

Theory prediction of ϵ_K

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Abstract. The parameter ϵ_K plays an important role in the phenomenology of the standard model and its extensions. It is very sensitive to high-energy scales and can be predicted with remarkable precision. I review the theory prediction of ϵ_K in the standard model, with an emphasis on the theoretical and parametrical uncertainties. In addition, I present the results of our recent NNLO QCD calculation of η_{ct} , the charm-top-quark contribution to the effective $|\Delta S| = 2$ Hamiltonian, leading to $|\epsilon_K| = (1.90 \pm 0.26) \times 10^{-3}$.

1. Introduction

Neutral Kaon mixing proceeds via the quark-level flavour-changing neutral current (FCNC) $s - d$ transition. It is forbidden in the standard model of particle physics (SM) at tree level and induced by higher-order weak interactions, and therefore naturally sensitive to short-distance physics. The parameter ϵ_K describes indirect CP violation in the neutral Kaon system and is dominated by virtual top quarks (see Fig. 1). The coupling of the top quark to W bosons is proportional to a small Cabibbo-Kobayashi-Maskawa (CKM) matrix element, hence ϵ_K can set very stringent bounds in particular on models of “new physics” (NP) which deviate from the CKM flavour structure of the SM, i.e. which go beyond minimal flavour violation. In order to exploit this potential, a reliable theory prediction is mandatory.

After a general introduction to indirect CP violation in the neutral Kaon system in Sec. 2, we discuss the short-distance (Sec. 3) and long-distance (Sec. 4) contributions to ϵ_K in some detail. Sec. 5 contains the numerical prediction for ϵ_K using our NNLO value for η_{ct} .

2. Indirect CP violation in the neutral Kaon system

The time evolution¹ of the strangeness eigenstates $|K^0(t)\rangle$ and $|\bar{K}^0(t)\rangle$, where $|K^0(0)\rangle = |K^0\rangle$ and $|\bar{K}^0(0)\rangle = |\bar{K}^0\rangle$, is given in the Wigner-Weisskopf approximation (see Ref. [2]) by

$$i \frac{d}{dt} \begin{pmatrix} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{pmatrix} = \left[\begin{pmatrix} M_{11} & M_{12} \\ M_{12}^* & M_{11} \end{pmatrix} - \frac{i}{2} \begin{pmatrix} \Gamma_{11} & \Gamma_{12} \\ \Gamma_{12}^* & \Gamma_{11} \end{pmatrix} \right] \begin{pmatrix} |K^0(t)\rangle \\ |\bar{K}^0(t)\rangle \end{pmatrix}. \quad (1)$$

The diagonal entries of the Hermitian, time-independent mass matrix M and decay matrix Γ are equal as a result of CPT invariance, and the off-diagonal entries are induced by the $|\Delta S| = 2$ transition. Eq. (1) is diagonal in the basis of mass eigenstates

$$|K_S\rangle = p|K^0\rangle + q|\bar{K}^0\rangle \quad \text{and} \quad |K_L\rangle = p|K^0\rangle - q|\bar{K}^0\rangle. \quad (2)$$

¹ The discussion in this section adopts the conventions of Ref. [1], where also some missing details can be found.

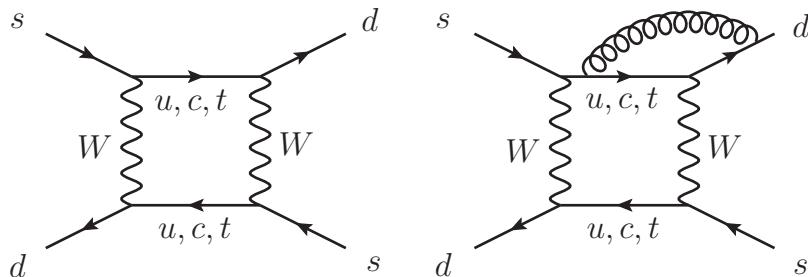


Figure 1. LO (left) and NLO (right) box diagrams inducing the $|\Delta S| = 2$ transition in the SM.

Their mass and width difference is given by $\Delta m_K = M_L - M_S$ and $\Delta\Gamma_K = \Gamma_S - \Gamma_L$, where M_L (M_S) and Γ_L (Γ_S) are the mass and the decay width of the longer-(shorter-)lived mass eigenstate.

We now define ϵ_K by

$$\epsilon_K = \frac{\eta_{00} + 2\eta_{+-}}{3} \quad (3)$$

in terms of the experimentally measured ratios $\eta_{ij} = \langle \pi^i \pi^j | K_L \rangle / \langle \pi^i \pi^j | K_S \rangle$. It can be expressed as the ratio of the decay amplitudes of a K_L and a K_S into an isospin-zero eigenstate,

$$\epsilon_K = \frac{\langle (\pi\pi)_{I=0} | K_L \rangle}{\langle (\pi\pi)_{I=0} | K_S \rangle} \left(1 + \mathcal{O}\left(\frac{A_2^2}{A_0^2}\right) \right) \quad (4)$$

up to corrections proportional to the second power of the ratio $|A_2|/|A_0|$, where $A_I = \langle (\pi\pi)_I | K^0 \rangle$. The subscript I denotes the isospin of the state. The $\Delta I = 1/2$ rule implies that this ratio is small, $|A_2|/|A_0| \approx 1/22$.

The parameter ϵ_K vanishes in case of exact CP symmetry. As a consequence of Watson's theorem [3], each amplitude A_I is characterised by a single strong phase. This phase drops out in the ratio (4), hence there is no direct CP violation in ϵ_K .

Using the explicit expressions for p and q in Eq. (4) leads to [1]

$$\epsilon_K = e^{i\phi_\epsilon} \sin \phi_\epsilon \left(\frac{\text{Im}M_{12}}{\Delta m_K} + \xi \right), \quad (5)$$

where $\phi_\epsilon = \arctan(2\Delta m_K/\Delta\Gamma_K) \approx 45^\circ$, and $\xi = \text{Im}A_0/\text{Re}A_0$ is a correction of $\mathcal{O}(5\%)$. We have neglected terms proportional to the second power of the small quantities $\phi = \arg(-M_{12}/\Gamma_{12})$ and $|A_2|/|A_0|$, and to the first power in Γ_L/Γ_S . Taking Δm_K and ϕ_ϵ from experiment, we obtain a theory prediction for ϵ_K by computing $\text{Im}M_{12}$ and ξ .

To this end, we make use of the effective field theory formalism. In this framework we can systematically separate short-distance from long-distance scales. The information about short-distance scales is contained in the Wilson coefficients of the weak effective Hamiltonian; they can in principle be calculated to any desired precision within perturbation theory. The main contributions of long-distance scales are contained in the hadronic matrix element of the effective Hamiltonian, and are governed by low-energy, non-perturbative QCD. They can be evaluated reliably by means of lattice QCD.

The matrix element M_{12} is given in terms of the effective Hamiltonian by $2\Delta m_K M_{12} = \langle \bar{K}^0 | \mathcal{H}^{|\Delta S|=2} | K^0 \rangle$ up to a small correction which will be discussed, together with the parameter ξ , in Sec. 4.

3. Short-distance contributions to ϵ_K

The effective Hamiltonian below the charm-quark scale, inducing the $|\Delta S| = 2$ transitions in the SM, is given by

$$\mathcal{H}^{\Delta S=2} = \frac{G_F^2}{4\pi^2} M_W^2 [\lambda_c^2 \eta_{cc} S(x_c) + \lambda_t^2 \eta_{tt} S(x_t) + 2\lambda_c \lambda_t \eta_{ct} S(x_c, x_t)] b(\mu) \tilde{Q}_{S2} + \text{H.c.} + \dots, \quad (6)$$

where G_F is the Fermi constant, M_W the W -boson mass, and $x_i = m_i^2/M_W^2$ (m_i being the quark masses). The factor $b(\mu)$ contains the remaining scale dependence, which cancels against the corresponding scale dependence of the hadronic matrix elements (see Sec. 4). $\tilde{Q}_{S2} = (\bar{s}_L \gamma_\mu d_L) \otimes (\bar{s}_L \gamma^\mu d_L)$ is the leading local four-quark operator that induces the $|\Delta S| = 2$ transition, defined in terms of the left-handed s - and d -quark fields. Higher-dimensional operators are estimated to contribute less than 1% to $\text{Im}M_{12}$ [4].

The short-distance contributions are contained in the Wilson coefficient of \tilde{Q}_{S2} , which is induced in the SM at LO by the box diagram of Fig. 1 (left side). The couplings of the W bosons to the fermions are proportional to the CKM matrix elements, collected in the parameters $\lambda_i = V_{is}^* V_{id}$. Using the unitarity relation $\lambda_u = -\lambda_c - \lambda_t$, the effective Hamiltonian $\mathcal{H}^{|\Delta S|=2}$ can be split into three terms as in Eq. (6). The loop functions S , proportional to the squares of the quark masses due to the Glashow-Iliopoulos-Maiani (GIM) mechanism, denote the contribution of the LO SM box diagrams. The coefficients η_{ij} comprise the higher-order QCD corrections.

The calculation of ϵ_K involves the imaginary part of M_{12} ; accordingly, the interplay between $\text{Im}\lambda_i \lambda_j$ and the loop functions S determines the relative size of the individual contributions. In contrast to B -meson mixing, where only the top-quark contribution is relevant, the smallness of λ_t^2 lets all three contributions compete in size in the case of Kaon mixing. We now discuss them in turn.

The term proportional to λ_t^2 is dominant and contributes approximately +72% to $\mathcal{H}^{|\Delta S|=2}$. The QCD corrections can be computed by a fixed-order matching calculation at the top-quark scale; there is no large logarithm. A next-to-leading-order (NLO) QCD calculation yields $\eta_{tt} = 0.5765(65)$ [5], with a small remaining theory uncertainty of about 1%.

The smallest contribution of about -14% is proportional to λ_c^2 . The GIM mechanism causes the absence of a large logarithm $\log(m_c/M_W)$. A matching calculation at the charm-quark scale at NLO QCD yields $\eta_{cc} = 1.43(23)$ [6]. The relatively large remaining scale dependence is mitigated by the absolute smallness of this contribution.

The term proportional to $\lambda_c \lambda_t$ contributes roughly +42% to the effective Hamiltonian. It contains a large logarithm $\log(m_c/M_W)$, and the theoretical prediction requires a full renormalisation group analysis with double insertions of $|\Delta S| = 1$ operators (see Fig. 2). We have performed a next-to-next-to-leading-order (NNLO) QCD calculation with the result $\eta_{ct} = 0.496(47)$ [7].

In order to obtain a reliable theory prediction it is essential to estimate the theory uncertainty. The latter is a consequence of the truncation of the perturbation series. Terms proportional to powers of $\alpha_s \log(\mu/m_c)$, where μ is an arbitrary, unphysical renormalisation scale, are summed to all orders by the renormalisation group, but only incompletely cancelled in the matching calculation. η_{ct} is a scale- and scheme-independent quantity and therefore μ -independent if calculated to all orders. The residual μ -dependence gives us a handle on neglected higher-order terms. The large NNLO anomalous dimensions and an accidental cancellation of corresponding terms at leading order partly explain the still sizeable remaining scale dependence of the NNLO result.

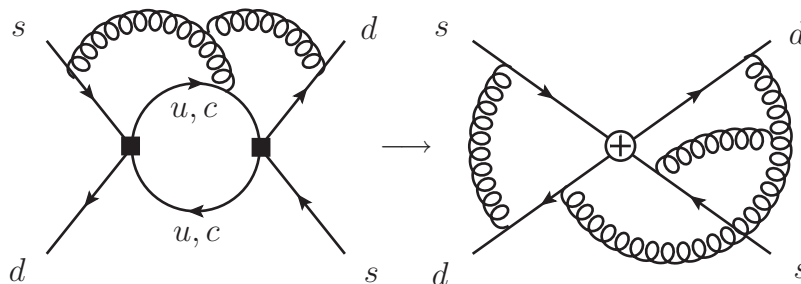


Figure 2. SM diagrams contributing to the mixing of $|\Delta S| = 1$ operators into Q_{S2} at NNLO QCD.

4. Long-distance contributions to ϵ_K

The hadronic matrix elements of the operator \tilde{Q}_{S2} include the major part of the long-distance contributions to ϵ_K . They are parameterised by the bag factor

$$\hat{B}_K = \frac{3}{2} b(\mu) \frac{\langle \bar{K}^0 | \tilde{Q}_{S2} | K^0 \rangle}{f_K^2 M_K^2}, \quad (7)$$

where f_K is the Kaon decay constant, and $b(\mu)$ is factored out of Eq. (6) in such a way that \hat{B}_K is a renormalisation-group invariant quantity. The determination of the bag factor was the main impediment to an accurate prediction of ϵ_K until a couple of years ago.

Nowadays, the most precise predictions for \hat{B}_K come from lattice QCD calculations, where the chiral and continuum extrapolation as well as finite volume effects are now well under control [8]. A fully unquenched calculation results in $B_K = 0.725(26)$ [9], and the authors identify the transition from the non-perturbative renormalisation scheme on the lattice to the $\overline{\text{MS}}$ scheme, which is known only at one-loop, as the main source of the uncertainty. This value is consistent with the upper bound $\hat{B}_K \leq 0.75$, derived in the framework of large- N QCD [10].

There are additional long-distance contributions which are not contained in \hat{B}_K , proportional to the dispersive and absorptive parts of the amplitude $\int d^4x \langle \bar{K}^0 | \mathcal{H}^{\Delta S=1}(x) \mathcal{H}^{\Delta S=1}(0) | K^0 \rangle$, respectively. The parameter ξ comprises the absorptive part and has been estimated by relating it to the ratio ϵ'_K/ϵ_K [1, 11]², yielding a -6% correction to ϵ_K . $\text{Im}M_{12}$ receives contributions from the dispersive part of the amplitude. These have been estimated using chiral perturbation theory in [12] and enhance ϵ_K by 2.4%. The two contributions are combined together with the experimental value of ϕ_ϵ into the phenomenological parameter $\kappa_\epsilon = 0.94(2)$, which multiplies the expression (5), in which we then set $\xi = 0$ and $\phi_\epsilon = 45^\circ$.

5. Numerics and input parameters

The parameter ϵ_K is measured with high accuracy: the value quoted by the Particle Data Group is $\epsilon_K = (2.228 \pm 0.011) \times 10^{-3} \times e^{i(43.5 \pm 0.7)^\circ}$ [13]. In order to compare this with the theoretical prediction, we compute ϵ_K using the following formula³ [18, 11],

$$|\epsilon_K| = \kappa_\epsilon C_\epsilon \hat{B}_K |V_{cb}|^2 \lambda^2 \bar{\eta} [|V_{cb}|^2 (1 - \bar{\rho}) \eta_{tt} S(x_t) + \eta_{ct} S(x_c, x_t) - \eta_{cc} S(x_c)], \quad (8)$$

where

$$C_\epsilon = \frac{G_F^2 F_K^2 M_{K^0} M_W^2}{6\sqrt{2}\pi^2 \Delta M_K}. \quad (9)$$

² $\epsilon'_K = (\eta_{+-} - \eta_{00})/3$ measures direct CP violation in the neutral Kaon system.

³ A term proportional to $\text{Re}\lambda_t/\text{Re}\lambda_c = \mathcal{O}(\lambda^4)$ has been neglected in Eq. (8) (see Ref. [18]).

Table 1. Input parameters used in our numerical analysis.

Parameter	Value	Ref.	Parameter	Value	Ref.
M_W	80.399(23) GeV	[13]	$\sin 2\beta$	0.671(23)	[13]
$m_t(m_t)$	163.7(1.1) GeV	[14]	F_K	156.1(8) MeV	[15]
$m_b(m_b)$	4.163(16) GeV	[16]	G_F	$1.166\,367(5) \times 10^{-5} \text{ GeV}^{-2}$	[13]
$m_c(m_c)$	1.279(13) GeV	[16]	λ	0.2255(7)	[15]
M_K	497.614(24) MeV	[13]	$ V_{cb} $	$4.06(13) \times 10^{-2}$	[13]
κ_ϵ	0.94(2)	[12]	M_{B_d}	5.2795(3) GeV	[13]
ΔM_K	5.292(9)/ns	[13]	M_{B_s}	5.3663(6) GeV	[13]
ΔM_d	0.507(5)/ps	[13]	ΔM_s	17.77(12)/ps	[13]
ξ_s	1.243(28)	[17]	η_{tt}	0.5765(65)	[5]
\hat{B}_K	0.725(26)	[17]	η_{cc}	1.43(23)	[6]

We write $\bar{\eta} = R_t \sin \beta$ and $1 - \bar{\rho} = R_t \cos \beta$, where R_t is given by

$$R_t \approx \frac{\xi_s}{\lambda} \sqrt{\frac{M_{B_s}}{M_{B_d}}} \sqrt{\frac{\Delta M_d}{\Delta M_s}} \quad (10)$$

and $\xi_s = (F_{B_s} \sqrt{\hat{B}_s}) / (F_{B_d} \sqrt{\hat{B}_d})$ is a ratio of B -meson decay constants and bag factors that can be computed on the lattice with high precision [17]. Using the NNLO value $\eta_{ct} = 0.496(47)$ and the numerical values given in Tab. 1, we obtain

$$|\epsilon_K| = (1.90 \pm 0.04_{\eta_{cc}} \pm 0.02_{\eta_{tt}} \pm 0.07_{\eta_{ct}} \pm 0.11_{\text{LD}} \pm 0.22_{\text{parametric}}) \times 10^{-3}, \quad (11)$$

where all errors have been added in quadrature.

The first three errors correspond to η_{cc} , η_{tt} , η_{ct} , respectively. While the error on η_{ct} is still dominating, a NNLO QCD calculation could reduce the error on η_{cc} .

The error indicated by LD originates from the long-distance contributions, namely ξ_s , \hat{B}_K , and κ_ϵ , which account for 40%, 37%, and 22% of the long-distance error, respectively. As mentioned before, the error on \hat{B}_K could be reduced significantly by calculating higher order corrections to the perturbative matching to the $\overline{\text{MS}}$ scheme.

The main share of the parametric error stems from $|V_{cb}|$ (59%) and $\sin(2\beta)$ (19%), while all other contributions are well below 10%. The dominance of $|V_{cb}|$ can easily be understood from Eq. (8): we see that the dominant top-quark contribution is proportional to the fourth power of $|V_{cb}|$. At the same time, the experimental error on $|V_{cb}|$ is quite large, not least because of the two only marginally consistent values obtained from inclusive and exclusive semileptonic B -meson decays, respectively [13]. The clarification of this issue would certainly be most helpful.

6. Conclusion

The parameter ϵ_k is an important precision observable: it is very sensitive to high-energy scales and can be computed reliably. In this talk I reviewed the status of the SM prediction for ϵ_K . The recent progress in the calculation of the long- and short-distance contributions results in the numerical value $|\epsilon_K| = (1.90 \pm 0.26) \times 10^{-3}$.

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