = \overline{f(\beta,\alpha)}$, $\forall \alpha,\beta \in G$, and $f(\alpha,\alpha) = \pm 1$, then either

a) f has trivial diagonal; or

b) $G = \mathbb{Z}_2 \underset{\perp}{\times} \overline{G}$, with $f(\tau,\tau) = -1$ for the generator τ of the $\mathbb{Z}_2$-part, and f is nondegenerate and has trivial diagonal on $\overline{G}$, or

c) there is some (unique) $m \geq 1$, such that $G = (\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m}) \underset{\perp}{\times} \overline{G}$, where f is given on the generators ξ and η of the $(\mathbb{Z}_{2^m} \times \mathbb{Z}_{2^m})$ part by

$$f(\xi,\eta) = e^{\frac{2\pi i}{2^m}}, \quad f(\xi,\xi) = 1 \quad \text{but} \quad f(\eta,\eta) = -1 \,. \qquad (7.86)$$

Furthermore, f is nondegenerate with trivial diagonal on $\overline{G}$.

We shall not give a detailed proof of this fact here but satisfy the reader's curiosity with a few remarks. The first part is a standard exercise in normal forms, using the invariant-divisor form $G = \mathbb{Z}_{\nu_1} \times \cdots \times \mathbb{Z}_{\nu_n}$, $\nu_i \mid \nu_{i+1}$, of the group and the nondegeneracy of f. If f

is only skew, then $\alpha \to f(\alpha,\alpha)$ is a homomorphism $G \to \mathbb{Z}_2$, so there is some $\tau \in G$, with $f(\tau,\alpha) = f(\alpha,\alpha)$, $\forall \alpha \in G$ and $\tau^2 = 1$. In case a) we have $\tau = 1$, in case b) $\tau \neq 1$, with $f(\tau,\tau) = -1$, and $\overline{G}$ is simply given by $\overline{G} = \tau^{\perp}$. The complications arise when $\tau \neq 1$ and $f(\tau,\tau) = 1$. Then τ is contained in some maximal $\mathbb{Z}_{2^m}$ with generator ξ, so $\tau = \xi^{2^{(m-1)}}$.

The relevance of studying nondegenerate, bihomomorphic forms becomes clear in the next lemma.

Lemma 7.3.3 *Suppose that, for an irreducible object ρ, of a C^*-quantum category,* $\mathrm{supp}(\rho \circ \bar{\rho}) = \mathrm{stab}(\rho)$. *Then*

i) the multiplicity of $\sigma \in \rho \circ \bar{\rho}$ is one, for all $\sigma \in \mathrm{stab}(\rho)$, and $d_\rho^2 = |\mathrm{stab}(\rho)|$.

ii) Let $f(\alpha,\beta) := R^{\pm}(\rho,\alpha,\beta,\rho)_\rho^\rho$ for all $\alpha,\beta \in \mathrm{stab}(\rho)$. Then f is bihomomorphic and skewsymmetric, and

$$f(\alpha,\alpha) = e^{\pm 2\pi i \theta_\alpha} \in \mathbb{Z}_2 \,. \qquad (7.87)$$

All selfconjugate elements $\alpha \in \mathrm{stab}(\rho)$, i.e., $\alpha^2 = 1$, are real.

iii) f is nondegenerate.

Proof:

i) We first repeat an argument given in the proof of Proposition 7.3.1. We have that $\sigma \circ \rho = \rho$ is irreducible. Thus $1 = N_{\sigma \circ \rho,\rho} = N_{\sigma,\rho \circ \bar{\rho}}$. From $\rho \circ \bar{\rho} = \sum_{\sigma \in \mathrm{stab}(\rho)} \sigma$ we have that

$$d_\rho \cdot d_{\bar{\rho}} = \sum_{\sigma \in \mathrm{stab}(\rho)} 1 \; = |\mathrm{stab}(\rho)| \,,$$

as $d_\sigma = 1$.

ii) The number $f(\alpha,\beta)\delta_{k\rho}\delta_{\rho\rho} := R^{\pm}(\rho,\alpha,\beta,\rho)_\ell^k \in U(1)$ is well defined because $\dim(\mathrm{Int}\,(\rho \circ \alpha \circ \beta, \rho)) = 1$. The claim of Lemma 7.3.3 is that the sectors in $\mathrm{stab}(\rho)$ obey trivial statistics, so $m(\alpha,\beta) = \varepsilon(\beta,\alpha)\,\varepsilon(\alpha,\beta) = 1$, which on the level of R-matrices means $R^+(\rho,\beta,\alpha,\rho)_\rho^\rho R^+(\rho,\alpha,\beta,\rho)_\rho^\rho = 1$. But this is just expressed by the monodromy $m(\rho,\sigma) = \varepsilon^+(\sigma,\rho)\varepsilon^+(\rho,\sigma) = U^-(\sigma)^* U^+(\sigma) = e^{2\pi i\vartheta_\sigma}$ defines a character on stab(ρ).

We can define a second operator $\overline{U}(\sigma)$, which is different from $U^\pm(\sigma)$, for $\rho \neq \bar\rho$, by setting

$$\overline{U}(\sigma) := d_\rho \, \bar\rho \circ \sigma \left(\Gamma^*_{\rho\circ\bar\rho,1}\right) \, \Gamma_{\bar\rho\circ\rho,1} \,, \tag{7.76}$$

so that $\overline{U}(\sigma) \in \mathrm{Int}(\bar\rho \circ \sigma, \rho)$. Also, since $\sigma \circ \rho = \rho$ and by (7.76), we have that $\overline{U}(\sigma) \in \bar\rho \circ \rho(\mathfrak{M})' \cap \mathfrak{M}$, and

$$\overline{U}(\sigma) \, \Gamma_{\bar\rho\circ\rho,1} = \Gamma_{\bar\rho\circ\rho,1} \,. \tag{7.77}$$

From the irreducibility of $\bar\rho$ and $\bar\rho \circ \sigma$ and from (7.76) it follows that $\overline{U}(\sigma)$ is unitary. Thus $\sigma_{\overline{U}(\sigma)} \circ \bar\rho = \bar\rho \circ \sigma$. This shows that $\sigma \to \overline{U}(\sigma)$ is a unitary representation of stab(ρ), since, by (7.77), no 2-cocycles (as in (7.73)) can arise. Therefore we can write $\overline{U}(\sigma)$ in the form

$$\overline{U}(\sigma) = \sum_{k,\alpha=1}^{N_{\bar\rho\circ\rho,k}} (h_k(\sigma))^\alpha_\beta \; \Gamma_{\bar\rho\circ\rho,k}(\alpha) \, \Gamma^*_{\bar\rho\circ\rho,k}(\beta) \,, \tag{7.78}$$

where $h_k : \mathrm{stab}(\rho) \to \mathrm{End}\left(\mathbb{C}^{N_{\bar\rho\circ\rho,k}}\right)$, $\sigma \mapsto h_k(\sigma)$, is a unitary representation of stab(ρ) on $\mathbb{C}^{N_{\bar\rho\circ\rho,k}}$, and $h_1(\sigma) = 1$.

The left inverse, $\varphi_{\bar\rho}$, of $\bar\rho$, defined by

$$\varphi_{\bar\rho}(A) = \Gamma^*_{\rho\circ\bar\rho,1} \rho(A) \Gamma_{\rho\circ\bar\rho,1} \,, \quad A \in \mathfrak{M} \,, \tag{7.79}$$

maps $\mathrm{Int}(\bar\rho \circ \sigma, \bar\rho)$ to $\mathrm{Int}(\sigma, 1)$. It therefore follows from Schur's Lemma that

$$\varphi_{\bar\rho}\left(\overline{U}(\sigma)\right) = 0 \,, \qquad \text{for} \quad \sigma \neq 1 \,. \tag{7.80}$$

Note that, by the "generalized" Temperley-Lieb-relations (7.57), we also have that

$$\varphi_{\bar\rho}\left(e_\sigma(\bar\rho \circ \rho)\right) = \beta^{-1} \,, \tag{7.81}$$

for all $\sigma \in$ stab(ρ).

In the case of interest, $\bar\rho \circ \rho = 1 \oplus \sigma \oplus \psi$, (7.78) specializes (with $\Gamma_{\bar\rho\circ\rho,\psi}\Gamma^*_{\bar\rho\circ\rho,\psi} = 1 - e_1(\bar\rho,\rho) - e_\sigma(\bar\rho,\rho)$) to

$$\overline{U}(\sigma) = h_\psi(\sigma) + (1 - h_\psi(\sigma)) \; e_1(\bar\rho,\rho) + (h_\sigma(\sigma) - h_\psi(\sigma)) \; e_\sigma(\bar\rho,\rho) \,. \tag{7.82}$$

If we apply the left inverse to (7.82) and use (7.80) and (7.81) this yields the following equation for β:

$$\beta = 2 - \frac{1 + h_\sigma(\sigma)}{h_\psi(\sigma)} \,. \tag{7.83}$$

Here $h_\sigma(\sigma)$ and $h_\psi(\sigma)$ are characters of stab(ρ) and, because of the equation

$$\beta = d_\rho^2 = d_1 + d_\sigma + d_\psi = 2 + d_\psi \,,$$

they are constrained to satisfy $-\frac{1+h_\sigma(\sigma)}{h_\psi(\sigma)} = d_\psi \geq 1$. If ψ is an automorphism stab(ρ) is of order three, thus isomorphic to $\mathbb{Z}_3$, and therefore $\psi \equiv \bar\sigma$. Also, we have that $h_\sigma(\sigma) = (h_\psi(\sigma))^{-1}$ is a third root of unity, and it follows that

$$\beta = 3 \,.$$

For a ψ which is not an automorphism, we show that stab$(\rho) \cong \mathbb{Z}_2$, so that $h_\sigma(\sigma)$, $h_\psi(\sigma) \in \{1, -1\}$. The only solution of (7.83) with $\beta > 3$ is therefore $h_\psi(\sigma) = -1$, $h_\sigma(\sigma) = 1$, and we obtain that

$$\beta = 4 \,.$$

This completes the proof of Proposition 7.3.1. □

The statement of next lemma can also be expressed as the fact that all sectors in stab(ρ) are either fermionic or bosonic and obey trivial statistics relations among each other. The superselection structure of stab(ρ) may be realized by any finite, abelian group. This changes if we assume that the automorphisms stabilizing ρ constitute the entire decomposition of $\rho \circ \rho$, i.e., if we assume supp$(\rho \circ \bar\rho) =$ stab(ρ) and $\rho \circ \rho$ contains at least one invertible element. Still there exist fusion rule algebras for any abelian group G such that $G \cong$ stab(ρ), but if we require this fusion rule algebra to describe a quantum category (resp. a local quantum field theory) these automorphisms are given by the representations of a finite, abelian reflection group, i.e., stab$(\rho) \cong (\mathbb{Z}_2)^N$ for some N. The best known examples are those for $N = 1$ which arises in the quantum category constructed from $U_q^{\mathrm{red}}(s\ell_2)$, $q = e^{\frac{i\pi}{4}}$, with A_3-fusion rules, realized by the $\widehat{SU}(2)_{k=2}$ WZNW-model (or any other $c = \frac{1}{2}$-RCFT) or the critical Ising model, and for $N = 2$ where the category is obtained from the dihedral group, $D_2 \subset SU(2)$, with $D_4^{(1)}$-fusion rules, and realized by the $SU(2)/D_2$-orbifold model at $c = 1$ or a 4-state Potts model.

More precisely, for $H_{q_\rho,\infty} = \mathbf{C}[B_\infty]/ I^H_{q_\rho}$, $\mathcal{A}_{\beta,\infty} = \mathbf{C}[B_\infty]/ I^{\mathcal{A}}_\beta$, we have that $I^H_{q_\rho} \subset I^{\mathcal{A}}_\beta$, i.e., $\mathcal{A}_{\beta,\infty}$ is a quotient of $H_{q_\rho,\infty}$, if and only if (7.68) holds. From this we obtain the possible values of d_ρ:

$$d_\rho = 2\cos\frac{\pi}{N}, \tag{7.69}$$

which, in particular, shows that $\mathrm{Ind}(\rho) \leq 4$.

We remark here that, for the situation where $\rho \circ \rho \cong \sigma \oplus \psi$, the Temperley-Lieb relations for the projections $e_n := \rho^n(e_\sigma(\rho,\rho))$ imply that σ is an automorphism. This is most easily verified by computing $\varepsilon^+(\sigma,\sigma)$ from $\varepsilon^+(\rho,\rho)$ with the help of the polynomial equations and the cabelling procedure. It turns out that

$$\varepsilon^+(\sigma,\sigma) = z_\rho^4\, q_\rho\, \mathbf{1}. \tag{7.70}$$

However, a result in [19] tells us that if $\varepsilon^+(\sigma,\sigma)$ is proportional to the identity σ is an automorphism.

Finally, for case ii) we only assume that $\rho \circ \bar{\rho} = 1 \oplus \sigma \oplus \psi$ and show that $\mathrm{Ind}(\rho) = 4$ follows. The peculiarity we exploit here is that the decomposition of $\rho \circ \bar{\rho}$ yields an automorphism σ, with $\sigma \circ \rho \cong \rho$, which, in the language used above, means that the subgroup $\mathrm{stab}(\rho) \subset \mathrm{Out}(\Phi)$ is nontrivial ($\cong \mathbb{Z}_2$). At the level of a local algebra, a stabilizer subgroup of $\mathrm{Aut}(\mathcal{C})$ can be defined similarly, by $\mathrm{stab}(\mathfrak{A}) := \{\sigma \in \mathrm{Aut}(\mathcal{C}) : \sigma(A) = A,\ \forall A \in \mathfrak{A}\}$, where $\mathfrak{A} \subset \mathfrak{M}$. If we restrict the projection π' of $\mathrm{Aut}(\mathcal{C})$ onto the quotient $\mathrm{Aut}(\mathcal{C})/\,\mathrm{Int}(\mathcal{C}) \cong \mathrm{Out}(\Phi)$, as discussed in Section 2.5.3, to $\mathrm{stab}(\rho(\mathfrak{M}))$ it is clear that its image lies in $\mathrm{stab}(\rho)$; i.e., we have a group homorphism π given by

$$\begin{tikzcd}
\pi :\ \mathrm{stab}(\rho(\mathfrak{M})) \arrow[r] \arrow[d] & \mathrm{stab}(\rho) \arrow[d] \\
\mathrm{Aut}(\mathcal{C}) \arrow[r, two heads] & \mathrm{Out}(\Phi).
\end{tikzcd}$$

For a representative $\sigma' \in \mathrm{Aut}(\mathcal{C})$ of $[\sigma'] \in \mathrm{stab}(\rho)$, there exists a unitary operator $\Gamma_{\sigma'\circ\rho,\rho} \in U(\mathcal{C})$, with $\sigma' \circ \rho(A)\Gamma_{\sigma'\circ\rho,\rho} = \Gamma_{\sigma'\circ\rho,\rho}\rho(A)$. Thus $\sigma := \sigma_{\Gamma^*_{\sigma'\circ\rho,\rho}} \circ \sigma'$ is an element in $\mathrm{stab}(\rho(\mathfrak{M}))$ with $[\sigma] = [\sigma']$, showing that π is surjective. Since ρ is irreducible, it also follows from $\rho(\mathfrak{M})' \cap U(\mathcal{C}) = \mathbf{C}\mathbf{1}$ that π is injective. Hence

$$\mathrm{stab}(\rho(\mathfrak{M})) \cong \mathrm{stab}(\rho). \tag{7.71}$$

In particular, this implies that $\mathrm{stab}(\rho(\mathfrak{M}))$ is an abelian group, although its elements are in general not causally independent, and the group extension

$$0 \to \mathit{Int}(\mathcal{C}) \longrightarrow \mathcal{G} \longrightarrow \mathit{stab}(\rho) \to 0$$

splits. Here, $\mathcal{G} \subset \mathit{Aut}(\mathcal{C})$ is the respective preimage of $\mathit{stab}(\rho)$.

The one-dimensional space $\mathrm{Int}(\sigma \circ \rho, \rho \circ \sigma) = \mathrm{Int}(\rho, \rho \circ \sigma)$, is spanned by either $U^+(\sigma) := \varepsilon^+(\rho,\sigma)$ or $U^-(\sigma) := \varepsilon^-(\rho,\sigma)$, so $U^+(\sigma) = e^{2\pi i\vartheta_\sigma} U^-(\sigma)$ and by the definition of the statistics operator, we have that

$$\rho(\Gamma_{\sigma^\pm,\sigma}) = \Gamma_{\sigma^\pm,\sigma}\, U^\pm(\sigma), \tag{7.72}$$

where $\sigma^\pm \times \rho$ and $\mathrm{as}(\sigma^\pm) \gtrless \mathrm{as}(\rho)$.

Since $\rho \circ \sigma = \sigma_{U^\pm(\sigma)} \circ \rho$, we find that $\sigma \to U^\pm(\sigma)$ defines two projective representations of $\mathrm{stab}(\rho)$ in $U(\mathcal{C})$. Thus there are 2-cocycles $\gamma^\pm \in B^2(\mathrm{stab}(\rho))$, with

$$U^\pm(\sigma)\, U^\pm(\mu) = \gamma^\pm(\sigma,\mu)\, U^\pm(\sigma \circ \mu), \tag{7.73}$$

where $\gamma^+ \sim \gamma^-$ by $\gamma^- = \gamma^+ \cdot \delta\left(e^{2\pi i\vartheta}\right)$

If we let $\Gamma_{\mu^\pm\circ\sigma^\pm,\mu\circ\sigma} = \mu^\pm(\Gamma_{\sigma^\pm,\sigma})\,\Gamma_{\mu^\pm,\mu}$ be the charge transport operator for the composed automorphism $\sigma \circ \mu = \mu \circ \sigma \in \mathrm{stab}(\rho(\mathfrak{M}))$, we can relate these cocycles to the charge transporters, by inserting (7.72) in (7.73). This yields

$$\mu(\Gamma_{\sigma^\pm,\sigma}) = \gamma^\pm(\sigma,\mu)\,\Gamma_{\sigma^\pm,\sigma}. \tag{7.74}$$

Applying μ to (7.72), it follows that

$$\mu\left(U^\pm(\sigma)\right) = \overline{\gamma^\pm(\sigma,\mu)}\, U^\pm(\sigma). \tag{7.75}$$

From (7.73) and (7.75) it follows immediately that $\gamma^\pm(\sigma,\mu)$ is a homomorphism in both arguments separately, and, by (7.61), we have that $\varepsilon^\pm(\sigma,\mu) = \gamma^\pm(\sigma,\mu)\mathbf{1}$. Since $U^+(\sigma)$ is proportional to $U^-(\sigma)$, we conclude from (7.75) that $\gamma^+(\sigma,\mu) = \gamma^-(\sigma,\mu)$. In other words, the sectors in $\mathrm{stab}(\rho(\mathfrak{M}))$ obey ordinary Fermi-Bose statistics among themselves, i.e., $\varepsilon^+(\sigma,\mu) = \varepsilon^-(\sigma,\mu) =: \gamma(\sigma,\mu)$. Moreover, it follows that $\delta\left(e^{2\pi i\vartheta}\right) = 1$, so the value of

The equation (7.101) follows from the monodromy spectrum $m(\rho,\rho) = \varepsilon(\rho,\rho)^2 = \sum_{\alpha\in\mathrm{stab}(\rho)} e^{2\pi i(2\theta_\rho - \theta_{\sigma_1\circ\alpha})} e_{\sigma_1\circ\alpha}(\rho,\rho)$ and relation (7.118) of Lemma 7.4.1, below. If we attempt to compute the statistical parameter of, ρ, we find from

$$\lambda_\rho := \frac{1}{d_\rho} e^{-2\pi i\theta_\rho} = \rho\left(\Gamma^*_{\rho\circ\bar{\rho},1}\right)\ \varepsilon(\rho,\rho)\ \rho\left(\Gamma^*_{\rho\circ\bar{\rho},1}\right) ,$$

and the generalized Temperley-Lieb relation (7.57), that

$$A^{-1} = \frac{1}{d_\rho} \sum_{\alpha\in\mathrm{stab}(\rho)} q(\alpha), \tag{7.112}$$

where $d_\rho = \pm\sqrt{|\mathrm{stab}(\rho)|}$, the sign depending on the reality of ρ. In order to obtain the more detailed information on the braid matrices, given in part iv), we have to use the presentation of $\mathrm{stab}(\rho)$ and f in Lemma 7.3.2. We shall restrict our attention first to case a), where f has trivial diagonal or, equivalently, all sectors in $\mathrm{stab}(\rho)$ are bosonic. This implies that $q^2 \equiv 1$. $\overline{G}$ has the decomposition $\overline{G} = (\mathbb{Z}_2\times\mathbb{Z}_2) \underset{\perp}{\times} \ldots \underset{\perp}{\times} (\mathbb{Z}_2\times\mathbb{Z}_2)$, with generators η_i, ξ_i in each factor, and $f(\eta_i,\eta_j) = f(\xi_i,\xi_j) = 1$ and $f(\eta_i,\xi_j) = (-1)^{\delta_{i,j}}$. Thus from (7.111) the value of q on a general element in $\overline{G}$ can be computed from $q(\eta_i)$ and $q(\xi_i) \in \{+1,-1\}$ as

$$\begin{aligned} q\left(\prod_{i=1}^N \xi_i^{\varepsilon_i}\,\eta_i^{\delta_i}\right) &= \prod_{i=1}^N q\left(\xi_i^{\varepsilon_i}\,\eta_i^{\delta_i}\right) = \prod_{i=1}^N (-1)^{\varepsilon_i\delta_i}\, q\left(\xi_i^{\varepsilon_i}\right)\, q\left(\eta_i^{\delta_i}\right) \\ &= (-1)^{\sum_{i=1}^N \varepsilon_i\delta_i} \prod_{i=1}^N q(\xi_i)^{\varepsilon_i}\, q(\eta_i)^{\delta_i} . \end{aligned} \tag{7.113}$$

To prove (7.103) we have to show that σ_1 can be chosen such that $q(\xi_i) = q(\eta_i) = 1$. Clearly any map q from the generators of $\mathrm{stab}(\rho)$ to $\mathbb{Z}_2$ extends uniquely to a homomorphism $\tilde{q} : \mathrm{stab}(\rho) \to \mathbb{Z}_2$, such that $\tilde{q}(\xi_i) = q(\xi_i)$, $\tilde{q}(\eta_i) = q(\eta_i)$, (but in general $\tilde{q} \neq q$). Since f is nondegenerate there exists some $\alpha_q \in \mathrm{stab}(\rho)$ with $f(\alpha_q, g) = \tilde{q}(g)$. If we now set $\sigma_1' = \alpha_q \circ \sigma_1$ we find from (7.102) that $q'(\xi_i) = f(\alpha_q,\xi_i)\, q(\xi_i) = \tilde{q}(\xi_i)\, q(\xi_i) = 1$ and also $q'(\eta_i) = 1$. Thus, for a given choice of generators ξ_i and η_i of $\mathrm{stab}(\rho)$, σ_1 is in fact uniquely determined by $q(\xi_i) = q(\eta_i) = 1$.

Using

$$\sum_{\{\varepsilon,\delta\}} (-1)^{\sum_{i=1}^N \varepsilon_i\delta_i} = 2^N , \qquad \varepsilon_i, \delta_i \in \{0,1\}$$

and $|\mathrm{stab}(\rho)| = 4^N$ we find from (7.112) that $A = \pm 1$. Inserting this into (7.101) and using $\theta_{\sigma_1\circ\alpha} = \theta_{\sigma_1} + \theta_\alpha = \theta_{\sigma_1}$ mod 1 in the bosonic case we arrive at (7.104).

For the cases b) and c) we can repeat the above procedure on the $\overline{G}$-parts. Agai from orthogonality, $\tau^\perp = \overline{G}$, we have in case b) $q(\tau^\varepsilon g) = q(\tau)^\varepsilon q(g)$, that $\varepsilon \in \{0,1\}$, and since $q(\tau)^2 = f(\tau,\tau) = -1$ and because of the freedom to change the sign of $q(\tau)$ by replacing σ_1 by $\tau\circ\sigma_1$ we can choose σ_1 such that $q(\tau) = -i$, and $q(g)$ as in (7.103) on $\overline{G}$ From the equation $\sum_{\alpha\in\mathrm{stab}(\rho)} q(\alpha) = \sum_{\varepsilon=0,1,g\in\overline{G}} (-i)^\varepsilon q(g) = (1-i)2^N$, and $|\mathrm{stab}(\rho)| = 2^{(2N+1}$ we find the value of A. This yields (with (7.101)) $4\theta_\rho = \frac{1}{4} + \theta_{\sigma_1}$, and since $\theta_{\tau^\varepsilon\circ g\circ\sigma_1} =$ $\theta_{\tau^\varepsilon\circ g} + \theta_{\sigma_1} = \frac{\varepsilon}{2} + \theta_{\sigma_1}$, we find (7.105).

Similarly we can choose σ_1 in case c) (with $m = 1$) such that $q(\tau) = 1$, $q(b) = i$ an q on $\overline{G} \cong (\mathbb{Z}_2\times\mathbb{Z}_2)^N$ as in a). We then find $\sum_{\alpha\in\mathrm{stab}(\rho)} q(\alpha) = 2^{(N+1)}$ and $|\mathrm{stab}(\rho)| = 4^N$, s that $A = \pm 1$. Similarly as in b), this, together with $\theta_{\tau^\varepsilon b^\delta g} = \frac{\delta}{2}$ mod 1, $g \in \overline{G}$, implie formulae (7.107).

We can now use this result and the previous ones on fusion rule algebras, in order to obtai a sharper version of Proposition 7.3.1 in the case where $d(\rho) < 2$. This restriction on th dimension eliminates the possibilities iii) and iv) of Proposition 7.3.1. The decompositio under ii) belongs to only one inclusion, namely D_4. The associated fusion rule algebra given in (3.128),(3.129) and (3.130) of Theorem 3.4.11, with $\mathrm{stab}(\rho) = \mathbb{Z}_3$, we can b excluded, by Proposition 7.3.4, to be associated to any C^*-quantum category. If we als require $d(\rho) > 1$ the only remaining case is the two channel decomposition in i). Th ratio of the two eigenvalue of the monodromy $m(\rho,\rho) = \varepsilon(\rho,\rho)^2$ is q^2, related to the inde by (7.68). Thus, we can exclude the $d(\rho) = 2$ cases in i) if we require the monodromy t be nonscalar. To summarize, we have:

Proposition 7.3.5 *Suppose that ρ is an irreducible object of a C^*-quantum categor Then the following are equivalent:*

i)

$$1 < d(\rho) < 2$$

ii) We have a decomposition

$$\rho \circ \rho = \sigma + \psi \,,$$

where ψ is an irreducible and σ an invertible object, and the monodromy of ρ is nonscalar, i.e.,

$$m(\rho, \rho) = \varepsilon(\rho, \rho)^2 \notin \mathbb{C}1_{\rho\circ\rho} \,.$$

iii) The 0- and 1- graded part of the fusion rule algebra are finite and the restriction of the fusion rule matrix N_ρ corresponds to one of the following bicolorable graphs:

$$A_l \, (l \geq 3), \ D_{2l} \, (l \geq 3), \ E_6, \ E_8 \,. \tag{7.114}$$

The results proven above also lead to the exclusion of various fusion rule algebras at $d_\rho = 2$. For instance, if we consider the series of fusion rule algebras obtained from $D^{(1)}_{2p'+2}$, $p' \in \mathbb{N}$, (see Lemma 3.4.5 ii) (3.93)) by the procedure given in Proposition 3.3.2, we find for the element $f := (0, \omega_{p'})$, that $f \circ f = \sum_{\alpha \in G} \sigma_1 \circ \alpha$ if $\mathrm{grad}(f) = 1$ and $f \circ f = \sum_{\alpha \in G} \alpha$ if $\mathrm{grad}(f) = 0$, where $G = \mathrm{Out}\,(\Phi_0) = \mathrm{stab}(f)$. In the list of possible fusion rule algebras, Theorem 3.4.11 ii), the cases $G = \mathbb{Z}_2 \times \mathbb{Z}_2$, (3.146)-(3.150), for any ρ, and $G = \mathbb{Z}_4$, (3.151)-(3.158), are both represented. For $p = 2p'$, the existence of the sub-quantum category with generator f and Proposition 7.3.4 imply that only the fusion rule algebras with $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ are admissible. Comparing this to Lemma 3.4.9, (3.111) and (3.112), we then find as a result that all fusion rule algebras with selfconjugate generator, ρ, of dimension $d_\rho = 2$ which describe a quantum-category are in fact realized by a compact subgroup of $SU(2)$ or $O(2)$. At the $d_\rho = 2$-threshold we also encounter the first two examples of fusion rule algebras, specified in Lemma 3.4.10, which cannot be deduced from a selfconjugate version. However if, as in the case of $D^{(4)}_1 \left(A^{(1)}_1\right)^{(a-2)}$, $a \geq 3$, $\rho \circ \rho = 2\tau$, then the monodromy $\varepsilon(\rho, \rho)^2$, clearly has to be scalar, so either $\varepsilon(\rho, \rho) = e^{2\pi i\left(\theta_\rho - \frac{1}{2}\theta_\tau\right)}\mathbf{1}$ or $\varepsilon(\rho, \rho) = e^{2\pi i\left(\theta_\rho - \frac{1}{2}\theta_\tau\right)}\left(e^1_\tau(\rho, \rho) - e^2_\tau(\rho, \rho)\right)$. For these two possibilities,the statistics parameter $\lambda_\rho := \rho\left(\Gamma^*_{\rho_3, \bar\rho, 1}\right) \varepsilon(\rho, \rho)\, \rho\left(\Gamma_{\rho\circ\bar\rho, 1}\right)$ is either a phase, i.e., $|\lambda_\rho| = 1$, or $\lambda_\rho = 0$, both contradictory $|\lambda_\rho| = \frac{1}{|d_\rho|} = \frac{1}{2}$. A similar argument applies to exclude the algebras $E_6 \left(A^{(1)}_3\right)^{(a-2)}$ and descendents, (3.159)-(3.162), from those consistent with a quantum category.

7.4 Balancing Phases

In this section we compute the possible balancing or statistical phases for the fusionrules determined in Theorem 3.4.11, assuming that they are associated to some C^*-quantum category. This computation, based on the situation described in Proposition 7.3.5, imposes consistency conditions by which the E- and D- algebras and certain twisted A- algebras can be excluded. It will be seen in Chapter 8 that the remaining fusionrules are in fact all realized as object algebras of a C^*-quantum category. In the derivation of these results we again use the language of local quantum theories which can be easily translated into the general categorial formulation.

As we have seen in Lemma 7.1.2, any C^*-quantum category admits a natural balancing. The balancing endomorphisms, in this case, are all unitary and are determined by their values on the irreducible objects. We thus have phases $\theta_\rho \in \mathbb{R}/\mathbb{Z}$, defined by

$$\sigma_o(\rho) = \pm R(1, \rho, \bar\rho, 1)^{\rho 11}_{\bar\rho 11} =: e^{2\pi i \theta_\rho} = e^{2\pi i \theta_{\bar\rho}} \,, \tag{7.115}$$

where the sign is as in (7.20). These phases will be called spins or statistical phases, in reference to the spin-statistics theorem for relativistic local quantum field theories. For some simple quantum categories, the spins can be computed directly from the fusion rules, without any further knowledge of the category beyond its existence. In doing so, we encounter consistency relations by which most of the exceptional fusion rule algebras from Theorem 3.4.11 can be excluded as building blocks for quantum categories.

One of the main tools used to determine spins comes from the analysis of the braiding relations involving *invertible* objects σ, i.e., $\sigma \in \mathrm{Out}(\Phi)$. Since, for any $\sigma \in \mathrm{Out}(\Phi)$ and irreducible $\phi \in \Phi$, $\sigma \circ \phi$ is irreducible, too, we find that

$$\varepsilon(\sigma, \phi) \circ \varepsilon(\phi, \sigma) =: m(\sigma, \phi) = m(\phi, \sigma) = e^{2\pi i \Theta_\sigma(\phi)} \cdot \mathbf{1} \,, \tag{7.116}$$

with

$$\Theta_\sigma(\phi) = \theta_\phi + \theta_\sigma - \theta_{\sigma\circ\phi} \bmod 1 \,. \tag{7.117}$$

The properties of the phases $\Theta_\sigma(\phi)$ that can be obtained from the polynomial equations have already been mentioned at the end of Section 3.3. We give a more complete summary in the following lemma.

Lemma 7.4.1 *Suppose Φ is the fusion rule algebra of a C^*-quantum category, and let*

$$\Theta_\sigma(\phi) \in \mathbb{R}/\mathbb{Z}, \qquad \phi \in \Phi, \qquad \sigma \in \mathrm{Out}(\Phi)$$

be defined as in (7.117). Then the following statements hold.

i) For any $\sigma \in \mathrm{Out}(\Phi)$, the map

$$\Theta_\sigma : \Phi \to \mathbb{R}/\mathbb{Z}$$

is a grading, i.e., there exists a character,

$$\Theta'_\sigma : \mathrm{Grad}(\Phi) \to \mathbb{R}/\mathbb{Z},$$

such that

$$\Theta_\sigma = \Theta'_\sigma \circ \mathrm{grad}\,. \tag{7.118}$$

ii) The assignment

$$\begin{array}{rcl} \Theta' : \mathrm{Out}(\Phi) & \to & \widehat{\mathrm{Grad}}(\Phi) \\ \sigma & \to & \Theta'_\sigma \end{array}$$

is a group homomorphism.

iii) If $i^* : \widehat{\mathrm{Grad}(\Phi)} \to \widehat{D(\Phi)}$ is the pull back of the inclusion $D(\Phi) \subset \mathrm{Grad}(\Phi)$ then

$$\Theta' : \mathrm{Out}\,(\Phi_0) \to \ker i^* \cong (\widehat{\mathrm{Grad}(\Phi)}/\, D(\Phi))\,.$$

Thus there exists a homomorphism

$$\begin{array}{rrcl} & \Theta'' : D(\Phi) & \to & \widehat{D(\Phi)} \\ \textit{with} & \Theta''_{g_1}(g_2) & = & \Theta''_{g_2}(g_1) \qquad (7.119) \\ \textit{such that} & i^* \circ \Theta'_\sigma & = & \Theta''_{\mathrm{grad}\,(\sigma)}\,. \qquad (7.120) \end{array}$$

Proof. If $\psi_3 \in \psi_1 \circ \psi_2$, i.e., there exists an isometry $\Gamma_{\psi_1\circ\psi_2,\psi_3} \neq 0$, it follows from

$$\begin{aligned} & e^{2\pi i(\Theta_\sigma(\psi_1)+\Theta_\sigma(\psi_2))}\,\Gamma_{\psi_1\circ\psi_2,\psi_3} \\ &= \psi_1(\varepsilon(\psi_2,\sigma))\,\varepsilon(\psi_1,\sigma)\,\varepsilon(\sigma,\psi_1)\,\psi_1(\varepsilon(\sigma,\psi_2))\,\Gamma_{\psi_1\circ\psi_2,\psi_3} \qquad (7. \\ &= \psi_1(\varepsilon(\psi_2,\sigma))\,\varepsilon(\psi_1,\sigma)\,\sigma(\Gamma_{\psi_1\circ\psi_2,\psi_3})\,\varepsilon(\sigma,\psi_3) \\ &= \Gamma_{\psi_1\circ\psi_2,\psi_3}\,\varepsilon(\psi_3,\sigma) = \Gamma_{\psi_1\circ\psi_2,\psi_3}\,e^{2\pi i\Theta_\sigma(\psi_3)}\,, \end{aligned}$$

that $\Theta_\sigma(\psi_3) = \Theta_\sigma(\psi_1) + \Theta_\sigma(\psi_2)$ mod 1, i.e. Θ_σ is a grading. Here we use the nota $\psi(I)$, as in the frame work of local field theories, instead of the more conventional nota $\mathbf{1}_\psi \circ I$, for an object ψ and an arrow I. Similarly, we have that $\Theta_{\sigma_1\circ\sigma_2}(\psi) = \Theta_{\sigma_1}(\psi$ $\Theta_{\sigma_2}(\psi)$ mod 1, using the fact, that $\sigma_1(\varepsilon(\sigma_2,\psi))\ \varepsilon(\sigma_1,\psi)$ is equivalent to $\varepsilon(\sigma_1 \circ \sigma_2,$ This shows that Θ_σ is a grading, and hence, by the considerations of Section 3.3, be expressed by the homomorphism Θ'_σ, and $\sigma \to \Theta'_\sigma$ is also a homomorphism. Clea we have that $\Theta_{\sigma_1}(\sigma_2) = \Theta_{\sigma_2}(\sigma_1)$, which implies $\tilde{\Theta}'_{\sigma_1}(\mathrm{grad}\,(\sigma_2)) = \tilde{\Theta}'_{\sigma_2}(\mathrm{grad}\,(\sigma_1))$, a therefore, since $\mathrm{grad}\,(\sigma) = 1$, $\forall\sigma \in \mathrm{Out}\,(\Phi_0)$, statement iii) of Lemma 7.4.1 follows.

In the $\mathbb{Z}_a$-graded case, the most general expression for $\Theta_\sigma(\phi)$ can be found without di culty:

Lemma 7.4.2 *Assume Φ is the fusion rule algebra of a C^*-quantum category a* $\mathrm{Grad}(\Phi) = \mathbb{Z}_a$. *Let r be given by $D(\Phi) \cong \mathbb{Z}_r$ and the inclusion $D(\Phi) \subset \mathrm{Grad}(\Phi)$ $a''\mathbb{Z}_r \subset \mathbb{Z}_a$, where $a = r \cdot a''$. Then there is a homomorphism*

$$\eta : \mathrm{Out}\,(\Phi_0) \to \mathbb{Z}_{a''}$$

and, for any fixed $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\mathrm{grad}\,(\sigma_1) = a''$, a number $h_{\sigma_1} \in \mathbb{Z}_a$, with

$$h_{\sigma_1} \equiv \eta\,(\sigma_1^r) \bmod a''\,, \qquad (7.12$$

such that

$$\Theta_{\sigma_1^k\circ\beta}(\phi) = \left(k\,\frac{h_{\sigma_1}}{a} + \frac{\eta(\beta)}{a''}\right)\mathrm{grad}(\phi) \bmod 1\,, \qquad (7.12$$

for all $\beta \in \mathrm{Out}\,(\Phi_0)$, $\phi \in \Phi$ and $k \in \mathbb{Z}$.

Proof. Clearly every $\Theta \in \widehat{\mathrm{Grad}(\Phi)} = \hat{\mathbb{Z}}_a$ is determined by some number $h_\Theta \in \mathbb{Z}_a$, so th

$$\Theta(\mathrm{grad}(\phi)) = \frac{h_\Theta}{a}\,\mathrm{grad}(\phi) \bmod 1\,.$$

A character Θ is in $\ker i^*$ iff it annihilates $a''\mathbb{Z}_r \subset \mathbb{Z}_a$, i.e., iff h_Θ is a multiple of r. Hence

$$\ker i^* = \left\{\Theta : \exists \eta_\Theta \in \mathbb{Z}_{a''} : \Theta(\mathrm{grad}(\phi)) = \frac{\eta_\Theta}{a''}\,\mathrm{grad}(\phi) \bmod 1\right\} .$$

The homomorphism $\Theta' : \mathrm{Out}\,(\Phi_0) \rightarrow \ker i^*$ from Lemma 7.4.1 iii), is then determined by the homomorphism $\eta : \mathrm{Out}\,(\Phi_0) \rightarrow \mathbb{Z}_{a''}$, with $\Theta'_\beta(1) = \frac{\eta(\beta)}{a''}$. Furthermore, by Lemma 7.4.1 ii), $h : \mathrm{Out}\,(\Phi_0) \rightarrow \mathbb{Z}_a$ with $\Theta'_\sigma(1) = \frac{h_\sigma}{a}$ is a homomorphism. Therefore, for some fixed $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\mathrm{grad}\,\sigma_1 = a''$,

$$\Theta'_{\sigma_1^k \circ \beta}(1) = k\Theta'_{\sigma_1}(1) + \Theta'_\beta(1) = k\,\frac{h_{\sigma_1}}{a} + \frac{\eta(\beta)}{a''} .$$

So far this is the general form of a character on $\mathbb{Z} \times \mathrm{Out}\,(\Phi_0)$. However, in order to be a character on $\mathrm{Out}(\Phi)$, we have to make sure that it vanishes on the kernel of the projection $\sigma_1^k \times \beta \rightarrow \sigma_1^k \circ \beta$, which is generated by $\sigma_1^r \times \sigma_1^{-r}$. The latter yields the condition $h_{\sigma_1} = \eta\,(\sigma_1^r) \bmod 1$. Together with Lemma 7.4.1 i) we obtain the assertion for $\Theta_{\sigma_1^k \circ \beta}(\phi)$ from the formula for $\Theta'_{\sigma_1^k \circ \beta}(1)$. □

It is clear that the above result gives an exhaustive description of the homomorphisms, $\sigma \rightarrow \Theta_\sigma$, since $\mathbb{Z} \times \mathrm{Out}\,(\Phi_0) \rightarrow \mathrm{Out}(\Phi) : k \times \beta \rightarrow \sigma_1^k \circ \beta$ is surjective for any $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\mathrm{grad}\,(\sigma_1) = a''$. The choice of h_{σ_1} depends on σ_1 as

$$h_{\sigma_1 \circ \beta} - h_{\sigma_1} = r\eta(\beta) \bmod a . \tag{7.124}$$

In the case where $\sigma_1 = (1,1)$ is the canonical automorphism of the presentation $\Phi = \tau_\alpha\,(\mathbb{Z}_r * \Phi'')$, with $\mathrm{Grad}(\Phi'') \cong \mathbb{Z}_{a''}$, then h_{σ_1} is constrained by $h_{\sigma_1} \equiv \eta(\alpha) \bmod a''$, as $\sigma_1^r = a$. The relevance of Lemma 7.4.2 can be understood if we rewrite equation (7.123) in terms of the spins:

$$\theta_\phi - \theta_{\sigma_1^k \circ \beta \circ \phi} = \left(k\,\frac{h_{\sigma_1}}{a} + \frac{\eta(\beta)}{a''}\right)\,\mathrm{grad}(\phi) - \theta_{\sigma_1^k \circ \beta} \bmod 1 . \tag{7.125}$$

Suppose we know the spins of elements in $\mathrm{Out}(\Phi)$. Then (7.125) gives the change of the spin-value of an arbitrary representation ϕ under the multiplicative action of $\mathrm{Out}(\Phi)$ on Φ. The determination of the values θ_σ, $\sigma \in \mathrm{Out}(\Phi)$ is the content of the next result.

Lemma 7.4.3 *Suppose Φ is a $\mathbb{Z}_a$-graded fusion rule algebra of a C^*-quantum category, and let a'' and r be as above. Then there are homomorphisms*

$$\begin{array}{rrcl} & \eta : \mathrm{Out}\,(\Phi_0) & \rightarrow & \mathbb{Z}_{a''} \\ \textit{and} & \delta : \mathrm{Out}\,(\Phi_0) & \rightarrow & \mathbb{Z}_2 , \end{array}$$

and, further, for any fixed $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\mathrm{grad}\,(\sigma_1) = a''$, constants

$$h_{\sigma_1} \in \mathbb{Z}_{2a} \qquad \textit{and} \qquad \varepsilon_{\sigma_1} \in \mathbb{Z}_2 ,$$

constrained by

$$\begin{array}{rrcll} & \eta\,(\sigma_1^r) & = & h_{\sigma_1} & \bmod a'' \\ \textit{and} & \delta\,(\sigma_1^r) & = & r\,(\varepsilon_{\sigma_1} + h_{\sigma_1}) & \bmod 2 , \end{array} \tag{7.126}$$

such that

$$\theta_{\sigma_1^k \circ \beta} = \frac{\delta(\beta)}{2} - \frac{k^2}{2r}\,(h_{\sigma_1} + r\varepsilon_{\sigma_1}) \bmod 1 , \tag{7.127}$$

for all $\beta \in \mathrm{Out}\,(\Phi_0)$ and $k \in \mathbb{Z}$, and equation (7.125) holds for any $\phi \in \Phi$.

Proof. If we insert $\phi = \sigma_1^{k'} \circ \beta'$ into (7.125) and use $\mathrm{grad}\left(\sigma_1^{k'} \circ \beta'\right) = a'' \cdot k'$ we obtain that

$$\theta_{\sigma_1^k \circ \beta} + \theta_{\sigma_1^{k'} \circ \beta'} = \theta_{\sigma_1^{k+k'} \circ \beta \circ \beta'} + \frac{h_{\sigma_1}}{r}\,k \cdot k' \bmod 1 . \tag{7.128}$$

In particular, we find, for $k = k' = 0$, that $\mathrm{Out}\,(\Phi_0) \rightarrow \mathbb{R}/\mathbb{Z} : \beta \rightarrow \theta_\beta$ is a homomorphism. Since for spins we have $\theta_\beta = \theta_{\bar{\beta}} = \theta_{\beta^{-1}} = -\theta_\beta \bmod 1$ the range of this map is in $\frac{1}{2}\mathbb{Z}/\mathbb{Z}$, i.e. $2\theta_\beta = 0 \bmod 1$, $\forall \beta \in \mathrm{Out}\,(\Phi_0)$. The spins on $\mathrm{Out}\,(\Phi_0)$ are therefore given by

$$\theta_\beta = \frac{1}{2}\,\delta(\beta) \bmod 1 \tag{7.129}$$

where $\delta : \mathrm{Out}\,(\Phi_0) \rightarrow \mathbb{Z}_2$ is a homomorphism. Setting $\beta' = 1$ and $k = 0$ we obtain the decomposition $\theta_{\sigma_1^k \circ \beta} = p_k + \frac{1}{2}\delta(\beta)$. The numbers $p_k \in \mathbb{R}/\mathbb{Z}$, $k \in \mathbb{Z}$, are defined by $p_k := \theta_{\sigma_1^k}$ and satisfy $p_0 = 0$, $p_k = p_{-k}$ and, by (7.128), $p_k + p_{k'} = p_{k+k'} + \frac{h_{\sigma_1}}{r}\,kk' \bmod 1$. The most general solution of these equations is given by $p_k = -\frac{qk^2}{2r}$, where $q \in \mathbb{Z}_{2r}$ obeys $q = h_{\sigma_1} \bmod r$. The latter constraint is solved if we pick some $h_{\sigma_1} \in \mathbb{Z}_{2a}$ such that its image under the projection $\mathbb{Z}_{2a} \rightarrow \mathbb{Z}_{2a}/\,a\mathbb{Z}_2 = \mathbb{Z}_a$ is the original h_{σ_1}, and set

$$q = h_{\sigma_1} + r \cdot \varepsilon_{\sigma_1} \bmod (2r)$$

with $\varepsilon_{\sigma_1} \in \mathbb{Z}_2$. (If a'' is even this is also well defined for the original $h_{\sigma_1} \in \mathbb{Z}_a$). Finally we have to make sure that $\theta_{\sigma_1^k \circ \beta}$ as given in terms of the above decomposition, is well defined, i.e., we have to impose the condition $\frac{1}{2}\delta(\sigma_1^r) = p_r = -\frac{rq}{2} = -\frac{r}{2}(h_{\sigma_1} + r\varepsilon_{\sigma_1}) \bmod 1$, which is just condition (7.126). This, together with Lemma 7.4.2, proves the claim of Lemma 7.4.3. □

For convenience and later applications we give a more detailed description in special cases:

Corollary 7.4.4 *Let Φ be as in Lemma 7.4.2*

i) If $a'' = 1$ then there is some $h \in \mathbb{Z}_{2a}$ such that for

$$\theta^\circ_\phi := \theta_\phi + \frac{h}{2a}\,\mathrm{grad}(\phi)(a + \mathrm{grad}(\phi))\,,$$

we have

$$\theta^\circ_\phi + \theta^\circ_\sigma = \theta^\circ_{\sigma\circ\phi}\,, \qquad \forall \sigma \in \mathrm{Out}(\Phi)\,, \quad \phi \in \Phi\,. \tag{7.130}$$

In particular, $\sigma \to \theta^\circ_\sigma$ is a homomorphism of $\mathrm{Out}(\Phi)$ to $\mathbb{Z}_2$ whose kernel contains all α^2, $\alpha \in \mathrm{Out}(\Phi)$, and $\mathrm{stab}(\phi)$ for any $\phi \in \Phi$. If it also contains $\mathrm{Out}(\Phi_0)$, i.e., $\theta^\circ_\sigma = 0$, $\forall \sigma \in \mathrm{Out}(\Phi_0)$, then h can be chosen such that θ°_σ vanishes for all invertible elements.

ii) If $a'' = 2$, and, for $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\mathrm{grad}(\sigma_1) = 2$, there is some $\rho \in \Phi$ with $\sigma_1 \circ \bar{\rho} = \rho$ and $\mathrm{grad}(\rho) = 1$, then there is some $h_{\sigma_1} \in \mathbb{Z}_{2r}$ and homomorphisms

$$\eta, \delta : \mathrm{Out}(\Phi) \to \mathbb{Z}_2$$

obeying

$$\eta(\sigma_1^r) = h_{\sigma_1} \bmod 2 \qquad \textit{and} \qquad \delta(\sigma_1^r) = r h_{\sigma_1} \bmod 2$$

such that

$$\theta_{\sigma_1^k \circ \beta} = \frac{\delta(\beta)}{2} - \frac{k^2 h_{\sigma_1}}{2r} \bmod 1 \tag{7.131}$$

and

$$\theta_\phi - \theta_{\sigma_1^k \circ \beta \circ \phi} = \frac{h_{\sigma_1}}{2r}\,k(k + \mathrm{grad}(\phi)) + \frac{\delta(\beta)}{2} + \frac{\eta(\beta)}{2}\,\mathrm{grad}(\phi) \bmod 1\,. \tag{7.132}$$

If h_{σ_1} is even, i.e., there is some $\gamma_{\sigma_1} \in \mathbb{Z}_r$ with $h_{\sigma_1} = 2\gamma_{\sigma_1}$, then

$$\theta^\circ_\phi := \theta_\phi + \frac{\gamma_{\sigma_1}}{4r}\,\mathrm{grad}(\phi)^2 \bmod 1$$

is well defined and there exists a homomorphism $\bar{\eta} : \mathrm{Out}(\Phi) \to \mathbb{Z}_2$ with $\bar{\eta}(\sigma_1) =$ such that

$$\theta^\circ_\sigma + \theta^\circ_\phi - \theta^\circ_{\sigma\circ\phi} = \frac{\bar{\eta}(\sigma)}{2}\,\mathrm{grad}(\phi) \qquad (7.1$$

and $\sigma \to \theta^\circ_\sigma$ is a homomorphism $\mathrm{Out}(\Phi) \to \frac{1}{2}\mathbb{Z}/\mathbb{Z}$, with $\theta^\circ_{\sigma_1} = 0$. In particular, have

$$\theta_{\sigma_1^r \circ \phi} = \theta_\phi\,, \qquad \textit{for all} \quad \phi \in \Phi\,. \qquad (7.1$$

If h_{σ_1} is odd, then we have $\sigma_1^r \neq 1$. If $\mathrm{Out}(\Phi_0) = \{1, \sigma_1^r\}$, with $\sigma_1^r \neq 1$, and $\mathrm{Out}(\Phi) \to \mathbb{Z}_{2r}$ is the canonical isomorphism, with $\pi(\sigma_1) = 1$, then

$$\theta_\phi - \theta_{\sigma\circ\phi} = \frac{h_{\sigma_1}}{2r}\,(\pi(\sigma) + \mathrm{grad}(\phi))\,, \qquad (7.13$$

for any $\sigma \in \mathrm{Out}(\Phi)$ and $\phi \in \Phi$.

Thus for odd h_{σ_1}, $\sigma_1^r \in \mathrm{stab}(\phi)$ implies $\mathrm{grad}(\phi) \equiv r \bmod 2$.

Proof.

i) Clearly, for $a'' = 1$, η does not appear in the formula and $h = h_{\sigma_1}$ is independe of σ_1. The equation (7.130) then follows immediately from Lemma 7.4.3 and impl the remaining remarks in i).

ii) From (7.125) and Lemma 7.4.3 we obtain, for the case $a'' = 2$ and $\sigma_1 \circ \bar{\rho} = \rho$ wi $\mathrm{grad}(\rho) = 1$, that

$$\begin{aligned} 0 &= \theta_{\bar{\rho}} - \theta_\rho = \theta_{\bar{\rho}} - \theta_{\sigma_1\circ\bar{\rho}} = \frac{h_{\sigma_1}}{a}\,\mathrm{grad}(\bar{\rho}) - \theta_{\sigma_1} = \\ &= -\frac{h_{\sigma_1}}{a} + \frac{1}{2r}\,(h_{\sigma_1} + r\varepsilon_{\sigma_1}) = \frac{\varepsilon_{\sigma_1}}{2}\,, \quad \text{so} \quad \varepsilon_{\sigma_1} = 0\,. \end{aligned}$$

The first part of Corollary 7.4.4 ii) is obtained simply by specializing Lemma 7. to $a'' = 0$ and inserting $\varepsilon_{\sigma_1} = 0$. The following statements are again immedi consequences of (7.132).

In the proof of Proposition 7.3.1 the unitary representations of the braid groups and the Temperley-Lieb algebra, that arise in local quantum field theories have been considered. It follows by straightforward translation that all of the statements made there also hold for a general C^*-quantum category. In particular, we have Temperley-Lieb projectors $e_\sigma(\rho,\rho)$, for any invertible $\sigma \in \rho \circ \rho$, which satisfy the generalized Temperley-Lieb equations (7.57) and (7.58), and, for a two-channel-situation $\rho \circ \rho = \sigma \oplus \psi$, the decomposition (7.67) of the statistics operator.

The restriction of the possible values of q_ρ to $q_\rho = e^{\pm \frac{2\pi i}{N}}$, where N is the Coxeter number of the inclusion graph of the tower discussed in Section 3.4, evidently has to imply certain restrictions on the possible values of spins. These are given in the next lemma. Here we also distinguish the situations corresponding to the two signs in $\Gamma^*_{\rho\circ\rho,1}\rho(\Gamma_{\rho\circ\rho,1}) = \pm\frac{1}{d_\rho}$, $d_\rho > 0$, for ρ selfconjugate. If the positive sign holds ρ will be called real, for negative sign ρ is called pseudoreal.

Lemma 7.4.5 *Suppose that for an object $\rho \in \Phi$ of a C^*-quantum category*

$$\rho \circ \rho = \sigma_1 + \psi\,,$$

with $\sigma_1 \in \mathrm{Out}(\Phi)$ and ψ irreducible, and assume, further, that its monodromy, $m(\rho,\rho) = \varepsilon(\rho,\rho)^2$, is not a multiple of the identity. Let a be given by $\mathbb{Z}_a \cong \mathrm{Grad}\left(\Phi_{[\rho]}\right)$, so $a'' = 1$ or 2. Then there exists some $t \in \mathbb{Z}_{4a}$, some $N \in \mathbb{N}$ and a sign such that

$$\begin{array}{rcl} \pm\,\theta_{\sigma_1} & = & \dfrac{t}{a} \bmod 1 \\ \textit{and} & & \\ \pm\theta_\rho & = & \dfrac{3}{4N} + \dfrac{t}{4a} \bmod 1\,. \end{array} \tag{7.136}$$

Here N is the Coxeter number of the inclusion $\mathbb{N}_\rho$, i.e., $\|\mathbb{N}_\rho\| = |d_\rho| = 2\cos\left(\frac{\pi}{N}\right)$, and, comparing with Corollary 7.4.4 we have

$$\begin{array}{rclll} t & = & \pm 2h & \mathrm{mod}\, a\,, & \textit{for} \quad a'' = 1\,, \\ t & = & \pm h_{\sigma_1} & \mathrm{mod}\, a\,, & \textit{for} \quad a'' = 2\,. \end{array} \tag{7.137}$$

For the representation $\psi' := \bar\sigma_1 \circ \psi$, with $\psi' \in \rho \circ \bar\rho$, we have

$$\pm\,\theta_{\psi'} = \frac{2}{N} \bmod 1\,, \tag{7.138}$$

independent of t. In the selfconjugate case, i.e., if $\sigma_1 = 1$, we find $t = \delta a$, with $\delta \in \mathbb{Z}_4$ and

$$\begin{array}{ll} \delta \textit{ even}\,, & \textit{for } \rho \textit{ pseudoreal} \\ \delta \textit{ odd}\,, & \textit{for } \rho \textit{ real.} \end{array} \tag{7.139}$$

Proof. With the decomposition (7.67) of the statistics operator, we can compute the statistical parameter:

$$\begin{array}{rcl} \lambda_\rho 1 & := & \rho\left(\Gamma^*_{\rho\circ\bar\rho,1}\right)\,\varepsilon^+(\rho,\rho)\,\rho\left(\Gamma_{\rho\circ\bar\rho,1}\right) \\ & = & z_\rho\left[(q_\rho+1)\;\rho\left(\Gamma^*_{\rho\circ\bar\rho,1}\right)\;e_\sigma(\rho,\rho)\;\rho\left(\Gamma_{\rho\circ\bar\rho,1}\right) - 1\right] \\ & = & z_\rho\left[\dfrac{q_\rho+1}{\beta_\rho} - 1\right] \end{array}$$

by the generalized Temperley-Lieb equation (7.57). Using (7.68) we obtain

$$\lambda_\rho = -\frac{z_\rho}{1+q_\rho} \tag{7.140}$$

as in the self-conjugate case of [23]. Comparing with the expression in [15]

$$\lambda_\rho^2 = \frac{1}{d_\rho^2}\,e^{-4\pi i\theta_\rho} = \beta_\rho^{-1}\,e^{-4\pi i\theta_\rho}$$

we find

$$e^{4\pi i\theta_\rho} = q_\rho\, z_\rho^{-2}\,.$$

Further, the monodromy $m(\rho,\rho)$ satisfies

$$m(\rho,\rho)\,\Gamma_{\rho\circ\rho,\sigma} = z_\rho^2\,q_\rho^2\,\Gamma_{\rho\circ\rho,\sigma}\,,$$

which has to coincide with a similar equation, where the eigenvalue is expressed in terms of spins, i.e.,

$$z_\rho^2\,q_\rho^2 = e^{2\pi i\left(2\theta_\rho - \theta_{\sigma_1}\right)}\,.$$

Combining these equations we have

$$e^{2\pi i\left(4\theta_\rho - \theta_{\sigma_1}\right)} = q_\rho^3\,. \tag{7.141}$$

With $q_\rho = e^{\pm\frac{2\pi i}{N}}$ equations (7.136) follow from (7.141). In terms of t and N we also find

$$z_\rho^2 = e^{\pm i\pi\left(\frac{1}{N} + \frac{t}{a}\right)}\,. \tag{7.142}$$

In the selfconjugate case we can use the polynomial equation

$$\rho\left(\Gamma^*_{\rho\circ\rho,1}\right)\ \varepsilon^+(\rho,\rho)\,\rho\left(\varepsilon^+(\rho,\rho)\right)=\Gamma^*_{\rho\circ\rho,1}$$

to obtain

$$\lambda_\rho=\Gamma^*_{\rho\circ\rho,1}\,\rho\left(\varepsilon^-(\rho,\rho)\,\Gamma_{\rho\circ\rho,1}\right)=\varepsilon\,\frac{1}{d_\rho}\,z_\rho^{-1}\,q_\rho^{-1} \tag{7.143}$$

with $\varepsilon=1$ for real ρ and $\varepsilon=-1$ for pseudoreal ρ, and $d_\rho=2\cos\frac{\pi}{N}$. Together with (7.140) this yields

$$z_\rho^2=-\varepsilon\,e^{\pm\frac{i\pi}{N}}\,.$$

The statement on reality and pseudoreality of ρ now follows by comparison with (7.142). For θ°_ρ, defined in Corollary 7.4.4, we have $0=\theta^\circ_\rho-\theta^\circ_{\bar\rho}=\theta_{\sigma_1\circ\bar\rho}-\theta^\circ_{\bar\rho}=\theta^\circ_{\sigma_1}$ ·, hence $\theta_{\sigma_1}=-\frac{2h}{a}$ mod 1. Equations (7.137) are thus found by inserting the expressions of Corollary 7.4.4. It follows from (7.67) that q_ρ^2 is the ratio of the eigenvalues of the monodromy $m(\rho,\rho)$. In terms of the spins, this ratio is expressed as $e^{2\pi i\left(\theta_\psi-\theta_{\sigma_1}\right)}=e^{2\pi i\theta_{\psi'}}\Theta_{\sigma_1}(\psi')=e^{2\pi i\theta_{\psi'}}$, since ψ' is trivially graded. Thus equation (7.138) follows from a comparison of these phases. □

The special situation in which the generating object ρ has a two-channel decomposition, $\rho\circ\rho=\sigma_1+\psi$, allows us to determine the spin for each object by an inductive procedure. Although the following arguments and computations apply to the general framework of a quantum category with arbitrary, compatible fusion rules, they are closely related to the analysis of exchange algebras in conformal field theories presented in [55]. First, we shall give a formula relating the matrices $R^+(k,p,q,\ell)$ and $R^-(k,p,q,\ell)$ which is derived in [15] for general local quantum field theories, using the spatial rotation group in $\mathbb{M}^3$ and the actual spins. However, the proof given below only uses elementary identities of the categories under consideration, so that only statistical phases appear in the statement.

Lemma 7.4.6 *For any C^*-quantum category let the unitary maps*

$$R^\pm(k,p,q,\ell):\sum_i\mathbb{C}^{N_{kp,i}}\otimes\mathbb{C}^{N_{iq,\ell}}\to\sum_j\mathbb{C}^{N_{kq,j}}\otimes\mathbb{C}^{N_{jp,\ell}}$$

with $R^-(k,q,p,\ell)=\left(R^+(k,p,q,\ell)\right)^{-1}$ be defined as usual. Then the following equation for the matrix elements holds

$$R^+(k,p,q,\ell)^{j\nu'\mu'}_{i\nu\mu}=e^{2\pi i(\theta_i+\theta_j-\theta_\ell-\theta_k)}\,R^-(k,p,q,\ell)^{j\nu'\mu'}_{i\nu\mu}\,, \tag{7.144}$$

for any orthonormal basis of arrows or intertwiners.

Proof. Fixing an orthonormal basis, $\Gamma_{k\circ q,i}(\nu)$, $\nu=1,\ldots,N_{kp,i}$, we consider the composition

$$I:=\varepsilon^+(q,k)\,\varepsilon^+(k,q)\,k\left(\varepsilon^+(p,q)\right)\ \Gamma_{k\circ p,i}(\nu)\,\Gamma_{i\circ q,\ell}(\mu)\,.$$

The definition of the R-matrices yields

$$\begin{aligned} I &= \sum_{j\nu'\mu'}R^+(k,p,q,\ell)^{j\nu'\mu'}_{i\nu\mu}\ \varepsilon^+(q,k)\,\varepsilon^+(k,q)\,\Gamma_{k\circ q,j}(\nu')\,\Gamma_{j\circ p,\ell}(\mu')\\ &= \sum_{j\nu'\mu'}e^{2\pi i(\theta_q+\theta_k-\theta_j)}\,R^+(k,p,q,\ell)^{j\nu'\mu'}_{i\nu\mu}\ \Gamma_{k\circ q,j}(\nu')\,\Gamma_{j\circ p,\ell}(\mu'). \end{aligned}$$

using the fact that the $\Gamma_{k\circ q,j}$'s diagonalize the monodromy $m(k,q)$. Alternatively, w evaluate I using the polynomial identity for $\Gamma_{k\circ p,i}(\nu)$:

$$\begin{aligned} I &= \varepsilon^+(q,k)\,q\left(\Gamma_{k\circ p,i}(\nu)\right)\ \varepsilon^+(i,q)\,\Gamma_{i\circ q,\ell}(\mu)\\ &= e^{2\pi i(\theta_i+\theta_q-\theta_\ell)}\,\varepsilon^+(q,k)\,q\left(\Gamma_{k\circ p,i}(\nu)\right)\ \varepsilon^-(i,q)\,\Gamma_{i\circ q,\ell}(\mu)\\ &= e^{2\pi i(\theta_i+\theta_q-\theta_\ell)}\,\varepsilon^+(q,k)\,\varepsilon^-(k,q)\,k\left(\varepsilon^-(p,q)\right)\ \Gamma_{k\circ p,i}(\nu)\,\Gamma_{i\circ q,\ell}(\mu)\\ &= e^{2\pi i(\theta_i+\theta_q-\theta_\ell)}\ \sum_{j\nu'\mu'}R^-(k,p,q,\ell)^{j\nu'\mu'}_{i\nu\mu}\ \Gamma_{k\circ q,j}(\nu')\,\Gamma_{i\circ p,\ell}(\mu')\,, \end{aligned}$$

where $\varepsilon^-(p,q)=\left(\varepsilon^+(q,p)\right)^{-1}$. The identity (7.144) is now obtained by comparing th coefficients of the two expressions given for I.

Note that (7.144) is not a proportionality relation among R-matrices, but it is a relatio of R-matrices and diagonal maps on the path space, similar to the ones used in (7.60 In special cases, however, where we can show that the R-matrix is in some sense bloc diagonal, (7.144) implies strong restrictions on the values of spins and the possible for of the F-matrix isomorphisms. The precise statement is given in the next corollary.

Corollary 7.4.7 *Suppose we have irreducible objects k,ℓ,ρ, so that the statistics operat is block-diagonal on* $\mathrm{Int}(k\circ\rho\circ\rho,\ell)$, *in the sense that*

$$R^\pm(k,\rho,\rho,\ell)\in\sum_j^\oplus\mathrm{End}\left(\mathbb{C}^{N_{k\rho,j}}\otimes\mathbb{C}^{N_{j\rho,\ell}}\right)\subset\mathrm{End}\left(\sum_j^\oplus\mathbb{C}^{N_{k\rho,j}}\otimes\mathbb{C}^{N_{j\rho,\ell}}\right)$$

(or, more specifically, $R^{\pm}(k,\rho,\rho,\ell)^{j,\nu\mu}_{i,\nu'\mu'} = 0$, for $i \neq j$). Then for any $\xi \in \operatorname{supp}(\rho \circ \rho) \cap \operatorname{supp}(\ell \circ \bar{k})$ the spins obey

$$\theta_\ell + \theta_k - 2\theta_j = \theta_\xi - 2\theta_\rho \bmod 1\,, \tag{7.145}$$

whenever the corresponding block of the F-matrix

$$\hat{F}(k,\rho,\rho,\ell)^{\xi,\dots}_{j,\dots} : \mathbb{C}^{N_{k\rho,j}} \otimes \mathbb{C}^{N_{j\rho,\ell}} \to \mathbb{C}^{N_{k\xi,\ell}} \otimes \mathbb{C}^{N_{\rho\rho,\xi}}$$

is non zero.

If, for irreducible objects, k, ℓ and ρ, we have that there is a single object, j, with

$$\operatorname{supp}(\rho \circ k) \cap \operatorname{supp}(\bar{\rho} \circ \ell) = \{j\}$$

then the equation (7.145) holds, without any assumption on the R- and F-matrices, and θ_ξ is independent of ξ for all $\xi \in \operatorname{supp}(\rho \circ \rho) \cap \operatorname{supp}(\ell \circ \bar{k})$.

Proof. Assume $R^{\pm}(k,\rho,\rho,\ell)$ has the proposed form and consider the block-matrices $R^{+}(k,\rho,\rho,\ell)^{j}_{j} \in U\left(\mathbb{C}^{N_{k\rho,j}} \otimes \mathbb{C}^{N_{j\rho,\ell}}\right)$, with $R^{-}(k,\rho,\rho,\ell)^{j}_{j} = \left(R^{+}(k,\rho,\rho,\ell)^{j}_{j}\right)^{-1}$. If we specialize (7.144) to $p = q = \rho$ and $i = j$ we find the equation

$$R^{+}(k,\rho,\rho,\ell)^{j}_{j} = e^{2\pi i(2\theta_j - \theta_\ell - \theta_k)}\, R^{-}(k,\rho,\rho,\ell)^{j}_{j}$$

thus

$$M(k,\rho,\rho,\ell)^{j}_{i} = \underset{ji}{\delta}\; e^{2\pi i(2\theta_j - \theta_\ell - \theta_k)}\, \mathbf{1}_{\mathbb{C}^{N_{k\rho,j}\, N_{j\rho,\ell}}}\,. \tag{7.146}$$

As remarked earlier, the isomorphism $\hat{F}(k,\rho,\rho,\ell) : \sum_j^{\oplus} \mathbb{C}^{N_{k\rho,j}} \otimes \mathbb{C}^{N_{j\rho,\ell}} \to \sum_\xi^{\oplus} \mathbb{C}^{N_{k\xi,\ell}} \otimes \mathbb{C}^{N_{\rho\rho,\xi}}$, diagonalizes the monodromy, in the sense that, for

$$\tilde{M}(k,\rho,\rho,\ell) = \hat{F}(k,\rho,\rho,\ell)\, M(k,\rho,\rho,\ell)\, \hat{F}(k,\rho,\rho,\ell)^{-1} \in \operatorname{End}\left(\sum_\xi^{\oplus} \mathbb{C}^{N_{k\xi,\ell}} \otimes \mathbb{C}^{N_{\rho\rho,\xi}}\right),$$

we have

$$\tilde{M}(k,\rho,\rho,\ell)^{\xi}_{\xi'} = \delta_{\xi\xi'}\, e^{2\pi i\left(2\theta_\rho - \theta_\xi\right)}\, \mathbf{1}_{\mathbb{C}^{N_{k\xi,\ell}\, N_{\rho\rho,\xi}}}\,. \tag{7.147}$$

It follows, that (7.146) is equivalent to

$$\hat{F}(k,\rho,\rho,\ell)^{\xi}_{j}\, e^{2\pi i\left(2\theta_\rho - \theta_\xi\right)} = \hat{F}(k,\rho,\rho,\ell)^{\xi}_{j}\, e^{2\pi i(2\theta_j - \theta_\rho - \theta_k)}\,,$$

for all $\xi \in \operatorname{supp}(\rho \circ \rho) \cap \operatorname{supp}(\ell \circ \bar{k})$ and $j \in \operatorname{supp}(k \circ \rho) \cap \operatorname{supp}(\bar{\rho} \circ \ell)$, which implies the assertion.

If $\operatorname{supp}(k \circ \rho) \cap \operatorname{supp}(\bar{\rho} \circ \ell)$ consists only of one object, j, then the prerequisite on the block-form of the R-matrix is void. Moreover, in this case the F-matrix provides an isomorphism of $\mathbb{C}^{N_{k\ell,j}} \otimes \mathbb{C}^{N_{j\rho,\ell}} \cong \sum^{\oplus} \mathbb{C}^{N_{k\xi,\ell}} \otimes \mathbb{C}^{N_{\rho\rho,\xi}}$, so none of the different blocks can be zero, if $\xi \in \operatorname{supp}(\rho \circ \rho) \cap \operatorname{supp}(\ell \circ \bar{k})$. Hence equation (7.145) holds without further assumptions. □

If $d_\rho < 2$ it is possible to find situations in which Corollary 7.4.7 is applicable:

Corollary 7.4.8 *Suppose for ρ and ψ irreducible and $\sigma_1 \in \mathrm{Out}(\Phi)$ we have $\rho \circ \rho = \sigma_1 + \psi$ and let the spins be given by the expressions in Lemma 7.4.5.*

i) If, for irreducible $k, \ell \in \Phi$,

$$\ell \in \rho \circ \rho \circ k \quad \textit{and} \quad \ell \neq \sigma_1 \circ k$$

then

$$\theta_k + \theta_\ell - 2\theta_j = \pm\left(\frac{1}{2N} + \frac{t}{2a}\right) \bmod 1 \tag{7.148}$$

holds for all $j \in \operatorname{supp}(k \circ \rho) \cap \operatorname{supp}(\ell \circ \bar{\rho})$.

ii) If for an irreducible object $k \in \Phi$ also $j := k \circ \rho$ is irreducible then their spins are related by

$$\pm\, 2(\theta_j - \theta_k) = \frac{t}{2a}\,(1 + 2\,\mathrm{grad}(k)) + \frac{3}{2N} \bmod 1\,. \tag{7.149}$$

Proof.

i) In the two-channel case, the $\hat{F}$-matrix diagonalizes $R^{\pm}(k,\rho,\rho,\ell)$ in the same way it diagonalizes the monodromy, using the fact that the multiplicities in the decomposition are at most one. If, in addition, we choose k and ℓ such that $\ell \neq \sigma_1 \circ k$ we have an isomorphism

$$\hat{F}(k,\rho,\rho,\ell) : \sum_j^{\oplus} \mathbb{C}^{N_{k\rho,j}} \otimes \mathbb{C}^{N_{j\rho,\ell}} \to \mathbb{C}^{N_{k\psi,\ell}}$$

as $N_{k\sigma_1,\ell} = 0$ and $N_{\rho\rho,\psi} = 1$.

Clearly the action of the statistics operator $\varepsilon^+(\rho,\rho)$ on the intertwiner space $\cong \mathbb{C}^{N_{k\psi,\ell}} \otimes \mathbb{C}^{N_{\rho\rho,\psi}}$ is given by $\tilde{R}(k,\rho,\rho,\ell) = -z_\rho \mathbf{1}$, so that also $R^\pm(k,\rho,\rho,\ell)$ is a multiple of identity and, in particular, block-diagonal. Furthermore, since $\hat{F}$ are isomorphisms, $\hat{F}((k,\rho,\rho,\ell)^\psi_j \neq 0$, for all $j \in \mathrm{supp}(k \circ \rho) \cap \mathrm{supp}(\ell \circ \bar{\rho})$, and thus, by Corollary 7.4.7, $\theta_\ell + \theta_k - 2\theta_j = \theta_\psi - 2\theta_\rho = \theta_{\sigma_1} + \theta_{\psi'} - 2\theta_\rho$ mod 1. Inserting here the expressions from Lemma 7.4.5 gives (7.148).

ii) The final statement of Corollary 7.4.7 applies to this situation if we set $\ell := \sigma_1 \circ k$, so that $\bar{\rho} \circ \ell = \bar{\rho} \circ \sigma_1 \circ k = \rho \circ k = j$. Clearly $\sigma_1 \in \ell \circ \bar{k} = \sigma_1 \circ k \circ \bar{k}$, so that (7.145) holds for $\xi = \sigma_1$ and can be written as

$$2\,(\theta_j - \theta_k) = 2\theta_\rho - \Theta_{\sigma_1}(k)\,,$$

where Θ_{σ_1} is the gradation given in (7.123). From $\Theta_{\sigma_1}(\bar{\rho}) = \theta_{\sigma_1}$ we find that $\Theta_{\sigma_1}(k) = \mp\frac{t}{a}\mathrm{grad}(k)$, and (7.149) is obtained from the values given in Lemma 7.4.5. □

The relation (7.148) among the spin values can be used as a recursion formula for the spins of certain sequences of objects. For any maximal sequence of this type we then find from (7.149) that its length has to be a multiple of the Coxeter number N. This observation excludes most of the exceptional fusion rule algebras. The solution to the recursion and the precise termination-condition are given in the next lemma:

Lemma 7.4.9 *Assume $\rho, \psi \in \Phi$ are irreducible and $\sigma_1 \in \mathrm{Out}(\Phi)$ with $\rho \circ \rho = \sigma_1 + \psi$. Let ξ_j, $j = 1, \ldots, L$, be a sequence of objects satisfying*

$$\begin{aligned} &\xi_1 = 1\,, \qquad \xi_2 = \rho \\ \textit{and} \quad &\xi_{2j} \in \rho \circ \xi_{2j-1}\,, \quad \xi_{2j+1} \in \bar{\rho} \circ \xi_{2j}\,, \end{aligned} \tag{7.150}$$

such that

$$\xi_{j-1} \neq \xi_{j+1} \qquad \textit{for all } j = 1, \ldots, L\,. \tag{7.151}$$

i) If t and N are as in Lemma 7.4.5, then the spins are given by

$$\pm\,\theta_{\xi_j} = \frac{j^2 - 1}{4N} + \frac{t}{4a}\,\mathrm{grad}\,(\xi_j) \;\mathrm{mod}\; 1\,. \qquad (7.15$$

Here we have that $\mathrm{grad}\,(\xi_j) = 0$, for j odd, and $\mathrm{grad}\,(\xi_j) = 1$, for j even.

ii) Suppose the sequence cannot be continued after L steps, i.e.,

$$\begin{aligned} \bar{\rho} \circ \xi_L &= \xi_{L-1} \qquad \textit{if } L \textit{ is even} \\ \rho \circ \xi_L &= \xi_{L-1} \qquad \textit{if } L \textit{ is odd.} \end{aligned}$$

Then $L + 1$ is a multiple of the Coxeter number N.

Proof.

i) To compute the spins of the sequence ξ_j it is convenient to use another sequence, of objects given by $\gamma_{2(j+1)} := \sigma_1^j \circ \xi_{2(j+1)}$ and $\gamma_{2j+1} := \sigma_1^j \circ \xi_{2j+1}$, $j = 1, \ldots, L$. F these we have, with $\gamma_1 = 1$, the simpler recursion relations $\gamma_{j+1} \in \rho \circ \gamma_j$ and γ_{j+1} $\sigma_1 \circ \gamma_{j-1}$.

Equation (7.148) of Corollary 7.4.8 is now applicable to the triple $k = \gamma_{j-1}$, $j =$ $\ell = \gamma_{j+1}$, for any j, i.e., we have

$$\theta_{\gamma_{j+1}} + \theta_{\gamma_{j-1}} - 2\theta_{\gamma_j} = \pm\left(\frac{1}{2N} + \frac{t}{2a}\right) \mathrm{mod}\; 1\,.$$

With the initial data, $\pm\theta_{\gamma_1} = 0$ and $\pm\theta_{\gamma_2} = \frac{3}{4N} + \frac{t}{4a}$ mod 1 this can be easily int grated to

$$\pm\,\theta_{\gamma_j} = \frac{j^2 - 1}{4N} + \frac{(j-1)^2}{4a}\,t \quad \mathrm{mod}\; 1\,. \qquad (7.15$$

From Corollary 7.4.4 we see that, for any σ_1 with $\sigma_1 \circ \bar{\rho} = \rho$, where $\mathrm{grad}(\rho) = 1$, t following relation holds for any ϕ:

$$\theta_\phi = \theta_{\sigma_1^n \circ \phi} - \theta_{\sigma_1} n(\mathrm{grad}(\phi) + n)\,. \qquad (7.15$$

This allows us to compute the spins θ_{ξ_j} from the spins θ_{γ_j} given in (7.153). Inserti the value of θ_{σ_1} given in Lemma 7.4.5 we obtain equation (7.152). Finally (7.15

can also be used to find the spins of all compositions $\sigma_1^r \circ \xi_j$ which can be expressed as follows:

$$\pm\,\theta_{\sigma_1^n \circ \xi_j} = \frac{j^2-1}{4N} + \frac{t}{4a}\,(\mathrm{grad}\,(\xi_j) + 2n)^2\,. \tag{7.155}$$

Note that, by identification of the 1 in $\mathrm{grad}\,(\xi_j) \in \{0,1\}$ as the conventional generator of $\mathbb{Z}_{4a}$, the above equation is meaningful, however the squared term in (7.155) cannot be substituted by $\mathrm{grad}\,(\sigma_1^n \circ \xi_j) \in \mathbb{Z}_a$.

ii) It is again convenient to work with the sequence γ_j for which the termination condition is $\bar\rho \circ \gamma_L = \gamma_{L-1}$ or $\rho \circ \gamma_L = \sigma_1 \circ \gamma_{L-1}$. We can now use the formulae from the proof of part i) to compute

$$\begin{aligned} \pm 2\left(\theta_{\sigma_1 \circ \gamma_{L-1}} - \theta_{\gamma_L}\right) &= \pm 2\left(\theta_{\gamma_{L-1}} - \theta_{\gamma_L}\right) \pm 2\theta_{\sigma_1}(L-1) \\ &= -\frac{(L-1)}{N} + \frac{3}{2N} + \frac{t}{2a}\,(1 + 2\mathrm{grad}\,(\gamma_L)) \bmod 1 \end{aligned}$$

where $\mathrm{grad}\,(\gamma_L) = L-1$. If we compare this to (7.149) in Corollary 7.4.8, with $j = \sigma_1 \circ \gamma_{L-1}$ and $k = \gamma_L$, we find as a condition on L: $\frac{L+1}{N} = 0 \bmod 1$. This is just the assertion. □

Note that not all fusion rule algebras with a generator of dimension $d_\rho < 2$ have a two-channel decomposition to which the above analysis applies, namely those obtained from the D_4-algebra. For these, however, we have that $\rho \circ \rho$ and $\rho \circ \bar\rho$ decompose entirely into invertible objects, i.e., $\mathrm{supp}(\bar\rho \circ \rho) = \mathrm{stab}(\rho)$. In order to discuss the possibility of finding spins and eventually quantum categories for fusion rule algebras of this kind, we first elaborate on the observation, already made in the proof of Proposition 7.3.1 that the objects in $\mathrm{stab}(\rho) := \{\sigma : \sigma \circ \rho = \rho\} \subset \mathrm{Out}(\Phi)$ have half integer spin.

Lemma 7.4.10 *Let Φ be a $\mathbb{Z}_a$-graded fusion rule algebra of a C^*-quantum category, and a'', r as in Lemma 7.4.2.*

Then we find for any $\rho \in \Phi$, with $\mathrm{grad}(\rho) = 1$, that

$$\theta_\beta \equiv 0\,, \quad \forall \beta \in \mathrm{stab}(\rho)\,, \quad \text{if } a'' \text{ is odd}\,,$$

and, for a'' even, we have a homomorphism

$$\theta : \mathrm{stab}(\rho) \to \mathbb{Z}_2 : \beta \to \theta_\beta\,.$$

The gradation Θ_β for elements $\beta \in \mathrm{stab}(\rho)$ is given by

$$\Theta_\beta(\phi) = \theta_\beta\,\mathrm{grad}(\phi) \bmod L\,.$$

Proof. For any ρ and $\beta \in \mathrm{stab}(\rho)$, we clearly have that $\Theta_\beta(\rho) = \theta_\beta$. Since $\beta \to \Theta_\beta \in \widehat{\mathrm{Grad}}(\Phi)$ is a homomorphism, $\beta \to \theta_\beta$ is one, too. Since $\mathrm{stab}(\rho) = \mathrm{stab}(\bar\rho)$, we find from the gradation of Θ_β that $0 = \Theta_\beta(\rho) + \Theta_\beta(\bar\rho) = 2\theta_\beta \bmod 1$, so that $\theta_\beta \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. Assume now that ρ has $\mathrm{grad}(\rho) = 1$ in a $\mathbb{Z}_a$-graded fusion rule algebra. Then we find from Lemma 7.4.2 that $\Theta_\beta(\rho) = \frac{\eta(\beta)}{a''}$, and therefore $a''\theta_\beta = 0 \bmod 1$. This shows that $\theta_\beta \equiv 0$, for odd a''. The general form of Θ_β follows from the same lemma. □

The formulae and constraints obtained in the previous lemmas, especially in Lemma 7.4.9, allow us to discard from the list of fusion rule algebras in Theorem 3.4.11 those which are not realized as object algebras of a C^*-quantum category. Together with Proposition 7.3.5 we can summarize the results of Sections 7.3 and 7.4 in the following proposition.

Proposition 7.4.11 *Suppose ρ is an irreducible object of a C^*-quantum category. Then*

i) The statistical dimension of ρ obeys $d_\rho < 2$ if and only if we have that

$$\rho \circ \rho = \sigma \oplus \psi\,,$$

where σ is invertible and ψ irreducible, and, furthermore, $m(\rho,\rho) = \varepsilon(\rho,\rho)^2$ is non-scalar.

ii) If i) holds for ρ, and ρ generates the fusion rule algebra, Φ, of the C^*-quantum category (or if we restrict our consideration to the subcategory associated to the fusion rule subalgebra generated by ρ) then Φ and the statistical phases are restricted to the following possibilities:

a) Φ is a fusion rule subalgebra of some $A_n \times \mathbb{Z}_r$ (the crossed product being the same as in Lemma 3.3.3), namely (3.117) or (3.120) of Theorem 3.4.11. The *inclusion, i, of Φ is given as follows:*

$$\begin{aligned} \text{for}\quad \Phi &= \overline{A}_n * \mathbb{Z}_r\,, \qquad n \geq 1\,, \\ i:\Phi &\hookrightarrow A_{2n} \times \mathbb{Z}_r \end{aligned} \qquad (7.156)$$

is the inclusion (3.77) from Corollary 3.4.3, multiplied with the identity on the $\mathbb{Z}_r$-factor; for $\Phi = A_{2n-1} * \mathbb{Z}_r$, $n \geq 2$, we have

$$\begin{aligned} i:\Phi &\hookrightarrow A_{2n-1} \times \mathbb{Z}_{2r} \\ (\xi, k) &\rightarrow \xi \otimes \alpha^{\mathrm{grad}(\xi,k)}\,, \end{aligned} \qquad (7.157)$$

where $\xi \in A_{2n-1}$, $\mathrm{grad}(\xi) \in \{0,1\}$, and α generates $\mathbb{Z}_{2r}$.

b) The fusion rule algebra is given by either (3.121), with n odd, or (3.125), with n even, i.e.,

$$\Phi = \tau_\alpha\left(A_{2n-1} * \mathbb{Z}_r\right)\,,$$

with $n \geq 3$, and

$$r \equiv n + 1 \bmod 2\,. \qquad (7.158)$$

iii) Let ρ_j, $j = 0, ..., n-1$ denote the irreducible elements of the A_n- fusion rule algebra as defined in Lemma 3.4.2 i) with fusionrules (3.75). The possible statistical phases can be given in terms of the standard spins *of A_n-fusion rules,*

$$\theta^{\mathrm{st.}}_{\rho_j} := \frac{(j+1)^2 - 1}{4(n+1)} \mathrm{mod}\,,$$

and the set of possible statistical phases, $\{\theta^\tau\}$, of the fusion rules corresponding to $\mathbb{Z}_r$ are labelled by $\tau \in \mathbb{Z}_{2r}$, with $\tau r \equiv 0 \bmod 2$, and are determined by

$$\theta^\tau_{\alpha^\ell} \equiv \frac{\tau \ell^2}{2r} \bmod 1\,, \qquad (7.159)$$

where α is the generator of $\mathbb{Z}_r$.

a) If Φ is a fusion rule subalgebra of $A_n \times \mathbb{Z}_r$ and $i : \Phi \hookrightarrow A_n \times \mathbb{Z}_r$ then every choice of statistical phases is given by

$$\pm\theta_k = \tilde{\theta}_{i(k)} \bmod 1\,, \qquad k \in \Phi\,,$$

where $\tilde{\theta} : A_n \times \mathbb{Z}_r \rightarrow \mathbb{R}/\mathbb{Z}$ is given by

$$\tilde{\theta}_{\xi\otimes\alpha^\ell} = \theta^{\mathrm{st.}}_\xi + \theta^\tau_{\alpha^\ell} \bmod 1\,, \qquad \text{(7.16}$$

for some τ as above.

b) For $\Phi = \tau_\alpha\left(A_{2m-1} * \mathbb{Z}_r\right)$ the possible phases are given by

$$\pm\,\theta_{(\rho_j,k)} = \theta^{\mathrm{st.}}_{\rho_j} + \frac{2\tau+1}{8r}\,\left(\mathrm{grad}\left(\rho_j\right) + 2k\right)^2 \bmod 1\,, \qquad \text{(7.161}$$

for $k = 0, \ldots, r-1$, and some $\tau \in \mathbb{Z}_{4r}$.

Proof. First we shall use the previous results to exclude all fusion rule algebras not liste in Proposition 7.4.11 from those realized in a C^*-quantum category. The most importan tool here is Lemma 7.4.9 ii). It states that if Λ is the matrix-block of $\mathbb{N}_\rho$ restricted to Φ and we consider the bicolored graph associated to it, every path in this graph starting at for which two succeeding vertices of one coloration are distinct and which terminates a a point of edge degree one (i.e., an end point of an "external" leg) has to have a length with the property that N divides $(L+1)$. Since all bicolored graphs with norm less tha two are trees, any such path is without self intersection, thus reresents an A_L-subgrap with Coxeter number $L+1$. By monotonicity of the norm with respect to subgraphs follows that $L+1 \leq N$, and therefore by Lemma 7.4.9 ii)

$$N = L + 1\,.$$

Again, monotonicity implies that the A_L-graph is already the entire graph.

This fact can also be verified by finding paths in the E- and D-graphs violating t condition $N/(L+1)$. For $d_\rho < 2$ and $a'' = 2$ in Theorem 3.4.11, this excludes the algebr (3.118), (3.119), (3.122), (3.123) and (3.127) with two-channel decompositions of $\rho\circ$ The only admissible algebra with $a'' = 2$ is the one in (3.117), since the bicolored grap associated to Λ is the Coxeter graph $A_{\ell n}$. From Proposition 7.3.4 ii) and the followi remarks we learned that the D_4-algebras (3.128), (3.129) and (3.130) are not admissib either. The additional constraint (7.158) will be obtained in the following calculation the spins.

From Lemma 7.4.3, (7.120), we find the form (7.159) by specializing to $Out(\Phi_0) = 1$ and $a'' = 1$, so that both η and δ are trivial. Setting $\tau = -(h_{\sigma_1} + r\varepsilon_{\sigma_1})$, the constraint $\tau r \equiv 0 \bmod 1$ is equivalent to (7.119). In order to treat the case $\Phi = \overline{A}_n \times \mathbb{Z}_r$, with $a'' = 1$, we use Corollary 7.4.4, i). As Out $(\Phi_0) = 1$, we have $\theta^0_\sigma = 0$, so that formula (7.123) yields

$$\theta_{\xi \cdot \alpha^j} = \theta_\xi + \theta^\tau_{\alpha^j} , \tag{7.162}$$

for $\xi \in \Phi_0$, (i.e. grad$(\xi) = 0$ and $\theta^0_\xi = \theta_\xi$), where α is the generator of $\mathbb{Z}_r$, with grad$(\alpha) = 1$, and $\tau = h(a+1) \bmod 2a$; (this form is equivalent to $\tau a = 0 \bmod 2$).

As suggested above, in the computation of the $\overline{A}_n$-spins, we mainly make use of Lemma 7.4.9 i). For the selfconjugate case, $\rho \circ \rho = 1 + \psi$, this has to be specialized to $t = \delta a$, with $\delta \in \mathbb{Z}_4$, as described in Lemma 7.4.5 and we obtain using that $\frac{1}{4}\text{grad}\,(\xi_j) = -\frac{1}{4}(j^2 - 1) \bmod 1$.

$$\pm\, \theta_{\xi_j} = \frac{j^2 - 1}{4} \left(\frac{1}{N_{\text{Cox.}}} - \delta \right) \quad \bmod 1 . \tag{7.163}$$

Let us choose a basis of the $\overline{A}_n$-fusion rule algebra : $\{\varphi_1 = 1,\, \varphi_2 = \rho, \ldots, \varphi_n\}$. Then $\mathbb{N}_\rho$ is given by

$$\begin{aligned} \rho \circ \varphi_i &= \varphi_{i-1} + \varphi_{i+1} \quad \text{, for } i = 2, \ldots, n-1 , \\ \text{and} \quad \rho \circ \varphi_n &= \varphi_n + \varphi_{n-1} . \end{aligned}$$

The only path, $\{\xi_j\}$, in the $\overline{A}_n$-graph, which satisfies the prerequisites of Lemma 7.4.9 is the following

$$\begin{aligned} \xi_i &= \varphi_i && \text{for } i = 1, \ldots, n , \\ \text{and} \quad \xi_i &= \varphi_{(2n+1)-i} && \text{for } i = (n+1), \ldots, 2n , \end{aligned} \tag{7.164}$$

so that

$$N_{\text{Cox.}} = L + 1 = 2n + 1 .$$

Evidently we have the consistency requirement that $\theta_{\xi_j} = \theta_{\xi_{N-j}}$ $,\forall j = 1, \ldots, L$, which turns out to be equivalent to $\delta = -N_{\text{Cox.}} \bmod 4$. We find

$$\pm\, \theta_{\xi_j} = \frac{j^2 - 1}{4} \left(\frac{1}{N_{\text{Cox.}}} - N_{\text{Cox.}} \right) = \begin{cases} \dfrac{j^2 - 1}{4 N_{\text{Cox.}}} , & j \text{ odd} \\[2ex] \dfrac{(N_{\text{Cox.}} - j)^2 - 1}{4 N_{\text{Cox.}}} , & j \text{ even} \end{cases} \tag{7.165}$$

using that $N_{\text{Cox.}}$ is odd.

Comparing (7.165) to the explicit formula for the inclusion

$$\begin{aligned} i(\varphi_j) &= \rho_{j-1} , && j \text{ odd} , \\ i(\varphi_j) &= \rho_{N-j-1} , && j \text{ even} , \end{aligned}$$

we can summarize (7.165) in the formula

$$\pm\, \theta_\varphi = \theta^{\text{st.}}_{i(\varphi)} \qquad , \forall \varphi \in \overline{A}_n . \tag{7.166}$$

This proves the assertion of Proposition 7.4.11, ii) a): $\Phi = \overline{A}_n * \mathbb{Z}_r$.

For the cases $\Phi = A_{2n-1} * \mathbb{Z}_r$ and $\Phi = \tau_\alpha\,(A_{2n-1} * \mathbb{Z}_r)$, the path we have to consider is clearly $\xi_j = (\rho_{j-1}, 0)$. Here the relevant formula to find the possible values of spins is given by (7.150). If $\Phi = A_{2n-1} * \mathbb{Z}_r$ we have that $\sigma_1^r = 1$, so $\theta_{\sigma_1^\tau \circ \xi_j} = \theta_{\xi_j}$, which is the same as requiring t to be even. With $t = 2\tau$ and $a = 2r$ we obtain

$$\pm\, \theta_{(\rho_j, k)} = \theta^{\text{st.}}_{\rho_j} + \frac{\tau}{4r} \left(\text{grad}\,(\rho_j, k)\right)^2 \quad \bmod 1 , \tag{7.167}$$

and this expression is now well defined for grad$\,(\rho_j, k) \in \mathbb{Z}_{2r}$. The second term in (7.167) has precisely the form (7.159) for the spins of $\Phi = \mathbb{Z}_{2r}$, the contraint $(2r)\tau \equiv 0 \bmod 2$ being automatically fulfilled.

Finally we consider (7.150) for $\Phi = \tau_\alpha\,(A_{2n-1} * \mathbb{Z}_r)$. Since $\theta_{\sigma_1^\tau \circ \rho_j} = \theta_{\rho_{2n-2} \circ \rho_j} = \theta_{\rho_{N-j-2}}$ we obtain additional contraints on t, r and N, which are given by:

$$t \equiv 1 \bmod 2 \tag{7.168}$$

$$\text{and} \quad n \equiv r + 1 \bmod 2 \tag{7.169}$$

To show this we use (7.155) and we replace $a = 2r$ and $t = 2\tau + 1$ to find (7.161). Proposition 7.4.11 is thereby proven. □

Let us add a few remarks concerning the *reality* of selfconjugate objects, ρ, with $\rho \circ \rho = 1 + \psi$. From Lemma 7.4.5, (7.132), we see that the value of θ_ρ already determines whether ρ is a real or a pseudoreal object. For instance, for $\rho \in \overline{A}_n$, it follows from $\delta \equiv -N \bmod 4$ and $N \equiv 1 \bmod 2$ that ρ is real. This is what we expect, since the reality property provides a $\mathbb{Z}_2$-grading, Grad $\left(\overline{A}_n\right) = 1$. However, if $\rho \in \dot{A}_{N-1}$ has the standard spin, $\theta^{\text{st.}}_{\rho_2}$, as for the fundamental representation of $U_q(s\ell_2)$, it is pseudoreal. The remaining possible values of spin for a selfconjugate ρ can be derived from Proposition 7.4.11 as follows:

The possible spins of A_{2n-1} are found from the inclusion $i : A_{2n-1} \hookrightarrow A_{2n-1} \times \mathbb{Z}_2$. For a given $\tau \in \mathbb{Z}_4$, as in (7.159), this yields the general expression of Lemma 3.4.10 for the selfconjugate case with

$$\delta = \tau \bmod 4 . \tag{7.170}$$

The remaining fusion rule algebra with selfconjugate generator is A_{2n}, $n \geq 1$. It appears in the classification as $\overline{A}_n \times \mathbb{Z}_2$, where the isomorphism is given by

$$\varphi_j \otimes \alpha^\varepsilon \to \begin{cases} \rho_j & j \not\equiv \varepsilon \bmod 2 \\ \rho_{N-j} & j \equiv \varepsilon \bmod 2 \end{cases} \tag{7.171}$$

where α is the generator of $\mathbb{Z}_2$, $\varepsilon \in \{0,1\}$, $j = 1, \ldots, n$. Following Proposition 7.4.11, ii) we can for $\overline{A}_n \times \mathbb{Z}_2$, we can determine the spins for some choice of $\tau \in \mathbb{Z}_4$. This induces spins on A_{2n}, reproducing the formula in Lemma 7.4.5 with

$$\delta \equiv N + \tau \bmod 4 . \tag{7.172}$$

The observation made in this discussion is that a selfconjugate sector ρ, with $\rho \circ \rho = 1 + \psi$, can be changed from real to pseudoreal and vice versa by tensoring it with a semion, whereas its reality properties are unchanged if it is tensored with a boson or a fermion.

We note that all the fusion rule algebras with selfconjugate-generator are contained in part a) of Proposition 7.4.11 ii), i.e., they do not involve any τ_α-operation. We also notice that the only enclosing algebras $A_{N-1} \times \mathbb{Z}_r$ listed in part a) are those with r even. However, for odd r, i.e., $r = 2r' + 1$, we have, by virtue of Lemma 3.3.3, an isomorphism

$$\begin{aligned} \cong : \quad A_{N-1} \times \mathbb{Z}_r &\to A_{N-1} * \mathbb{Z}_r \\ \xi \otimes \alpha &\to (\xi,\ \ell + r' \cdot \mathrm{grad}(\xi)) . \end{aligned}$$

The canonical generator of gradation is therefore $\rho = \rho_2 \otimes \alpha^{1+r'}$ and the parameter t from Lemma 7.4.9, (7.147) is related to τ in (7.159) by

$$t \equiv 4\tau(r'+1)^2 \bmod 8r . \tag{7 173}$$

It is not hard to show that the list of fusion rule algebras in Proposition 7.4.11 is n redundant, i.e., no two fusion rule algebras are isomorphic to each other. The transfor mation of spins under fusion rule algebraautomorphisms are given by automorphisms $\mathbb{Z}_{2r}$, changing the constant τ. The sign ambiguity in the determination of the spins reflec the fact that we can obtain from any braided category a second, in general inequivalen one by replacing the statistics operator ε by ε^{-1} everywhere.

7.5 Theta - Categories

In this section we present a complete analysis of categories for which all irreducible objects are invertible. In reference to what is known as θ *- (or abelian) statistics in quantum field theory we call these categories* $\underline{\theta - categories}$*. The fusion rule algebra,* Φ*, associated to a* θ *- category is thus entirely described by an abelian group,* G*, namely* $\Phi = \mathbb{N}^G$*, where the composition law on* Φ *is induced by that on* G*. The classification of* θ *- categories can be reduced entirely to a problem in group cohomology. The relevant classifying constructions are obtained from are the Eilenberg - MacLane spaces,* $H(G,n)$*, which are the homology groups of complexes denoted by* $A(G,n)$*.*

In the following discussion we shall not consider the most general aspects of this construction, but rather exemplify it for the complex $A(G,2)$ which is obtained by starting from the ordinary inhomogeneous chain complex over G, here denoted by $A(G,1)$. We provide the basic tools, e.g., a chain equivalence for cyclic groups, the Künneth formula and the universal coefficient theorem, allowing us to compute the homology- and cohomology groups of $A(G,1)$ and $A(G,2)$ in low dimensions. (For details, generalizations and proofs we refer the reader to the textbooks [59]). To begin with, we review the definition of the complex $A(G,1)$:

This complex has a grading, $A(G,1) = \bigoplus_{n \geq 0} A_n(G,1)$, where each $A_n(G,1)$ is a free $\mathbb{Z}$-module, and a canonical $\mathbb{Z}$-basis is given by cells, $c_n = [g_1 \mid \ldots \mid g_n]$, $g_i \in G$, $g_i \neq e$, where e is the unit element in G. We use the convention that $c_n = 0$ if $g_i = e$, for some $i = 1, \ldots, n$. The boundary, $\partial \in \mathrm{End}(A(G,1))$, is a map of degree -1, with $\partial^2 = 0$, and has the form

$$\partial \ [g_1 \mid \ldots \mid g_n] = \\ [g_2 \mid \ldots \mid g_n] + \textstyle\sum_{j=1}^{n-1} (-1)^j [g_1 \mid \ldots \mid g_j \cdot g_{j+1} \mid \ldots \mid g_n] + (-1)^n [g_1 \mid \ldots \mid g_{n-1}] \,. \tag{7.174}$$

The resulting sequence of maps of the chain complex is commonly summarized in a diagram

$$0 \longleftarrow \mathbb{Z} \quad \xleftarrow{\partial_1 = 0} \quad A_1(G,1) \ \cong \ \mathbb{Z}[G]/1 \cdot \mathbb{Z} \quad \xleftarrow{\partial_2} \quad A_2(G,1) \quad \xleftarrow{\partial_3} \ldots \tag{7.175}$$

We use the notation $B_k(G,1) := \mathrm{im}\,\partial_{k+1} = \mathrm{im}\,\partial \cap A_k(G,1)$, for the boundaries, for the cycles we write $Z_k(G,1) := \ker \partial \cap A_k(G,1)$, and the homology groups are denoted by $H_k(G,1) = Z_k(G,1) \,/\, B_k(G,1)$. For small k and abelian G, the homologies can be readily computed. Of course, we have

$$H_0(G,1) = \mathbb{Z}\,. \tag{7.176}$$

Since $Z_1(G,1) = A_1(G,1)$, and $\partial[g \mid h] = [g] + [h] - [gh]$, $H_1(G,1)$ is the abelian group with generators $[g]$ and relations $[g] + [h] = [gh]$, so

$$\begin{array}{rcl} i_1 : G & \longrightarrow & H_1(G,1) \\ g & \longrightarrow & [g] \end{array} \tag{7.177}$$

is an epimorphisms, and, for abelian groups G, an isomorphism. For finite cyclic groups, $G \cong \mathbb{Z}_a$, all homology groups are known,

$$\begin{array}{rcl} & H_{2m}(\mathbb{Z}_a,1) \ \cong \ 0\,, \\ \text{and} & \\ & H_{2m+1}(\mathbb{Z}_a,1) \ \cong \ \mathbb{Z}_a\,. \end{array} \tag{7.178}$$

This result is obtained from a simpler chain complex, $M(a,1)$, which is homologically isomorphic to $A(\mathbb{Z}_a,1)$. It is a free $\mathbb{Z}$-module with grading, $M(a,1) = \bigoplus_{n \geq 0} M_n(a,1)$, and each $M_n(a,1)$ is one-dimensional. Hence there are generators v_m and w_m such that $M_{2m}(a,1) = \mathbb{Z} v_m$, $m = 1, 2, \ldots$, and $M_{2m+1}(a,1) = \mathbb{Z} w_m$, $m = 0, 1, \ldots$. The boundary, ∂, is given by

$$\partial v_m = a w_{m-1}\,, \qquad \text{and} \qquad \partial w_m = 0\,. \tag{7.179}$$

Clearly this is the simplest chain complex producing the homology groups (7.178). In order to define a chain equivalence, we introduce, for some fixed generator $1 \in \mathbb{Z}_a$, the cochain $\beta \in \mathrm{Hom}\,(A_1(\mathbb{Z}_a,1), \mathbb{Z})$, given by

$$\beta(i) = i\,, \qquad \text{for} \qquad 0 \leq i < a\,, \tag{7.180}$$

and the cocycle $\gamma \in \mathrm{Hom}\,(A_2(\mathbb{Z}_a,1), \mathbb{Z})$, (with $\delta(\gamma) = \gamma \circ \partial = 0$) by

$$\gamma(i,j) = \begin{cases} 1 & a \leq i + j < 2a\,, \quad 0 \leq i,j < a \\ 0 & 0 \leq i + j < a\,. \end{cases} \tag{7.181}$$

We note that

$$\delta\beta = a\gamma \,. \tag{7.182}$$

The two complexes are related by chain transformations $I : M(a,1) \to A(\mathbb{Z}_a, 1)$ and $P : A(\mathbb{Z}_a, 1) \to M(a,1)$, i.e., I and P have degree zero and intertwine the boundary by

$$P\partial_A = \partial_M P \qquad \text{and} \qquad I\partial_M = \partial_A I \,. \tag{7.183}$$

The explicit formulae for P and I read

$$\begin{aligned} P_{2m}([i_1 \mid j_1 \mid \ldots \mid i_m \mid j_m]) &= \left(\prod_{s=1}^{m} \gamma(i_s, j_s)\right) v_m \\ P_{2m+1}([k \mid i_1 \mid j_1 \mid \ldots \mid i_m \mid j_m]) &= \left(\beta(k) \prod_{s=1}^{m} \gamma(i_s, j_s)\right) w_m \end{aligned} \tag{7.184}$$

$$\begin{aligned} I_{2m}(v_m) &= \sum_{i_1,\ldots,i_m \in \mathbb{Z}_a} [i_1 \mid 1 \mid \ldots \mid i_m \mid 1] \\ I_{2m+1}(w_m) &= \sum_{i_1,\ldots,i_m \in \mathbb{Z}_a} [1 \mid i_1 \mid \ldots \mid i_m \mid 1] \end{aligned} \tag{7.185}$$

from which (7.183) can be verified easily. Here 1 is a fixed generator of $\mathbb{Z}_a$. The situation is summarized in the diagram (the maps Φ are defined below):

$$\begin{tikzcd} 0 & \mathbb{Z} \arrow[l] \arrow[d, shift right] & A_1(\mathbb{Z}_a,1) \arrow[l, "\partial"'] \arrow[r, shift right, "\Phi"'] \arrow[d, shift right, "P_1"'] & A_2(\mathbb{Z}_a,1) \arrow[l, shift right, "\partial"'] \arrow[r, shift right, "\Phi"'] \arrow[d, shift right, "P_2"'] & A_3(\mathbb{Z}_a,1) \arrow[l, shift right, "\partial"'] \arrow[r, shift right] \arrow[d, shift right, "P_3"'] & {} \arrow[l, shift right] \\ 0 & \mathbb{Z} \arrow[l] \arrow[u, shift right] & \mathbb{Z}w_0 \arrow[l, "0"'] \arrow[u, shift right, "I_1"'] & \mathbb{Z}v_1 \arrow[l, "\cdot a"'] \arrow[u, shift right, "I_2"'] & \mathbb{Z}w_1 \arrow[l, "0"'] \arrow[u, shift right, "I_3"'] & {} \arrow[l, "\cdot a"'] \end{tikzcd} \qquad \ldots \tag{7.186}$$

for which (7.183) expresses the fact that each square involving either P or I commutes. Equation (7.183) also implies that P and I map boundaries and cycles onto one another. Hence they induce maps of the homology groups $H(P) : H(\mathbb{Z}_a) \to H(M(a,1))$, and $H(I) : H(M(a,1)) \to H(\mathbb{Z}_a, 1)$. It is shown in [57] that there exists a homotopy $\Phi : A(\mathbb{Z}_a,1) \to A(\mathbb{Z}_a,1)$ for $IP \approx 1$, which proves I and P to be the injection and the projection of a contraction, respectively, i.e., we have that

$$\begin{aligned} PI &= 1 \,, & \partial\Phi + \Phi\partial &= 1 - IP \,, \\ \Phi I &= 0 \,, & P\Phi &= 0 \,. \end{aligned} \tag{7.187}$$

From this one sees that $H(P)$ and $H(I)$ are isomorphisms of the homology group with $H(P) = H(I)^{-1}$. A popular strategy to compute the homologies for an arbitra abelian group consists of the repeated application of the Künneth formula which express $H_k(G_1 \oplus G_2, 1)$ in terms of $H_r(G_1)$ and $H_s(G_2)$, $r, s \leq k$, starting from the results cyclic groups. We carry out this exercise for the group $H_2(G,1)$. We consider the cycl $[x \mid y] - [y \mid x] \in Z_2(G,1)$ and their classes in $H_2(G,1)$,

$$\{g \mid h\} = \overline{[g \mid h]} - \overline{[h \mid g]} \,. \qquad (7.18$$

Using the relations in $H_2(G,1)$ given by the boundaries,

$$\partial[g \mid h \mid k] = [h \mid k] - [gh \mid k] + [g \mid hk] - [g \mid h] \,, \qquad (7.18$$

we show that $\{g \mid h\}$ is bilinear which means that we have a homomorphism

$$\begin{aligned} i_2 : \Lambda^2 G &\to H_2(G,1) \\ g \wedge h &\to \{g \mid h\} \,. \end{aligned} \qquad (7.19$$

The Künneth theorem for $H_2(G,1)$ asserts that the map

$$\zeta : H_2(G_1,1) \oplus H_2(G_2,1) \oplus G_1 \otimes G_2 \to H_2(G_1 \oplus G_2, 1) \qquad (7.19$$

is an isomorphism, since $\mathrm{Tor}(\mathbb{Z}, G_i) = 0$ and by (7.177), where ζ is induced on $H_2(G_1,$ simply by the inclusion of cycles and, on $G_1 \otimes G_2$, we define ζ by

$$\zeta(g_1 \otimes g_2) = \{g_1 \mid g_2\} \,. \qquad (7.19$$

Having a natural decomposition of $\Lambda^2(G_1 \oplus G_2)$ with mixed term $G_1 \otimes G_2$, we obtain t commuting diagram:

$$\begin{tikzcd} \Lambda^2 G_1 \oplus \Lambda^2 G_2 \oplus G_1 \otimes G_2 \arrow[r, "\cong"] \arrow[d, "i_{2,G_1} \oplus i_{2,G_2} \oplus \mathrm{id}_{G_1 \otimes G_2}"] & \Lambda^2(G_1 \oplus G_2) \arrow[d, "i_{2,(G_1 \oplus G_2)}"] \\ H_2(G_1,1) \oplus H_2(G_2,1) \oplus G_1 \otimes G_2 \arrow[r, "\zeta"] & H_2(G_1 \oplus G_2, 1) \,. \end{tikzcd} \qquad (7.19$$

It demonstrates that if i_{2,G_1} and i_{2,G_2} are isomorphisms then the same is true for $i_{2,G_1 \oplus G}$ Since, by (7.178), we have that $\Lambda^2 \mathbb{Z}_a = H_2(\mathbb{Z}_a, 1) = 0$, we conclude that (7.190) yields a isomorphism for an arbitrary abelian group G.

Similarly, $\Lambda^3 G$ appears as a subgroup of $H_3(G,1)$, with inclusion

$$g_1 \wedge g_2 \wedge g_3 \hookrightarrow \sum_{\pi \in S_3} \operatorname{sgn}(\pi) \left[g_{\pi(1)} \mid g_{\pi(2)} \mid g_{\pi(3)} \right] , \tag{7.194}$$

but, due to non-trivial torsion, Tor (G_1, G_2), present in the Künneth formula, and because $H_3(\mathbb{Z}_a, 1) \neq 0$, this is obviously not an isomorphism.

As originally intended, we shall now proceed with the construction of the complex $A(G,2)$, for an abelian group G. To begin with, it is essential to remark that $A(G,2)$ can be equipped with the structure of a differential, graded, augmented (DGA-)algebra. This structure manifests itself in the existence of an associative, graded product, $*$, defined on pairs of cells, which obeys the Leibnitz-rule, i.e.,

$$\begin{aligned} \deg(c_1 * c_2) &= \deg(c_1) + \deg(c_2) \\ \text{and} \quad \partial(c_1 * c_2) &= (\partial c_1) * c_2 + (-1)^{\deg(c_1)} c_1 * (\partial c_2) . \end{aligned} \tag{7.195}$$

On $A(G,1)$, $*$ is given by

$$[g_1 \mid \dots \mid g_p] * [g_{p+1} \mid \dots \mid g_{p+q}] = \sum_{\pi \in S_{p,q}} \operatorname{sgn}(\pi) \left[g_{\pi(1)} \mid \dots \mid g_{\pi(p+q)} \right] \tag{7.196}$$

where $S_{p,q} \subset S_{p+q}$ is the subgroup of all permutations, called (p,q)-shuffles, with

$$\begin{aligned} \pi(i) < \pi(j) , \qquad & \text{for} \quad 1 \leq i < j \leq p \\ & \text{and for} \quad p+1 \leq i < j \leq p+q . \end{aligned} \tag{7.197}$$

For cells of dimension less than two (7.196) yields

$$[g] * [h] = [g \mid h] - [h \mid g] = -[h] * [g] , \tag{7.198}$$

and

$$\begin{aligned} [g] * [h \mid k] &= [g \mid h \mid k] - [h \mid g \mid k] + [h \mid k \mid g] \\ &= [h \mid k] * [g] , \end{aligned} \tag{7.199}$$

for any $g, h, k \in G$.

The first step in the construction of $A(G,2)$ is the definition of a doubly graded, free $\mathbb{Z}$-module, $A(G,2) = \bigoplus_{n,m} A_{(n,m)}(G,2)$.

A $\mathbb{Z}$-basis of $A_{(n,m)}(G,2)$ is given by elements $[c_1 \mid \dots \mid c_m]$, where $c_k \in A(G,1)$ are cells with $\sum_{k=1}^{m} \deg(c_k) = n$. The total degree of a cell in $A(G,2)$ is then

$$\deg([c_1 \mid \dots \mid c_m]) = m + \sum_{k=1}^{m} \deg(c_k) . \tag{7.200}$$

Since $A(G,2)$ has a differential and a multiplicative structure, there are two possible boundary operators: One is defined similarly to the boundary (7.174) on $A(G,1)$, namely

$$\begin{aligned} \partial' : A_{(n,m)}(G,2) &\rightarrow A_{(n,m-1)}(G,2) \\ \partial'([c_1 \mid \dots \mid c_m]) &= \sum_{j=1}^{m-1} (-1)^{\deg([c_1 \mid \dots \mid c_j])} [c_1 \mid \dots \mid c_j * c_{j+1} \mid \dots \mid c_m] . \end{aligned} \tag{7.201}$$

The other one is obtained by extending ∂ on $A(G,2)$ to a derivation,

$$\begin{aligned} \partial'' : A_{(n,m)}(G,2) &\rightarrow A_{(n-1,m)}(G,2) \\ \partial''([c_1 \mid \dots \mid c_m]) &= \sum_{j=1}^{m} (-1)^{\deg([c_1 \mid \dots \mid c_{j-1}])} [c_1 \mid \dots \mid \partial c_j \mid \dots \mid c_m] . \end{aligned} \tag{7.202}$$

Besides the conditions $(\partial')^2 = (\partial'')^2 = 0$, one can also prove from (7.195) and (7.201) that

$$\partial' \partial'' + \partial'' \partial' = 0 . \tag{7.203}$$

Thus $(A(G,2), \partial', \partial'')$ is a *double complex*, and we can define $A(G,2)$ to be the corresponding condensed complex, where the grading, $A(G,2) = \bigoplus_{n \geq 0} A_n(G,2)$, is given by

$$A_n(G,2) = \bigoplus_{j=0}^{n} A_{(n-j,j)}(G,2) ,$$

and the boundary, $\partial : A_n(G,2) \rightarrow A_{n-1}(G,2)$, by

$$\partial = \partial' + \partial'' . \tag{7.204}$$

(In the generalized form of this construction, one can also obtain $A(G,1)$ systematically from the complex $A(G,0) : 0 \leftarrow \mathbb{Z}[G] \overset{\partial_1}{\leftarrow} 0 \leftarrow 0 \dots$, and define complexes $A(G,n)$ inductively, for arbitrary n.)

We remark that

$$\begin{aligned} S : A(G,1) &\rightarrow A(G,2) \\ c &\rightarrow [c] \end{aligned} \tag{7.205}$$

for any cell c, is a chain transformation, i.e., $\partial S = S\partial$, of degree one. The induced homomorphism $S_* : H(G,1) \to H(G,2)$ of the homology groups of degree one is called the *suspension*. In order to describe the cells of $A(G,2)$, we adopt the convention to replace double brackets by double bars, e.g. $[[g_1|g_2|g_3] \mid [g_4] \mid [g_5|g_6]] \equiv [g_1|g_2|g_3\|g_4\|g_5|g_6] \in A_q(G,2)$. A $\mathbb{Z}$-basis of $A(G,2)$ is given, up to dimension five, by

$$\begin{aligned}
A_0(G,2) &= \mathbb{Z}, \\
A_1(G,2) &= 0, \\
A_2(G,2) &= \mathbb{Z}[G] = S(A_1(G,1)), \\
A_3(G,2) &= S(A_2(G,1)) \\
A_4(G,2) &= S(A_3(G,1)) \oplus \bigoplus_{g,h\in\dot{G}} \mathbb{Z}[g\|h], \\
A_5(G,2) &= S(A_4(G,1)) \oplus \bigoplus_{g,h,k\in\dot{G}} \mathbb{Z}[g|h\|k] \oplus \bigoplus_{g,h,k\in\dot{G}} \mathbb{Z}[g\|h|k]
\end{aligned} \tag{7.206}$$

where $\dot{G} := G\backslash\{e\}$.

Obviously the homology groups of dimension not greater than two remain unchanged, i.e., we have

$$H_0(G,2) = \mathbb{Z}, \qquad H_1(G,2) = 0, \tag{7.207}$$

and

$$S_* \circ i_1 : G \to H_2(G,2) \tag{7.208}$$

is an isomorphism, where i_1 is as in (7.177) and S_* is the suspension. Also the cycles $Z_3(G,2) = S(Z_2(G,1))$ are the same, so S_* is onto, but we have to add the boundaries

$$\partial[g\|h] = [g|h] - [h|g] \tag{7.209}$$

to $S(B_2(G,1))$, in order to obtain $B_3(G,2)$. From (7.209) it follows that $\{g|h\} \in \ker S_*$, and, by (7.190), $S_* \circ i_2 = 0$. Since the latter map is surjective, we conclude

$$H_3(G,2) = 0. \tag{7.210}$$

The equations (7.189) and

$$\partial[g_1|g_2|g_3|g_4] = [g_2|g_3|g_4] - [g_1g_2|g_3|g_4] + [g_1|g_2g_3|g_4] - [g_1|g_2|g_3g_4] + [g_1|g_2|g_3] \tag{7.211}$$

hold also for cells in $A(G,2)$, because S is a chain transformation. The remainin generators of $B_4(G,2)$ are given by

$$\begin{aligned}
\partial[g|h\|k] &= -[[g|h] * k] + [\partial[g|h] \mid [k]] \\
&= -[g|h|k] + [g|k|h] - [k|g|h] + [h\|k] - [g \cdot h|k] + [g\|k]
\end{aligned} \qquad (7.212$$

and

$$\begin{aligned}
\partial[g\|h|k] &= [[g] * [h|k]] - [[g] \mid \partial[h|k]] \\
&= [g|h|k] - [h|g|k] + [h|k|g] + [g\|h] - [g\|h \cdot k] + [g\|k].
\end{aligned} \qquad (7.213$$

From (7.209) we see that $[g\|g]$ and $[g\|h] + [h\|g]$ are cycles. Using the relations (7.212 and (7.213), we find that they are not independent in $H_4(G,2)$:

$$\begin{aligned}
\{g\|h\} &:= \overline{[g\|h]} + \overline{[h\|g]} \\
&= \overline{[g \cdot h\|g \cdot h]} - \overline{[g\|g]} - \overline{[h\|h]}.
\end{aligned} \qquad (7.214$$

Further manipulations with (7.212) and (7.213) prove that $\{g\|h\}$ is bilinear which, b (7.214), is the same as saying that $\overline{[g\|g]}$ is quadratic. To be more precise, we introduc the abelian group $\Gamma_4(G)$, with generators $\{g\}$, $g \in G$, and relations

$$\begin{aligned}
\{g \cdot h \cdot k\} - \{g \cdot k\} - \{h \cdot k\} - \{g \cdot k\} + \{g\} + \{h\} + \{k\} = 0 \\
\text{and} \qquad \{g\} = \{g^{-1}\}.
\end{aligned} \qquad (7.21$$

Then the previous observations imply that there exists a homomorphism

$$\begin{aligned}
\gamma_4 : \Gamma_4(G) &\to H_4(G,2) \\
\text{with} \quad \gamma_4(\{g\}) &= \overline{[g\|g]}.
\end{aligned} \qquad (7.21$$

For cyclic groups $G = \mathbb{Z}_a$, the chain contraction (7.187) to the complex $M(a,1)$ can used to prove that γ_4 is an isomorphism. This depends crucially on the existence a multiplication on $M(a,1)$ for which P and I are homomorphisms. Then the ma $P^\#([c_1 \mid \ldots \mid c_m]) := [P(c_1) \mid \ldots \mid P(c_m)]$ and $I^\#([c_1 \mid \ldots \mid c_m]) := [I(c_1) \mid \ldots \mid I(c$ define a contraction of $A(G,2)$ to the complex $M(a,2)$ which is constructed similar The homology groups in $M(a,2)$ can be computed easily, and we find that

$$\Gamma_4(\mathbb{Z}_a) \cong H_4(\mathbb{Z}_a,2) \cong \mathbb{Z}_{(2,a)a} \qquad (7.21$$

where $\overline{[1\|1]} = \overline{I}^{\#}\left(\overline{[w_0|w_0]}\right)$ is a generator if $1 \in \mathbb{Z}_a$ is a generator. The proof that γ_4 in (7.216) is an isomorphism, for general, abelian groups G, now follows the same lines as the one for i_2 in (7.190). Using that $H_2(G,2) \cong G$ and $H_1(G,2) = H_3(G,2) = 0$, the Künneth formula yields an *isomorphism*

$$\zeta : H_4(G_1,2) \oplus H_4(G_2,2) \oplus G_1 \otimes G_2 \to H_4(G_1 \oplus G_2, 2) \tag{7.218}$$

which, on $H_4(G_i,2)$, is given by the inclusion of cycles and, on $G_1 \otimes G_2$, is given by

$$\zeta(g_1 \otimes g_2) = \overline{[g_1\|g_2]} + \overline{[g_2\|g_1]} = \{g_1\|g_2\} \,. \tag{7.219}$$

Notice that, besides $\Gamma_4(G_k)$ with inclusion $i_k^\Gamma : \Gamma_4(G_k) \hookrightarrow \Gamma_4(G_1 \oplus G_2)$, $k = 1,2$, $\Gamma_4(G_1 \oplus G_2)$ also contains a crossed term given by the image of

$$\tau : G_1 \otimes G_2 \hookrightarrow \Gamma_4(G_1 \oplus G_2) : g_1 \otimes g_2 \to \{g_1 \cdot g_2\} - \{g_1\} - \{g_2\} \,. \tag{7.220}$$

If we compare formulae (7.214), (7.219) and (7.220) we obtain the following commutative diagram

$$\begin{tikzcd}
\Gamma_4(G_1) \oplus \Gamma_4(G_2) \oplus G_1 \otimes G_2 \arrow[r, "\cong", "i_1^\Gamma \oplus i_2^\Gamma \oplus \tau"'] \arrow[d, "\gamma_{4,G_1} \oplus \gamma_{4,G_2} \oplus id_{G_1 \otimes G_2}"] & \Gamma_4(G_1 \oplus G_2) \arrow[d, "\gamma_{4(G_1 \oplus G_2)}"] \\
H_4(G_1,2) \oplus H_4(G_2,2) \oplus G_1 \otimes G_2 \arrow[r, "\cong", "\zeta"'] & H_4(G_1 \oplus G_2, 2) \,.
\end{tikzcd} \tag{7.221}$$

Thus, with (7.217), this implies, that $\gamma_{4,G}$ is an isomorphism , for arbitrary G. We note here that the suspension

$$S_* : H_3(G,1) \to H_4(G,2) \tag{7.222}$$

vanishes on $\Lambda^3 G \subset H_3(G,1)$, generated by the expressions in (7.194), by the symmetry of (7.212) in g and h. Moreover, $\Gamma_4(G)$ is closely related to the symmetric part of $G \otimes G$ by homomorphisms

$$\begin{aligned} D : \Gamma_4(G) &\to G \otimes G : \{g\} \to g \otimes g \quad \text{and} \\ Q : G \otimes G &\to \Gamma_4(G) : g \otimes h \to \{g \cdot h\} - \{g\} - \{h\} \,. \end{aligned} \tag{7.223}$$

The maps D and Q satisfy $QD = 2$, and $2 - DQ = 1 - T$, with $T(g \otimes h) = h \otimes g$. From $\mathrm{im}(D) = \ker(1-T)$ and $D(\mathrm{im}\, Q) = \mathrm{im}(1+T)$ we obtain a map

$$\overline{D} : \Gamma_4(G)/\mathrm{im}\, Q \twoheadrightarrow \ker(1-T)/\mathrm{im}(1+T) \cong G/2G \,, \tag{7.224}$$

where the isomorphism on the right hand side is induced by $G \to G \otimes G/\mathrm{im}(1+T) : g \to \overline{g \otimes g}$. The group on the left hand side is given in terms of generators $\{g\}$, $g \in G$, and relations, $\{g \cdot h\} = \{g\} + \{h\}$ and $2\{g\} = 0$, and hence is equal to $G/2G$. Since $\overline{D}$ is onto this yields $\ker D \subset \mathrm{im}\, Q$, and, by $DQ = 1 + T$, we have $\ker D = Q(\ker(1+T))$. Also, we have $\ker Q = \mathrm{im}(1-T) \subset \ker(1+T)$, so that

$$\overline{Q} : {}_2G \cong \ker(1+T)/\mathrm{im}(1-T) \to \ker D \tag{7.225}$$

is an isomorphism.

In particular, we find that

$$D \circ \gamma_4^{-1} \circ S_* \equiv 0 \,, \tag{7.226}$$

where we use that $D \circ \gamma_4^{-1}$ is the restriction of $\eta : A_k/B_k \to G \otimes G$, with $\eta\left(\overline{[g\|h]}\right) = g \otimes h$ and $\eta([g|k|h]) = 0$, to H_k.

Let M be any abelian coefficient group. The cochains $(A^*(G,n;M), \delta)$, $n = 1,2$, with $A^k(G,n;M) = \mathrm{Hom}(A_k(G,n), M)$ and $\delta = \partial^*$, define cohomology groups which we denote by $H^*(G,n;M)$. We write

$$B^k(G,n;M) \subset Z^k(G,n;M) \subset A^k(G,n;M) \,,$$

for coboundaries and cocycles. The main link between the homology groups determined above and cohomology groups is provided by the universal coefficient theorem which asserts that, for $n = 1,2$,

$$0 \to \mathrm{Ext}(H_{k-1}(G,n), M) \overset{\delta}{\hookrightarrow} H^k(G,n;M) \overset{\alpha}{\twoheadrightarrow} \mathrm{Hom}(H_k(G,n), M) \to 0 \tag{7.227}$$

is exact and splits. Here the epimorphism, α, is naturally induced by $Z^k(G,n;M) \cong \mathrm{Hom}(A_k(G,n)/B_k(G,n)\,; M) \xrightarrow{i^*} \mathrm{Hom}(H_k(G,n); M)$. The left term in (7.227) arises from the identity

$$\mathrm{Ext}(H_k(G,n), M) \cong \mathrm{Hom}(B_k(G,n), M)/\mathrm{Hom}(Z_k(G,n), M) \,,$$

and δ is induced by $\partial^* : \mathrm{Hom}(B_{k-1}(G,n), M) \to Z^k(G,n;M)$. If G is torsion-free, or if M is a $\mathbb{Q}$-module, e.g., $M = \mathbb{R}, \mathbb{Q}, \mathbb{R}/\mathbb{Z}\ldots$, then $\mathrm{Ext}(G,n) = 0$, and α is an isomorphism.

Note that the map from (7.205) also induces a suspension

$$S^\bullet : H^k(G,1;M) \to H^{k+1}(G,2;M),$$

for cohomology. Among the immediate consequences of (7.227) are

$$\begin{aligned} H^0(G,1;M) &= H^0(G,2;M) = \mathrm{Hom}(\mathbb{Z},M) \cong M \\ H^1(G,1;M) &= 0 \\ H^2(G,2;M) &\xrightarrow[S^\bullet]{\cong} H^1(G,1;M) \xrightarrow[i_1^\bullet \circ \alpha]{\cong} \mathrm{Hom}(G,M). \end{aligned} \tag{7.228}$$

With the homologies (7.177), (7.190) and (7.210) at our disposal, we can readily compute the cohomology groups for the next higher dimensions:

$$H^2(G,1;M) \xrightarrow[(\delta \circ i_1^\bullet)^{-1} \oplus i_2^\bullet \circ \alpha]{\cong} \mathrm{Ext}(G,M) \oplus \mathrm{Hom}\left(\Lambda^2 G, M\right), \tag{7.229}$$

and

$$H^3(G,2;M) \xrightarrow[(\delta^\bullet \circ \delta \circ i_1^\bullet)^{-1}]{\cong} \mathrm{Ext}(G,M). \tag{7.230}$$

Thus $S^\bullet : H^3(G,2;M) \hookrightarrow H^2(G,1;M)$ is just the inclusion of $\mathrm{Ext}(G,M)$.

The cocycle condition, $\mu \in Z^2(G,1;M)$ for some $\mu : G \times G \to M : (g,h) \mapsto \mu([g|h])$, can be derived explicitly from (7.189) as

$$0 = (\delta\mu)(g,h,k) = \mu(h,k) - \mu(gh,k) + \mu(g,hk) - \mu(g,h), \tag{7.231}$$

and the additional condition for μ to be in $S^\bullet(Z^3(G,2;M)) \subset Z^2(G,1;M)$ takes the form

$$\mu^t(g,h) := \mu(h,g) = \mu(g,h), \tag{7.232}$$

by (7.209). Here we denote $\mu(g,h) \equiv \mu([g \mid h])$.

The coboundaries are given, for any $\lambda : G \to M$, by

$$(\delta\lambda)(g,h) = \lambda(g) + \lambda(h) - \lambda(g \cdot h). \tag{7.233}$$

Thus, in a fashion more accessible to calculations, the formal identities (7.229) and (7.230) can be restated as follows: The map $\hat{\alpha}$ which assigns to each $\mu : G \times G \to M$, with (7.231), a skew-bilinear form in $\mathrm{Hom}(\Lambda^2 G, M)$, by

$$\hat{\alpha}(\mu) = \mu - \mu^t, \tag{7.234}$$

is surjective and vanishes on boundaries. For any symmetric cocycle, μ, there exists an abelian group $E \supset M$, with $E/M \cong G$, and a section $\psi : G \to E$, with $\pi \circ \psi = id_G$, such that $\mu(g,h) = \psi(g \cdot h) - \psi(g) - \psi(h) \in M$. If $\mathrm{Ext}(G,M) = 0$, then we have $\psi(g) =$ $g + \lambda(g) \in G \oplus M = E$; hence $\mu = \delta\lambda$, for any $\mu \in \ker\hat{\alpha}$. In the last considerations we made use of the well-known one-to-one correspondence between $\mathrm{Ext}(G,M)$ and the inequivalent, *abelian* extensions of M over G.

There is another interpretation for $H^2(G,1;M)$ in terms of *central extensions* of M over G. The aim of our discussion is now to find interpretations for $H^3(G,1;M)$ and $H^4(G,2;M)$, at least when $M = \mathbb{R}/\mathbb{Z}$, and investigate how they are related by the suspension. Contrary to the previous example, $S^\bullet$ is going to be very different from a mere injection. From (7.227), (7.216), (7.210) and (7.190) we find

$$H^3(G,1;M) \xrightarrow[i_2^\bullet \circ \delta^{-1} \oplus \alpha]{\cong} \mathrm{Ext}\left(\Lambda^2 G, M\right) \oplus \mathrm{Hom}\left(H_3(G,1);M\right) \qquad (7.235$$

and

$$H^4(G,2;M) \xrightarrow[\gamma_4^\bullet \circ \alpha]{\cong} \mathrm{Hom}\left(\Gamma_4(G),M\right). \qquad (7.236$$

For later applications, we give a more detailed description of the relations (7.235) and (7.236) and the associated complexes. The elements of $A^3(G,1;M)$ can be given as functions, $f : \dot{G} \times \dot{G} \times \dot{G} \to M : [g|h|k] \to F(g,h,k)$, $(\dot{G} = G \backslash \{e\})$, and the cocycle condition, $f \in Z^3(G,1;M)$, becomes, with (7.211),

$$\begin{aligned} 0 &= (\delta f)(g_1,g_2,g_3,g_4) \\ &= f(g_2,g_3,g_4) - f(g_1 g_2,g_3,g_4) + f(g_1,g_2 g_3,g_4) - f(g_1,g_2,g_3 g_4) + f(g_1,g_2,g_3), \end{aligned} \qquad (7.237$$

and the coboundaries are as in (7.231). Denote by $[\lambda]$ the generators of $A^3(G,2;M$ where $\lambda : \dot{G} \times \dot{G} \to M$ and $S^\bullet([\lambda]) = \lambda \in A^2(G,1;M)$. The elements of $A^4(G,2;M)$ ca then be given as pairs $[f,r]$, with $f : \dot{G} \times \dot{G} \times \dot{G} \to M$ and $r : \dot{G} \times \dot{G} \to M$, so th $[f,r]([g|h|k]) = f(g,h,k)$ and $[f,r]([g||h]) = r(g,h)$. The suspension is induced by t omission

$$S^\bullet([f,r]) = f, \qquad (7.2$$

and we find from (7.209) that

$$\delta[\lambda] = \left[\delta\lambda, \lambda - \lambda^t\right], \qquad (7.2$$

for the coboundaries in $B^2(G,2;M)$.

Since, by (7.238) and $S^\bullet\delta = \delta S^\bullet$, we have that

$$(\delta[f,r])([g_1|g_2|g_3|g_4]) = (\delta f)(g_1,g_2,g_3,g_4)\,,$$

the cocycle condition, $[f,r] \in Z^4(G,2;M)$, is given by $f \in Z^3(G,1;M)$ and we obtain the two equations

$$\begin{aligned} 0 &= (\delta[f,r])([g|h\|k]) \\ &= -f(g,h,k) + f(g,k,h) - f(k,g,h) + r(h,k) - r(g\cdot h,k) + r(g,k) \end{aligned} \tag{7.240}$$

and

$$\begin{aligned} 0 &= (\delta[f,r])([g\|h|k]) \\ &= f(g,h,k) - f(h,g,k) + f(h,k,g) + r(g,h) - r(g,h\cdot k) + r(g,k)\,. \end{aligned} \tag{7.241}$$

The definition of $\Gamma_4(G)$ in terms of the relations (7.215) allows us to identify the space $\mathrm{Hom}\,(\Gamma_4(G),M)$ in (7.236) with the set of M-valued quadratic functions, θ, i.e., with all functions $\theta:\Gamma_4(G)\to M$, with

$$\begin{aligned} \theta(ghk) - \theta(gh) - \theta(gk) - \theta(hk) + \theta(g) + \theta(h) + \theta(k) = 0 \\ \text{and} \qquad \theta(g) = \theta(g^{-1})\,. \end{aligned} \tag{7.242}$$

The isomorphism of (7.236) is then given by

$$\theta(g) := \gamma_4^\bullet \circ \alpha([f;r]) = r(g,g)\,. \tag{7.243}$$

In particular, (7.236) implies that a cocycle $[f;r]$ is a coboundary iff the diagonal of r is zero, and, conversely, to any quadratic function θ, there corresponds a cocycle with (7.243). We now claim that

$$0 \to \mathrm{Hom}\left(\Lambda^2 G, M\right) \overset{\pi^\bullet}{\hookrightarrow} \mathrm{Hom}(G\otimes G,M) \xrightarrow{D^\bullet} \mathrm{Hom}\,(\Gamma_4(G),M) \xrightarrow{S^\bullet\circ(\gamma_4^{-1})^\bullet} H^3(G,1;M) \tag{7.244}$$

is exact, where π is the projection onto $\Lambda^2 G$, and D is given in (7.223). The definition of D implies exactness at $\mathrm{Hom}(G\otimes G,M)$, and the composition of maps at $\mathrm{Hom}\,(\Gamma_4(G),M)$ is zero by (7.226). Suppose now that $\theta \in \ker\left(S^\bullet \circ \left(\gamma_4^{-1}\right)^\bullet\right)$, for some quadratic function θ. Then there is a representing cocycle $[f;r]$ with (7.243), and $S^\bullet([f,r]) = \delta\lambda \in B^3(G,1;M)$,

so $f = \delta\lambda$. The function θ is then also represented by the cocycle $[f,r] - \delta[\lambda] = [0,\rho]$, $\rho = r - (\lambda - \lambda^t)$. The cocycle conditions (7.240) and (7.241) show that ρ is bilinear and therefore extends to $G\otimes G$. For $\theta \in \ker\left(S^\bullet \circ \left(\gamma_4^{-1}\right)^\bullet\right)$, we find

$$\theta(g) = \rho(g\otimes g) = D^\bullet(\rho)(g)\,, \quad \text{for some } \rho \in \mathrm{Hom}(G\otimes G,M)\,, \tag{7.245}$$

which proves exactness of (7.244).

In order to extend results on the cohomology of cyclic groups to arbitrary abelian groups, we consider the dual version of (7.221):

$$\begin{tikzcd}
H^4\,(G_1\oplus G_2,2;M) \arrow[r, "\cong"] \arrow[d, "\cong"', "\gamma^\bullet_{4,(G_1\oplus G_2)}"] & \oplus_{i=1}^2 H^4\,(G_i,2;M) \oplus \mathrm{Hom}\,(G_1\otimes G_2,M) \arrow[d, "\cong"', "\gamma^\bullet_{4,G_1}\oplus\gamma^\bullet_{4,G_2}\oplus id"] \\
\mathrm{Hom}\,(\Gamma_4\,(G_1\oplus G_2)\,,M) \arrow[r, "\cong"] & \oplus_{i=1}^2 \mathrm{Hom}\,(\Gamma_4\,(G_i)\,,M) \oplus \mathrm{Hom}\,(G_1\otimes G_2,M)\,.
\end{tikzcd} \tag{7.246}$$

The horizontal arrows in (7.246) that project onto the direct summands of the spaces $H^4\,(G_1\oplus G_2,2;M)$ and $\mathrm{Hom}\,(\Gamma_4\,(G_1\oplus G_2)\,,M)$ are obtained from the inclusions in (7.221). Thus, to every quadratic function θ on $G_1\oplus G_2$, we associate unique elements $\theta_i \in \mathrm{Hom}\,(\Gamma_4\,(G_i)\,;M)$, defined by the restrictions of θ, and some $q = \tau^\bullet(\theta) \in \mathrm{Hom}\,(G_1\otimes G_2,M)$, where τ is given in (7.220), such that

$$\theta\,((g_1,g_2)) = \theta_1\,(g_1) + \theta_2\,(g_2) + q\,(g_1\otimes g_2)\,. \tag{7.247}$$

If we set $K_G := \ker\left(S^\bullet\circ\left(\gamma_4^{-1}\right)^\bullet\right) = \mathrm{im}\,(D^\bullet) \subset \mathrm{Hom}\,(\Gamma_4(G),M)$ the composition

$$K_{(G_1\oplus G_2)} = K_{G_1} \oplus K_{G_2} \oplus \mathrm{Hom}\,(G_1\otimes G_2,M) \tag{7.248}$$

holds in the sense that K_{G_i} are subspaces of $\mathrm{Hom}\,(\Gamma_4\,(G_i)\,,M)$ in (7.246). To see this, we define $\rho \in \mathrm{Hom}\,((G_1\oplus G_2)\otimes(G_1\oplus G_2)\,,M)$ to be equal to q on $G_1\otimes G_2$ and zero on all other $G_i\otimes G_j$. Then $\rho\,((g_1,g_2)\otimes(g_1,g_2)) = q\,(g_1\otimes g_2)$, and (7.247) implies that $\mathrm{Hom}\,(G_1\otimes G_2,M) \subset K_{(G_1\oplus G_2)}$. So, if $\theta \in K_{(G_1\oplus G_2)}$ then $(\theta - D^\bullet(\rho)) = \Sigma\theta_i \in K_{(G_1\oplus G_2)}$, and therefore there is some $\tilde\rho$ with $\tilde\rho\,((g_1,g_2)\otimes(g_1,g_2)) = \theta_1\,(g_1) + \theta_2\,(g_2)$. Setting $g_2 = 0$ yields $\theta_1 = D^\bullet(\tilde\rho) \restriction G_1\otimes G_1 \in K_{G_1}$, and (7.248) follows.

The image of $S^\bullet$ in $H^3(G_1 \otimes G_2, 1; M)$ is thus described by

$$\begin{aligned} S^\bullet \circ \left(\gamma_4^{-1}\right)^* (\mathrm{Hom}(\Gamma_4(G_1 \oplus G_2), M)) &= \\ \bigoplus_{i=1,2} S^\bullet \circ \left(\gamma_4^{-1}\right)^* (\mathrm{Hom}(\Gamma_4(G_i), M)) &\cong \bigoplus_{i=1,2} \mathrm{Hom}(\Gamma_4(G_i), M) / K_{G_i} \,. \end{aligned} \tag{7.249}$$

The complete image of $S^\bullet$ can now be easily determined by starting from (7.217) and iterating (7.249). Note that $D^\bullet Q^\bullet = 2$, found from (7.223), implies

$$2\,\mathrm{Hom}(\Gamma_4(G), M) \subset K_G\,, \tag{7.250}$$

so that all elements in im$S^\bullet$ are of order two.

This observation leads us to consider cohomology with $\mathbb{Z}_2$-coefficients. Since reductions of coefficients strongly depend on the original group M, we shall avoid complications by restricting our attention to the case $M = \mathbb{R}/\mathbb{Z}$ (in which we are actually interested). First, we remark that there is an involution, $\mathcal{F}$, on $A^4(G, 2; \mathbb{R}/\mathbb{Z})$ with

$$\mathcal{F}([f;r]) = \left[-f; r^t\right]. \tag{7.251}$$

One immediately verifies that it maps cocycles to cocycles, that $\mathcal{F}\delta[\lambda] = -\delta[\lambda]$ and that the induced map $\overline{\mathcal{F}}$ is the identity on $H^4(G, 2; M)$. It follows that $1 - \mathcal{F}$ maps any cocycle $[f;r]$ to a coboundary. Since we have coefficients $\mathbb{R}/\mathbb{Z}$ we can choose this as

$$(1 - \mathcal{F})([f;r]) = 2\delta[\mu]\,,$$

where $\mu \in A^3(G, 1; \mathbb{R}/\mathbb{Z})$. Another representative of the cohomology class of $[f;r]$ is then given by $[\tilde{f};\tilde{r}] := [f, r] - \delta[\mu]$, which, by the last formula, is fixed by $\mathcal{F}$. This means that, in every cohomology class, we have a representative with

$$2\tilde{f} \equiv 0 \bmod 1 \qquad \text{and} \qquad \tilde{r} = \tilde{r}^t\,. \tag{7.252}$$

We denote the space of cocycles obeying (7.252) by $Z^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z})$. The restricted projection $Z^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z}) \twoheadrightarrow H^4(G, 2; \mathbb{R}/\mathbb{Z})$ is still onto, and its kernel is given by $B^4_{\mathrm{symm}} := Z^4_{\mathrm{symm}} \cap B^4$. Since $\mathcal{F}$ acts as -1 on the boundaries, the $\mathcal{F}$-invariant set is given by ${}_2(B^4(G, 2; \mathbb{R}/\mathbb{Z}))$, where we use the notation ${}_pG = \{g \in G : g^p = 1\}$. But for $[\lambda]$, with $\delta[\lambda] \in B^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z})$, this implies $2[\lambda] \in Z^3(G, 2; \mathbb{R}/\mathbb{Z})$. Since by (7.230) we have that $H^3(G, 2; \mathbb{R}/\mathbb{Z}) = 0$, we can find some $\mu \in A^2(G, 2; \mathbb{R}/\mathbb{Z})$, such that $2[\lambda] = 2\delta\mu$. Fo $[\lambda'] = [\lambda] - \delta\mu$, we then have $\delta[\lambda'] = \delta[\lambda]$ and $\lambda'(g, h) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. We conclude that

$$B^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z}) = {}_2\left(B^4(G, 2; \mathbb{R}/\mathbb{Z})\right) = B^4\left(G, 2; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right). \qquad (7.253$$

Similar to $S^\bullet$ in (7.238), we have a well defined suspension of cocycles

$$\begin{aligned} S^\bullet_{\mathrm{symm}} : Z^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z}) &\rightarrow Z^3\left(G, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \\ [f;r] &\rightarrow f\,. \end{aligned} \qquad (7.254$$

By (7.253), it has the property

$$S^\bullet_{\mathrm{symm}}\left(B^4_{\mathrm{symm}}(G, 2; \mathbb{R}/\mathbb{Z})\right) = B^3\left(G, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right).$$

Together with $Z^4_{\mathrm{symm}} \big/ B^4_{\mathrm{symm}} \cong H^4(G, 2; \mathbb{R}/\mathbb{Z})$ this induces a homomorphism

$$\overline{S}^\bullet_{\mathrm{symm}} : H^4(G, 2; \mathbb{R}/\mathbb{Z}) \rightarrow H^3\left(G, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right). \qquad (7.255$$

The connection of $\overline{S}^\bullet_{\mathrm{symm}}$ and $S^\bullet$ is obtained by considering the short exact sequence o coefficients

$$0 \rightarrow \tfrac{1}{2}\mathbb{Z}/\mathbb{Z} \overset{i}{\hookrightarrow} \mathbb{R}/\mathbb{Z} \xrightarrow{2} \mathbb{R}/\mathbb{Z} \rightarrow 0 \qquad (7.256$$

and the associated long exact sequence

$$\rightarrow H^2(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\bar{\delta}} H^3\left(G, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \xrightarrow{\bar{i}} H^3(G, \mathbb{R}/\mathbb{Z}) \xrightarrow{\bar{2}} H^3(G, \mathbb{R}/\mathbb{Z})\,, \qquad (7.257$$

where $\bar{\delta}$ is the connecting homomorphism, and $\bar{i}$ and $\bar{2}$ the maps induced from (7.256 We find the following commuting diagram

$$\begin{tikzcd} H^4(G, 2; \mathbb{R}/\mathbb{Z}) \arrow[rr, "S^\bullet"] \arrow[dr, "\overline{S}^\bullet_{\mathrm{symm}}"'] & & H^3(G, 1; \mathbb{R}/\mathbb{Z}) \\ & H^3\left(G, 1, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \arrow[ur, "\bar{i}"] & \end{tikzcd} \qquad (7.25$$

For general abelian groups, working with this substitution of the coefficients tends to rather awkward. However, for cyclic groups, the decomposition of $S^\bullet$ according to (7.25 turns out to be pertinent. First, we observe that, for $G = \mathbb{Z}_a$, $H^2(G, \mathbb{R}/\mathbb{Z}) = 0$ impl

$\bar{\delta} = 0$ in the exact sequence (7.257), and $\bar{\imath}$ is injective. From (7.217) and (7.236) we find that

$$H^4(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_{(2,a)a} \tag{7.259}$$

and the generating quadratic function ,θ_a, is given by

$$\theta_a(j) = \frac{j^2}{(2a)a} \bmod 1\,, \qquad \forall j \in \mathbb{Z}_a\,. \tag{7.260}$$

Moreover, since $\mathbb{Z}_a \otimes \mathbb{Z}_a = \mathbb{Z}_a$, the bilinear functions are generated from

$$\rho(i \otimes j) = \frac{ij}{a} \bmod 1\,, \qquad \forall i, j \in \mathbb{Z}_a\,. \tag{7.261}$$

By (7.244), the kernel of $S^\bullet$ (which is, with injective $\bar{\imath}$, also the kernel of $\overline{S}^\bullet_{\mathrm{symm}}$) is given by $\mathbb{Z}_a$ and has generator $(2,a)\theta_a = D^\bullet(\rho)$. Hence

$$\operatorname{im} S^\bullet \cong \operatorname{im} \overline{S}^\bullet_{\mathrm{symm}} \cong \mathbb{Z}_{(2,a)}\,. \tag{7.262}$$

Comparing this to

$$H^3\left(\mathbb{Z}_a, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \cong \operatorname{Hom}\left(\mathbb{Z}_a, \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \cong \mathbb{Z}_{(2,a)}\,, \tag{7.263}$$

which follows from $H_2(\mathbb{Z}_a, 1) = 0$, (7.178) and (7.227), we infer that $\overline{S}^\bullet_{\mathrm{symm}}$ is surjective, and hence

$$\operatorname{im} S^\bullet = \operatorname{im} \bar{\imath} = \ker \bar{2} = {}_2\!\left(H^3(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z})\right)\,. \tag{7.264}$$

For odd orders a, the groups (7.263) and (7.264) are trivial and $\theta_a = D^\bullet(\rho)$, so that the representing cocycle in Z^4_{symm} of the class of θ_a is

$$\begin{aligned} r(i,j) &= \rho(i \otimes j)\,, \qquad \forall i, j \in \mathbb{Z}_a\,, \\ f &\equiv 0\,. \end{aligned} \tag{7.265}$$

For even order $a =: 2a'$, the groups (7.263) and (7.264) are $\mathbb{Z}_2$ and the generator θ_a is mapped to the non-trivial element in $H^3\left(\mathbb{Z}_a, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right)$.

We shall use the special dependence given in (7.258), with $\bar{\imath}$ mapping into and $\overline{S}^\bullet_{\mathrm{symm}}$ onto, in the way, that, for any representative $f \in Z^3\left(\mathbb{Z}_a, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right)$ of the non-trivial cohomology class, we can adjoin some (unique) $r : G \times G \to \mathbb{R}/\mathbb{Z}$, such that

$$[f : r] \in Z^4_{\mathrm{symm}}(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z}) \quad \text{and} \quad r(j,j) = \theta_a(j)\,.$$

In order to determine a cocycle f of this kind, we employ a chain contraction of the cochain complex $A^\bullet(\mathbb{Z}_a, 1; M)$ onto the cochain complex $M^\bullet(a, 1; M)$, where $M^k(a, 1; M) = \operatorname{Hom}(M_k(a,1), M)$, $M_\bullet(a,1)$ as in (7.179) and $\delta = \partial^\bullet$. The projection and injection are $I^\bullet$ and $P^\bullet$, from (7.184) and (7.185), and the homotopy is $\Phi^\bullet$, and we obtain a diagram as in (7.186) with all arrows reversed.

The cohomology groups of $\mathbb{Z}_a$ can be computed directly from $M^\bullet(a, 1; M)$ as follows:

Since ∂ is zero on $M_{2m+1}(a,1)$, δ vanishes on $M^{2m}(a, 1; M)$, and we have

$$B^{2m+1}(a, 1; M) = 0\,, \tag{7.266}$$

$$Z^{2m}(a, 1; M) = \operatorname{Hom}(M_{2m}(a,1); M) \cong M\,. \tag{7.267}$$

Furthermore, it follows from (7.179) that

$$B^{2m}(a, 1; M) = a \cdot \operatorname{Hom}(M_{2m}(a,1); M) \cong aM\,, \tag{7.268}$$

and

$$Z^{2m+1}(a, 1; M) = {}_a(\operatorname{Hom}(M_{2m+1}(a,1); M)) \cong {}_aM\,. \tag{7.269}$$

Finally

$$H^{2m}(a, 1; M) \cong M/aM\,, \tag{7.270}$$

and

$$H^{2m+1}(a, 1; M) \cong {}_aM\,. \tag{7.271}$$

In particular, for odd dimensions, two cocycles are cohomologous only if they are equal.

Equation (7.271) confirms that $H^3\left(a, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right) \cong {}_a(\mathbb{Z}_a) = \mathbb{Z}_2$ for even a, and the non-trivial cocycle is

$$\begin{aligned} q : M_3(a,1) &\to \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\,, \\ \text{with} \qquad q(w_1) &= \tfrac{1}{2} \bmod 1\,. \end{aligned} \tag{7.272}$$

Thus, a non-trivial cocycle $f \in Z^3\left(\mathbb{Z}_a, 1; \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\right)$ is given by

$$f = P^\bullet_3(q)\,, \tag{7.273}$$

where P has been defined in (7.184). The explicit expression is then found from (7.184) as

$$\begin{aligned} f(i,j,k) &= q\left(P_3([i|j|k])\right) \\ &= \tfrac{1}{2}\,\beta(i)\,\gamma(j,k) \quad \bmod 1\,, \end{aligned} \tag{7.274}$$

for all $i, j, k \in \mathbb{Z}_a$, and with β, γ as in (7.180) and (7.181). To find the cocycle $[f; r] \in Z^4_{\text{symm}}(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z})$ representing the class of the generator $\theta_a \in \widehat{\Gamma_4(\mathbb{Z}_a)}$, we have to solve the following set of equations for $r : \mathbb{Z}_a \times \mathbb{Z}_a \to \mathbb{R}/\mathbb{Z}$:

$$\begin{aligned} r(j,j) &= \frac{j^2}{2a} \bmod 1 , \\ r(i,j) &= r(j,i) , \end{aligned} \tag{7.275}$$

and

$$r(i,j) + r(i,k) - r(i, j+k) = \tfrac{1}{2}\,\beta(i)\,\gamma(j,k) \bmod 1 .$$

Here we used that f is symmetric in the last two arguments and $f = -f$. One easily verifies that

$$r(i,j) = \frac{\beta(i)\,\beta(j)}{2a} \bmod 1 \tag{7.276}$$

is a solution, by viewing the left hand side of (7.276) as a 2-coboundary for fixed i and using (7.182). In a more systematic approach, this particular cocycle can also be obtained from the chain complex $M_\bullet(a,2)$ that we mentioned previously as being homologically equivalent to $A_\bullet(\mathbb{Z}_a, 2)$. Starting from $\tilde{q} \in Z^4(a, 2; \mathbb{R}/\mathbb{Z}) \subset \widehat{M_4(a,2)}$, with $\tilde{q}([w_0 \mid w_0]) = \frac{1}{2a} \bmod 1$ and $\tilde{q}([w_1]) = \frac{1}{2} \bmod 1$, $[f; r]$ is the same as $\left(P^\#\right)^\bullet(\tilde{q})$. More precisely we have

$$\begin{aligned} r(i,j) &= \tilde{q}\left(P^\#([i \| j])\right) &&= \tilde{q}([P([i]) \mid P([j])]) \\ f(i,j,k) &= \tilde{q}\left(P^\bullet([i|j|k])\right) &&= \tilde{q}([P([i|j|k])]) , \end{aligned} \tag{7.277}$$

which reproduces (7.274) and (7.276). We interrupt our line of arguments with a summary on cohomology of cyclic groups.

Lemma 7.5.1 *For any $a \in \mathbb{N}$, we have*

$$H^4(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_{(2,a)a} .$$

(Here $(2,a) = 1$ if a is odd and $(2,a) = 2$ i a is even.) A symmetric cocycle $[f; r] \in Z^4_{\text{symm}}(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z})$ with the property that $\overline{[f; r]}$ generates $H^4(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z})$, is given by

$$\begin{aligned} r(i,j) &= \frac{1}{(2,a)a}\,\beta(i)\,\beta(j) \bmod 1 \\ f(i,j,k) &= \frac{1}{(2,a)}\,\beta(i)\,\gamma(j,k) \bmod 1 \end{aligned} \tag{7.278}$$

for all $i, j, k \in \mathbb{Z}_a$. For the suspension

$$S^\bullet : H^4(\mathbb{Z}_a, 2; \mathbb{R}/\mathbb{Z}) \to H^3(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z}) \cong \mathbb{Z}_a$$

we have

$$\begin{aligned} \operatorname{im} S^\bullet &= {}_2(H^3(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z})) &&\cong \mathbb{Z}_{(2,a)} \\ \text{and} \qquad \ker S^\bullet &= 2H^4(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z}) &&\cong \mathbb{Z}_a . \end{aligned} \qquad (7.2$$

In particular, for even a, the cocycle $f \in Z^3(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z})$ from (7.278) represents a n trivial cohomology class in $H^3(\mathbb{Z}_a, 1; \mathbb{R}/\mathbb{Z})$.

The technology presented so far allows us to generalize Lemma 7.5.1 to arbitrary fin abelian groups

$$G = \mathbb{Z}_{a_1} \oplus \ldots \oplus \mathbb{Z}_{a_n} . \qquad (7.28$$

First, the quadratic forms of G are decomposed by iterating the lower horizontal map (7.246):

$$\operatorname{Hom}(\Gamma_4(G), \mathbb{R}/\mathbb{Z}) \cong \bigoplus_{i=1}^{n} \operatorname{Hom}(\Gamma_4(\mathbb{Z}_{a_i}), \mathbb{R}/\mathbb{Z}) \oplus \bigoplus_{1 \le i < j \le n} \operatorname{Hom}\left(\mathbb{Z}_{a_i} \otimes \mathbb{Z}_{a_j}, \mathbb{R}/\mathbb{Z}\right) . \qquad (7.28$$

For any $\theta \in \operatorname{Hom}(\Gamma_4(G), \mathbb{R}/\mathbb{Z})$ and any $g \in G$, given by $g = g_1 \ldots g_n$, $g_i \in \mathbb{Z}_{a_i}$, we can u (7.247) to write the components of θ in (7.281) in the form

$$\theta(g) = \sum_{i=1}^{n} \theta_i(g_i) + \sum_{i<j} \rho_{ij}(g_i \otimes g_j) , \qquad (7.28$$

where $\theta_i \in \operatorname{Hom}(\Gamma_4(\mathbb{Z}_{a_i}), \mathbb{R}/\mathbb{Z})$ is given by $\theta_i = \theta \restriction \mathbb{Z}_{a_i}$ and $\rho_{ij} \in \operatorname{Hom}\left(\mathbb{Z}_{a_i} \otimes \mathbb{Z}_{a_j}, \mathbb{R}/\right.$ by $\rho_{ij}(g_i \otimes g_j) = \theta(g_i g_j) - \theta(g_i) - \theta(g_j)$. More explicitly, we have

$$\operatorname{Hom}(\Gamma_4(G), \mathbb{R}/\mathbb{Z}) \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{(2,a_i)a_i} \oplus \bigoplus_{1 \le i < j \le n} \mathbb{Z}_{(a_i,a_j)} \qquad (7.28$$

in the sense that, for some given generators ξ_i of $\mathbb{Z}_{a_i} \subset G$, $i = 1, \ldots, n$, we have

$$\theta(\xi_1^{\nu_1} \ldots \xi_n^{\nu_n}) = \sum_{i=1}^{n} \frac{\tau_i}{(2,a_i)\,a_i}\,\nu_i^2 + \sum_{1 \le i < j \le n} \frac{\tau_{ij}}{(a_i, a_j)}\,\nu_i\,\nu_j \bmod 1 , \qquad (7.28$$

where $\tau_i \in \mathbb{Z}_{(2,a_i)a_i}$ and $\tau_{ij} \in \mathbb{Z}_{(a_i,a_j)}$. The decomposition (7.248) together with the spec result (7.279) put us in the position to determine which of the functions θ from (7.284) ha bilinear extensions to $G \otimes G$, i.e., $\theta \in \operatorname{im} D^*$, and are thus annihilated by the suspension map in (7.244). The condition is

$$\theta \in K_G \qquad \text{iff} \qquad (2, a_i) \mid \tau_i\,, \quad \forall i = 1, \ldots, n\,. \tag{7.285}$$

From the two short exact sequences

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{im} D^* \arrow[r, hook] & \operatorname{Hom}(\Gamma_4(G), \mathbb{R}/\mathbb{Z}) \arrow[r, two heads, "i^*"] & \operatorname{Hom}(\ker D, \mathbb{R}/\mathbb{Z}) \arrow[r] & 0 \\ 0 \arrow[r] & \ker S^* \arrow[r, hook] \arrow[u, "\cong"', "\gamma_4^*"] & H^4(G, 2; \mathbb{R}/\mathbb{Z}) \arrow[r, "S^*"] \arrow[u, "\cong"', "\gamma_4^*"] & \operatorname{im} S^* \subset H^3(G, 1; \mathbb{R}/\mathbb{Z}) \arrow[r] \arrow[u, "\cong"', "\overline{\gamma_4}"] & 0 \end{tikzcd} \tag{7.286}$$

we can derive the unique isomorphism $\overline{\gamma_4}$, which, together with (7.225), yields

$$\operatorname{im} S^* \cong \widehat{\ker D} \cong {}_2\hat{G}\,. \tag{7.287}$$

For G as in (7.280) this group is $\mathbb{Z}_{(a_1,2)} \oplus \ldots \oplus \mathbb{Z}_{(a_n,2)}$, and the map i^* can be explicitly given, once we pick $\alpha_i = a_i(\xi_i)$ as the generators of $\ker D$, which are of order two, for even a_i, and zero, for odd a_i. We have

$$i^*(\theta)(\alpha_i) = \frac{\tau_i}{(2, a_i)} \bmod 1\,. \tag{7.288}$$

For the computation of representing cocycles for the associated cohomology classes we notice that by the commutativity of (7.246), the following short exact sequence is a canonical presentation of $H^4(G, 2; \mathbb{R}/\mathbb{Z})$ in terms of cocycles and coboundaries, compatible with the decomposition (7.281):

$$\begin{tikzcd} 0 \arrow[r] & \bigoplus_{i=1}^{n} B^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z}) \arrow[r, hook] & \bigoplus_{i=1}^{n} Z^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z}) \oplus \bigoplus_{i \leq 1 < j \leq n} \operatorname{Hom}\left(\mathbb{Z}_{a_i} \otimes \mathbb{Z}_{a_j}, \mathbb{R}/\mathbb{Z}\right) & \\ & B^4(G, a; \mathbb{R}/\mathbb{Z}) \arrow[u, phantom, "\cap"] & Z^4(G, a; \mathbb{R}/\mathbb{Z}) \arrow[u, phantom, "\cap"] & \\ & {} \arrow[r, two heads] & H^4(G, a; \mathbb{R}/\mathbb{Z}) \arrow[r] & 0 \end{tikzcd} \tag{7.289}$$

Here the surjection onto $H^4(G, a; \mathbb{R}/\mathbb{Z})$ is given for the crossed terms by the identification

$$\begin{array}{ccc} \operatorname{Hom}\left(\mathbb{Z}_{a_i} \otimes \mathbb{Z}_{a_j}, \mathbb{R}/\mathbb{Z}\right) & \longrightarrow & Z^4(G, 2; \mathbb{R}/\mathbb{Z}) \\ \rho_{ij} & \longmapsto & [0; \rho]\,, \end{array}$$

where $\rho \in \operatorname{Hom}(G \otimes G, \mathbb{R}/\mathbb{Z}) = \bigoplus_{rs} \operatorname{Hom}(\mathbb{Z}_{a_r} \otimes \mathbb{Z}_{a_s}, \mathbb{R}/\mathbb{Z})$ is equal to f_{ij} on the summand with $r = i$ and $s = j$ and zero for all other r and s.

Furthermore, the inclusion $Z^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z}) \subset Z^4(G, 2; \mathbb{R}/\mathbb{Z})$ is naturally given by $\left(\pi_i^\#\right) : A^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z}) \to A^4(G, 2; \mathbb{R}/\mathbb{Z})$, where $\pi_i^\# : A_*(G, 2) \to A_*(\mathbb{Z}_{a_i}, 2)$ is the chain map obtained from the projection $\pi_i : G \to \mathbb{Z}_{a_i}$, with

$$\pi_i^\#([g_1 \mid \ldots \mid g_n]) = [\pi_i(g_i) \mid \ldots \mid \pi_i(g_n)]\,,$$

$g_i \in G$. Explicitly, $[f_i, r_i] \in Z^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z})$ is identified with $[f; r] \in Z^4(G, 2; \mathbb{R}/\mathbb{Z})$ by

$$\begin{aligned} r(g, k) &= r_i(\pi_i(g), \pi_i(k)) \\ \text{and} \qquad f(g, h, k) &= f_i(\pi_i(g), \pi_i(h), \pi_i(k))\,. \end{aligned} \tag{7.290}$$

Exactness of (7.289) also implies that two cocycles with a decomposition of this form are cohomologous iff their contributions in each $\operatorname{Hom}\left(\mathbb{Z}_{a_i} \otimes \mathbb{Z}_{a_j}, \mathbb{R}/\mathbb{Z}\right)$, $1 \leq i < j \leq n$, are equal and the respective components in $Z^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z})$ have the same class in the space $H^4(\mathbb{Z}_{a_i}, 2; \mathbb{R}/\mathbb{Z})$, for all $i = 1, \ldots, n$. Suppose now we have a quadratic function, θ, given by (7.284), with coefficients $\tau_i \in \mathbb{Z}_{(2,a_i)a_i}$ and $\tau_{ij} \in \mathbb{Z}_{(a_i,a_j)}$. Then we can use the compatibility of (7.289) with (7.281), the canonical representatives for the mixed terms and the explicit formulae for the cocycles (7.290), given in Lemma 7.5.1, to obtain a representing cocycle for the class associated to θ. It is given by $[f; r]$, where

$$\begin{aligned} r(\xi_1^{\nu_1} \ldots \xi_n^{\nu_n}, \xi_1^{\mu_1} \ldots \xi_n^{\mu_n}) &= \sum_{i=1}^{n} \frac{\tau_i}{(2, a_i)\, a_i} \beta(\nu_i)\, \beta(\mu_i) \\ &\quad + \sum_{1 \leq i < j \leq n} \frac{\tau_{ij}}{(a_i, a_j)} \nu_i\, \mu_j \bmod 1\,, \end{aligned} \tag{7.291}$$

and

$$f(\xi_1^{\nu_1} \ldots \xi_n^{\nu_n}, \xi_1^{\mu_1} \ldots {}^{\mu_n}_{n}, \xi_1^{\eta_1} \ldots \xi_n^{\eta_n}) = \sum_{i=1}^{n} \frac{\tau_i}{(2, a_i)} \beta(\nu_i)\, \gamma(\mu_i, \eta_i) \bmod 1\,.$$

The advantage of this normalization is that $\overline{[f; r]} \in \ker S^*$ if $f \equiv 0$ (instead of just $f \equiv \delta\lambda$).

Alternatively, we can find from these expressions representatives in $Z^4_{\text{symm}}(G, 2; \mathbb{R}/\mathbb{Z})$, defined in (7.252). They are obtained from $[\tilde{f}; \tilde{r}] = [f; r] - \delta[\lambda]$, with

$$\lambda(\xi_1^{\nu_1} \ldots \xi_n^{\nu_n}, \xi_1^{\mu_1} \ldots {}^{\mu_n}_{n}) = \sum_{1 \leq i < j \leq n} \frac{\tau_{ij}}{2\,(a_i, a_j)} \beta(\nu_i)\, \beta(\mu_j)\,,$$

so that

$$\bar{\tau}(\xi_1^{\nu_1}\dots\xi_n^{\nu_n},\xi_1^{\mu_1}\dots{}_n^{\mu_n}) = \sum_{i=1}^{n}\frac{\tau_i}{(2,a_i)\,a_i}\,\beta(\nu_i)\,\beta(\mu_i) + \sum_{1\leq i<j\leq n}\frac{\tau_{ij}}{2\,(a_i,a_j)}\,(\beta(\nu_i)\,\beta(\mu_j)+\beta(\nu_j)\,\beta(\mu_i))$$

and

$$\bar{f}(\xi_1^{\nu_1}\dots\xi_n^{\nu_n},\xi_1^{\mu_1}\dots{}_n^{\mu_n},\xi_1^{\eta_1}\dots\xi_n^{\eta_n}) = \sum_{i=1}^{n}\frac{\tau_i}{(2,a_i)}\,\beta(\nu_i)\,\gamma(\mu_i,\eta_i) + \sum_{1\leq i<j\leq n}\frac{\tau_{ij}a_j}{2\,(a_i,a_j)}\,\beta(\nu_i)\,\gamma(\mu_j,\eta_j) - \sum_{1\leq i<j\leq n}\frac{\tau_{ij}a_i}{2\,(a_i,a_j)}\,\gamma(\nu_i,\mu_i)\,\beta(\eta_j)\,. \tag{7.292}$$

Given these normal forms, we end here our discussion of the algebraic properties of the cohomology groups $H^4(G,2;\mathbb{R}/\mathbb{Z})$ and turn to their interpretation in the context of θ-categories.

In general, if a cohomology group, $H^k(G,n;M)$, with $k>n\geq 1$, admits an interpretation (e.g., in terms of a classification of certain algebraic objects), we expect that there exists a similar interpretation of the group $H^{k+1}(G,n+1;M)$, which is related to $H^k(G,n;M)$ by the suspension $S^* : H^{k+1}(G,n+1;M)\rightarrow H^k(G,n;M)$, and, further, that there is a connection between these interpretations which is parallel to S^*. We already encountered the example $S^* : H^3(G,2;M)\rightarrow H^2(G,1;M)$, where the suspension could be interpreted as the inclusion of the group of *abelian* extensions of M over G into the group of *central* extensions of M over G. A similar relation can be found for $H^3(G,1;\mathbb{R}/\mathbb{Z})$ and $H^4(G,2;\mathbb{R}/\mathbb{Z})$.

The group $H^3(G,1;\mathbb{R}/\mathbb{Z})$ can be naturally interpreted as the classifying object of inequivalent, relaxed, monoidal C^*-categories with fusion rule algebra $\Phi=\mathbb{N}^G$. Analogous results have been obtained in slightly different contexts, with possibly nonabelian G, like in the classification of WZW-actions with gauge group G [60], or in the guise of quasitriangular quasi-Hopf algebras, $\mathcal{A}=\mathbb{C}[G]\bowtie C(G)$, with certain restrictions [33]. Nevertheless we shall recall the derivation in a purely categorial language. For a category of the type

specified above, the composition of two irreducible objects is again irreducible, hence associativity isomorphism,

$$\alpha_{g,h,k}\in \mathrm{Mor}(g\circ(h\circ k),(g\circ h)\circ k) \qquad (7.2$$

for irreducible g,h and k, is irreducible, too, and, as the arrow space in (7.293) is o dimensional, we can consider it to be a scalar. A realization as a linear map is obtai if we choose a basis, $\Gamma_{goh,g\cdot h}\in \mathrm{Mor}(g\cdot h, g\circ h)$, and let α act on these arrows by multiplication, i.e.

$$\alpha_{g,h,k}\,(1_g\times\Gamma_{hok,h\cdot k})\,\Gamma_{go(h\cdot k),g\cdot h\cdot k} = \breve{\varphi}(g,h,k,g\cdot h\cdot k)^{g\cdot h}_{h\cdot k}\,(\Gamma_{goh,g\cdot h}\times 1_k)\,\Gamma_{g\cdot hok,g\cdot h\cdot k}\,, \qquad (7.2$$

the $\breve{\varphi}$-matrices are numbers. We shall use the simpler notation

$$\breve{\varphi}(g,h,k,g\cdot h\cdot k)^{g\cdot h}_{h\cdot k} =: e^{2\pi i f(g,h,k)}\,. \qquad (7.29$$

Clearly the numerical data from (7.295) and a choice of basis determines α uniquely. order for α to determine a monoidal category, it has to satisfy the pentagonal equatio meaning that the following diagram has to commute

$$\begin{tikzcd}
g_1\circ(g_2\circ(g_3\circ g_4)) \arrow[r, "\alpha_{g_1,g_2,g_3\circ g_4}"] \arrow[d, "1_{g_1}\times\alpha_{g_2,g_3,g_4}"] & (g_1\circ g_2)\circ(g_3\circ g_4) \arrow[r, "\alpha_{g_1\circ g_2,g_3,g_4}"] & ((g_1\circ g_2)\circ g_3)\circ g \\
g_1\circ((g_2\circ g_3)\circ g_4) \arrow[rr, "\alpha_{g_1,g_2\circ g_3,g_4}"] & & (g_1\circ(g_2\circ g_3))\circ g \arrow[u, "\alpha_{g_1,g_2,g_3}\times 1_{g_4}"']
\end{tikzcd}$$

(7.29

In terms of $f : G\times G\times G\rightarrow\mathbb{R}/\mathbb{Z}$, this is equivalent to

$$f(g_1,g_2,g_3\cdot g_4)+f(g_1\cdot g_2,g_3,g_4) = f(g_2,g_3,g_4)+f(g_1,g_2\cdot g_3,g_4)+f(g_1,g_2,g_3)\,.$$

If we consider f as an element of $A^3(G,1;\mathbb{R}/\mathbb{Z})$ and compare this to (7.237), (7.296 reexpressed as

$$f\in Z^3(G,1;\mathbb{R}/\mathbb{Z})\,. \qquad (7.2$$

We may now ask when two categories $\mathcal{C}$ and $\mathcal{C}'$ with identical objects, $\Phi=\mathbb{N}^G$, and defi by cocycles f and f' are isomorphic. An isomorphism maps the spaces $\mathrm{Mor}(g\circ h, g$

onto each other. Thus if $\left\{\Gamma'_{g\circ h,gh}\right\}$ is the image in $\mathcal{C}$ of the basis chosen in $\mathcal{C}'$ then there obviously has to exist $\lambda : G \times G \to \mathbb{R}/\mathbb{Z}$ with

$$\Gamma'_{g\circ h,gh} = e^{-2\pi i\lambda(g,h)}\,\Gamma_{g\circ h,gh}\,, \tag{7.298}$$

and f' is the cocycle determined in the basis (7.298) instead of $\{\Gamma_{g\circ h,gh}\}$. From (7.294) we see that they are related by

$$\begin{aligned} f(g,h,k) - f'(g,h,k) &= -\lambda(g,h) - \lambda(gh,k) + \lambda(h,k) + \lambda(g,h\cdot k) \\ &= (\delta\lambda)(g,h,k)\,. \end{aligned} \tag{7.299}$$

Thus f and f' define isomorphic categories iff

$$f - f' \in B^3(G,1;\mathbb{R}/\mathbb{Z})\,. \tag{7.300}$$

Hence the possible associativity arrows and thereby the possible inequivalent monoidal categories with $\Phi = \mathbb{N}^G$ are identified with elements in $H^3(G,1;\mathbb{R}/\mathbb{Z})$.

An analogous interpretation can be found for $H^4(G,2;\mathbb{R}/\mathbb{Z})$ if we require that the (relaxed) monoidal C^*-categories, with $\Phi = \mathbb{N}^G$, in addition admit a braided structure. We call a braided category of this type a θ-category. The statistics operators of a θ-category

$$\varepsilon(g,h) \in \mathrm{Mor}(g\circ h, h\circ g) \tag{7.301}$$

are determined, for irreducible objects $g, h \in G$, and a fixed basis $\{\Gamma_{g\circ h,h\cdot g}\}$, by some $r : G \times G \to \mathbb{R}/\mathbb{Z}$, so that

$$\varepsilon(g,h)\,\Gamma_{g\circ h,h\cdot g} = e^{2\pi i r(g,h)}\,\Gamma_{g\circ h,h\cdot g}\,. \tag{7.302}$$

For general objects X and Y, $\varepsilon(X,Y)$ has to satisfy the isotropy and the hexagonal equation, which can be summarized in the polynomial equations.

We shall use them here in the form of Theorem 2.3.4, where the R-matrices are defined by

$$\begin{aligned} &\alpha_{g,k,h}\,(1_g \times \varepsilon(h,k))\;\alpha^*_{g,h,k}\,(\Gamma_{g\circ h,g\cdot h} \times 1_k)\;\Gamma_{g\cdot h\circ k,g\cdot h\cdot k} \\ &= R(g,h,k,g\cdot h\cdot k)^{g\cdot k}_{g\cdot h}\,(\Gamma_{g\circ h,g\cdot h} \times 1_h)\;\Gamma_{g\cdot h\circ k,g\cdot h\cdot k}\,. \end{aligned} \tag{7.303}$$

Combining (7.294), (7.295) and (7.302) we find

$$R(g,h,k,g\cdot h\cdot k)^{g\cdot k}_{g\cdot h} = e^{2\pi i(r(h,k)+f(g,k,h)-f(g,h,k))}\,. \tag{7.304}$$

From this, together with the $\hat{\varphi}$-matrices

$$\hat{\varphi}(g,h,k,g\cdot h\cdot k)^{h\cdot k}_{g\cdot h} = e^{-2\pi i f(g,h,k)}\,, \tag{7.305}$$

we can reduce the first polynomial equation

$$\begin{aligned} &R^+(\ell\cdot g,h,k,\ell\cdot g\cdot h\cdot k)^{\ell\cdot g\cdot k}_{\ell\cdot g\cdot h}\,R^+(\ell,g,k,\ell\cdot g\cdot k)^{\ell\cdot k}_{\ell\cdot g}\,\hat{\varphi}(\ell\cdot k,g,k,\ell\cdot g\cdot h\cdot k)^{\ell\cdot k\cdot g}_{g\cdot h} \\ &= \hat{\varphi}(\ell,g,h,\ell\cdot g\cdot h)^{\ell\cdot g}_{g\cdot h}\,R^+(\ell,g\cdot h,k,\ell\cdot g\cdot h\cdot k)^{\ell\cdot k}_{\ell\cdot g\cdot h} \end{aligned} \tag{7.306}$$

to the condition

$$\begin{aligned} &f(g,h,k) - f(g,k,h) + f(k,g,h) - r(h,k) + r(g\cdot h,k) - r(g,k) \\ &= (\delta f)(\ell,g,h,k) - (\delta f)(\ell,g,k,h) + (\delta f)(\ell,k,g,h) \end{aligned} \tag{7.307}$$

on the functions f and $-r$. Since the pentagonal equation also holds for θ-categories, the right hand side of (7.307) vanishes by (7.297). We recognize the resulting equation as the cocycle condition (7.240). Similarly, we obtain (7.241) from the second polynomial equation. Thus, a pair of functions f and r defines via (7.295) and (7.302) a θ-category if and only if

$$[f;r] \in Z^4(G,2;\mathbb{R}/\mathbb{Z})\,. \tag{7.308}$$

Again $[f;r]$ and $[f';r']$ define the same category iff they differ by a rescaling of the basis as in (7.298). Besides (7.299), we obtain from (7.302)

$$r(g,h) - r'(g,h) = \lambda(g,h) - \lambda(h,g)\,. \tag{7.309}$$

Comparison with (7.239) then shows that the θ-categories constructed from $[f;r]$ and $[f';r']$ are isomorphic if and only if

$$[f;r] - [f';r'] \in B^4(G,2;\mathbb{R}/\mathbb{Z})\,. \tag{7.310}$$

This establishes the interpretation of $H^4(G,2;\mathbb{R}/\mathbb{Z})$ as the class of θ-categories with $\Phi = \mathbb{N}^G$.

Notice that we have by (7.302)

$$\varepsilon(g,g) = e^{2\pi i r(g,g)} 1_{g\circ g} \tag{7.311}$$

showing that $r(g,g)$ is a basis-independent quantity. For a θ-category the dimensions of irreducible objects are all one, so that the statistical phase, $\theta(g)$, of an irreducible object g is equal to its statistical parameter. Hence, we obtain from (7.311) the identification

$$\theta(g) = r(g,g) \bmod 1\,. \tag{7.312}$$

Let us also introduce the (basis-dependent) function $\gamma : G \rightarrow \mathbb{R}/\mathbb{Z}$ by

$$\gamma(g) = r(g,g) + r\left(g,g^{-1}\right) = f\left(g^{-1},g,g^{-1}\right) \bmod 1\,. \tag{7.313}$$

We easily find that

$$\begin{aligned} \gamma(g) &= -\gamma(g^{-1}) \bmod 1 \\ \text{and} \quad \gamma(g)-\gamma'(g) &= \lambda(g,g^{-1})\,\lambda(g^{-1},g) \bmod 1\,. \end{aligned} \tag{7.314}$$

Hence, for elements $g \in {}_2G$ of order two, $\gamma(g)$ is an invariant and $\gamma(g) \in \frac{1}{2}\mathbb{Z}/\mathbb{Z}$. In other words: γ distinguishes among the selfconjugate elements ${}_2G$ the *real* $(\gamma(g) = 0)$ and the *pseudoreal* $(\gamma(g) = \frac{1}{2})$ ones. Furthermore $\gamma : {}_2G \rightarrow \frac{1}{2}\mathbb{Z}/\mathbb{Z}$ is a homomorphism.

For the following considerations let us denote by Cat$\,(G)$ the class of θ-categories with $\Phi = \mathbb{N}^G$. So far, we have achieved an identification of Cat$\,(G)$ and $H^4(G,2;\mathbb{R}/\mathbb{Z})$ only as sets. Apparently Cat$\,(G)$ also carries a group structure induced by this correspondence which we want to describe more directly.

To this end we define a composition of θ-categories associated with two abelian groups G and H.

$$\begin{aligned} \text{Cat}\,(G) \times \text{Cat}\,(H) &\rightarrow \text{Cat}\,(G \oplus H)\,, \\ (\mathcal{C}_G, \mathcal{C}_H) &\rightarrow \mathcal{C}_G \oplus \mathcal{C}_H\,. \end{aligned} \tag{7.315}$$

The objects in $\mathcal{C}_G \oplus \mathcal{C}_H$ are given by $\mathbb{N}^G \otimes \mathbb{N}^H = \mathbb{N}^{(G\oplus H)}$ with composition

$$(g_1,h_1)\circ(g_2,h_2) = (g_1\circ g_2, h_1 \circ h_2)$$

and the arrows are given by

$$\text{Mor}\,((g_1,h_1),(g_2,h_2)) = \text{Mor}\,(g_1,g_2) \otimes \text{Mor}\,(h_1,h_2)\,,$$

with correspondingly factorized arrows α and ε.

In the cohomological translation, this corresponds to the embedding of the terms in (7.246), $H^4(G,2;M) \oplus H^4(H,2;M) \hookrightarrow H^4(G\oplus H,2;M)$. If G contains a group $\tilde{G}$, with inclusion $i : \tilde{G} \hookrightarrow G$, then we have the natural map

$$i^* : \text{Cat}\,(G) \rightarrow \text{Cat}\,(\tilde{G})\,,$$

which restricts all arrows to the objects in $\mathbb{N}^{\tilde{G}}$ and obviously corresponds to

$$\bar{i}^{\#*} \;:\; H^4(G,2;\mathbb{R}/\mathbb{Z}) \longrightarrow H^4(\tilde{G},2;\mathbb{R}/\mathbb{Z})\,.$$

Let us choose this injection to be diag : $G \hookrightarrow G\oplus G : g \rightarrow (g,g)$ and consider the com sition

$$\begin{aligned} \text{Cat}\,(G) \times \text{Cat}\,(G) \;&\rightarrow\; \text{Cat}\,(G\oplus G) \;\rightarrow\; \text{Cat}\,(G) \\ (\mathcal{C}^1_G, \mathcal{C}^2_G) \;&\rightarrow\; \mathcal{C}^1_G \oplus \mathcal{C}^2_G \;\rightarrow\; \mathcal{C}^1_G \cdot \mathcal{C}^2_G = \text{diag}^*\,(\mathcal{C}^1_G \oplus \mathcal{C}^1_G)\,. \end{aligned} \qquad (7.31$$

This is by construction precisely the multiplication induced by $H^4(G,2;\mathbb{R}/\mathbb{Z})$. Theref the correspondence between θ-categories and group-cohomology is in fact a group hom morphism, once Cat$\,(G)$ is endowed with the group structure given in (7.316). The u element in Cat$\,(G)$ is the ordinary representation category of G, where the statistics ope ator is just the flip, and thanks to the special properties of $H^4(G,2;\mathbb{R}/\mathbb{Z})$, especially th $\text{im}\,S^* \subset {}_2(H^3(G,1;\mathbb{R}/\mathbb{Z}))$, the inverse, $\mathcal{C}'$, of a category $\mathcal{C} \in$ Cat$\,(G)$ can be obtained setting $\varepsilon'(g,h) = \varepsilon(h,g)^*$ and $\alpha' = \alpha$. (For general, monoidal C^*-categories with $\Phi =$ N the definition of an inverse requires a choice of basis.) As the key observation of discussion on θ-categories, let us record their correspondence with cohomology groups the following proposition:

Proposition 7.5.2 *For* $\mathcal{C} \in$ Cat$\,(G)$ *and a given arrow-basis, let the* R- *and* $\check{\varphi}$-*matri be defined as in (7.294) and (7.303). Then the assignment*

$$(\check{\varphi}, R) \rightarrow [f;r],$$

specified in (7.295) and (7.304, yields an identification of C and its basis with a cocycle in $Z^4(G,2;\mathbb{R}/\mathbb{Z})$. The category C is trivial iff $[f;r]$ is a coboundary. The induced map

$$\mathrm{Cat}\,(G) \to H^4(G,2;\mathbb{R}/\mathbb{Z})$$

is an isomorphism of abelian groups, where the multiplication in $\mathrm{Cat}\,(G)$ is given by (7.316).

The isomorphismexplained in Proposition 7.5.2 serves as a tool to translate the results on the properties $H^4(G,2;\mathbb{R}/\mathbb{Z})$ into the context of the group $\mathrm{Cat}\,(G)$. They are gathered in the next proposition:

Proposition 7.5.3

i) For a θ-category $C \in \mathrm{Cat}\,(G)$, the function $\theta_C : G \to \mathbb{R}/\mathbb{Z}; g \to \theta_C(g)$, defined by the statistical phases $\theta_C(g)$, is quadratic (see (7.242)) and yields an invariant for each C which is separating in $\mathrm{Cat}\,(G)$. Conversely, to every quadratic function $\theta \in \Gamma_4\widehat{(G)}$, there exists a unique category $C \in \mathrm{Cat}\,(G)$ such that $\theta = \theta_C$. Hence

$$\mathrm{Cat}\,(G) \to \Gamma_4\widehat{(G)} : C \to \theta_C \tag{7.317}$$

is a group-isomorphism.

ii) Let G and H be finite abelian groups, $C_G \in \mathrm{Cat}\,(G)$ and $C_H \in \mathrm{Cat}\,(H)$ two corresponding θ-categories, with statistical phase functions θ_G and θ_H, and $q \in \mathrm{Hom}(H \otimes G, \mathbb{R}/\mathbb{Z})$ a bilinear function. Then there is a unique θ-category

$$C = C_G \oplus_q C_H \in \mathrm{Cat}\,(G \oplus H) \tag{7.318}$$

called the sum of C_G and C_H with "statistical interaction" q, such that the objects and arrows of C' are as in the sum (7.315), and

$$\alpha_{(g_1,h_1),(g_2,h_2),(g_3,h_3)} := \alpha_{g_1,g_2,g_3} \otimes \alpha_{h_1,h_2,h_3}$$

but

$$\varepsilon\left((g_1,h_1),(g_2,h_2)\right) := e^{2\pi i q(h_1,g_2)}\,\varepsilon\left(g_1,g_2\right) \otimes \varepsilon\left(h_1,h_2\right). \tag{7.319}$$

The statistical phases of C are given by

$$\theta_C((g,h)) = \theta_G(g) + \theta_H(h) + q(h,g). \tag{7.320}$$

Every θ-category $C' \in \mathrm{Cat}\,(G \oplus H)$ is isomorphic to a category given in the form (7.318), where $q \in \mathrm{Hom}(G \otimes G, \mathbb{R}/\mathbb{Z})$ is unique, and the categories C_G and C_H are unique up to isomorphisms. If two θ-categories $C^i \in \mathrm{Cat}\,(G \oplus H)$, $i = 1,2$, have a presentation of the form (7.318), in terms of $C^i_G \in \mathrm{Cat}\,(G)$, $C^i_H \in \mathrm{Cat}\,(H)$, and $q_i \in \mathrm{Hom}(H \otimes G, \mathbb{R}/\mathbb{Z})$, the product in $\mathrm{Cat}\,(G \oplus H)$ can be expressed as

$$C^1 \cdot C^2 = \left(C^1_G \cdot C^2_G\right) \oplus_{(q_1+q_2)} \left(C^1_H \cdot C^2_H\right). \tag{7.321}$$

Also we have that

$$C_G \oplus_q C_H \cong C_H \oplus_{q^t} C_G. \tag{7.322}$$

Suppose $C \in \mathrm{Cat}\ (G_1 \oplus G_2 \oplus G_3)$ is decomposed in two ways

$$(C_{G_1} \oplus_{q_{12}} C_{G_2}) \oplus_{(q_{23}+q_{13})} C_{G_3} \cong C'_{G_1} \oplus_{\left(q'_{12}+q'_{13}\right)} \left(C'_{G_1} \oplus_{q'_{23}} C'_{G_3}\right) \tag{7.323}$$

where $(q_{23} + q_{13}) \in \mathrm{Hom}\,(G_3 \otimes (G_1 \oplus G_2), \mathbb{R}/\mathbb{Z})$ is written as the sum of $q_{i3} \in \mathrm{Hom}\,(G_3 \otimes G_i, \mathbb{R}/\mathbb{Z})$, $i = 1,2$, and similarly $(q'_{12} + q'_{13})$, then we have

$$\begin{aligned} q_{ij} &= q'_{ij} \\ C_{G_i} &\cong C'_{G_i}. \end{aligned} \tag{7.324}$$

Hence, for any $C \in \mathrm{Cat}\,\left(\bigoplus_{i=1}^{n} G_i\right)$, there is a unique, well defined presentation of C as a sum of θ-categories, $C_i \in \mathrm{Cat}\ (G_i)$, with statistical interactions given by $q_{ij} \in \mathrm{Hom}\,(G_j \otimes G_i, \mathbb{R}/\mathbb{Z})$, $i < j$, denoted by

$$C = \bigoplus_{i=1}^{n}{}_{(q_{ij},i<j)}\, C_i\,, \tag{7.325}$$

such that the statistical phases are given by

$$\theta_C\,(g_1 \ldots g_n) = \sum_{i=1}^{n} \theta_{C_i}\,(g_i) + \sum_{1 \le i < j \le n} q_{ij}\,(g_j, g_i)\,, \tag{7.326}$$

where $g_i \in G_i$.

iii) Let $\mathrm{Cat}^0(G)$ *be the set of monoidal* C^**-categories with* $\Phi = \mathbb{N}^G$ *and*

$$\sigma : \mathrm{Cat}\,(G) \to \mathrm{Cat}^0(G) \tag{7.327}$$

the identification of θ*-category as a category in* $\mathrm{Cat}^0(G)$ *by omission of the braided structure, i.e.,* ε*. If* $\mathrm{Cat}^0(G)$ *is equipped with the same multiplication (7.316), so that* σ *is a homomorphism, we have* $\mathrm{Cat}^0(G) \cong H^3(G, 1; \mathbb{R}/\mathbb{Z})$*, and the unit element,* $C_0 \in \mathrm{Cat}^0(G)$*, is characterized by the fact that there is an orthonormal basis of arrows such that*

$$\alpha_{g,h,k} = (\Gamma_{goh,g\cdot h} \times 1_k)\; \Gamma_{g\cdot hok,g\cdot h\cdot k}\; \Gamma^*_{go(h\cdot k),g\cdot h\cdot k}\; (1_g \times \Gamma_{hok,h\cdot k})^* \,, \tag{7.328}$$

and it is realized by the ordinary representation category of G.

For a θ*-category* $C \in \mathrm{Cat}\,(G)$ *the corresponding category in* $\mathrm{Cat}^0(G)$ *is trivial, i.e.,* $\sigma(C) = C_0$*, iff* θ_C *extends to a bilinear form* $\rho \in \mathrm{Hom}(G \otimes G, \mathbb{R}/\mathbb{Z})$*, meaning that* $\theta_C(g) = \rho(g \otimes g)$*, or equivalently, iff* θ_C *vanishes on* $\ker D \cong {}_2G$*, where* D *is given in (7.223).*

Further, we have that

$${}_2\hat{G} \cong \mathrm{im}\,\sigma \subset \mathrm{Cat}^0(G) \tag{7.329}$$

and clearly

$$\sigma\,(C_G \oplus_q C_H) = \sigma\,(C_G) \oplus \sigma\,(C_H)\,. \tag{7.330}$$

iv) If we define, for a θ*-category* $C \in \mathrm{Cat}\,(G)$ *the function on* ${}_2G$ *given by*

$$\gamma_C = 2\theta_C \restriction {}_2G \tag{7.331}$$

this is a character $\gamma_C \in {}_2\hat{G}$ *with*

$$\gamma_C(g) = f(g,g,g) \in \tfrac{1}{2}\mathbb{Z}/\mathbb{Z}\,, \tag{7.332}$$

for any $g \in {}_2G$ *A selfconjugate object* $g \in {}_2G$ *is real if* $\gamma_C(g) = 0$ *and pseudoreal if* $\gamma_G(g) = \frac{1}{2}$.

In part i) of Proposition 7.5.3 we merely put the isomorphism (7.236) into the languag of θ-categories, using the identification of the statistical phases (7.312) with the quadrati functions in (7.243).

Part ii) is an application of the Künneth formula (7.246), where the spin formul (7.320) is a repetition of (7.247). In the construction of (7.319) we use that the representing cocycle of the mixed term can be chosen in the form $[0; \rho]$. The direct sur decomposition of the cohomology groups entails, as elementary consequences, equation (7.321)-(7.324) which by iteration yield (7.325) and (7.326).

The map σ, which is investigated in part iii), is, in cohomological terms, just th suspension S^* from (7.238). The kernel of σ, $\sigma^{-1}(\{C_0\})$, is found from the exact sequenc (7.244) or (7.245), whereas the formula for the image (7.328) follows from (7.287). Th obvious relation (7.330) corresponds to (7.248). In part iv), the properties of γ fror (7.313), evaluated on elements of order two, are summarized. Finally, we combine th correspondence of Proposition 7.5.2 and formula (7.291) to provide a normal form θ-categories, for a fixed choice of generators of the underlying group G.

Proposition 7.5.4 *Let* G *be a finite abelian group with generators* ξ_i*,* $i = 1, \ldots, n$*, suc that*

$$G = \mathbb{Z}_{a_1}(\xi_1) \oplus \cdots \oplus \mathbb{Z}_{a_n}(\xi_n)\,. \qquad (7.33$$

Then

i) the group of θ*-categories over* G *is given by*

$$\mathrm{Cat}\,(G) \cong \bigoplus_{i=1}^{n} \mathbb{Z}_{(2,a_i)a_i} \oplus \bigoplus_{1\le i<j\le n} \mathbb{Z}_{(a_i,a_j)}\,, \qquad (7.33$$

and, for a given category

$$C = (\tau_i,\ 1 \le i \le n\,,\ \ \tau_{ij},\ 1 \le i < j \le n) \qquad (7.33$$

with $\tau_i \in \mathbb{Z}_{(2,a_i)a_i}$ *and* $\tau_{ij} \in \mathbb{Z}_{(a_i,a_j)}$*, the statistical phase function is given by*

$$\theta_C\,(\xi^{\nu_1} \ldots \xi^{\nu_n}) = \sum_{i=1}^{n} \frac{\tau_i}{(2,a_i)\,a_i}\,\nu_i + \sum_{1\le i<j\le n} \frac{\tau_{ij}}{(a_i,a_j)}\,\nu_i\nu_j \bmod 1\,. \qquad (7.33$$

The set $\ker\sigma$ *of categories which are trivial as monoidal* C^**-categories is characterized by the condition*

$$(2, a_i) \mid \tau_i \,, \qquad i = 1, \ldots, n\,. \tag{7.337}$$

The image of σ*, i.e., the set of monoidal* C^**-categories that can be equipped with a braided structure, is given by*

$$\sigma(\mathrm{Cat}\,(G)) \cong \mathbb{Z}_{(2,a_1)} \oplus \cdots \oplus \mathbb{Z}_{(2,a_n)}\,. \tag{7.338}$$

More explicitly, there are categories $\mathcal{D}_i \in \mathrm{Cat}^{\,0}(G)\,,\ \ i = 1, \ldots, n\,,$ *such that* $\{\mathcal{D}_i\}_{a_i = \mathrm{even}}$ *are independent generators of order two, and*

$$\sigma(C) = \tau_1 \mathcal{D}_1 \oplus \cdots \oplus \tau_n \mathcal{D}_n \tag{7.339}$$

where C *is as defined in (7.334) and the sum is as in (7.316).*

ii) There exists a choice of arrows such that the R- and F-matrices are given as follows

$$\breve{\varphi}_C(\nu, \mu, \eta, \nu + \mu + \eta)^{(\nu+\mu)}_{(\mu+\eta)} \;=\; (-1)^{\sum\limits_{i=1}^{n} \frac{2\tau_i}{(2,a_i)} \beta(\nu_i)\gamma(\mu_i,\eta_i)} \tag{7.340}$$

and

$$R^+_C(\nu, \mu, \eta, \nu + \mu + \eta)^{(\nu+\eta)}_{(\nu+\mu)} \;=\; e^{2\pi i \left(\sum\limits_{i=1}^{n} \frac{\tau_i}{(2,a_i)a_i} \beta(\mu_i)\beta(\eta_i) + \sum\limits_{1 \le i < j \le n} \frac{\tau_{ij}}{(a_i,a_j)} \mu_i \eta_i \right)} \tag{7.341}$$

Here we abbreviated $\nu \equiv \xi_1^{\nu_1} \ldots \xi_n^{\nu_n}$ *and used the functions* β *and* γ *defined in (7.180) and (7.181). The remaining matrices are given by*

$$\begin{aligned} \hat{\varphi}(\nu, \mu, \eta, \nu + \mu + \eta)^{(\mu+\eta)}_{(\nu+\mu)} \;&=\; \overline{\breve{\varphi}(\nu, \mu, \eta, \nu + \mu + \eta)^{(\nu+\mu)}_{(\mu+\eta)}} \\ \textit{and} \qquad & \\ R^-(\nu, \mu, \eta, \nu + \mu + \eta)^{(\nu+\eta)}_{(\nu+\mu)} \;&=\; \overline{R_+(\nu, \mu, \eta, \nu + \mu + \eta)^{(\nu+\mu)}_{(\nu+\eta)}}\,. \end{aligned} \tag{7.342}$$

For this normalization, $\breve{\varphi}_C$ *is in* $\{\pm 1\}$ *and* R^+_C *in (7.341) is independent of* ν*. Further,* $\breve{\varphi}_C \equiv 1$ *holds if and only if* $C \in \ker\sigma$*. The normalization (7.340) provides a homomorphism*

$$\sigma(C) \to \breve{\varphi}_C \tag{7.343}$$

into the group of possible associativity structures of a category with a fixed basis, which is a right inverse to the map assigning to each set of $\breve{\varphi}$*-matrices the* equivalence class *in* Cat (G) *of categories they define.*

iii) The composition of arrows depicted in (2.56), which appears in the axiom of conjugate elements, is given, in the normalization (7.340), by :

$$(\Gamma_{g^{-1}\circ g,1} \times 1)^* \;\, \alpha_{g^{-1},g,g^{-1}} \;\, (1 \times \Gamma_{g\circ g^{-1},1}) = e^{2\pi i \gamma_C(g)}\, \varepsilon\left(g^{-1}, 1\right) \tag{7.344}$$

where

$$\gamma_C(\nu) = \sum_{i=1}^{n} \frac{\tau_i}{(2, a_i)}\, \nu_i \quad \mathrm{mod}\ 1\,. \tag{7.345}$$

γ_C *only depends on* $\sigma(C)$ *and*

$$\sigma(C) \to \gamma_C \in \mathrm{Hom}(G, \tfrac{1}{2}\,\mathbb{Z}/\mathbb{Z}) \tag{7.346}$$

is a homomorphism.

If $\kappa_i := \xi_i^{\frac{a_i}{(2,a_i)}}$ *denote the generators of the subgroup* ${}_2G = \mathbb{Z}_{(2,a_i)} \oplus \ldots \oplus \mathbb{Z}_{(2,a_n)}$*, we have, with*

$$\gamma_C\left(\kappa_1^{\varepsilon_1} \ldots \kappa_n^{\varepsilon_n}\right) = (-1)^{\sum\limits_i \tau_i\, \varepsilon_i\, \rho(a_i)} \tag{7.347}$$

where $\rho(a) = 1$*, for* $a \equiv 2 \bmod 4$*, and* $\rho(a) = 0$ *otherwise, that, for arbitrary* C *all elements in* $2G \cap {}_2G$ *are real (*γ_C *is zero), and, with* $H := {}_2G\,/2G \cap {}_2G$*, the map*

$$\begin{aligned} \mathrm{Cat}\,(G) \;&\to\; \widehat{H} \\ C \;&\to\; \overline{\gamma}_C \end{aligned} \tag{7.348}$$

is surjective.

In the first part of Proposition 7.5.4, the isomorphism (7.317) from Proposition 7.5.3 and the formula for quadratic functions, (7.283) and (7.284), are combined, so that the condition (7.337) corresponds to (7.285). In (7.340) and (7.341), we inserted the expressions from (7.291) into (7.295) and (7.304), using that $2f \equiv 0 \bmod 1$ and that f is symmetric in its last two arguments. In part iii), the function γ_C from (7.313) has been evaluated, yielding a basis-independent statement on the reality of selfconjugate elements.

Given the classification of and the normal forms for θ-categories, we anticipate to find some conceptual insights by addressing the question of duality. In fact, the duality problem, as posed in Chapter 7.1, can be solved for θ-categories in a straightforward manner. However, since we also included categories in our discussion that are not equivalent to any *strict* monoidal category, it is necessary to extend the range of dual objects from coassociative to quasi-coassociative Hopf algebras, first introduced by Drinfel'd. We recall how the properties described in Chapter 4.1 have to be altered, in order to yield the definition in [4]. In the first place, coassociativity (4.2) is abandoned and replaced by the weaker condition (4.5), for some invertible element $\phi \in \mathcal{K}^{\otimes 3}$. The latter is subject to the pentagon equation

$$(id \otimes id \otimes \Delta)(\phi)(\Delta \otimes id \otimes id)(\phi) = (1 \otimes \phi)(id \otimes \Delta \otimes id)(\phi)(\phi \otimes 1)\,. \tag{7.349}$$

The Hopf-algebra axioms (4.6) and (4.7) remain valid. Also the commutation relation (4.1) is assumed to hold, but the condition (4.9) becomes

$$\begin{aligned} (\Delta \otimes id)(\mathcal{R}) &= \phi_{312}\,\mathcal{R}_{13}\,\phi_{132}^{-1}\,\mathcal{R}_{23}\,\phi \\ (id \otimes \Delta)(\mathcal{R}) &= \phi_{231}^{-1}\,\mathcal{R}_{13}\,\phi_{213}\,\mathcal{R}_{12}\,\phi^{-1}\,. \end{aligned} \tag{7.350}$$

For quasi-coassociative Hopf algebras, the notion of equivalence is given by so-called twist-transformations: For any invertible element $F \in \mathcal{K}^{\otimes 2}$, another quasitriangular quasi Hopf algebra is defined by the coproduct

$$\Delta^F(a) = F\Delta(a)F^{-1} \quad , \tag{7.351}$$

the $\mathcal{R}$-matrix is then given by

$$\mathcal{R}^F = \sigma(F) \cdot \mathcal{R} \cdot F^{-1} \tag{7.352}$$

and the coassociativity isomorphism by

$$\phi^F = (1 \otimes F) \cdot (id \otimes \Delta)(F) \cdot \phi \cdot (\Delta \otimes id)\left(F^{-1}\right) \cdot \left(F^{-1} \otimes 1\right)\,. \tag{7.353}$$

On the dual space, $\mathcal{K}^*$, we still have a product induced by Δ for which, by lack of associativity, basic properties, like the uniqueness of inverses, may fail to hold. However, if we assume that two-sided inverses in $\mathcal{K} \otimes (\mathcal{K}^*)^{\otimes n}$ are unique then the antipode on $\mathcal{K}$ is unique and antihomomorphic, although it is in general not anticohomomorphic. With this assumption on $\mathcal{K}$ and $\mathcal{K}^F$, the twist transformations are not entirely arbitrary. The

algebra can be equipped with a proper counit, E^F, and an antipode, S^F, only if t elements

$$1 \otimes E(F) \qquad \text{and} \qquad E \otimes 1(F)$$ (7.35

are central, and if

$$q_F = m(1 \otimes S)(F) \quad \text{and} \quad p_F = m(S \otimes 1)\left(F^{-1}\right)$$ (7.35

are invertible and

$$p_F \cdot q_F$$

is central. In this case we have

$$E^F = E$$

and

$$S^F(a) = q_F\,S(a)\,q_F^{-1}\,.$$ (7.35

If a quantum category has *integer dimensions* we can always realize it, in the naïve sens as the representation category of some *semisimple* quasi-Hopf algebra, $\mathcal{K}$. The unitari constraints on the category then make it possible to choose $\mathcal{K}$ to be a quasitriangula quasicoassociative $*$-Hopf algebra.

The $*$-prefix signifies that $\mathcal{K}$ admits an antilinear antiinvolution, $*$, such that

$$\begin{aligned} \Delta * &= * \otimes * \Delta \\ * \otimes * \mathcal{R} &= \mathcal{R}^{-1} \\ * \otimes * \otimes * \phi &= \phi^{-1}\,. \end{aligned}$$ (7.35

The twists are therefore restricted to those with

$$F^* = F^{-1}\,.$$

If the unitary representations of an algebra $\mathcal{K}$ of this kind obey the selection rules $\Phi = \mathbb{N}^G$ then we have

$$\begin{aligned} \mathcal{K} &\cong \mathbb{C}[\widehat{G}] = C(G)\,, \\ \Delta(\sigma) &= \sigma \otimes \sigma\,, \quad \sigma \in \hat{G} \\ \text{and} \qquad \sigma^* &= \sigma^{-1}\,. \end{aligned}$$ (7.35

The elements $\mathcal{R} \in \mathcal{K}^{\otimes 2}$ and $\phi \in \mathcal{K}^{\otimes 3}$ can be considered to be functions $\mathcal{R} \in C(G \times G)$ and $\phi \in C(G \times G \times G)$ on the discrete commutative space defined by the fusion rules. Using (7.357) we can set

$$\begin{aligned} \phi(g \otimes h \otimes k) &= e^{-2\pi i f(g,h,k)} && (7.359) \\ \mathcal{R}(g \otimes h) &= e^{2\pi i r(g,h)} && (7.360) \end{aligned}$$

with functions $f : G \times G \times G \to \mathbb{R}/\mathbb{Z}$ and $r : G \times G \to \mathbb{R}/\mathbb{Z}$. Conversely, given functions f and r we can express the elements of $\mathcal{K}$ by

$$\mathcal{R} = \frac{1}{|G|^2} \sum_{g_i \in G, \sigma_i \in \hat{G}} e^{2\pi i r(g_1,g_2)}\, \overline{\sigma_1(g_1)}\, \overline{\sigma_2(g_2)}\, \sigma_1 \otimes \sigma_2 \qquad (7.361)$$

and

$$\phi = \frac{1}{|G|^3} \sum_{g_i \in G, \sigma_i \in \hat{G}} e^{-2\pi i f(g_1,g_2,g_3)}\, \overline{\sigma_1(g_1)}\, \overline{\sigma_2(g_2)}\, \overline{\sigma_3(g_3)}\, \sigma_1 \otimes \sigma_2 \otimes \sigma_3 . \qquad (7.362)$$

Thus all the conditions on $\mathcal{R}$ and ϕ to define a quasitriangular quasi-Hopf algebra can be translated into conditions on r and f.

Since $\mathcal{K}$ is commutative and accidentally cocommutative and coassociative, the commutation relations (4.1) and (4.5) are automatically true. Not surprisingly, the pentagon equation (7.349) reduces to the cocycle condition (7.237) on f and the axioms (7.350) turn out to be equivalent to equations (7.240) and (7.241). If we choose as a twist-transformation

$$F(g \otimes h) = e^{2\pi i \lambda(g,h)} \qquad (7.363)$$

we find, for the functions f' and r' that determine ϕ^F and $\mathcal{R}^F$, that

$$[f';r'] - [f;r] = \delta[\lambda] \in B^4(G,2;\mathbb{R}/\mathbb{Z}) .$$

The coproduct remains the same, since $\mathcal{K}$ is commutative

From this we infer a statement analogous to that of Proposition 7.5.2, namely that (7.359) and (7.360) induce an isomorphism of $H^4(G,2;\mathbb{R}/\mathbb{Z})$ onto the group of twistinequivalent, quasitriangular, quasiassociative $*$-Hopf algebras, whose unitary representations obey the fusion rules $\Phi = \mathbb{N}^G$.

A quadratic function θ on G can be identified, setting

$$V(g) = e^{-2\pi i \theta(g)} , \qquad (7.364)$$

with some element $V \in \mathcal{K}$, which satisfies

$$\begin{aligned} \Delta^2(V)\, V \otimes V \otimes V &= (\Delta(V))_{13}\, (\Delta(V) \otimes 1)\, (1 \otimes \Delta(V)) \\ \text{and} \qquad\qquad\qquad\qquad & && (7.365) \\ S(V) &= V , \end{aligned}$$

and, conversely, if (7.365) holds for some V the function θ given in (7.364) is quadratic. For the abelian algebra $\mathcal{K}$ we notice that $m(\mathcal{R})$, given by

$$m(\mathcal{R}) = e^{2\pi i r(g,g)} , \qquad (7.366)$$

is a twist-invariant.The assertion for Hopf-algebras corresponding to Proposition 7.5.3 now reads as follows: If $\mathcal{K}$ is the $*$-Hopf algebra from (7.358) then, to every unitary element $V \in \mathcal{K}$ which obeys equations (7.365), there exists an (up to twist-equivalence unique) quasitriangular quasi Hopf algebra structure $(\mathcal{R}, \phi)$ such that

$$V^{-1} = m(\mathcal{R}) . \qquad (7.367)$$

We observe that V is precisely the central element of a ribbon-graph-Hopf algebra as defined in (6.94) and (6.95). The element $U = m(S \otimes 1)\sigma(\mathcal{R})$ is then

$$U(g) = e^{2\pi i r(g,g^{-1})} \qquad (7.368)$$

and

$$G(g) = \left(UV^{-1}\right) = e^{2\pi i \gamma(g)} , \qquad (7.369)$$

with γ defined in (7.313). Note that G is grouplike (i.e., γ is a homomorphism) if we are in the coassociative case, $\phi \equiv 1$, or if we have chosen the normalization yielding (7.345). The case where $[f;r] \in \ker S^*$ occurs iff $\mathcal{K}$ is twist equivalent to a properly coassociative, quasitriangular Hopf algebra. The corresponding condition $\theta \in \operatorname{im} D^*$ simply means that, for $V \in \mathcal{K}$, there exist some $\tilde{\mathcal{R}} \in \mathcal{K}^{\otimes 2}$ such that the equations (4.9) hold for $\tilde{\mathcal{R}}$, and V is given in terms of $\tilde{\mathcal{R}}$ by (7.367). The group structure induced by $H^4(G,2;\mathbb{R}/\mathbb{Z})$ is just given by the multiplication $(\phi_1, \mathcal{R}_1)(\phi_2, \mathcal{R}_2) = (\phi_1\phi_2, \mathcal{R}_1\mathcal{R}_2)$, and the direct sums from (7.246) correspond to the direct sums of the Hopf algebras, with ϕ and $\mathcal{R}$ defined analogous to (7.319).

The description of the isomorphisms (7.236) in this language suggests that quasitriangular quasi Hopf algebras are the appropriate object for which a *nonabelian* generalization of (7.236) should exist. Thus, given some associative algebra $\mathcal{K}$, with a list of representations $\mathcal{C}$, a fusion rule algebra $\Phi = \mathbb{N}^{\mathcal{C}}$, and some "quadratic" element $V \in \mathcal{K} \cap \mathcal{K}'$, one may hope to find conditions such that V determines, up to twist equivalence, a unique structure $(\Delta, \mathcal{R}, \phi)$ such that V is the twist-invariant ribbon-graph element of $\mathcal{K}$. We shall leave this as an open problem.

Chapter 8

The Quantum Categories with a Generator of Dimension less than Two

8.1 Product Categories and Induced Categories

In the first part of this section we introduce the notion of product categories. We d
fine an action of the group, $H^4(Grad(\mathcal{O}bj), 2; U(1))$, of θ-categories on the set of qua
tum categories with object (fusion rule) algebra $\mathcal{O}bj$. It is denoted $C \to C^q$, for q
$H^4(Grad\mathcal{O}bj, 2; U(1))$, and C^q is a diagonal subcategory in the product of C with t
respective θ-category.

Next, we define the class of fusion rule algebra homomorphisms to which the subs
quent definition of induced categories applies, namely the irreducible, coherent or grad
homomorphisms, $f : \mathcal{O}bj_1 \to \mathcal{O}bj_2$. They are equivalently described by a subgroup
invertible objects, $\ker f = f^{-1}(1)$, whose action on the irreducible objects, $J_1 \subset \mathcal{O}b$
by multiplication is free and $\mathcal{O}bj_2$ is given by the orbits of $\ker f$. For a given cohere
homomorphism, $f : \mathcal{O}bj_1 \to \mathcal{O}bj_2$, and a quantum category C_2, with object algebra $\mathcal{O}b$
we show that there exists a unique quantum category C_1, with objects $\mathcal{O}bj_1$, such that
extends to a compatible tensorfunctor. We say that C_1 is induced by C_2 and f.

We prove that for a coherent homomorphism $f : Obj_1 \rightarrow Obj_2$ and a quantum category $\mathcal{C}_1$ with objects Obj_1 there exists a quantum category $\mathcal{C}_2$ with objects Obj_2 such that $\mathcal{C}_1$ is induced by $\mathcal{C}_2$ and f if and only if the subcategory associated to $\ker f$ is trivial and the monodromies of elements in $\ker f$ with all other objects vanish.

The remainder of this section is devoted to the question for which categories $\mathcal{C}_1$ it is possible to find a θ-category, $q \in H^4(Grad(Obj_1, 2; U(1))$, such that $\mathcal{C}_1^q$ is an induced category with respect to some graded homomorphism f defined on Obj_1. We find the relevant obstruction to lie in $H^5(Grad(Obj)/(\ker f), 2; \mathbb{Z}_2)$. We derive explicit expressions for the case where $Grad(Obj)$ is a cyclic group.

In the previous chapters various results on fusion rule algebras have been obtained by using the special properties of invertible elements of a category. More specifically, we showed in section 3.3 that nontrivially graded, invertible elements allow us to describe fusion rule algebras in terms of smaller ones. In section 7.3, we learned that this leads, for the case of fusion rule algebras with generators of small dimensions, to the situation where the generator is selfconjugate. Finally the categories that contain only invertible objects have been characterized in section 7.4. The purpose of this chapter is to combine and extend these techniques, in order to describe categories with nontrivially graded, invertible elements in terms of simpler ones. This requires the definition of a number of relations between categories, namely "subcategory", "products of categories" and, most important, "induced categories". We start by explaining what we mean by a product category. For two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ with objects in Obj_1 and Obj_2, we introduce a category, $\mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2$, whose object set is $\mathbb{N}^{Obj_1 \times Obj_2}$, i.e., a general element has the form $\sum_{X_i \in Obj_i} n_{X_1X_2}(X_1, X_2)$, and its sets of morphisms are given by

$$Mor\Big(\sum n_{X_1X_2}(X_1,X_2),\ \sum m_{Y_1Y_2}(Y_1,Y_2)\Big) :=$$
$$\bigoplus_{X_i,Y_i \in Obj_i} Hom_{\mathbb{C}}\left(\mathbb{C}^{n_{X_1X_2}}, \mathbb{C}^{n_{Y_1Y_2}}\right) \otimes Mor_1(X_1,Y_1) \otimes Mor_2(X_2,Y_2) \tag{8.1.1}$$

equipped with the obvious composition law. The tensor product for the objects is the linear extension of $(X_1,X_2) \circ (Y_1,Y_2) = (X_1 \circ Y_1, X_2 \circ Y_2)$, and the tensor product of morphisms is the one naturally induced from $Vec_{\mathbb{C}}, \mathcal{C}_1$ and $\mathcal{C}_2$. The special isomorphisms ε and α are obtained from ε_i and α_i in $\mathcal{C}_i$, $i = 1, 2$. One easily verifies from (8.1.1) that there are isomorphisms among the objects, $A + B \approx A \oplus B$, $(X, Y_1) + (X, Y_2) \approx (X, Y_1 \oplus Y_2)$ and $(X_1, Y) + (X_2, Y) \approx (X_1 \oplus X_2, Y)$. It is possible to define a quantum-category, $\mathcal{C}_1 \otimes \mathcal{C}_2$, whose objects are the equivalence classes of $\mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2$, given by $Obj_1 \otimes Obj_2 = \mathbb{N}^{(J_1 \times J_2)}$ of $Obj_i = \mathbb{N}^{J_i}$, and for which there exists an injective tensor functor $\mathcal{C}_1 \otimes \mathcal{C}_2 \hookrightarrow \mathcal{C}_1 \tilde{\otimes} \mathcal{C}_2$. In the language of structural data which we have used in previous sections, the underlying fusion rule algebra for $\mathbb{N}^{(J_1 \times J_2)}$ is given by the constants

$$N_{(i_1i_2)(j_1j_2),(k_1k_2)} = N^1_{i_1j_1,k_1}\, N^2_{i_2j_2,k_2}, \tag{8.1.2}$$

accounting for the dimensions of

$$Mor((k_1k_2),(i_1i_2) \circ (j_1j_2)) = Mor_1(k_1, i_1 \circ j_1) \otimes Mor_2(k_2, i_2 \circ j_2)\,. \tag{8.1.3}$$

The fusion matrices can be expressed as

$$\begin{aligned} F((i_1,i_2),(j_1,j_2),(k_1,k_2),(l_1,l_2)) &= T_{23}(F(i_1,j_1,k_1,l_1) \otimes F(i_2,j_2,k_2,l_2)T_{23} : \\ \bigoplus_{s_1s_2} Mor((s_1,s_2),(j_1,j_2)\circ(k_1,k_2)) &\otimes Mor((l_1,l_2),(i_1,i_2)\circ(s_1,s_2)) \\ \rightarrow \bigoplus_{s_1s_2} Mor((s_1,s_2),(i_1,i_2)\circ(j_1,j_2)) &\otimes Mor((l_1,l_2),(s_1,s_2)\circ(k_1,k_2)) \end{aligned} \tag{8.1.4}$$

where we used the identification (8.1.3) and the transposition, T_{23}, of the second and third factor of the resulting fourfold tensorproduct. Furthermore, the fundamental braid matrices, with $r(i,j,k) := R^+(1,i,j,k)$, take the following form

$$\begin{aligned} r((i_1,i_2),(j_1,j_2),(k_1,k_2)) &= r(i_1,j_1,k_1) \otimes r(i_2,j_2,k_2) : \\ Hom((k_1,k_2),(i_1,i_2)\circ(j_1,j_2)) &\longrightarrow Hom((k_1,k_2),(j_1,j_2)\circ(i_1,i_2))\,. \end{aligned} \tag{8.1.5}$$

The intrinsic invariants of the product category, the statistical phases and the statistical dimensions, are given by

$$\begin{aligned} d_{(i_1,i_2)} &= d_{i_1}\, d_{i_2} \\ \theta_{(i_1,i_2)} &= \theta_{i_1} + \theta_{i_2} \ mod\, 1\,. \end{aligned} \tag{8.1.6}$$

Clearly the gradation of $\mathcal{C}_1 \otimes \mathcal{C}_2$ is given by

$$\begin{aligned} grad\ :\ J_1 \times J_2 &\longrightarrow Grad(\mathcal{C}_1 \otimes \mathcal{C}_2) \ = \ Grad(\mathcal{C}_1) \oplus Grad(\mathcal{C}_2) \\ (i_1, i_2) &\longrightarrow (grad_1(i_1),\ grad_2(i_2))\ . \end{aligned} \qquad (8.1.7)$$

For categories with invertible elements, we already used this structure: If $\theta_i \in Hom(\Gamma_4(J_i), U(1))$ are the statistical phases of $\mathcal{C}_1$ and $\mathcal{C}_2$, $\theta_1 + \theta_2 \in Hom(\Gamma_4(J_1 \oplus J_2), U(1))$ is the statistical phase of $\mathcal{C}_1 \otimes \mathcal{C}_2$. The procedure of taking products of categories is, of course, associative, i.e., $(\mathcal{C}_1 \otimes \mathcal{C}_2) \otimes \mathcal{C}_3 \cong \mathcal{C}_1 \otimes (\mathcal{C}_2 \otimes \mathcal{C}_3)$.

The notion of a subcategory has already been used on various occasions in the previous chapters. If $J' \subset J$ is a subset of irreducible objects closed under tensor products, so that $Obj' = \mathbb{N}^{J'} \subset Obj = \mathbb{N}^{J}$ is a fusion rule subalgebra, then we find a subcategory, $\mathcal{C}'$, by restriction of the objects to Obj' and the morphisms to those between elements in Obj'. The braid- and fusion matrices are obtained by restricting their arguments to Obj'.

Suppose $\mathcal{C}$ is a category with gradation Grad$(\mathcal{C})$. Then we have a fusion rule algebra monomorphism

$$\zeta\ :\ J\ \hookrightarrow\ J\ \times\ Grad(\mathcal{C})\ :\ j\ \rightarrow\ (j,\ grad(j)),$$

identifying J as a fuison rule subalgebra of $J \times$ Grad$(\mathcal{C})$. Let $q \in$ Hom$(\Gamma_4($Grad$(\mathcal{C}))$, $\mathbb{R}/\mathbb{Z})$, defining a θ-category, $\mathcal{C}_{Grad(\mathcal{C}),q}$, with object set $\mathbb{N}^{Grad(\mathcal{C})}$, and braid- and fusion matrices given by $[f_q, r_q] \in H^4$ (Grad$(\mathcal{C}), 2; \mathbb{R}/\mathbb{Z})$, as in section 7.4. We then consider the product category $\mathcal{C} \otimes \mathcal{C}_{Grad(\mathcal{C}),q}$ with fusion rule algebra $J \times$ Grad$(\mathcal{C})$ which, by the above inclusion ζ, contains a category $\mathcal{C}^q$ with fusion rule algebra $\mathbb{N}^J$. For two quadratic forms q_1 and q_2 on Grad$(\mathcal{C})$, the category $(\mathcal{C}^{q_1})^{q_2}$ is the subcategory of $(\mathcal{C} \otimes \mathcal{C}_{Grad(\mathcal{C}),q_1}) \otimes \mathcal{C}_{Grad(\mathcal{C}),q_2}$, whose irreducible elements are $(j$, grad(j), grad$(j))$, $j \in J$.

By associativity of the category product and the fact that $g \hookrightarrow g \otimes g$ defines the inclusion of the subcategory, we have that

$$\mathcal{C}_{Grad(\mathcal{C}),(q_1+q_2)}\ \hookrightarrow\ \mathcal{C}_{Grad(\mathcal{C}),q_1}\ \otimes\ \mathcal{C}_{Grad(\mathcal{C}),q_2}\ ,$$

as in (7.369). This yields immediately the canonical isomorphism $(\mathcal{C}^{q_1})^{q_2} \cong \mathcal{C}^{q_1+q_2}$.

For the group of quadratic forms on the universal grading group of a fusion rule algebra, this procedure defines, therefore, a free action, $\mathcal{C} \rightarrow \mathcal{C}^q$, on the set of categories realizing this fusion rule algebra. The braid- and fusion matrices, r^q and F^q, of $\mathcal{C}^q$ can be given in terms of the original data as follows:

$$\begin{aligned} F^q(i,j,k,l) &= e^{-2\pi i\, f_q(grad(i), grad(j), grad(k))}\ F(i,j,k,l), \\ r^q(i,j,k) &= e^{2\pi i\, r_q(grad(i), grad(j))}\ r(i,j,k)\ , \end{aligned} \qquad (8.1.8)$$

and the statistical phases and dimensions of $\mathcal{C}^q$ are found from

$$\begin{aligned} d_j^q &= d_j, \\ \theta_j^q &= \theta_j\ +\ q(grad(j)) \bmod 1, \end{aligned} \qquad (8.1.9)$$

for all $j \in J$. In this formula, one application of our manipulations becomes apparent: Suppose $H \subset J$ is a subgroup of the set of invertible elements, Out$(\mathbb{N}^J)$, and grad:$H \hookrightarrow$ Grad$(\mathcal{C})$ is <u>injective</u>. The restriction of the category to $\mathbb{N}^H$ yields a θ-category and hence determines an element $\tilde{q} \in$ Hom$(\Gamma_4(H), \mathbb{R}/\mathbb{Z})$, where, by assumption, $\Gamma_4(H)$ is a subgroup of $\Gamma_4($Grad$(\mathcal{C}))$. For coefficients $\mathbb{R}/\mathbb{Z}$, the character $\tilde{q}$ can be extended to $\Gamma_4($Grad$(\mathcal{C}))$, i.e., to a quadratic form, q, on Grad$(\mathcal{C})$. If we started from $\mathcal{C}^{-q}$ the subcategory on H would be trivial, and, conversely, using that $(\mathcal{C}^{-q})^q = \mathcal{C}$, we can think of $\mathcal{C}$ as being included in the product of a category with the same fusion rules but trivial statistical phases for the objects in H, with a θ-category in which H is contained, too, but which carries the statistical phases given for $\mathcal{C}$. If H is a direct summand of Grad$(\mathcal{C})$ this θ-category can be assumed to consist of H only.

Next, we explain an important tool for the analysis of the gradation reduction of categories analogous to that for fusion rule algebras, namely induced categorial structures. To be more specific, we consider a fusion rule algebra epimorphism $\zeta : Obj_1 \twoheadrightarrow Obj_2$ and a category $\mathcal{C}_2$ with object set Obj_2. A category $\mathcal{C}_1$ with object set Obj_1 is then called <u>induced</u> by ζ and $\mathcal{C}_2$ if ζ extends to a tensor functor from $\mathcal{C}_1$ to $\mathcal{C}_2$.

In the following discussion we shall find conditions on ζ such that a unique, induced category $\mathcal{C}_1$ exists for every category $\mathcal{C}_2$, and we shall also determine those categories $\mathcal{C}_1$ which are induced by some $\mathcal{C}_2$, given ζ. The first simplification we make is to confine our attention to "irreducible" fusion rule algebra homomorphisms, meaning that ζ shall map irreducible objects to irreducible objects. In this case, $\zeta : \mathbb{N}^{J_1} \twoheadrightarrow \mathbb{N}^{J_2}$, is already given by $\zeta : J_1 \twoheadrightarrow J_2$. The structure of irreducible fusion rule algebra epimorphisms can be conveniently described as in the next lemma.

LEMMA 8.1.1

Suppose $\zeta : J_1 \rightarrow J_2$ *extends to an irreducible fusion rule algebra homomorphism, and let*

$$ker\,\zeta \;:=\; \{\sigma \in J_1 \;:\; \zeta(\sigma) = 1\}\,. \tag{8.1.10}$$

Then

(i) $ker\zeta$ *is a subgroup of invertible objects.*

(ii) *The action of* $ker\zeta$ *on* J_1 *by multiplication is free, and different orbits of* $ker\zeta$ *are mapped to different objects in* J_2.

(iii) *If* R *is a subgroup of invertible elements in a fusion rule algebra* $\mathbb{N}^J$ *which acts freely (by multiplication) on* J, *then the Perron-Frobenius algebra,* $\mathbb{N}^J/\mathbb{N}^R$, *(see section 3.2) is a fusion rule algebra,* $\mathbb{N}^{(J/R)}$, *where the irreducible objets,* J/R, *are the orbits of* R. *The projection* $\pi_R : J \rightarrow J/R$ *extends to an irreducible fusion rule algebra epimorphism.*

(iv) *For* ζ *as above, there exists an injection* $i : J_1/ker\,\zeta \hookrightarrow J_2$, *extending to a fusion rule algebra monomorphism, such that*

$$\zeta \;=\; i \,\circ\, \pi_{ker\,\zeta}. \tag{8.1.11}$$

Proof: We remark that, for fusion rule algebra homomorphisms ζ, with $\zeta(1) = 1$ and $\zeta(X^\vee) = \zeta(X)^\vee$ – in particular, for irreducible ones and ones that extend to tensor functors of categories – we have that $\zeta(X)$ is invertible (irreducible) only if X is already invertible (irreducible). To see this, we may write $X^\vee \circ X = Y + \|X\|^2 1$, so that $\zeta(X)^\vee \circ \zeta(X) = \zeta(Y) + \|X\|^2 1$. If $\zeta(X)$ is invertible we have that $\zeta(Y) = 0$ and $\|X\| = 1$. Hence $Y = 0$, and X is invertible. This immediately implies the assertion in i). Also, if $\zeta(j)$ is irreducible and $\zeta(i) = \zeta(j)$ then

$$\begin{aligned} 1 \;&=\; \|\zeta(j)\|^2 \;=\; \big(\zeta(j), \zeta(i)\big) \;=\; \varepsilon\big(\zeta(j \circ i^\vee)\big) \;= \\ &=\; \sum_{\sigma \in ker\zeta} N_{ji^\vee,\sigma} \;=\; \sum_{\sigma \in ker\zeta} N_{i\sigma,j} \;=\Big|\,\{\sigma \in ker\zeta \;:\; j = \sigma \circ i\}\,\Big|, \end{aligned} \tag{8.1.12}$$

where ε (the evaluation) is defined as in section 3.1.

This equation shows that two irreducible elements which are mapped by ζ to the same object differ by multiplication by an object in $ker\,\sigma$, (the converse being trivially true). Furthermore, the invertible object is unique, which implies statement ii).

In order to show iii), we use the definitions in Lemma 3.2.2, denoting by $[j] \in J/R$ (or $\mathcal{C}^{[j]} \subset J$) the orbit of $j \in J$ under the action of R. For the dimensions we find, with $\sigma \in R$, $j \in J$, that

$$d_{(\sigma \circ j)} \;=\; d_\sigma\, d_j \;=\; d_j \;=:\; d_{[j]}, \tag{8.1.13}$$

i.e., they depend only on orbits. Thus, the component of the dimension vector corresponding to an orbit $[j]$ is given by

$$\vec{d}^{[j]} \;=\; \sum_{j \in \mathcal{C}_{[j]}} d_j \phi_j \;=\; d_{[j]} \sum_{j \in \mathcal{C}_{[j]}} \phi_j\,, \tag{8.1.14}$$

which has the norm $\|\vec{d}^{[j]}\| = d_{[j]}\sqrt{|\,R\,|}$, since $|\,\mathcal{C}_{[j]}\,| = |\,R\,|$.

For the constants in (3.24), we thus obtain

$$\kappa_{[j]} \;=\; d_{[j]}. \tag{8.1.14a}$$

Using (8.1.13) and (8.1.14a) we see that the dimensions in (3.25) cancel and, by (3.29), we obtain for the fusion rules of $\mathbb{N}^J/\mathbb{N}^R$

$$\overline{N}_{[i][j],[k]} \;=\; \sum_{k \in \mathcal{C}_{[k]}} N_{ij,k} \tag{8.1.14b}$$

for arbitrary representatives $i \in C_{[i]}, j \in C_{[j]}$. Since these are integers, $\mathbb{N}^J/\mathbb{N}^R = \mathbb{N}^{(J/R)}$ is a fusion rule algebra.

With (3.26) and (3.27), we also find the corresponding vectors in $(\mathbb{R}^+)^J$

$$\vec{\delta}^{[j]} = \frac{1}{|R|} \sum_{j \in C_{[j]}} \phi_j = \phi_j \circ \vec{\delta}^{[1]}, \; j \in C_{[j]}. \tag{8.1.14c}$$

Clearly the projection $\pi_R : J \to J/R : j \to [j]$ extends to an irreducible fusion rule algebra epimorphism and $ker\,\pi_R = R$. The claim in iv) is a direct consequence of the previous statements.

□

Given an exact sequence of irreducible homomorphisms,

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook, "i"] & J \arrow[r, shift left, "\pi_R"] & \overline{J} \arrow[l, shift left, "\gamma"] \arrow[r] & 0, \end{tikzcd} \tag{8.1.15}$$

where R consists only of invertible objects, we can describe J, in analogy to groups, as an extension of $\overline{J}$ over R. For this purpose, we choose a map $\gamma : \overline{J} \to J$, with $\pi_R \circ \gamma = id$ and $\gamma([1]) = 1$. Then

$$\Gamma : \overline{J} \times R \to J, \quad \text{defined by } ([j], g) \mapsto \gamma([j]) \circ g, \tag{8.1.16}$$

is one to one, since R acts freely on J. The "cocycle" of the extension is given by a map

$$A : \overline{J}^3 \to \mathbb{N}^R : ([i],[j],[k]) \mapsto A_{[i][j],[k]}, \tag{8.1.17}$$

determined by

$$\gamma([i]) \circ \gamma([j]) = \sum_{[k] \in \overline{J}} A_{[i][j],[k]} \circ \gamma([k]), \tag{8.1.18}$$

using the isomorphism Γ of (8.1.16).

For the objects in (8.1.17) we infer the relations

$$A_{[i][j],[k]} = A_{[j][i],[k]}, \tag{8.1.18a}$$

$$A_{[j][j]^\vee,[1]} \in R, \; A_{[i][j]^\vee,[1]} = 0, \text{ for } [i] \neq [j], \tag{8.1.18b}$$

$$A_{[i][j][k],[l]} := \sum_{[s]} A_{[i][j],[s]} \circ A_{[s][k],[l]} = \sum_{[t]} A_{[i][t],[l]} \circ A_{[j][k],[t]}, \tag{8.1.18c}$$

and, furthermore,

$$\pi_R \left(A_{[i][j],[k]}\right) = \overline{N}_{[i][j],[k]} \cdot 1 \,. \tag{8.1.18d}$$

The data needed for the extension of $\overline{J}$ over R can thus be viewed as $\mathbb{N}^R$-valued (instead of $\mathbb{N}$-valued) <u>fusion rules</u>. Due to the ambiguity in our choice of γ, we have a natural notion of equivalence:

$$A \approx A' \quad \text{if and only if there exists a map} \quad \sigma : \overline{J} \to R,$$

with

$$A'_{[i][j],[k]} = \sigma([i]) \circ \sigma([j]) \circ \sigma([k])^{-1} A_{[i][j],[k]}. \tag{8.1.18e}$$

For example, the sequence (8.1.15) splits. In other words, $J = \overline{J} \times R$, as fusion rule algebras, and i, π_R are the canonical maps, iff $A \approx 1$.

Conversely, given $\overline{J}$ and R, a "cocycle" A as in (8.1.17), obeying the relations (8.1.18a), (8.1.18b) and (8.1.18c), defines a fusion rule algebra, $J = \overline{J} \times_A R$, which yields a sequence of homomorphisms as in (8.1.15), and the sequences for A and A' are isomorphic iff $A \approx A'$.

For an adequate definition of induced categories, it is necessary to impose an additional requirement on the fusion rule algebra homomorphisms that shall be considered. In order to arrive at such a definition, the following notion is useful: The free action of the subgroup of invertible elements R on J is called <u>coherent</u> iff the objects $A_{[i][j],[k]} \in \mathbb{N}^R$, as well as the objects $A_{[i][j][k],[l]} \in \mathbb{N}^R$ in (8.1.18c), are of the form $N\sigma$, where $N \in \mathbb{N}$ and $\sigma \in R$.

By (8.1.18d), this implies the existence of invertible objects $\sigma_{[i][j],[k]} \in R$, with

$$A_{[i][j],[k]} = \overline{N}_{[i][j],[k]} \, \sigma_{[i][j],[k]}, \tag{8.1.19}$$

and the constraints (8.1.18a)-(8.1.18c) reduce to

$$\sigma_{[i][j],[k]} = \sigma_{[j][i],[k]} \tag{8.1.20}$$

and

$$\sigma_{[i][j],[s]} \circ \sigma_{[s][k],[l]} = \sigma_{[i][t],[l]} \circ \sigma_{[j][k],[t]} \tag{8.1.21}$$

where the right hand side (the left hand side) of (8.1.21) is independent of $[t]$ ($[s]$, resp.), as long as the fusion rules are obeyed. We say that an irreducible fusion rule algebra homomorphism, $\zeta : J_1 \rightarrow J_2$, is coherent if $ker\zeta$ has a free, coherent action on J_1. We wish to express this property in a second, different way:

For a given $\gamma : \overline{J} \rightarrow J$, we introduce a function

$$\psi_{[k]} \,:\, J \times J \rightarrow \mathcal{C}_{[k]}$$

with ($\mathcal{C}_{[k]} = \pi_R^{-1}([k])$, as in section 3.2) by setting

$$\psi_{[k]}(\gamma([i]), \gamma([j])) = \sigma_{[i][j],[k]} \circ \gamma([k]), \tag{8.1.22}$$

and requiring the covariance condition

$$\psi_{[k]}(\sigma \circ i, \mu \circ j) = \sigma \circ \mu \circ \psi_{[k]}(i,j), \quad \forall \sigma, \mu \in R. \tag{8.1.23}$$

These functions relate the fusion rules of J and $\overline{J}$ through the equation

$$N_{ij,k} = \delta_{k,\psi_{[k]}(i,j)} \overline{N}_{[i][j],[k]}. \tag{8.1.24}$$

It follows that the restriction

$$\zeta \,:\, \mathrm{supp}(i \circ j) \hookrightarrow \mathrm{supp}([i] \circ [j]) \tag{8.1.25}$$

is injective. The fusion rule relations accompanying these functions are as follows:

$$\psi_{[k]}(i,j) = \psi_{[k]}(j,i), \tag{8.1.26}$$

and

$$\psi_{[l]}(\psi_{[s]}(i,j),k) = \psi_{[l]}(i,\psi_{[t]}(j,k)) =: \psi_{[l]}(i,j,k), \tag{8.1.27}$$

where the objects in (8.1.27) are independent of $[s]$ and $[t]$.

In our applications we shall encounter only a special case of irreducible, coherent homomorphisms, $\zeta : J_1 \rightarrow J_2$, namely graded ones. They are characterized by the property that

$$grad_1 \,:\, ker\zeta \hookrightarrow Grad(\mathbb{N}^{J_1}) \tag{8.1.28}$$

is an injection, or, equivalently, that ζ is a fusion rule algebra monomorphism if restricted to the trivially graded subalgebra, $(\mathbb{N}^{J_1})_0$. It easily follows from (8.1.28) that graded, irreducible fusion rule algebra homomorphisms are coherent, and

$$grad_1(\sigma_{[i][j],[k]}) = grad_1(\gamma([i]))\, grad_1(\gamma([j]))\, grad_1(\gamma([k]))^{-1}\,. \tag{8.1.29}$$

We note that, for any irreducible graded homomorphism ζ, there exists a unique group homomorphism, $\zeta^{\#}$, such that the diagram

$$\begin{tikzcd} J_1 \arrow[r, "\zeta"] \arrow[d, "grad_1"'] & J_2 \arrow[d, "grad_2"] \\ Grad(\mathbb{N}^{J_1}) \arrow[r, "\zeta^{\#}"'] & Grad(\mathbb{N}^{J_2}) \end{tikzcd} \tag{8.1.30}$$

commutes. Moreover, if ζ maps onto the trivially graded component $(\mathbb{N}^{J_2})_0$ then we have that

$$ker\,\zeta^{\#} = grad_1(ker\,\zeta). \tag{8.1.31}$$

Hence, for a graded π_R, we have the exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook] & Grad(\mathbb{N}^J) \arrow[r, two heads, "\pi_R^{\#}"] & Grad(\mathbb{N}^J) \arrow[r] & 0 \end{tikzcd} \tag{8.1.32}$$

With (8.1.15), it follows immediately that two graded extensions (8.1.15) are equivalent if and only if the corresponding sequences (8.1.32) are equivalent. (In particular, (8.1.15) splits iff (8.1.32) splits).

For irreducible, coherent fusion rule algebra homomorphism, the existence of corresponding induced categories is guaranteed by the following proposition.

PROPOSITION 8.1.2

Suppose that $\zeta : Obj_1 \rightarrow Obj_2$ is an irreducible, coherent fusion rule algebra homomorphism, and that C_2 is a quantum category with object set Obj_2.

(i) *Then there exists a category C_1, unique up to natural isomorphism, whose object set is Obj_1 and for which ζ can be extended to a tensor functor from C_1 to C_2.*

(ii) *The θ-subcategory of C_1, given by the fusion rule subalgebra $\mathbb{N}^{ker\zeta} \subset Obj_1$, is trivial.*

Proof.

We first comment on some properties of a general tensor functor, $(\zeta, \mathcal{F}, C)$, extending an irreducible fusion rule algebra homomorphism ζ. By $\mathcal{F} : Mor_1(X,Y) \rightarrow Mor_2(\zeta(X), \zeta(Y))$, we denote the map between morphisms with the properties that $\mathcal{F}(\mathbb{1}_X) = \mathbb{1}_{\zeta(X)} \in End_2(\zeta(X))$ and that, for the isomorphisms $C(X,Y) \in Mor_2(\zeta(X) \circ \zeta(Y), \zeta(X \circ Y))$,

$$\mathcal{F}(I \circ J)\, C(X,Y) \;=\; C(X',Y')\, (\mathcal{F}(I) \circ \mathcal{F}(J))\,, \tag{8.1.33}$$

for arbitrary $I \in Mor_1(X,X')$ and $J \in Mor_1(Y,Y')$. For the restrictions

$$\mathcal{F} \,:\, \bigoplus\nolimits_{k \in C_{[k]}} Mor_1(k,X) \;\longrightarrow\; Mor_2([k], \zeta(X)), \tag{8.1.34}$$

and

$$\mathcal{F} \,:\, \bigoplus\nolimits_{k \in C_{[k]}} Mor_1(X,k) \;\longrightarrow\; Mor_2(\zeta(X), [k]), \tag{8.1.35}$$

we note that the spaces on the left hand sides (right hand sides) of (8.1.34) and (8.1.35) are dual to each other by multiplication on X (on $\zeta(X)$). From the functoriality of $\mathcal{F}$ and the fact that $\mathcal{F}(\mathbb{1}_k) = \mathbb{1}_{[k]}$ it follows that the maps in (8.1.34) and (8.1.35) preserve the contraction and are thus injective. To the decomposition of the semisimple algebras $End_1(X) = \bigoplus_{k \in Obj_1} End(X)_k$ and $End_2(\zeta(X)) = \bigoplus_{[k] \in Obj_2} End_2(\zeta(X))_{[k]}$ into sums of simple algebras ($\overset{\cong}{\text{e.g.}}$ $Mat_{N_{X,k}}(\mathbb{C})$), according to the channels k, we can associate a partition of $\mathbb{1}_X$ and $\mathbb{1}_{\zeta(X)}$ into minimal central projections, $\pi_k(X) \in End_1(X)$ and $\pi_{[k]}(\zeta(X)) \in End_2(\zeta(X))$. Using the fact that the representation of $End_1(X)$ on the space $\bigoplus_k Mor(k,X)$ by multiplication on X is faithful, we find from injections (8.1.34) and (8.1.35) that

$$\mathcal{F} \,:\, \bigoplus\nolimits_{k \in C_{[k]}} End_1(X)_k \;\longrightarrow\; End_2\big(\zeta(X)\big)_{[k]} \tag{8.1.36}$$

is an inclusion of algebras. For $im\,\mathcal{F} \subset Mor_2(\zeta(X), [k])$, we also have that $\mathcal{F}(a)I = 0$ if $a \in End_1(X)$ and $I \in (im\,\mathcal{F})^{\perp} \subset Mor_2([k], \zeta(X))$. But since we require that $\mathcal{F}(\mathbb{1}_X) = \mathbb{1}_{\zeta(X)}$, it follows that $(im\,\mathcal{F})^{\perp} = 0$. Hence the maps $\mathcal{F}$ in (8.1.34) and (8.1.35) are, in fact, isomorphisms. The induced direct sum decomposition of $Mor_2([k], \zeta(X))$ is given by a refinement of the partition of unity,

$$\pi_{[k]}(\zeta(X)) \;=\; \sum_{k \in C_{[k]}} \bar{\pi}_k(X), \tag{8.1.37}$$

where we define $\bar{\pi}_k(X) = \mathcal{F}(\pi_k(X)) \in End_2(\zeta(X))_{[k]}$. Counting ranks and dimensions we recover the equation

$$N_{\zeta(X),[k]} \;=\; \sum_{k \in C_{[k]}} N_{X,k}. \tag{8.1.38}$$

Similar to the "*End*-spaces", the "*Mor*-spaces" can be decomposed according to channels given by $k \in J$, and we have an injection

$$\mathcal{F} \,:\, \bigoplus\nolimits_{k \in C_{[k]}} Mor_1(X,Y)_k \;\hookrightarrow\; Mor_2\big(\zeta(X), \zeta(Y)\big)_{[k]} \tag{8.1.39}$$

for all $X, Y \in Obj_1$. As a consequence of semisimplicity, the image of the map in (8.1.39) is given by

$$im\,\mathcal{F} \;=\; \big\{ I \in Mor_2\big(\zeta(X), \zeta(Y)\big) \;:\; \bar{\pi}_k(Y) I \;=\; I \bar{\pi}_k(X),\ \forall\, k \in C_{[k]} \big\}. \tag{8.1.40}$$

The compatibility of these decompositions with the tensor product is expressed by the formula

$$\bar{\pi}_k(X \circ Y) \;=\; \sum_{i,j \,:\, \psi_{[k]}(i,j)=k} \pi_{[k]}\big(\zeta(X \circ Y)\big)\, C(X,Y) \big(\bar{\pi}_i(X) \circ \bar{\pi}_j(Y)\big)\, C(X,Y)^{-1}, \tag{8.1.41}$$

where the functions $\psi_{[k]} : J \times J \to \mathcal{C}_{[k]}$ are defined in (8.1.22), (8.1.23), for coherent homomorphisms. The image of the (i,j)-th projection in the sum (8.1.41) in its representation on $Mor_2\big([k], \zeta(X \circ Y)\big)$ is given by the image of

$$Mor_1(i,X) \otimes Mor_1(j,Y) \otimes Mor_2([k],[i]\circ[j]) \hookrightarrow Mor_2\big([k],\zeta(X \circ Y)\big)$$
$$I \otimes J \otimes K \to \mathcal{F}(I \circ J)K \qquad (8.1.42)$$

and has dimension $N_{X,i}N_{Y,j}\overline{N}_{[i][j],[k]} = N_{X,i}N_{Y,j}N_{ij,\psi_{[k]}(i,j)}$. Summation over i and j yields $N_{X\circ Y,k} = \sum_{ij} N_{X,i}N_{Y,j}N_{ij,k} = \sum_{ij} N_{X,i}N_{Y,j}\overline{N}_{[i][j],[k]}\,\delta_{k,\psi_{[k]}(ij)}$ as the total rank of $\bar{\pi}_k(X \circ Y)$. As for general tensor functors, the braid- and associativity isomorphisms are related by

$$\alpha_2\big(\zeta(X),\zeta(Y),\zeta(Z)\big) =$$
$$= \big(C(X,Y)^{-1} \circ \mathbb{I}\big)C(X\circ Y,Z)^{-1}\mathcal{F}\big(\alpha_1(X,Y,Z)\big)C(X,Y\circ Z)\big(\mathbb{I}\circ C(Y,Z)\big) \qquad (8.1.43)$$

and

$$\epsilon_2\big(\zeta(X),\zeta(Y)\big) = C(Y,X)^{-1}\mathcal{F}\big(\epsilon_1(X,Y)\big)C(X,Y). \qquad (8.1.44)$$

For the proof of existence and uniqueness of induced categories it is useful to introduce, for a (not necessarily irreducible) fusion rule algebra homomorphism $\zeta : Obj_1 \to Obj_2$, the natural notion of a pulled back category, $\mathcal{C}_2^\zeta$, where $\mathcal{C}_2$ is an arbitrary braided tensor category with object set Obj_2: The object set of $\mathcal{C}_2^\zeta$ is given by Obj_1, with the same tensor product. The morphism spaces of $\mathcal{C}_2^\zeta$ are defined in such a way that there are isomorphisms:

$$\mathcal{D} : Mor_2^\zeta(X,Y) \xrightarrow{\cong} Mor_2\big(\zeta(X),\zeta(Y)\big), \quad \forall\, X,Y \in Obj_1. \qquad (8.1.45)$$

The composition- and tensor-products of morphisms are defined to be the ones induced by $\mathcal{D}$, and the braid- and monoidal isomorphisms are given by $\epsilon_2^\zeta(X,Y) := \epsilon_2\big(\zeta(X),\zeta(Y)\big)$ and $\alpha_2^\zeta(X,Y,Z) := \alpha_2\big(\zeta(X),\zeta(Y),\zeta(Z)\big)$.

Note that, in contrast to the categories $\mathcal{C}_1$ and $\mathcal{C}_2$, there exist, in $\mathcal{C}_2^\zeta$, pairs of different objects which are equivalent. More precisely, $X \approx Y$, in $\mathcal{C}_2^\zeta$, iff $\zeta(X) = \zeta(Y)$ in Obj_2,

since for such objects $Mor_2^\zeta(X,X) \cong End\big(\zeta(X)\big)$ contains the isomorphism $\mathcal{D}^{-1}(\mathbb{I}_{\zeta(X)})$. The equivalence classes of objects in $\mathcal{C}_2^\zeta$ are identified with $im\,\zeta \cong Obj_2$.

The two categories are related by a tensor functor

$$(\zeta,\mathcal{D},\mathbb{I}) : \mathcal{C}_2^\zeta \longrightarrow \mathcal{C}_2.$$

This allows us to factor any tensor functor, $(\zeta,\mathcal{F},C) : \mathcal{C}_1 \to \mathcal{C}_2$, by the unique functor $(id,\widetilde{\mathcal{F}},\widetilde{C}) : \mathcal{C}_1 \to \mathcal{C}_2^\zeta$, such that the diagram

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[rr, "{(id,\widetilde{\mathcal{F}},\widetilde{C})}"] \arrow[dr, "{(\zeta,\mathcal{F},C)}"'] & & \mathcal{C}_2^\zeta \arrow[dl, "{(\zeta,\mathcal{D},\mathbb{I})}"] \\
& \mathcal{C}_2 &
\end{tikzcd} \qquad (8.1.46)$$

commutes.

To prove uniqueness, we show that, to every pair of categories, $\mathcal{C}_1$ and $\mathcal{C}_1'$, with functors $(\zeta,\mathcal{F},C)$ and $(\zeta,\mathcal{F}',C')$ to $\mathcal{C}_2$, one can associate isomorphisms $(id,\mathcal{G}_1,A) : \mathcal{C}_1 \to \mathcal{C}_1'$ and $(id,\mathcal{G}_2,B) : \mathcal{C}_2^\zeta \to \mathcal{C}_2^\zeta$ such that the following diagram commutes:

$$\begin{tikzcd}
\mathcal{C}_1 \arrow[r, "{(id,\widetilde{\mathcal{F}},\widetilde{C})}"] \arrow[d, "{(id,\mathcal{G}_1,A)}"'] & \mathcal{C}_2^\zeta \arrow[d, "{(id,\mathcal{G}_2,B)}"] \\
\mathcal{C}_1' \arrow[r, "{(id,\widetilde{\mathcal{F}}',\widetilde{C}')}"] & \mathcal{C}_2^\zeta
\end{tikzcd} \quad , \qquad (8.1.47)$$

For the endomorphism algebras in $\mathcal{C}_2^\zeta$ we have the decomposition into simple subalgebras, $End_2^\zeta(X) = \bigoplus_{[k]\in im\,\zeta} End_2^\zeta(X)_{[k]}$, induced by $\mathcal{D}$, with minimal, central projections $\pi_{[k]}(X) := \mathcal{D}^{-1}\big(\pi_{[k]}(\zeta(X))\big)$. The refinement of the partition of unity, analogous to (8.1.37), is given by the projections $\bar{\pi}_k(X) := \widetilde{\mathcal{F}}\big(\pi_k(X)\big) = \mathcal{D}^{-1}\big(\bar{\pi}_k(X)\big) \in End_2^\zeta(X)_{[k]}$. The equation (8.1.41) for products also holds true in $End_2^\zeta(X \circ Y)_{[k]}$.

We now first determine the functor $(id,\mathcal{G}_2,B)$ of $\mathcal{C}_2^\zeta$ onto itself. A large class of such functors, exhaustive for θ-categories and most other examples in this work, is given by the “coboundaries” of a set of isomorphisms, $U(X) \in End_2^{\zeta}(X)$, $X \dot{\in} Obj_1$:

$$\mathcal{G}_2(I) := U(Y)\, I\, U(X)^{-1}, \quad I \in Mor_2^{\zeta}(X,Y),$$

and

$$B(X,Y) := U(X \circ Y)\, U(X)^{-1} \circ U(Y)^{-1}. \tag{8.1.48}$$

One easily verifies (8.1.33), (8.1.43) and (8.1.44), with $\mathcal{F} = \mathcal{G}_2$, $C = B$, $\zeta = id$, $\varepsilon_2 = \varepsilon_1 = \varepsilon_2^{\zeta}$, $\alpha_2 = \alpha_1 = \alpha_2^{\zeta}$. As in (8.1.40), we have that

$$\tilde{\mathcal{F}}(Mor_1(X,Y)) = \{I \in Mor_2^{\zeta}(X,Y) : \tilde{\pi}_k(Y) I = I \tilde{\pi}_k(X), \quad \forall\, k \in C_{[k]}\} \tag{8.1.49}$$

and similarly for $\tilde{\mathcal{F}}'(Mor_1(X,Y))$. Since, for a given $X \in Obj_1$ and $[k] \in im\,\zeta$, $\tilde{\pi}_k$ and $\tilde{\pi}'_k$, $k \in C_{[k]}$, form partitions of unity in $End_2^{\zeta}(X)_{[k]}$ of equal rank, there exist invertible maps $U(X)$ such that

$$U(X)\tilde{\pi}_k(X)U(X)^{-1} = \tilde{\pi}'_k(X), \quad \forall\, k \in J_1. \tag{8.1.50}$$

For a functor $(id, \mathcal{G}_2, B)$ defined, as in (8.1.48), for a collection of isomorphisms $U(X)$ satisfying (8.1.50), we immediately find from (8.1.49) that

$$\mathcal{G}_2 : \tilde{\mathcal{F}}(Mor_1(X,Y)) \xrightarrow{\cong} \tilde{\mathcal{F}}'(Mor'_1(X,Y)),$$

i.e., that $\mathcal{G}_2$ provides an isomorphism between the images of $\tilde{\mathcal{F}}$ and $\tilde{\mathcal{F}}'$, for any given pair $X, Y \in Obj_1$. This shows that the map $\mathcal{G}_1 : Mor_1(X,Y) \xrightarrow{\cong} Mor'_1(X,Y)$ is well defined and unique if (8.1.47) is required to commute, in the sense of abelian categories. In order to examine the tensor product structure, we consider the endomorphisms

$$a(X,Y) := U(X \circ Y)\, \tilde{C}(X,Y)\, U(X)^{-1} \circ U(Y)^{-1}\, \tilde{C}'(X,Y)^{-1} \tag{8.1.51}$$

in $End_2^{\zeta}(X \circ Y)$. Using the decomposition (8.1.41) for $\tilde{\pi}_k(X \circ Y)$ and $\tilde{\pi}'_k(X \circ Y)$, it is a straightforward computation to show that

$$a(X,Y)\, \tilde{\pi}'_k(X \circ Y) = \tilde{\pi}'_k(X \circ Y)\, a(X,Y). \tag{8.1.52}$$

Hence, by (8.1.49), there exists a unique isomorphism $A(X,Y) \in End'_1(X \circ Y)$ such that

$$\tilde{\mathcal{F}}'(A(X,Y)) = a(X,Y). \tag{8.1.53}$$

For the functor $(id, \mathcal{G}_1, A) : \mathcal{C}_1 \to \mathcal{C}'_1$, properties (8.1.33), (8.1.43) and (8.1.44) are then verified by a computation, without difficulty. This proves the uniqueness of induced categories. For the proof of existence, we again consider the pull back, $\mathcal{C}_2^{\zeta}$, of $\mathcal{C}_2$ with respect to $\zeta : Obj_1 \to Obj_2$. In our previous discussion, we remarked that, for the minimal, central projections of $End_2^{\zeta}(X) = \oplus_{[k]} End_2^{\zeta}(X)_{[k]}$, we can express the rank in terms of the multiplicities of $X \in Obj_1$ by

$$rk\big(\pi_{[k]}(X)\big) = \sum_{k \in C_{[k]}} N_{X,k}\,. \tag{8.1.54}$$

An induced category $\mathcal{C}_1$ can now be defined, for any partition of unity in $End_2^{\zeta}(X)_{[k]}$ as in (8.1.37), provided the projections, $\pi_k(X), k \in J_1$, satisfy the condition

$$rk\big(\pi_k(X)\big) = N_{X,k}\,. \tag{8.1.55}$$

By (8.1.54), we can always find such a partition.

The morphism spaces of $\mathcal{C}_1$ are then defined by

$$Mor_1(X,Y) := \{I \in Mor_2^{\zeta}(X,Y) : I\pi_k(X) = \pi_k(Y)I,\ \forall k \in C_{[k]}\}. \tag{8.1.56}$$

They obviously close under the composition induced by $\mathcal{C}_2^{\zeta}$. The projections π_k yield a direct sum decomposition,

$$Mor_2^{\zeta}(k,X) = Mor_2([k],\zeta(X)) \cong \bigoplus_{k \in C_{[k]}} Mor_1(k,X),$$

which must be preserved by any morphism. Hence $X \approx Y$ in $\mathcal{C}_1$ iff $dim\big(Mor_1(k,X)\big) = dim\big(Mor_1(k,Y)\big)$ which holds iff $N_{X,k} = N_{Y,k}$, $\forall k \in J_1$, i.e., iff $X = Y$.

For tensor products, the decomposition of $Mor_2^\zeta(k, X \circ Y)$ can be written as follows, using a natural isomorphism, as in (8.1.42):

$$\begin{aligned} Mor_2^\zeta(k, X \circ Y) &= Mor_2\big([k], \zeta(X \circ Y)\big) \\ &\cong \bigoplus_{[i],[j] \in im\,\zeta} Mor_2\big([i], \zeta(X)\big) \otimes Mor_2\big([j], \zeta(Y)\big) \otimes Mor_2\big([k], [i] \circ [j]\big) \\ &\cong \bigoplus_{i,j \in J_1} Mor_1(i, X) \otimes Mor_1(j, Y) \otimes Mor_1\big(\psi_{[k]}(i,j), i \circ j\big) \\ &\cong \bigoplus_{k' \in C_{[k]}} \bigoplus_{i,j:\psi_{[k]}(i,j)=k'} Mor_1(i, X) \otimes Mor_1(j, Y) \otimes Mor_1(k', i \circ j), \end{aligned} \tag{8.1.57}$$

and the projection on the k'-th summand in (8.1.57) is given by

$$\pi^0_{k'}(X,Y) \;=\; \pi_{[k]}(X \circ Y) \sum_{i,j:\psi_{[k]}(i,j)=k'} \pi_i(X) \circ \pi_j(Y). \tag{8.1.58}$$

Its rank is given by $\sum_{ij} N_{X,i} N_{Y,j} N_{ij,k'} = N_{X \circ Y, k'}$. It is thus equal to the rank of $\pi_{k'}(X \circ Y)$. Hence there exist isomorphisms $C(X,Y)_{[k]} \in End_2^\zeta(X \circ Y)_{[k]}$, and therefore isomorphisms $C(X,Y) = \oplus_{[k]} C(X,Y)_{[k]} \in End_2^\zeta(X \circ Y)$, such that

$$C(X,Y)\pi^0_k(X,Y)C(X,Y)^{-1} \;=\; \pi_k(X \circ Y), \;\; \forall\, k \in J_1; \tag{8.1.59}$$

compare to (8.1.41). Now we may define the tensor product of morphisms $I \in Mor_1(X, X')$, $J \in Mor_1(Y, Y')$:

$$I \circ_1 J \;:=\; C(X', Y')(I \circ J)C(X,Y)^{-1}. \tag{8.1.60}$$

By (8.1.58) and (8.1.59), $I \circ_1 J$ lies in $Mor_1(X \circ Y, X' \circ Y')$, as defined in (8.1.56). Furthermore, we define braiding and associativity isomorphisms in $Mor_2^\zeta(X \circ Y, Y \circ X)$ and in $Mor_2^\zeta(X \circ (Y \circ Z), (X \circ Y) \circ Z)$ by setting

$$\varepsilon_1(X,Y) \;:=\; C(Y,X)\; \varepsilon_2^\zeta(X,Y)\, C(X,Y)^{-1}, \tag{8.1.61}$$

and

$$\alpha_1(X,Y,Z) := C(X \circ Y, Z)\big(C(X,Y) \circ \mathbb{I}\big)\alpha_2^\zeta(X,Y,Z)\big(\mathbb{I} \circ C(Y,Z)^{-1}\big)C(X, Y \circ Z)^{-1}. \tag{8.1.62}$$

From (8.1.59) we immediately find that

$$\varepsilon_1(X,Y)\,\pi_k(X \circ Y) \;=\; \pi_k(Y \circ X)\,\varepsilon_1(X,Y), \quad \text{i.e.,}\;\; \varepsilon_1(X,Y) \in Mor_1(X \circ Y, Y \circ X),$$

using (8.1.26). Condition (8.1.27) is needed to prove an analogous property for α_1. Using (8.1.27) and applying (8.1.59) repeatedly, we find that the projections $\pi_k\big((X \circ Y) \circ Z\big)$ and $\pi_k\big(X \circ (Y \circ Z)\big)$ are given by

$$\pi_k\big((X \circ Y) \circ Z\big) = C(X \circ Y, Z)\big(C(X,Y) \circ \mathbb{I}\big)\pi^0_k(X, Y \mid Z)\big(C(X,Y)^{-1} \circ \mathbb{I}\big)\; C(X \circ Y, Z)^{-1}, \tag{8.1.63}$$

with

$$\pi^0_k(X, Y \mid Z) \;=\; \sum_{\substack{r,s,j\,: \\ \psi_{[k]}(r,s,j)=k}} \pi_{[k]}\big((X \circ Y) \circ Z\big)\; \pi_r(X) \circ \pi_s(Y) \circ \pi_j(Z),$$

and

$$\pi_k\big(X \circ (Y \circ Z)\big) \;=\; C(X, Y \circ Z)\big(\mathbb{I} \circ C(Y,Z)\big)\; \pi^0_k(X \mid Y, Z)\big(\mathbb{I} \circ C(Y,Z)^{-1}\big)\; C(X, Y \circ Z)^{-1}, \tag{8.1.64}$$

with

$$\pi^0_k(X \mid Y, Z) \;=\; \sum_{\substack{r,s,j\,: \\ \psi_{[k]}(r,s,j)=k}} \pi_{[k]}\big(X \circ (Y \circ Z)\big)\; \pi_r(X) \circ \pi_s(Y) \circ \pi_j(Z).$$

Clearly we have that $\alpha_2^\zeta(X,Y,Z)\pi^0_k(X \mid Y, Z) = \pi^0_k(X, Y \mid Z)\alpha_2^\zeta(X,Y,Z)$, so that $\alpha_1(X,Y,Z)\pi_k\big(X \circ (Y \circ Z)\big) = \pi_k\big((X \circ Y) \circ Z\big)\; \alpha_1(X,Y,Z)$, and hence it follows that $\alpha_1(X,Y,Z) \in Mor_1\big(X \circ (Y \circ Z), (X \circ Y) \circ Z\big)$.

On the category $\mathcal{C}_1$ constructed from these data, we have a tensor functor to $\mathcal{C}_2$:

$$(\zeta, \mathcal{D}, C) \;:\; \mathcal{C}_1 \longrightarrow \mathcal{C}_2\,, \tag{8.1.65}$$

where $\mathcal{D}$ is the restriction of the morphism map in (8.1.45) to the subspaces $Mor_1(X,Y) \subset Mor_2^\zeta(X,Y)$. This completes the proof of assertion i) of Proposition 8.1.2.

In order to prove part ii) of Proposition 8.1.2, i.e., triviality of the θ-category associated with $ker\zeta \subset J_1$, we establish an explicit relation among the braid-matrices of the two categories. Recall that, for $k \neq \psi_{[k]}(i,j)$, we have that $Mor_1(k, i \circ j) = 0$. Hence, by (8.1.34), there is an isomorphism

$$\begin{aligned} H^{ij}_{[k]} \; : \; Mor_1\big(\psi_{[k]}(i,j), i \circ j\big) &\xrightarrow{\cong} Mor_2\big([k], [i] \circ [j]\big), \\ I &\longrightarrow C(i,j)^{-1}\, \mathcal{F}(I)\,. \end{aligned} \tag{8.1.66}$$

For the braid matrices r, given by

$$r_1(i,j,k) \;:\; Mor_1(k, i \circ j) \longrightarrow Mor_1(k, j \circ i) \;:\; I \rightarrow \varepsilon_1(i,j) I, \tag{8.1.67}$$

and similarly for $r_2\big([i],[j],[k]\big)$, the following diagram commutes:

$$\begin{tikzcd} Mor_1(k, i \circ j) \arrow[r, "r_1(i,j,k)"] \arrow[d, "H^{ij}_{[k]}"] & Mor_1(k, j \circ i) \arrow[d, "H^{ji}_{[k]}"'] \\ Mor_2([k], [i] \circ [j]) \arrow[r, "r_2([i],[j],[k])"] & Mor_2([k], [j] \circ [i]) \end{tikzcd} \tag{8.1.68}$$

Since $r_2(1,1,1) = r_2\big([\sigma],[\sigma],[\sigma^2]\big) = 1$, it follows that $r_1(\sigma,\sigma,\sigma^2) = e^{2\pi i\theta(\sigma)} = 1$, for all $\sigma \in ker\zeta$. Here θ is the quadratic form which, by Proposition 7.4.3, determines the category of $ker\zeta$ uniquely. Thus $\theta = 0$ mod 1, and this implies part ii) of Proposition 8.1.2.

□

As a supplement to our discussion of braid matrices presented in the proof of Proposition 8.1.2, we wish to give the explicit relations between the fusion matrices F_1 and F_2, for the case that $\mathcal{C}_1$ is induced by $\mathcal{C}_2$. Since the fusion rule algebra homomorphism $\zeta : Obj_1 \rightarrow Obj_2$ is assumed to be coherent, we have that $Mor_1(l, i \circ j \circ k) = 0$, for $i,j,k,l \in J_1$, unless $l = \psi_{[l]}(i,j,k)$. In this case, we infer from (8.1.34) that there are two natural isomorphisms

$$P^{i(jk)}_{[l]}, P^{(ij)k}_{[l]} \;:\; Mor_1\big(\psi_{[l]}(i,j,k), i \circ j \circ k\big) \xrightarrow{\cong} Mor_2\big([l], [i] \circ [j] \circ [k]\big),$$

defined by

$$P^{i(jk)}_{[l]}(I) \;:=\; \big(\mathbb{I} \circ C(j,k)^{-1}\big)\, C(i, j \circ k)^{-1} \mathcal{F}(I), \tag{8.1.69}$$

and

$$P^{(ij)k}_{[l]}(I) \;:=\; \big(C(i,j)^{-1} \circ \mathbb{I}\big)\, C(i \circ j, k)^{-1} \mathcal{F}(I). \tag{8.1.70}$$

We introduce the following notation for the usual isomorphisms decomposing the space $Mor(l, i \circ j \circ k)$ into the basic spaces $Mor(k, i \circ j)$:

$$\begin{aligned} \mu^{i(jk)}_1 \;:\; \bigoplus{}_s\, Mor_1(s, j \circ k) \otimes Mor_1(l, i \circ s) \;&\rightarrow\; Mor_1(l, i \circ j \circ k) \\ I \,\otimes\, J \;&\mapsto\; (\mathbb{I} \circ I) J, \end{aligned} \tag{8.1.71}$$

and

$$\begin{aligned} \mu^{(ij)k}_1 \;:\; \bigoplus{}_s\, Mor_1(s, i \circ j) \otimes Mor_1(l, s \circ k) \;&\rightarrow\; Mor_1(l, i \circ j \circ k) \\ I \,\otimes\, J \;&\mapsto\; (I \circ \mathbb{I}) J. \end{aligned} \tag{8.1.72}$$

The isomorphisms $\mu_2^{[i]([j][k])}$ and $\mu_2^{([i][j])[k]}$ are defined similarly. The decomposed spaces on the left hand sides of (8.1.71) and (8.1.72) associated with the two categories $\mathcal{C}_1$ and $\mathcal{C}_2$ can be related to each other directly by using the isomorphisms given in (8.1.66). By (8.1.27), we can write, for $l = \psi_{[l]}(i,j,k)$:

$$\begin{aligned} H^{\otimes 2} : & \bigoplus_s Mor_1(s, j \circ k) \otimes Mor_1(l, i \circ s) \\ & \quad = \bigoplus_{[s] \in im\zeta} Mor_1\big(\psi_{[s]}(j,k), j \circ k\big) \otimes Mor_1\big(\psi_{[l]}\big(i, \psi_{[s]}(j,k)\big), i \circ \psi_{[s]}(j \circ k)\big) \\ & \quad \xrightarrow{\bigoplus_{[s]} H^{jk}_{[s]} \otimes H^{i\psi_{[s]}(jk)}_{[l]}} \bigoplus_{[s] \in im\zeta} Mor_2\big([s], [j] \circ [k]\big) \otimes Mor_2\big([l], [i] \circ [s]\big) \end{aligned} \tag{8.1.73}$$

which provides an isomorphism that factors. On the decomposition given in (8.1.72) $H^{\otimes 2}$ is defined in the same way. We consider the following diagram of isomorphisms, assuming that $l = \psi_{[l]}(i,j,k)$:

$$\begin{tikzcd}
\bigoplus_{s} Mor_1(s, j \circ k) \otimes Mor_1(l, i \circ s) \arrow[r, "F_1(i,j,k,l)"] \arrow[d, "\mu_1^{i(jk)}"] \arrow[ddd, bend right=60, "H^{\otimes 2}"'] & \bigoplus_{s} Mor_1(s, i \circ j) \otimes Mor_1(l, s \circ k) \arrow[d, "\mu_1^{(ij)k}"] \arrow[ddd, bend left=60, "H^{\otimes 2}"] \\
Mor_1(l, i \circ (j \circ k)) \arrow[r, "\alpha_1(i,j,k)"] \arrow[d, "p^{i(jk)}"] & Mor_1(l, (i \circ j) \circ k) \arrow[d, "p^{(ij)k}"] \\
Mor_2([l], [i] \circ ([j] \circ [k])) \arrow[r, "\alpha_2([i],[j],[k])"] & Mor_2([l], ([i] \circ [j]) \circ [k]) \\
\bigoplus_{[s]} Mor_2([s], [j] \circ [k]) \otimes Mor_2([l], [i] \circ [s]) \arrow[r, "F_2([i],[j],[k],[l])"] \arrow[u, "\mu_2^{[i]([j][k])}"] & \bigoplus_{[s]} Mor_2([s], [i] \circ [j]) \otimes Mor_2([l], [s] \circ [k]) \arrow[u, "\mu_2^{([i][j])[k]}"']
\end{tikzcd}$$

(8.1.74)

Here the squares on top and at the bottom of the diagram commute as a consequence of the definition of F-matrices. From (8.1.43) we find that the square in the center commutes, where α_1 and α_2 act on $i \circ (j \circ k)$ and $[i] \circ ([j] \circ [k])$, respectively. Commutativity of the squares on the left and on the right of (8.1.74) can be verified by a direct computation. We summarize the resulting relations between the fusion matrices F_1 and F_2 in the formula

$$F_1(i,j,k,l) \;=\; (H^{\otimes 2})^{-1}\, F_2([i],[j],[k],[l])\, H^{\otimes 2}. \tag{8.1.75}$$

This formula is consistent with the relation following from (8.1.68), i.e.,

$$r_1(i,j,k) \;=\; H^{-1} r_2([i],[j],[k]) H. \tag{8.1.76}$$

If we use bases in the spaces $Mor_1(k, i \circ j)$ obtained from some choice of bases in $Mor_2([k], [i] \circ [j])$ by application of H, we infer from (8.1.75) and (8.1.76) that

$$F_1(\sigma \circ i, \mu \circ j, \nu \circ k, \sigma \circ \mu \circ \nu \circ l)^{\mu \circ \nu \circ t}_{\sigma \circ \mu \circ s} \;=\; F_1(i,j,k,l)^{t}_{s}, \tag{8.1.77}$$

and

$$r_1(\sigma \circ i, \mu \circ j,\; \sigma \circ \mu \circ k) \;=\; r_1(i,j,k), \tag{8.1.78}$$

where $\sigma, \mu, \nu \in ker\zeta$ (so that, by (8.1.23), $[\sigma \circ i] = [i]$). In our analysis we have not, so far, considered the special balancing elements $\sigma(X) \in End(X)$, with $\varepsilon(Y,X)\varepsilon(XY) = \sigma(X \circ Y)\sigma(X)^{-1} \circ \sigma(Y)^{-1}$, which, in our context, are given by $\sigma(X) \mid Mor_1(k, \dot{X}) = e^{2\pi i \theta_k}$, $k \in J_1$, for statistical phases (or spins) θ_k. If we consider balanced tensor categories and tensor-functors between balanced tensor categories – which, in addition obey $\mathcal{F}\big(\sigma(X)\big) \;=\; \sigma\big(\zeta(X)\big)$ – then all of the results above still hold. The condition analogous to (8.1.77) and (8.1.78) is then given by

$$\theta_{\sigma \circ j} \;=\; \theta_j, \quad \forall \sigma \in R,\ \forall j \in J_1. \tag{8.1.78a}$$

The next question we wish to address is whether the triviality of the θ-category of $ker\zeta$ is also sufficient for a category C_1 to be induced by a category C_2, for a given $\zeta : Obj_1 \to Obj_2$. As a first step, we show that in this case the equations (8.1.75) and (8.1.76) can be solved on the level of structural data.

LEMMA 8.1.3

Suppose C_1 is a quantum category, $R \subset Obj_1$ a subgroup of invertible elements with a free and coherent action on J_1, and the θ-subcategory associated with R is trivial up to isomorphism. Assume further that the balancing elements, θ_j, of C_1 are R-invariant, i.e., equation (8.1:78a) holds. Then there exist matrices F_2 and r_2 defined on vector spaces modelled on basic spaces $Mor_2([i], [j] \circ [k]) \cong \mathbb{C}^{\tilde{N}_{[j],[k],[i]}}$, as in (8.1.68) and (8.1.74), bottom lines, and corresponding isomorphisms

$$H^{ij}_{[k]} \;:\; Mor_1\big(\psi_k(i,j), i \circ j\big) \;\longrightarrow\; \mathbb{C}^{N_{[i],[j],[k]}}, \tag{8.1.79}$$

such that equations (8.1.75) and (8.1.76) hold.

Proof.

We first make a choice, corresponding to a map $\gamma : \overline{J}_1 = J_1/R \to J_1 : [k] \to \gamma([k])$, with $\pi_R \circ \gamma = id_{\overline{J}_1}$, of representatives in the classes of $\overline{J}_1$. We further introduce $\overline{N}_{[i],[j],[k]}$-dimensional spaces $Mor_2([k],[i]\circ[j])$ with "canonical" elements $\bar{\mathbb{I}}_{[k]} \in End_2([k])$.

The fact that the θ-category associated to R is trivial implies, for the structural data, that there exist numbers $\lambda_{\sigma,\mu} \in \mathbb{C}$ (of modulus one, for C^*-categories) such that

$$\begin{aligned} F_1(\sigma,\mu,\nu,\sigma\circ\mu\circ\nu)\mathbb{I}_{\sigma\circ\mu}\circ\mathbb{I}_{\sigma\circ\mu\circ\nu} &= \frac{\lambda_{\sigma,\mu}\lambda_{\sigma\circ\mu,\nu}}{\lambda_{\mu,\nu}\lambda_{\sigma,\mu\circ\nu}}\,\mathbb{I}_{\mu\circ\nu}\circ\mathbb{I}_{\sigma\circ\mu\circ\nu}, \\ r_1(\sigma,\mu,\sigma\circ\mu)\mathbb{I}_{\sigma\circ\mu} &= \frac{\lambda_{\sigma,\mu}}{\lambda_{\mu,\sigma}}\,\mathbb{I}^{\mu\circ\sigma}, \\ \lambda_{1,\sigma} = \lambda_{\sigma,1} &= 1\,. \end{aligned} \tag{8.1.80}$$

Hence, for i,j and k in R, we can solve eqs. (8.1.75) and (8.1.76) by setting

$$H^{\sigma,\mu}_{[1]}(\mathbb{I}_{\sigma\circ\mu}) = \lambda_{\sigma,\mu}\,\bar{\mathbb{I}}_{[1]}, \quad \forall\sigma,\mu\in R, \tag{8.1.81}$$

and

$$F_1([1],[1],[1],[1]) := id, \quad r_2([1],[1],[1]) := id\,. \tag{8.1.82}$$

Next, we attempt to find a convenient normalization of the maps

$$H^{\sigma\circ\gamma([j]),\mu}_{[j]}, H^{\sigma,\mu\circ\gamma([j])}_{[j]} : End_1(\sigma\circ\mu\circ\gamma([j])) \to End_2([j]), \;\; [j]\neq[1]\,. \tag{8.1.83}$$

For a given choice of these maps, we define numbers

$$\varphi_{[j]}(\sigma,\mu,\nu),\, \varphi'_{[j]}(\sigma,\mu,\nu) : End_2([j]) \to End_2([j]) \tag{8.1.84}$$

by setting

$$\begin{aligned} (H^{\sigma\circ\gamma([j]),\mu}_{[j]} \otimes H^{\sigma\circ\mu\circ\gamma([j]),\nu}_{[j]}) &F_1(\sigma\circ\gamma([j]),\mu,\nu,\sigma\circ\mu\circ\nu\circ\gamma([j])) \\ &= \varphi_{[j]}(\sigma,\mu,\nu)(H^{\mu,\nu}_{[1]} \otimes H^{\sigma\circ\gamma([j]),\mu\circ\nu}_{[j]}), \end{aligned} \tag{8.1.85}$$

and

$$\begin{aligned} (H^{\mu,\nu}_{[1]} \otimes H^{\mu\circ\nu,\sigma\circ\gamma([j])}_{[j]}) &F_1(\mu,\nu,\sigma\circ\gamma([j]),\sigma\circ\mu\circ\nu\circ\gamma([j])) \\ &= \varphi'_{[j]}(\sigma,\mu,\nu)(H^{\nu,\sigma\circ\gamma([j])}_{[j]} \otimes H^{\mu,\nu\circ\sigma\circ\gamma([j])}_{[j]}). \end{aligned} \tag{8.1.85a}$$

For arbitrary assignments $a_{[j]}, b_{[j]} : R \to \mathbb{C}$, U(1), resp., with $a_{[j]}(1) = b_{[j]}(1) = 1$, $[j]\neq 1$, we define the maps $H^{\sigma\circ\gamma([j]),\mu}_{[j]}$ and $H^{\sigma,\mu\circ\gamma([j])}$ of (8.1.83) as follows:

If we set

$$\begin{aligned} H^{\gamma([j]),\mu}_{[j]}(\mathbb{I}_{\mu\circ\gamma([j])}) &= a_{[j]}(\mu)\,\bar{\mathbb{I}}_{[j]}, \\ H^{\mu,\gamma([j])}_{[j]}(\mathbb{I}_{\mu\circ\gamma([j])}) &= b_{[j]}(\mu)\,\bar{\mathbb{I}}_{[j]}, \end{aligned} \tag{8.1.86}$$

all other maps are uniquely determined by (8.1.85), with $\sigma = 1$, provided we assume that

$$\varphi_{[j]}(1,\mu,\nu) = \varphi'_{[j]}(1,\mu,\nu) = 1\,. \tag{8.1.87}$$

Note that (8.1.87) is consistent with (8.1.86) for $\mu = 1$, or $\nu = 1$, because $F_1(i,1,j,k) = F_1(i,j,1,k) = id$. With this normalization, we consider the pentagon equation

$$\begin{aligned} \Big(F_1(\gamma([j]),\sigma,\mu,\sigma\circ\mu\circ\gamma([j]))\otimes\mathbb{I}\Big)&\Big(\mathbb{I}\otimes F_1(\gamma([j]),\sigma\circ\mu,\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big) \\ &\Big(F_1(\sigma,\mu,\nu,\sigma\circ\mu\circ\nu)\otimes\mathbb{I}\Big), \\ = \Big(\mathbb{I}\otimes F_1(\sigma\circ\gamma([j]),\mu,\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)&T^{12} \\ &\Big(\mathbb{I}\otimes F_1(\gamma([j]),\sigma,\mu\circ\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big), \end{aligned} \tag{8.1.88}$$

and conjugate it by $H^{\otimes 3}$. Combining this identity with (8.1.82) and (8.1.87), we find the resulting equation on $End_2([j])$ to be

$$\varphi_{[j]}(\sigma,\mu,\nu) = 1, \tag{8.1.89}$$

and

$$\varphi'_{[j]}(\sigma,\mu,\nu) = 1,$$

by a similar argument. Thus, if we put

$$F_2([j],[1],[1],[j]) \;=\; F_2([1],[1],[j],[j]) \;=\; id \qquad (8.1.90)$$

we have a solution for (8.1.75), provided either $i, j \in R$ or $j, k \in R$. Denoting by $\overset{\circ}{H}{}^{j,\sigma}_{[j]}$ and $\overset{\circ}{H}{}^{\sigma,j}_{[j]}$ the isomorphisms defined by setting $a_{[j]} = b_{[j]} \equiv 1$, we find that, in the general case,

$$H^{\mu\circ\gamma([j]),\nu}_{[j]} \;=\; \frac{a_{[j]}(\mu\circ\nu)}{a_{[j]}(\mu)}\;\overset{\circ}{H}{}^{\mu\circ\gamma([j]),\nu}_{[j]}$$

and

$$H^{\mu,\nu\circ\gamma([j])}_{[j]} \;=\; \frac{b_{[j]}(\nu\circ\mu)}{b_{[j]}(\nu)}\;\overset{\circ}{H}{}^{\mu,\nu\circ\gamma([j])}_{[j]}\;. \qquad (8.1.91)$$

In order to determine the coefficients $a_{[j]}(\mu)$ and $b_{[j]}(\mu)$ in such a way that eq. (8.1.90) can be extended to $F_2([1],[j],[1],[j])$, we define numbers $\psi_{[j]}(\sigma \mid \mu,\mu)$, $\rho_{[j]}(\sigma,\mu)$ and $\tilde{\rho}_{[j]}(\sigma,\mu)$ in $End_2([j])$ by setting

$$\begin{aligned}(H^{\mu,\sigma\circ\gamma([j])}_{[j]} \otimes H^{\mu\circ\sigma\circ\gamma([j]),\nu}_{[j]})\; & F_1(\mu,\sigma\circ\gamma([j]),\nu,\mu\circ\nu\circ\sigma\circ\gamma([j]))\\ &= \psi_{[j]}(\sigma \mid \mu,\nu)\,(H^{\sigma\circ\gamma([j]),\nu}_{[j]} \otimes H^{\mu,\nu\circ\sigma\circ\gamma([j])}_{[j]})\end{aligned} \qquad (8.1.92)$$

and

$$\begin{aligned}H^{\mu\circ\gamma([j]),\sigma}_{[j]}\,r_1(\sigma,\mu\circ\gamma([j]),\sigma\circ\mu\circ\gamma([j])) \;&=\; \rho_{[j]}(\sigma,\mu)\;H^{\sigma,\mu\circ\gamma([j])}_{[j]},\\ H^{\sigma,\mu\circ\gamma([j])}_{[j]}\,r_1(\mu\circ\gamma([j]),\sigma,\mu\circ\sigma\circ\gamma([j])) \;&=\; \tilde{\rho}_{[j]}(\sigma,\mu)\;H^{\mu\circ\gamma([j]),\sigma}_{[j]}\;.\end{aligned} \qquad (8.1.93)$$

In order to derive relations for the constants $\psi_{[j]}$ introduced in (8.1.92), we consider the following two special cases of the pentagonal equation:

$$\begin{aligned}&\Big(F_1(\sigma,\gamma([j]),\mu,\sigma\circ\mu\circ\gamma([j]))\otimes\mathbb{1}\Big)\Big(\mathbb{1}\otimes F_1(\sigma,\mu\circ\gamma([j]),\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)\\ &\qquad\qquad\Big(F_1(\gamma([j]),\mu,\nu,\mu\circ\nu\circ\gamma([j]))\otimes\mathbb{1}\Big)\\ &= \Big(\mathbb{1}\otimes F_1(\sigma\circ\gamma([j]),\mu,\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)T_{12}\\ &\qquad\qquad(\mathbb{1}\otimes F_1(\sigma,\gamma([j]),\nu\circ\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big),\end{aligned} \qquad (8.1.94)$$

and

$$\begin{aligned}&\Big(F_1(\sigma,\mu,\gamma([j]),\sigma\circ\mu\circ\gamma([j]))\otimes\mathbb{1}\Big)\Big(\mathbb{1}\otimes F_1(\sigma,\mu\circ\gamma([j]),\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)\\ &\qquad\qquad\Big(F_1(\mu,\gamma([j]),\nu,\mu\circ\nu\circ\gamma([j]))\otimes\mathbb{1}\Big)\\ &= \Big(\mathbb{1}\otimes F_1(\sigma\circ\mu,\gamma([j]),\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)T_{12}\\ &\qquad\qquad\Big(\mathbb{1}\otimes F_1(\sigma,\mu,\nu\circ\gamma([j]),\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big)\;.\end{aligned}$$

Conjugating these equations by $H^{\otimes 3}$ we find, using (8.1.90):

$$\begin{aligned}\psi_{[j]}(1 \mid \sigma,\mu)\;\psi_{[j]}(\mu \mid \sigma,\nu) \;&=\; \psi_{[j]}(1 \mid \sigma,\mu\circ\nu),\\ \psi_{[j]}(\mu \mid \sigma,\nu)\;\psi_{[j]}(1 \mid \mu,\nu) \;&=\; \psi_{[j]}(1 \mid \sigma\circ\mu,\nu)\;.\end{aligned} \qquad (8.1.95)$$

In particular, since the two equations defining $\psi_{[j]}(\mu \mid \sigma,\nu)$ have to be compatible with each other, we conclude that

$$\psi_{[j]}(1 \mid \cdot,\cdot) \;\in\; Z^2(R,U(1)), \qquad (8.1.96)$$

and, moreover, that every $\psi_{[j]}(\mu \mid \cdot,\cdot)$ is a 2-boundary.

Next, we study the implications of the hexagonal equation

$$\begin{aligned}&\Big(r_1(\mu,\sigma\circ\gamma([j]),\sigma\circ\mu\circ\gamma([j]))\otimes\mathbb{1}\Big)F_1(\mu,\sigma\circ\gamma([j]),\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\\ &\qquad\qquad\Big(r_1(\nu,\sigma\circ\gamma([j]),\nu\circ\sigma\circ\gamma([j]))\otimes\mathbb{1}\Big)\\ &= F_1(\sigma\circ\gamma([j]),\mu,\nu,\sigma\circ\mu\circ\nu\circ\gamma([j]))\Big(\mathbb{1}\otimes r(\nu\circ\nu,\sigma\circ\gamma([j])),\mu\circ\nu\circ\sigma\circ\gamma([j])\Big)\\ &\qquad\qquad F_1(\mu,\nu,\sigma\circ\gamma([j]),\sigma\circ\mu\circ\nu\circ\gamma([j]))\end{aligned} \qquad (8.1.97)$$

which, upon conjugation with $H^{\otimes 2}$, takes the form

$$\rho_{[j]}(\mu,\sigma)\;\psi_{[j]}(\sigma \mid \mu,\nu)\;\rho_{[j]}(\nu,\sigma) \;=\; \rho_{[j]}(\mu\circ\nu,\sigma). \qquad (8.1.98)$$

From (8.1.98) we see immediately that $\psi_{[j]}(1 \mid \cdot,\cdot)$ is a symmetric 2-cocycle and, therefore, lies in the kernel of the isomorphism

$$\hat{\alpha} \;:\; Z^2(R,U(1)) \;\rightarrow\; Hom(\Lambda^2 R,U(1)), \qquad (8.1.99)$$

as defined in (7.190). Hence, since $Ext(R, U(1)) = 0$, we have that $\psi_{[j]}(1 \mid \cdot, \cdot) \in B^2(R, U(1))$, i.e., there exists a function $\beta_{[j]} : R \to U(1) : \sigma \mapsto \beta_{[j]}(\sigma)$, such that

$$\psi_{[j]}(1 \mid \sigma, \mu) = \frac{\beta_{[j]}(\sigma)\beta_{[j]}(\mu)}{\beta_{[j]}(\sigma \circ \mu)} . \tag{8.1.100}$$

Denoting by $\psi^0_{[j]}$ and β^0 the constants defined in (8.1.92) for the choice of isomorphisms $\overset{\circ}{H}$ as given in (8.1.91), we deduce from (8.1.92) the relation

$$\psi_{[j]}(1 \mid \mu, \nu) = \frac{a_{[j]}(\mu \circ \nu)}{a_{[j]}(\mu)\, a_{[j]}(\nu)} \frac{b_{[j]}(\nu)\, b_{[j]}(\mu)}{b_{[j]}(\mu \circ \nu)} \psi^0_{[j]}(1 \mid \mu, \nu) . \tag{8.1.101}$$

Thus if we require the normalization to be of the form

$$a_{[j]}(\mu) = \xi_{[j]}(\mu)\tau_{[j]}(\mu), \quad b_{[j]}(\mu) = \beta^0_{[j]}(\mu)\xi_{[j]}(\mu), \tag{8.1.102}$$

for some maps $\xi_{[j]} : R \to U(1)$ and $\tau_{[j]} \in Hom(R, U(1)) \equiv \hat{R}$, we obtain that $\psi_{[j]}(1 \mid \mu, \nu) = 1$, and, by (8.1.95), $\psi_{[j]}(\mu \mid \sigma, \nu) = 1$. Therefore, setting

$$F_2([1], [j], [1], [j]) = id, \tag{8.1.103}$$

this choice of H-isomorphisms provides a solution of (8.1.75), whenever $i, k \in R$. Suppose $\rho''_{[j]}(\sigma, \mu)$ is the constant determined in (8.1.93) for the case $\xi = \tau \equiv 1$. Then the general form of $\rho_{[j]}$ is described by

$$\rho_{[j]}(\sigma, \mu) = \tau(\sigma)\, \rho''_{[j]}(\sigma, \mu), \tag{8.1.104}$$

independent of ξ. Another special case of the hexagonal equation is given by

$$\begin{aligned}\Big(r(\sigma \circ \gamma([j]), \sigma \circ \gamma([j])) \otimes \mathbb{I}\Big)\, &F(\mu, \sigma, \gamma([j]), \sigma \circ \mu \circ \gamma([j]))^{-1} \left(r(\sigma, \mu, \sigma \circ \mu) \otimes \mathbb{I}\right) \\ = F(\mu, \gamma([j]), \sigma, \sigma \circ \mu \circ \gamma([j]))^{-1} &\Big(\mathbb{I} \otimes r(\sigma, \mu \circ \gamma([j]), \sigma \circ \mu \circ \gamma([j]))\Big) \\ &\qquad F(\sigma, \mu, \gamma([j]), \sigma \circ \mu \circ \gamma([j]))^{-1} .\end{aligned} \tag{8.1.105}$$

After conjugation with $H^{\otimes 2}$, this equation becomes

$$\rho_{[j]}(\sigma, 1) = \rho_{[j]}(\sigma, \mu), \tag{8.1.106}$$

i.e., $\rho_{[j]}$ is independent of μ. Furthermore, we see that, since $\psi_{[j]}(\sigma \mid \mu, \nu) = 1$, (8.1.98) implies that $\sigma \to \rho_{[j]}(\sigma, 1)$ is a homomorphism. We can therefore choose $\tau(\sigma) = \rho''_j(\sigma, 1)^{-1} \in \hat{R}$, and hence $\rho_{[j]}(\sigma, \mu) = 1$. The fact that the balancing elements, $j \mapsto \theta_j$, are invariant under the action of R yields the equation

$$r(\sigma, \mu \circ \gamma([j]), \sigma \circ \mu \circ \gamma([j]))r(\mu \circ \gamma([j]), \sigma, \sigma \circ \mu \circ \gamma([j])) = e^{2\pi i(\theta_\sigma + \theta_{\mu \circ \gamma([j])} - \theta_{\sigma \circ \mu \circ \gamma([j])})} = 1, \tag{8.1.107}$$

so that, by conjugating with $H^{\mu \circ \gamma([j]), \sigma}_{[j]}$ and using (8.1.93), we find that

$$\tilde{\rho}_{[j]}(\sigma, \mu) = \rho_{[j]}(\sigma, \mu)^{-1} = 1 .$$

If we set

$$r_2([j], [1], [j]) = r_2([1], [j], [j]) = 1 \tag{8.1.108}$$

the H-isomorphisms determined so far also yield a solution to eq. (8.1.76), for $i \in R$ or $j \in R$.

For a given choice of $\widetilde{H}^{i,j}_{[k]}$, consistent with our normalizations for $i \in R$ or $j \in R$, we introduce invertible linear maps

$$\widetilde{F}_L\begin{pmatrix}[i][j], [k] \\ \mu\nu, \sigma\end{pmatrix}, \ \widetilde{F}_R\begin{pmatrix}[i][j], [k] \\ \mu\nu, \sigma\end{pmatrix} \in End_{\mathbb{C}}(Mor_2([k], [i] \circ [j]))$$

as the transforms of the F-matrices, i.e.,

$$\begin{aligned}&\left(\widetilde{H}^{\sigma, \mu \circ \gamma([i])}_{[i]} \otimes \widetilde{H}^{\sigma \circ \mu \circ \gamma([i])}_{[k]}\right) F_1(\sigma, \mu \circ \gamma([i]), \nu \circ \gamma([j]), \sigma \circ \mu \circ \nu \circ \sigma_{[i][j],[k]} \circ \gamma([k])) \\ &= \widetilde{F}_L\begin{pmatrix}[i][j], [k] \\ \mu\nu, \sigma\end{pmatrix} \left(\widetilde{H}^{\mu \circ \gamma([i]), \nu \circ \gamma([j])}_{[k]} \otimes \widetilde{H}^{\sigma, \mu \circ \nu \circ \sigma_{[i][j],[k]} \circ \gamma([k])}_{[k]}\right),\end{aligned} \tag{8.1.109}$$

and

$$\begin{aligned}&\left(\widetilde{H}^{\mu \circ \gamma([i]), \nu \circ \gamma([j])}_{[k]} \otimes \widetilde{H}^{\mu \circ \nu \circ \sigma_{[i][j],[k]} \circ \gamma([k]), \sigma}_{[k]}\right) \\ &\qquad\qquad F_1(\mu \circ \gamma([i]), \nu \circ \gamma([j]), \sigma, \sigma \circ \mu \circ \nu \circ \sigma_{[i][j],[k]} \circ \gamma([k])) \\ &= \widetilde{F}_R\begin{pmatrix}[i][j], [k] \\ \mu\nu, \sigma\end{pmatrix} \left(\widetilde{H}^{\nu \circ \gamma([j]), \sigma}_{[j]} \otimes \widetilde{H}^{\mu \circ \gamma([i]), \sigma \circ \nu \circ \gamma([j])}_{[k]}\right).\end{aligned} \tag{8.1.110}$$

Here we are using the invertible objects $\sigma_{[i][j],[k]} \in R$, defined, for a coherent action of R, in eq. (8.1.19). Moreover, we are identifying $Hom_{\mathrm{C}}(Mor_2([k],[i]\circ[j])\otimes End_2([k]), End_2([i])\otimes Mor_2([k],[i]\circ[j]))$ and $Hom_{\mathrm{C}}(End_2([j])\otimes Mor_2([k],[i]\circ[j]), Mor_2([k],[i]\circ[j])\otimes End_2([k]))$ with $End_{\mathrm{C}}(Mor_2([k],[i]\circ[j]))$ by using the canonical elements $\bar{\mathbb{I}}_k \in End_2([k])$. The pentagonal equation for $k := \nu\circ\sigma_{[i][j],[k]}\circ\gamma([k])$,

$$\begin{aligned}&\Big(F_1(\mu,\sigma,\gamma([i]),\mu\circ\sigma\circ\gamma([i]))\otimes\mathbb{I}\Big)\Big(\mathbb{I}\otimes F_1(\mu,\sigma\circ\gamma([i]),\nu\circ\gamma([j]),\sigma\circ\mu\circ k)\Big)\\&\qquad\qquad\Big(F_1(\sigma,\gamma([i]),\nu\circ\gamma([j]),\sigma\circ k)\otimes\mathbb{I}\Big)\\&=\Big(\mathbb{I}\otimes F_1(\mu\circ\sigma,\gamma([i]),\nu\circ\gamma([j]),\sigma\circ\mu\circ k)\Big)T_{12}(\mathbb{I}\otimes F_1(\mu,\sigma,\nu\circ k,\mu\circ\sigma\circ k)),\end{aligned}\tag{8.1.111}$$

yields a factorization of $\widetilde{F}_L$ of the form

$$\widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ \sigma\nu,\mu\end{pmatrix} = \widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ 1\,\nu,\mu\circ\sigma\end{pmatrix}\widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ 1\nu,\sigma\end{pmatrix}^{-1}. \tag{8.1.112}$$

Similarly, we find that

$$\widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ \mu\nu,\sigma\end{pmatrix} = \widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ \mu\,1,\nu\end{pmatrix}^{-1}\widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ \mu\,1,\sigma\circ\nu\end{pmatrix}. \tag{8.1.113}$$

Finally, from the equation

$$\begin{aligned}&\Big(F_1(\mu,\gamma([i]),\gamma([j]),\mu\circ k')\otimes\mathbb{I}\Big)(\mathbb{I}\otimes F_1(\mu,k',\nu,\mu\circ\nu\circ k'))\\&\qquad\qquad\Big(F_1(\gamma([i]),\gamma([j]),\nu,\nu\circ k')\otimes\mathbb{I}\Big)\\&=\Big(\mathbb{I}\otimes F_1(\mu\circ\gamma([i]),\gamma([j]),\nu,\mu\circ\nu\circ k')\Big)T_{12}\Big(\mathbb{I}\otimes F_1(\mu,\gamma([i]),\nu\circ\gamma([j]),\mu\circ\nu\circ k')\Big)\end{aligned}\tag{8.1.114}$$

where $k' = \sigma_{[i][j],[k]}\circ\gamma([k])$, we obtain the relation

$$A\begin{pmatrix}[i][j],[k]\\ \mu,\nu\end{pmatrix} := \widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ 1\,\nu,\mu\end{pmatrix}\widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ 1\,1,\nu\end{pmatrix}^{-1} = \widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ \mu\,1,\nu\end{pmatrix}^{-1}\widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ 1\,1,\mu\end{pmatrix}, \tag{8.1.115}$$

so that eqs. (8.1.112) and (8.1.113) can be rewritten as

$$\begin{aligned}\widetilde{F}_L\begin{pmatrix}[i][j],[k]\\ \sigma\nu,\mu\end{pmatrix} &= A\begin{pmatrix}[i][j],[k]\\ \mu\circ\sigma,\nu\end{pmatrix}A\begin{pmatrix}[i][j],[k]\\ \sigma,\nu\end{pmatrix}^{-1},\\ \widetilde{F}_R\begin{pmatrix}[i][j],[k]\\ \mu\nu,\sigma\end{pmatrix} &= A\begin{pmatrix}[i][j],[k]\\ \mu,\nu\end{pmatrix}A\begin{pmatrix}[i][j],[k]\\ \mu,\sigma\circ\nu\end{pmatrix}^{-1}.\end{aligned}\tag{8.1.116}$$

Replacing the isomorphisms, $\widetilde{H}^{i,j}_{[k]}$, by the maps

$$H^{\sigma\circ\gamma([i]),\mu\circ\gamma([j])}_{[k]} = A\begin{pmatrix}[i][j],[k]\\ \sigma,\mu\end{pmatrix}^{-1}\widetilde{H}^{\sigma\circ\gamma([i]),\mu\circ\gamma([j])}_{[k]}, \tag{8.1.117}$$

corresponds to replacing the "structure constants" $\widetilde{F}_L$ and $\widetilde{F}_R$ by $F_L\binom{[i][j],[k]}{\sigma\nu,\mu} = F_R\binom{[i][j],[k]}{\mu\nu,\sigma} = 1$, as follows from eqs. (8.1.116), (8.1.117) and (8.1.109), (8.1.110).

Note that $\widetilde{F}_L = \widetilde{F}_R = \mathbb{I}$, and hence $A = \mathbb{I}$, when $[i] = 1$ or $[j] = 1$, so that the isomorphisms determined previously remain unchanged in this case. Thus, setting

$$F_2([i],[j],1,[k]) = F_2(1,[i],[j],[k]) = id, \tag{8.1.118}$$

we have found a solution to (8.1.75) when either i or k are restricted to R. A complete solution to (8.1.75) can be found by using the hexagonal equation

$$\begin{aligned}&\Big(r_1(\sigma\circ\gamma([i]),\mu,\mu\circ\sigma\circ\gamma([i]))\otimes\mathbb{I}\Big)F_1(\sigma\circ\gamma([i]),\mu,\nu\circ\gamma([j]),\mu\circ k'')\\&\qquad\qquad\Big(r_1(\nu\circ\gamma([j]),\mu,\mu\circ\nu\circ\gamma([j]))\otimes\mathbb{I}\Big)\\&= F_1(\mu,\sigma\circ\gamma([i]),\nu\circ\gamma([j]),\mu\circ k'')(\mathbb{I}\otimes r_1(k'',\mu,\mu\circ k''))\\&\qquad\qquad F_1(\sigma\circ\gamma([i]),\nu\circ\gamma([j]),\mu,\mu\circ k''),\end{aligned}\tag{8.1.119}$$

with $k'' = \sigma\circ\nu\circ\sigma_{[i][j],[k]}\circ\gamma([k])$. With (8.1.108) and (8.1.118), we derive from (8.1.119) the equation

$$\begin{aligned}&\Big(H^{\sigma\circ\gamma([i]),\mu}_{[i]}\otimes H^{\mu\circ\sigma\circ\gamma([i]),\nu\circ\gamma([j])}_{[k]}\Big)F_1(\sigma\circ\gamma([i]),\mu,\nu\circ\gamma([j]),\mu\circ k'')\\&\qquad\qquad = H^{\mu,\nu\circ\gamma([j])}_{[j]}\otimes H^{\sigma\circ\gamma([i]),\mu\circ\nu\circ\gamma([j])}_{[k]}.\end{aligned}$$

Setting

$$F_2([i],1,[j],[k]) = id \tag{8.1.120}$$

we thus find a solution to eq. (8.1.75) if only j is restricted to R.

In the remainder of the proof we show that, for the choice of H's satisfying (8.1.108), (8.1.118) and (8.1.120), we can find F_2's that provide a complete solution of (8.1.75).

For this purpose, we define maps

$$\hat{F}_2(i,j,k,[l]) \ : \ \bigoplus_{[s]} Mor_2([s],[j]\circ[k]) \otimes Mor_2([l],[i]\circ[s]) \\ \rightarrow \bigoplus_{[s]} Mor_2([s],[i]\circ[j]) \otimes M_2([l],[s]\circ[k])$$

through the equation

$$\left(H^{i,j}_{[s]} \otimes H^{\psi_{[s]}(i,j),k}_{[l]}\right) F_1\left(i,j,k,\psi_{[l]}(i,j,k)\right)^{\psi_{[t]}(j,k)}_{\psi_{[s]}(i,j)} \\ = \hat{F}_2(i,j,k,[l])^{[t]}_{[s]} \left(H^{j,k}_{[t]} \otimes H^{i,\psi_{[t]}(j,k)}_{[l]}\right), \qquad (8.1.121)$$

with $\hat{F}_2(i,j,k,[l]) = id$ if i, j or k belongs to R.

For $\sigma \in R$, $i,j,k \in J_1$ and $l = \psi_{[l]}(i,j,k)$, we may consider the following special case of the pentagonal equation:

$$\left(\bigoplus_{[s]} F_1(\sigma,i,j,\sigma\circ\psi_{[s]}) \otimes \mathbb{I}\right)\left(\bigoplus_{[s]} \mathbb{I} \otimes F_1(\sigma,\psi_{[s]}(i,j),k,\sigma\circ l)\right)(F_1(i,j,k,l)\otimes\mathbb{I}) \\ = (\bigoplus_{[s]} \mathbb{I} \otimes F_1(\sigma\circ i,j,k,\sigma\circ l))T_{12}\left(\bigoplus_{[s]} \mathbb{I} \otimes F(\sigma,i,\psi_{[s]}(j,k),\sigma\circ l)\right). \qquad (8.1.122)$$

The transformed equation for the $\hat{F}_2$'s simply reads

$$\hat{F}_2(i,j,k,[l]) \ = \ \hat{F}_2(\sigma\circ i,j,k,[l]). \qquad (8.1.123)$$

By considering the equations obtained by replacing (σ,i,j,k) by (i,σ,j,k) and (i,j,σ,k), we also find that

$$\hat{F}_2(i,\sigma\circ j,k,[l]) \ = \ \hat{F}_2(i,j,\sigma\circ k,[l]) \ = \ \hat{F}_2(i,j,k,[l]),$$

for all $\sigma \in R$. Hence, one can assign, in a well defined manner, linear maps F_2 to every quadruple of objects in J_1/R such that

$$\hat{F}_2(i,j,k,[l]) \ = \ F_2([i],[j],[k],[l]). \qquad (8.1.124)$$

These maps provide us with a general solution to (8.1.75). Similarly, we introduce functions $\hat{r}_2$ by setting

$$H^{j,i}_{[k]} r_1\left(i,j,\psi_{[k]}(i,j)\right) \ = \ \hat{r}_2(i,j,[k]) H^{i,j}_{[k]}. \qquad (8.1.125)$$

The hexagonal equation

$$\left(r_1(\sigma,k,\sigma\circ k)\otimes\mathbb{I}\right)F_1\left(\sigma,k,j,\sigma\circ\psi_{[l]}(k,j)\right)\left(r_1\left(j,k,\psi_{[l]}(k,j)\right)\otimes\mathbb{I}\right) \\ = F_1\left(k,\sigma,j,\sigma\circ\psi_{[l]}(k,j)\right)\left(\mathbb{I}\otimes r_1\left(\sigma\circ j,k,\sigma\circ\psi_{[l]}(k,j)\right)\right)F_1\left(\sigma,j,k,\sigma\circ\psi_{[l]}(k,j)\right)$$

yields the equation

$$\hat{r}_2(j,k,[l]) \ = \ \hat{R}_2(\sigma\circ j,k,[l]), \qquad (8.1.126)$$

and an analogous equation, with k and j exchanged, proves invariance under the action by $\sigma \in R$ on the second argument. Hence we can write

$$\hat{r}_2(i,j,[k]) \ =: \ r_2([i],[j],[k]), \qquad (8.1.127)$$

and r_2 is a solution to (8.1.76).

Finally, the assumed invariance of the "balancing phases" under $\sigma \in R$ allows us to define such phases on J_1/R by setting

$$\theta_j \ =: \ \theta_{[j]} \quad mod\, 1\,. \qquad (8.1.128)$$

Clearly, for the structural data r_2, F_2 and θ just constructed, the pentagonal-, hexagonal- and balancing equations can be derived directly from the corresponding equations in C_1, via (8.1.75) and (8.1.76). This completes the proof of Lemma 8.1.3.

□

This result leads us to a formulation of the basic criterion for the existence of induced categories.

__PROPOSITION 8.1.4__

Suppose that C_1 is a quantum category with object set Obj_1, and let $R \subset Obj_1$ be a group of invertible objects with free and coherent action on J_1, so that we have a fusion rule algebra epimorphism $\pi_R : Obj_1 \rightarrow \overline{Obj}_1 = \mathbb{N}^{(J_1/R)}$. Then there exists a category C_2 with object set $\overline{Obj}_1$ such that C_1 is the category induced by C_2 and π_R if and only if the following two conditions are met in C_1:

(i) *The θ-subcategory associated to R is trivial.*

(ii) *The balancing elements (statistical phases) are R-invariant, i.e.,*

$$\theta_j = \theta_{\sigma \circ j}, \quad \forall \sigma \in R.$$

Proof:

As a first step in constructing $\mathcal{C}_2$ we build a certain category $\overline{\mathcal{C}}_1$, related to $\mathcal{C}_1$ by a tensor functor

$$(id, \mathcal{F}, \mathbb{I}) \; : \; \mathcal{C}_1 \to \overline{\mathcal{C}}_1. \tag{8.1.129}$$

The object set of $\overline{\mathcal{C}}_1$ is the same as that of $\mathcal{C}_1$. However, two objects X and Y in $\overline{\mathcal{C}}_1$ are equivalent ($X \approx Y$) iff $\pi_R(X) = \pi_R(Y)$, i.e., modulo equivalence, the object set of $\overline{\mathcal{C}}_1$ is $\overline{Obj}_1$.

From the building blocks

$$M([k], X) := \bigoplus_{k \in \mathcal{C}_{[k]}} Mor_1(k, X) \tag{8.1.130}$$

we define the spaces of morphisms

$$\overline{Mor}_1(X, Y) := \bigoplus_{[k]} Hom_{\mathbb{C}}(M([k], X), \;\; M([k], Y)), \tag{8.1.131}$$

equipped with the obvious composition of morphisms.

For $I \in Mor_1(X, Y)$, we define the action of $\mathcal{F}(I)$ on $M([k]X)$ into $M([k]Y)$ by left multiplication on X, i.e., for $v = \sum_{k \in \mathcal{C}_{[k]}} v_k \in M([k], X)$, with $v_k \in Mor_1(k, X)$, we set

$$\mathcal{F}(I)(v) = \sum_{k \in \mathcal{C}_{[k]}} I \, v_k, \qquad I \, v_k \in Mor_1(k, Y). \tag{8.1.132}$$

In order to find the (unique) tensor product on $\overline{\mathcal{C}}_1$ such that a functor (8.1.129) exists, we use the collection of isomorphisms

$$\Gamma^{[k]}_{X \circ Y} \; : \; \bigoplus_{[i],[j] \in J_1/R} M([i], X) \otimes M([j], Y) \otimes Mor_2([k], [i] \circ [j]) \\ \longrightarrow M([k], X \circ Y),$$

which, for $v_i \in Mor_1(i, X) \subset M([i], X)$, $v_j \in Mor_1(j, Y) \subset M([j]Y)$ and $w \in Mor_2([k], [i] \circ [j])$, are given by

$$\Gamma^{[k]}_{X \circ Y}(v_i \otimes v_j \otimes w) = (v_i \circ v_j)\Big(H^{ij}_{[k]}\Big)^{-1}(w) \\ \in Mor_1\big(\psi_{[k]}(i, j), X \circ Y\big) \subset M([k], X \circ Y). \tag{8.1.133}$$

The tensor product of two morphisms $I \in \overline{Mor}_1(X, X')$ and $J \in \overline{Mor}_1(Y, Y')$ is then given by

$$(I \bar{\circ} J)\Gamma^{[k]}_{X,Y} = \Gamma^{[k]}_{X',Y'}(I \otimes J \otimes \mathbb{I}). \tag{8.1.134}$$

It is immediately clear from (8.1.133) that

$$\mathcal{F}(I \circ J) = \mathcal{F}(I) \bar{\circ} \mathcal{F}(J), \tag{8.1.135}$$

for arbitrary $I \in Mor_1(X, X')$ and $J \in Mor_1(Y, Y')$. If the isomorphisms in (8.1.133) are chosen as proposed in Lemma 8.1.3 we conclude that $\overline{\mathcal{C}}_1$, equipped with the following braiding and associativity isomorphisms

$$\overline{\varepsilon}_1(X, Y) := \mathcal{F}(\varepsilon_1(X, Y)), \\ \overline{\alpha}_1(X, Y, Z) := \mathcal{F}(\alpha_1(X, Y, Z)), \tag{8.1.136}$$

is a quantum-category, and $(id, \mathcal{F}, \mathbb{I})$ is a trensor functor. Since the pentagonal and hexagonal equations follow easily from (8.1.136), we are left with proving the isotropy equations

$$\overline{\varepsilon}_1(X', Y')(I \bar{\circ} J) = (J \bar{\circ} I)\overline{\varepsilon}_1(X, Y), \tag{8.1.137}$$

and

$$\overline{\alpha}_1(X', Y', Z')(I \bar{\circ}(J \bar{\circ} K)) = ((I \bar{\circ} J)\bar{\circ} K) \; \overline{\alpha}_1(X, Y, Z), \tag{8.1.138}$$

for $I \in \overline{Mor}_1(X, X')$, $J \in \overline{Mor}_1(Y, Y')$ and $K \in \overline{Mor}_1(Z, Z')$.

From the corresponding isotropy equations in $\mathcal{C}_1$ and from relations (8.1.67) and (8.1.68) we obtain that

$$\varepsilon_1(X, Y)(v_i \circ v_j)\Big(H^{i,j}_{[k]}\Big)^{-1}(w) = (v_j \circ v_i)\Big(H^{j,i}_{[k]}\Big)^{-1}(r_2([i], [j], [k])w),$$

for $v_i \in Mor_1(i, X)$, $v_j \in Mor_1(j, Y)$ and $w \in Mor_2([k], [i] \circ [j])$. Hence

$$\bar{\varepsilon}_1(X,Y)\Gamma^{[k]}_{X,Y} = \Gamma^{[k]}_{Y,X}\Big(\bigoplus_{[i][j]} T_{12} \otimes r_2([i],[j],[k])\Big), \tag{8.1.139}$$

where $T_{12} : M([i], X) \otimes M([j], Y) \to M([j], Y) \otimes M([i], X)$ is the flip of factors. From (8.1.139) and the definition (8.1.134) of the tensor product $\bar{\circ}$, we deduce (8.1.137). Similarly, (8.1.75) and the commutativity of the top of the total square in (8.1.74) imply that

$$\begin{aligned}&\alpha_1(X,Y,Z)\big(v_i \circ (v_j \circ v_k)\big)\mu^{i(jk)}\,(H^{\otimes 2})^{-1}(z) = \\ &\big((v_i \circ v_j) \circ v_k\big)\mu^{(ij)k}\,(H^{\otimes 2})^{-1}\big(F_2([i],[j],[k],[l])(z)\big),\end{aligned} \tag{8.1.140}$$

for $z \in \bigoplus_{[s]} Mor_2([s], [j] \circ [k]) \otimes M_2([l], [i] \circ [s])$. In terms of the isomorphisms Γ introduced in (8.1.132), this relation reads as follows:

$$\begin{aligned}&\bar{\alpha}_1(X,Y,Z)\Gamma^{X,Y\circ Z}_{[l]}\Big(\bigoplus_{[s]} \mathbb{I} \otimes \Gamma^{Y,Z}_{[s]} \otimes \mathbb{I}\Big) \\ &= \Gamma^{X\circ Y,Z}_{[l]}\Big(\bigoplus_{[s]} \Gamma^{X,Y}_{[s]} \otimes \mathbb{I}^{\otimes 2}\Big)T_{34}\Big(\bigoplus_{[i][j][k]} \mathbb{I}^{\otimes 3} \otimes F_2([i],[j],[k],[l])\Big).\end{aligned} \tag{8.1.141}$$

From (8.1.141) we derive (8.1.138) in the same way as we found (8.1.137) from (8.1.139). This establishes existence of a category $\overline{\mathcal{C}}_1$ and of a functor (8.1.129), with the property that $X \approx Y$ iff $\pi_R(X) = \pi_R(Y)$.

For some choice of a map $\gamma : \overline{Obj}_1 \longrightarrow Obj_1$, with $\pi_R \circ \gamma = id$, we then define $\mathcal{C}_2$, as an abelian category, to be the subcategory of $\overline{\mathcal{C}}_1$ with

$$Mor_2(\bar{X}, \bar{Y}) = \overline{Mor}_1(\gamma(\bar{X}), \gamma(\bar{Y})). \tag{8.1.141a}$$

Furthermore, for each X with $\pi_R(X) = \bar{X}$, we select a particular isomorphism $Q(X) \in \overline{Mor}_1(\gamma(\bar{X}), X)$, with $Q(\gamma(\bar{X})) = 1$. We define a functor between abelian categories, $(\pi_R, \mathcal{G}) : \overline{\mathcal{C}}_1 \to \mathcal{C}_2$, by setting

$$\mathcal{G}(I) := Q(Y)^{-1} I Q(X), \tag{8.1.141b}$$

for $I \in \overline{Mor}_1(X, Y)$. The tensor product of two morphisms $\bar{I} \in Mor_2(\bar{X}, \bar{X}')$ a $\bar{J} \in Mor_2(\bar{Y}, \bar{Y}')$ is defined by

$$\bar{I} \circ_2 \bar{J} := Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y}')\big)^{-1}(\bar{I}\bar{\circ}\bar{J})Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y})\big). \qquad (8.1.141$$

Defining $C(X,Y) \in End_2(\bar{X} \circ \bar{Y})$ by

$$C(X,Y) := Q(X \circ Y)^{-1}\big(Q(X)\bar{\circ}Q(Y)\big)Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y})\big), \qquad (8.1.141c$$

then, for the functor

$$(\pi_R, \mathcal{G}, C) : \overline{\mathcal{C}}_1 \longrightarrow \mathcal{C}_2, \qquad (8.1.141e$$

the compatibility condition (8.1.33) is readily verified. For the braiding- and associa tivity isomorphisms defined by

$$\varepsilon_2(\bar{X}, \bar{Y}) := Q\big(\gamma(\bar{Y}) \circ \gamma(\bar{X})\big)^{-1}\bar{\varepsilon}_1\big(\gamma(\bar{X}), \gamma(\bar{Y})\big)Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y})\big),$$

and

$$\begin{aligned}&\alpha_2(\bar{X}, \bar{Y}, \bar{Z}) := \\ &Q\big(\gamma(\bar{X} \circ \bar{Y}) \circ \gamma(\bar{Z})\big)^{-1}\Big(Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y})\big)^{-1}\bar{\circ}\mathbb{I}\Big)\bar{\alpha}_1\big(\gamma(\bar{X}), \gamma(\bar{Y}), \gamma(\bar{Z})\big) \\ &\qquad\Big(\mathbb{I}\bar{\circ}Q\big(\gamma(\bar{Y}), \gamma(\bar{Z})\big)\Big)Q\big(\gamma(\bar{X}) \circ \gamma(\bar{Y} \circ \bar{Z})\big)\end{aligned} \tag{8.1.141f}$$

we also find relations (8.1.43) and (8.1.44). Thus (8.1.141e) is, in fact, a tensor functor o quantum categories. Proposition 8.1.4 follows by considering the composition of tenso functors

$$(\pi_R, \mathcal{G} \circ \mathcal{F}, C) : \mathcal{C}_1 \longrightarrow \mathcal{C}_2.$$

□

Application of Proposition 8.1.4 requires that the subgroup, $R = ker\zeta$, of invertibl elements has trivial categorial properties, in the very strict sense that the braided monoidal category associated to it is trivial, and all monodromies with other objects o the total categories vanish. (This can be expressed here by the invariance of statistica ases.) In many situations, however, this information is not available, but only the iviality of the monoidal category associated to R is known. The following discussion devoted to the question to what extent this suffices to conclude that, to a category $\mathcal{C}$ ith objects Obj, one can associate a category $\overline{\mathcal{C}}$ with objects $\overline{Obj} = Obj/R$ such that is induced by $\overline{\mathcal{C}}$ and π_R.

We first recall some notation and some simple facts that have been used earlier.

We assume that $R \subset Obj$ is a subgroup of invertible objects with a free, coherent ction on the irreducible objects, $J \subset Obj$, of a rigid, braided, monoidal category, $\mathcal{C}$. We denote by

$$\pi_R \,:\, Obj \;\to\; \overline{Obj} \;=\; \mathbb{N}^{\bar{J}}, \quad j \;\mapsto\; [j], \tag{8.1.142}$$

with $\bar{J} := J/R$, the fusion rule algebra homomorphism onto the Perron-Frobenius fusion rule algebra $\overline{Obj}$, whose irreducible objects, $\bar{J}$, are the orbits of R in J. There is a universal gradation, grad, assigning to each irreducible element an element of Grad(Obj), see end of Chapter 3.3. Defining

$$R_0 \;:=\; \{\mu \in R \,:\, grad(\mu) \;=\; 1\}, \tag{8.1.143}$$

we have the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[r] & R \arrow[r, hook] \arrow[d, "\cong"] & Obj \arrow[r, two heads, "\pi_R"] \arrow[d, "grad"] & \overline{Obj} \arrow[r] \arrow[d, "\overline{grad}"] & 0 \\
0 \arrow[r] & R_0 \arrow[r, hook] & R \arrow[r, "grad"] & Grad(Obj) \arrow[r, "\pi_R^{\#}"] & Grad(\overline{Obj}) \arrow[r] & 0
\end{tikzcd} \tag{8.1.144}$$

in which the rows are exact sequences. We define

$$\bar{R} \;:=\; R/R_0 \;\cong\; grad(R) \;\subset\; Grad(Obj). \tag{8.1.145}$$

For any choice of γ, as in (8.1.15), the algebra Obj can be described by the fusion rules of $\overline{Obj}$ and, with (8.1.18) and (8.1.19), by elements $\sigma_{[i][j],[k]} \in R$ satisfying (8.1.20) and

(8.1.21). To any map $\gamma : \bar{J} \to J$, we associate a unique map $\eta : J \to R$ characterized by

$$j \;=\; \eta(j) \circ \gamma([j]), \qquad \text{for} \quad j \in J, \tag{8.1.146}$$

such that

$$\eta(\mu \circ j) \;=\; \mu \circ \eta(j), \qquad \eta\big(\gamma([j])\big) \;=\; 1, \qquad \text{for} \quad \mu \in R,$$

and, furthermore,

$$\sigma_{[i][j],[k]} \;=\; \frac{\eta(k)}{\eta(i)\,\eta(j)}, \qquad \text{whenever} \quad k \in i \circ j\,. \tag{8.1.147}$$

Our first result on induced monoidal categories is a simple modification of Proposition 8.1.2. The fusion rule algebra of a category without braided structure can be non-abelian. For the notion of a coherent action of R on J to be meaningful, we shall then have to assume that

$$\mu \circ j \;=\; j \circ \mu, \qquad \text{for} \quad \mu \in R,\; j \in J\,. \tag{8.1.148}$$

As a consequence, the Perron-Frobenius algebra $\overline{Obj}$ and the elements $A_{[i][j],[k]}$ are well defined, and we impose conditions (8.1.19) and (8.1.21), but omit (8.1.20).

PROPOSITION 8.1.5

Suppose that $\zeta : Obj_1 \to Obj_2$ is a coherent fusion rule algebra homomorphism; (Obj_i is possibly non-abelian). Assume that there is a semisimple, monoidal category, $\mathcal{C}_2$, with objects Obj_2.

(i) *Then there is a monoidal category, $\mathcal{C}_1$, unique up to natural isomorphisms, such that there exists a tensor functor,*

$$(\zeta, \mathcal{F}, C) \,:\, \mathcal{C}_1 \;\longrightarrow\; \mathcal{C}_2 \tag{8.1.149}$$

compatible with the associativity constraint and extending ζ.

(ii) *The monoidal subcategory associated with R is trivial.*

Proof: We can adapt the proof of Proposition 8.1.2 word by word, discarding the definition of ε for the pull back category and omitting the definition in (8.1.61). This will work, since these constraints were only used for the verification of the compatibility condition (8.1.44) which can be ignored for a monoidal functor, as in (8.1.149).

Moreover, commutativity of the fusion rules, with the exception of condition (8.1.148), has nowhere been used in the construction of the functor (8.1.149) and of the associativity constraint in the proof of Proposition 8.1.2. Part ii) of Proposition 8.1.5 is obvious.

$\square$

Next, we wish to formulate a result analogous to that of Lemma 8.1.3, concerning the dependence of the structure matrices on the action of R. Although the monoidal subcategory corresponding to R is assumed to be trivial, it is, in general, not possible to eliminate the R-dependence of the associativity constraint by an appropriate definition of isomorphisms, $H^{i,j}_{[k]}$.

Yet, if we assume that the category is equipped with a braided structure, a convenient general form of the $\hat{r}_2$- and $\hat{F}_2$-matrices can be derived, following the lines of reasoning in the proof of Lemma 8.1.3. But first we study an invariant for braided categories which was already used extensively in Section 7.4.

__LEMMA 8.1.6__

Suppose $R \subset J$ is any subgroup of invertible objects of a quantum category, C.

(i) *Then there exists an invariant of C, given by a character*

$$\bar{m} \in Hom(\bar{R} \otimes Grad(Obj), U(1)), \tag{8.1.150}$$

($\bar{R} = Grad(R)$) such that

$$\varepsilon(\sigma, j)\varepsilon(j, \sigma) = \bar{m}(grad(\sigma), grad(j))\, 1_{\sigma \circ j},$$

for $\sigma \in R$, $j \in J$. The restriction of $\bar{m}$ to $\bar{R} \otimes \bar{R}$ is symmetric.

(ii) *Let $E := i^*\Big(Hom\big(Grad(Obj) \otimes_s Grad(Obj), U(1)\big)\Big)$ be the subgroup of characters defined in (8.1.150) with symmetric restriction to $\bar{R} \otimes \bar{R}$, extending symmetrically to bilinear forms on $Grad(Obj)$. Here i^* is the pull back of $i : \bar{R} \otimes Grad(Obj) \rightarrow Grad(Obj) \otimes_s Grad(Obj)$. We denote by*

$$[\bar{m}] \in Hom\big(\bar{R} \otimes Grad(Obj),\ U(1)\big)/E \tag{8.1.151}$$

the class of $\bar{m}$ in the quotient.

Then $[\bar{m}]$ is unchanged if C is replaced by C^q, defined in (8.1.7) and (8.1.8), with $q \in Hom\Big(\Gamma_4\big(Grad(Obj)\big), U(1)\Big)$. For any $\bar{m}' \in [\bar{m}]$ there exists some q such that $\bar{m}'$ is the invariant (8.1.150) of C^q. If $Grad(Obj)$ is cyclic then the r.h.s. of (8.1.151) is trivial, and $\bar{m} = 0$, for some C^q.

Proof:

For each $\mu \in R$ and $X \in Obj$, we define the endomorphism $m(\mu, X) \in End(X)$ by

$$\varepsilon(X, \mu)\varepsilon(\mu, X) =: \mathbb{I}_\mu \circ m(\mu, X). \tag{8.1.152}$$

Clearly m is isotropic, i.e., $m(\mu, Y)I = Im(\mu, X)$, for any $I \in Mor(X, Y)$. Using the hexagonal equations,

$$\alpha(\mu, X, Y)\varepsilon(X \circ Y, \mu)^{\pm}\alpha(X, Y, \mu) = \big(\varepsilon(X, \mu)^{\pm} \circ \mathbb{I}\big)\alpha(X, \mu, Y)(\mathbb{I} \circ \varepsilon(Y, \mu)^{\pm}),$$

with $\varepsilon(X, Y)^{-} = \big(\varepsilon(Y, X)^{+}\big)^{-1}$, we easily find that

$$m(\mu, X \circ Y) = m(\mu, X) \circ m(\mu, Y),$$

i.e., $m(\mu, \cdot)$ is a grading. We thus have that, for $j \in J$, $m(\mu, j) = \bar{m}(\mu, \mathrm{grad}(j))\mathbb{I}_j$, with $\bar{m}(\mu, \cdot) \in Hom\big(Grad(Obj), U(1)\big)$. By a similar hexagonal constraint, we obtain that

$$m(\mu, X)\, m(\nu, X) = m(\mu \circ \nu, X),$$

for $X \in Obj$, and $\mu, \nu \in R$. These properties of m, together with the symmetry obvious from definition (8.1.152), imply the general form (8.1.150).

From (8.1.8) and (7.267) we have that, for any $\mu \in \bar{R}$ and $g \in \mathrm{Grad}(Obj)$,

$$\bar{m}^q(\mu, g) = \bar{m}(\mu, g)\, \delta q(\mu, g),$$

where $\delta q(\mu,g) \;:=\; q(\mu g)q(g)^{-1}q(\mu)^{-1}$, and $\bar{m}^q$ is the invariant of C^q. It is clear from (7.295) that $\delta q \in E$, so that $[\bar{m}^q] = [\bar{m}]$. Conversely, assume that $\bar{m} \in E$. Then we may use a result from Section 7.4, namely that the map

$$\begin{aligned} \bar{Q} \;:\; G \otimes_s G \;&\to\; \Gamma_4(G), \\ \overline{[g\mid h]} \;&\mapsto\; \{gh\} - \{g\} - \{h\}, \end{aligned} \tag{8.1.153}$$

with $G \otimes_s G := G \otimes G/im(1-T) = G \otimes G/([g \mid h] - [h \mid g])$, as in (7.276), is injective. Hence

$$\bar{Q}^\star \;:\; Hom(\Gamma_4(G), U(1)) \;\twoheadrightarrow\; Hom(G \otimes_s G, U(1)),$$

is onto, and thus, given $\bar{m} \in E$, there exists a $q \in Hom\Big(\Gamma_4\big(\mathrm{Grad}(Obj)\big), U(1)\Big)$, with

$$\bar{m}(g,h) \;=\; \bar{Q}^\star(q)(g,h) \;=\; \delta q(g,h) \;=\; q(gh)q(g)^{-1}q(h)^{-1}.$$

We have $\bar{m} = \bar{m}^q$ iff

$$q(g) \;=\; \hat{q}\big(\pi(g)\big)\, \varepsilon(g), \tag{8.1.154}$$

where π is the projection: $\mathrm{Grad}(Obj) \twoheadrightarrow \bar{G} := \mathrm{Grad}(Obj)/\bar{R}$, $\hat{q} \in Hom(\Gamma_4(\bar{G}),\, U(1))$, and $\varepsilon \in Hom(G, \mathbb{Z}_2)$. The fact that the map

$$i' \;:\; \bar{R} \otimes Grad(Obj) \;\to\; Grad(Obj) \otimes Grad(Obj), \tag{8.1.155}$$

induced by the inclusion $\bar{R} \subset \mathrm{Grad}(Obj)$, is into, for a cyclic $\mathrm{Grad}(Obj)$, and that the right hand side is already symmetric implies the last assertion in part ii) of Lemma 8.1.6. Note that, for general $\bar{R}$ and $\mathrm{Grad}(Obj)$, the group (8.1.151) is non-trivial, and (8.1.155) may have a kernel.

□

We are now in a position to prove the following generalization of Lemma 8.1.3.

LEMMA 8.1.7

Suppose that C is a quantum category, with objects Obj, and $R \subset Obj$ is a subgroup of invertible elements with a free, coherent action on $J \subset Obj$. Assume, furthermore, that

the category associated to R is trivial as a monoidal category. Denote by r_1 and F_1 the usual structure matrices of C and by

$$\pi_R \;:\; Obj \;\to\; \overline{Obj}$$

the fusion rule algebra homomorphism defined in (8.1.142). Finally, let $\gamma : \bar{J} \to J$ be an arbitrary map with $\pi_R \circ \gamma = id$ from which $\eta : J \to R$ and $\sigma_{[i][j],[k]}$ are defined as in (8.1.146) and (8.1.147). Then

(i) *there exist vector spaces $Mor_2([k], [i] \circ [j]) \cong \mathbb{C}^{\tilde{N}_{[i][j],[k]}}$ and isomorphisms $H^{i,j}_{[k]}$, as in (8.1.79), such that the matrices $\hat{r}_2$ and $\hat{F}_2$, defined by (8.1.121) and (8.1.125), satisfy the "gauge-constraints"*

$$\hat{F}_2(\mu, i, j, [k]) \;=\; \hat{F}_2(i, j, \mu, [k]) \;=\; 1, \tag{8.1.156}$$

and

$$\hat{r}_2(\gamma([j]), \mu) \;=\; 1, \tag{8.1.157}$$

for $i, j, k \in J$ and $\mu \in R$.

(ii) *The residual "gauge freedom" preserving the constraints (8.1.156) and (8.1.157) is generated through transformations of the R-category preserving (8.1.156), for $i, j \in R$, by natural transformations of the Mor_2-spaces. More precisely, if $H^{i,j}_{[k]}$ is a set of isomorphisms consistent with (8.1.156) and (8.1.157) then any other such set is given by*

$$\left(H^{i,j}_{[k]}\right)' \;=\; A^{i,j}_{[k]}\, H^{i,j}_{[k]}, \tag{8.1.158}$$

where $A^{i,j}_{[k]} \in End_{\mathbb{C}}\big(Mor_2([k], [i] \circ [j])\big)$ has the form

$$A^{i,j}_{[k]} \;=\; \omega\big(\eta(i) \circ \sigma_{[i][j],[k]}, \eta(j)\big)\omega\big(\eta(i), \sigma_{[i][j],[k]}\big)\, \frac{\xi(k)}{\xi(i)\,\xi(j)}\, a^{[i][j]}_{[k]}, \tag{8.1.159}$$

with $k = \psi_{[k]}(i,j)$, $\xi : J \to U(1)$ (or $\mathbb{C}$), $a^{[i][j]}_{[k]} \in End_{\mathbb{C}}\big(Mor_2([k], [i] \circ [j])\big)$, and $\omega \in Z^2(R, 1; U(1))$.

(iii) *If the "gauge constraints" are obeyed the* $\hat{F}_2$*- and* $\hat{r}_2$ *matrices can be expressed by matrices* $\hat{\hat{F}}_2$ *and* $\hat{\hat{r}}_2$*, whose indices only depend on the classes in* $\overline{Obj}$*, and by*

$$\rho := \hat{r}_2 \restriction_{R\times R} \in Hom(R \otimes R, U(1)), \tag{8.1.160}$$

as follows:

$$\hat{r}_2(j, k, [l]) =$$
$$\rho(\eta(j), \eta(k))\rho\big(\sigma_{[j][k],[l]}, \eta(k)\eta(j)^{-1}\big)\bar{m}\Big(grad(\eta(j)), grad(\gamma([j]))\Big)\hat{\hat{r}}_2([j], [k], [l]), \tag{8.1.161}$$

$$\begin{aligned}
&\hat{F}_2(i, j, k, [l]) = \\
&= \Big(\bigoplus_s \rho\big(\sigma_{[i][j],[s]}, \eta(j)^{-1}\big) \otimes \mathbb{I}_{N_{sk,l}}\Big) \hat{\hat{F}}_2([i], [j], [k], [l]) \\
&\qquad\qquad\qquad\qquad \Big(\bigoplus_s \mathbb{I}_{N_{jk,s}} \otimes \rho\big(\sigma_{[i][s],[l]}, \eta(j)\big)\Big) \\
&= \Big(\bigoplus_s \mathbb{I}_{N_{ij,s}} \otimes \rho\big(\sigma_{[s][k],[l]}, \eta(j)\big)\Big) \hat{\hat{F}}_2([i], [j], [k], [l]) \\
&\qquad\qquad\qquad\qquad \Big(\bigoplus_s \rho\big(\sigma_{[j][k],[s]}, \eta(j)\big)^{-1} \otimes \mathbb{I}_{N_{is,l}}\Big).
\end{aligned} \tag{8.1.162}$$

The matrices $\hat{\hat{r}}_2$ *and* $\hat{\hat{F}}_2$ *are unity if* $[i] = 1$*,* $[j] = 1$ *or* $[k] = 1$*, but, in general, they do not satisfy the pentagon- and hexagon equations.*

(iv) *If* $\hat{F}_2'$ *and* $\hat{r}_2'$ *are the structure matrices in a new gauge,* $\left(H^{ij}_{[k]}\right)'$*, as in (8.1.158), then they are given by the same fomrulae (8.1.161) and (8.1.162), inserting*

$$\rho'(\mu, \nu) = \frac{\omega(\nu, \mu)}{\omega(\mu, \nu)}\, \rho(\mu, \nu), \tag{8.1.162a}$$

$$\hat{\hat{r}}_2'([j], [k], [l]) = a^{[k][j]}_{[l]}\, \hat{\hat{r}}_2([j], [k], [l])(a^{[j][k]}_{[l]})^{-1}, \tag{8.1.162b}$$

$$\begin{aligned}
\hat{\hat{F}}_2'([i], [j], [k], [l]) = &\Big(\bigoplus_{[s]} \omega\big(\sigma_{[i],[j],[s]}, \sigma_{[s][k],[l]}\big) a^{[i][j]}_{[s]} \otimes a^{[j][k]}_{[l]}\Big) \\
&\hat{\hat{F}}_2([i], [j], [k], [l]) \\
&\Big(\bigoplus_{[s]} \omega\big(\sigma_{[i][s],[l]}, \sigma_{[j][k],[s]}\big) a^{[j][k]}_{[s]} \otimes a^{[i][s]}_{[l]}\Big)^{-1}.
\end{aligned} \tag{8.1.162c}$$

Proof:

(i) The proof of the first part of Lemma 8.1.7 is merely a recapitulation of those arguments of the proof of Lemma 8.1.3 that do not require the triviality but only the existence of a braided structure. As in (8.1.80) and (8.1.82), triviality of the monoidal category associated to R implies that there exists $H^{\sigma,\mu}_{[1]}$, $\sigma, \mu \in R$, such that

$$\hat{F}_2(\sigma, \mu, \nu, [1]) = 1,$$

for $\sigma, \mu, \nu \in R$. Imposing (8.1.86), $\hat{F}_2\big(\mu, \nu, \gamma([j]), [j]\big) = 1$, and $\hat{F}_2\big(\gamma([j]), \mu, \nu, [j]\big) = 1$, we derive from the pentagonal constraint (8.1.88) the invariance corresponding to (8.1.89) and (8.1.90), namely

$$\hat{F}_2(\mu, \nu, j, [j]) = \hat{F}_2(j, \mu, \nu, [j]) = 1, \tag{8.1.163}$$

for $\mu, \nu \in R$ and $j \in J$. We retain the "gauge freedom" expressed by (8.1.91). From the pentagonal equations (8.1.94) the cocycle condition (8.1.96) for

$$\psi_{[j]}(1 \mid \mu, \nu) = \hat{F}_2\big(\mu, \gamma([j]), \nu, [j]\big)$$

is derived. Assuming only the existence of a braided structure, we find from (8.1.97) the constraint (8.1.98), with $\rho_{[j]}(\mu, \sigma) = \hat{r}_2\big(\mu, \sigma \circ \gamma([j]), [j]\big)$. Hence $\psi_{[j]}(1 \mid \cdot, \cdot)$ is symmetric and therefore a coboundary. Having a solution $\beta_{[j]} : \sigma \to \beta_{[j]}(\sigma)$ to (8.1.100), we can therefore find a gauge such that $\psi_{[j]}(1 \mid \cdot, \cdot) = 1$. This implies, with (8.1.94) and (8.1.95), that

$$\hat{F}_2(\mu, j, \nu, [j]) = 1, \tag{8.1.164}$$

for $\mu, \nu \in R$ and $j \in J$; (compare to (8.1.103)). We see from (8.1.102) that if we impose (8.1.163) and (8.1.164) and keep the $H^{\mu,\nu}_{[1]}$, for $\mu, \nu \in R$, fixed, then the remaining freedom in choosing $H^{\mu,j}_{[j]}$ and $H^{j,\mu}_{[j]}$ is given by the "gauge transformations"

$$\begin{aligned}
H^{\mu,j}_{[j]} &\longrightarrow \frac{\xi(\mu \circ j)}{\xi(j)}\, H^{\mu,j}_{[j]}, \\
H^{j,\mu}_{[j]} &\longrightarrow \tau_{[j]}(\mu)\, \frac{\xi(\mu \circ j)}{\xi(j)}\, H^{j,\mu}_{[j]},
\end{aligned} \tag{8.1.165}$$

where $\xi : J \to U(1)$ (or $\mathbb{C}$) is any function, with $\xi \mid R \equiv 1$, and $\tau_{[j]} \in Hom(R, U(1))$, with $\tau_{[1]} = 1$. For the $\hat{r}_2$-matrices with arguments in R the transformation law then reads

$$\begin{aligned} \hat{r}_2(j,\mu,[j]) &\longrightarrow \frac{1}{\tau_{[j]}(\mu)}\,\hat{r}_2(j,\mu,[j]), \\ \hat{r}_2(\mu,j,[j]) &\longrightarrow \tau_{[j]}(\mu)\,\hat{r}_2(\mu,j,[j]). \end{aligned} \tag{8.1.166}$$

Considering (8.1.97) and an analogous equation for the inverse r-matrices, we see that, in any gauge consistent with (8.1.163) and (8.1.164),

$$\hat{r}_2(j,\cdot,[j]),\ \hat{r}_2(\cdot,j,[j]) \in Hom(R,U(1)),$$

for all $j \in J$, i.e.,

$$\hat{r}_2(j,\mu\circ\nu,[j]) = \hat{r}_2(j,\mu,[j])\,\hat{r}_2(j,\nu,[j]), \tag{8.1.167}$$

and similarly for $\hat{r}_2(\cdot,j,[j])$.

Setting $\tau_{[j]} := \hat{r}_2(\gamma([j]),\cdot,[j])$, a transformation of the form (8.1.165) produces the desired constraint (8.1.157), as follows from (8.1.166).

Imposing the normalization conditions discussed above, we next consider the special $\hat{F}_2$-matrices defined in (8.1.109) and (8.1.110). The pentagonal equations (8.1.111) and (8.1.114) yield the relations (8.1.112), (8.1.113) and (8.1.115). Performing a gauge transformation as in (8.1.117), we finally find a set of isomorphisms such that the $\hat{F}_2$-matrices fulfill (8.1.156).

(ii) For a general gauge transformation

$$H^{i,j}_{[k]} \longrightarrow A^{i,j}_{[k]}\,H^{i,j}_{[k]}, \tag{8.1.167a}$$

with $A^{i,j}_{[k]} \in Gl(Mor_2([k],[i]\circ[j]))$, the conditions (8.1.156) and (8.1.157) yield the constraints:

$$A^{i,j}_{[k]} \otimes A^{\mu,k}_{[k]} = A^{\mu,i}_{[k]} \otimes A^{\mu\circ i,j}_{[k]} \tag{8.1.168}$$

$$A^{i,j}_{[k]} \otimes A^{k,\mu}_{[k]} = A^{j,\mu}_{[k]} \otimes A^{i,\mu\circ j}_{[k]} \tag{8.1.169}$$

$$A^{\mu,\gamma([j])}_{[k]} = A^{\gamma([j]),\mu}_{[j]}, \tag{8.1.170}$$

where $k = \psi_{[k]}(i,j)$. The most general solution to these equations can be found by first specializing to $i \in R$ or $j \in R$ and determining the form of $A^{\mu,j}_{[j]}$ and $A^{j,\mu}_{[j]}$. The result is

$$A^{\mu,j}_{[j]} = A^{j,\mu}_{[j]} = \omega(\mu,\eta(j))\,\frac{\xi(\mu\circ j)}{\xi(\mu)\,\xi(j)}, \tag{8.1.171}$$

where $\omega \in Z^2(R,1;U(1))$ (see Section 7.4) and $\xi : J \to U(1)$ (or $\mathbb{C}$). Here we may assume that $\xi \mid R \equiv 1$, since we can substitute $\xi'(j) = \xi(j)\Big(\xi\big(\eta(j)\big)\Big)^{-1}$ and $\omega' = \omega(\delta\xi)^{-1}$ without changing $A^{\mu,j}_{[j]}$. Equations (8.1.168), (8.1.169) and (8.1.171) give a complete description of how the transformations $A^{i,j}_{[k]}$ depend on the objects i in an orbit $[i]$ and j in an orbit $[j]$. This dependence can be absorbed into the prefactor of $a^{[i][j]}_{[k]}$ in (8.1.159), using identities (8.1.147) and $\delta\omega = 1$.

(iii) We assume that equations (8.1.156) and (8.1.157) hold true. The hexagonal equation (8.1.105) and the inverse version thereof provide us with the following formula for the action of R:

$$\begin{aligned} \hat{r}_2(\mu\circ j,\nu,[j]) &= \hat{r}_2(\mu,\nu,[1])\,\hat{r}_2(j,\nu,[j]), \\ \hat{r}_2(\nu,\mu\circ j,[j]) &= \hat{r}_2(\nu,\mu,[1])\,\hat{r}_2(\nu,j,[j])\,. \end{aligned} \tag{8.1.172}$$

In particular (8.1.172) and (8.1.167) show that the restriction of $\hat{r}_2$ to $R \times R$ is a bihomomorphism, justifying our definition (8.1.160) of ρ.

We immediately find, with (8.1.157), (8.1.167) and (8.1.172), the general form

$$\begin{aligned} \hat{r}_2(j,\nu,[j]) &= \rho(\eta(j),\nu), \\ \hat{r}_2(\nu,j,[j]) &= \rho(\nu,\eta(j))\,\bar{m}\Big(grad(\nu),\ grad\big(\gamma([j])\big)\Big). \end{aligned} \tag{8.1.173}$$

If we insert (8.1.156) and (8.1.173) into the hexagonal equation (8.1.119) and use (8.1.147) we arrive at

$$\hat{F}_2(i,\mu,j,[k]) = \rho(\sigma_{[i][j],[k]},\mu)\,. \tag{8.1.174}$$

The expressions (8.1.173) and (8.1.174) for $\hat{r}_2$- and $\hat{F}_2$-matrices with arguments in R enable us to find the general transformation properties of the structure matrices under the action of R: The hexagonal equation preceding (8.1.126) yields

$$\begin{aligned}\hat{r}_2(\mu \circ j, k, [l]) &= \\ &= \rho(\mu, \eta(k))\bar{m}\Big(grad(\mu), grad(\gamma([k]))\Big)\rho\big(\sigma_{[j][k],[l]}, \mu\big)^{-1}\hat{r}_2(j, k, [l]),\end{aligned} \qquad (8.1.175)$$

and, similarly, from the inverse hexagonal equation

$$\hat{r}_2(j, \mu \circ k, [l]) = \rho\big(\eta(j), \mu\big)\rho\big(\sigma_{[j][k],[l]}, \mu\big)\hat{r}_2(j, k, [l]) . \qquad (8.1.176)$$

The solution to (8.1.175) and (8.1.176) is given precisely by the expression in (8.1.161), where $\hat{\hat{r}}_2$ only depends on the classes of the objects i, j, k in $\overline{Obj}$. The dependence of the $\hat{F}_2$-matrices on the first and third entry under the action of R has already been determined in the proof of Lemma 8.1.3. With the help of the pentagonal-equation (8.1.122), the invariance (8.1.123) was inferred, so that by using a similar argument for the third index we can write

$$\hat{F}_2(i, j, k, [l]) = \hat{F}_2^0([i], j, [k], [l]) . \qquad (8.1.177)$$

The pentagonal-equations

$$\begin{aligned}&\Big(\bigoplus_s \hat{F}_2\big(i, \mu, j, [s]\big) \otimes \mathbb{1}_{N_{sk,l}}\Big)\hat{F}_2\big(i, \mu \circ j, k, [l]\big)\Big(\bigoplus_s \hat{F}_2\big(\mu, j, k, [s]\big) \otimes \mathbb{1}_{N_{is,l}}\Big) \\ &= \hat{F}_2\big(\mu \circ i, j, k, [l]\big)\Big(\bigoplus_s \mathbb{1}_{N_{jk,s}} \otimes \hat{F}_2\big(i, \mu, s, [l]\big)\Big)\end{aligned}$$

and

$$\begin{aligned}&\Big(\bigoplus_s \hat{F}_2\big(i, j, \mu, [s]\big) \otimes \mathbb{1}_{N_{sk,l}}\Big)\hat{F}_2\big(i, \mu \circ j, k, [l]\big)\Big(\bigoplus_s \hat{F}_2\big(j, \mu, k, [s]\big) \otimes \mathbb{1}_{N_{is,l}}\Big) \\ &= \Big(\bigoplus_s \mathbb{1}_{N_{ij,s}} \otimes \hat{F}_2\big(s, \mu, k, [l]\big)\Big)\hat{F}_2\big(i, j, \mu \circ k, [l]\big)\end{aligned} \qquad (8.1.178)$$

yield the following formula for the action of R on the second index

$$\begin{aligned}&\hat{F}_2^0\big([i], \mu \circ j, [k], [l]\big) = \\ &= \Big(\bigoplus_s \rho\big(\sigma_{[i][j],[s]}, \mu\big)^{-1} \otimes \mathbb{1}_{N_{sk,l}}\Big)\hat{F}_2^0\big([i], j, [k], [l]\big)\Big(\bigoplus_s \mathbb{1}_{N_{jk,s}} \otimes \rho\big(\sigma_{[i][s],[l]}, \mu\big)\Big) \\ &= \Big(\bigoplus_s \mathbb{1}_{N_{ij,s}} \otimes \rho\big(\sigma_{[s][k],[l]}, \mu\big)\Big)\hat{F}_2^0\big([i], j, [k], [l]\big)\Big(\bigoplus_s \rho\big(\sigma_{[j][k],[s]}\big) \otimes \mathbb{1}_{N_{is,l}}\Big).\end{aligned} \qquad (8.1.179)$$

The general solution to (8.1.177) and (8.1.179) is given by (8.1.162). Note that, by (8.1.21), the two expressions given in (8.1.162) and (8.1.179), respectively, are equivalent.

(iv) The gauge dependence given in (8.1.162a) is derived by applying a natural transformation to the defining equation (8.1.160). By the very construction of the $A^{ij}_{[k]}$, this corresponds to adding a coboundary $\delta\omega \in B^4(R, 2; U(1))$ to the θ-category associated with $R \subset Obj$. Formulae (8.1.162b) and (8.1.162c) are obtained by applying a gauge-transformation

$$\begin{aligned}\hat{F}_2(i, j, k, [l]) &= \Big(\bigoplus_{[s]} A^{i,j}_{[s]} \otimes A^{\psi_{[s]}(i,j),k}_{[l]}\Big)\, \hat{F}_2(i, j, k, [l])\Big(\bigoplus_{[s]} A^{j,k}_{[s]} \otimes A^{i,\psi_{[s]}(j,k)}_{[l]}\Big)^{-1} \\ \hat{r}_2(j, k, [l]) &= A^{k,j}_{[l]}\, \hat{r}_2(j, k, [l])\Big(A^{j,k}_{[l]}\Big)^{-1}\end{aligned} \qquad (8.1.180)$$

to the identities

$$\begin{aligned}\hat{\hat{F}}_2\big([i], [j], [k], [l]\big) &= \hat{F}_2\big(\gamma([i]), \gamma([j]), \gamma([k]), [l]\big) \\ \hat{\hat{r}}_2\big([j], [k], [l]\big) &= \hat{r}_2\big(\gamma([j]), \gamma([k]), [l]\big) .\end{aligned} \qquad (8.1.181)$$

□

Until now, we have considered the general case of coherent fusion rule algebra homomorphisms. This structure has turned out to suffice to conclude the existence of induced monoidal categories, in Proposition 8.1.5, and to derive the general dependence of the structural data, braid- and fusion matrices, on the group action, (i.e., the action of R on $J \subset Obj$) in Lemma 8.1.7. In order to give a characterization, analogous to the one in Proposition 8.1.4, of those categories that are induced, as monoidal categories, by smaller ones, we need to find more convenient expressions for the R-dependence of r- and F-matrices from which the structural data of a smaller, braided, monoidal category can be extracted. This problem can be subdivided, in a natural way, into two steps: First, we discuss the action of the subgroup, R_0, of elements in R with trivial grading (see (8.1.143) for the definition). Subsequently, we determine the dependence of the structural data of the reduced category on the action of the graded subgroup $\bar{R} = R/R_0$; (see (8.1.145)). The advantage of working with graded fusion rule algebra homomorphisms is that formulae (8.1.162) simplify considerably. As a consequence, the $\hat{\bar{r}}_2$- and $\hat{\bar{F}}_2$-matrices will then satisfy pentagonal- and hexagonal equations, up to scalar multiples. For the first step, we make use of Lemma 8.1.6 which implies that all monodromies with entries in R_0 vanish. (Note, however, that, since we have no evidence for the existence of coherent, non-graded fusion rule algebras with $R_0 \neq 1$, the following discussion could turn out to be superfluous.)

LEMMA 8.1.8

Let C be a braided tensor category and $R \subset Obj$ a subgroup of invertible objects with a free, coherent action on $J \subset Obj$.

(i) *The subgroup, R_0, of R defined in (8.1.143) also has a free, coherent action on $J \subset Obj$. The Perron-Frobenius algebra, $Obj' = Obj/R_0$, contains $\bar{R}$ as a subgroup of invertible objects with a free, graded action on $J' = J/R_0$. The situation is summarized in the following commutative diagram:*

$$\begin{tikzcd}
Obj \arrow[r, two heads, "\pi_{R_0}"] \arrow[d, "grad"] \arrow[rr, bend left, "\pi_R"] & Obj' \arrow[r, two heads, "\pi_{\bar{R}}"] \arrow[d, "grad'"] & \overline{Obj} \arrow[d, "\overline{grad}"] \\
Grad(Obj) \arrow[r, "\pi^{\#}_{R_0}", "\cong"'] \arrow[rr, bend right, "\pi^{\#}_{R}"'] & Grad(Obj') \arrow[r, two heads, "\pi^{\#}_{\bar{R}}"] & Grad(\overline{Obj})
\end{tikzcd} \tag{8.1.182}$$

(ii) *The subcategory associated with R_0 has abelian permutation group statistics. It is trivial as a monoidal category. There is a "bosonic" subgroup $R_0^+ \subset R_0$, and either $R_0^+ = R_0$ or $R_0/R_0^+ \cong \mathbb{Z}_2$, with the property that the braided tensor category of R_0^+ is trivial, and C is induced, as a braided tensor category, by a category on $Obj'' := Obj/R_0^+$. Moreover, C is induced by a category on Obj' iff $R_0 = R_0^+$.*

(iii) *Let us suppose that C is induced, as a monoidal category, by π_R and a category $\overline{C}$ with object set $\overline{Obj}$. Let us assume, moreover, that C and $\overline{C}$ are equipped with a braided structure. Then there exists a braided, monoidal category C' with objects Obj', such that C is induced, as a braided category, by C' and π_{R_0}, and C' is induced, as a monoidal category, by $\overline{C}$ and $\pi_{\bar{R}}$. In particular, we always have that $R_0 = R_0^+$. Up to automorphisms of $\overline{C}$, the functor from C to $\overline{C}$ is therefore given by the composition*

$$C \xrightarrow{(\pi_{R_0}, \mathcal{F}^0, C^0)} C' \xrightarrow{(\pi_{\bar{R}}, \overline{\mathcal{F}}, \overline{C})} \overline{C}, \tag{8.1.183}$$

where the first functor is compatible with the commutativity constraint. If C is induced by $\overline{C}$, as a braided category, then also C' is induced by $\overline{C}$, as a braided category, and the functor $(\pi_{\bar{R}}, \overline{\mathcal{F}}, \overline{C})$ is compatible with the associativity constraint.

Proof:

(i) As a subgroup of a freely acting group, R_0 clearly also has a free action on J. Hence $Obj' = Obj/R_0$ is a fusion rule algebra, and

$$\pi_{R_0} : Obj \twoheadrightarrow Obj' : j \mapsto \{j\}$$

is a fusion rule algebra homomorphism. Clearly, π_{R_0} maps invertible objects to invertible objects, and the restriction, $\pi_{R_0} : R \twoheadrightarrow \bar{R} \subset Obj'$, is the ordinary projection amounting to taking the quotient by R_0. The fact that R acts freely on J implies that $\bar{R}$ also acts freely on J'. Hence $\pi_{\bar{R}}$ is well defined, assigning to $\{j\}$ the class $[j] \equiv [\{j\}]$, (where, on the left, we may pick any representative $j \in \{j\}$). The composition of $\pi_{\bar{R}}$ with π_{R_0} is just π_R, as indicated in the top row of (8.1.182).

Let us now suppose that we have chosen a map $\gamma : \bar{J} \to J$, along with the corresponding map $\eta : J \to R$, and elements $\sigma_{[i][j],[k]} \in R$. To any section $\psi : \bar{R} \to R$, with $\pi_{R_0} \circ \psi = id_{\bar{R}}$, we associate a choice of a map $\gamma^0 : J' \to J$ as follows. It is clear that there is a unique map $\bar{\eta} : J' \to \bar{R}$ such that the diagram

$$\begin{tikzcd}
J \arrow[r, two heads, "\eta"] \arrow[d, two heads, "\pi_{R_0}"'] & R \arrow[d, two heads, "\pi_{R_0}"] \\
J' \arrow[r, two heads, "\bar{\eta}"'] & \bar{R}
\end{tikzcd} \tag{8.1.184}$$

commutes, and that $\bar{\eta}(\{\mu\} \circ \{j\}) = \{\mu\} \circ \bar{\eta}(\{j\})$, for any $\mu \in R$, with $\{\mu\}$ the corresponding element of $\bar{R}$. We define

$$\gamma^0(\{j\}) := \psi(\bar{\eta}(\{j\})) \circ \gamma([\{j\}])\,. \tag{8.1.185}$$

For $j \in \{j\} \equiv \pi_{R_0}(j)$, it then follows from

$$\begin{aligned}
\pi_{R_0}(\gamma^0(\{j\})) &= \bar{\eta}(\{j\}) \circ \pi_{R_0}(\gamma([\{j\}])) \\
&= \pi_{R_0}(\eta(j)) \circ \pi_{R_0}(\gamma([j])) \\
&= \pi_{R_0}(j) \;=\; \{j\}
\end{aligned}$$

that γ^0 is an admissible selection of representatives in the classes of J'. Since $j = \eta^0(j) \circ \gamma^0(\{j\})$, and by (8.1.185) we find that the map $\eta^0 : J \to R_0$, with $\eta^0(\sigma \circ j) = \sigma \circ \eta(j)$, is determined by

$$\eta(j) = \eta^0(j) \circ \psi(\bar{\eta}(\{j\})). \tag{8.1.186}$$

From (8.1.185) or (8.1.186) we see that

$$A_{\{i\}\{j\},\{k\}} = N_{\{i\}\{j\},\{k\}}\, \sigma_{\{i\}\{j\},\{k\}}, \tag{8.1.187}$$

where we use the definitions of (8.1.18) and (8.1.19) with respect to R_0 and γ^0. The invertible object σ in (8.1.187) is given by

$$\sigma_{\{i\}\{j\},\{k\}} = \psi(\bar{\eta}(\{i\})) \circ \psi(\bar{\eta}(\{j\})) \circ \psi(\bar{\eta}(\{k\}))^{-1} \circ \sigma_{[\{i\}][\{j\}],[\{k\}]}. \tag{8.1.188}$$

It follows immediately from this expresion that π_{R_0} is coherent whenever π_R is coherent.

From (8.1.31), and since $\mathrm{grad}(R_0) = 1$, we infer that $\pi^{\#}_{R_0}$ is an isomorphism. Using the properties of π_{R_0} and the commutativity of (8.1.182) this proves that the restriction

$$grad' : \bar{R} \hookrightarrow Grad(Obj') \tag{8.1.189}$$

is an injection. Hence $\pi_{\bar{R}}$ is a graded and thus coherent fusion rule algebra homomorphism. We remark that $\bar{\eta}$, as defined in (8.1.184), corresponds to the choice of $\bar{\gamma} : \bar{J} \to J'$ given by

$$\bar{\gamma}([j]) := \pi_{R_0}(\gamma([j])). \tag{8.1.190}$$

The elements in $\bar{R}$ corresponding to this choice are given by

$$\bar{\sigma}_{[i][j],[k]} = \pi_{R_0}(\sigma_{[i][j],[k]}).$$

They can be uniquely determined from

$$grad'(\bar{\sigma}_{[i][j],[k]}) = \pi^{\#}_{R_0}\Big(grad(\gamma([i]))\; grad(\gamma([j]))\; grad(\gamma([k]))^{-1}\Big), \tag{8.1.191}$$

using that (8.1.189) is injective. This proves part i) of Lemma 8.1.8.

(ii) From Lemma 8.1.6, i) we see that

$$\varepsilon(\sigma, X) = \varepsilon(X, \sigma)^{-1}, \qquad \text{for} \quad \sigma \in R_0. \tag{8.1.192}$$

For $X = \mu \in R_0$, this proves that the category determined by R_0 has permutation group statistics. For the quadratic function $q(\sigma)\mathbb{1}_{\sigma\circ\sigma} := \varepsilon(\sigma, \sigma)$, this implies, using that $q(\sigma\mu)\, q(\sigma)^{-1}\, q(\mu)^{-1} = \varepsilon(\sigma, \mu)\, \varepsilon(\mu\sigma)$:

$$q \in Hom(R_0, \mathbb{Z}_2). \tag{8.1.193}$$

Hence we can define a subgroup $R_0^+ := ker\ q$ for which the associated θ-category is trivial. Let α be the non-trivial element in $Hom(\mathbb{Z}_2 \otimes \mathbb{Z}_2, U(1))$, and consider the function $\hat{r}_2(\mu, \sigma)\alpha(q(\mu), q(\sigma))^{-1}$. The logarithm of this function is skew symmetric and vanishes on the diagonal. Hence $\hat{r}_2(\mu, \sigma)\alpha(q(\mu), q(\sigma))^{-1}$ can be written as

$\xi(\mu,\sigma)\xi(\sigma,\mu)^{-1}$ for some function $\xi : R_0 \times R_0 \to U(1)(\mathbb{C})$. Gauging the $H_{[1]}^{\mu,\sigma}$ with ξ, i.e., adding the coboundary $\delta\xi \in B^4(R_0, 2; U(1))$ to the structure constants of the θ-category determined by R_0, we can achieve that

$$\hat{r}_0(\mu,\sigma) \;=\; \alpha\big(q(\mu), q(\sigma)\big). \tag{8.1.194}$$

With (8.1.193), this implies that $\hat{r}_2 \in Hom\big(R_0 \otimes R_0, U(1)\big)$. Thus the monoidal category determined by R_0 is trivial. If we set $R = R_0^+$ in Lemma 8.1.7, iii) we infer from equations (8.1.161) and (8.1.162), using that $\rho = \hat{r}_2 \restriction_{R_0 \times R_0} = 1$ and $\mathrm{grad}\big(\eta(j)\big) = 1$, for all $j \in J$, that there is a choice of H's such that $\hat{r}_2 = \hat{\hat{r}}_2$ and $\hat{F}_2 = \hat{\hat{F}}_2$ are invariant under the action of R_0^+. As described in the proof of Proposition 8.1.4, this implies the existence of a braided category, $\mathcal{C}''$, with objects $Obj'' = Obj/R_0^+$ and a functor

$$\big(\pi_{R_0}^+, \mathcal{F}, \mathcal{C}\big) \;=\; \mathcal{C} \longrightarrow \mathcal{C}'',$$

i.e., $\mathcal{C}$ is induced by $\mathcal{C}''$ and $\pi_{R_0}^+$. In $\mathcal{C}''$ we have that $R_0'' = R_0/R_0^+$, so that, for $q \neq 1$, we conclude that $R_0'' \cong \mathbb{Z}_2$, where R_0'' is generated by a fermionic object σ, with $\varepsilon''(\sigma,\sigma) = -\mathbb{1}$.

Since $q \restriction_{R_0}$ is an invariant, we conclude that $\mathcal{C}$ is induced by a category on Obj' only if $q \equiv 1$ on R_0, i.e., $R_0 = R_0^+$. If this is the case the previous argument shows, in particular, that $\mathcal{C}$ is induced by π_{R_0} and a category on Obj'. This proves part ii) of the lemma.

(iii) Suppose that $\mathcal{C}$ is induced by $\overline{\mathcal{C}}$ and π_R and that both, $\mathcal{C}$ and $\overline{\mathcal{C}}$, are equipped with a braided structure. We then have a collection of isomorphisms, $H_{[k]}^{i,j} : Mor_1(k, i \circ j) \to Mor_2([k], [i] \circ [j])$, such that (8.1.75) holds. We may consider the category $\mathcal{C}'$, with objects Obj', which is induced by $\overline{\mathcal{C}}$ and $\pi_{\bar{R}}$. A choice of $\bar{R}$-invariant structure matrices, $\hat{F}_2'$, can be found for any collection

$$H_{[k]}^{\{i\}\{j\}} : Mor_2'(\{k\}, \{i\} \circ \{j\}) \xrightarrow{\cong} Mor_2([\{k\}], [\{i\}] \circ [\{j\}]),$$

by use of (8.1.85). Setting

$$H_{\{k\}}^{i,j} := \Big(H_{[k]}^{\{i\}\{j\}}\Big)^{-1} H_{[k]}^{i,j} : Mor_1(k, i \circ j) \to Mor_2'(\{k\}, \{i\} \circ \{j\}), \tag{8.1.195}$$

we obviously have a solution to eq. (8.1.75), relating the structure matrices F_1 and $\hat{F}_2'$ of $\mathcal{C}$ and $\mathcal{C}'$. By the R_0-invariance of the $\hat{F}_2'$-matrices, it follows from (8.1.174) that, for $\hat{r}_2'$ defined by (8.1.76) in terms of the r_1-matrices, one has that $\hat{r}_2'\,(\sigma_{\{i\}\{j\},\{k\}}, \mu) = 1$. Hence, in particular, we conclude that

$$q(\sigma_{\{i\}\{j\},\{k\}}) \;=\; 1\,. \tag{8.1.196}$$

However, (8.1.147), (8.1.193) and (8.1.196) imply that

$$j \;\rightarrow\; q\big(\eta^0(j)\big) \tag{8.1.197}$$

is a $\mathbb{Z}_2$-grading on Obj. By definition of R_0, this has to be trivial on R_0. This means that $q \restriction_{R_0} = 1$, i.e., the subcategory R_0^+ of R_0 is trivial as a braided category, as well. Thus, there exists a unique braided monoidal category $\mathcal{C}'$ with objects Obj' such that $\mathcal{C}$ is induced, as a braided category, by $\mathcal{C}'$ and π_{R_0}. This proves (8.1.183), with $(\pi_{\bar{R}}, \overline{\mathcal{F}}, \bar{\mathcal{C}})$ a functor compatible with the associativity constraint and constructed from the isomorphisms $H_{[k]}^{\{i\}\{j\}}$.

If we assume that $\mathcal{C}$ is induced by $\overline{\mathcal{C}}$, as a braided category, then we find structure-matrices for $\mathcal{C}$, $\mathcal{C}'$ and $\overline{\mathcal{C}}$ such that, for suitable isomorphisms $H_{[k]}^{i,j}$, the data (r_1, F_1) of $\mathcal{C}$ and the data $(\hat{r}_2, \hat{F}_2)$ of $\overline{\mathcal{C}}$ are related by (8.1.75) and (8.1.76). Furthermore, for suitable isomorphisms $H_{\{k\}}^{i,j}$, the data (r_1, F_1) of $\mathcal{C}$ are related to the data $(\hat{r}_2', \hat{F}_2')$ by the same equations. It follows immediately that the isomorphisms $H_{[k]}^{\{i\}\{j\}}$ defined by (8.1.195) provide a solution to eqs. (8.1.75) and (8.1.76) if we insert the structure matrices of the categories $\mathcal{C}'$ and $\overline{\mathcal{C}}$. Using the arguments of Proposition 8.1.4, this is seen to imply that we can choose $(\pi_{\bar{R}}, \overline{\mathcal{F}}, \bar{\mathcal{C}})$ to be compatible with the commutativity constraint. ☐

If we assume that R_0 does not contain a fermionic object, i.e, $R_0 = R_0^+$, then Lemma 8.1.8 shows that it is suffiicient to study graded fuison-rule algebra homomorphisms ($R_0 = 1$), in order to get a complete characterization of induced categories. In fact, in most applications, we will have graded fusion rule algebra homomorphisms right from the beginning.

The advantage gained from a graded action of R on J ist that, for a convenient choice of γ, and by use of (8.1.29), the structure constants $\hat{r}_2$ and $\hat{F}_2$, as presented in (8.1.161) and (8.1.162), will be proportional to $\hat{\hat{r}}_2$ and $\hat{\hat{F}}_2$, and the corresponding factors of proportionality do not depend on the arguments of $\hat{r}_2$ and $\hat{F}_2$ but only on their gradings. Let us recall some basic facts on graded fusion rule algebras and present simplified versions of eqs. (8.1.161) and (8.1.162). If R has a graded action (8.1.144) reduces to a pair of short exact sequences:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r, hook] \arrow[d, "\cong"] & J \arrow[r, two heads, "\pi_R"] \arrow[d, "grad"] & \bar{J} \arrow[r] \arrow[d, "\overline{grad}"] \arrow[l, bend right, "\gamma"'] & 0 \\
0 \arrow[r] & R \arrow[r, hook, "grad"] & Grad(Obj) \arrow[r, two heads, "\pi_R^{\#}"] & Grad(\overline{Obj}) \arrow[r] \arrow[l, bend left, "\bar{\psi}"] & 0
\end{tikzcd} \qquad (8.1.198)$$

where the squares in the middle commute. Here we also require a section $\bar{\psi}$: Grad$(\overline{Obj})$ $\hookrightarrow$ Grad(Obj), with $\pi_R^{\#} \circ \bar{\psi} = id$. With any such $\bar{\psi}$ we can associate a symmetric cocycle

$$\xi \in Z^2\left(Grad(\overline{Obj}), 1;\ R\right),$$

by setting

$$grad\left(\xi(g,h)\right) = \bar{\psi}(g)\, \bar{\psi}(g \cdot h)^{-1}\, \bar{\psi}(h), \qquad (8.1.199)$$

where we use exactness of the lower row in (8.1.198). The ambiguity in choosing $\bar{\psi}$, corresponding to multiplication by a function λ : Grad$(\overline{Obj}) \to R$, implies that ξ is

only given up to boundaries $\delta\lambda$. As explained in the analysis following (7.287), the possible ξ's correspond to the classes

$$[\xi] \in \mathrm{Ext}\big(Grad(\overline{Obj}), R\big) \subset H^2\left(Grad(\overline{Obj}), 1; R\right) \qquad (8.1.200)$$

which describe the possible extensions of R over Grad$(\overline{Obj})$, given by Grad(Obj) and the short exact sequence in (8.1.198).

The circumstance that, from two groups, R and Grad$(\overline{Obj})$, and an extension $[\xi]$, one finds a new group, Grad(Obj), containing R and Grad$(Obj)/R \cong$ Grad$(\overline{Obj})$ motivates the following generalization, where the gradation groups are replaced by fusion rule algebras. We assume that $\overline{Obj}$ is a fusion rule algebra, R an abelian group and $[\xi] \in$ Ext(Grad$(\overline{Obj}), R$). Then the algebra $\overline{Obj}\ \text{ⓔ}_{[\xi]}\ R$ is defined as follows: The objects are of the form $\sum_{\mu\in R}(X_\mu, \mu)$, with $(X+Y,\mu) = (X,\mu)+(Y,\mu)$, for $X_\mu, X, Y \in \overline{Obj}$. Thus the irreducible objects are given by $J = \{(j,\mu)\}_{j\in\bar{J},\mu\in R}$. The tensor product is defined by

$$(i,\mu)\circ(j,\nu) = \Big(i \circ j, \mu\circ\nu\circ\xi\big(grad(i), grad(j)\big)\Big), \qquad (8.1.201)$$

where we have chosen some representative $\xi \in [\xi]$.

Up to isomorphism, this fusion rule algebra is independent of the particular choice of a representative in the class $[\xi]$, because $(j,\mu) \mapsto \big(j, \mu\circ\lambda(grad(j))\big)$ provides an isomorphism from the algebra defined with the help of $\xi\cdot\delta\lambda$ to the algebra defined with the help of ξ.

The universal grading is given by the group Grad(Obj) associated to the extension $[\xi]$ of R over Grad$(\overline{Obj})$. An injection $R \hookrightarrow$ Grad(Obj) and a choice of some section $\bar{\psi}$: Grad$(\overline{Obj}) \hookrightarrow$ Grad(Obj) satisfying $\pi_R \circ \bar{\psi} = id$ and eq. (8.1.199) determines the grading to be given by

$$grad\big((j,\mu)\big) = \bar{\psi}\big(grad(j)\big)\circ\mu\,. \qquad (8.1.202)$$

The algebra $\overline{Obj}\ \text{ⓔ}_{[\xi]}\ R$ contains R as a subgroup of invertible objects with a free, graded action on J. The quotient of $\overline{Obj}\ \text{ⓔ}_{[\xi]}\ R$ by R is precisely $\overline{Obj}$, with a homomorphism

π_R given by

$$\pi_R((j,\mu)) \;=\; j\,. \tag{8.1.203}$$

In fact, these properties determine the algebra $\overline{Obj}\ \textcircled{e}_{[\xi]}\, R$ completely.

LEMMA 8.1.9

Assume that $R \subset Obj$ is a subgroup of invertible objects with a free, graded action on J. Let $\overline{Obj} := Obj/R$ be the Perron-Frobenius quotient by R, and denote by $[\xi] \in Ext\big(Grad(\overline{Obj}), R\big)$ the extension of gradation groups induced by (8.1.182). Then the following statements hold true:

(i) *For any $\bar\psi \in \{Grad(\overline{Obj}) \to Grad(Obj)\}$, with $\pi_R^{\#} \circ \bar\psi = id$, there is a unique choice of a map $\gamma : \bar J \to J$, with $\pi_R \circ \gamma = id$, such that*

$$\bar\psi\big(\overline{grad}([j])\big) \;=\; grad\big(\gamma([j])\big)\,, \tag{8.1.204}$$

i.e., the right, outer square in (8.1.198) commutes. The corresponding map $\eta : J \to R$, and the group elements, $\sigma_{[i][j],[k]}$ are given by

$$grad\big(\eta(j)\big) \;=\; grad(j)\Big(\bar\psi\big(\overline{grad}([j])\big)\Big)^{-1}, \tag{8.1.205}$$

$$\sigma_{[i][j],[k]} \;=\; \xi\big(\overline{grad}([i]), \overline{grad}([j])\big), \tag{8.1.206}$$

where $\xi \in [\xi]$ is the representative obtained from $\bar\psi$.

(ii) *Furthermore,*

$$Obj \;\cong\; \overline{Obj}\ \textcircled{e}_{[\xi]}\, R \tag{8.1.207}$$

as a fusion rule algebra, i.e., $\overline{Obj}, R$ and $[\xi]$ determine Obj completely. If, for some $\xi \in [\xi]$, the tensor product $\overline{Obj}\ \textcircled{e}_{[\xi]}\, R$ is given by (8.1.201) then an explicit isomorphism of fusion rule algebras is given by

$$\begin{aligned} &([j],\mu) \;\longrightarrow\; \mu \circ \gamma([j]), \\ \textit{with inverse}\qquad &j \;\longrightarrow\; ([j],\eta(j))\,, \end{aligned} \tag{8.1.208}$$

where γ and η are the maps associated with ξ and some section $\bar\psi$ by (8.1.204) and (8.1.205).

Proof. i) We first consider the expression on the right hand side of (8.1.205). Using that $\pi_R^{\#} \circ \bar\psi = id$, it follows that the expression lies in the kernel of $\pi_R^{\#}$. Hence, by the exactness of the lower sequence in (8.1.198), we find a function $\eta : J \to R$ such that (8.1.205) holds. The covariance condition $\eta(\sigma \circ j) = \sigma \circ \eta(j)$ is obvious from (8.1.205). Hence the map $\gamma([j])$ is well defined by setting $j = \eta(j) \circ \gamma([j])$. Inserting $\gamma([j])$ in (8.1.205), we arrive at (8.1.204). Equation (8.1.206) is found by combining (8.1.29), (8.1.199) and (8.1.204).

Part ii) of the lemma can be verified directly by using the results of part i).

□

We remark that the map η from J to R can be expressed in terms of the function

$$\eta' \;:\; Grad(Obj) \;\to\; R, \quad g \;\to\; g\big(\bar\psi \circ \pi_R^{\#}(g)\big)^{-1} \tag{8.1.209}$$

as

$$\eta \;=\; \eta' \circ grad\,. \tag{8.1.210}$$

In the next lemma we evaluate the expressions for the structure matrices found in Lemma 8.1.7, using the special forms of γ, η and $\sigma_{[i][j],[k]}$ given above. The problem of extracting a braided tensor category with object set $\overline{Obj}$ from a category on Obj can then be translated into a problem of group cohomology.

LEMMA 8.1.10

Suppose that $R \subset Obj$ is as above and that C is a braided, monoidal category, with objects Obj, which is trivial on R as a monoidal category.

Let $\bar\psi : Grad(\overline{Obj}) \to Grad(Obj)$ be an arbitrary section and $\xi \in Z^2\big(Grad(\overline{Obj}), 1; R\big)$ the associated, symmetric cocycle. Then the following statements hold true:

(i) *There exist vector spaces $Mor_2([k],[i] \circ [j]) \cong \mathbb{C}^{\bar N_{[i][j],[k]}}$, and isomorphisms $H^{i,j}_{[k]}$, as in (8.1.79), such that the matrices $\hat r_2$ and $\hat F_2$, defined by (8.1.121), can be expressed by the phase factors ρ introduced in (8.1.160) and by matrices $\hat{\hat F}_2$ and*

$\hat{\hat{r}}_2$, *in the following way:*

$$\hat{F}_2(i,j,k,[l]) =$$
$$= \rho\Big(\xi\big(\overline{grad}([i]) \circ \overline{grad}([j]), \overline{grad}([k])\big) \cdot \xi\big(\overline{grad}([j]), \overline{grad}([k])\big)^{-1}, \eta(j)\Big)$$
$$\hat{\hat{F}}_2([i],[j],[k],[l]) \tag{8.1.211}$$

$$\hat{r}_2(j,k,[l]) =$$
$$= \rho(\eta(j),\eta(k))\rho\Big(\xi\big(\overline{grad}([j]), \overline{grad}([k])\big), \eta(k)\cdot\eta(j)^{-1}\Big)\bar{m}\big(grad(\eta(j)); \bar{\psi}\big(\overline{grad}([j])\big)\big)$$
$$\hat{\hat{r}}_2([j],[k],[l]), \tag{8.1:212}$$

where η is given in (8.1.205) and $\bar{m}$ in (8.1.150).

(ii) *We define $\omega \in A^5(Grad(\overline{Obj}), 2; M)$ (with $M = \mathbb{C}$ or $U(1)$, and $A^*(G,n;M)$ as in Chapter 7.3) by the following formulae:*

$$\omega([g_1 \mid g_2 \mid g_3 \mid g_4]) \equiv \omega^0(g_1,g_2,g_3,g_4) :=$$
$$= \rho\big(\xi(g_1\cdot g_2\cdot g_3, g_4)\xi(g_2\cdot g_3, g_4)^{-1}, \xi(g_2,g_3)\big) \tag{8.1.213}$$

$$\omega([g_1 \mid g_2 \| g_3]) \equiv \omega^+(g_1,g_2 \mid g_3) :=$$
$$= \rho\big(\xi(g_1\cdot g_2, g_3), \xi(g_1,g_2)\big)\bar{m}\big(\xi(g_1,g_2), \bar{\psi}(g_3)\big)^{-1} \tag{8.1.214}$$

$$\omega([g_3 \| g_1 \mid g_2]) \equiv \omega^-(g_1,g_2 \mid g_3) :=$$
$$= \rho\big(\xi(g_1\cdot g_2, g_3),\ \xi(g_1,g_2)\big)^{-1}\,. \tag{8.1.215}$$

Then ω is a cocycle, i.e.,

$$\omega \in Z^5\left(Grad(\overline{Obj}), 2; M\right). \tag{8.1.216}$$

(iii) *The reduced structure matrices obey the following modified categorial equations.*

Pentagonal equations:

$$\Big(\bigoplus_{[s]} \hat{\hat{F}}_2([i],[j],[k],[s]) \otimes \mathbb{I}_{N_{sl,t}}\Big)\Big(\bigoplus_{[s]} \mathbb{I}_{N_{jk,s}} \otimes \hat{\hat{F}}_2([i],[s],[l],[t])\Big)$$
$$\Big(\bigoplus_{[s]} \hat{\hat{F}}_2([j],[k],[l],[s]) \otimes \mathbb{I}_{N_{is,t}}\Big)$$
$$= \omega^0(g_i,g_j,g_k,g_l)^{-1}\Big(\bigoplus_{[s]} \mathbb{I}_{N_{ij,s}} \otimes \hat{\hat{F}}_2([s],[k],[l],[t])\Big)T_{12}\Big(\bigoplus_{[s]} \mathbb{I}_{N_{kl,s}}$$
$$\otimes \hat{\hat{F}}_2([i],[j],[s],[t])\Big) \tag{8.1.217}$$

Hexagonal, +:

$$\Big(\bigoplus_{[l]} \hat{\hat{r}}_2([i],[k],[l]) \otimes \mathbb{I}_{N_{lj,t}}\Big)\hat{\hat{F}}_2([i],[k],[j],[t])\Big(\bigoplus_{[l]} \hat{\hat{r}}_2([j],[k],[l]) \otimes \mathbb{I}_{N_{il,t}}\Big)$$
$$= \omega^+(g_i,g_j \mid g_k)^{-1}\hat{\hat{F}}_2([k],[i],[j],[t])\Big(\bigoplus_{[l]} \mathbb{I}_{N_{ij,l}} \otimes \hat{\hat{r}}_2([l],[k],[t])\Big)\hat{\hat{F}}_2([i],[j],[k],[t]) \tag{8.1.218}$$

Hexagonal, -:

$$\Big(\bigoplus_{[l]} \hat{\hat{r}}_2([k],[i],[l])^{-1} \otimes \mathbb{I}_{N_{lj,t}}\Big)\hat{\hat{F}}_2([i],[k],[j],[t])\Big(\bigoplus_{[l]} \hat{\hat{r}}_2([k],[j],[l])^{-1} \otimes \mathbb{I}_{N_{il,t}}\Big)$$
$$= \omega^-(g_i,g_j \mid g_k)\hat{\hat{F}}_2([k],[i],[j],[t])\Big(\bigoplus_{[l]} \mathbb{I}_{N_{ij,l}} \otimes \hat{\hat{r}}_2([k],[l],[t])^{-1}\Big)\hat{\hat{F}}_2([i],[j],[k],[t])\,. \tag{8.1.219}$$

Here we are using the abbreviations $g_i = \overline{grad}([i])$, etc..

(iv) *For any $\lambda \in A^4(Grad(Obj), 2; M)$, we set*

$$F_2([i],[j],[k],[l]) = \lambda([g_i \mid g_j \mid g_k])\hat{\hat{F}}_2([i],[j],[k],[l]),$$
$$r_2([i],[j],[k]) = \lambda([g_i \| g_j])\hat{\hat{r}}_2([i],[j],[k])\,. \tag{8.1.220}$$

Then the matrices F_2 and r_2 satisfy the modified categorial relations (8.1.217), (8.1.218) and (8.1.219), where ω is replaced by

$$\omega' = \omega(\delta\lambda)^{-1}\,. \tag{8.1.221}$$

Hence, the obstruction against finding a solution to the usual categorial equations by rescalings, as in (8.1.220), lies in

$$H^5(Grad(Obj), 2; M)\,. \tag{8.1.222}$$

Proof.

The assertions made in Lemma 8.1.10 are verified by straightforward computations which we will not reproduce here. Nevertheless, we shall assist the readers' task with the following remarks and formulae.

i) In order to obtain (8.1.211) and (8.1.212) we insert (8.1.206) into the relations (8.1.161) and (8.1.162). Since $\sigma_{[i][j],[k]}$ only depends on the grading of its indices, and since the grading of the summation indices in (8.1.162) is fixed by g_i, g_j and g_k, the diagonal matrices in (8.1.162) are, in fact, multiples of the identity which combine to the factor in (8.1.211).

ii) The cocycle condition (8.1.216) is given by the following five equations:

$$\omega^0 \in Z^4(Grad(Obj), 1; M)\ , \tag{8.1.223}$$

and

$$\begin{aligned}\omega^0(g_1,g_2,g_3,g_4)\omega^0(g_2,g_1,g_3,g_4)^{-1}\omega^0(g_2,g_3,g_1,g_4)\omega^0(g_2,g_3,g_4,g_1)^{-1} &= \\ = \omega^-(g_3,g_4 \mid g_1)^{-1}\omega^-(g_2g_3,g_4 \mid g_1)\omega^-(g_2,g_3g_4 \mid g_1)^{-1}\omega^-(g_2,g_3 \mid g_1)\end{aligned} \tag{8.1.224}$$

$$\begin{aligned}\omega^0(g_1,g_2,g_3,g_4)\omega^0(g_1,g_2,g_4,g_3)^{-1}\omega^0(g_1,g_4,g_2,g_3)\omega^0(g_4,g_1,g_2,g_3)^{-1} &= \\ = \omega^+(g_1,g_2 \mid g_4)\omega^+(g_1,g_2g_3 \mid g_4)^{-1}\omega^+(g_1g_2,g_3 \mid g_4)\omega^+(g_2,g_3 \mid g_4)^{-1}\end{aligned} \tag{8.1.225}$$

$$\omega^+(g_1,g_2 \mid g_3)\omega^+(g_2,g_1 \mid g_3)^{-1} = \omega^-(g_3,g_2 \mid g_1)\omega^-(g_2,g_3 \mid g_1)^{-1} \tag{8.1.226}$$

$$\begin{aligned}&\omega^0(g_1,g_2,g_3,g_4)\omega^0(g_1,g_3,g_2,g_4)^{-1}\omega^0(g_3,g_1,g_2,g_4)\, \cdot \\ &\qquad \cdot\, \omega^0(g_1,g_3,g_4,g_2)\omega^0(g_3,g_1,g_4,g_2)^{-1}\omega^0(g_3,g_4,g_1,g_2) \\ &= \omega^+(g_1,g_2 \mid g_3)^{-1}\omega^+(g_1,g_2 \mid g_4)^{-1}\, \cdot \\ &\qquad \cdot\, \omega^-(g_3,g_4 \mid g_2)\omega^-(g_3,g_4 \mid g_1g_2)^{-1}\omega^-(g_3,g_4 \mid g_1)\ .\end{aligned} \tag{8.1.227}$$

For the verification of (8.1.216) it is useful to observe that the special function ω, given in (8.1.213) - (8.1.215), has the symmetry properties

$$\begin{aligned}\omega^\pm(g_1,g_2 \mid g_3) &= \omega^\pm(g_2,g_1 \mid g_3) \\ \omega^0(g_1,g_2,g_3,g_4) &= \omega^0(g_1,g_3,g_2,g_4) = \omega^0(g_4,g_2,g_3,g_1)\ .\end{aligned} \tag{8.1.228}$$

Parts iii) and iv) simply follow by inserting formulae (8.1.211) and (8.1.212) into the usual categorial equations and formulae (8.1.220) into the modified categorial equations (8.1.217) - (8.1.219). The expressions for $\delta\lambda$ are given by (7.290), (7.293) and (7.294).

☐

The strategy we are pursuing here for expressing categories with graded subgroups by smaller ones involves the concept of induced categories, combined with the operation $\mathcal{C} \to \mathcal{C}^q$, for $q \in Hom\Big(\Gamma_4\big(Grad(Obj)\big), U(1)\Big)$, described at the beginning of this chapter. In the examples we are interested in, the categories associated with the subgroups of invertible elements can be converted into categories with permutation statistics. Thus, the remaining obstruction to trivialize such a category is the extendability of the relevant quadratic forms, i.e., the signatures, to the entire universal grading group. As a starting point to a more detailed analysis of this situation we make the following definition:

Consider the map

$$i_2 : R/2R \longrightarrow Grad(Obj)/2\, Grad(Obj) \tag{8.1.229}$$

defined by requiring commutativity of the diagram

$$\begin{tikzcd} 0 \arrow[r] & 2R \arrow[r, hook] \arrow[d, hook, "i"] & R \arrow[r, two heads, "p^R"] \arrow[d, hook, "i"] & R/2R \arrow[r] \arrow[d, "i_2"] & 0 \\ 0 \arrow[r] & 2\, Grad(Obj) \arrow[r, hook] & Grad(Obj) \arrow[r, two heads, "p^G"] & Grad(Obj)/2\, Grad(Obj) \arrow[r] & 0 \end{tikzcd} \tag{8.1.230}$$

The groups in (8.1.229) contain only elements of prime order two and thus give rise to vector spaces over the field $\mathbb{Z}_2$, (with scalar multiplication $(\varepsilon, g) \to g^{\varepsilon}, \varepsilon \in \mathbb{Z}_2$). We can therefore find a space complementary to the kernel, $\left(R \cap 2Grad(Obj)\right)/2R$, of i_2. Its preimage, $\tilde{R}$, in R is characterized by the properties

$$2R \subset \tilde{R} \subset R$$

$$R/2R \; = \; \left(R \cap 2Grad(Obj)\right)/2R \,\oplus\, \tilde{R}/2R\,. \qquad (8.1.231)$$

Definition.

We shall call a subgroup $\tilde{R} \subset R$ satisfying (8.1.231) a <u>maximal</u>, <u>signature-extendable</u> subgroup (for reasons that become clear below).

LEMMA 8.1.11

Let C be a braided, monoidal category with objects Obj, $R \subset Obj$ a subgroup of invertible elements with a free, graded action on Obj, and $\tilde{R} \subset R$ a maximal, signature-extendable subgroup thereof.

Assume that $\bar{m} \in Hom\left(R \otimes Grad(Obj), U(1)\right)$ (see (8.1.150)) has a symmetric extension to $Grad(Obj)^{\otimes 2}$, i.e., the class $[\bar{m}]$, as in (8.1.151), is trivial: $[\bar{m}] = 0$.

Then we have the following results:

(i) *There exists a quadratic function $q \in Hom\Big(\Gamma_4\big(Grad(Obj)\big), U(1)\Big)$ such that C^q is induced, as a braided category, by some category $\tilde{C}$, with objects $\widetilde{Obj} := Obj/\tilde{R}$ and homomorphism, $\pi_{\tilde{R}}$.*

(ii) *The subgroup $\bar{R} : \pi_{\tilde{R}}(R) \cong R/\tilde{R}$, of invertible elements in $\widetilde{Obj}$ obeys*

$$\bar{R} \subset 2\Big({}_4\big(Grad(\widetilde{Obj})\big)\Big)\,. \qquad (8.1.232)$$

Here, the quadratic form q can be chosen such that the subcategory of $\tilde{C}$ associated with $\bar{R}$ is trivial, as a monoidal category, and has permutation statistics. This enables us to find, for some gauge, an element

$$\rho \;\in\; Hom(\bar{R} \otimes \bar{R}, \mathbb{Z}_2)\,. \qquad (8.1.233)$$

Moreover, we have that

$$\bar{m} \;=\; 0, \qquad on\ \tilde{C}\,. \qquad (8.1.234)$$

Proof.

(i) We take it from Lemma 8.1.6 that there exists a quadratic function $\bar{q}^0 \in Hom\Big(\Gamma_4\,\big(Grad(Obj)\big), U(1)\Big)$ such that

$$\bar{m}^{\bar{q}^0} \;\equiv\; 1\,, \qquad (8.1.235)$$

as defined in (8.1.152). In particular, the monodromies on R vanish, and hence the quadratic function q^0, given by $q^0(g) := \rho(g, g)$, $g \in R$, satisfies

$$q^0 \;\in\; Hom(R, \mathbb{Z}_2)\,. \qquad (8.1.236)$$

The quadratic function $\bar{q}^0$ can always be multiplied by an expression of the form (8.1.154) without changing (8.1.235). Hence q_0 can always be replaced by

$$q \;=\; q^0\,\varepsilon^{-1}\,, \qquad (8.1.237)$$

with

$$\varepsilon \;\in\; i^{\star}\,\Big(Hom\big(Grad(Obj),\ \mathbb{Z}_2\big)\Big)\,,$$

(i.e., ε is extendable to $Grad(Obj)$). Next, we show that, for any given subgroup $\tilde{R} \subset R$ satisfying (8.1.231), we can find an ε such that $\tilde{R}$ is in the kernel of the quadratic form q.

Since the map i_2 in (8.1.229) gives rise to a linear map between vector spaces over the field $\mathbb{Z}_2$, we can find a homomorphism

$$\psi \;:\; Grad(Obj)/2\; Grad(Obj) \;\longrightarrow\; R/2R \qquad (8.1.238)$$

such that $\psi \circ i_2$ is the projection onto the summand $\tilde{R}/2R$ in the decomposition (8.1.231), i.e.,

$$\psi \circ i_2 \restriction_{\tilde{R}/2R} = \;id\,. \qquad (8.1.239)$$

Clearly, $q^0 = \hat{q}^0 \circ p^R$, for some $\hat{q}^0 \in Hom(R/2R, U(1))$.

Setting

$$\varepsilon \ = \ \hat{q}^0 \circ \psi \circ p^G, \tag{8.1.240}$$

it follows from the equation $\varepsilon \circ i = \hat{q}^0 \circ \psi \circ p^G \circ i = \hat{q}^0 \circ \psi \circ i_2 \circ p^R$ and from (8.1.239) that $\varepsilon \circ i \restriction_{\tilde{R}}$. Inserting this choice of ε into (8.1.237) we obtain that

$$q \restriction_{\tilde{R}} \equiv 1, \tag{8.1.241}$$

in the category $\mathcal{C}^{\tilde{q}}$, with $\tilde{q} = \bar{q}^0 \varepsilon^{-1}$.

Thus, we can find a gauge in which

$$\rho \ := \ \hat{r}_2 \restriction_{\tilde{R} \times \tilde{R}} = 1, \tag{8.1.242}$$

where the $\hat{r}_2$-matrices are the ones computed for $\mathcal{C}^{\tilde{q}}$. Together with (8.1.235), this shows that the $\hat{F}_2$- and $\hat{r}_2$-matrices in (8.1.161) and (8.1.162) are $\tilde{R}$-invariant, and hence $\mathcal{C}^{\tilde{q}}$ is induced by some category $\tilde{\mathcal{C}}$ with objects $\widetilde{Obj}$.

(ii) We remark that the direct sum decomposition in (8.1.231) is equivalent to the conditions,

$$\tilde{R} \cap 2\, Grad(Obj) \ = \ 2R$$

$$\text{and} \qquad R \subset \tilde{R} \ + \ 2\, Grad(Obj)\,. \tag{8.1.243}$$

If we take (8.1.243) modulo $\tilde{R}$ and use the fact that $\text{Grad}(\widetilde{Obj}) = \pi_{\tilde{R}}(Grad(Obj))$ we find that $\tilde{R} \subset 2Grad(\widetilde{Obj})$. However, $2R \subset \tilde{R}$ also implies that $2\tilde{R} = \{1\}$. This, in summary, yields the inclusion (8.1.232). Of course, we still have that $\tilde{m} = 0$ for the category $\widetilde{\mathcal{C}}$, so that (8.1.233) follows by the same arguments as in part i).

□

The special situation to which the study of braided, monoidal categories is reduced in Lemma 8.1.11 allows us to find particularly simple representatives in the cohomology class of the cocycle introduced in Lemma 8.1.10 ii). To this end, we propose to make choices γ, as in (8.1.198), such that the associated extension (see (8.1.199)) factorizes. The relevant group-theoretical lemma in this context is the following one.

LEMMA 8.1.12

Let G be a finite, abelian group, and let R be a subgroup with

$$R \subset 2({}_4G) \subset {}_2G\,. \tag{8.1.244}$$

Define π and $\bar{G}$ by the short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & R \arrow[r, hook, "i"] & G \arrow[r, two heads, "\pi"] & \bar{G} \arrow[r] & 0\,. \end{tikzcd} \tag{8.1.245}$$

Then there exists a section $\bar{\psi} : \bar{G} \to G$ and presentations of the groups R and G

$$R \ = \ \mathbb{Z}_2(c_1) \oplus \cdots \oplus \mathbb{Z}_2(c_k) \tag{8.1.246}$$

$$\bar{G} \ = \ \mathbb{Z}_{2^{n_1}}(\bar{b}_1) \oplus \cdots \oplus \mathbb{Z}_{2^{n_k}}(\bar{b}_k) \oplus H \tag{8.1.247}$$

with generators $c_1, \cdots, c_k \in R, \bar{b}_1, \cdots \bar{b}_k \in \bar{G}$, and $H \subset \bar{G}$, such that the extension $\xi \in Ext(\bar{G}, R) \subset H^2(\bar{G}, 1; R)$ is given by

$$\xi\big(h\bar{b}_1^{\nu_1} \cdots \bar{b}_k^{\nu_k},\, g\bar{b}_1^{\mu_1} \cdots \bar{b}_k^{\mu_k}\big) \ = \ \prod_{j=1}^{k} c_j^{\gamma_j(\nu_j, \mu_j)}\,. \tag{8.1.248}$$

Here

$$\gamma_j(\nu, \mu) \ = \ \begin{cases} 0 \ , & 0 \le \nu_j \ + \ \mu_j \ < \ 2^{n_j}, \\ 1 \ , & \quad\ \ \nu_j \ + \ \mu_j \ \ge \ 2^{n_j}. \end{cases} \tag{8.1.249}$$

Proof.

The first step is to present G as a sum of cyclic groups, $\mathbb{Z}_{p^n}$, whose orders are powers of primes. It is clear that any element of order two lies entirely in the direct sum of the $\mathbb{Z}_{2^n}$-subgroups. Hence we can write

$$G \ = \ \mathbb{Z}_{2^{m_1}}(b_1^0) \oplus \cdots \oplus \mathbb{Z}_{2^{m_l}}(b_l^0) \oplus H^0$$

$$\text{with} \qquad {}_2G \ = \ \mathbb{Z}_2\big((b_1^0)^{2^{(m_1-1)}}\big) \oplus \cdots \oplus \mathbb{Z}_2\big((b_l^0)^{2^{(m_l-1)}}\big)$$

for generators $b_1^0, \cdots, b_l^0 \in G$, and $m_1 \leq m_2 \leq \cdots \leq m_l$. The subgroup $2({}_4G)$ is given by the direct sum of cyclic subgroups with $m_j \geq 2$. Given the generators b_j^0, we can define characters $\alpha_i \in Hom({}_2G, \mathbb{Z}_2)$ by setting $\alpha_i((b_j^0)^{2^{(m_j-1)}}) = (-1)^{\delta_{ij}}$. Their pull backs are given by $i^\star(\alpha_j) \in Hom(R, \mathbb{Z}_2)$. Let $j_1, 1 \leq j_1 \leq l$, be the smallest integer such that $i^\star(\alpha_{j_1}) \neq 1$, and let $c_1 \in R$ be such that $i^\star(\alpha_{j_1})(c_1) = -1$. It follows that

$$R \subset \mathbb{Z}_2((b_{j_1}^0)^{2^{(m_{j_1}-1)}}) \oplus \cdots \oplus \mathbb{Z}_2((b_l^0)^{2^{(m_l-1)}}),$$

and that

$$i(c_1) = (b_{j_1})^{2^{(m_{j_1}-1)}},$$

where b_{j_1} is of the form

$$b_{j_1} = b_{j_1}^0 \prod_{i>j_1}^{l} (b_i^0)^{x_i \cdot 2^{(m_i - m_{j_1})}},$$

for some $x_i \in \mathbb{N}$. In particular, b_{j_1} has order $2^{m_{j_1}}$, and we can replace $b_{j_1}^0$ by b_{j_1} as a generator of G. Since R can be seen as a vector space over $\mathbb{Z}_2$, we can write

$$R = \mathbb{Z}_2(c_1) \oplus ker(i^\star(\alpha_{j_1})) .$$

The image of $R' := ker(i^\star(\alpha_{j_1}))$ under i lies entirely in the subgroup of G generated by $b_{j_1+1}^0, \cdots, b_l^0$. Repeating the above argument for the inclusion of R' in this subgroup we obtain generators c_2, b_{j_2}, and so forth. If we add the cyclic groups with $i^\star(\alpha_j) = 1$ to H^0 and use that $m_j \geq 2$ we find that the groups R and G of (8.1.244) have the following presentations:

$$G = \mathbb{Z}_{2(n_1+1)}(b_1) \oplus \cdots \oplus \mathbb{Z}_{2(n_k+1)}(b_k) \oplus H, \tag{8.1.250}$$

and R has the form (8.1.246), with the property that the inclusion $i : R \hookrightarrow G$ is given by

$$i(c_j) = b_j^{2^{n_j}} . \tag{8.1.251}$$

The presentation (8.1.247) of $\bar{G}$ follows immediately, and the projection $\pi : G \twoheadrightarrow \bar{G}$ is given by setting $\pi(b_j) = \bar{b}_j$ and $\pi(h) = h$, for $h \in H$.

We now define a section $\bar{\psi} : \bar{G} \to G$, with $\pi \circ \bar{\psi} = id$, by setting

$$\bar{\psi}(\bar{b}_1^{\nu_1} \cdots \bar{b}_k^{\nu_k} h) = b_1^{f_1(\nu_1)} \cdots b_k^{f_k(\nu_k)} h , \tag{8.1.252}$$

where $f_j : \mathbb{Z}_{2^{n_j}} \to \mathbb{Z}_{2^{n_j+1}}$ is the function $f_j(\nu) = \nu$, for $\nu = 0, \cdots, 2^{n_j} - 1$. In analogy with eq. (7.235) for the quantities given in (7.233) and (7.234), we have that

$$\delta f_j = 2^{n_j} \gamma_j . \tag{8.1.253}$$

Hence the extension defined by

$$i(\xi(a,b)) = \bar{\psi}(a) \, \bar{\psi}(b) \, \bar{\psi}(a \cdot b)^{-1} \tag{8.1.254}$$

is the one given in (8.1.248)

□

In the special situation described in Lemma 8.1.11. ii) it is possible to eliminate the prefactors ω^0 and ω^+ in equations (8.1.217) and (8.1.218) by a substitution of the form (8.1.220). Moreover, one can find a simple, factorized form of ω^- in (8.1.219). This, however, requires some basic knowledge of the group $H_5(G,2)$ which has been computed in [57]. The cycle

$$\begin{aligned} \zeta(g) &:= \frac{1}{2} \partial [g \mid g \| g \mid g] \\ &= -[g \mid g \mid g \mid g] + [g \| g \mid g] - [g \mid g \| g], \end{aligned} \tag{8.1.255}$$

for $g \in {}_2G$, i.e., $g^2 = 1$, plays a crucial role in this analysis, since the homomorphism

$$\Delta = \Gamma_4({}_2G) \longrightarrow H_5(G,2) : \{g\} \longrightarrow \zeta(g) \tag{8.1.256}$$

describes the torsion-free part of the homology group. Furthermore, using that

$$Hom(H_5(G,2), M) \cong H^5(G,2;M), \quad \text{for} \quad M = U(1), \mathbb{C},$$

ζ induces the dual homomorphism

$$\Delta^\star : H^5(G,2;M) \longrightarrow Hom(\Gamma_4({}_2G); M), \tag{8.1.257}$$

defined, for a cocycle $\omega \in Z^5(G,2;M)$, by

$$\Delta^\star(\omega)(g) \;=\; \omega^0(g,g,g,g)^{-1}\,\omega^-(g,g\mid g)\,\omega^+(g,g\mid g)^{-1}\,. \tag{8.1.258}$$

We easily check that, for $g \in {}_2G$, the expression (8.1.258) depends, in fact, only on the cohomology class of ω. Since, with the help of Lemma 8.1.12, we can find a decomposition of G into cyclic groups for which the cocycle considered here factorizes, we only need to know the groups $H_5(\mathbb{Z}_{2^n}, M)$. It has been shown in [57] that, for these groups the map Δ defined in (8.1.256) is onto, and the kernel is generated by $\{gh\} - \{g\} - \{h\}$. This shows that $\Delta^\star$, as defined in (8.1.257), is injective, and its image is $Hom({}_2G, M)$. Hence

$$\begin{aligned} H^5(\mathbb{Z}_{2^n},2;M) &\;\rightarrow\; \mathbb{Z}_2,\\ \omega \qquad &\;\mapsto\; \Delta^\star(\omega)(2^{n-1}), \end{aligned} \tag{8.1.259}$$

is an isomorphism. The non-trivial cohomology class can, for example, be represented by the cocycle

$$\begin{aligned} \omega^0 \equiv 1 \quad &, \quad \omega^+ \equiv 1,\\ \omega^-(j,k\mid l) &= exp\left(\frac{2\pi i}{2^n}\, l\gamma(j,k)\right), \end{aligned} \tag{8.1.260}$$

where $j,k,l \in \mathbb{Z}_{2^n}$, and γ is as in (8.1.249), with $n = n_j$. For the special cocycle in Lemma 8.1.10, ii), the invariant

$$\Delta^\star(\omega) \;\in\; Hom\Big(\Gamma_4\big({}_2(Grad(\overline{Obj}))\big),\, \mathbb{Z}_2\Big)$$

is given by

$$\Delta^\star(\omega)(g) \;=\; \rho\big(\xi(g,g),\xi(g,g,)\big)^{-1}\,\bar{m}\big(\xi(g,g),\bar{\psi}(g)\big), \tag{8.1.261}$$

for $g \in {}_2Grad(\overline{Obj})$. In the case where $2R = 1$ (i.e., $R = {}_2R$), we easily see that ${}_2Grad(Obj) \rightarrow R : g \mapsto \xi(g,g)$ is a homomorphism. If we assume, furthermore, that $\bar{m} = 0$, it follows that

$$\Delta^\star(\omega) \;\in\; Hom\big({}_2Grad(\overline{Obj}),\, \mathbb{Z}_2\big)\,. \tag{8.1.262}$$

The results on $H^5(G,2;M)$ cited above, together with the normal form for extensions given in Lemma 8.1.12, allow us to find a particularly simple representative in the cohomology class of the cocycle ω in Lemma 8.1.10 ii), assuming that the conditions (8.1.232), (8.1.233) and (8.1.234) in Lemma 8.1.11, ii), hold. More precisely, we have the following result:

LEMMA 8.1.13

Let C be a quantum category and $R \subset Obj$ a graded subgroup of invertible objects with

$$R \;\subset\; 2\Big({}_4(Grad(Obj))\Big)\,. \tag{8.1.263}$$

Assume that all monodromies with objects in R vanish, i.e.,

$$\bar{m} \;=\; 1\,. \tag{8.1.264}$$

Suppose that R and $Grad(\overline{Obj}) \cong Grad(Obj)/R$ are presented as in eqs. (8.1.246) and (8.1.247) of Lemma 8.1.12, and let $\xi \in Ext\big(Grad(\overline{Obj}), R\big)$ be the extension given in (8.1.248). Let $\omega \in Z^5\big(Grad(\overline{Obj}),2;M\big)$ be the cocycle defined in terms of ξ as in Lemma 8.1.10, ii), for a choice of gauge of $H^{i,j}_{[k]}$ for which $\hat{F}_2(\sigma,\mu,\nu,[1]) = 1$, for $\sigma,\mu,\nu \in R$, so that $\rho \in Hom\big(R\otimes R, U(1)\big)$. Let $\varepsilon_j \in \mathbb{Z}_2$ be the invariants given by

$$\varepsilon_j \;:=\; \Delta^\star(\omega)\Big((\bar{b}_j)^{2^{(n_j-1)}}\Big) \;=\; \rho(c_j,c_j)\,. \tag{8.1.265}$$

Then:

(i) *The cocycle ω is cohomologous to the cocycle $\hat{\omega}$ given by*

$$\begin{aligned} \hat{\omega}^0 &\equiv 1\,,\\ \hat{\omega}^+ &\equiv 1\,,\\ \hat{\omega}^-(a,b\mid c) &= exp\Big(2\pi i \sum_{i,\varepsilon_j=-1} 2^{-n_j}\,\pi_j(c)\,\gamma_j(\pi_j(a),\pi_j(b))\Big)\,, \end{aligned} \tag{8.1.266}$$

where the π_j's are the projections onto the cyclic factors in (8.1.247), i.e.,

$$\pi_j \;:\; Grad(\overline{Obj}) \;\longrightarrow\; \mathbb{Z}_{2^{n_j}}(\bar{b}_j)\,,$$

and γ_j is as in (8.1.249).

(ii) *The F_2- and r_2-matrices defined in (8.1.220), where $\lambda \in A^4(Grad(\overline{Obj}), 2; M)$ is such that $\hat{\omega} = \omega(\delta\lambda)^{-1}$, satisfy the usual pentagonal equations and also one of the hexagonal equations. The only categorial equation that is modified is the second hexagonal equation:*

$$\begin{aligned}&\Big(\bigoplus_{[l]} r_2([k],[i],[l])^{-1} \otimes \mathbb{1}\Big) F_2\big([i],[k],[j],[t]\big)\Big(\bigoplus_{[l]} r_2([k],[j],[l])^{-1} \otimes \mathbb{1}\Big)\\ &= \hat{\omega}(g_i, g_j \mid g_k) F_2\big([k],[i],[j],[t]\big)\Big(\bigoplus_{[l]} \mathbb{1} \otimes r_2([k],[l],[t])^{-1}\Big) F_2\big([i],[j],[k],[t]\big)\,.\end{aligned} \tag{8.1.267}$$

(iii) *Let R^+ be the subgroup generated by $\{c_j : \varepsilon_j = 1\}$. Then C is induced as a braided, monoidal category by some category $\overline{C}$, with object set $\overline{Obj} := Obj/R^+$, and projection π_{R^+}.*

Proof.

If $\bar{m} \equiv 0$ it follows that the quadratic function in $Hom(\Gamma_4(R), U(1))$ characterizing the category C_R associated to R has values in $\mathbb{Z}_2$, so that C_R is trivial as a monoidal category. Hence there exists a gauge in which $\hat{F}_2(\sigma, \mu, \nu, [1]) = 1$, for $\sigma, \mu, \nu \in R$, and, as $R \otimes R$ has only elements of order two, with $\rho \in Hom(R \otimes R, \mathbb{Z}_2)$. Let $c_1, \cdots, c_k$ be the generators of R in the presentation (8.1.246) that are used for the factorized form, eq. (8.1.248), of the extension. We define $\beta \in Hom(R \otimes R, \mathbb{Z}_2) \subset Z^2(R, 1; U(1))$ by setting

$$\beta(c_i, c_j) = \begin{cases} \rho(c_i, c_j) & , \quad \text{for } \; i < j\,, \\ 1 \quad , & \quad \text{otherwise}\,. \end{cases} \tag{8.1.268}$$

If we perform a gauge transformation with

$$A^{i,j}_{[k]} := \beta\big(\eta(i) \circ \xi(g_1, g_j), \eta(j)\big)\, \beta\big(\eta(i), \xi(g_i, g_j)\big), \tag{8.1.269}$$

as in eq. (8.1.159), the braid matrix on R, $\rho' := \beta^t \beta^{-1} \rho$, defined as in eq. (8.1.162a), is diagonal in the generators c_j, i.e.,

$$\rho'(c_i, c_j) = \varepsilon_j^{\delta_{ij}}, \tag{8.1.270}$$

as follows by using that $\rho\rho^t = \bar{m} = 1$. We have that $\rho' \restriction_{R^+ \times R^+} \equiv 1$. By using that $\bar{m} \equiv 1$, we find from Lemma 8.1.7, iii), that, in this gauge, the $\hat{F}_2$- and $\hat{r}_2$-matrices are independent of the R^+-action. This implies part iii) of Lemma 8.1.13. The cocycle ω' constructed from ρ' and ξ differs from ω, as determined by ρ and ξ, by a coboundary. This can be seen from (8.1.162b) and (8.1.162c), where the gauge-transformation (8.1.269) corresponds to rescaling the $\hat{\hat{r}}_2$- and $\hat{\hat{F}}_2$-matrices by some λ_1, with

$$\begin{aligned}\lambda_1([g\|h]) &= 1\,,\\ \lambda_1([g \mid h \mid k]) &= \frac{\beta\big(\xi(g,h),\, \xi(gh,k)\big)}{\beta\big(\xi(g,hk),\, \xi(h,k)\big)}\,.\end{aligned} \tag{8.1.271}$$

We therefore have that

$$\omega' = \omega(\delta\lambda_1)^{-1}\,. \tag{8.1.272}$$

Inserting expression (8.1.248) for ξ into the formulae for the cocycle ω' in Lemma 8.1.10, ii), and using the special form (8.1.270) of ρ', we find that

$$\omega^0(g_1, g_2, g_3, g_4) = \prod_{j=1}^{k} \omega_j^0\big(\pi_j(g_1), \pi_j(g_2), \pi_j(g_3), \pi_j(g_4)\big),$$

with

$$\begin{aligned}\omega_j^0(k,l,m,n) &= \varepsilon_j^{\left[\gamma_j(l,m)\left(\gamma_j(k+l+m,n) - \gamma_j(l+m,n)\right)\right]}, \\ \omega^{\pm}(g_1, g_2 \mid g_3) &= \prod_{j=1}^{k} \omega_j^{\pm}\big(\pi_j(g_1), \pi_j(g_2) \mid \pi_j(g_3)\big)\,,\end{aligned} \tag{8.1.273}$$

and

$$\omega_j^{\pm}(k,l \mid m) = \varepsilon_j^{\left[\gamma_j(k+l,m)\,\gamma_j(k,l)\right]}. \tag{8.1.274}$$

Thus ω factorizes completely into cocycles over the cyclic subgroups, $\mathbb{Z}_{2^{n_j}}$, each of which is cohomologous to the cocycle given in eqs. (8.1.260) if $\varepsilon_j = -1$ and to the trivial cocycle if $\varepsilon_j = 1$. Therefore $\omega \sim \omega' \sim \hat{\omega}$, as defined in (8.1.266). This proves part i) of Lemma 8.1.13. The statement in part ii) is a direct consequence of Lemma 8.1.10, iv).

□

We already found that $\Delta^*(\omega)$, as defined in eq. (8.1.261), is independent of the particular choice of gauge we have made. It is straightforward to check that $\Delta^*(\omega)$ is alsc independent of the particular section $\bar{\psi} : Grad(\overline{Obj}) \to Grad(Obj)$, with $\pi^{\#} \circ \bar{\psi} = id$, we use to define the extension ξ. Thus $\Delta^{\star}(\omega)$ is a true invariant of the category C. Furthermore, one easily verifies that $\Delta^{\star}(\omega)$ does not change if C is replaced by C^q, for some $q \in Hom\Big(\Gamma_4\big(Grad(Obj)\big), U(1)\Big)$. In particular, if C is of the form $C \cong \hat{C}^q$, where $\hat{C}$ is induced by some category on $\overline{Obj}$, the obstruction $\Delta^{\star}(\omega)$ has to vanish.

We conclude this section with a summary of results for a cyclic grading group.

COROLLARY 8.1.14

Let C be a quantum category with objects Obj, and let $R \subset Obj$ be a graded subgroup. Assume that $Grad(Obj)$ is cyclic, so that, for some numbers $n, m \in \mathbb{N}$, $Grad(Obj) = \mathbb{Z}_{nm}(g)$ and $R = \mathbb{Z}_m(\sigma)$, and $grad : R \hookrightarrow Grad(Obj) : \sigma \mapsto g^n$.

(i) *If $\Delta^{\star}(\omega) \neq 1$, then m and n are even, i.e., $m = 2m'$ and $n = 2n'$. In this case, we conclude that*

  (a) *there exists a quadratic form $q \in Hom\Big(\Gamma_4\big(Grad(Obj)\big), U(1)\Big)$, and, defining $R' := \mathbb{Z}_{m'}(\sigma^2) \subset \mathbb{Z}_m(\sigma) = R$, a quantum-category, $\widetilde{C}$, on the objects $\widetilde{Obj} = Obj/R'$ such that $C \cong \hat{C}^q$, where $\hat{C}$ is the category induced by $\widetilde{C}$ and $\pi_{R'}$, and the monodromy $\bar{m}$ vanishes on $\widetilde{C}$. The subgroup, $\bar{R} = \pi_{R'}(R) = \mathbb{Z}_2(\bar{\sigma})$, of invertible elements in $\widetilde{Obj}$ is embedded into $Grad(\widetilde{Obj}) = \mathbb{Z}_{2n}(\bar{g})$ by $\widetilde{grad}(\bar{\sigma}) = \bar{g}^b$.*

  (b) *There exists a gauge for the structure constants of $\widetilde{C}$ such that*

$$\hat{F}_2(i,j,k,[l]) =$$

$$= (-1)^{\eta'\left(\widetilde{grad}(j)\right)\left[\gamma(g_i+g_j,g_k)-\gamma(g_i,g_k)\right]}\ \hat{\hat{F}}_2([i],[j],[k],[l])$$

$$\hat{r}_2(j,k,[l]) =$$

$$= (-1)^{\eta'\left(\widetilde{grad}(j)\right)\eta'\left(\widetilde{grad}(k)\right)}(-1)^{\gamma(g_j,g_k)\left[\eta'\left(\widetilde{grad}(k)\right)-\eta'\left(\widetilde{grad}(j)\right)\right]}\hat{\hat{r}}_2([j],[k],[l]). \tag{8.1.275}$$

Here $\gamma \in Z^2(\mathbb{Z}_n, 1; \mathbb{Z}_2)$ is as in eq. (7.234), and

$$\eta'(\bar{g}^{\nu}) = \begin{cases} 1 \ , & \text{for} \quad n \leq \nu < 2n \, , \\ 0 \ , & \text{for} \quad 0 \leq \nu < n \, . \end{cases}$$

We denote by $[j] \in \overline{Obj} = \widetilde{Obj}/\bar{R}$ the $\bar{R}$-orbit of order two of $j \in \widetilde{Obj}$ and by $g_j = \overline{grad}([j])$ the grading in $Grad(\overline{Obj}) = \mathbb{Z}_n(\bar{g})$, with $\bar{g} := \pi^{\#}_{\bar{R}}(\bar{g})$.

  (c) *The $\hat{\hat{F}}$- and $\hat{\hat{r}}$-matrices satisfy the modified categorial equations (8.1.217) - (8.1.219), where the cocycle $\omega \in Z^5\big(\mathbb{Z}_n, 2; U(1)\big)$ is of the form given in (8.1.273) and (8.1.274), with γ_j replaced by γ and $\varepsilon_j = -1$. There exists a function $\lambda \in A^4\big(Grad(\overline{Obj}), 2;\ U(1)\big)$ so that the r_2- and F_2-matrices defined by (8.1.220) satisfy the ordinary pentagonal equations for a monoidal category on $\overline{Obj}$ and the hexagonal equations*

$$\begin{aligned}
&\Big(\bigoplus_{[l]} r_2([i],[k],[l]) \otimes \mathbb{I}\Big) F_2([i],[k],[j],[t]) \Big(\bigoplus_{[l]} r_2([j],[k],[l]) \otimes \mathbb{I}\Big) \\
&= F_2([k],[i],[j],[t]) \Big(\bigoplus_{[l]} \mathbb{I} \otimes r_2([l],[k],[t])\Big) F_2([i],[j],[k],[t]) \\
&\Big(\bigoplus_{[l]} r_2([k],[i],[l])^{-1} \otimes \mathbb{I}\Big) F_2([i],[k],[j],[t]) \Big(\bigoplus_{[l]} r_2([k],[j],[l])^{-1} \otimes \mathbb{I}\Big) \\
&= exp\Big(\frac{2\pi i}{n}\, g_k \gamma(g_i, g_j)\Big) F_2([k],[i],[j],[t]) \Big(\bigoplus_{[l]} \mathbb{I} \otimes r_2([k],[l],[t])^{-1}\Big) \\
&\qquad F_2([i],[j],[k],[t]) \, .
\end{aligned} \tag{8.1.276}$$

(ii) *If $\Delta^{\star}(\omega) = 1$, then $C \cong \hat{C}^q$, where $\hat{C}$ is induced by some category with objects $\overline{Obj} = Obj/R$ and projection π_R.*

Proof.

We first remark that $\bar{m}$ always has a symmetric extension, since, for a pair of cyclic groups $H \subset G$, the induced map $H \otimes G \to G \otimes G = G \otimes_s G$ is injective and all extensions over U(1) are tivial. If the integer m is odd we have that $R = 2R$, and if n is odd we find that i_2, as defined in (8.1.229) and (8.1.230), is an isomorphism. In both cases it follows that R is a maximal signature-extendable group. By Lemma 8.1.11, ii), the category C is of the form described in part ii) of the corollary. In particular, we have that $\Delta^{\star}(\omega) \equiv 1$.

In the case where both integers, n and m, are even one finds that $i_2 \equiv 0$, and the maximal signature-extendable group is $R' = 2R$. Using Lemma 8.1.11, i), we can describe $\mathcal{C}$ as $\mathcal{C} \cong \hat{\mathcal{C}}^q$, where $\hat{\mathcal{C}}$ is induced by a category $\tilde{\mathcal{C}}$ with objects $\widetilde{Obj} = Obj/R'$, and projection $\pi_{R'}$. If $\hat{\rho} \in Hom(R \otimes R, \mathbb{Z}_2)$ is the basic braid matrix of $\hat{\mathcal{C}}$, see eq. (8.1.160), and $\tilde{\rho} \in Hom\big(\mathbb{Z}_2(\tilde{\sigma}) \otimes \mathbb{Z}_2(\tilde{\sigma}), \mathbb{Z}_2\big)$ the braid matrix of $\tilde{\mathcal{C}}$ then, since $\bar{m} \equiv 0$ in both categories,

$$\begin{aligned} \Delta^\star(\omega)_{\mathcal{C}}\,(\bar{g}^{n'}) &= \Delta^\star(\omega)_{\hat{\mathcal{C}}}\,(\bar{g}^{n'}) \\ &= \hat{\rho}(\sigma^{m'}, \sigma^{m'}) \\ &= \tilde{\rho}(\tilde{\sigma}, \tilde{\sigma}) \;=\; \Delta^\star(\omega)_{\tilde{\mathcal{C}}}(\bar{g}^{n'})\,. \end{aligned} \tag{8.1.277}$$

Hence if $\Delta^\star(\omega)_{\mathcal{C}} \equiv 1$ the same equation holds for $\Delta^\star(\omega)_{\tilde{\mathcal{C}}}$. It then follows from Lemma 8.1.13, iii) that $R^+ \subset \tilde{R}$, i.e., $\tilde{\mathcal{C}}$ is induced by some category $\bar{\mathcal{C}}$ on $\overline{Obj} = \widetilde{Obj}/\tilde{R} = Obj/R$ and $\pi_{\tilde{R}}$. This implies that $\hat{\mathcal{C}}$ is induced by $\bar{\mathcal{C}}$ and that $\pi_R = \pi_{\tilde{R}} \circ \pi_{R'}$, proving part ii) of Corollary 8.1.14. If $\Delta^\star_{\mathcal{C}}(\omega)(\bar{g}^{n'}) = \Delta^\star_{\tilde{\mathcal{C}}}(\omega)(\bar{g}^{n'}) = -1$, then the formulae for the structure constants, eqs. (8.1.275) and (8.1.276), immediately follow from Lemma 8.1.10 and the fact that ω is cohomologous to the cocycle (8.1.260), where 2^n is replaced by n. The section $\bar{\psi} : Grad(\overline{Obj}) = \mathbb{Z}_n(\bar{g}) \longrightarrow Grad(\widetilde{Obj}) = \mathbb{Z}_{2n}(\tilde{g})$ for which the expressions in (8.1.275) have been computed, is defined by

$$\bar{\psi}(\bar{g}^\nu) \;=\; \tilde{g}^\nu, \quad \text{with} \quad \nu \;=\; 0, \cdots, n-1\,. \tag{8.1.278}$$

This completes the proof of the corollary.

□

8.2 The A_n - Categories and Main Results

In the first part of this section we present a classification of semisimple, monoidal as well as quantum categories with A_n- fusionrules. In particular, we show that the monoidal categories are uniquely determined by the statistical dimension of the generating object, ρ, and, for braided categories, by the eigenvalues of $\varepsilon(\rho, \rho)$. In both cases they are realized by the category $\overline{Rep(U_q(sl_2))}$, as described in Chapter 7.1.

We show that in the case of $Obj_2 = A_n$ fusionrules the $H^5(Grad(Obj_2), 2; \mathbb{Z}_2)$-obstruction discussed at the end of the previous chapter vanishes. This is used to show that the quantum categories with $\mathbb{Z}_r \star A_{2n-1}$- and $\mathbb{Z}_r \star \bar{A}_n$- fusionrules are isomorphic to subcategories of a product of a θ- category with group $\mathbb{Z}_r$ and a $\overline{Rep(U_q(sl_2))}$- category. The quantum categories with $\tau_\alpha(\mathbb{Z}_r \star A_{2n-1})$- fusionrules are described in terms of the categories they induce by the graded homomorphism $f^\star : \mathbb{Z}_{2r} \star A_{2n-1} \twoheadrightarrow \tau_\alpha(\mathbb{Z}_r \star A_{2n-1})$.

Combining these resultes with the restrictions on fusion rule algebras and statistical dimensions obtained in Proposition 7.4.11 we arrive at the classification of C^* - quantum categories which are generated by an object of statistical dimension less than two.

In this section we shall be concerned with proofs of uniqueness of some simple categories. Together with the existence guaranteed by the explicit constructions based on quantum groups and θ-categories, this allows us to give a classification of quantum categories with a generator of dimension less than two.

We begin with a proof of existence and uniqueness for monoidal categories with A_n-fusion rules, disregarding any braided structure. For this purpose, we need to gather some basic facts concerning these categories.

Suppose that $\mathcal{C}$ is a semi-simple, rigid, monoidal category with A_{k+1}-fusion rules, for $k \geq 1$. We denote the objects of $\mathcal{C}$ by $\rho_0 = 1, \rho_1, \dots, \rho_k$, as in Lemma 7.3.2. i). We choose a pair of morphisms $\vartheta_1 \in Mor(1, \rho_1 \circ \rho_1)$ and $\vartheta_1^\dagger \in Mor(\rho_1 \circ \rho_1, 1)$ such that

$$(\vartheta_1^\dagger \circ 1)\alpha(\rho_1,\rho_1,\rho_1)(1 \circ \vartheta_1) \;=\; (1 \circ \vartheta_1^\dagger)\alpha(\rho_1,\rho_1,\rho_1)^{-1}(\vartheta_1 \circ 1) \;=\; 1\,. \tag{8.2.1}$$

We define a sequence of numbers $d_j, j = 0, 1, \dots$:

$$\begin{aligned} d_0 \;=\; 1, \qquad d_1 \;&=\; \vartheta_1^\dagger \vartheta_1\,, \\ \text{and} \qquad d_{j+1} \;+\; d_{j-1} \;&=\; d_1 d_j\,. \end{aligned} \tag{8.2.2}$$

For a given $\vartheta_1^\dagger$, we introduce two bilinear forms on one-dimensional spaces, as follows:

$$\begin{aligned} p_j \;&: Mor(\rho_{j+1}, \rho_j \circ \rho_1) \otimes Mor(\rho_j, \rho_{j+1} \circ \rho_1) \to \mathbb{C} \\ &I \otimes J \to (1 \circ \vartheta_1^\dagger)\alpha(\rho_j, \rho_1, \rho_1)^{-1}(I \circ 1)J \end{aligned} \tag{8.2.3}$$

and

$$\begin{aligned} q_j \;&: Mor(\rho_j, \rho_{j+1} \circ \rho_1) \otimes Mor(\rho_{j+1}, \rho_j \circ \rho_1) \to \mathbb{C} \\ &I \otimes J \to (1 \circ \vartheta_1^\dagger)\alpha(\rho_{j+1}, \rho_1, \rho_1)^{-1}(I \circ 1)J\,, \end{aligned} \tag{8.2.4}$$

where $j = 0, 1, \dots, k-1$.

We have the following results concerning these quantities.

LEMMA 8.2.1

Let $\mathcal{C}$ be a semi-simple, rigid, monoidal category with A_{k+1}-fusion rules.

(i) *The number d_1 (and thus every d_j) is an invariant of $\mathcal{C}$ independent of the choice of ϑ_1 and $\vartheta_1^\dagger$. There exists some $l \in \mathbb{Z}_{2(k+2)}$ with $(l, k+2) = 1$ such that*

$$d_1 \;=\; 2\cos\Big(\frac{l\pi}{k+2}\Big) \;=\; (2)_q\,, \tag{8.2.5}$$

with $q = e^{\frac{i\pi l}{k+2}}$. Furthermore,

$$d_j \;=\; (j+1)_q \;\neq\; 0\,,$$

for $j = 0, \dots, k$, and

$$d_{k+1} \;=\; 0\,.$$

(ii) *For $j = 0, 1, \dots, k-1$, the bilinear forms in (8.2.3) and (8.2.4) are non-degenerate and related by*

$$p_j \;=\; \frac{d_{j+1}}{d_j}\, q_j^t\,. \tag{8.2.6}$$

(iii) *If $\mathcal{C}$ is a C^*-category, then*

$$l \;\equiv\; \pm 1 \quad mod\,(k+2)\,. \tag{8.2.7}$$

<u>Proof</u>.

From the pentagon equation

$$\alpha(\rho_j \circ \rho_1, \rho_1, \rho_1)\,\alpha(\rho_j, \rho_1, \rho_1 \circ \rho_1) = \big(\alpha(\rho_j, \rho_1, \rho_1) \circ 1\big)\,\alpha(\rho_j, \rho_1 \circ \rho_1, \rho_1)\big(1 \circ \alpha(\rho_1, \rho_1, \rho_1)\big)$$

and (8.2.1) we immediately derive the identity

$$1 = \big((1_j \circ 1_1) \circ \vartheta_1^\dagger\big)\,\alpha(\rho_j \circ \rho_1, \rho_1, \rho_1)^{-1}\big(\alpha(\rho_j, \rho_1, \rho_1) \circ 1\big)\big((1 \circ \vartheta_1) \circ 1\big). \tag{8.2.8}$$

From the isomorphism $\mu_1^{(\rho_j\rho_1)\rho_1}$, as defined in equ. (8.72), we find sequences of morphisms $I_j^\varepsilon \in Mor(\rho_{j+\varepsilon}, \rho_j \circ \rho_1)$ and $J_j^\varepsilon \in Mor(\rho_j, \rho_{j+\varepsilon} \circ \rho_1)$, $\varepsilon = \pm 1$, for $j = 0, \dots, k-1$ when $\varepsilon = 1$, and $j = 1, \dots k$ when $\varepsilon = -1$, such that

$$\alpha(\rho_j, \rho_1, \rho_1)(1 \circ \vartheta_1) \;=\; \sum_\varepsilon (I_j^\varepsilon \circ 1)\; J_j^\varepsilon\,, \tag{8.2.9}$$

where we sum over $\varepsilon = \{\pm 1\}$ whenever the morphisms are defined. Inserting (8.2.8) we find that

$$1_{j \circ 1} \;=\; \sum_\varepsilon I_j^\varepsilon\,(1 \circ \vartheta_1^\dagger)\,\alpha(\rho_{1+\varepsilon}, \rho_1, \rho_1)^{-1}(J_j^\varepsilon \circ 1) \tag{8.2.10}$$

which is just the partition of $1_{j \circ 1} \in End(\rho_j \circ \rho_1)$ into the minimal projections associated to the channels $\rho_{j+\varepsilon}$. Since I_j^ε is the corresponding injection, we obtain that

$$1_{j+\varepsilon} \;=\; (1 \circ \vartheta_1^\dagger)\,\alpha(\rho_{j+\varepsilon}, \rho_1, \rho_1)^{-1}\,(J_j^\varepsilon \circ 1)\, I_j^\varepsilon\,.$$

In terms of the forms defined in (8.2.3) and (8.2.4) this is expressed as

$$1 = q_j(J_j^+, I_j^+) = p_j(J_{j+1}^-, I_{j+1}^-) \tag{8.2.11}$$

for $j = 0, \dots, k-1$. This equation already implies that none of the q_j's is degenerate and that $J_j^\epsilon \otimes I_j^\epsilon \neq 0$, whenever defined. For the map

$$\begin{aligned} \bar{p}_j \ : Mor(\rho_{j+1}, \rho_j \circ \rho_1) &\rightarrow Mor(\rho_j, \rho_{j+1} \circ \rho_1)^* = Mor(\rho_{j+1} \circ \rho_1, \rho_1), \\ I &\mapsto p_j(I, \cdot) = (1 \circ \vartheta_1^\dagger)\, \alpha(\rho_j, \rho_1, \rho_1)^{-1}(I \circ 1), \end{aligned}$$

the inverse is explicitly given by

$$\bar{p}_j^{-1}(\bar{I}) = (\bar{I} \circ 1)\, \alpha(\rho_{j+1}, \rho_1, \rho_1)(1 \circ \vartheta_1) \, . \tag{8.2.12}$$

Similarly, the inverse of

$$\bar{q}_j(I) = q_j(I, \cdot) = (1 \circ \vartheta_1^\dagger)\, \alpha(\rho_{j+1}, \rho_1, \rho_1)^{-1}(I \circ 1)$$

is given by

$$\bar{q}_j^{-1}(\bar{I}) = (\bar{I} \circ 1)\, \alpha(\rho_j, \rho_1, \rho_1)(1 \circ \vartheta_1) \, . \tag{8.2.13}$$

If we apply $(1 \circ \vartheta_1^\dagger)\, \alpha(\rho_j, \rho_1, \rho_1)^{-1}$ to (8.2.9) from the left we obtain that

$$d_1 = p_j(I_j^+, J_j^+) + q_{j-1}(I_j^-, J_j^-) \, , \tag{8.2.14}$$

for $j = 1, \dots, k-1$, and, in addition, that

$$d_1 = p_0(I_0^+, J_0^+), \quad d_1 = q_{k-1}(I_k^-, J_k^-) \, . \tag{8.2.15}$$

Since both forms, p_j and q_j^t, are non-zero and lie in the same one-dimensional space, there exist $\xi_j \in \mathbb{C}^*$, $j = 0, \dots, k-1$, such that $p_j = \xi_j \, q_j^t$. From (8.2.11), (8.2.14) and (8.2.15) we find that

$$d_1 = \xi_0 = \xi_{k-1}^{-1} \, ,$$

$$\text{and} \qquad d_1 = \xi_j + \xi_{j-1}^{-1} \, , \tag{8.2.16}$$

so that, by (8.2.2),

$$d_j = \xi_{j-1} \cdot \ldots \cdot \xi_0 \, .$$

The existence of a solution to (8.2.16) thus implies that

$$d_j \neq 0, \quad \text{for} \quad j = 0, \dots, k, \quad \text{and} \quad d_{k+1} = 0 \, . \tag{8.2.17}$$

It is straightforward to verify that (8.2.17) holds if and only if d_1 is of the form stated in (8.2.5). The fact that d_1 is an invariant follows from (8.2.1) which constrains rescalings to be of the form $\bar{\vartheta}_1 = \lambda \vartheta_1, \bar{\vartheta}_1^\dagger = \frac{1}{\lambda} \vartheta_1^\dagger$, so that d_1 in (8.2.2) is unchanged. We therefore have proven that p_j and q_j^t are invariantly related to each other as in (8.2.6), with a factor only depending on d_1.

If we are considering a C^*-category we can choose ϑ_1 and $\vartheta_1^\dagger$ such that

$$\vartheta_1^* = sgn(d_1)\, \vartheta_1^\dagger \, .$$

With this normalization, we find that

$$\bar{q}_j(I^*) = sgn(d_1)\, \bar{p}_j^{-1}(I^*) \, . \tag{8.2.17a}$$

For $I \in Mor(\rho_j, \rho_{j+1} \circ \rho_1)$, we find from (8.2.17)

$$\begin{aligned} 0 \leq I^* I = p_j(\bar{p}_j^{-1}(I^*), I) &= \xi_j \, q_j(I, \bar{p}_j^{-1}(I^*)) \\ &= \xi_j \, sgn(d_1)\, \bar{q}_j(I)\, \bar{q}_j(I)^* \, , \end{aligned}$$

and hence

$$sgn(d_1) = sgn(\xi_j) = \frac{sgn(d_{j+1})}{sgn(d_j)} \, . \tag{8.2.18}$$

Using the explicit expressions for d_j, i.e., $d_j = (j+1)_q$, we see that (8.2.18) holds if and only if l satisfies the constraint (8.2.7).

□

The relations found in Lemma 8.2.1, now serve us as a tool to consistently define isomorphisms between the $Mor(k, i \circ j)$-spaces and the $Mor'(k, i \circ j)$-spaces, of two ategories C and C' of this type, providing an equivalence of the F-matrices of C and $'$. In our next lemma we derive this equivalence for a certain type of associativity onstraint. This will be sufficient to prove the equivalence of the monoidal categories C nd C'.

LEMMA 8.2.2

Suppose that C and C' are two monoidal categories with A_{k+1}-fusion rules and with the same value, d_1, for the invariant dimension of the generator.

(i) *For any sequence of isomorphisms*

$$H_{\rho_{j+1}}^{\rho_j,\rho_1} \,:\, Mor(\rho_{j+1},\rho_j \circ \rho_1) \,\rightarrow\, Mor'(\rho_{j+1},\rho_j \circ \rho_1),$$

with $j = 0,\dots,k-1$, and $H_{\rho_1}^{1,\rho_1}(1_1) = 1_1$, there exists a unique sequence of isomorphisms

$$H_{\rho_j}^{\rho_{j+1},\rho_1} \,:\, Mor(\rho_j,\rho_{j+1} \circ \rho_1) \,\rightarrow\, Mor'(\rho_j,\rho_{j+1} \circ \rho_1),$$

with $j = 0,\dots,k-1$, such that, simultaneously,

$$p'_j(H_{\rho_{j+1}}^{\rho_j,\rho_1} \,\otimes\, H_{\rho_j}^{\rho_{j+1},\rho_1}) \;=\; p_j \qquad (8.2.19)$$

and

$$q'_j(H_{\rho_j}^{\rho_{j+1},\rho_1} \,\otimes\, H_{\rho_{j+1}}^{\rho_j,\rho_1}) \;=\; q_j\,. \qquad (8.2.20)$$

For these, we have that $H_1^{\rho_1,\rho_1}(\vartheta_1) = \vartheta'_1$.

(ii) *For any given choice of isomorphisms H_j^{i,ρ_1}, as in i), and with $H_j^{j,1}(1_j) := 1_j$, there exists a unique completion of the choice of isomorphisms*

$$H_k^{i,j} \,:\, Mor(k,i \circ j) \,\rightarrow\, Mor'(k,i \circ j)$$

such that the diagram

$$\begin{tikzcd}
\bigoplus_s Mor(s,j \circ \rho_1) \otimes Mor(k,i \circ s) \arrow[r, "F(i,j,\rho_1,k)"] \arrow[d, "H^{\otimes 2}"] & \bigoplus_s Mor(s,i \circ j) \otimes Mor(k,s \circ \rho_1) \arrow[d, "H^{\otimes 2}"] \\
\bigoplus_s Mor'(s,j \circ \rho_1) \otimes Mor'(k,i \circ s) \arrow[r, "F'(i,j,\rho_1,k)"] & \bigoplus_s Mor'(s,i \circ j) \otimes Mor'(k,s \circ \rho_1)
\end{tikzcd} \qquad (8.2.21)$$

commutes, for all $i,j,k \in J$.

Proof.

It is clear that, by the non-degeneracy of p_j and p'_j, there exists a unique sequence of $H_{\rho_j}^{\rho_{j+1},\rho_1}$ such that (8.2.19) holds. Since $d_1 = d'_1$, and thus $d_j = d'_j$, we immediately find from (8.2.6) in Lemma 8.2.1 that (8.2.20) is automatically fulfilled. From (8.2.20),

$$q_0(\vartheta_1,1) \;=\; 1\,, \qquad (8.2.21a)$$

and $H_{\rho_1}^{1,\rho_1}(1) = 1$, we obtain that $H_1^{\rho_1,\rho_1}(\vartheta_1) = \vartheta'_1$.

For a given choice of H_b^{a,ρ_1} we now show part ii) of the lemma. The proof will proceed by induction in n. Assuming that we have defined

$$H_l^{i,\rho_j} \,:\, Mor(l,i \circ \rho_j) \,\rightarrow\, Mor'(l,i \circ \rho_j),$$

for all $j = 0,\dots,n$, and that

$$\begin{tikzcd}
\bigoplus_s Mor(s,\rho_j \circ \rho_1) \otimes Mor(l,i \circ s) \arrow[r, "F(i,\rho_j,\rho_1,l)"] \arrow[d, "H^{\otimes 2}"] & \bigoplus_s Mor(s,i \circ \rho_j) \otimes Mor(l,s \circ \rho_1) \arrow[d, "H^{\otimes 2}"] \\
\bigoplus_s Mor'(s,\rho_j \circ \rho_1) \otimes Mor'(l,i \circ s) \arrow[r, "F'(i,\rho_j,\rho_1,l)"] & \bigoplus_s Mor'(s,i \circ \rho_j) \otimes Mor'(l,s \circ \rho_1)
\end{tikzcd} \qquad (8.2.22)$$

commutes, for $j = 0, \dots, n-1$, we can find unique isomorphisms, $H_l^{i,\rho_{n+1}}$, for $i, l \in J$, such that (8.2.22) commutes for $j = n$.

To this end, we have to show that, independently of the choice of $H_j^{i,\rho_{n+1}}$, the diagram for the restrictions to one summand on the left hand side:

$$\begin{tikzcd}
\begin{matrix} Mor(\rho_{n-1}, \rho_n \circ \rho_1) \\ \otimes Mor(\rho_t, i \circ \rho_{n-1}) \end{matrix} \arrow[r, "F(i,\rho_n,\rho_1,\rho_t)"] \arrow[d, "H^{\otimes 2}"] & \begin{matrix} \bigoplus_{\varepsilon=\pm 1} Mor(\rho_{t+\varepsilon}, i \circ \rho_n) \\ \otimes Mor(\rho_{t+\varepsilon}, \rho_t \circ \rho_1) \end{matrix} \arrow[d, "H^{\otimes 2}"] \\
\begin{matrix} Mor'(\rho_{n-1}, \rho_n \circ \rho_1) \\ \otimes Mor'(\rho_t, i \circ \rho_{n-1}) \end{matrix} \arrow[r, "F'(i,\rho_n,\rho_1,\rho_t)"] & \begin{matrix} \bigoplus_{\varepsilon=\pm 1} Mor'(\rho_{t+\varepsilon}, i \circ \rho_n) \\ \otimes Mor'(\rho_{t-\varepsilon}, \rho_t \circ \rho_1) \end{matrix}
\end{tikzcd} \tag{8.2.23}$$

commutes, whenever $l = \rho_t \in i \circ \rho_{n-1}, t = 0, \dots, k$. We show this by expressing the matrix elements of these maps by matrix elements of $F(i, \rho_{n-1}, \rho_1, \rho_t)$ and the isomorphisms $\bar{q}_j$ and $\bar{p}_j$ from Lemma 8.2.1 (which are mapped, under the action of H, into $\bar{q}'_j$ and $\bar{p}'_j$). In order to derive a useful relation, we consider the pentagonal equation

$$\begin{aligned} &\alpha(i, \rho_n, \rho_1 \circ \rho_1)\big(1_i \circ \alpha(\rho_n, \rho_1, \rho_1)^{-1}\big)\, \alpha(i, \rho_n \circ \rho_1, \rho_1)^{-1} \\ &\quad = \alpha(i \circ \rho_n, \rho_1, \rho_1)^{-1}\big(\alpha(i, \rho_n, \rho_1) \circ 1_1\big)\,. \end{aligned} \tag{8.2.24}$$

Now, choose $I \in Mor(\rho_t, i \circ \rho_{n-1})$, $J \in Mor(\rho_{n-1}, \rho_n \circ \rho_1)$, $L' \in Mor(\rho_{t+\varepsilon}, \rho_t \circ \rho_1)$, and $K \in Mor(i \circ \rho_n, \rho_{t+\varepsilon})$, and multiply (8.2.24) with $\Big(\big((1_i \circ J)I\big) \circ 1_1\Big)L'$ from the right and with $K \circ \vartheta_1^\dagger$ from the left. This yields

$$\begin{aligned} &K\big(1_i \circ \big[(1 \circ \vartheta_1^\dagger)\,\alpha(\rho_n, \rho_1, \rho_1)^{-1}(J \circ 1_1)\big]\big)\, \alpha(i, \rho_{n-1}, \rho_1)^{-1}(I \circ 1_1)L' \\ &= (1_{\rho_{t+\varepsilon}} \circ \vartheta_1^\dagger)\,\alpha(\rho_{t+\varepsilon}, \rho_1, \rho_1)^{-1}\big(\big[(K \circ 1_1)\,\alpha(i, \rho_n, \rho_1)(1_i \circ J)I\big] \circ 1_1\big)L', \end{aligned} \tag{8.2.25}$$

using only the isotropy (8.1.38). The term in square brackets on the left hand side is found to be $\bar{q}_{n-1}(J) \in Mor(\rho_n \circ \rho_1, \rho_{n-1})$, and the right hand side is identified with one of the bilinear forms (8.2.3), or (8.2.4) between L' and the term in square brackets, depending on $\varepsilon = \pm 1$. With an appropriate substitution of L', and using identity (8.2.6), we obtain the following explicit formula: For $\varepsilon = 1$,

$$\begin{aligned} &L(K \circ 1_1)\, \alpha(i, \rho_n, \rho_1)(1 \circ J)I = \\ &\qquad = \frac{d_{t+1}}{d_t}\, K\big(1 \circ \bar{q}_{n-1}(J)\big)\, \alpha(i, \rho_{n-1}, \rho_1)^{-1}(I \circ 1)\,\bar{p}_t^{-1}(L)\,, \end{aligned} \tag{8.2.26}$$

with $L \in Mor(\rho_{t+1} \circ \rho_1, \rho_t)$, identifying, with $1_j \to 1 \in \mathbb{C}$, both sides of (8.2.26) with $\mathbb{C}$-numbers. In terms of F-matrices, this equation can be rewritten as

$$\begin{aligned} &\langle K \otimes L, F(i, \rho_n, \rho_1, \rho_t)\; J \otimes I \rangle = \\ &\qquad = \frac{d_{t+1}}{d_t}\, \langle \bar{q}_{n-1}(J) \otimes K,\; F(i, \rho_{n-1}, \rho_1, \rho_{t+1})^{-1}\, I \otimes \bar{p}_t^{-1}(L) \rangle\,, \end{aligned} \tag{8.2.27}$$

where we view $K \in Mor(\rho_{t+1}, i \circ \rho_n)^*$ and $L \in Mor(\rho_1, \rho_{t+1} \circ \rho_1)^*$.

Similarly, we find, for $\varepsilon = -1$,

$$\begin{aligned} &\langle K \otimes L, F(i, \rho_n, \rho_1, \rho_t)\; J \otimes I \rangle = \\ &\qquad = \frac{d_{t-1}}{d_t}\, \langle \bar{q}_{n-1}(J) \otimes K,\; F(i, \rho_{n-1}, \rho_1, \rho_{t-1})^{-1}\, I \otimes \bar{q}_{t-1}^{-1}(L) \rangle\,, \end{aligned} \tag{8.2.28}$$

with $K \in Mor(\rho_{t-1}, i \circ \rho_n)^*$ and $L \in Mor(\rho_t, \rho_{t-1} \circ \rho_1)^*$. Note that the equations (8.2.19) and (8.2.20) can also be expressed as

$$\big(H_{\rho_j}^{\rho_{j+1},\rho_1}\big)^*\, \bar{p}'_j \;=\; \bar{p}_j\, \big(H_{\rho_{j+1}}^{\rho_j,\rho_1}\big)^{-1} \tag{8.2.29}$$

and

$$\big(H_{\rho_{j+1}}^{\rho_j,\rho_1}\big)^*\, \bar{q}'_j \;=\; \bar{q}_j\, \big(H_{\rho_j}^{\rho_{j+1},\rho_1}\big)^{-1}, \tag{8.2.30}$$

and that (8.2.22) commutes for $j = n-1$, by our induction hypothesis. This allows us to relate the matrix elements of $F(i, \rho_n, \rho_1, \rho_t)$ to the ones of $F'(i, \rho_n, \rho_1, \rho_t)$, using formulae (8.2.27) and (8.2.28), and to prove commutativity of (8.2.23) whenever the morphism spaces are non-empty.

Next, we assume that $\rho_t \in i \circ \rho_{n+1}$ and derive a second set of relations among F-matrix elements. For this purpose we consider the pentagon equation

$$\begin{aligned} &\alpha(i, \rho_{n-1}, \rho_1 \circ \rho_1)\big(1_i \circ \alpha(\rho_{n-1}, \rho_1, \rho_1)^{-1}\big)\, \alpha(i, \rho_{n-1} \circ \rho_1, \rho_1)^{-1} = \\ &\qquad = \alpha\big(i \circ \rho_{n-1}, \rho_1, \rho_1\big)^{-1}\, \big(\alpha(i, \rho_{n-1}, \rho_1) \circ 1_1\big)\,. \end{aligned} \tag{8.2.31}$$

We choose $I \in Mor(\rho_{t+\varepsilon}, i \circ \rho_n)$, $J' \in Mor(\rho_n, \rho_{n-1} \circ \rho_t)$, and we multiply (8.2.31) by $K \circ \vartheta_1^\dagger$ from the left, and by $\Big(((1_i \circ J')I) \circ 1_1\Big)L$ from the right.

This yields

$$\begin{aligned} & K\big(1_i \circ \big[(1_{n-1} \circ \vartheta_1^\dagger)\alpha(\rho_{n-1}, \rho_1, \rho_1)^{-1}(J' \circ 1_1)\big]\big)\, \alpha(i, \rho_n, \rho_1)^{-1}(I \circ 1)L \\ & \quad = \; (1_t \circ \vartheta_1^t)\, \alpha(\rho_t, \rho_1, \rho_1)^{-1} \left(\big[(K \circ 1_1)\alpha(i, \rho_{n-1}, \rho_1)(1_i \circ J')I\big] \circ 1_1\right) L\,. \end{aligned} \tag{8.2.32}$$

Substituting $J = \bar{p}_{n-1}(J') \in Mor(\rho_n \circ \rho_1, \rho_{n-1}) = Mor(\rho_{n-1}, \rho_n \circ \rho_1)^*$, we obtain from (8.2.32), in F-matrix language, the equation

$$\begin{aligned} & \langle J \otimes K,\; F(i, \rho_n, \rho_1, \rho_t)^{-1} I \otimes L\rangle \;= \\ & \quad = \frac{d_{t+1}}{d_t}\, \langle K \otimes \bar{q}_t(L), F(i, \rho_{n-1}, \rho_1, \rho_{t+1})\bar{p}_{n-1}^{-1}(J) \otimes I\rangle, \quad \text{for } \varepsilon = 1, \\ & \quad = \frac{d_{t-1}}{d_t}\, \langle K \otimes \bar{p}_{t-1}(L), F(i, \rho_{n-1}, \rho_1, \rho_{t-1})\bar{p}_{n-1}^{-1}(J) \otimes I\rangle, \quad \text{for } \varepsilon = -1, \end{aligned} \tag{8.2.33}$$

for $J \otimes K \in Mor(\rho_{n-1}, \rho_n \circ \rho_1)^* \otimes Mor(\rho_t, i \circ \rho_{n-1})^*$ and I, L arbitrary. By similar arguments as for (8.2.23), we see that (8.2.33) implies the commutativity of

$$\begin{tikzcd}
\begin{matrix} Mor(\rho_{n-1}, \rho_n \circ \rho_1) \\ \otimes Mor(\rho_t, i \circ \rho_{n-1})^* \end{matrix} \arrow[r, hook, "(F(i, \rho_n, \rho_1, \rho_t)^{-1})^*"] & \begin{matrix} \bigoplus_s Mor(s, i \circ \rho_j)^* \\ \otimes Mor(\rho_t, s \circ \rho_1)^* \end{matrix} \\
\begin{matrix} Mor'(\rho_{n-1}, \rho_n \circ \rho_1)^* \\ \otimes Mor'(\rho_t, i \circ \rho_{n-1})^* \end{matrix} \arrow[r, hook, "(F'(i, \rho_n, \rho_1, \rho_t)^{-1})^*"] \arrow[u, "H^* \otimes H^*"'] & \begin{matrix} \bigoplus_s Mor'(s, i \circ \rho_j)^* \\ \otimes Mor'(\rho_t, s \circ \rho_1)^* \end{matrix} \arrow[u, "H^* \otimes H^*"']
\end{tikzcd} \tag{8.2.34}$$

Since the space $\bigoplus_s Mor(s, i \circ \rho_j) \otimes Mor(\rho_t, s \circ \rho_1)$ is at most two-dimensional, the image of $Mor(\rho_{n+1}, \rho_n \circ \rho_1) \otimes Mor(\rho_t, i \circ \rho_{n+1})$ in it may be expressed as follows

$$\begin{aligned} Q_n \; &:= \; F(i, \rho_n, \rho_1, \rho_t)\big(Mor(\rho_{n+1}, \rho_n \circ \rho_1) \otimes Mor(\rho_t, i \circ \rho_{n+1})\big) \\ &= \; F(i, \rho_n, \rho_1, \rho_t)\Big(\big(Mor(\rho_{n-1}, \rho_n \circ \rho_1)^* \otimes Mor(\rho_t, i \circ \rho_{n-1})^*\big)^{\perp}\Big) \\ &= \; \Big[\big(F(i, \rho_n, \rho_1, \rho_t)^{-1}\big)^*\big(Mor(\rho_{n-1}, \rho_n \circ \rho_1)^* \otimes Mor(\rho_t, i \circ \rho_{n-1})^*\big)\Big]^{\perp}. \end{aligned} \tag{8.2.35}$$

From (8.2.35) and (8.2.34) we conclude that $H^{\otimes 2}$ maps Q_n onto Q'_n. We may then consider the following diagram:

$$\begin{tikzcd}
Mor(\rho_{n+1}, \rho_n \circ \rho_1) \otimes Mor(\rho_t, i \circ \rho_{n+1}) \arrow[r, "F(i, \rho_n, \rho_1, \rho_t)", "\cong"'] \arrow[d, "H^{\rho_n, \rho_1}_{\rho_{n+1}} \otimes H^{i, \rho_{n+1}}_{\rho_t}"] & Q_n \arrow[d, "\cong"', "H^{\otimes 2}"] \\
Mor'(\rho_{n+1}, \rho_n \circ \rho_1) \otimes Mor'(\rho_t, i \circ \rho_{n+1}) \arrow[r, "F'(i, \rho_n, \rho_1, \rho_t)", "\cong"'] & Q'_n
\end{tikzcd} \tag{8.2.36}$$

Since, for $1 \leq n < k$ and $\rho_t \in i \circ \rho_{n+1}$, all other isomorphisms between one-dimensional spaces are already determined, there exists a unique $H^{i, \rho_{n+1}}_{\rho_t}$ such that (8.2.36) commutes. Combining the commutativity of (8.2.23) and (8.2.36), we obtain the commutativity of (8.2.22), with $j = n$. Since $F(i, 1, \rho_1, \rho_t) = id$, and since the isomorphisms H^{a, ρ_1}_b are already defined, the claim for $n = 1$ is clear. For $n = k$, the commutativity of (8.2.22) is identical to that of (8.2.23), since $\rho_k \circ \rho_1 = \rho_{k-1}$, and the induction can be terminated without any further definitions of H's . This completes the proof of the lemma.

□

Incidentally, the uniqueness of the isomorphisms in Lemma 8.2.2., ii) allows us to show that all natural transformations that leave the F-matrices of an A_{k+1}-category invariant have to be trivial. More precisely, a natural transformation is defined by a set of isomorphisms, $h^{i,j}_k$, of the spaces $Mor(k, i \circ j)$, $i, j, k \in J$, which, for A_n-fusion rules with $N_{ij,k} \in \{0, 1\}$, can be given by $\mathbb{C}$-numbers. A natural transformation leaves the F-matrices invariant if

$$h^{\otimes 2} F(i, j, k, l) \;=\; F(i, j, k, l)\, h^{\otimes 2}, \tag{8.2.37}$$

and a family of natural transformations obeying (8.2.37), called trivial, is given by

$$h^{i,j}_k \;=\; \frac{\lambda_i\, \lambda_j}{\lambda_k}\,, \tag{8.2.38}$$

for some function $\lambda : J \to \mathbf{C}$. (We use the conventions $h_i^{i,1} = h_i^{1,i} = id$, $\lambda_1 = 1$). To show that (8.2.38) holds in our example, we first find λ_{ρ_1} such that $h_1^{\rho_1,\rho_1} = \lambda_{\rho_1}^2$. We then define

$$\lambda_{\rho_n} := \prod_{j=1}^{n} \frac{\lambda_{\rho_1}}{h_{\rho_n}^{\rho_{n-1},\rho_1}}, \qquad \text{for} \quad n = 2, \ldots, k. \tag{8.2.39}$$

This implies equation (8.2.38), for $i = \rho_n$, $j = \rho_1$ and $k = \rho_{n+1}$. Since the maps p_j and q_j also represent F-matrix elements, we have to satisfy (8.2.19) and (8.2.20) with $p_n' = \lambda_{\rho_1}^{-2} p_n$ and $q_n' = \lambda_{\rho_1}^{-2} q_n$, yielding $h_{\rho_{n+1}}^{\rho_n,\rho_1}\, h_{\rho_n}^{\rho_{n+1},\rho_1} = \lambda_{\rho_1}^2$ and hence equation (8.2.39), for $i = \rho_{n+1}$, $j = \rho_1$ and $k = \rho_n$. By Lemma 8.2.2, the completion of the h_j^{i,ρ_1}'s compatible with (8.2.37) is unique, and hence the expression (8.2.38) is the only one possible. This observation, made on the level of structural data, can be put into the formal language of categories as follows:

$$\begin{aligned} & C(X,Y) \in End(X \circ Y), \quad \text{with} \\ (C(X,Y) \circ 1)\, C(X \circ Y, Z)\, \alpha(X,Y,Z) &= \alpha(X,Y,Z)\, C(X, Y \circ Z)(1 \circ C(Y,Z)), \\ \text{and} \qquad C(X',Y') I \circ J &= I \circ J\, C(X,Y), \end{aligned} \tag{8.2.40}$$

with $I \in Mor(X,X')$ and $J \in Mor(Y,Y')$, can be epxressed by a collection of isomorphisms, $\Lambda(X) \in End(X)$, which are isotropic, i.e., $\Lambda(X')I = I\Lambda(X)$, for all $I \in Mor(X,X')$:

$$C(X,Y) = \Lambda(X \circ Y)^{-1} (\Lambda(X) \circ \Lambda(Y)). \tag{8.2.41}$$

For a monoidal A_n-category, there exist exactly two solutions to (8.2.41) differing by the $\mathbb{Z}_2$-grading of the A_n-fusion rules. We can interpret the expressions in (8.2.40) and (8.2.41) as non-commutative generalizations of cocycle- and coboundary conditions, i.e., we can interpret (8.2.40) and (8.2.41) as triviality of a generalized second cohomology group.

Lemma 8.2.1 and Lemma 8.2.2 now put us in a position to prove the first result on the classification of categories.

PROPOSITION 8.2.3

For every $l = 1, \ldots, k+1$, with $(l, k+1) = 1$, there exists a semisimple, rigid, monoi category, unique up to natural equivalence, with A_{k+1}-fusion rules, such that

$$d_1 = -2\cos\left(\frac{l\pi}{k+2}\right). \qquad (8.2.4$$

It is given by the semisimple quotient of the representation category of $U_q(sl_2)$, wi $q = e^{\pm\frac{i\pi l}{k+2}}$. It is isomorphic to a C^*-category if and only if

$$l \in \{1, k+1\}. \qquad (8.2.4$$

This is the complete list of monoidal categories with A_{k+1}-fusion rules. Categorie corresponding to different values of l (i.e., different d_1) are inequivalent.

Remark: This result is generalized in [63], using the representation theory of Heck algebras. More precisely, it is shown that the monoidal categories with $U_q(sl_n)$ - fu sionrules, with $n > 2$, are precisely the $U_q(sl_n)$ - categories and that they are uniquel determined by the statistical dimension of the fundamental representation.

Proof.

The first step in the proof of Proposition 8.2.3 is to extend the commutativity of (8.2.2 to arbitrary representations and use this to prove uniqueness of an A_{k+1}-category, fo a given d_1. For this purpose, we define

$$\begin{aligned} F''(i,j,k,l) &:= (H^{\otimes 2})^{-1}\, F'(i,j,k,l) H^{\otimes 2} \\ &: \bigoplus_s Mor(s, j \circ k) \otimes Mor(l, i \circ s) \longrightarrow \bigoplus_s Mor(s, i \circ j) \otimes Mor(l, s \circ k), \end{aligned} \qquad (8.2.4$$

where the F- and F'-matrices are the structural data of two categories C and C' wi the same d_1, and $H_k^{i,j}$ are the isomorphisms specified in Lemma 8.2.2. To show that

$$\begin{tikzcd}
\bigoplus_s Mor(s, j \circ k) \otimes Mor(l, i \circ s) \arrow[r, "F(i,j,k,l)"] \arrow[d, "H^{\otimes 2}"] & \bigoplus_s Mor(s, i \circ j) \otimes Mor(l, s \circ k) \arrow[d, "H^{\otimes 2}"] \\
\bigoplus_s Mor'(s, j \circ k) \otimes Mor'(l, i \circ s) \arrow[r, "F'(i,j,k,l)"] & \bigoplus_s Mor'(s, i \circ j) \otimes Mor'(l, s \circ k)
\end{tikzcd}$$

(8.2.45)

ommutes is equivalent to showing $F = F''$, by (8.2.44). By assumption, we have that oth maps, F and F'', satisfy the pentagon equation, and, by Lemma 8.2.2, that

$$F''(i,j,\rho_1,l) \;=\; F(i,j,\rho_1,l)\,, \tag{8.2.46}$$

for all $i,j,l \in J$. Substituting (8.2.46) into a pentagonal equation for F'', we obtain

$$\begin{aligned}
&\bigoplus_{\varepsilon=\pm1} \mathbb{I} \otimes F''(i,j,\rho_{n+\varepsilon},t) \;= \\
&= T_{12}\Big(\bigoplus_s \mathbb{I} \otimes F(s,\rho_n,\rho_1,t)^{-1}\Big)\Big(\bigoplus_s F''(i,j,\rho_n,s) \otimes \mathbb{I}\Big) \\
&\qquad\quad \Big(\bigoplus_s \mathbb{I} \otimes F(i,s,\rho_1,t)\Big)\Big(\bigoplus_s F(j,\rho_n,\rho_1,s) \otimes \mathbb{I}\Big)\,.
\end{aligned} \tag{8.2.47}$$

From (8.2.47) and the pentagonal equation for F we see that if

$$F''(i,j,\rho_m,l) \;=\; F(i,j,\rho_m,l) \tag{8.2.48}$$

holds, for $m = 1, \ldots, n$, it also holds for $m = n+1$. Hence (8.2.48) follows by induction which proves (8.2.45).

In order to construct the explicit functor of equivalence, $(id, \mathcal{F}, C) : \mathcal{C} \to \mathcal{C}'$, we proceed in the same fashion as in similar constructions in section 8.1. We first fix an rbitrary set of isomorphisms

$$\mathcal{F} \,:\, Mor(i,X) \longrightarrow Mor'(i,X)\,. \tag{8.2.49}$$

This extends by functoriality and, since $Mor(X,Y) \cong \bigoplus_i Hom(Mor(i,X), Mor(i,Y))$, o a unique functor of abelian categories. Using that $Mor(k, X \circ Y) \cong \bigoplus_{ij} Mor(i,X) \otimes$

$Mor(j,Y) \otimes Mor(k, i \circ j)$, as specified in (8.2.42), we can define $C(X,Y) \in End'(X \circ Y)$ uniquely by the formula

$$C(X,Y)(\mathcal{F}(\hat{I}) \circ \mathcal{F}(\hat{J}))H_k^{i,j}(\hat{K}) \;=\; \mathcal{F}((\hat{I} \circ \hat{J})\hat{K})\,, \tag{8.2.50}$$

where $\hat{I} \in Mor(i,X)$, $\hat{J} \in Mor(j,Y)$ and $\hat{K} \in Mor(k, i \circ j)$. The compatibility with tensor products of morphisms in (8.2.33) follows immediately from the form of (8.2.50), using the fact that, by semisimplicity, it suffices to check (8.2.33) when it is multiplied by some $(\mathcal{F}(\hat{I}) \circ \mathcal{F}(\hat{J}))H_k^{i,j}(\hat{K})$ from the right. The verification of (8.2.43) is done similarly, multiplying

$$\Big(\mathcal{F}(\hat{I}) \circ \big((\mathcal{F}(\hat{J}) \circ \mathcal{F}(\hat{K}))H_s^{j,k}(\hat{S})\big)\Big)H_t^{i,s}(\hat{T}) \tag{8.2.51}$$

from the right, with $\hat{I} \in Mor(i,X)$, $\hat{J} \in Mor(j,Y)$, $\hat{K} \in Mor(k,Z)$, $\hat{S} \in Mor(s, j \circ k)$, and $\hat{T} \in Mor(t, i \circ s)$. Here we need to employ isotropy, eq. (8.1.38), of both α and α' and, furthermore, commutativity of (8.2.45).

We may now consider the monoidal representation category of $U_q(sl_2)$, with $q = e^{\pm\frac{i\pi l}{k+2}}$, $l = 1, \ldots, k+1$, $(l, k+2) = 1$.

We restrict the set of objects to those generated by the two-dimensional fundamental representation with highest weight $\lambda = 1$, i.e., to all integral highest weight representations, $V_{\lambda+1}$, $\lambda = 0, 1, \ldots, k$, and to the indecomposable projective modules $W_i, i \in \mathbb{Z}$, as defined in section 5.3.

We pass from this category to its semi-simple quotient. Hence we have exactly $k+1$ irreducible objects left over, and, by Theorem 5.3.1, these satisfy the A_{k+1}-fusion rules. If we use $\{v_0, v_1\}$, as a basis for the representation space V_2 of highest weight $\lambda = 1$, as in Proposition 5.2.1, and let $\{l_0, l_1\}$ be its dual basis in V_2^*, with $l_i v_j = \delta_{ij}$, the invariant tensors ϑ_1 and $\vartheta_1^\dagger$ are of the form

$$\vartheta_1 \;=\; \alpha(v_0 \otimes v_1 - q\, v_1 \otimes v_0) \;\in\; Hom_{U_q(sl_2)}(1, V_2 \otimes V_2),$$

and

$$\vartheta_1^\dagger \;=\; \beta(l_1 \otimes l_0 - q^{-1}\, l_0 \otimes l_1) \;\in\; Hom_{U_q(sl_2)}(V_2 \otimes V_2, 1)\,; \tag{8.2.52}$$

From the equation

$$(\mathbb{I} \otimes \vartheta_1^\dagger)(\vartheta_1 \otimes \mathbb{I}) \;=\; (\vartheta_1^\dagger \otimes \mathbb{I})(\vartheta_1 \otimes \mathbb{I}) \;=\; \alpha\beta\mathbb{I}$$

we see that (8.2.1) is satisfied iff $\alpha = \beta^{-1}$, so that

$$d_1 \;=\; \vartheta_1\vartheta_1^\dagger \;=\; -(q+q^{-1}) \;=\; -(2)_q\,. \tag{8.2.53}$$

Comparing (8.2.53) to (8.2.5) in Lemma 8.2.1 (with $d_1 \to -d_1$, for $l \to k+2-l$), we see that, for all admissible values of d_1, there exists a realization of an A_{k+1}-category obtained from the representation category of some $U_q(sl_2)$. Having proven uniqueness, for each value of d_1, this completes the classification of monoidal A_{k+1}-categories.

Finally, we wish to prove the result concerning a C^*-structure. In Lemma 8.2.1 we already found that $l = 1$ or $k+1$ are the only compatible values. In order to see that we can implement a C^*-structure in both cases, we first show that there exists an inner product on the $Mor(k, i \circ j)$-spaces such that the F-matrices define unitary maps. We have proven in Lemma 6.3.3 that, for $l = 1$, there exists an inner product such that the braid matrices are unitary. From the hexagonal equations, as expressed in Lemma 6.2.1, we see that the F-matrices can be written as products of unitary braid matrices and are therefore unitary with repsect to the given inner product, too. This system of F-matrices can be multiplied by the trivial 3-cocycle $f \in Z^3(\mathbb{Z}_2, 1; \mathbb{R}/\mathbb{Z})$, as described in (8.2.8), preserving the pentagonal equation and unitarity. For the invariant d_1' associated with these data, we find

$$\frac{1}{d_1'} \;=\; e^{2\pi i\, f(\alpha,\alpha,\alpha)}\, F(\rho_1\rho_1, \rho_1, \rho_1)_1^1 \;=\; -F(\rho_1, \rho_1, \rho_1, \rho_1)_1^1 \;=\; -\frac{1}{d_1}, \tag{8.2.54}$$

where $\alpha = \mathrm{grad}(\rho_1)$ is the non-trivial element in $\mathbb{Z}_2$, so that d_1' is precisely the invariant for $l = k+1$, and the resulting structural data are equivalent to those of $U_q(sl_2)$, with $q = -e^{\pm\frac{i\pi}{k+2}}$.

Once we have unitary F-matrices, we can implement a C^*-structure as follows: We define a positive definite inner product on each of the basic spaces, $Mor(k, X)$, with

$k \in J$ and $X \in Obj$, and denote by $*$, with

$$* \,:\, Mor(k,X) \longrightarrow Mor(k,X)^* \;=\; Mor(X,k), \tag{8.2.5}$$

the associated involution. This involution extends uniquely to $* \,:\, Mor(X,Y)$ – $Mor(Y,X)$, by $(IJ)^* = J^*I^*$, yielding a C^*-structure on C. We consider the ma $P(X,Y) \in End(X \circ Y)$ defined by the equation

$$P(X,Y)(I \circ J)K \;=\; \left(K^*(I^* \circ J^*)\right)^*, \tag{8.2.56}$$

for $I \in Mor(i,X)$, $J \in Mor(j,Y)$ and $K \in Mor(k, i \circ j)$. It is immediate from (8.2.56) that $P(i,j) = \mathbb{I}_{i \circ j}$, for $i,j \in J$. For $\bar{I} \in Mor(i',X)$, $\bar{J} \in Mor(j',Y)$, and $\bar{K} \in Mor(k', i' \circ j')$, we obtain the relation

$$((\bar{I} \circ \bar{J}) \circ \bar{K})^* P(X,Y)(I \circ J)K \;=\; \delta_{ii'}\delta_{jj'}\delta_{kk'}\langle \bar{I}, I\rangle\langle \bar{J}, J\rangle\langle \bar{K}, K\rangle, \tag{8.2.57}$$

so that $P(X,Y) > 0$ as an element of the C^*-algebra $End(X \circ Y)$. Hence there are isomorphisms $C(X,Y) \in End(X,Y)$, with $P(X,Y) = C(X,Y)^*C(X,Y)$ and $C(i,j) = \mathbb{I}_{i \circ j}$, for $i,j \in J$. If we apply the natural transformation $(id, \mathbb{I}, C)$ to this category we find that (8.2.56) holds with $P = 1$, and, by semi-simplicity, we conclude that

$$(A \circ B)^* \;=\; A^* \circ B^*\,, \tag{8.2.58}$$

for any $A \in Mor(X,X')$ and $B \in Mor(Y,Y')$. Since $C(i,j) = \mathbb{I}$, the F-matrices do not change under this change of tensor product. Thus, if the inner product chosen on $Mor(k,X)$ coincides, for $X = i \circ j$, with the one determined previously, the F-matrices are also unitary in the new category, based on (8.2.58). With these two ingredients, it is now easy to show that $\alpha(X,Y,Z)$ is unitary, too.

From the explicit formula (5.23) for highest weight vectors in tensor products we see that, for $j_1 = j_2$ and $j = 0$,

$$T\vartheta_\lambda \;=\; (g^{-1} \otimes 1)\vartheta_\lambda\,, \tag{8.2.59}$$

ith $g = (-q)^{-h}$, $S^2(a) = gag^{-1}$, $\vartheta_\lambda \in Hom_{U_q(sl_2)}(1, V_{\lambda+1} \otimes V_{\lambda+1})$ and $T(v \otimes w) =$ $\otimes v$. For the element $\vartheta_\lambda^\dagger \in Hom_{U_q(sl_2)}(V_{\lambda+1} \otimes V_{\lambda+1}, 1)$, with

$$(1 \otimes \vartheta_\lambda^\dagger)(\vartheta_\lambda \otimes 1) \;=\; (\vartheta_\lambda^\dagger \otimes 1)(1 \otimes \vartheta_\lambda) \;=\; \mathbb{I}_\lambda \,, \tag{8.2.60}$$

we find from (8.2.59) that

$$d_\lambda \;:=\; \vartheta_\lambda^\dagger \vartheta_\lambda \;=\; tr_{V_{\lambda+1}}(g^{-1}) \;=\; (-1)^\lambda (\lambda+1)_q \;=\; (\lambda+1)_{-q} \,. \tag{8.2.61}$$

Hence these quantities coincide with the ones defined by the recursion (8.2.2). This could also be derived from the existence of a balanced, braided structure, and thus of cyclic traces, compatible with the tensor product. It is of special interest to observe that

$$d_{k-j} \;=\; (-1)^{l+k+1}\, d_j \,. \tag{8.2.62}$$

If we denote by $C_{k,l}$ the category obtained from $U_q(sl_2)$, with $q = e^{\pm\frac{i\pi l}{k+2}}$, $l = 1, \ldots, k+1$, $(l, k+2) = 1$, the uniqueness assertion shows that there exists an isomorphism

$$(id, \mathcal{F}, C) \;:\; C_{k,l} \longrightarrow C_{k',l'}$$

only if $k = k'$ and $l = l'$. However, in order to prove that all $C_{k,l}$ are inequivalent, we have to consider isomorphisms

$$(\zeta, \mathcal{F}, C) \;:\; C_{k,l} \longrightarrow C_{k',l'} \tag{8.2.63}$$

where ζ is an arbitrary fusion rule algebra isomorphism. Clearly, this is only possible for $k = k'$. Also we need to have that $d_j = d'_{\zeta(j)}$. So if $\zeta(1) = 1$ we also have that $d_1 = d'_1$, and hence $l = l'$. The isomorphism ζ also has to preserve the Perron-Frobenius eigenvalue, $d_j^{P.F.}$, of the fusion rule matrix, i.e., $d_j^{P.F.} = d_{\zeta(j)}^{P.F.}$. This implies that $\zeta(j) \in \{j, k-j\}$. Moreover, ζ has to preserve the gradation, i.e., $\zeta(j) \equiv j \; mod 2$. For odd k, $\zeta(j) = j$ is therefore the only possibility. For even k, we also have $\zeta(\rho_j) = \rho_k^j \circ \rho_j$, as fusion rule algebra isomorphism. In the last case, l has to be odd. Hence, by (8.2.62), $d'_1 = d'_{k-1} = d'_{\zeta(1)}$. Since the existence of (8.2.63) implies that $d'_{\zeta(j)} = d_j$, we find that $d'_1 = d_1$, and thus $l = l'$. This proves that all categories $C_{k,l}$, for different pairs (k, l), are inequivalent.

□

Next, we supplement the classification of monoidal categories with A_n-fusion rules by an investigation of the possible braided structures for these categories. More precisely, we show that if the fusion rule algebra $\overline{Obj}$ is generated by an irreducible object, ρ, with $\rho \circ \rho = 1 + \psi$, $\psi \in J$, then the obstruction possibly present in the modified hexagonal equations and described by $H^5(\mathrm{Grad}(\overline{Obj}), 2; U(1))$ vanishes. Furthermore, we show that the possible fusion- and braid matrices for the fundamental object ρ can all be obtained from $U_q(sl_2)$. A general argument, often referred to as "cabeling", then shows that the entire braided category is isomorphic to the semisimple category obtained from $U_q(sl_2)$. The first result is obtained by solving a set of simple, algebraic equations.

LEMMA 8.2.4

Suppose C is a semisimple, monoidal category with objects Obj, and let $\rho \in J \subset Obj$ be an irreducible object with

$$\rho \circ \rho \;=\; 1 + \psi \,, \tag{8.2.64}$$

where $\psi \in J$. Denote by

$$F(\rho,\rho,\rho,\rho) \;:\; \bigoplus_{s=1,\psi} Mor(s, \rho\circ\rho) \otimes Mor(\rho, \rho\circ s) \;\rightarrow\; \bigoplus_{s=1,\psi} Mor(s, \rho\circ\rho) \otimes Mor(\rho, s\circ\rho) \tag{8.2.65}$$

the fundamental fusion matrix. Consider the modified hexagonal equations:

$$\begin{aligned}\Big(\bigoplus_{s=1,\psi} r(\rho,\rho,s) \otimes \mathbb{I}\Big)\, F(\rho,\rho,\rho,\rho) \Big(\bigoplus_{s=1,\psi} r(\rho,\rho,s) \otimes \mathbb{I}\Big) \;&= \\ =\; F(\rho,\rho,\rho,\rho) \Big(\bigoplus_{s=1,\psi} \mathbb{I} \otimes r(s,\rho,\rho)\Big)\, F(\rho,\rho,\rho,\rho) \,,&\end{aligned} \tag{8.2.66}$$

$$\Big(\bigoplus_{s=1,\psi} r(\rho,\rho,s)^{-1} \otimes \mathbb{I}\Big)\, F(\rho,\rho,\rho,\rho)\, \Big(\bigoplus_{s=1,\psi} r(\rho,\rho,s)^{-1} \otimes \mathbb{I}\Big) =$$

$$= \hat{\omega}^{-}\, F(\rho,\rho,\rho,\rho)\Big(\bigoplus_{s=1,\psi} \mathbb{I} \otimes r(\rho,s,\rho)^{-1}\Big)\, F(\rho,\rho,\rho,\rho)\,. \tag{8.2.66a}$$

These equations have a solution with $r(\rho,\rho,s) \in End\big(Mor(s,\rho\circ\rho)\big)$, *with* $r(s,\rho,\rho)$, $r(\rho,s,\rho)^{-1} \in Hom\big(Mor(\rho,s\circ\rho), Mor(\rho,\rho\circ s)\big)$, *and* $r(1,\rho,\rho)(1_\rho) = r(\rho,1,\rho)(1_\rho) = 1_\rho$ *if and only if*

$$\hat{\omega}^{-} = 1\,. \tag{8.2.67}$$

Up to natural gauge transformations, the solution is uniquely determined by the invariant $t \in \mathbb{C}^*$ *defined by*

$$r(\rho,\rho,\psi) =: t^{-1}\, \mathbb{I}_{Mor(\psi,\rho\circ\rho)}\,. \tag{8.2.68}$$

A solution to the modified hexagonal equations exists for $t \in \mathbb{C}^*$ *iff* $t^4 \neq -1$. *There exists a gauge and a choice of basis in the morphism spaces such that the matrix elements of the r's and F's are given by the following formulas:* $r(\rho,\rho,\psi)$ *is given by (8.2.68), and*

$$\begin{aligned}
r(\rho,\rho,1) &= t^3,\\
r(\psi,\rho,\rho) = r(\rho,\psi,\rho) &= -\,t^4,\\
F(\rho,\rho,\rho,\rho)^1_1 = -\,F(\rho,\rho,\rho,\rho)^\psi_\psi &= -\,\frac{1}{(2)_{t^2}}\,,\\
F(\rho,\rho,\rho,\rho)^1_\psi &= 1\,,\\
\textit{and} \qquad F(\rho,\rho,\rho,\rho)^\psi_1 &= \frac{(3)_{t^2}}{\big((2)_{t^2}\big)^2}\,.
\end{aligned} \tag{8.2.69}$$

Proof.

We begin by recalling some properties of the linear transformation $F(\rho,\rho,\rho,\rho)$ given in (8.2.65). As before, we may use the canonical element $1_\rho \in End(\rho)$ to associate to the matrix block $\big(F(\rho,\rho,\rho,\rho)\big)^1_1$ a unique element in $End\big(Mor(1,\rho\circ\rho)\big)$ and thus a $\mathbb{C}$-number. Rigidity, eq. (8.2.1), implies that this number is non-zero. Hence we can define an invariant $d_\rho \in \mathbb{C}^*$ of the category C by the equation

$$d_\rho^{-1}\mathbb{I}_{Mor(1,\rho\circ\rho)} = \big(F(\rho,\rho,\rho,\rho)\big)^1_1 = \big(F(\rho,\rho,\rho,\rho)^{-1}\big)^1_1\,, \tag{8.2.70}$$

where the second equality in (8.2.70) follows from (8.2.1). Concerning the bilinear form p_1, defined by

$$\begin{aligned}
p_1 \,:\, Mor(\psi,\rho\circ\rho) \otimes Mor(\rho,\psi\circ\rho) &\to \mathbb{C}\\
I \otimes J &\mapsto (1 \otimes \vartheta_1^\dagger)\, F(\rho,\rho,\rho,\rho)^{-1}(I \otimes J)\,,
\end{aligned} \tag{8.2.71}$$

as in (8.2.3), we know from the proof of lemma 8.2.1 that it is non-degenerate, with an explicit inverse given by (8.2.12). This shows that the matrix block $\big(F(\rho,\rho,\rho,\rho)^{-1}\big)^1_\psi \neq 0$, and hence the linear transformation

$$F(\rho,\rho,\rho,\rho)^1_\psi \,:\, Mor(\psi,\rho\circ\rho) \otimes Mor(\rho,\rho\circ\psi) \xrightarrow{\cong} Mor(1,\rho\circ\rho) \otimes End(\rho)\,, \tag{8.2.72}$$

does not vanish. Furthermore, the isomorphism $F(\rho,\rho,\rho,\rho)$ is constrained by the pentagonal equation

$$\big(F(\rho,\rho,\rho,\rho) \otimes \mathbb{I}\big)\Big(\bigoplus_{s=1,\psi} \mathbb{I} \otimes F(\rho,s,\rho,1)\Big)\big(F(\rho,\rho,\rho,\rho) \otimes \mathbb{I}\big) =$$

$$= \Big(\bigoplus_{s=1,\psi} \mathbb{I} \otimes F(s,\rho,\rho,1)\Big)\, T_{12}\, \Big(\bigoplus_{s=1,\psi} \mathbb{I} \otimes F(\rho,\rho,s,1)\Big)\,, \tag{8.2.73}$$

which, incidentally, also implies (8.2.70). We define isomorphisms $\Phi_s \,:\, Mor(\rho,s\circ\rho) \to Mor(\rho,\rho\circ s)$ by setting

$$F(\rho,s,\rho,1) = \Phi_s \otimes \mathbb{I}_{Mor(1,\rho\circ\rho)}\,, \tag{8.2.74}$$

so that $\Phi_1 \equiv \mathbb{I}_{End(\rho)}$. Furthermore, we define an isomorphism $F \in End\big(\bigoplus_s Mor(s,\rho\circ\rho) \otimes Mor(\rho,s\circ\rho)\big)$ by

$$F := F(\rho,\rho,\rho,\rho)\Big(\bigoplus_s \mathbb{I} \otimes \Phi_s\Big)\,. \tag{8.2.75}$$

From the pentagonal equation (8.2.73) we conclude that

$$F^2 = \bigoplus_{s=1,\psi} \lambda_s\, \mathbb{I}_s \tag{8.2.76}$$

is diagonal with respect to the one-dimensional subspaces corresponding to the channels $s = 1,\psi$. Clearly we have that $\lambda_1 \equiv 1$, since $F(1,\rho,\rho,1) = F(\rho,1,\rho,1) =$

$F(\rho,\rho,1,1) = \mathbb{I}$. Of course, the diagonal matrix (8.2.76) has to commute with the F-matrix which by (8.2.72) is non-diagonal. We conclude that (8.2.76) has to be a multiple of the identity, i.e., since $\lambda_1 = 1$, we have that $F^2 = 1$, or

$$F(\rho,\rho,\rho,\rho)^{-1} = \Big(\bigoplus_s \mathbb{I} \otimes \Phi_s\Big)\, F(\rho,\rho,\rho,\rho)\Big(\bigoplus_s \mathbb{I} \otimes \Phi_s\Big) . \tag{8.2.77}$$

Inverting eq. (8.2.66) and inserting (8.2.77) yields:

$$\begin{aligned}\Big(\bigoplus_s r(\rho,\rho,s)^{-1} \otimes \mathbb{I}\Big)\, F(\rho,\rho,\rho,\rho)\Big(\bigoplus_s r(\rho,\rho,s)^{-1} \otimes \mathbb{I}\Big)&\\ = F(\rho,\rho,\rho,\rho)\Big(\bigoplus_s \mathbb{I} \otimes \big(\Phi_s\, r(s,\rho,\rho)^{-1} \Phi_s\big)\Big)\, F(\rho,\rho,\rho,\rho) .&\end{aligned}$$

If we compare this to (8.2.66a), we find that

$$\hat{\omega}^-\, r(s,\rho,\rho) = \Phi_s\, r(\rho,s,\rho)\, \Phi_s . \tag{8.2.78}$$

For $s = 1$, (8.2.78) implies (8.2.67), i.e., triviality of the $H^5(\mathrm{Grad}(\overline{Obj}),2;U(1))$-obstruction if ρ generates $\overline{Obj}$. Conversely, for $\hat{\omega}^- = 1$, and with eqs. (8.2.77) and (8.2.78), any solution to (8.2.66) turns out to also be a solution to (8.2.66a). Besides the invariants $r(\rho,\rho,s)$ and d_ρ, we introduce a fourth invariant, $y \in \mathbb{C}^*$, by setting

$$y\, \mathbb{I}_{Mor(\rho,\psi\circ\rho)} := \Phi_\psi^{-1}\, r(\psi,\rho,\rho) = r(\rho,\psi,\rho)\, \Phi_\psi . \tag{8.2.79}$$

With the diagonal matrices $D, Q \in End\big(\bigoplus_s Mor(s,\rho\circ\rho) \otimes Mor(\rho,s\circ\rho)\big)$ given by $D := diag\big(r(\rho,\rho,1),\, r(\rho,\rho,\psi)\big)$ and $Q := diag(1,y)$, we can write the hexagonal equation as an equation between endomorphisms:

$$D\,F\,D = F\,Q\,F . \tag{8.2.80}$$

Using that $F^2 = 1$, we infer from this equation that DQ commutes with F, and since F is non-diagonal, DQ is a multiple of the identity, i.e.,

$$r(\rho,\rho,1) = y\, r(\rho,\rho,\psi) . \tag{8.2.81}$$

Using that $F \neq \pm\mathbb{I}$, we also find that

$$det(F) = -1 \quad \text{and} \quad tr(F) = 0 . \tag{8.2.82}$$

Hence, from (8.2.80),

$$\begin{aligned}&& det(D)^2 &= -det(Q) ,\\ \text{or} && y &= -r(\rho,\rho,1)^2\, r(\rho,\rho,\psi)^2 .\end{aligned} \tag{8.2.83}$$

The general solution to (8.2.81) and (8.2.83) can be parametrized by a number $t \in \mathbb{C}^*$ with the property that

$$\begin{aligned} r(\rho,\rho,1) &= t^3, & r(\rho,\rho,\psi) &= -t^{-1},\\ r(\psi,\rho,\rho) &= -t^4\Phi_\psi, & r(\rho,\psi,\rho) &= -t^4\Phi_\psi^{-1} .\end{aligned} \tag{8.2.84}$$

From (8.2.70) and (8.2.82) we find that

$$F_{11} = -F_{\psi\psi} = \frac{1}{d_\rho} . \tag{8.2.85}$$

If we take the trace on both sides of (8.2.80) we obtain, with (8.2.85) and $F^2 = 1$, the relation

$$\frac{1}{d_\rho}\big(r(\rho,\rho,1)^2 - r(\rho,\rho,\psi)^2\big) = 1 + y , \tag{8.2.86}$$

which, by (8.2.84), yields the expression

$$d_\rho = -(2)_{t^2} . \tag{8.2.87}$$

For arbitrary $\vartheta_\rho \in Mor(1,\rho\circ\rho)$ and $\vartheta_\psi \in Mor(1,\psi\circ\psi)$, we next determine basis vectors $I \in Mor(\rho,\psi\circ\rho)$, $J \in Mor(\rho, rho\circ\psi)$ and $K \in Mor(\psi,\rho\circ\rho)$ such that $F(\rho,\rho,\rho,\rho)$ has the matrix elements given in eq. (8.2.69), and, in addition, that

$$F(\psi,\rho,\rho,1)_\rho^\psi = F(\rho,\psi,\rho,1)_\rho^\rho = F(\rho,\rho,\psi,1)_\psi^\rho = 1 . \tag{8.2.88}$$

The morphisms I, J and K are unique up to a change of sign, $I, J, K \to -I, -J, -K$. We first determine K and I from the equations

$$F(\psi,\rho,\rho,1)(K \otimes \vartheta_\psi) = I \otimes \vartheta_\rho$$

$$F(K \otimes I) = -\frac{1}{d_\rho} K \otimes I + \vartheta_\rho \otimes 1_\rho . \qquad (8.2.89)$$

These equations have unique solutions K and I, up to a sign. The last matrix element of F in the basis $\{\vartheta_\rho \otimes 1_\rho,\ K \otimes I\}$,

$$F_{\psi 1} = 1 - \frac{1}{d_{\rho^2}}, \qquad (8.2.90)$$

is obtained from $det(F) = -1$. The condition that $F(\rho,\psi,\rho,1)^\rho_\rho = 1$ means that

$$J = \Phi_\psi(I), \qquad (8.2.91)$$

which, together with (8.2.75), yields the formulas for the matrix elements of $F(\rho,\rho,\rho,\rho)$ given in the lemma. Using (8.2.91) in (8.2.84), we also find the formulas for the r-matrices. Finally, the equation $F(\rho,\rho,\psi,1)^\rho_\psi = 1$ follows from (8.2.88). The fact that these matrices provide a solution to the hexagonal equation (8.2.66) can be verified by direct computation or by the observation that these data are identical to the ones for $U_q(sl_2), q = t^2$.

□

The observation, made in Lemma 8.2.4, that the braid- and fusion matrices of the fundamental representation ρ coincide with those of $U_q(sl_2)$ is, in fact, sufficient to infer that the entire category is isomorphic to the one obtained from $U_q(sl_2)$. This insight is based on the following cabeling argument which is an easy consequence of the hexagonal equation.

LEMMA 8.2.5

Suppose $\mathcal{C}$ and $\mathcal{C}'$ are braided tensor categories for which there exists an isomorphism

$$(\zeta, \mathcal{F}, C) : \mathcal{C} \to \mathcal{C}' \qquad (8.2.92)$$

between monoidal categories. Assume that $T \subset J$ is a set of irreducible objects which generate Obj and for which the equation

$$\mathcal{F}\big(\varepsilon(t,s)\big) = C(s,t)\,\varepsilon'\big(\zeta(t),\zeta(s)\big)\,C(t,s)^{-1}, \quad \forall\, t,s \in T, \qquad (8.2.93)$$

holds.

Then (8.2.92) is also an isomorphism between braided categories.

Proof.

In order to prove Lemma 8.2.5, we need to verify that

$$\mathcal{F}\big(\varepsilon(X,Y)\big) = C(Y,X)\,\varepsilon'\big(\zeta(X),\zeta(Y)\big)\,C(X,Y)^{-1} \qquad (8.2.94)$$

holds for each pair, (X,Y), of objects. Since both isomorphisms, ε and ε', are isotropic, it follows that, for subobjects $\tilde{X} \subset X$ and $\tilde{Y} \subset Y$, (8.2.94) holds for $(\tilde{X},\tilde{Y})$ whenever it is true for (X,Y). Conversely, if $W = X \oplus Y$ and (8.2.94) holds for (Z,X) and (Z,Y) it also holds for (Z,W). If we apply $\mathcal{F}$ to the hexagonal equation

$$\varepsilon(X \circ Y, Z) =$$

$$= \alpha(Z,X,Y)^{-1}\big(\varepsilon(X,Z)\circ 1\big)\,\alpha(X,Z,Y)\big(1\circ\varepsilon(Y,Z)\big)\,\alpha(X,Y,Z)^{-1} \qquad (8.2.94)$$

and use the fact that $(\zeta, \mathcal{F}, C)$ is a monoidal functor, so that ε' satisfies an equation analogous to (8.2.94), we find that (8.2.94) holds for the pair $(X \circ Y, Z)$ if it holds for (X,Z) and (Y,Z). Similarly, the pairs solving (8.2.94) close under taking tensor products in the second arguments. Thus, if by assumption (t,s) is admissible, for $t, s \in T$, then we can build any object X from $s \in T$ by a succession of steps which preserve the validity of (8.2.94). Hence (t, Y) is admissible, for every $t \in T$ and $Y \in Obj$. Applying the same argument to the first argument, we can prove (8.2.94) for all pairs. This completes the proof of the lemma.

□

Combining Proposition 8.2.3, Lemma 8.2.4 and Lemma 8.2.5, we arrive at the following result on braided tensor categories with A_n-fusion rules.

PROPOSITION 8.2.6

For every $l \in \mathbb{Z}_{4(k+2)}$, with $(l, k+2) = 1$, there exists a unique quantum category $C_{k,l}$ with A_{k+1}-fusion rules ($k > 1$), and satisfying

$$r(\rho_1, \rho_1, \rho_2) \;=\; exp\left(-2\pi i \, \frac{l}{4(k+2)}\right). \qquad (8.2.95)$$

It is isomorphic to the semi-simple category obtained from $U_q(sl_2)$, with $q^{1/2} = exp\left(2\pi i \, \frac{l}{4(k+2)}\right)$.

Two categories, $C_{k,l}$ and $C_{k,l'}$, are isomorphic as braided categories iff $l = l'$. They are isomorphic as monoidal categories iff

$$l' \;\equiv\; \pm\, l \quad mod\, 2(k+2)\,. \qquad (8.2.96)$$

The category $C_{k,l}$ is isomorphic to a C^*-category iff

$$l \;\equiv\; \pm\, 1 \quad mod\,(k+2)\,. \qquad (8.2.97)$$

This is the complete list of quantum categories with A_{k+1}-fusion rules.

The category $C_{k,l}$ has the invariants

$$r(\rho_j, \rho_j, 1) \;=\; exp\left(\frac{2\pi i\, l}{4(k+2)} \cdot j(j+2)\right) \;=:\; e^{2\pi i\, \theta_j}\,, \qquad (8.2.98)$$

for $j = 0, \ldots, k$. The θ_j's are balancing phases for $C_{k,l}$. The only further balancing structure is given by the phases

$$\theta'_j \;\equiv\; \theta_j \;+\; \frac{j}{2}\; mod\, 1\,. \qquad (8.2.99)$$

Proof.

From Lemma 8.2.4 (with $\rho = \rho_1$ and $\psi = \rho_2$) we know that, for some given $r(\rho_1, \rho_1, \rho_2)$ $= -t^{-1}$, the matrices $F(\rho_1, \rho_1, \rho_1, \rho_1)$ and $r(\rho_1, \rho_1, 1)$ are uniquely determined up to natural equivalence. In particular, for the invariant d_1 of Lemma 8.2.1, we have that

$$d_1 \;=\; -\,(2)_{t^2}\,.$$

The restriction on d_1 given in eq. (8.2.5) of that lemma is equivalent to the condition (8.2.95) for the value of t. Hence, for any of these values of t, we find, according to Proposition 8.2.3, a unique monoidal category. With given eigenvalues, $r(\rho_1, \rho_1, 1)$ and $r(\rho_1, \rho_1, \rho_2)$, of $\varepsilon(\rho_1, \rho_1) \in End(\rho_1 \circ \rho_1) \cong \mathbb{C}1 \oplus \mathbb{C}1$, we see that (8.2.93) holds for $T = \{\rho_1\}$. Since ρ_1 generates all objects of the category we conclude from Lemma 8.2.5 that, for a given value of t, one can find at most one braided structure on the given monoidal category. Thus, for a given $r(\rho_1, \rho_1, \rho_2)$ as in (8.2.95), there exists at most one braided tensor category. Each of these possible categories does in fact exist and can be obtained from the representation category of $U_q(sl_2)$, for the given value of $q^{1/2}$. This is easily verified by applying the transformation $T\mathcal{R}$, (where $\mathcal{R}$ is the universal R-matrix of $U_q(sl_2)$) to the highest weight vector $\xi^2_{1/2} \otimes \xi^2_{1/2} \in V_1 \otimes V_1$ corresponding to the eigenvalue $q^{1/2}$, i..e., $t = -q^{1/2}$. This proves existence and uniqueness of the categories $C_{k,l}$. In order to compute the invariants $r(\rho_j, \rho_j, 1)$ we simply compute the eigenvalue of $T\mathcal{R}$ for the invariant vector $\xi^1_0 \in V_j \otimes V_j$, given by

$$\vartheta_j \;\equiv\; \xi^1_0 \;=\; \sum_{m=-j/2}^{j/2} \;(-q)^{(j/2-m)}\, \xi^{j+1}_m \otimes \xi^{j+1}_{-m}\,; \qquad (8.2.100)$$

(compare to (5.23) for highest weight vectors). Using the equation $(a \otimes 1)\vartheta_j = 1 \otimes S^{-1}(a)$ and eq. (8.2.59) for an element g satisfying (6.94), we see that

$$T\mathcal{R}\vartheta_j \;=\; T(1 \otimes u)\vartheta_j \;=\; (ug^{-1} \otimes \mathbb{I})\vartheta_j\,,$$

where u is as in the definition of a ribbon-graph Hopf-algebra; see (6.92) and (6.93). It follows that the special central element $v = ug^{-1}$ acts on V_j like $r(\rho_j, \rho_j, 1)\mathbb{I}$. By (6.93), this implies that the phases θ_j given in (8.2.94) are indeed balancing, i.e., that

$$r(\rho_i, \rho_j, \rho_k)\, r(\rho_j, \rho_i, \rho_k) \;=\; e^{2\pi i(\theta_i + \theta_j - \theta_k)}\,.$$

If the braided category $C_{k,l}$ is a C^*-category then the corresponding monoidal category is a C^*-category, too, and hence condition (8.2.43) of Proposition 8.2.3 must hold. If

$\mathcal{C}_{k,l}$ is a C^*-category, as a monoidal category, then the projections in $End(\rho_1 \circ \rho_1)$ have to be selfconjugate, and, using that $| r(\rho_1, \rho_1, 1) | = | r(\rho_1, \rho_1, \rho_2) | = 1$, it follows that $\varepsilon(\rho_1, \rho_1)$ is unitary. From the iterative construction of all the other isomorphisms $\varepsilon(X, Y)$ obtained from the cabeling formula (8.2.94) and orthogonal decompositions of objects, we find that all $\varepsilon(X, Y)$ are automatically unitary. Thus, for the values of l given in (8.2.97), which is consistent with (8.2.7) of Lemma 8.2.1, the category $\mathcal{C}_{k,l}$ is a C^*-category as a braided tensor category. This completes the proof of the proposition.

□

The example $k = 1$ has already been studied in section 7.4 by observing that the A_2-algebra is just a $\mathbb{Z}_2$-fusion rule algebra and by noting that θ-categories are classified by $Hom(\Gamma_4(\mathbb{Z}_2),\ U(1)) \cong \mathbb{Z}_4$. A $\mathbb{Z}$-algebra is also contained in $\mathcal{C}_{k,l}$, for a general $k > 1$, which contains the invertible object ρ_k. The structural data of the corresponding subcategory are given by

$$\theta \equiv \frac{kl}{4}\, mod 1\,,$$

$$\text{and} \qquad F(\rho_k, \rho_k, \rho_k, \rho_k) = d_k^{-1} = (-1)^{kl}\,. \tag{8.2.101}$$

The results stated in Proposition 8.2.3 and Proposition 8.2.6 can be used to find all the categories with $\bar{A}_n$-fusion rules. To this end, we observe that $A_{2n} \cong \bar{A}_n \times \mathbb{Z}_2$. The corresponding graded projection $\bar{\zeta}_n : A_{2n} \twoheadrightarrow \bar{A}_n$, and the injection $i : \bar{A}_n \hookrightarrow A_{2n}$, with $\bar{\zeta}_n \circ i = id$, are given in Lemma 7.3.4, ii). We have that $ker(\bar{\zeta}_n) = \{1, \rho_{2n-1}\}$. Suppose now that $\bar{\mathcal{C}}$ is a (braided) monoidal category with $\bar{A}_n$-fusion rules. Then there is a unique number l, $l = 1, \ldots, 2n$ ($l \in \mathbb{Z}_{4(2n+1)}$, with $(l, 2n+1) = 1$), respectively, such that $\bar{\mathcal{C}}$ and $\bar{\zeta}_n$ induce $\mathcal{C}_{k,l}$ as a (braided) monoidal category, for $k = 2n-1$. The $\mathbb{Z}_2$-subcategory of the induced category has to be trivial. Having the explicit data (8.2.101), and with $k = 2n - 1$, this property can be expressed in terms of l as follows:

$$\begin{aligned} l &\equiv 0 \quad mod 2, && \text{for monoidal categories ;} \\ l &\equiv 0 \quad mod 4, && \text{for braided categories .} \end{aligned} \tag{8.2.102}$$

Conversely, if, for $\mathcal{C}_{2n-1,l}$, the $\mathbb{Z}_2$-subcategory is trivial as a monoidal category, then we can use formulae (8.2.11) and (8.2.12) for the dependence of the r- and F-matrices on the $\mathbb{Z}_2$-action. Since $\mathrm{Grad}(\bar{A}_n) = 1$, we also have that $\bar{\psi} \equiv 1$, $\xi \equiv 1$, and $\gamma : \bar{A}_n \to A_{2n}$ is precisely the injection i of fusion rule algebras, and, finally, $\eta(j) = \rho_k^{\mathrm{grad}(j)}$. It follows immediately from equ. (8.2.11) that $\mathcal{C}_{2n-1,l}$ is induced, as a monoidal category, by some category with $\bar{A}_n$-fusion rules and $\bar{\zeta}_n$. If, in addition, the $\mathbb{Z}_2$-subcategory of $\mathcal{C}_{2n-1,l}$ is trivial as a braided category it follows from equ. (8.2.12) that $\mathcal{C}_{2n-1,l}$ is also induced as a braided category by some category $\bar{\mathcal{C}}$ with $\bar{A}_n$-fusion rules. We thus have established a one-to-one correspondence between categories $\bar{\mathcal{C}}$ with $\bar{A}_n$-fusion rules and categories $\mathcal{C}_{2n-1,l}$ with A_{2n}-fusion rules, where l is constrained by (8.2.102). Clearly, every category $\mathcal{C}_{2n-1,l}$ contains a subcategory $\bar{\mathcal{C}}$ with $\bar{A}_n$-fusion rules, as $\bar{A}_n \subset A_{2n}$. If $\mathcal{C}_{2n-1,l}$ is also induced by some $\bar{\mathcal{C}}'$, i.e., if there is a functor $(\bar{\zeta}_n, \mathcal{F}, C) : \mathcal{C} \to \bar{\mathcal{C}}'$, then, since the restriction of $\bar{\zeta}_n$ to $\bar{A}_n \subset A_{2n}$ is the identity, the restriction of the functor to $\bar{\mathcal{C}}$ yields an isomorphism $\bar{\mathcal{C}} \cong \bar{\mathcal{C}}'$. Hence the $\bar{A}_n$-category associated to $\mathcal{C}_{2n-1,l}$, where l obeys (8.2.102), can be identified with the corresponding subcategory. We denote by $\bar{\mathcal{C}}_{n,\bar{l}}$ the braided category with $\bar{A}_n$-fusion rules which induces $\mathcal{C}_{2n-1,4\bar{l}}$, with $\bar{l} \in \mathbb{Z}_{2n+1}$, $(\bar{l}, 2n+1) = 1$ and $n = 1, 2, \ldots$. The relation between $\bar{\mathcal{C}}_{n,\bar{l}}$ and $\mathcal{C}_{2n-1,4\bar{l}}$ can be written compactly as

$$\mathcal{C}_{2n-1,4\bar{l}} \cong \bar{\mathcal{C}}_{n,\bar{l}} \otimes \mathcal{C}_{\mathbb{Z}_2, q=0}\,, \tag{8.2.103}$$

where the functor yielding (8.2.103) extends the isomorphism $A_{2n} \cong \bar{A}_n \times \mathbb{Z}_2$. Any monoidal category $\bar{\mathcal{C}}$ with $\bar{A}_n$-fusion rules induces a monoidal category $\mathcal{C}_{2n-1,l}$ with A_{2n}-fusion rules, where, by eq. (8.2.102), $l = 2\bar{\bar{l}}\ mod 4(2n+1)$, with $\bar{\bar{l}} \in \mathbb{Z}_{2(2n+1)}$. Following eq. (8.2.96) of Proposition 8.2.6, this category (viewed as a monoidal category) is equivalent to the one with $l = 2(\bar{\bar{l}} + (2n+1))$, so that l may always be chosen to be a multiple of four, i.e., $l = 4\bar{l}$. However, the category $\mathcal{C}_{2n-1,4\bar{l}}$ is induced by $\bar{\mathcal{C}}_{n,\bar{l}}$ also as a monoidal category. By the uniqueness of inducing categories, this implies that $\bar{\mathcal{C}} \cong \bar{\mathcal{C}}_{n,\bar{l}}$. Hence all monoidal categories with $\bar{A}_n$-fusion rules can be obtained from a braided monoidal category by omission of the braided structure. It is obvious from (8.2.103) that $\bar{\mathcal{C}}_{n,\bar{l}} \cong \bar{\mathcal{C}}_{n,\bar{l}'}$, as (braided) monoidal categories, if and only if $\mathcal{C}_{2n-1,4\bar{l}} \cong \mathcal{C}_{2n-1,4\bar{l}'}$. Proposition 8.2.6 implies that this is the case if and only if $\bar{l} = \bar{l}'$, for the braided situation, and $\bar{l} \equiv \pm\bar{l}'\ mod(2n+1)$ if we consider only the monoidal structure. Moreover, (8.2.103) shows that $\bar{\mathcal{C}}_{n,\bar{l}}$ is a C^*-category iff $\mathcal{C}_{2n-1,4\bar{l}}$ is one. Finally, we remark that, by the invariance of the $\mathbb{Z}_2$-action, the invariants of $\mathcal{C}_{2n-1,4\bar{l}}$ satisfy

$$r(\rho_j, \rho_j, \rho_s) = r(\rho_{2n-1-j}, \rho_{2n-1-j}, \rho_s), \quad \text{for } s = 0, 2, \ldots 2\,min\,(j, 2n-1-j)\,.$$

In particular, $\varepsilon(\rho_1, \rho_1)$ has the same spectrum as $\varepsilon(\rho_{2(n-1)}, \rho_{2(n-1)})$, where $\rho_{2(n-1)}$ is the generator of $\bar{A}_n$ with Perron-Frobenius dimension less than two. We summarize these conclusions, derived from Proposition 8.2.6, in the following corollary.

COROLLARY 8.2.7

Let ρ be the canonical generator of the $\bar{A}_n$-fusion rules, with $\rho \circ \rho = 1 + \psi$.

(i) *For every $\bar{l} \in \mathbb{Z}_{2n+1}$, with $(\bar{l}, 2n+1) = 1$, there exists a unique quantum category, $\bar{\mathcal{C}}_{n,\bar{l}}$, such that*

$$r(\rho, \rho, \psi) = exp\left(-\,2\pi i \,\frac{\bar{l}}{2n+1}\right). \tag{8.2.104}$$

This gives the complete list of quantum categories with $\bar{A}_n$-fusion rules. They are C^*-categories iff

$$\bar{l} \equiv \pm\, n^2, \qquad mod\,(2n+1)\,. \tag{8.2.105}$$

For each $\bar{\mathcal{C}}_{n,\bar{l}}$, there is a unique set of balancing phases, θ_α, given by

$$e^{2\pi i \theta_\alpha} = r(\alpha, \alpha, 1)\,; \tag{8.2.105a}$$

e.g., $e^{2\pi i \theta_\rho} = exp(6\pi i\,\bar{l}/(2n+1))$.

(ii) *Every rigid, monoidal category with $\bar{A}_n$-fusion rules is obtained from a quantum category by omission of the braided structure. We have that $\bar{\mathcal{C}}_{n,\bar{l}} \cong \bar{\mathcal{C}}_{n,\bar{l}'}$, as monoidal categories, iff*

$$\bar{l} \equiv \pm\,\bar{l}' \qquad mod\,(2n+1)\,. \tag{8.2.106}$$

They are C^*-categories iff (8.2.105) holds.

(iii) *The category $\bar{\mathcal{C}}_{n,\bar{l}}$ is isomorphic to the subcategory of the semisimple quotient of the representation category of $U_q(sl_2)$, $q^{1/2} = exp(2\pi i\bar{l}/(2n+1))$, generated by the $(2n-1)$-dimensional representation $\rho = V_{2(n-1)}$.*

At this point we have all the technical insights that allow us to classify all possible quantum categories with untwisted fusion rule algebras given by $\mathbb{Z}_r * \bar{A}_n$ and $\mathbb{Z}_r * A_{2n-1}$ as subcategories of products of $U_q(sl_2)$-categories and θ-categories of cyclic groups. The simplest result is the following theorem.

THEOREM 8.2.8

Let $\mathbb{Z}_r * \bar{A}_n \cong \mathbb{Z}_r \times \bar{A}_n$, with $r, n \geq 1$, be the fusion rule algebra specified in eq. (7.127) of Theorem 7.3.11.

For every $\bar{l} \in \mathbb{Z}_{2n+1}$ with $(\bar{l}, 2n+1) = 1$ and every $q \in Hom(\Gamma_4(\mathbb{Z}_r), U(1))$, we can define a quantum category

$$\bar{\mathcal{C}}_{n,r}(\bar{l}, q) := \mathcal{C}_{\mathbb{Z}_r,q} \otimes \bar{\mathcal{C}}_{n,\bar{l}} \tag{8.2.107}$$

with the fusion rules specified in the hypothesis.

(i) *The categories $\bar{\mathcal{C}}_{n,r}(\bar{l}, q)$ constitute the complete list of quantum categories with $\mathbb{Z}_r * \bar{A}_n$-fusion rules. If there is an isomorphism of quantum categories*

$$(\zeta, \mathcal{F}, C) : \bar{\mathcal{C}}_{n,r}(\bar{l}, q) \to \bar{\mathcal{C}}_{n,r}(\bar{l}', q') \tag{8.2.108}$$

then ζ is uniquely determined by its restriction, $\zeta_0 : \mathbb{Z}_r \overset{\cong}{\to} \mathbb{Z}_r$, to the subgroup of invertible objects. Furthermore,

$$\bar{l} = \bar{l}',$$

$$\text{and} \qquad q = \zeta_0^*\,(q')\,. \tag{8.2.109}$$

(ii) *There exists an isomorphism of the form given in (8.2.108) between monoidal categories if and only if*

$$\bar{l} \equiv \pm\,\bar{l}' \qquad mod\,(2n+1)\,,$$

$$\text{and} \qquad S^* \circ (\gamma_4^{-1})^*(q) = S^* \circ (\gamma_4^{-1})^*(q')\,. \tag{8.2.110}$$

where S and γ_4 are as in section 7.4, and the monoidal structures,

$$im\ S^* \;=\; {}_2\big(H^3(\mathbb{Z}_r,1;\mathbb{R}/\mathbb{Z})\big) \;\cong\; \mathbb{Z}_{(2,r)}\,,$$

of the two categories are identified by the unique isomorphism between them.

(iii) *The category $\bar{\mathcal{C}}_{n,r}(\bar{l},q)$ is C^* iff*

$$\bar{l} \;\equiv\; \pm\, n^2 \qquad mod\,(2n+1)\,. \tag{8.2.111}$$

It is always balanced, and the possible balancing phases are given by $\mathbb{Z}_2$-gradings, $\varepsilon \in Hom(\mathbb{Z}_r,\mathbb{Z}_2)$, of the group of invertible elements. For an irreducible object $j \in \bar{A}_n$ and a $\sigma \in \mathbb{Z}_r$, they are given by

$$exp(2\pi i\; \theta^{\varepsilon}_{(\sigma,j)}) \;=\; r(j,j,1)\; q(\sigma)\; \varepsilon(\sigma)\,. \tag{8.2.112}$$

Proof.

For the graded subgroup, $R \cong \mathbb{Z}_r$, of invertible objects we have that grad: $R \to$ Grad(Obj) is an isomorphism, i.e., Grad($\overline{Obj}$) = 1. In particular, we have that the obstruction $\Delta^*(\omega)$ from equ. (8.2.61) is always trivial. Thus, if $\mathcal{C}$ is a category with $\mathbb{Z}_r * \bar{A}_n$-fusion rules it follows from Corollary 8.1.14, ii) that there exists a quadratic function $q \in Hom(\Gamma_4(\mathbb{Z}_r),U(1))$ such that $\mathcal{C} \cong \hat{\mathcal{C}}^q$, and $\hat{\mathcal{C}}$ is induced by a category $\bar{\mathcal{C}}_{n,\bar{l}}$ with objects $\overline{Obj} = Obj/R = \bar{A}_n$ and a homomorphism $\pi_R : \mathbb{Z}_r * \bar{A}_n \to \bar{A}_n : (\sigma,j) \to j$. From formulae (8.2.4) and (8.2.5) for the structure constants of a product of categories we see that the r- and F-matrices of $\bar{\mathcal{C}}_{n,r}(\bar{l}, q \equiv 1)$ are invariant under the $\mathbb{Z}_r$-action. Hence $\bar{\mathcal{C}}_{n,r}(\bar{l},1)$ is also induced by π_R and a category on $\bar{A}_n$ which, by comparison of structural data, e.g., of $r(\rho,\rho,\psi)$, has to be $\bar{\mathcal{C}}_{n,\bar{l}}$. By the uniqueness of induced categories, it follows that $\hat{\mathcal{C}} \cong \bar{\mathcal{C}}_{n,r}(\bar{l},1)$. Clearly, we have that

$$\bar{\mathcal{C}}_{n,r}(\bar{l},q_1)^{q_2} \;\cong\; \bar{\mathcal{C}}_{n,r}(\bar{l},q_1\cdot q_2)\,. \tag{8.2.113}$$

Hence, in particular, $\mathcal{C}$ is of the form (8.2.107). An isomorphism $\zeta : \mathbb{Z}_r \times \bar{A}_n \to \mathbb{Z}_r \times \bar{A}_n$ has to map the ungraded subalgebras $\bar{A}_n$ onto each other. Since all objects in $\bar{A}_n$ have different Perron-Frobenius dimensions, this map from $\bar{A}_n$ to $\bar{A}_n$, denoted by f, is uniquely determined. Moreover, ζ has to map invertible objects to invertible objects. Hence its restriction to $\mathbb{Z}_r$, $\zeta_0 : \mathbb{Z}_r \to \mathbb{Z}_r$, is a well defined group isomorphism. It follows that, for $j \in \bar{A}_n$ and $\sigma \in \mathbb{Z}_r$, $\zeta\big((\sigma,j)\big) = \big(\zeta_0(\sigma), f(j)\big)$, i.e., ζ is unique for a given ζ_0.

For the canonical generator ρ of the ungraded $\bar{A}_n$-subalgebra satisfying $\rho \circ \rho = 1 + \psi$, the fact that $(\zeta, \mathcal{F}, C)$ is an isomorphism of braided categories implies that $r(\rho,\rho,\psi) = r'\big(f(\rho), f(\rho), f(\psi)\big)$ (see (8.2.104)), and hence that $\bar{l} = \bar{l}'$. Furthermore, the isomorphism (8.2.108) imposes on the quadratic, invariant functions q and q' the equation $q(\sigma) = q'\big(\zeta_0(\sigma)\big)$, for all invertible objects σ, i.e., $q = \zeta_0^*(q')$. Conversely, if (8.2.109) holds we have (according to section 7.4) an isomorphism $(\zeta_0, \mathcal{F}_0, C_0) : \mathcal{C}_{\mathbb{Z}_r,q} \to \mathcal{C}_{\mathbb{Z}_r,q'}$ which, when tensored with the identity on $\bar{\mathcal{C}}_{n,\bar{l}}$, yields the isomorphism (8.2.108) for the product categories.

For the proof of part ii) of the theorem it is sufficient, as in the case of braided categories, to show that there exist isomorphisms for the categories associated to the trivially graded objects and for the categories associated to the invertible objects. As a first condition we obtain eq. (8.2.106) of Corollary 8.2.7. If $\zeta_0 : \mathbb{Z}_r \to \mathbb{Z}_r$ is the restriction of ζ to the invertible objects it induces an isomorphism, $\zeta_0^{\#} : im\ S^* \circ \gamma_4^{-1}(\hat{\Gamma}_4(\mathbb{Z}_r)) \to im\ S^* \circ \gamma_4^{-1}(\hat{\Gamma}_4(\mathbb{Z}_r))$, and the two categories are isomorphic iff $\zeta_0^{\#}(S^* \circ \gamma_4^{-1}(q)) = S^* \circ \gamma_4^{-1}(q')$. Since the group on which $\zeta_0^{\#}$ is defined, is either $\{1\}$ or $\mathbb{Z}_2$, it is independent of ζ_0. Hence the requirement in (8.2.110) is also independent of ζ.

To prove part (iii) we remark that $\bar{\mathcal{C}}_{n,r}(\bar{l},q)$ is a C^*-category if and only if $\bar{\mathcal{C}}_{n,\bar{l}}$ and $\mathcal{C}_{\mathbb{Z}_r,q}$ are C^*-categories. Since θ-categories always carry a C^*-structure, we are left with condition (8.2.111), as in eq. (8.2.105) of Corollary 8.2.7, i). A set of balancing phases of a product category is given by the product of balancing phases of the individual categories, e.g., by the phases given in eq. (8.2.105a) of Corollary 8.2.7, i), for the $\bar{\mathcal{C}}_{n,\bar{l}}$-factor, and the quadratic function (7.296), for the $\mathcal{C}_{\mathbb{Z}_r,q}$-factor. Taking into account that distinct sets of balancing phases can only differ by $\mathbb{Z}_2$-gradings, we arrive at (8.2.112).

□

Notice that the fusion rule algebra isomorphism

$$\zeta \;:\; \mathbb{Z}_2 \times \bar{A}_n \;\rightarrow\; A_{2n} \;:\; (\varepsilon, j) \;\mapsto\; \rho_k^{\varepsilon} \circ j \quad (h = 2n-1)$$

extends to an isomorphism of braided categories,

$$(\zeta, \mathcal{F}, C) \;:\; \bar{\mathcal{C}}_{n,2}(\bar{l}, q) \;\rightarrow\; \mathcal{C}_{2n-1,l}\,,$$

if and only if

$$\bar{l} \;\equiv\; n^2 l \quad mod\,(2n+1)\,,$$

$$\text{and} \qquad q(j) \;=\; exp\left(\frac{(2n-1)l}{4}\, j^2\right), \quad j \,\in\, \mathbb{Z}_2\,. \tag{8.2.114}$$

The basic strategy to describe the categories associated to the $\mathbb{Z}_r * A_{2n-1}$-fusion rules relies on the fact that

$$\begin{aligned} i \;:\; \mathbb{Z}_r * A_{2n-1} \;&\hookrightarrow\; \mathbb{Z}_{2r}(g) \times A_{2n-1} \\ (k,\rho) \;&\mapsto\; (grad((k,\rho)), \rho)\,, \end{aligned} \tag{8.2.115}$$

is an inclusion of fusion rule algebras, see (7.255). Here $grad((k,\rho)) \;=\; g^{2k+\epsilon}$, where $\epsilon \,=\, 1$ if ρ is graded non-trivially, and $\epsilon \,=\, 0$ otherwise. A large class of braided tensor categories with $\mathbb{Z}_r * A_{2n-1}$ - fusion rules is therefore provided by the subcategories of the product categories $\mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}_{2(n-1),l}$. For a given $q \,\in\, Hom(\Gamma_4(\mathbb{Z}_{2r}), U(1))$, and $l \,\in\, \mathbb{Z}_{8n}$ with $(l, 2n) \,=\, 1$, we denote this subcategory by $\mathcal{C}_{n,r}(l, q)$. It is obvious from the definitions that

$$\mathcal{C}_{n,r}(l,\, q_1)^{q_2} \;=\; \mathcal{C}_{n,r}(l,\, q_1 \cdot q_2)\,, \tag{8.2.116}$$

for any pair $q_1\,, q_2 \;\in\; Hom(\Gamma_2(\mathbb{Z}_{2r}), U(1))$. The subcategory associated to the graded fusion rule subalgebra, $\mathbb{Z}_r \;\subset\; \mathbb{Z}_r * \bar{A}_n$, is characterized by the restriction, $i^*(q) \,\in\, Hom(\Gamma_4(\mathbb{Z}_r), U(1))$, of the qudratic function q, where $i \,:\, \mathbb{Z}_r \hookrightarrow \mathbb{Z}_{2r}$ is the monomorphism obtained from (8.2.115).

Notice that, for a quadratic function $\omega \,\in\, Hom(\Gamma_4(\mathbb{Z}_2), U(1))$, given by

$$\omega(j) \;=\; exp\left(2\pi i \frac{\tau}{4} j^2\right) \quad (h = 2n-2), \tag{8.2.117}$$

for some $\tau \,\in\, \mathbb{Z}_4$, we have by composition with the invariant (8.2.95)

$$\left(\mathcal{C}_{k,l}\right)^{\omega} \;\cong\; \mathcal{C}_{k,(l-\tau(k+2))}\,, \;(k = 2n-2) \tag{8.2.118}$$

This isomorphism can be used to generate an equivalence of the higher graded categories. To this end, we consider the following commutative diagram of inclusions, where we only assign the fusion rule algebra monomorphisms to the arrows :

$$\begin{tikzcd}
\mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}_{\mathbb{Z}_2,\omega} \otimes \mathcal{C}_{2(n-1),l} \arrow[r, leftrightarrow, "id \otimes (grad^o, id)"] & \mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}^{\omega}_{2(n-1),l} \\
\mathcal{C}_{\mathbb{Z}_{2r},\, q\cdot\pi^*(\omega)} \otimes \mathcal{C}_{2(n-1),l} \arrow[u, "\delta \otimes id"'] & \mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}_{2(n-1),(l-2n\tau)} \arrow[u, "\cong"'] \\
\mathcal{C}_{n,r}(l\,;\, q\cdot\pi^*(\omega)) \arrow[u, "i \,=\, (grad, p_2)"'] \arrow[r, leftrightarrow, "\cong"] & \mathcal{C}_{n,r}(l - 2n\tau\,,\, q) \arrow[u, "i \,=\, (grad, p_2)"']
\end{tikzcd} \tag{8.2.119}$$

Here, i is as in (8.2.115), with $p_2((k,\rho)) \;=\; \rho$. Furthermore, the projection map, $\pi \;:\; \mathbb{Z}_{2r} \twoheadrightarrow \mathbb{Z}_2$, yields the quotient by ${}_2(\mathbb{Z}_{2r}) \;\cong\; \mathbb{Z}_r$, and we use the notation $\delta(g) \;:=\; g \otimes \pi(g)$. Using that $\pi \circ grad \;=\; grad^o \circ p_2$, we see that this diagram commutes for the fusion rule algebra homomorphisms, and all but the bottom line can be extended to inclusion functors of braided categories. Therefore, the two categories in the bottom line are isomorphic to the same subcategory and thus isomorphic to each other.

Further equivalences of categories can be obtained from the automorphisms of $\mathbb{Z}_r * A_{2n-1}$. The only non-trivial fusion rule algebra automorphism of A_{2n-1} is given by γ_n which, in Lemma 7.3.4,i)a), is defined by

$$\gamma_n(\rho_j) \;=\; \rho_j \circ \left(\rho_{2(n-1)}\right)^j \tag{8.2.120}$$

We denote by $\alpha_\nu \,:\, g \rightarrow g^\nu$, with $(\nu, 2r) \,=\, 1$, the automorphisms of $\mathbb{Z}_{2r}$. The group of automorphisms of $\mathbb{Z}_r * A_{2n-1}$ is then generated by $\bar{\gamma}_n$ and $\bar{\alpha}_\nu$ which can be uniquely defined by the commutative diagram

$$\begin{tikzcd}
\mathbb{Z}_r * A_{2n-1} \arrow[r, hook, "i"] \arrow[d, "\bar{\alpha}_\nu \circ \bar{\gamma}_n^\epsilon"', "\cong"] & \mathbb{Z}_{2r} \times A_{2n-1} \arrow[d, "\alpha_\nu \times \gamma_n^\epsilon"', "\cong"] \\
\mathbb{Z}_r * A_{2n-1} \arrow[r, hook, "i"] & \mathbb{Z}_{2r} \times A_{2n-1}
\end{tikzcd} \qquad (8.2.121)$$

with $\epsilon = 0,1$ and $(\nu, 2r) = 1$. More explicitly, $\bar{\alpha}_\nu$ and $\bar{\gamma}_n$ are defined by

$$\begin{aligned}
\bar{\alpha}_\nu((k,\rho_j)) &= \Big(k + (\frac{\nu-1}{2}) grad((k,\rho_j)) \, , \, \rho_j\Big) \, , \\
\bar{\gamma}_n((k,\rho_j)) &= \big(k \, , \, \gamma_n(\rho_j)\big) \, ,
\end{aligned} \qquad (8.2.122)$$

for $k \in \mathbb{Z}_r$, $\rho_j \in A_{2n-1}$.

Generalizing formulae (8.2.104) we find, for $j = 1, 2, \dots, k-1$, the values for the invariants of $C_{k,l}$ $(k = 2(n-1))$

$$r(\rho_j, \rho_j, \rho_2) = -exp\Big(2\pi i \frac{l}{4(k+2)} (j^2 + 2j - 4)\Big) \qquad (8.2.123)$$

from explicit computations of the spectrum of $R = T\mathcal{R}$ on $U_q(sl_2)$-representations. In particular, for $k = 2(n-1)$, we have that

$$r(\rho_{k-1}, \rho_{k-1}, \rho_2) = -exp(-2\pi i \frac{\epsilon_n l}{8n}) \, , \qquad (8.2.124)$$

where $\epsilon_n \in \mathbb{Z}_{8n}$, with $\epsilon_n^2 = 1$, is given by

$$\epsilon_n = \begin{cases} 1 & \text{if } n \text{ is even} \\ 1 + 4n & \text{if } n \text{ is odd} \, . \end{cases} \qquad (8.2.125)$$

This shows that there exist functors of braided categories extending γ_n between precisely the following pairs :

$$(\gamma_n, \mathcal{F}, C) : \mathcal{C}_{2(n-1),l} \xrightarrow{\cong} \mathcal{C}_{2(n-1),\, \epsilon_n \cdot l} \, . \qquad (8.2.126)$$

From the functors in (8.2.126) we obtain canonical isomorphisms between the categories $\mathcal{C}_{\mathbb{Z}_r, q} \otimes \mathcal{C}_{2(n-1),l}$ and $\mathcal{C}_{\mathbb{Z}_r, q} \otimes \mathcal{C}_{2(n-1)}, \epsilon_n l$, for fixed q, and thus, by completing the square in (8.2.121), the isomorphisms

$$(\bar{\gamma}_n, \mathcal{F}, C) : \mathcal{C}_{n,r}(l, q) \xrightarrow{\cong} \mathcal{C}_{n,r}(\epsilon_n \cdot l, q) \, . \qquad (8.2.127)$$

In the same way we obtain the isomorphisms

$$(\bar{\alpha}_\nu, \mathcal{F}, C) : \mathcal{C}_{n,r}(l, q) \xrightarrow{\cong} \mathcal{C}_{n,r}\big(l, \alpha_\nu^*(q)\big) \, . \qquad (8.2.128$$

With the definition of the $\mathcal{C}_{n,r}(l,q)$-categories at our disposal, we are in a position t describe the classification of categories which have the second type of untwisted A-fusio rules, namely the $\mathbb{Z}_r * A_{2n-1}$ fusion rule algebras.

THEOREM 8.2.9

Let $\mathbb{Z}_r * A_{2n-1}$, with $r \geq 1$, $n \geq 2$, be the fusion rule algebras specified in (7.130) an (7.134) of Theorem 7.3.11. Denote by ρ the canonical generator with $\rho \circ \rho = \sigma + \psi$ where σ is the invertible object of order r.

(i) *All quantum categories with $\mathbb{Z}_r * A_{2n-1}$-fusion rules are isomorphic to $\mathcal{C}_{n,r}(l,q)$, for some $l \in \mathbb{Z}_{8n}$, with $(l, 2n) = 1$, and some $q \in Hom(\Gamma_4(\mathbb{Z}_{2r}), U(1))$. For given ρ, l and q are determined - up to the equivalence described in (8.2.119) - by the formulae*

$$\begin{aligned}
r(\rho, \rho, \psi) &= -exp(-2\pi i \frac{l}{8n}) \, q\big(grad(\rho)\big) \, , \\
r(\sigma, \sigma, \sigma^2) &= q\big(grad(\rho)\big)^4 \, .
\end{aligned} \qquad (8.2.129)$$

The only isomorphisms between these categories are compositions of those given in (8.2.119), (8.2.127) and (8.2.128), and, for $n = 2$ and r even, one further functor.

(ii) *The category is a C^* - category if and only if*

$$l \equiv \pm 1 \quad mod(2n) \, . \qquad (8.2.130)$$

There are two possible sets of balancing phases for $\mathcal{C}_{n,r}(l,q)$:

$$exp(2\pi i \theta_{(s,\rho_j)}) = exp\big(2\pi i \frac{l}{8n} j(j+2)\big) \, q\big(grad(s, \rho_j)\big) \, \epsilon^j \qquad (8.2.131)$$

with $s \in \mathbb{Z}_r$, $\rho_j \in A_{2n-1}$, $j = 0, 1, \dots, 2(n-1)$, and $\epsilon = \pm 1$.

Proof.

The fusion rule algebra $\mathbb{Z}_r * A_{2n-1}$ has a graded subgroup, $R := \mathbb{Z}_r(\sigma)$, with generator $\sigma \equiv (1,1)$, which is included in $Grad(Obj) = \mathbb{Z}_{2r}(g)$ (with generator $g := grad(\rho)$) by the map $\sigma^s \to g^{2s}$. It defines the graded fusion rule algebra homomorphism

$$\begin{aligned} \pi_R \, : \, Obj \, = \, \mathbb{Z}_r * A_{2n-1} &\twoheadrightarrow \overline{Obj} \, = \, Obj/R \, = \, A_{2n-1} \, , \\ (s \, , \rho_j) &\longmapsto \rho_j \qquad , \end{aligned} \tag{8.2.132}$$

so that $\mathrm{Grad}(Obj) \cong \mathbb{Z}_r$.

We consider a braided tensor category $\mathcal{C}$ with $\mathbb{Z}_r * A_{2n-1}$ - fusion rules and compute the invariant (depending on R) $\Delta^*(\omega) \in Hom\big({}_2\mathrm{Grad}(\overline{Obj}),\, \mathbb{Z}_2\big)$. Corollary 8.1.14,i) states that if $\Delta^*(\omega)$ is non-trivial then $r = 2r'$ is even, and we can find a monoidal category on A_{2n-1} and braid matrices $r(i,j,k)$ such that the modified hexagonal equations (8.2.76) hold. If we identify all representation labels in (8.2.76) with the fixed A_{2n-1} - generator ρ_1, i.e., $[i] = [j] = [k] = [t] = \rho_1$, we arrive at the equations (8.2.66) and (8.2.66a) given in Lemma 8.2.4. For $n = 2$, $g_i = g_j = g_k = \overline{grad}(\rho) = 1$ (in additive writing), and $\gamma(g_i, g_j) = 1$, we obtain for the prefactor in (8.2.66a) the equation

$$\hat{\omega}^- = -1 \quad .$$

This contradicts the assertion (8.2.67) of Lemma 8.2.4. It follows that

$$\Delta^*(\omega) \equiv 1 \quad , \tag{8.2.133}$$

for all braided tensor categories with $\mathbb{Z}_r * A_{2n-1}$ - fusion rules. Hence, by Corollary 8.1.14, ii), there exists a quadratic function $q \in Hom(\Gamma_4(Grad(Obj)),\, U(1))$ such that $\mathcal{C} \cong \hat{\mathcal{C}}^q$, and $\hat{\mathcal{C}}$ is induced by some category $\mathcal{C}_{2(n-1),l}$ with A_{2n-1}-fusion rules and by π_R. For the category, $\mathcal{C}_{\mathbb{Z}_{2r},q=0} \otimes \mathcal{C}_{2(n-1),l}$, with $\mathbb{Z}_{2r} \times A_{2n-1}$ - fusion rules, the subgroup $G = \{(\sigma,1)\}_{\sigma \in \mathbb{Z}_{2r}} \cong \mathbb{Z}_{2r}$ of invertible objects fullfills the hypotheses of Proposition 8.1.4, since the braid matrices of tensor product of categories have no mixed terms, i.e., the monodromy, $\bar{m} \in Hom(R \otimes Grad(Obj),\, U(1))$, as defined in Lemma 8.1.6, is trivial : $\bar{m} \equiv 1$. There therefore exists a category, $\tilde{\mathcal{C}}$, with A_{2n-1} - fusion rules such that $\mathcal{C}_{\mathbb{Z}_{2r},q=0} \otimes \mathcal{C}_{2(n-1),l}$ is induced by $\tilde{\mathcal{C}}$ and by the graded fusion rule algebra homomorphism $\pi : \mathbb{Z}_{2r} \times A_{2n-1} \to A_{2n-1} : g \otimes \rho_j \mapsto \rho_j$. By virtue of the inclusion $i_G : A_{2n-1} \hookrightarrow \mathbb{Z}_{2r} \times A_{2n-1} : \rho_j \mapsto 1 \otimes \rho_j$, ($1$ = neutral element) of fusion rule algebras, $\mathcal{C}_{2(n-1),l}$ is a subcategory of $\mathcal{C}_{\mathbb{Z}_{2r},q=0} \otimes \mathcal{C}_{2(n-1),l}$, and since $\pi_G \circ i_G = id_{A_{2n-1}}$, the composition of the corresponding functors yields $\tilde{\mathcal{C}} \cong \mathcal{C}_{2(n-1),l}$. The inclusion $i : \mathbb{Z}_{2r} * A_{2n-1} \hookrightarrow \mathbb{Z}_{2r} \times A_{2n-1}$, given in (8.2.115), then extends to an inclusion of the braided tensor category $\mathcal{C}_{n,r}(l,\, q \equiv 1)$ into $\mathcal{C}_{\mathbb{Z}_{2r},q=0} \otimes \mathcal{C}_{2(n-1),l}$. Since $\pi_R = \pi_G \circ i$, we find that, by composition of this inclusion with the functor onto $\mathcal{C}_{2(n-1),l}$, $\mathcal{C}_{n,r}(l,1)$ is induced, as a braided tensor category, by $\mathcal{C}_{2(n-1),l}$ and π_R. From the uniqueness of induced categories we conclude that $\hat{\mathcal{C}} \cong \mathcal{C}_{n,r}(l,1)$, and finally, with (8.2.116), we find that

$$\mathcal{C} \cong \mathcal{C}_{n,r}(l,q) \quad ,$$

proving the first assertion of the theorem. The invariants in (8.2.129) are simply those inherited from $\mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}_{2(n-1),l}$. If we denote by r_q and r_o the braid matrices of the two factors, then (8.2.5) implies that

$$\begin{aligned} r(\rho,\rho,\psi) &= r_q\big(grad(\rho), grad(\rho), grad(\psi)\big)\, r_o\big(\pi_R(\rho), \pi_R(\rho), \pi_R(\psi)\big) \, , \\ \text{and} \qquad r(\sigma,\sigma,\sigma^2) &= r_q\big(grad(\sigma), grad(\sigma), grad(\sigma)^2\big) \, . \end{aligned}$$

Hence, setting $\rho_1 = \pi_R(\rho)$, $\rho_2 = \pi_R(\psi)$, $grad(\sigma) = grad(\psi) = grad(\rho)^2$, and with the help of formula (8.2.95), we obtain (8.2.129).

A generator ρ of the $\mathbb{Z}_r * A_{2n-1}$ - algebra, in the sense of Theorem 7.3.11, is characterized by the facts that $grad(\rho)$ is invertible in (i.e., a generator of) $Grad(Obj) \cong \mathbb{Z}_{2r}$ and that $d_\rho = 2cos(\frac{\pi}{2n})$. If $n \neq 2$ the only automorphism of $\mathbb{Z}_r * A_{2n-1}$ which maps such a generator to itself is the identity, since tensor products with ρ have at most two irreducible summands and the equation $\rho \circ \rho = \sigma + \psi$ implies that, since ψ is not invertible, σ is mapped to itself. The only exception from this implication occurs for $n = 2$

and r even. In this case, $\rho_2 \in A_3$ is invertible, and an automorphism ζ on $\mathbb{Z}_r * A_{2n-1}$ can be defined from the equations $\zeta((s,\rho_1)) = (s,\rho_1)$ and $\zeta((s,\alpha)) = (s,\rho_2 \circ \alpha)$, for $s \in \mathbb{Z}_r$ and $\alpha \in \{1,\rho_2\}$. This fusion rule algebra homomorphism extends to a functor

$$(\zeta,\mathcal{F},C) \,:\, \mathcal{C}_{2,r}(l,q) \xrightarrow{\cong} \mathcal{C}_{2,r}(-l,q')\,, \tag{8.2.134}$$

with

$$q'(grad(\rho)) \;=\; -\,exp(2\pi i\,\frac{l}{8})\,q(grad(\rho))\,,$$

for any even r, any odd $l \in \mathbb{Z}_{16}$ and any $q \in Hom(\Gamma_4(Grad(Obj)),\,U(1))$.

Thus, a general automorphism ζ on $\mathbb{Z}_r * A_{2n-1}$ is, for $n \neq 2$, uniquely determined by the image, $\zeta(\rho)$, of the generator ρ. Since the group $\{\bar{\alpha}_\nu \,:\, \nu \in \mathbb{Z}_{2n}, (\nu,2r) = 1\}$ of automorphisms on $\mathbb{Z}_r * A_{2n-1}$ acts transitively on the invertible elements in the ring, $Grad(Obj) \cong \mathbb{Z}_{2r}$, and each graded component contains at most two objects with dimension $2cos(\frac{\pi}{2n})$ which are mapped onto each other by $\bar{\gamma}_n$, we see that the group of automorphisms, defined in (8.2.121), acts transitively on the set of generators. This proves that every automorphism on $\mathbb{Z}_r * A_{2n-1}$ is of the form (8.2.121) and, for $n = 2$ and r even, can also be composed with the special automorphism ζ defined above.

The categories $\mathcal{C}_{n,r}(l,q)$ are those with a generator ρ and an invertible object σ, with $\rho \circ \rho = \sigma + \psi$. Let us assume that there is an isomorphism

$$(\zeta,\mathcal{F},C) \,:\, \mathcal{C}_{n,r}(l,q) \xrightarrow{\cong} \mathcal{C}_{n,r}(l',q')$$

between two such categories. We can always write such a functor as a composition of the functors given in equations (8.2.127), (8.2.128) and, for $n = 2$, (8.2.134) with a further functor for which ζ maps the objects ρ and σ - and thereby all elements of $\mathbb{Z}_r * A_{2n-1}$ generated by ρ and σ - onto each other. For the latter, it follows from (8.2.129) that $(q)^4 = (q')^4$. A quadratic function $\bar{q}$ on the cyclic group Grad(Obj), with $\bar{q}^4 = 1$, is always of the form $\bar{q} = \pi^*(\omega)$, where $\omega \in Hom(\Gamma_4(\mathbb{Z}_2),\,U(1))$ is as in (8.2.117), and $\pi \,:\, \mathbb{Z}_{2r} \twoheadrightarrow \mathbb{Z}_2$ is the quotient by ${}_2(\mathbb{Z}_{2r})$. Hence, for $q' = q\cdot\pi^*(\omega)$, we

find from the first equation in (8.2.129) that $l' \equiv l + 2n\tau \; mod(8n)$. For any $\tau \in \mathbb{Z}_4$ we have already costructed the corresponding functors in (8.2.119). This completes the proof of part i) of the theorem.

The proof of the second part of Theorem 8.2.9 uses the facts that an induced category is a C^*-category if and only if the inducing category is C^* and that θ-categories are always C^*-categories. This shows that it is sufficient to verify the existence of a C^*-structure on the A_{2n-1}- category. Condition (8.2.130) is thus the same as (8.2.97).

The balancing phases recorded in (8.2.131) are simply those inherited from the category $\mathcal{C}_{\mathbb{Z}_{2r},q} \otimes \mathcal{C}_{2(n-1),l}$, multiplied with a $\mathbb{Z}_2$- grading, $(s,\rho_j) \to \epsilon^j$, which accounts for the only ambiguity in choosing the phases $\theta_{(2,\rho_j)}$, for a given braided tensor category .

□

The remaining A_n- categories we want to determine are those with $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ fusion rules, (see Sect. 3 for definition). The group R of invertible elements for this algebra is $\mathbb{Z}_{2r}$ and the induced grading, grad: $R \to$ Grad(Obj), has ${}_2R$ as a kernel .Thus, contrary to the previous cases, only the subgroups of R of odd order are graded, and hence, for $r = 2^p \cdot r'$, with r' odd, the order of Grad$(\overline{Obj})$, where $\overline{Obj}$ is the image of a graded homomorphism on Obj, is always a multiple of $2^{(p+1)}$. Fortunately, there is a second way to treat this situation:

We shall use the fact that there exists a graded homomorphism from an untwisted fusion rule algebra with a higher grading onto the twisted algebra under consideration. Before constructing this homomorphism, we must briefly recapitulate the definition of $\tau_\alpha(\mathbb{Z}_r * Obj)$ and the composition laws described in Definition 3.3.1. To this end we recall some notations used to describe extensions of cyclic groups. We consider the short exact sequence

$$\begin{tikzcd}
0 \arrow[r] & \mathbb{Z}_m(u) \arrow[r, hook, "i_{m,r}"] & \mathbb{Z}_{mr}(v) \arrow[r, two heads, "\pi_{m,r}"] & \mathbb{Z}_r(\bar{v}) \arrow[r] \arrow[l, bend right, "\beta_{m,r}"'] & 0
\end{tikzcd} \qquad (8.2.135)$$

of cylic groups with specified generators, homomorphisms $i_{m,r}(u) = v^r$ and $\pi_{m,r}(v) = \bar{v}$, and where $\beta_{m,r}$ is the section given by

$$\beta_{m,r} : \mathbb{Z}_r \to \mathbb{Z}_{rm} : \bar{v}^j \mapsto v^j , \quad j = 0, 1, \ldots , r-1, \qquad (8.2.136)$$

with $\pi_{m,r} \circ \beta_{m,r} = id$. Further, we define the map

$$\chi_{m,r} : \mathbb{Z}_{mr} \longrightarrow \mathbb{Z}_m$$

by

$$i_{m,r}(\chi_{m,r}(g)) = g\left(\beta_{m,r}(\pi_{m,r}(g))\right)^{-1} . \qquad (8.2.137)$$

When there is no confusion about the choice of generators we use an additive notation with generator 1; e.g., equation (8.2.137) can be written as

$$j \equiv \beta_{m,r}(\pi_{m,r}(j)) + i_{m,r}(\chi_{m,r}(j)) \qquad mod(mr) ,$$

for $j = 0, \ldots , mr-1$. We also define the cocycle $\gamma_a \in Z^2(\mathbb{Z}_a(g), \mathbb{Z})$ by

$$\gamma_a(i,j) \equiv \gamma_a(g^i, g^j) = \begin{cases} 1, & a \le i+j < 2a , \\ 0, & 0 \le i+j < a , \end{cases} \qquad (8.2.138)$$

with $i, j = 0, 1, \ldots , a-1$. Then

$$\delta\beta_{m,a} \equiv a\,\gamma_a \qquad mod(am) , \qquad (8.2.139)$$

(compare to (7.234) and (7.235)). For a fusion rule algebra $(Obj, \circ)$, the composition, $\circ_\alpha$, of $\tau_\alpha(Obj)$ is given by

$$x \circ_\alpha y = \alpha^{\gamma_a(grad(x), grad(y))} \circ x \circ y \qquad (8.2.140)$$

where γ_a is defined with respect to a given generator, g, of Grad$(Obj) \cong \mathbb{Z}_a(g)$, and $\alpha \in R_o = \{\sigma : \sigma \circ \sigma^\vee = 1, grad(\sigma) = 0\}$. The composition, $\circ_\gamma$, of the fusion rule algebra $\mathbb{Z}_r * Obj$ is given by

$$(k_1, x_1) \circ_\gamma (k_2, x_2) = \left(k_1 + k_2 + \gamma_a(grad(x_1), grad(x_2)), x_1 \circ x_2\right) , \qquad (8.2.141)$$

for $k_j \in \mathbb{Z}_r$ and $x_j \in Obj$. We choose a generator, $\bar{g}$, of Grad$(\mathbb{Z}_r * Obj) = \mathbb{Z}_{ar}(\bar{g})$ such that

$$grad((k, x)) = \bar{g}^{\left(ak + \beta_{r,a}(grad(x))\right)} . \qquad (8.2.142)$$

The product, $\circ_{\gamma,\alpha}$, for $\tau_\alpha(\mathbb{Z}_r * Obj)$ is therefore given by

$$\begin{aligned} (k_1, x_1) \circ_{\gamma,\alpha} (k_2, x_2) &= \qquad (8.2.143) \\ &= \left(k_1 + k_2 + \gamma_a(grad(x_1), grad(x_2)), \alpha^{\gamma_{ar}(grad(k_1,x_1), grad(k_2,x_2))} \circ x_1 \circ x_2\right) . \end{aligned}$$

Using the identity

$$\begin{aligned} r\,\gamma_{ar}(grad(k_1, x_1), grad(k_2, x_2)) &= \qquad (8.2.144) \\ &= \beta_{\infty,r}(k_1) + \beta_{\infty,r}(k_2) + \gamma_a(grad(x_1), grad(x_2)) \\ &\quad - \beta_{\infty,r}(k_1 + k_2 + \gamma_a(grad(x_1), grad(x_2))) \end{aligned}$$

we showed in (3.48) that

$$\mathbb{Z}_r * \tau_\alpha(Obj) \longrightarrow \tau_{\alpha^r}(\mathbb{Z}_r * Obj) : (k, x) \longmapsto (k, \alpha^{-\beta_{\infty,r}(k)} \circ x) \qquad (8.2.145)$$

is a fusion rule algebra homomorphism. Furthermore, we have the isomorphism

$$\mathbb{Z}_m * (\mathbb{Z}_r * Obj) \longrightarrow \mathbb{Z}_{mr} * Obj : (k, (l, x)) \longmapsto (rk + \beta_{m,r}(l), x) , \qquad (8.2.146)$$

for $k \in \mathbb{Z}_m$, $l \in \mathbb{Z}_r$, $x \in Obj$, which preserves the generators of the grading groups.

Suppose that $\alpha \in {}_mR_o$, (i.e., $\alpha \circ \alpha^\vee = 1$, $grad(\alpha) = 0$ and $\alpha^m = 1$). Then we may consider the composition of homomorphisms

$$\begin{aligned} f^* : \mathbb{Z}_{mr} * Obj &\rightarrowtail\!\!\twoheadrightarrow \mathbb{Z}^*_m(\mathbb{Z}_r * Obj) \rightarrowtail\!\!\twoheadrightarrow \tau_{\alpha^m}\left(\mathbb{Z}_m * (\mathbb{Z}_r * Obj)\right) \\ &\rightarrowtail\!\!\twoheadrightarrow \mathbb{Z}_m * \left(\tau_\alpha(\mathbb{Z}_r * Obj)\right) \longrightarrow\!\!\!\!\twoheadrightarrow \tau_\alpha(\mathbb{Z}_r * Obj) \qquad (8.2.147) \\ : (j, x) &\longrightarrow (\pi_{m,r}(j), \alpha^{\chi_{m,r}(j)} \circ x) \end{aligned}$$

Here we use that the inverse of (8.2.146) maps (j, x) to $\left(\chi_{m,r}(j), (\pi_{m,r}(j), x)\right)$ and that the last epimorphism in (8.2.147) maps (k, l) to x. The fusion rule algebra homomorphism f^* is irreducible and graded, and its kernel is given by

$$ker\, f^* \;=\; \{(rl,\, \alpha^{-l})\}_{l \in \mathbb{Z}_m} \;\cong\; \mathbb{Z}_m\,. \tag{8.2.148}$$

The existence of a graded homomorphism f^* allows us to identify a category, $\bar{\mathcal{C}}$, with $\tau_\alpha(\mathbb{Z}_r * Obj)$ - fusion rules with the category $\mathcal{C}$ with $\mathbb{Z}_{mr} * Obj$ - fusion rules, that is induced by $\mathcal{C}$ and f^*. The family of all balanced, braided tensor categories $\mathcal{C}$ which are of this form is characterized by conditions $i)$ and $ii)$ of Proposition 8.1.4, where $R \;=\; ker\, f^*$.

We specialize this result to the case, where $Obj \;=\; A_{2n-1}$, $a \;=\; 2$, $\alpha \;=\; \rho_{2(n-1)}$ and $m \;=\; 2$, i.e., we have

$$\begin{aligned} f^* \;:\; \mathbb{Z}_{2r} * A_{2n-1} &\twoheadrightarrow \tau_\alpha\left(\mathbb{Z}_r * A_{2n-1}\right) \\ (s,\, \rho_j) &\longmapsto \left(\bar{s},\, \rho_{2(n-1)}^{\chi_{2r}(s)} \circ \rho_j\right), \end{aligned} \tag{8.2.149}$$

with $\bar{s} \;=\; \pi_{2,r}(s)$, and

$$\chi_{2,r}(s) \;=\; \begin{cases} 0, & s = 0, 1, \ldots, r-1, \\ 1, & s = r, r+1, \ldots, 2r-1, \end{cases}$$

and the kernel of f^* is given by

$$ker\, f^* \;=\; \{1,\, \Sigma\} \;\cong\; \mathbb{Z}_2\,, \tag{8.2.150}$$

$$\text{with} \qquad \Sigma \;:=\; (r,\, \rho_{2(n-1)})\,,$$

$$\text{and} \qquad grad(\Sigma) \;=\; 2r \quad mod(4r).$$

The conditions for $\mathcal{C}_{n,2r}(l, q)$ to be induced by some category on $\tau_\alpha\left(\mathbb{Z}_r * A_{2n-1}\right)$ are, according to Proposition 8.1.4 :

$$i) \qquad r(\Sigma, \Sigma, 1) \;=\; 1 \tag{8.2.151}$$

$$\text{and} \qquad ii) \qquad \theta_{\Sigma \circ j} \;\equiv\; \theta_j \;\, mod\, 1,\; \forall j \in \mathbb{Z}_{2r} * A_{2n-1}\,. \tag{8.2.152}$$

To check i), we compute, using (8.2.101),

$$\begin{aligned} r(\Sigma, \Sigma, 1) \;&=\; r_q\left(grad(\Sigma), grad(\Sigma), 1\right) r_o\left([\sigma], [\Sigma], 1\right) \\ &=\; q(2r)\, r_o\left(\rho_{2(n-1)}\,,\, \rho_{2(n-1)}\,, 1\right) \\ &=\; (-1)^{\tau_o r}\, (-1)^{l(n-1)} \qquad\qquad = 1\,, \end{aligned} \tag{8.2.153}$$

where we define $\tau_o \in \mathbb{Z}_{8r}$ by $q(j) \;=\; exp\left(2\pi i\, \tau_o \frac{j^2}{8r}\right)$. Using that $\Sigma \circ (s,\, \rho_j) \;=\; \left(s + r,\, \rho_{2(n-1)-j}\right)$, and applying formula (8.2.131) for the balancing phases, condition (8.2.152) becomes :

$$q\left(2r + grad(s,\, \rho_j)\right) q\left(grad(s,\, \rho_j)\right)^{-1} \;=\; (-1)^{l(j+1-n)}\,.$$

Expressing q in terms of $\tau_o \in \mathbb{Z}_{8r}$, this is equivalent to

$$(-1)^{\tau_o(r+j)} \;=\; (-1)^{l(j+1-n)}\,,\; \text{for } j \;=\; 0, 1, \ldots, 2(n-1)\,,$$

which, for $j \;=\; 0$, is precisely the equation (8.2.153). Hence, with $(l, 2n) \;=\; 1$, i.e., $l \equiv 1\; mod\, 2$, (8.2.151) and (8.2.152) are equvalent to

$$\begin{aligned} &i) && \tau_o \;\equiv\; 1 && mod\, 2 \\ \text{and}\quad &ii) && r \;\equiv\; n + 1 && mod\, 2\,. \end{aligned} \tag{8.2.154}$$

It is remarkable that $ii)$ of (8.2.154) is a condition on the fusion rule algebra only. The first constraint is equivalent to the requirement that $\delta \in Hom\left(\mathbb{Z}_{4r} \otimes \mathbb{Z}_{4r}\,, U(1)\right)$ does not degenerate on ${}_2\left(\mathbb{Z}_{4r}\right)$, i.e., that $\delta q(2r, 1) \;=\; -1$. In particular, $i)$ is independent of the choice of generators and the natural $\mathbb{Z}_2$- ambiguity of the quadratic form. A form with this property shall be called an <u>odd</u> quadratic form on $\mathbb{Z}_{4r}$.

In order to describe the structure matrices, we introduce the choice map, in the sense of equ. (8.1.5),

$$\begin{aligned} \gamma^* \;:\; \tau_\alpha\left(\mathbb{Z}_r * A_{2n-1}\right) &\longrightarrow \mathbb{Z}_{2r} * A_{2n-1} \\ (\bar{s},\, \rho_j) &\longmapsto \left(\beta_{2,r}(\bar{s}),\, \rho_j\right), \end{aligned} \tag{8.2.155}$$

with $f^* \circ \gamma^* = id$. The map η from equation (8.1.46) is then given by

$$\eta \; : \; \mathbb{Z}_{2r} * A_{2n-1} \longrightarrow ker\, f^* \; : \; (s, \rho_j) \longmapsto \Sigma^{\chi_{2,r}(2)} \, , \tag{8.2.156}$$

which is of the form (8.2.10), i.e., γ^* is the choice defined by Lemma 8.1.9 for the section $\bar{\psi} = \beta_{2,2r}$. For an automorphism ζ of the fusion rule algebra $\mathbb{Z}_{2r} * A_{2n-1}$ for which

$$\zeta(\Sigma) = \Sigma \, , \tag{8.2.157}$$

we can define a unique automorphism, $\hat{\zeta}$, on $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ by requiring the following diagram to commute :

$$\begin{tikzcd}
\mathbb{Z}_{2r} * A_{2n-1} \arrow[r, "\zeta"] \arrow[d, "f^*"'] & \mathbb{Z}_{2r} * A_{2n-1} \arrow[d, "f^*"'] \\
\tau_\alpha(\mathbb{Z}_r * A_{2n-1}) \arrow[r, "\hat{\zeta}"] & \tau_\alpha(\mathbb{Z}_r * A_{2n-1})
\end{tikzcd} \tag{8.2.158}$$

We easily check that (8.2.157) holds for the automorphisms $\bar{\gamma}_n$ and $\bar{\alpha}_n$ defined in (8.2.122). The corresponding maps on $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ are :

$$\begin{aligned}
\hat{\gamma}_n((s, \rho_j)) \;&=\; (s, \rho_j \circ (\rho_{2(n-1)})^j) \, , \\
\text{and} \quad \hat{\alpha}_\nu((s, \rho_j)) &= (s + (\frac{\nu - 1}{2})\pi_{2,r}(grad(s, \rho_j)), \rho_j \circ (\rho_{2(n-1)})^{h_\nu(grad(s,\rho_j))}),
\end{aligned} \tag{8.2.159}$$

where

$$h_\nu \; : \; \mathbb{Z}_{2r} \to \mathbb{Z}_2 \; : \; g \mapsto \chi_{2,2r}(\nu\, \beta_{2,2r}(g)) \, ,$$

with $(\nu, 2r) = (\nu, 4r) = 1$, $s \in \mathbb{Z}_r$ and $j = 0, 1, \ldots, 2(n-1)$. Since the corresponding automorphisms $\hat{\alpha}_\nu^\#$ of the grading group, with $\hat{\alpha}_\nu^\# = g^\nu$, for $g \in Grad(Obj) \cong \mathbb{Z}_{2r}$, generate again the entire group $Aut(Grad(Obj))$, we have that the group of automorphisms generated by the elements in (8.2.159) acts transitively on the generators of $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$. From this we conclude, by the same arguments as for the untwisted algebras, that the automorphisms in (8.2.159) generate all automorphisms if $n > 2$.

For $n = 2$, categories exist only for odd r, in which case we find, with (3.48), that $\tau_\alpha(\mathbb{Z}_r * A_3) \cong \mathbb{Z}_r * \tau_\alpha(A_3)$. However $\tau_\alpha(A_3) \cong A_3$, since $\rho_2 \circ \rho_1 = \rho_1$, so that the fusion rule algebras $\tau_\alpha(\mathbb{Z}_r * A_3)$ are, in fact, untwisted algebras.

Having a one-to-one correspondence between the automorphisms of the fusion rule algebras $\mathbb{Z}_{2r} * A_{2n-1}$ and $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$, we can establish an analogous correspondence between equivalences of categories associated to these fusion rule algebras. We denote by $\hat{C}_{n,r}(l, q)$, with $l \in \mathbb{Z}_{8n}$, $(l, 2n) = 1$, $q \in Hom(\Gamma_4(\mathbb{Z}_{4r}), U(1))$, $q = odd$ and $n \equiv r + 1 \; mod\, 2$, the category induced by $C_{n,2r}(l, q)$ and f^*: There is a functor

$$(f^*, \mathcal{F}^*, C^*) \;=\; C_{n,2r}(l, q) \twoheadrightarrow \hat{C}_{n,r}(l, q) \, . \tag{8.2.160}$$

Suppose that ζ is an isomorphism of the $\mathbb{Z}_{2r} * A_{2n-1}$- algebra which extends to a functor

$$(\zeta, \mathcal{F}, C) \; : \; C_{n,2r}(l, q) \longrightarrow C_{n,2r}(l', q') \, , \tag{8.2.161}$$

for some l' and q'. The corresponding isomorphism $\hat{\zeta}$ defined by (8.2.158), also extends to a functor, $(\hat{\zeta}, \hat{\mathcal{F}}, \hat{C})$, from $\hat{C}_{n,r}(l, q)$ to some other category $\hat{C}_{n,r}(l'', q'')$ with fixed generator. It follows from (8.2.158) that $C_{n,2r}(l', r')$ is induced by $\hat{C}_{n,r}(l'', q'')$, and hence, by uniqueness of induced categories, we conclude that $l' = l''$, $q' = q''$, and there is a functor $(f^*, \mathcal{F}', C')$ such that

$$\begin{tikzcd}
C_{n,2r}(l, q) \arrow[r, "{(\zeta, \mathcal{F}, C)}"] \arrow[d, "{(f^*, \mathcal{F}, C)}"'] & C_{n,2r}(l', q') \arrow[d, "{(f^*, \mathcal{F}', C')}"] \\
\hat{C}_{n,r}(l, q) \arrow[r, "{(\hat{\zeta}, \hat{\mathcal{F}}, \hat{C})}"] & \hat{C}_{n,r}(l', q')
\end{tikzcd} \tag{8.2.162}$$

In particular, we have the isomorphisms

$$(id, \hat{\mathcal{F}}, \hat{C}) \; : \; \hat{C}_{n,r}(l, q) \longrightarrow \hat{C}_{n,r}(l + 2n\tau, \, q \cdot \pi^*_{2r,r}(\omega)) \, , \tag{8.2.163}$$

where ω is given in (8.2.117) and, as $\delta(\pi^*_{2r,r}(\omega))(2r, j) = 1$, $\forall j \in \mathbb{Z}_{4r}$, $q \cdot \pi^*_{2r,r}(\omega)$ is odd if q is odd. Moreover, from (8.2.127) and (8.2.128), we obtain the isomorphisms

$$(\hat{\gamma}_n, \hat{\mathcal{F}}, \hat{C}) \; : \; \hat{C}_{n,r}(l, q) \longrightarrow \hat{C}_{n,r}(\epsilon_n\, l, \, q) \, , \tag{8.2.164}$$

and, for $(\nu, 2r) = 1$,

$$(\hat{\alpha}_\nu, \hat{\mathcal{F}}, \hat{C}) \; : \; \hat{C}_{n,r}(l, q) \longrightarrow \hat{C}_{n,r}(l, \, \alpha^*_\nu(q)) \, . \tag{8.2.165}$$

From the invariants $r(i,i,j)$, with $N_{ii,j} = 1$, defined for the category $C_{n,2r}(l,q)$ we obtain the corresponding invariants $\hat{r}(f^*(i), f^*(i), f^*(j)) = r(i,i,j)$ on $\hat{C}_{n,r}(l,q)$. If the object ρ, satisfying $\rho \circ \rho = \sigma + \psi$, denotes the fixed generator of $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ then $\rho^o := \gamma^*(\rho)$ is the fixed generator of $\mathbb{Z}_{2r} * A_{2n-1}$, and it satisfies $\rho^o \circ \rho^o = \sigma^o + \psi^o$, where $f^*(\sigma^o) = \sigma$ and $f^*(\psi^o) = \psi$. Hence, the invariants defined in equation (8.2.129) of Theorem 8.2.9 for the objects, ρ^o and σ^o, yield invariants $\hat{r}(\rho,\rho,\psi) = r(\rho^o,\rho^o,\psi^o)$ and $\hat{r}(\sigma,\sigma,\sigma^2) = r(\sigma^o,\sigma^o,(\sigma^o)^2)$. From this it follows, by the same arguments as for Theorem 8.2.9, that for $n > 2$, the only isomorphisms among the $\hat{C}_{n,r}(l,q)$ - categories are given by compositions of those given in (8.2.163), (8.2.164) and (8.2.165). We thus obtain the following classification of categories with $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ - fusion rules.

__THEOREM 8.2.10__

Let $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$, $(r \geq 1, n \geq 2)$, be the fusion rule algebra specified in (7.131) and (7.135) of Theorem 7.3.11. Denote by ρ the fixed generator with the property that $\rho \circ \rho = \sigma + \psi$, where σ is the invertible object of order $2r$.

(i) *There exist quantum categories with $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ - fusion rules if and only if*

$$r \equiv n - 1 \quad mod(2)\,. \tag{8.2.166}$$

For $l \in \mathbb{Z}_{8n}$, with $(l,2n) = 1$, and every odd $q \in Hom(\Gamma_4(\mathbb{Z}_{4r}), U(1))$, there exists a quantum category, $\hat{C}_{n,r}(l,q)$, such that

$$r(\rho,\rho,\psi) = -exp(-2\pi i \frac{l}{8n})\, q(c)$$

$$and \qquad r(\sigma,\sigma,\sigma^2) = q(c)^4, \tag{8.2.167}$$

where $c := \beta_{2,2r}(grad(\rho))$ generates $\mathbb{Z}_{4r}$, and q is odd iff $\delta q(c^{2r}, c) = -1$. This category is induced by the category, $\hat{C}_{n,2r}(l,q)$ given in Theorem 8.2.9 and f^*. For $n = 2$ ($r \equiv 1\, mod 2$), we have that $\tau_\alpha(\mathbb{Z}_r * A_3) \cong \mathbb{Z}_r * A_3$. For $n > 2$, the only isomorphisms between these categories are those given in (8.2.163), (8.2.164) and (8.2.165). None of these categories is equivalent to a category with $\mathbb{Z}_r * A_{2n-1}$ - fusion rules.

(ii) *The category $\hat{C}_{n,r}(l,q)$ is isomorphic to a C^* - category if and only if*

$$l \equiv \pm 1 \qquad mod 2n\,. \tag{8.2.168}$$

There are two possible sets of balancing phases for $\hat{C}_{n,r}(l,q)$:

$$exp(2\pi i \theta_{(s,\rho_j)}) = \epsilon^j\, exp(2\pi i \frac{l}{8n} j(j+2))\, q\Big(\beta_{2,2r}(grad(2,\rho_j))\Big) \tag{8.2.169}$$

with $s \in \mathbb{Z}_r$, $\rho_j \in A_{2n-1}$, $j = 0,1,\ldots,2(n-1)$, and $\epsilon = \pm 1$.

If we combine the classification of categories in Theorems 8.2.8, 8.2.9 and 8.2.10 with the description of possible fusion rule algebras given in Proposition 7.3.25, we finally arrive at a characterization of braided, monoidal C^* - categories that are generated by a single object of statistical dimension less than two. It is remarkable to see that the constraints imposed by the monodromies $\bar{m} \in Hom(\bar{R} \otimes Grad(Obj), U(1))$, as in (8.2.150), with $m(\rho,\rho) = \varepsilon(\rho,\rho)^2$, are sufficient to single out precisely those fusion rules for which quantum categories exist. Moreover, a comparison of (7.2.58) with (8.2.112) and (8.2.131) and of (7.2.59) with (8.2.169), concerning the possible values of l and q, shows that all the statistical phases described in Proposition 7.3.25,ii), are realized in some quantum category.

Notice that, by use of the isomorphisms (8.2.118) and (8.2.163), we may always shift the parameter $l \in \mathbb{Z}_{8n}$, with $l \equiv \pm 1\, mod 2n$, labelling C^*-categories with $\mathbb{Z}_r * A_{2n-1}$- or $\tau_\alpha(\mathbb{Z}_r * A_{2n-1})$ - fusion rules, such that $l \equiv 1\, mod 8n$. According to the result on equivalences given in Theorems 8.2.9 and 8.2.10, an equivalence between two categories with $n > 2$ and the parameter l constrained in this way, mapping the distinguished generators onto each other, exists if and only if the quadratic functions are the same, and in this case the category is unique (up to isomorphism).

We have to formulate the main result of this work for C^*- categories only, since Proposition 7.3.25 has been proven under the assumption that a C^*- structure exists. There is, however, little doubt that our classification can easily be extended to the general, semisimple case.

THEOREM 8.2.11

Suppose C is an abelian, monoidal, rigid, braided, balanced C^*- category. Assume, further, that equivalent objects in the object set, Obj, of C are equal and that Obj - as a fusion rule algebra - is generated by a single, irreducible object, ρ. Let

$$d(\rho) := \lambda_\rho^{-1} e^{-2\pi i \vartheta_\rho} \quad \in \mathbb{R}$$

be the statistical dimension of the generator, where λ_ρ is the statistical parameter defined by

$$\lambda_\rho \vartheta_\rho^* \vartheta_\rho 1_\rho = (1_\rho \circ \vartheta_\rho^*)\, \alpha(\rho,\rho,\rho^\vee)^* \left(\varepsilon(\rho,\rho) \circ 1_{\rho^\vee}\right) \alpha(\rho,\rho,\rho^\vee)\, (1_\rho \circ \vartheta_\rho)$$

with $\vartheta_\rho \in Mor(1, \rho \circ \rho^\vee)$. Let θ_ρ be the balancing phase of the generator.

(i) *The following are equivalent :*

(a)

$$1 < |d(\rho)| < 2 ,$$

(b)

$$d(\rho) = \pm 2 cos(\frac{\pi}{N}) , \quad N = 4, 5, \ldots$$

(c)

$$\rho \circ \rho = \sigma + \psi \quad ,$$

where σ and ψ are irreducible, $m(\rho,\rho) = \varepsilon(\rho,\rho)^2$ is not a multiple of the identity, and σ is invertible.

(d) (*If C comes from a local quantum field theory*)

$$\rho \circ \rho = \sigma + \psi \quad ,$$

where σ and ψ are irreducible; the projections in $End((\rho \circ \rho) \circ \rho)$ given by

$$e_1 = e_\sigma(\rho,\rho) \circ 1_\rho \quad ,$$
$$e_2 = \alpha(\rho,\rho,\rho)\,(1_\rho \circ e_\sigma(\rho,\rho))\,\alpha(\rho,\rho,\rho)^* \quad ,$$

where $e_\sigma(\rho,\rho) \in End(\rho \circ \rho)$ is the projector corresponding to the subobject, σ, satisfy the Temperley - Lieb - equations,

$$\beta\, e_1 e_2 e_1 = e_1 \quad ,$$
$$\beta\, e_2 e_1 e_2 = e_2 \quad ,$$

with modulus, β, different from four (hence, $\beta < 4$).

(ii) *If one of the conditions in i) is fullfilled then the category C (without balancing) is equivalent to one of the following braided categories (defined with respect to the fixed generator ρ):*

(a) *For $n, r \in \mathbb{N}$, with $n \geq 2$, $r \geq 1$, and $q \in Hom(\Gamma_4(\mathbb{Z}_r), U(1))$,*

$$\bar{C}_{n,r}(\pm n^2, q) ,$$

which is defined and described in Theorem 8.2.8 as the product $C_{\mathbb{Z}_r,q} \otimes \bar{C}_{n,\pm n^2}$. It has fusionrules

$$\mathbb{Z}_r \star \bar{A}_n \quad ,$$

as in (3.117) of Theorem 3.4.11.

(b) *For $n, r \in \mathbb{N}$, with $n \geq 2$, $r \geq 1$, and $q \in Hom(\Gamma_4(\mathbb{Z}_{2r}), U(1))$,*

$$C_{n,r}(\pm 1, q) ,$$

defined as a subcategory of $C_{\mathbb{Z}_r,q} \otimes C_{2(n-1),\pm 1}$ by virtue of the inclusion in (8.2.114) and described in Theorem 8.2.9. It has the fusionrules

$$\mathbb{Z}_r \star A_{2n-1}$$

as in (3.120) and (3.124) of Theorem 3.4.11.

(c) *for* $n, r \in \mathbb{N}$, *with* $n \geq 3$, $r \geq 1$, $r \equiv n-1 \; mod(2)$, *and* $q \in Hom(\Gamma_4(\mathbb{Z}_{4r}), U(1))$, *with* q *odd,*

$$\hat{C}_{n,r}(\pm 1, q)\,,$$

defined as the category inducing $C_{n,2r}(\pm 1, q)$ *by the graded morphism in (8.2.147) and described in Theorem 8.2.10. It has fusionrules*

$$\tau_\alpha(\mathbb{Z}_r \star A_{2n-1})\,,$$

as in (3.121) and (3.125) of Theorem 3.4.11.

In a) and b) we include the possibility $r = \infty$ *for a torsion free grading group, with* $\Gamma_4(\mathbb{Z}) \cong \mathbb{Z}$. *For each of these categories balancing phases exist and are uniquely determined up to* $\mathbb{Z}_2$ *- gradings.*

(iii) *The categories in ii), for given* n, r, q *and a given sign in the 1-argument, are inequivalent as braided categories with a dustinguished generator* ρ, *with the single exception of*

$$(\zeta, \mathcal{F}, C) : C_{2,r}(\pm 1, q) \xrightarrow{\cong} C_{2,r}(\mp 1, q')\,,$$

where $q'(grad(\rho)) = -\frac{1 \pm i}{\sqrt{2}}\, q(grad(\rho))$. *In any of the cases a), b) and c), the group of the automorphisms* $Aut(Obj)$ *of the fusion rules (modulo the exceptional one) acts freely and transitively on the set of generators,* $\{j : d_j^{P.F.} = |d(\rho)|,\ grad(j) \text{ generates } Grad(Obj)\}$, *and can be extended to equvalences of categories. For case a) we have that* $Aut(Obj) \cong Aut(Grad(Obj))$, *and, for cases b), with* $n > 2$, *and c), that* $Aut(Obj) \cong \mathbb{Z}_2 \oplus Aut(Grad(Obj))$.

Appendix A

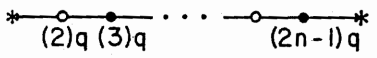
Undirected Graphs with Norm not Larger than Two

We give a list of all undirected, connected graphs with norm not larger than two. We distinguish between bicolorable and non-bicolorable graphs and indicate the possible bicolorations by white and black vertices. By Kronecker's theorem, the norm of such a graph is $2\cos\left(\frac{\pi}{N}\right)$, *where* $N = 3, 4, \ldots, \infty$ *is the Coxeter-number of the graph and is given below for graphs with norm less than two. The graphs with* $N = \infty$ *for which there exists a positive eigenvector with eigenvalue two are included. For each graph, the components of the Perron-Frobenius vector,* $\vec{d}$, *are given by the numbers indicated at the vertices which are expressed in terms of q-numbers* $(n)_q := \frac{q^n - q^{-n}}{q - q^{-1}}$, *with* $q = e^{\frac{i\pi}{N}}$, *and* N *is the Coxeter number of the graph. The vector* $\vec{d}$ *is normalized such that its smallest component on the graph is one, except when all vertices have edge degree two in which case we set* $\vec{d} := (2, 2, \ldots)$. *The sites where the Perron-Frobenius vector attains its minimum are marked by a "*", and the number, g, of such sites is indicated, (for each coloration separately, in the bicolorable case).*

.1 Bicolorable, finite graphs

a) $N < \infty$

$A_{2n},\ n \geq 1$:



$N = 2n + 1,$

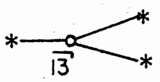
$g_o = 1,\ g_1 = 1$ (A1)

$A_{2n-1},\ n \geq 2$:

$N = 2n,$

$g_o = 2,\ g_1 = 0$ (A2)

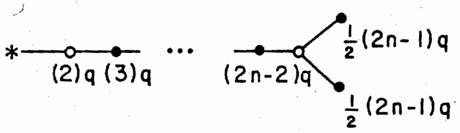
D_4 :



$N = 6,$

$g_o = 3,\ g_1 = 0$ (A3)

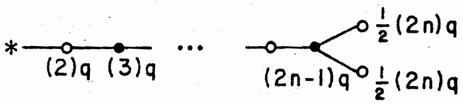
$D_{2n},\ n \geq 3$:



$N = 4n - 2,$

$g_o = 1,\ g_1 = 0$ (A4)

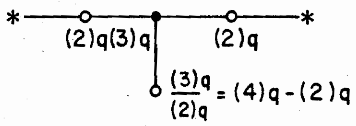
$D_{2n+1},\ n \geq 2$:



$N = 4n,$

$g_o = 1,\ g_1 = 0$ (A5)

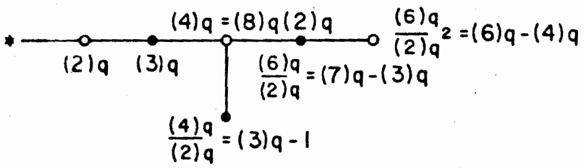
E_6 :



$N = 12,$

$g_o = 2,\ g_1 = 0$ (A6)

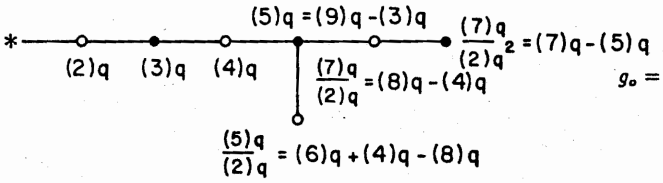
E_7 :



$N = 18,$

$g_o = 1,\ g_1 = 0$ (A7)

E_8 :

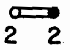


$N = 30,$

$g_o = 1,\ g_1 = 0$ (A8)

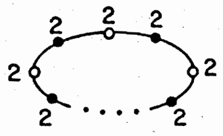
b) $N = \infty$

$A_1^{(1)}$:



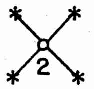
; $A_{2n-1}^{(1)},\ n \geq 2$:

$g_o = 0,\ g_1 = 0$



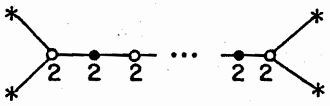
(A9)

$D_4^{(1)}$:



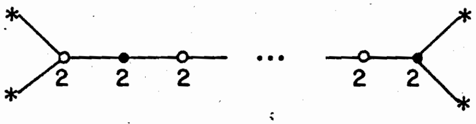
; $D_{2n}^{(1)},\ n \geq 3$:

$g_o = 4,\ g_1 = 0$



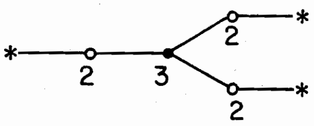
(A10)

$D_{2n+1}^{(1)},\ n \geq 2$:



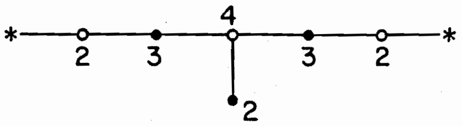
$g_o = 2,\ g_1 = 2$ (A11)

$E_6^{(1)}$:



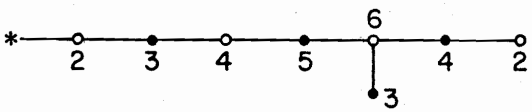
$g_0 = 3,\ g_1 = 0$ (A12)

$E_7^{(1)}$:



$g_0 = 2,\ g_1 = 0$ (A13)

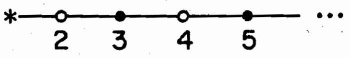
$E_8^{(1)}$:



$g_o = 1,\ g_1 = 0$ (A14)

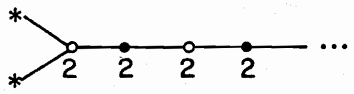
A.2 Bicolorable, infinite graphs (corresponding to $N = \infty$)

A_∞ :



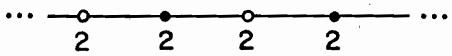
$g_o = 1,\ g_1 = 0$ (A15)

D_∞ :



$g_o = 2,\ g_1 = 0$ (A16)

$A_{\infty,\infty}$:



$g_o = 0,\ g_1 = 0$ (A17)

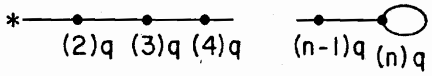
A.3 Non-bicolorable, finite graphs

a) $N < \infty$

$\overline{A}_1$:

$N = 3,\ g = 1$ (A1

$\overline{A}_n,\ n \geq 2$:

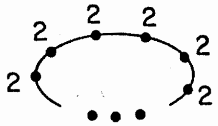


$N = 2n + 1,$

$g = 1$ (A1

b) $N = \infty$

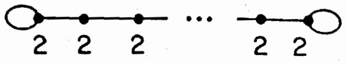
$A_{2n}^{(1)}\ n \geq 1$:



$g = 0$ (A2

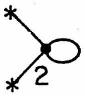
$\overline{\overline{A}}_1$:

$\overline{\overline{A}}_n,\ n \geq 2$:

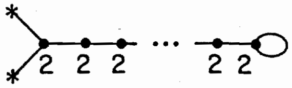


$g = 0$ (A2

$\overline{D}_3$:



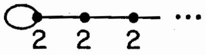
$\overline{D}_n$:



$g = 2$ (A2

A.4 Non-bicolorable, infinite graphs ($N = \infty$)

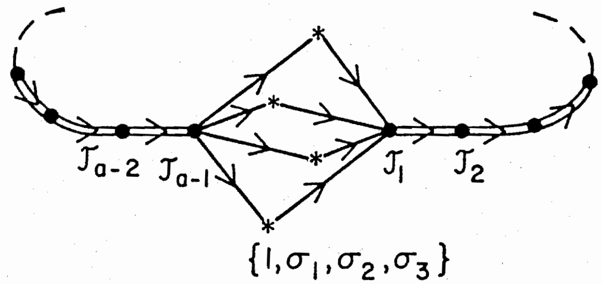
$\overline{A}_\infty$:



$g = 0$ (A23)

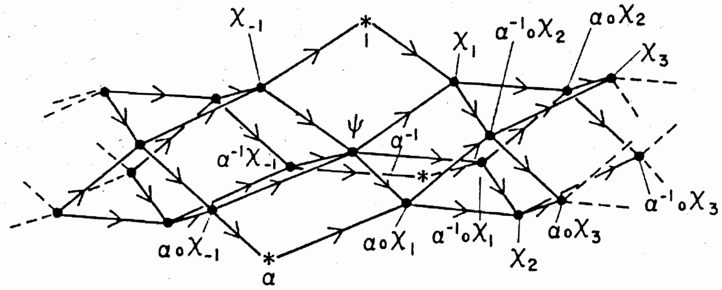
.5 The higher graded fusionrule algebras

i) The fusion graph for algebra $D_4^{(1)}\left(A_1^{(1)}\right)^{(a-2)}$:



(A24)

ii) The fusion graph for algebra $E_6^{(1)}\left(A_5^{(1)}\right)^{(a-2)}$:

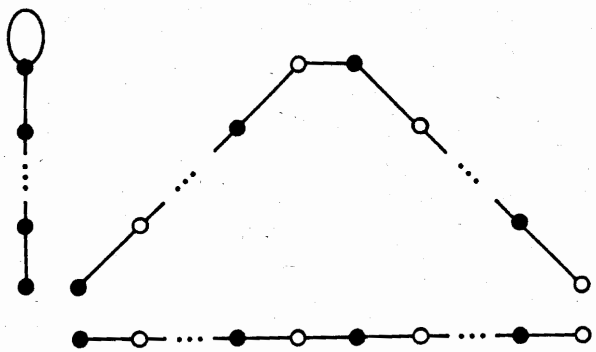


(A25)

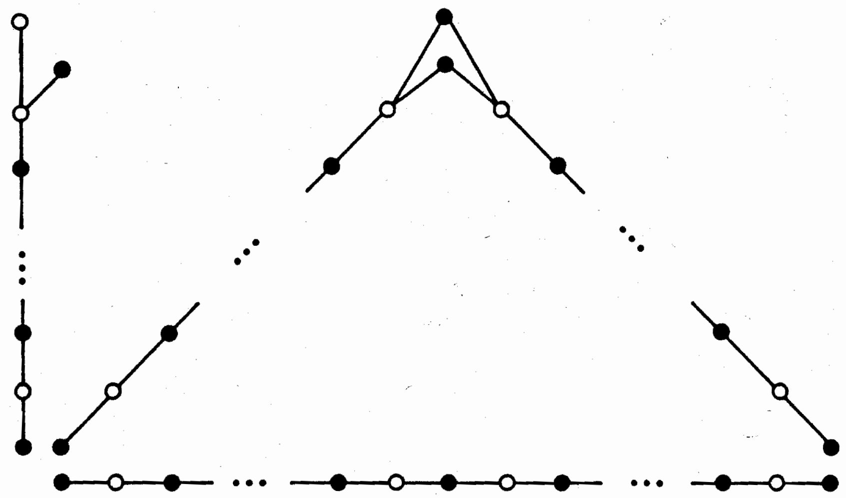
Appendix B

Fusion Rule Algebra Homomorphisms

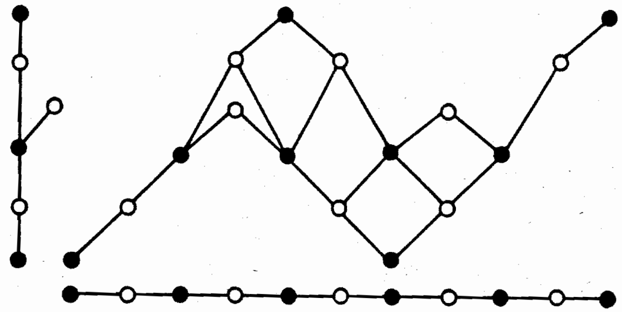
B.1 $\bar{\sigma}_n : A_{2n} \rightarrow \bar{A}_n$



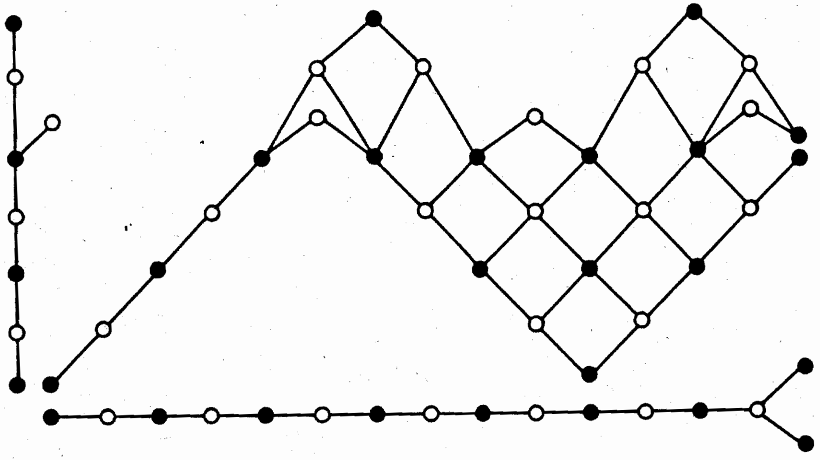
B.2 $\sigma_n^D : A_{4n-3} \rightarrow D_{2n}$



B.3 $\sigma^{E_6} : A_{11} \rightarrow E_6$



B.4 $\sigma^{DE} : D_{16} \to E_8$



Bibliography

[1] Woronowicz, S.L.: Compact matrix pseudogroups. Commun. Math. Phys. **111**, 613-665 (1987); Twisted $SU(2)$ group: an example of a non-commutative differential calculus. Publ. Res. Inst. Math. Sciences Kyoto Univ. **23**, 117-181 (1987).

[2] Jimbo, M.: A q-difference analogue of $U(\mathfrak{g})$ and the Yang-Baxter Equation. Lett. Math. Phys. **10**, 63-69 (1985); A q-analogue of $U(\mathfrak{g}(N+1))$, Hecke algebra, and the Yang-Baxter equation. Lett. Math. Phys. **11**, 247-252 (1986); Quantum R-matrix for generalized Toda system. Comm. Math. Phys. **102**, 537-547 (1986).

[3] Drinfel'd, V.G.: Quantum groups. In: Proc. of ICM Berkeley 1986. Ed. A.M. Gleason, Providence, R.I: Amer. Math. Soc. Publ. 1987, pp.798-820; Hopf algebras and the quantum Yang-Baxter equation. Soviet Math. Dokl. **32** (1985), No. 1.

[4] Drinfel'd, V.G.: Quasi-Hopf algebras and Knizhnik-Zamolodchikov equations. Preprint 1989.

[5] Drinfel'd, V.G.: On Almost Cocomutative Hopf Algebras. Leningrad Math. J.,**1**, 321-342(1990).

[6] Kerler, T.: Diploma thesis. Darstellungen der Quantengruppen und Anwendungen. ETH-1989, unpublished; Paper in preparation. Keller, G.: Lett. Math. Phys. **21**, 273-286 (1991) .

[7] Belavin, A., Polyakov, A.M., Zamolodchikov, A.B.: Infinite conformal symmetry in two-dimensional quantum field theory. Nucl. Phys. **B241**, 333-380 (1984).

[8] Witten, E.: Non-Abelian bosonization in two dimensions. Commun. Math. Phys. **92**, 455-472 (1984). Novikov, S.P.: Preprint. Gepner , D., Witten, E.: String theory on group manifolds. Nucl. Phys. **B278** (1986) 493-549.

[9] Knizhnik, V.G., Zamolodchikov, A.B.: Current algebra and Wess-Zumino model in two dimensions. Nucl. Phys. **B247**, 83-103 (1984). Tsuchiya, A., Kanie, V.: Vertex operators in the conformal field theory of P_1 and monodromy representations of the braid group. Lett. Math. Phys. **13**, 303-312 (1987). Kohno, T.: Quantized universal enveloping algebras and monodromy of braid groups. Preprint series No. 3 (1989). Felder, G., Fröhlich, J., Keller, G.: Braid matrices and structure constants for minimal conformal models. Commun. Math. Phys. **124**, 647-664 (1989).

[10] Zamolodchikov, A.B.: Landau Institute preprint (1989). Eguchi, T., Yang, S.K.: Deformations of conformal field theories and soliton equations. Phys. Lett. **B224**, 373-388 (1989). Le Clair, A.: Restricted Sine-Gordon theory and the minimal conformal series. Phys. Lett. **B230**, 103-107 (1989). Smirnov, F.A.: Leningrad preprint (1989). Bernanrd, D., Le Clair, A.: Cornell University preprint (1990). Reshetikhin, N.Yu., Smirnov, F.A.: Hidden quantum group symmetry and integrable perturbations of conformal field theories. Harvard Preprint (1990). HUTMP 90/B246.

[11] Fröhlich, J.: Statistics of fields, the Yang Baxter equation, and the theory of knots and links. In: Non-Perturbation Quantum Field Theory, G. t'Hooft et al. (eds.), New York: Plenum (1988).

[12] Zamolodchikov, A.B.: $\phi_{(1,3)}$ Perturbations of minimal models and braid statistics of the $\phi_{(3,1)}$ field. Landau-Institut. Preprint (1989)

[13] Siegel, W.: Unextended superfields in extended supersymmetry. Nucl. Phys. **B156**, 135-143 (1979)

[14] Pisarski, R.D., Rao, S.: Topologically massive chromodynamics in the perturbative regime. Phys. Rev. **32**, 2081-2096 (1985)

[15] Fröhlich, J., Gabbiani, F., Marchetti, P.A.: Superselection structure and statistics in three-dimensional local quantum theory. In: Knots, Topology and Quantum Field Theories. Proceedings of the $XIII^{th}$ John Hopkins Workshop on Current Problem in Particle Theory. L. Lusanna (ed.), Singapore, New Jersey: World Scientific Pres (1989). Braid statistics in three-dimensional local quantum theory. In: Proceedings the Workshop "Algebraic Theory of Superselection Sectors", Istituto Guccia, Palerm 1989, D. Kastler (ed.), World Scientific Press(1990).

[16] Witten, E.: Quantum field theory and the Jones polynomial. Commun. Math. Phys **121**, 351-399 (1989)

[17] Haag, R., Kastler, D.: An algebraic approach to quantum field theory. J. Math. Phys **5**, 848-861 (1963)

[18] Borchers, H.J.: On the vacuum state in quantum field theory, II. Commun. Mat Phys. **1**, 57-79 (1965). Local rings and the connection of spin with statistics. Commu Math. Phys. **1**, 281-307 (1965). Energy and momentum as observables in quantu field theory. Commun. Math. Phys. **2**, 49-54 (1966).

[19] Doplicher, S., Haag, R., Roberts, J.E.: Local observables and particle statistics I, I Commun. math. Phys. **23**, 199-230 (1971)/**35**, 49-85 (1974)

[20] Buchholz, D., Fredenhagen, K.: Locality and the structure of particle states. Com mun. math. Phys. **84**, 1-54 (1982). Fredenhagen, K.: Localizability of particle state FU Berlin. Preprint (1990).

[21] Gervais, J.-L., Neveu, A.: Nucl. Phys. B **238**, 125 (1984).

[22] Fredenhagen, K., Rehren, K.H., Schroer, B.: Superselection sectors with braid grou statistics and exchange algebras. Commun. Math. Phys. **125**, 201-226 (1989).

[23] Longo, R.: Index of subfactors and statistics of quantum fields I/II. Commun. Mat Phys. **126**, 217-247 (1989). Rome-preprint (1989)

[24] Fröhlich, J., Gabbiani, F.: Braid Statistics in Local Quantum Theory. Rev. Mat Phys. **2**, 251 - 353 (1990).

[25] Buchholz, D., Mack, G., Todorov, I.: Nucl. Phys. (Proc. Suppl.) **5B**, 20- (1988)

[26] Moore, G., Seiberg, N.: Classical and quantum conformal field theory. Commun. Math. Phys. **123**, 177-254 (1989). Polynomial equations for rational conformal field theories. Phys. Lett. **B212**, 451- (1988).

[27] Rehren, K.H., Schroer, B: Exchange algebra on the light-cone and order/disorder $2n$-point functions in the Ising field theory. Phys. Lett. **B198**, 84-88 (1987).

[28] Fröhlich, J., King, C.: Two-dimensional conformal field theory and three-dimensional topology. Int. J. of Mod. Phys. A, **20**, 5321-5399 (1989).

[29] Doplicher, S., Roberts, J.E.: Compact Lie groups associated with endomorphisms of C*-algebras. Bull. Am. Math. Soc. **11**, 333-338 (1984). A new duality theory for compact groups. Invent. math. **98**, 157-218 (1989). Endomorphisms of C*-algebras, cross products and duality for compact groups. Ann. of Math. **130**, 75-119 (1989).

[30] Mack, G., Schomerus, V.:Quasi Hopf Quantum Symmetry in Quantum Theory. Nucl. Phys. B **370** (1992) 185 - 230.

[31] Alvarez-Gaumé, L., Gomez, C., Sierra,G.: Nucl. Phys. B **319**, 155-186 (1989).
Bouwknegt, P., Mc Carthy,J., Pilch, K.: Phys. Letts. **234 B**, 297 - 303 (1990).
Gómez,C., Sierra,G.: Phys.Letts. **240 B**, 149-157 (1990).
Alekseev,A., Faddeev,L., Semenov-Tian-Shansky,M.: The Unraveling of the Quantum Group Structure in WZWN Theory. Hidden Quantum Group inside Kac-Moody Algebras. Preprints, St. Petersburg 1991.

[32] Reshetikhin, N. Yu., Moore, G.: A Comment on Quantum Group Symmetry in Conformal Field Theory. Nucl. Phys. B **328** (1989) 557 - 574.

[33] Dijkgraaf, R., Pasquier, V. and Roche, Ph.: Quasi-quantum groups related to orbifold models. To appear in the Proceedings of Int. Colloq. on Mod Q.F.T., Tata Inst. of Fund. Res. January 1990.

[34] Leinaas,M., Myrheim,J.: Nuovo Cimento B **37**,1 (1977).

[35] Goldin, G., Menikoff, R. and Sharp, D.: J. Math. Phys. **22**, 1664 (1981).
Goldin,G.,Sharp,D.H.: Phys. Rev. D **28**, 830 (1983).
Goldin,G., Menikoff,R., Sharp, D.H.: Phys. Rev. Letts. **54**, 603 (1985).

[36] Wilczek, F.: Phys. Rev. Letts. **48**, 1144 (1982); **49**, 957 (1982).

[37] Wilczek, F. and Zee, A.:Phys. Rev. Letts. **51**, 2250 (1983).

[38] Fröhlich, J. and Marchetti, P.-A.: Quantum Field Theory of Anyons, Lett. Math. Phys. **16**, 347 - 358 (1988); Quantum Field Theory of Vortices and Anyons, Comm. Math. Phys. **121** , 177 - 223 (1989).

[39] Bisognano, and Wichmann, E. On the duality condition for a Hermitian scalar field. J. Math. Phys. **16**, 985-1007 (1975).

[40] Fredenhagen, K. On the existence of antiparticles. Commun. Math. Phys. **79**, 141-151 (1981).

[41] Jones, V.F.R.: Index of subfactors. Invent. Math. **72**, 1-25 (1983). Hecke algebra representations of braid groups and link polynomials. Ann. of Math. **126**, 335-388 (1987). Wenzl, H.: Hecke algebra of type A_n and subfactors. Invent. math. **92**, 349-383 (1988). On sequences of projectons. C.R. Math. Rep. Acad. Sci. Canada-Vol. IX, **1**, 5-9 (1987). Quantum groups and subfactors of type B, C and D. San Diego-preprint (1989).

[42] Kerler, T.: Fusion Rule Algebras, a Classification for Small Indices. ETH-Preprint in preparation

[43] Reshetikhin, N. Yu.: Quantized universal enveloping algebras, the Yang-Baxter-equation and invariants of links. I. LOMI-preprint E-4-87 (1987), II. LOMI-preprint E-17-87 (1987).

[44] Reshetikhin, N. Yu., Turaev, V. G.: Ribbon graphs and their invariants derived from quantum groups. Commun. Math. Phys. **127**, 1-26 (1990)

[45] Goodman, F.M., de la Harpe, P., Jones, V.F.R.: Coxeter Graphs and Towers and Algebras. Springer Verlag, 1989.

[46] Ocneanu, A.: Quantized groups, string algebras and Galois theory for algebras. In: Operator Algebras and Applications. London Mathematical Society Lecture Note Series **136**, 119-172 (1988)

[47] Verlinde, E.: Fusion rules and modular transformations in $2D$ conformal field theory. Nucl. Phys. B **300**, 360-376 (1988)

[48] Rosso, M.: An analogue of P.B.W. theorem and the universal R-matrix for $U_h s\ell(N+1)$. Commun. Math. Phys. **124**, 307-318 (1989)

[49] Lusztig, G.: Modular representations and quantum groups. Contemp. Math. **82**, 58-77 (1989)

[50] Rosso, M.: Finite dimensional representations of the quantum analog of the enveloping algebra of a complex simple lie algebra. Comm. Math. Phys. **117**, 581-593 (1988)

[51] Arnaudon, D., Roche, Ph.: Irreducible representations of the quantum analogue of $SU(2)$. Lett. Math. Phys. **17**, 295-300 (1989)

[52] Pasquier, V., Saleur, H.: Symmetries of the XXZ chain and quantum-$SU(2)$. "Fields, strings and critical phenomena". Les Houches proceedings (1988)

[53] Pasquier, V.: Etiology of IRF-models. Commun. Math. Phys. **118**, 355 - 364 (1988).

[54] Wenzl., H.: Hecke algebras of type A_n and subfactors. Invent. Math. **92**, 349-383 (1988).

[55] Rehren, K.H.: Locality of conformal fields in two-dimensions: Exchange algebra on the light-cone. Commun. Math. Phys. **116**, 675-688 (1988).

[56] Deligne, P.: Catégories Tannakiennes. I.A.S.-preprint (1989), to appear in "Grothendieck Festschrift".

[57] Eilenberg, S., MacLane, S.: On the Groups $H(\pi, n)$, part I: Ann. of Math. **58**, 55-10
(1953). part II, Methods of Computation: Ann. of Math. **60**, 49-139 (1954).

[58] Cartan, H.: Algèbres d'Eilenberg-MacLane et Homotopie, Séminaire Henri Cartan
E.N.S., 7e année (1954/55).

[59] MacLane, S.: Homology, Springer-Verlag (Grundlehren), 1963. Hilton, P.D., Stamm
bach, U.: A Course in Homological Algebra, Springer-Verlag (Graduate Texts), 1971

[60] Dijkgraaf, R., Witten, E.: Topological Gauge Theories and Group Cohomology. Commun. Math. Phzs. **129**, 393 (1990).

[61] Kerler, T.: Non-Tannakian Categories in Quantum Field Theory. To appear in: Proceedings of the Cargesèse Summer Institute on "New Symmetry Principles in Quantum Field Theory", Cargèse, July 15-27; Plenum Press, 1991.

[62] Kazhdan,D., Lusztig,G.: Tensor Structures Arising from Affine Lie Algebras, I/I
Harvard Preprint 1992.

[63] Kazhdan,D., Wenzl,H.: Reconstructing Monoidal Categories. Harvard Preprint 199

Index