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HELICITY AMPLITUDES FOR MULTIPLE BREMSSTRAHLUNG IN MASSLESS NON-ABELIAN GAUGE THEORY(I): NEW DEFINITION OF POLARIZATION VECTOR AND FORMULATION OF AMPLITUDES IN GRASSMANN ALGEBRA*)

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ABSTRACT

We discuss the generalization of the method of helicity amplitudes proposed by CALKUL collaboration to the non-abelian case (QCD). In this paper we give a new definition of the polarization vector which is suitable for the non-abelian gauge theory, and show that this new definition formulates the amplitudes in Grassmann algebra.

1. Introduction

Due to the development of high energy accelerator experiments it is possible now to carry out measurements on multiple Bremsstrahlumg in high energy collisions. In these processes, as far as the electromagnetic and strong interactions are concerned, electrons, muons and light quarks can be considered massless. Thus the theoretical frameworks treating them are the massless gauge theories, namely massless QED and QCD. When perturbation theory is applicable, the problem is reduced to the calculation of the amplitudes of Feynman diagrams and the cross-sections for processes involving multiple Bremsstrahlung .

As is well known, the number of the Feynman diagrams increases more rapidly than the multiplicity does. This makes the calculation of the amplitudes a rather horrible work. Some computer programs have been developed[9] to aviod manual calculation, but sometimes the lengthy output printings of the computer dazzle people, and one can hardly find required results from them.

In view of this difficulty the CALKUL collaboration [2,3,4,6,7, 8] suggested the method of helicity ampitudes for massless gauge theories which can simplify the calculations remarkably. The general formulism in abelian case (QED) has been given and several examples such as $e^+e^- \to 3\gamma$, $e^+e^- \to 4\gamma$, $e^+e^- \to \mu^+\mu^-\gamma$ (Z_0 boson exchange included), $e^+e^- \to e^+e^-\gamma$ (Z_0 boson exchange included) and $e^+e^- \to \mu^+\mu^-\gamma\gamma$ have been worked out. The method shows indeed obvious advantages, making the manual calculation realistic, giving compact results which can easily be interpreted. In treating the non-abelian case (QCD) , however, we encounter a series of new problems. Although some QED-QCD processes like $e^+e^- \to \bar{q}qgg$ or $\bar{q}q\bar{q}'q'$ have been calculated [5] and some cross-sections of QCD processes like $\bar{q}q \to \bar{q}'q'g$ and $gg \to ggg$ have been given [1], the general formalism to treat the helicity amplitudes in non-abelian gauge theory is still an unsolved question.

What we would like to do is to generalize the method of helicity amplitudes to non-abelian case, especially for multiple Bremsstrahlung, and find some regularities governing those processes. This series of papers are organized as follows. In the first paper we give a modified definition of the polarization vector of emitted gauge boson and put the amplitudes into Grassmann algebra. In the second one we suggest a method to decompose the amplitudes into gauge-invariant subsets, making use of the advantages of the helicity amplitudes as much as possible. In the third paper we describe some significant features concerning the self-coupling vertices of the gluons. Finally in the fourth paper we calculate the amplitudes for the process $W \longrightarrow 2\ jets + 2\gamma$ as an example.

2. A Brief Review of Helicity Amplitudes in QED

Consider the annihilation of e^+ and e^- giving n photons:

$$e^+(q_+) + e^-(q_-) \longrightarrow \gamma(k_1) + \gamma(k_2) + \cdots + \gamma(k_n), \tag{2.1}$$

where

$$q_+^2 = q_-^2 = k_i^2 = 0. \quad (i=1,2,\ \ldots\ ,n) \tag{2.2}$$

The linear polarization vectors of a photon with momentum k are defined by referring to the fermion momenta q_+ and q_- [2,3]:

$$\varepsilon_\mu^{\parallel} = N'\left[(q_+k)\,q_{-\mu} - (q_-k)\,q_{+\mu}\right], \tag{2.3}$$

$$\varepsilon_\mu^{\perp} = N'\,\varepsilon_{\mu\alpha\beta\gamma}\,q_+^\alpha\, q_-^\beta\, k^\gamma, \tag{2.4}$$

where

$$N' = \left[2(q_+q_-)(q_+k)(q_-k)\right]^{-\frac{1}{2}}. \tag{2.5}$$

The polarization vectors satisfy the following relations:

$$(\varepsilon^{\parallel})^2 = (\varepsilon^{\perp})^2 = -1, \tag{2.6}$$

$$(k\varepsilon^{\parallel}) = (k\varepsilon^{\perp}) = (\varepsilon^{\parallel}\varepsilon^{\perp}) = 0. \tag{2.7}$$

The circular polarization vectors are then defined in terms of the linear ones:

$$\varepsilon_\mu^{\pm} = \frac{1}{\sqrt{2}}\left(\varepsilon_\mu^{\parallel} \pm i\,\varepsilon_\mu^{\perp}\right), \tag{2.8}$$

consequently

$$\not\varepsilon \equiv \varepsilon_\mu\gamma^\mu = N\left[\not k \not q_+ \not q_-(1\pm\gamma_5) - \not q_+ \not q_- \not k(1\mp\gamma_5) \mp 2(q_+q_-)\not k\gamma_5\right], \tag{2.9}$$

where

$$N = \left[16\,(q_+q_-)(q_+k)(q_-k)\right]^{-\frac{1}{2}}. \tag{2.10}$$

Th[illegible] that the last term of (2.9) can be discarded. Actual[illegible] at $\varepsilon = k$ the amplitudes will be zero due to the gauge-invariance of the theory, and the spinor of a massless fermion is an eigenstate of γ_5. So we write simply

$$\not\epsilon^{\pm}(k, q_+, q_-) = N[\not k \not q_+ \not q_- (1\pm\gamma_5) - \not q_+ \not q_- \not k (1\mp\gamma_5)]. \qquad (2.11)$$

On the other hand the fermion spinors satisfying

$$\not q\, u(q) = \not q\, v(q) = \bar{u}(q)\not q = \bar{v}(q)\not q = 0 \qquad (2.12)$$

can be specified by their helicites, denoted by $u_{\pm}(q)$ and $v_{\pm}(q)$:

$$(1\mp\gamma_5)\, u_{\pm} = (1\pm\gamma_5)\, v_{\pm} = \bar{u}_{\pm}(1\pm\gamma_5) = \bar{v}_{\pm}(1\mp\gamma_5) = 0. \qquad (2.13)$$

In fact we can put [3]

$$u_{\pm}(q) = v_{\mp}(q) \equiv |q\pm\rangle, \qquad (2.14)$$

$$\bar{u}_{\pm}(q) = \bar{v}_{\mp}(q) \equiv \langle q\pm|, \qquad (2.15)$$

and they satisfy the following equations:

$$\not q\,|q\pm\rangle = \langle q\pm|\,\not q = 0, \qquad (2.16)$$

$$(1\mp\gamma_5)\,|q\pm\rangle = \langle q\pm|\,(1\pm\gamma_5) = 0, \qquad (2.17)$$

$$\langle q\pm|\gamma_\mu|q\pm\rangle = 2q_\mu, \qquad (2.18)$$

$$|q\pm\rangle\langle q\pm| = \frac{1}{2}(1\pm\gamma_5)\not q, \qquad (2.19)$$

$$\langle p+|q+\rangle = \langle p-|q-\rangle = 0, \quad \text{for any p and q.} \qquad (2.20)$$

The amplitudes for process (2.1) can be characterized by the helicities of the particles, denonted by $M(\lambda_+, \lambda_-, \lambda_1, \lambda_2, \cdots)$ where $\lambda_+, \lambda_-, \lambda_1, \lambda_2, \cdots$ are the helicities of the positron, electron and the photons with momenta $k_1, k_2, \cdots$, respectively, and λ_+ and λ_- are related by

$$\lambda_+ \cdot \lambda_- = (-1) \qquad (2.21)$$

due to helicity conservation at vector vertices. In view of (2.14), (2.15) and (2.21) the helicity sign following q is called "helicity of the fermion line". It is the same as the fermion helicity and opposite to the antifermion helicity.

Taking helicity amplitudes with such a choice of polarization vectors leads to significant simplification in the calculations[2,3]:

(1) At least one of the two terms in the definition of $\not\epsilon^{\pm}$ (2.11) vanishes owing to (2.13).

(2) It happens that some helicity amplitudes vanish. For example when a photon of a given helicity is attached to the external incoming fermion (outgoing antifermion) of the same helicity, the amplitude is

identically zero. As a result the amplitudes for $e^+e^- \to n\gamma$ vanish when all the photons have identical helicities:

$$M(\pm, \mp, +, \cdots, +) = M(\pm, \mp, -, \cdots, -) = 0. \tag{2.22}$$

(3) In many cases some of the propagator denominators can be cancelled.

In this formulism simple and elegant expressions for the amplitudes and cross-sections of $e^+e^- \to 3\gamma$ and $e^+e^- \to 4\gamma$ were given [2,4,7].

In general, fermion lines associated with pair annihilation (contributing $\bar{v}u$), pair creation ($\bar{u}v$), scattering of fermion($\bar{u}u$) and scattering of antifermion($\bar{v}v$) can appear in the diagrams. We denote the momentum associated with an incoming line(u or v) by q_{in} and that with an outgoing line ($\bar{u}$ or $\bar{v}$) by q_{out}. The definition (2.11) for polarization vectors is generalized into

$$\not{\varepsilon}^{\pm} = N[\not{k}\not{q}_{out}\not{q}_{in}(1 \pm \gamma_5) - \not{q}_{out}\not{q}_{in}\not{k}(1 \mp \gamma_5)], \tag{2.23}$$

which leads to the vanishing of the amplitudes when a photon of given helicity is attached to an external outgoing fermion (incoming antifermion) of <u>opposite</u> helicity, in addition to the case discussed above, see Fig.1. We call these the helicity match rule.

Simplification of the calculation can be achieved only when the reference momenta in defining the polarization vector are associated with the fermion line to which the photon is attached. Problem arises when there are more fermion lines in a diagram, to any one of which the photon can be attached. In the process

$$e^+(p_+) + e^-(p_-) \to \mu^+(q_+) + \mu^-(q_-) + \gamma(k), \tag{2.24}$$

four Feynman diagrams conrtibute to the lowest order (Fig.2).

For diagrams (1) and (2) the appropriate polarization vector is

$$\not{\varepsilon}_q^{\pm} = N_q[\not{k}\not{q}_-\not{q}_+(1 \pm \gamma_5) - \not{q}_-\not{q}_+\not{k}(1 \mp \gamma_5)], \tag{2.25}$$

while for (3) and (4) it is

$$\not{\varepsilon}_p^{\pm} = N_p[\not{k}\not{p}_+\not{p}_-(1 \pm \gamma_5) - \not{p}_+\not{p}_-\not{k}(1 \mp \gamma_5)]. \tag{2.26}$$

But in fact we have to use a unique polarization vector to compute the amplitude to which all diagrams contribute. It appears that $\not{\varepsilon}_q$ and $\not{\varepsilon}_p$ are related by [3,4]

$$\not{\varepsilon}_q^{\pm} = e^{\pm i\phi}\not{\varepsilon}_p^{\pm} + \beta^{\pm}\not{k}, \tag{2.27}$$

where

$$e^{\pm i\phi} = - N_q N_p^* \, Tr\,[\not{q}_- \not{q}_+ \not{k} \not{p}_+ \not{p}_- \not{k}\ (1 \pm \gamma_5)], \qquad (2.28)$$

$$\beta^{\pm} = \frac{N_q}{(p_- k)}\ Tr\,[\not{p}_- \not{k} \not{q}_- \not{q}_+ \ (1 \pm \gamma_5)]. \qquad (2.29)$$

Owing to the identities

$$[M_{(1)} + M_{(2)}]_{\varepsilon = k} = [M_{(3)} + M_{(4)}]_{\varepsilon = k} = 0, \qquad (2.30)$$

we can use for instance ε_q for the amplitude and make use of (2.27) and (2.30) to get

$$M \equiv [M]_{\varepsilon = \varepsilon_q^{\pm}} = [M_{(1)} + M_{(2)}]_{\varepsilon = \varepsilon_q^{\pm}} + e^{\pm i\phi}\, [M_{(3)} + M_{(4)}]_{\varepsilon = \varepsilon_p^{\pm}}\ . \qquad (2.31)$$

The choice of polarization vectors in eq. (2.31) maintains all the simplification in helicity amplitude calculations as desired.

3. New Definition of Polarization Vector of Gauge Boson

The non-abelian theory is different from the abelian one in two important points:

(1) In the expressions of the scattering amplitudes the polarization vector of the gauge boson appears not only in the form of $\not{\varepsilon}$ but also in the form of inner-product $(\varepsilon p) \equiv \varepsilon_\mu p^\mu$, where p^μ is a momentum.

(2) In addition to the fermion-boson coupling vertex there are also the self-coupling vertices of gluons, shown in Fig. 3. Therefore ont only the quarks but also the virtual gluons can emit gluons (Fig.4).

The first point leads to the new definition of polarization vector, suitable both in QED and QCD, and the second point needs specific treatment to be discussed in subsequent papers[11].

In obtaining (2.11) a term $\not{k}\gamma_5$ in (2.9) was discarded. Therefore $\not{\varepsilon}$ (2.11) is <u>not</u> the contraction of a 4-vector with γ^μ. This expression cannot be used in QCD where inner-product of 4-vectors e.g. (εp) is involved. The expression for $\not{\varepsilon}$ (2.9) can, however, be rewritten as

$$\not{\varepsilon}^{\pm} = N\ [\not{k} \not{q}_+ \not{q}_- \ (1 \pm \gamma_5) + \not{q}_- \not{q}_+ \not{k}\ (1 \mp \gamma_5) - 2\,(q_+ q_-)\,\not{k}\,], \qquad (3.1)$$

where

$$N = [\,16\,(q_+ q_-)(q_+ k)(q_- k)\,]^{-\frac{1}{2}}. \qquad (3.2)$$

Again when used in calculating helicity amplitudes, the last term in (3.1) can be discarded, owing to the gauge invariance of the theory. Then $\not{\varepsilon}^{\pm}$ becomes

$$\not{\varepsilon}^{\pm}(k, g_1, g_2) = N\left[\not{k}\,\not{g}_1\not{g}_2\,(1 \pm \gamma_5) + \not{g}_2\not{g}_1\not{k}\,(1 \mp \gamma_5)\right], \tag{3.3}$$

which is the contraction with γ^{μ} of a 4-vector

$$\varepsilon^{\pm}_{\mu} = \frac{1}{\sqrt{2}}\left(\varepsilon^{\parallel}_{\mu} \pm i\,\varepsilon^{\perp}_{\mu}\right) + \frac{1}{2}\sqrt{\frac{(g_1 g_2)}{(g_1 k)(g_2 k)}}\; k_{\mu}. \tag{3.4}$$

When the reference momenta are changed from g_1, g_2 to p_1, p_2, the polarization vector $\varepsilon^{\pm}_{g}(k, g_1, g_2)$ and $\varepsilon^{\pm}_{p}(k, p_1, p_2)$ are still connected by

$$\not{\varepsilon}^{\pm}_{g} = e^{\pm i\phi}\,\not{\varepsilon}^{\pm}_{p} + \beta^{\pm}\not{k}, \tag{3.5}$$

where

$$e^{\pm i\phi} = -N_g N_p^{*}\, Tr\left[\not{g}_1\not{g}_2\not{k}\,\not{p}_1\not{p}_2\not{k}\,(1 \pm \gamma_5)\right]. \tag{3.6}$$

With the projection operator (2.19) it can be shown that

$$e^{\pm i\phi} = \sqrt{2}\,N_g \langle k\mp|\,\not{g}_1\not{g}_2\,|k\pm\rangle \cdot \sqrt{2}\,N_p^{*}\langle k\pm|\,\not{p}_2\not{p}_1\,|k\mp\rangle. \tag{3.7}$$

It is remarkable that the polarization vector $\varepsilon^{\pm}_{\mu}$ is complex, and its normalization is $(\varepsilon^{\pm}_{\mu})^{*}\varepsilon^{\pm\mu} = -1$. There is a phase freedom which can be exploited to simplify relations. It will be shown in the next section (eq.(4.13)) that $\langle g_1\pm|g_2\mp\rangle$ and $\sqrt{2}\,(g_1 g_2)^{\frac{1}{2}}$ differ only by phase. The normalization constant N (3.2) can be redefined by

$$\begin{aligned} N^{\pm} &= \pm\left[\sqrt{2}\,\langle g_1\pm|g_2\mp\rangle\langle k\mp|g_1\pm\rangle\langle g_2\mp|k\pm\rangle\right]^{-1} \\ &= \pm\left[\sqrt{2}\,\langle k\mp|\,\not{g}_1\not{g}_2\,|k\pm\rangle\right]^{-1}, \end{aligned} \tag{3.8}$$

such that (3.7) becomes

$$e^{\pm i\phi} = 1. \tag{3.9}$$

Consequently,

$$\not{\varepsilon}^{\pm}_{g} = \not{\varepsilon}^{\pm}_{p} + \beta^{\pm}\not{k}. \qquad (q-h+| = -(h-q+| \tag{3.10}$$

From (3.8) it follows that

$$(N^{+})^{*} = N^{-}. \tag{3.11}$$

With the new definition of $\not{\varepsilon}^{\pm}$ (3.3) and $N^{\pm}$ (3.8),

$$\begin{aligned} \not{\varepsilon}^{\pm}(k, g_1, g_2) &= 2N^{\pm}\big[\,|k\mp\rangle\langle k\mp|g_1\pm\rangle\langle g_1\pm|g_2\mp\rangle\langle g_2\mp| + \\ &\qquad + |g_2\pm\rangle\langle g_2\pm|g_1\mp\rangle\langle g_1\mp|k\pm\rangle\langle k\pm|\,\big] \\ &= \pm\frac{\sqrt{2}}{\langle g_2\mp|k\pm\rangle}\big[\,|k\mp\rangle\langle g_2\mp| + |g_2\pm\rangle\langle k\pm|\,\big], \end{aligned} \tag{3.12}$$

in which a relation $\langle p+|g-\rangle = -\langle g+|p-\rangle$ (4.8) has been used. Quite remarkably, g_1 drops out from the definition, inferring that the polarization vector can be defined in terms of one reference momentum. We write simply

$$\not{\varepsilon}^{\pm}(k,q) = \pm \frac{\sqrt{2}}{\langle q\mp|k\pm\rangle}\left[\,|k\mp\rangle\langle q\mp| + |q\pm\rangle\langle k\pm|\,\right]. \tag{3.13}$$

With the help of a relation given in the Appendix

$$2\,|B\pm\rangle\langle A\pm| = \langle A\pm|\gamma_\mu|B\pm\rangle\,\gamma^\mu\,\frac{1}{2}(1\mp\gamma_5), \tag{A.27}$$

the vector $\varepsilon_\mu^{\pm}$ can be transformed into

$$\varepsilon_\mu^{\pm} = \pm\frac{\langle q\mp|\gamma_\mu|k\mp\rangle}{\sqrt{2}\,\langle q\mp|k\pm\rangle}\,. \tag{3.14}$$

All the nice features of the polarization vector defined in [2] are preserved here. Actually, when $\not{\varepsilon}^{\pm}$ (3.13) is used in calculating helicity amplitudes, one of the two terms drops out, since the bras and kets in (3.13) are helicity eigenstates, and (2,20) is operative. Again when an external boson line is attached to an external fermion line, some of the helicity amplitudes vanish. This is, however, achieved by choosing the reference momentum in the following way. When the boson has the same (opposite) helicity as that of the fermion line to which it is attached, the reference momentum is chosen to be q_{in} (q_{out}). That the helicity match rules discussed in section 2 are preserved can be easily checked. For example $\not{\varepsilon}^{+}u_+(q_{in})$ corresponds to the case of same helicity of boson and fermion line and $\not{\varepsilon}^{+}(k,q_{in}) \propto |k-\rangle\langle q_{in}-| + |q_{in}+\rangle\langle k+|$ is to be used. Therefore $\not{\varepsilon}^{+}u_+(q_{in}) = \not{\varepsilon}^{+}(k,q_{in})|q_{in}+\rangle$ vanishes because $\langle q_{in}-|q_{in}+\rangle = 0$ and $\langle k+|q_{in}+\rangle = 0$. The case $\bar{v}_+(q_{out})\not{\varepsilon}^{+}$ corresponds to different helicities of the boson and fermion line , and $\bar{v}_+(q_{out})\not{\varepsilon}^{+}(k,q_{out}) = \langle q_{out}-|\not{\varepsilon}^{+}(k,q_{out})$ vanishes since $\langle q_{out}-|k-\rangle = 0$ and $\langle q_{out}-|q_{out}+\rangle = 0$.

The new definition has also the advantage that when the reference momentum is to be changed, no phase factor is required:

$$\not{\varepsilon}_q^{\pm} = \not{\varepsilon}_p^{\pm} + \beta^{\pm}\not{k}, \tag{3.15}$$

with

$$\beta^{\pm}(k,q,p) = \pm\frac{\sqrt{2}\,\langle p\mp|q\pm\rangle}{\langle q\mp|k\pm\rangle\langle p\mp|k\pm\rangle}, \tag{3.16}$$

$1/2\sqrt{2}$?

which follows directly from (3.13). In the example $e^+e^- \longrightarrow \mu^+\mu^-\gamma$ (Fig. 2 and eq.(2.31)), the amplitude is now

$$M = \left[M_{(1)} + M_{(2)}\right]_{\varepsilon=\varepsilon_q} + \left[M_{(3)} + M_{(4)}\right]_{\varepsilon=\varepsilon_p}. \tag{3.17}$$

The additional term $\beta^{\pm}\not{K}$ actually contributes nothing, since $M_{(1)}+M_{(2)}$ and $M_{(3)}+M_{(4)}$ are gauge-invariant subsets, independent of each other.

The new polarization vector is expressed in terms of the spinors, hence it gives the factorization of the amplitudes in a natural way. Essentially this is the formulation of the amplitudes in Grassmann algebra, to be discussed in detail in the next section.

More about the new polarization vector is given in the Appendix.

4. Factorization and Formulation of the Amplitudes in Grassmann Algebra

We first consider the process

$$e^{+}(q_{+})+e^{-}(q_{-})\longrightarrow \gamma(k_1)+\gamma(k_2) \tag{4.1}$$

as an example. From eq. (2.22) it follows that

$$M(\pm,\mp,+,+)=M(\pm,\mp,-,-)=0, \tag{4.2}$$

and the non-zero amplitudes are $M(\pm,\mp,+,-)$ and $M(\pm,\mp,-,+)$. As shown in Fig.5 there are two Feynman diagrams contributing to this process. For the amplitude $M(-,+,+,-)$ the reference momenta of $\not{\varepsilon}_1^{+}$ and $\not{\varepsilon}_2^{-}$ are chosen to be q_- and q_+, respectively, according to the rule explained in previous section:

$$\not{\varepsilon}_1^{+}=\not{\varepsilon}^{+}(k_1,q_-)=\frac{\sqrt{2}}{\langle q_- - | k_1 + \rangle}\left[\,|k_1-\rangle\langle q_- -| + |q_- +\rangle\langle k_1 +|\,\right], \tag{4.3}$$

$$\not{\varepsilon}_2^{-}=\varepsilon^{-}(k_2,q_+)=-\frac{\sqrt{2}}{\langle q_+ + | k_2 - \rangle}\left[\,|k_2+\rangle\langle q_+ +| + |q_+ -\rangle\langle k_2 -|\,\right]. \tag{4.4}$$

It follows that the diagram (1) does not contribute (the helicity match rule, Fig.1) and we have

$$\begin{aligned}
&M(-,+,+,-)=M_{(2)}(-,+,+,-)\\
&=\bar{v}_-(q_+)(-ie)\,\not{\varepsilon}_1^{+}\,\frac{i(\not{q}_- - \not{k}_2)}{(q_- - k_2)^2}(-ie)\,\not{\varepsilon}_2^{-}\,u_+(q_-)\\
&=-ie^2\frac{2}{\langle q_- - | k_1 + \rangle\langle q_+ + | k_2 - \rangle}\langle q_+ + | k_1 - \rangle\langle q_- - |\frac{(\not{q}_- - \not{k}_2)}{2(q_- k_2)}|q_+ -\rangle\langle k_2 - | q_- +\rangle\\
&=ie^2\frac{\langle q_+ + | k_1 - \rangle\langle q_- - | k_2 + \rangle\langle k_2 + | q_+ - \rangle\langle k_2 - | q_- + \rangle}{(q_- k_2)\langle q_- - | k_1 + \rangle\langle q_+ + | k_2 - \rangle}.
\end{aligned} \tag{4.5}$$

The amplitude is now expressed as a product of factors in the form

$$\langle p\pm | q\mp\rangle,$$

which we call the basic products of spinors, and which we use throughout.

In addition to the eqs. (2.16) —— (2.20) we choose a phase convention such that

$$(\,|q\pm\rangle\,)^{c}=C_1\,|q\pm\rangle^{*}=|q\mp\rangle, \tag{4.6}$$

where $(|\rangle)^c$ denotes the charge conjugation of the spinor $|\rangle$, and the matrix C_1 satisfies the equation

$$C_1 \gamma_\mu^* C_1^{-1} = -\gamma_\mu . \tag{4.7}$$

Indeed $(|q\pm\rangle)^c$ satisfies the same equations as $|q\mp\rangle$.
It is easy, then, to show that the four types of basic products of spinors satisfy the relations (see Appendix)

$$\langle p-|q+\rangle = (\langle q+|p-\rangle)^* = -\langle q-|p+\rangle = -(\langle p+|q-\rangle)^*. \tag{4.8}$$

Introducing the notation

$$\langle p-|q+\rangle = \langle pq\rangle , \tag{4.9}$$

we have

$$\langle q-|p+\rangle = \langle qp\rangle = -\langle pq\rangle , \tag{4.10}$$

$$\langle q+|p-\rangle = \langle pq\rangle^* , \tag{4.11}$$

$$\langle p+|q-\rangle = -\langle pq\rangle^* . \tag{4.12}$$

It is easy to show that

$$|\langle pq\rangle|^2 = 2(pq) \equiv 2p_\mu q^\mu . \tag{4.13}$$

We compare the basic product of spinors with the usual inner-product of momenta in Table 1. The product $\langle pq\rangle$ can be called the antisymmetric inner product of two momenta p and q . Actually the spinors $|q\pm\rangle$ are some kinds of representations of the momentum q, different from its 4-vector representation q_μ , with the fermion feature of the associated particle included. The antisymmetry of the product $\langle pq\rangle$ implies that the spinors are essentially Grassmann variables. Therefore the formulation given here expresses the helicity amplitudes in Grassmann algebra.

By means of eqs. (4.9) —(4.13) the amplitude (4.5) becomes

$$M(-,+,+,-) = -2ie^2 \frac{\langle q_-k_2\rangle \langle q_+k_1\rangle^*}{\langle q_-k_2\rangle^* \langle q_-k_1\rangle}$$

$$\hat{=} 2e^2 \sqrt{\frac{(q_+k_1)}{(q_-k_1)}} , \tag{4.14}$$

in which the symbol $\hat{=}$ denotes the identity modulo a phase factor, so that the norm of the amplitude is

$$|M(-,+,+,-)|^2 = 4e^4 \frac{(q_+k_1)}{(q_-k_1)} . \tag{4.15}$$

Similar calculation gives

$$|M(+,-,-,+)|^2 = |M(-,+,+,-)|^2 = 4e^4 \frac{(q_+ k_1)}{(q_- k_1)}, \tag{4.16}$$

$$|M(+,-,+,-)|^2 = |M(-,+,-,+)|^2 = 4e^4 \frac{(q_+ k_2)}{(q_- k_2)}, \tag{4.17}$$

and since

$$q_+ + q_- = k_1 + k_2,$$

we have (4.18)

$$(q_+ k_1) = (q_- k_2) \equiv -\frac{t}{2}, \qquad (q_- k_1) = (q_- k_2) \equiv -\frac{u}{2}. \tag{4.19}$$

The spin-averaged (for initial states) and polarization-summed (for final states) squared amplitude is

$$\overline{|M|^2} = 4e^4 \frac{t^2 + u^2}{tu}, \tag{4.20}$$

which is the well known result, but the intermediate steps in our calculation are much shorter.

Another simple example is the process

$$e^+(p_+) + e^-(p_-) \longrightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k), \tag{4.21}$$

of which the Feynman diagrams are shown in Fig.2. Let $M(\lambda_1, \lambda_2, \lambda_3, \lambda_4, \lambda_\gamma)$ denote the amplitude with helicities λ_1 of e^+, λ_2 of e^-, λ_3 of μ^+, λ_4 of μ^- and λ_γ of γ . We calculate $M(-,+,+,-,+)$ first. The polarization vectors are, according to the rule given in section 3,

$$\not{\varepsilon}_q^+ = \not{\varepsilon}^+(k, q_-) = \frac{\sqrt{2}}{\langle q_- - | k+ \rangle} \left[|k-\rangle\langle q_- -| + |q_- +\rangle\langle k+| \right], \tag{4.22}$$

$$\not{\varepsilon}_p^+ = \not{\varepsilon}^+(k, p_-) = \frac{\sqrt{2}}{\langle p_- - | k+ \rangle} \left[|k-\rangle\langle p_- -| + |p_- +\rangle\langle k+| \right], \tag{4.23}$$

and we have two non-contributing diagrams

$$[M_{(2)}]_{\varepsilon=\varepsilon_q} = [M_{(4)}]_{\varepsilon=\varepsilon_p} = 0. \tag{4.24}$$

The amplitude is then

$$M(\overset{e^+}{-},\overset{e^-}{+},\overset{\mu^+}{+},\overset{\mu^-}{-},\overset{\gamma}{+}) = [M_{(1)}]_{\varepsilon=\varepsilon_q} + [M_{(3)}]_{\varepsilon=\varepsilon_p}. \tag{4.25}$$

Before the calculation of eq. (4.25) we give some identities useful in the summation over the Lorentztensor indices in the amplitudes. The identities are to be deduced in the Appendix:

$$\langle A+|\gamma_\mu|B+\rangle\langle C-|\gamma^\mu|D-\rangle = 2\langle A+|D-\rangle\langle C-|B+\rangle, \tag{A.28}$$

and

$$\langle A+|\gamma_\mu|B+\rangle = \langle B-|\gamma_\mu|A-\rangle, \tag{A.24}$$

$$(1 \mp \gamma_5)\,|A\pm\rangle = 0, \tag{A.21}$$

and

$$(\,|A\pm\rangle)^C = |A\mp\rangle. \tag{A.22}$$

Now the calculation goes straightforward.

$$\begin{aligned}
&\left[M_{(1)}(-,+,+,-,+)\right]_{\varepsilon=\varepsilon_q^+} \\
&= (-ie)^3 \frac{-i}{(p_+ + p_-)^2} \langle p_+ + | \gamma_\mu | p_- + \rangle \langle q_- - | \gamma^\mu \frac{-i(\not{q}_+ + \not{k})}{(q_+ + k)^2} \not{\varepsilon}_q^+ | q_+ - \rangle \\
&= \frac{-2ie^3}{(p_+ + p_-)^2 (q_+ + k)^2} \langle q_- - | p_- + \rangle \langle p_+ + | (\not{q}_+ + \not{k}) \not{\varepsilon}_q^+ | q_+ - \rangle \\
&= \frac{2\sqrt{2}\, i e^3}{(p_+ + p_-)^2} \cdot \frac{\langle q_- - | p_- + \rangle}{\langle q_- - | k + \rangle} \cdot \frac{\langle q_+ + | k - \rangle}{2(q_+ k)} \langle p_+ + | (\not{q}_+ + \not{k}) | q_- + \rangle \\
&= \frac{2\sqrt{2}\, i e^3}{(p_+ + p_-)^2} \cdot \frac{\langle p_- - | q_- + \rangle}{\langle q_- - | k + \rangle \langle q_+ - | k + \rangle} \langle p_+ + | \not{p}_- | q_- + \rangle \\
&= -2\sqrt{2}\, i e^3 \frac{\langle p_- - | q_- + \rangle^2}{\langle p_+ - | p_- + \rangle \langle q_+ - | k + \rangle \langle q_- - | k + \rangle} \\
&= -2\sqrt{2}\, i e^3 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ k \rangle \langle q_- k \rangle},
\end{aligned} \tag{4.26}$$

similarly,

$$\left[M_{(3)}(-,+,+,-,+)\right]_{\varepsilon=\varepsilon_p^+} = 2\sqrt{2}\, i e^3 \frac{\langle p_- q_- \rangle^2}{\langle q_+ q_- \rangle \langle p_+ k \rangle \langle p_- k \rangle}, \tag{4.27}$$

in which we have made use of the energy-momentum conservation:

$$p_+ + p_- = q_+ + q_- + k. \tag{4.28}$$

Thus

$$\begin{aligned}
M(-,+,+,-,+) &= 2\sqrt{2}\, i e^3 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} \left(\frac{\langle p_+ p_- \rangle}{\langle p_+ k \rangle \langle p_- k \rangle} - \frac{\langle q_+ q_- \rangle}{\langle q_+ k \rangle \langle q_- k \rangle} \right) \\
&= 2\sqrt{2}\, i e^3 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (A_p - A_q),
\end{aligned} \tag{4.29}$$

where

$$A_p = \frac{\langle p_+ p_- \rangle}{\langle p_+ k \rangle \langle p_- k \rangle}, \qquad A_q = \frac{\langle q_+ q_- \rangle}{\langle q_+ k \rangle \langle q_- k \rangle}. \tag{4.30}$$

As for its norm we have

$$|A_p|^2 = \frac{1}{2} \frac{(p_+ p_-)}{(p_+ k)(p_- k)}, \quad |A_q|^2 = \frac{1}{2} \frac{(q_+ q_-)}{(q_+ k)(q_- k)}, \tag{4.31}$$

$$\begin{aligned}
A_p A_q^* + \text{c.c.} &= \frac{\langle p_+ p_- \rangle \langle q_+ q_- \rangle^*}{\langle p_+ k \rangle \langle p_- k \rangle \langle q_+ k \rangle^* \langle q_- k \rangle^*} + \text{c.c.} \\
&= \frac{1}{16 (p_+ k)(p_- k)(q_+ k)(q_- k)} \left(\langle p_+ p_- \rangle \langle q_+ k \rangle \langle q_- k \rangle \langle q_+ q_- \rangle^* \langle p_+ k \rangle^* \langle p_- k \rangle^* + \text{c.c.} \right) \\
&= \frac{-1}{16 (p_+ k)(p_- k)(q_+ k)(q_- k)} \mathrm{Tr}(\not{p}_+ \not{p}_- \not{k} \not{q}_+ \not{q}_- \not{k}) \\
&= -\frac{1}{2} \left[\frac{(p_+ q_+)}{(p_+ k)(q_+ k)} + \frac{(p_- q_-)}{(p_- k)(q_- k)} - \frac{(p_+ q_-)}{(p_+ k)(q_- k)} - \frac{(p_- q_+)}{(p_- k)(q_+ k)} \right],
\end{aligned} \tag{4.32}$$

and thus

$$|A_p - A_q|^2 = -\frac{1}{4}(v_p - v_q)^2, \tag{4.33}$$

in which

$$v_p = \frac{p_+}{(p_+k)} - \frac{p_-}{(p_-k)}, \qquad v_q = \frac{q_+}{(q_+k)} - \frac{q_-}{(q_-k)}. \tag{4.34}$$

Finally we get

$$|M(-,+,+,-,+)|^2 = -2e^6 \frac{t'^2}{ss'}(v_p - v_q)^2, \tag{4.35}$$

where the variables

$$\left.\begin{array}{ll} s=(p_+ + p_-)^2 = 2(p_+p_-), & s'=(q_+ + q_-)^2 = 2(q_+q_-), \\ t=(p_+ - q_+)^2 = -2(p_+q_+), & u=(p_+ - q_-)^2 = -2(p_+q_-), \\ t'=(p_- - q_-)^2 = -2(p_-q_-), & u'=(p_- - q_+)^2 = -2(p_-q_+), \end{array}\right\} \tag{4.36}$$

are used. The amplitudes for other helicity configurations have the similar structure:

$$|M(+,-,-,+,-)|^2 = |M(-,+,+,-,+)|^2 = -2e^6\frac{t'^2}{ss'}(v_p - v_q)^2, \tag{4.37}$$

$$|M(+,-,-,+,+)|^2 = |M(-,+,+,-,-)|^2 = -2e^6\frac{t^2}{ss'}(v_p - v_q)^2, \tag{4.38}$$

$$|M(+,-,+,-,+)|^2 = |M(-,+,-,+,-)|^2 = -2e^6\frac{u^2}{ss'}(v_p - v_q)^2, \tag{4.39}$$

$$|M(+,-,+,-,-)|^2 = |M(-,+,-,+,+)|^2 = -2e^6\frac{u'^2}{ss'}(v_p - v_q)^2. \tag{4.40}$$

For the unpolarized reaction we have finally

$$\overline{|M|^2} = -e^6\frac{t^2+u^2+t'^2+u'^2}{ss'}(v_p - v_q)^2. \tag{4.41}$$

Eq. (4.41) is identical with the results of [4]. The present method enables to further simplify the calculation.

5. Conclusion

The new definition of polarization vectors of gauge bosons is a step further based on the helicity amplitude formalism developed by CALKUL collaboration. It simplifies the calculation for QED processes, and is especially suitable for QCD processes to be discussed in subsequent papers[11].

Acknowledgements

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Appendix

1. General Convention

We use the metric and matrix convention as in the book of Bjorken and Drell [10].

2. Products of Spinors

Let

$$u_+(q) = v_-(q) = |q+\rangle, \qquad u_-(q) = v_+(q) = |q-\rangle, \tag{A.1}$$

$$\bar{u}_+(q) = \bar{v}_-(q) = \langle q+|, \qquad \bar{u}_-(q) = \bar{v}_+(q) = \langle q-|, \tag{A.2}$$

which satisfy

$$\not{q}\,|q\pm\rangle = \langle q\pm|\,\not{q} = 0, \tag{A.3}$$

$$(1 \mp \gamma_5)|q\pm\rangle = \langle q\pm|(1 \pm \gamma_5) = 0, \tag{A.4}$$

$$\langle q\pm|\gamma_\mu|q\pm\rangle = 2q_\mu, \tag{A.5}$$

$$|q\pm\rangle\langle q\pm| = \frac{1}{2}(1 \pm \gamma_5)\not{q}, \tag{A.6}$$

$$\langle q+|p+\rangle = \langle q-|p-\rangle = 0, \tag{A.7}$$

in which q and p are any 4-momenta satisfying $q^2 = p^2 = 0$.

We put as a convention

$$|q+\rangle^c = |q-\rangle, \qquad |q-\rangle^c = |q+\rangle, \tag{A.8}$$

where

$$|\,\rangle^c = C_1|\,\rangle^*, \qquad {}^c\langle\,| = -{}^*\langle\,|C_1^{-1}, \tag{A.9}$$

and

$$C_1 \gamma_\mu^* C_1^{-1} = -\gamma_\mu, \tag{A.10}$$

$$C_1 = C_1^\dagger = C_1^{-1} = C_1^* = C_1^T, \tag{A.11}$$

so,

$$\langle q-|p+\rangle = (\langle p+|q-\rangle)^* = {}^*\langle p+|q-\rangle^* = {}^*\langle p+|C_1^{-1}C_1|q-\rangle^* = -{}^c\langle p+|q-\rangle^c$$
$$= -\langle p-|q+\rangle,$$

that is

$$\langle p-|q+\rangle = -\langle q-|p+\rangle. \tag{A.12}$$

In defining

$$\langle p-|q+\rangle = \langle pq\rangle, \tag{A.13}$$

we have

$$\langle q+|p-\rangle = \langle pq\rangle^*, \tag{A.14}$$

$$\langle q-|p+\rangle = -\langle pq\rangle, \tag{A.15}$$

$$\langle p+|q-\rangle = -\langle pq\rangle^*, \tag{A.16}$$

and

$$|\langle pq\rangle|^2 = \langle p-|q+\rangle\langle q+|p-\rangle = \frac{1}{2}\,\mathrm{Tr}\,\not{p}\not{q}(1-\gamma_5) = 2(pq). \tag{A.17}$$

In general, when n is even,

$$(\langle p+|\gamma_{\mu_1}\cdots\gamma_{\mu_n}|q-\rangle)^* = {}^*\langle p-|\gamma_{\mu_1}^*\cdots\gamma_{\mu_n}^*|q-\rangle^*$$

$$= (-1)^n\,{}^*\langle p+|C_1^{-1}\gamma_{\mu_1}\cdots\gamma_{\mu_n}C_1|q-\rangle^* = (-1)^{n+1}\,{}^c\langle p+|\gamma_{\mu_1}\cdots\gamma_{\mu_n}|q-\rangle^c$$

$$= -\langle p-|\gamma_{\mu_1}\cdots\gamma_{\mu_n}|q+\rangle,$$

and at the same time

$$(\langle p+|\gamma_{\mu_1}\cdots\gamma_{\mu_n}|q-\rangle)^* = \langle q-|\gamma_{\mu_n}\cdots\gamma_{\mu_1}|p+\rangle,$$

therefore

$$\langle p-|\gamma_{\mu_1}\cdots\gamma_{\mu_n}|q+\rangle = -\langle q-|\gamma_{\mu_n}\cdots\gamma_{\mu_1}|p+\rangle,$$

or

$$\langle p-|\not{k}_1\cdots\not{k}_n|q+\rangle = -\langle q-|\not{k}_n\cdots\not{k}_1|p+\rangle \quad (\text{n even}) \tag{A.18}$$

Similarly,

$$\langle p+|\not{k}_1\cdots\not{k}_n|q-\rangle = -\langle q+|\not{k}_n\cdots\not{k}_1|p-\rangle \quad (\text{n even}) \tag{A.19}$$

and

$$\langle p+|\not{k}_1\cdots\not{k}_n|q+\rangle = \langle q-|\not{k}_n\cdots\not{k}_1|p-\rangle \quad (\text{n odd}) \tag{A.20}$$

where k_i (i=1, ... , n) is any momentum with or without $k_i^2 = 0$.

More generally, let $|A\pm\rangle$ be any spinor satisfying

$$(1\mp\gamma_5)|A\pm\rangle = 0, \tag{A.21}$$

$$|A+\rangle^c = |A-\rangle, \qquad |A-\rangle^c = |A+\rangle, \tag{A.22}$$

we have

$$\langle A\mp|B\pm\rangle = -\langle B\mp|A\pm\rangle, \tag{A.23}$$

$$\langle A+|\gamma_\mu|B+\rangle = \langle B-|\gamma_\mu|A-\rangle, \tag{A.24}$$

and if $|A\pm\rangle$ has the form

$$|A\pm\rangle = \begin{cases} \not{k}_1\cdots\not{k}_n|q\pm\rangle, & (\text{n even}) \\ \not{k}_1\cdots\not{k}_n|q\mp\rangle, & (\text{n odd}) \end{cases} \tag{A.25}$$

then

$$|A\pm\rangle^c = \begin{cases} \not{k}_1\cdots\not{k}_n|q\mp\rangle. & (\text{n even}) \\ -\not{k}_1\cdots\not{k}_n|q\pm\rangle. & (\text{n odd}) \end{cases} \tag{A.26}$$

3. Summation over Lorentz Tensor Indices

We want to calculate, say,

$$\langle A+|\gamma_\mu|B+\rangle\cdot\langle C-|\gamma^\mu|D-\rangle.$$

The matrix $|B+\rangle\langle A+|$ can be expanded into linear sum of 1, γ_μ, γ_5, $\gamma_\mu\gamma_5$ and $\gamma_\mu\gamma_\nu$ $(\mu\neq\nu)$:

$$2\,|B+\rangle\langle A+| = \langle A+|\gamma_\mu|B+\rangle\,\gamma^\mu\tfrac{1}{2}(1-\gamma_5). \tag{A.27}$$

Multiplying the both sides of (A.27) by $\langle C-|$ from left and $|D-\rangle$ from right and noting that

$$|D-\rangle = \tfrac{1}{2}(1-\gamma_5)|D-\rangle,$$

we get

$$\langle A+|\gamma_\mu|B+\rangle\cdot\langle C-|\gamma^\mu|D-\rangle = 2\langle A+|D-\rangle\cdot\langle C-|B+\rangle. \qquad (A.28)$$

Making use of eqs. (A.24), (A.25),(A.26) and (A.28) we can complete similar summation such as

$$\langle A+|\gamma_\mu|B+\rangle\cdot\langle C+|\gamma^\mu|D+\rangle$$

and

$$\langle A-|\gamma_\mu|B-\rangle\cdot\langle C-|\gamma^\mu|D-\rangle.$$

Actually this summation is a particular application of the Fiertz rearrangement theorem.

Another useful rearrangement of the spinor products is the identity

$$\langle A-|B+\rangle\langle C-|D+\rangle = \langle A-|D+\rangle\langle C-|B+\rangle + \langle A-|C+\rangle\langle B-|D+\rangle. \qquad (A.29)$$

The proof of (A.29) is as follows.

$$\begin{aligned}
&|A+\rangle\langle A+|B-\rangle\langle B-| + |B+\rangle\langle B+|A-\rangle\langle A-| \\
&= \tfrac{1}{4}\langle A+|\gamma_\mu|A+\rangle\gamma^\mu(1-\gamma_5)\cdot\tfrac{1}{4}\langle B-|\gamma_\nu|B-\rangle\gamma^\nu(1+\gamma_5) + \\
&\qquad + \tfrac{1}{4}\langle B+|\gamma_\mu|B+\rangle\gamma^\mu(1-\gamma_5)\cdot\tfrac{1}{4}\langle A-|\gamma_\nu|A-\rangle\gamma^\nu(1+\gamma_5) \\
&= \tfrac{1}{8}\langle A+|\gamma_\mu|A+\rangle\langle B-|\gamma_\nu|B-\rangle(\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu)(1+\gamma_5) \\
&= \tfrac{1}{4}\langle A+|\gamma_\mu|A+\rangle\langle B-|\gamma^\mu|B-\rangle(1+\gamma_5) \\
&= \langle A+|B-\rangle\langle B-|A+\rangle\tfrac{1}{2}(1+\gamma_5)
\end{aligned}$$

therefore

$$\langle A-|B+\rangle\tfrac{1}{2}(1+\gamma_5) = |B+\rangle\langle A-| - |A+\rangle\langle B-|. \qquad (A.30)$$

Multiplying the both sides of (A.30) by $\langle C-|$ from left and $|D+\rangle$ from right we get (A.29).

A Special case of (A.29) is that

$$\langle p_1p_2\rangle\langle p_3p_4\rangle = \langle p_1p_4\rangle\langle p_3p_2\rangle + \langle p_1p_3\rangle\langle p_2p_4\rangle, \qquad (A.31)$$

which enables us to sum up contributions form different diagrams in multiple Bremsstrahlung.

4. Properties of the New Polarization Vectors

Define

$$\not{\varepsilon}^\pm(k,q) = \pm\frac{\sqrt{2}}{\langle q\mp|k\pm\rangle}\left[\,|k\mp\rangle\langle q\mp| + |q\pm\rangle\langle k\pm|\,\right], \qquad (A.32)$$

we have

$$\varepsilon_\mu^\pm = \pm\frac{\langle q\mp|\gamma_\mu|k\mp\rangle}{\sqrt{2}\langle q\mp|k\pm\rangle}, \qquad (A.33)$$

$$(\varepsilon_\mu^+)^* = \varepsilon_\mu^-, \qquad (A.34)$$

$$(\varepsilon^\pm k) = (\varepsilon^\pm q) = 0, \qquad (A.35)$$

$$(\varepsilon^+\varepsilon^+) = (\varepsilon^-\varepsilon^-) = 0, \qquad (A.36)$$

$$(\varepsilon^+\varepsilon^-) = -1. \qquad (A.37)$$

If

$$\varepsilon_1 = \varepsilon(k_1, q) , \qquad \varepsilon_2 = \varepsilon(k_2, q), \tag{A.38}$$

then

$$(\varepsilon_1^+ \varepsilon_2^+) = (\varepsilon_1^- \varepsilon_2^-) = 0, \tag{A.39}$$

$$(\varepsilon_1^+ \varepsilon_2^-) = -\frac{\langle q k_1\rangle^* \langle q k_2\rangle}{\langle q k_1\rangle \langle q k_2\rangle^*} . \tag{A.40}$$

And if

$$\varepsilon_q = \varepsilon(k, q) , \qquad \varepsilon_p = \varepsilon(k, p), \tag{A.41}$$

then

$$(\varepsilon_p^+ \varepsilon_q^+) = (\varepsilon_p^- \varepsilon_q^-) = 0, \tag{A.42}$$

$$(\varepsilon_p^+ \varepsilon_q^-) = -1 , \tag{A.43}$$

$$\varepsilon_q^\pm = \varepsilon_p^\pm + \beta^\pm k, \tag{A.44}$$

where

$$\beta^\pm = \pm \frac{\sqrt{2}\, \langle p\mp | q\pm\rangle}{\langle p\mp | k\pm\rangle \langle q\mp | k\pm\rangle} . \tag{A.45}$$

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Table Captions

Table 1 Comparison of $\langle pq \rangle$ and (pq) .

Table 1

	basic product of spinors	inner-product of momenta
symmetry in permutation	antisymmetric, $\langle pq \rangle = - \langle qp \rangle$	symmetric, $(pq) = (qp)$
reality	complex	real
dimension	$[M]$	$[M^2]$
relation	$\lvert \langle pq \rangle \rvert^2 = 2(pq)$	

Figure Captions

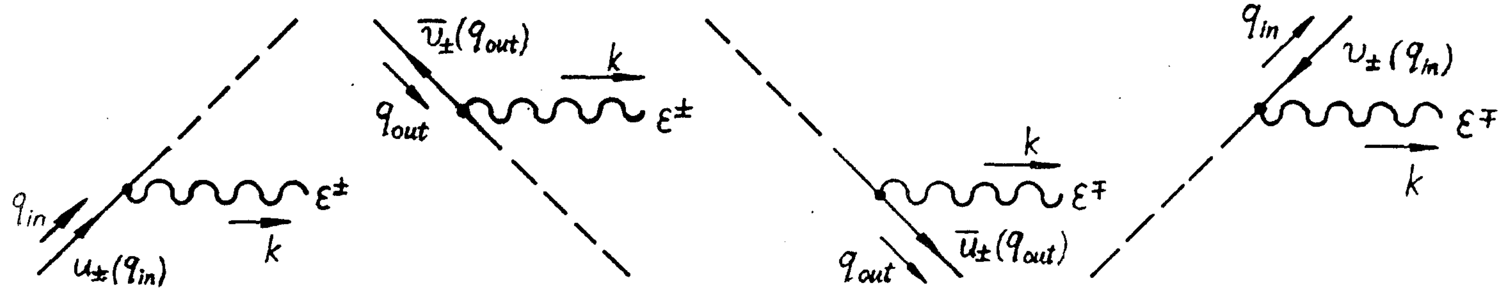
Fig.1 Vanishing configurations of helicities of bosons and fermions.

Fig.2 Feynman diagrams of $e^+e^- \to \mu^+\mu^-\gamma$.

Fig.3 Self-coupling vertices of gluons.

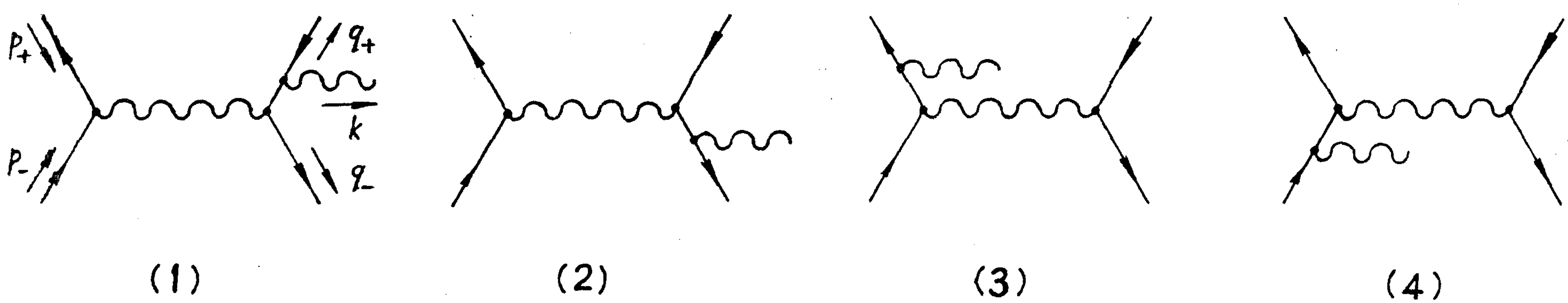
Fig.4 Examples of Feynman diagrams with the internal gluon line emitting gluons.

Fig.5 Feynman diagrams of $e^+e^- \to 2\gamma$.



q_{in}
$u_{\pm}(q_{in})$
k
$\varepsilon^{\pm}$
$\overline{v}_{\pm}(q_{out})$
q_{out}
k
$\varepsilon^{\pm}$
k
$\varepsilon^{\mp}$
q_{out}
$\overline{u}_{\pm}(q_{out})$
q_{in}
$v_{\pm}(q_{in})$
$\varepsilon^{\mp}$
k

Fig. 1



P_+
P_-
q_+
k
q_-
(1)
(2)
(3)
(4)

Fig. 2

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HELICITY AMPLITUDES FOR MULTIPLE BREMSSTRAHLUNG IN MASSLESS NON-ABELIAN GAUGE THEORY(II): DECOMPOSITION OF AMPLITUDES INTO GAUGE-INVARIANT SUBSETS *)

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ABSTRACT

In QED the Feynman diagrams are grouped into gauge invariant subsets according to the photon attachment to fermion lines. In QCD we show that contributions from the diagrams with gluon self-coupling can be decomposed into parts to be included in these subsets. In this way the calculations of helicity amplitudes can be simplified as far as possible.

The cross sections of multiple Bremsstrahlung processes at high energy in abelian, especially in non-abelian gauge theories are of interests for both experimentalists and theoreticians. The method given by CALKUL collaboration can simplify remarkably the calculations of the helicity amplitudes for these processes [2-8]. The key to this method is to refer the polarization vectors of gauge bosons to the momenta of fermion lines to which they are attached. In the previous paper [9] we proposed a modified definition of polarization vectors which can be used in both abelian (QED) and non-abelian (QCD) case and can further simplify the calculations. When more than one fermion lines appear in the Feynman diagrams, in QED the diagrams form gauge-invariant subsets naturally according to the photon attachment to fermion lines, thus we need not worry about the use of different reference momenta. But the existence of self-coupling vertices of gluons in QCD leads to complications compared with QED, since in some Feynman diagrams gluons are emitted from an internal gluon line due to the 3-gluon and 4-gluon vertices. In this paper we discuss the decomposition of these Feynman diagrams into gauge-invariant subsets for single and double Bremsstrahlung processes, making the results relatively simple. More general features of such diagrams will be discussed in the subsequent papers [10].

1. Single Bremsstrahlung in QCD

The four Feynman diagrams contributing to the amplitudes for the QED process

$$e^+(p_+) + e^-(p_-) \longrightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k) \tag{1.1}$$

are shown in Fig.1. We know that [4,9]

$$[M_{(1)} + M_{(2)}]_{\varepsilon=k} = [M_{(3)} + M_{(4)}]_{\varepsilon=k} = 0\,, \tag{1.2}$$

that is, the diagrams (1)+(2) and (3)+(4) form two independent gauge-invariant subsets of the Feynman diagrams. Because of eq.(1.2) we can write

$$M\,(e^+e^- \longrightarrow \mu^+\mu^-\gamma) = [M_{(1)} + M_{(2)}]_{\varepsilon=\varepsilon_q} + [M_{(3)} + M_{(4)}]_{\varepsilon=\varepsilon_p}\,, \tag{1.3}$$

if the new definition of polarization vectors [9] is used.

When we treat, however, the analog process in QCD

$$\bar{q}\,(p_+) + q\,(p_-) \longrightarrow \bar{q}'\,(q_+) + q'\,(q_-) + g\,(k)\,, \tag{1.4}$$

the situation turns out to be different. The Feynman diagrams of this process are shown in Fig.2. In addition to the QED-like diagrams, Fig.2(1)—(4), there is another diagram, Fig.2(5), in which a gluon is emitted by the virtual gluon due to 3-gluon vertex. Thus eq. (1.2) is no longer correct, and we have to consider the additional term due to the use of diferent polarization vectors [3,9]:

$$\begin{aligned}[M]_{\varepsilon=\varepsilon_q} = [M_{(1)} + M_{(2)}]_{\varepsilon=\varepsilon_q} + [M_{(3)} + M_{(4)}]_{\varepsilon=\varepsilon_p} &+ \\ + \beta\,[M_{(3)} + M_{(4)}]_{\varepsilon=k} + [M_{(5)}]_{\varepsilon=\varepsilon_q}&\,,\end{aligned} \tag{1.5}$$

in which the relation

$$\varepsilon_q = \varepsilon_p + \beta k \tag{1.6}$$

is taken into account. This brings complication in adding up the terms and makes the amplitude asymmetric with respect to the momenta p and q.

In order to make use of the advantages of helicity amplitudes as much as possible, we decompose the additional term $M_{(5)}$ into two parts:

$$M_{(5)} = M'_{(5)} + M''_{(5)}, \tag{1.7}$$

such that

$$[M_{(1)} + M_{(2)} + M'_{(5)}]_{\varepsilon=k} = [M_{(3)} + M_{(4)} + M''_{(5)}]_{\varepsilon=k} = 0. \tag{1.8}$$

Therefore we have a similar identity, comparable with eq.(1.3),

$$M(\bar{q}q \to \bar{q}'q'g) = [M_{(1)} + M_{(2)} + M'_{(5)}]_{\varepsilon=\varepsilon_q} + [M_{(3)} + M_{(4)} + M''_{(5)}]_{\varepsilon=\varepsilon_p}. \tag{1.9}$$

We refer to this method as the decomposition of amplitudes into gauge-invariant subsets.

The decomposition is rather straightforward in this case. We have

$$[M_{(1)} + M_{(2)}]_{\varepsilon=k} = -f_{abc}\, g^3 \frac{1}{(p_+ + p_-)^2} g^{\mu\nu} \bar{V}(p_+) \gamma_\mu T_a U(p_-) \cdot \bar{U}(q_-) \gamma_\nu T_b V(q_+), \tag{1.10}$$

$$[M_{(3)} + M_{(4)}]_{\varepsilon=k} = f_{abc}\, g^3 \frac{1}{(q_+ + q_-)^2} g^{\mu\nu} \bar{V}(p_+) \gamma_\mu T_a U(p_-) \cdot \bar{U}(q_-) \gamma_\nu T_b V(q_+), \tag{1.11}$$

and due to the gauge-invariance of the theory,

$$[M_{(5)}]_{\varepsilon=k} = -[M_{(1)} + M_{(2)}]_{\varepsilon=k} - [M_{(3)} + M_{(4)}]_{\varepsilon=k}, \tag{1.12}$$

where g is the coupling constant, $g^{\mu\nu}$ is the space-time metric tensor , T_a is the generator of the gauge (colour) group SU(N), f_{abc} is the structure constant of the group, U and V are the spinors with colour wave functions included, and c is the colour index of the emitted gluon (a,b,c=1,···, N^2-1). Now we put

$$M'_{(5)} = a M_{(5)}, \quad M''_{(5)} = (1-a) M_{(5)}, \tag{1.13}$$

where a is a parameter depending on the momenta concerned. It is determined by (1.8):

$$\left.\begin{aligned} a &= -\frac{Q^2}{P^2 - Q^2}, \\ 1 - a &= \frac{P^2}{P^2 - Q^2}, \end{aligned}\right\} \tag{1.14}$$

where

$$P = p_+ + p_- \, , \qquad Q = q_+ + q_- \, , \tag{1.15}$$

and

$$P^2 - Q^2 = 2(Pk) = 2(Qk) = ((P+Q)k), \tag{1.16}$$

because of the energy-momentum conservation:

$$P = Q + k. \tag{1.17}$$

Therefore,

$$M'_{(5)} = -\frac{Q^2}{P^2 - Q^2} M_{(5)}, \tag{1.18}$$

$$M''_{(5)} = \frac{P^2}{P^2 - Q^2} M_{(5)}. \tag{1.19}$$

Furthermore, since

$$T_a T_b = \frac{1}{2N} \delta_{ab} I + \frac{1}{2} (d_{abc} + i f_{abc}) T_c , \tag{1.20}$$

where I is the unit NXN matrix and d_{abc} is the symmetric tensor of the gauge group (see Appendix), the amplitudes consist of three parts depending on different tensors of the gauge group, i.e.

$$M(\bar{q} q \to \bar{q}' q' g) = M^{(e)} + M^{(d)} + M^{(f)}, \tag{1.21}$$

where $M^{(e)}$, $M^{(d)}$ and $M^{(f)}$ are proportional to $\delta_{ac} (\delta_{bc})$, d_{abc} and f_{abc} respectively.

It is easy to show that they are independently gauge-invariant:

$$[M^{(e)}]_{\varepsilon=k} = [M^{(d)}]_{\varepsilon=k} = [M^{(f)}]_{\varepsilon=k} = 0, \tag{1.22}$$

and are incoherent:

$$|M|^2 = |M^{(e)}|^2 + |M^{(d)}|^2 + |M^{(f)}|^2, \tag{1.23}$$

where the sum over colour indices is understood. Actually $M^{(e)}$ and $M^{(d)}$ are simple extensions of the corresponding amplitudes in QED, $M(e^+e^- \to \mu^+\mu^-\gamma)$, to which $M_{(5)}$ does not contribute:

$$M^{(e)} = \frac{1}{2N} T_c(q) I(q') [M_{(1)} + M_{(2)}]^{(QED)} + \frac{1}{2N} I(q) T_c(q') [M_{(3)} + M_{(4)}]^{(QED)}, \tag{1.24}$$

$$M^{(d)} = \frac{1}{2} d_{abc} T_a(q) T_b(q') \cdot M^{(QED)}, \tag{1.25}$$

in which

$$T_a(q) = \varphi^{\dagger}(\bar{q}) T_a \chi(q), \tag{1.26}$$

$$T_b(q') = \chi^{\dagger}(q') T_b \varphi(\bar{q}'), \tag{1.27}$$

$$I(q) = \varphi^{\dagger}(\bar{q}) \chi(q), \tag{1.28}$$

$$I(q') = \chi^{\dagger}(q') \varphi(\bar{q}'), \tag{1.29}$$

and $\chi(\varphi)$ is the colour wave function of the quark (antiquark) such that

$$U(p_-) = u(p_-) \chi(q), \qquad U(q_-) = u(q_-) \chi(q'), \tag{1.30}$$

$$V(p_+) = v(p_+) \varphi(\bar{q}), \qquad V(q_+) = v(q_+) \varphi(\bar{q}'), \tag{1.31}$$

and $M^{(QED)}$ is the amplitude for QED process $e^+e^- \to \mu^+\mu^-\gamma$ with coupling constant e replaced by g. $M^{(f)}$ is the particular one in QCD to get contribution from $M_{(5)}$. Let us calculate the helicity amplitude $M(-,+,+,-,+)$ in which the signs in the bracket indicate the helicities of $\bar{q}$, q, $\bar{q}'$, q' and g respectively. We choose the polarization vectors as, according to the rule explained in [9] ,

$$\not{\varepsilon}_q^+ = \not{\varepsilon}^+(k, q_-) = \frac{\sqrt{2}}{\langle q_- | k+\rangle} \left[\, |k-\rangle\langle q_-| + |q_+\rangle\langle k+| \,\right], \tag{1.32}$$

$$\not{\varepsilon}_p^+ = \not{\varepsilon}^+(k, p_-) = \frac{\sqrt{2}}{\langle p_- | k+\rangle} \left[\, |k-\rangle\langle p_-| + |p_+\rangle\langle k+| \,\right]. \tag{1.33}$$

Then it follows that

$$[M_{(2)}]_{\varepsilon=\varepsilon_q^+} = [M_{(4)}]_{\varepsilon=\varepsilon_p^+} = 0, \tag{1.34}$$

$$\begin{aligned}[M_{(1)}]^{(f)}_{\varepsilon=\varepsilon_q^+} &= -\frac{i}{2} f_{abc} T_a(q) T_b(q') \cdot [M_{(1)}]^{(QED)}_{\varepsilon=\varepsilon_q^+} \\ &= f_{abc} T_a(q) T_b(q') \cdot (-\sqrt{2}\, g^3) \frac{\langle p_- q_-\rangle^2}{\langle p_+ p_-\rangle \langle q_+ k\rangle \langle q_- k\rangle},\end{aligned} \tag{1.35}$$

$$\begin{aligned}[M_{(3)}]^{(f)}_{\varepsilon=\varepsilon_p^+} &= -\frac{i}{2} f_{abc} T_a(q) T_b(q') \cdot [M_{(3)}]^{(QED)}_{\varepsilon=\varepsilon_p^+} \\ &= f_{abc} T_a(q) T_b(q') \cdot \sqrt{2}\, g^3 \frac{\langle p_- q_-\rangle^2}{\langle q_+ q_-\rangle \langle p_+ k\rangle \langle p_- k\rangle},\end{aligned} \tag{1.36}$$

in which the notation for the basic product of spinors [9] is used. On the other hand,

$$M_{(5)} = f_{abc}\, T_a(q)\, T_b(q') \cdot g^3 \frac{1}{P^2 Q^2} \langle p_+ + | \gamma_\mu | p_- + \rangle \langle q_- - | \gamma_\nu | q_+ - \rangle \cdot$$

$$\cdot \left[- (\varepsilon(P+Q))\, g^{\mu\nu} - 2k^\mu \varepsilon^\nu + 2\varepsilon^\mu k^\nu \right]. \qquad (1.37)$$

Due to eqs. (1.32), (1.33) and (1.17) we have

$$\langle p_+ + | \not\varepsilon_p^+ | p_- + \rangle = \langle q_- - | \not\varepsilon_q^+ | q_+ - \rangle = 0, \qquad (1.38)$$

$$(\varepsilon(P+Q)) = 2(\varepsilon P) = 2(\varepsilon Q), \qquad (1.39)$$

$$2(\varepsilon_q^+ Q) = 2(\varepsilon_q^+ q_+) = \frac{\sqrt{2}\, \langle q_- - | q_+ + \rangle \langle q_+ + | k - \rangle}{\langle q_- - | k + \rangle}, \qquad (1.40)$$

$$2(\varepsilon_p^+ P) = 2(\varepsilon_p^+ p_+) = \frac{\sqrt{2}\, \langle p_- - | p_+ + \rangle \langle p_+ + | k - \rangle}{\langle p_- - | k + \rangle}, \qquad (1.41)$$

therefore,

$$[M'_{(5)}]_{\varepsilon = \varepsilon_q^+}$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot \frac{2g^3}{P^2(P^2 - Q^2)} \left[\langle p_+ + | q_+ - \rangle \langle q_- - | p_- + \rangle (\varepsilon_q^+ (P+Q)) - \right.$$

$$\left. - \langle p_+ + | \not\varepsilon_q^+ | p_- + \rangle \langle q_- - | \not k | q_+ - \rangle \right]$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot \sqrt{2}\, g^3 \frac{2}{P^2(P^2 - Q^2)} \frac{1}{\langle q_- - | k + \rangle} [\langle p_+ + | q_+ - \rangle \langle p_- - | q_- + \rangle \langle q_+ - | q_- + \rangle \langle q_+ + | k - \rangle +$$

$$+ \langle p_+ + | k - \rangle \langle p_- - | q_- + \rangle \langle k - | q_- + \rangle \langle q_+ + | k - \rangle]$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot \sqrt{2}\, g^3 \frac{2}{P^2(P^2 - Q^2)} \frac{\langle p_- - | q_- + \rangle \langle q_+ + | k - \rangle}{\langle q_- - | k + \rangle} \langle p_+ + | (\not q_+ + \not k) | q_- + \rangle$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot \sqrt{2}\, g^3 \frac{2(q_+ k)}{(q_+ k) + (q_- k)} \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ k \rangle \langle q_- k \rangle}. \qquad (1.42)$$

Similarly,

$$[M''_{(5)}]_{\varepsilon = \varepsilon_p^+}$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot \sqrt{2}\, g^3 \frac{(-2)(p_+ k)}{(p_+ k) + (p_- k)} \frac{\langle p_- q_- \rangle^2}{\langle q_+ q_- \rangle \langle p_+ k \rangle \langle p_- k \rangle}. \qquad (1.43)$$

Adding up eqs. (1.34) —— (1.36) and (1.42) —— (1.43) results in

$$M^{(f)}(-,+,+,-,+)$$

$$= f_{abc}\, T_a(q)\, T_b(q') \cdot (-\sqrt{2}\, g^3) \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (\lambda_p A_p - \lambda_q A_q); \tag{1.44}$$

where

$$\lambda_p = \frac{(p_+ k) - (p_- k)}{(p_+ k) + (p_- k)}, \qquad \lambda_q = \frac{(q_+ k) - (q_- k)}{(q_+ k) + (q_- k)}, \tag{1.45}$$

and

$$A_p = \frac{\langle p_+ p_- \rangle}{\langle p_+ k \rangle \langle p_- k \rangle}, \qquad A_q = \frac{\langle q_+ q_- \rangle}{\langle q_+ k \rangle \langle q_- k \rangle}. \tag{1.46}$$

$M^{(f)}$ (1.44) is similar in structure to $M^{(e)}$ and $M^{(d)}$:

$$M^{(e)}(-,+,+,-,+)$$

$$= (i\sqrt{2}\, g^3) \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} \left[\frac{1}{N} I(q)\, T_c(q') A_p - \frac{1}{N} T_c(q)\, I(q')\, A_q \right], \tag{1.47}$$

$$M^{(d)}(-,+,+,-,+)$$

$$= d_{abc}\, T_a(q)\, T_b(q') \cdot (i\sqrt{2}\, g^3) \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (A_p - A_q). \tag{1.48}$$

Since

$$|A_p|^2 = -\frac{1}{4} v_p^2, \qquad |A_q|^2 = -\frac{1}{4} v_q^2, \tag{1.49}$$

$$|A_p - A_q|^2 = -\frac{1}{4} (v_p - v_q)^2, \tag{1.50}$$

$$|\lambda_p A_p - \lambda_q A_q|^2 = -\frac{1}{4} (\lambda_p v_p - \lambda_q v_q)^2, \tag{1.51}$$

where

$$v_p = \frac{p_+}{(p_+ k)} - \frac{p_-}{(p_- k)}, \qquad v_q = \frac{q_+}{(q_+ k)} - \frac{q_-}{(q_- k)}, \tag{1.52}$$

and

$$|I(q)|^2 = |I(q')|^2 = N, \tag{1.53}$$

$$|T_c(q)|^2 = |T_c(q')|^2 = \frac{1}{2}(N^2-1), \tag{1.54}$$

$$|f_{abc}\, T_a(q)\, T_b(q')|^2 = \frac{1}{4} f_{abc} f_{abc} = \frac{1}{4N} N^2 (N^2-1), \tag{1.55}$$

$$|d_{abc}\, T_a(q)\, T_b(q')|^2 = \frac{1}{4} d_{abc} d_{abc} = \frac{1}{4N} (N^2-1)(N^2-4), \tag{1.56}$$

$$I(q)\, T_c^*(q) = I(q')\, T_c^*(q') = 0. \tag{1.57}$$

in which the sum over the colour degrees of freedom of quarks and gluons is understood (see Appendix), the squares of the amplitudes are

$$|M^{(e)}(-,+,+,-,+)|^2 = \frac{1}{8N}(N^2-1)(-2g^6)\frac{t'^2}{ss'}(v_p^2+v_q^2), \tag{1.58}$$

$$|M^{(d)}(-,+,+,-,+)|^2 = \frac{1}{16N}(N^2-1)(N^2-4)(-2g^6)\frac{t'^2}{ss'}(v_p-v_q)^2, \tag{1.59}$$

$$|M^{(f)}(-,+,+,-,+)|^2 = \frac{1}{16N}N^2(N^2-1)(-2g^6)\frac{t'^2}{ss'}(\lambda_p v_p-\lambda_q v_q)^2, \tag{1.60}$$

and finally we get

$$|M(-,+,+,-,+)|^2 = -2g^6\frac{t'^2}{ss'}\cdot\Big\{\frac{(N^2-1)^2}{16N}\left[(\lambda_p v_p-\lambda_q v_q)^2+(v_p-v_q)^2\right]+ + \frac{N^2-1}{16N}\left[(\lambda_p v_p-\lambda_q v_q)^2-(v_p-v_q)^2+4(v_p v_q)\right]\Big\}. \tag{1.61}$$

It is easy then to get the unpolarized, colour-averaged norm squared:

$$\overline{|M|^2} = -g^6\frac{t^2+u^2+t'^2+u'^2}{ss'}\cdot\Big\{C_1\left[(\lambda_p v_p-\lambda_q v_q)^2+(v_p-v_q)^2\right]+ + C_2\left[(\lambda_p v_p-\lambda_q v_q)^2-(v_p-v_q)^2+4(v_p v_q)\right]\Big\}, \tag{1.62}$$

where [1]

$$\left.\begin{aligned} s &= (p_+ + p_-)^2 = 2(p_+p_-), & s' &= (q_+ + q_-)^2 = 2(q_+q_-), \\ t &= (p_+ - q_+)^2 = -2(p_+q_+), & u &= (p_+ - q_-)^2 = -2(p_+q_-), \\ t' &= (p_- - q_-)^2 = -2(p_-q_-), & u' &= (p_- - q_+)^2 = -2(p_-q_+), \end{aligned}\right\} \tag{1.63}$$

$$C_1 = \frac{(N^2-1)^2}{16N^3}, \qquad C_2 = \frac{N^2-1}{16N^3}. \tag{1.64}$$

Actually the first term in the curly bracket is much greater than the second one since $(N^2-1) \gg 1$. The average norm (1.62) is comparable with that for QED process [4.9]:

$$\overline{|M|^2} = -e^6\frac{t^2+u^2+t'^2+u'^2}{ss'}(v_p-v_q)^2, \tag{1.65}$$

and that for the process $q_1q_2 \to q_1q_2g$ given in [1].

2. Double Bremsstrahung in QED

To the lowest order there are twenty Feynman diagrams contributing to the double Bremsstrahlung process in QED [8]

$$e^+(p_+) + e^-(p_-) \longrightarrow \mu^+(q_+) + \mu^-(q_-) + \gamma(k_1) + \gamma(k_2) \tag{2.1}$$

as shown in Fig.3.

It is easy to prove that the diagrams in Fig.3 form four independent gauge-invariant subsets:

$$[(1) + \cdots + (6)]_{\varepsilon=k} = [(7) + \cdots + (12)]_{\varepsilon=k} =$$

$$= [(13) + \cdots + (16)]_{\varepsilon=k} = [(17) + \cdots + (20)]_{\varepsilon=k} = 0, \tag{2.2}$$

in which (1) is the abbreviation for $M_{(1)}$, etc., and $\varepsilon=k$ means that $\varepsilon_1 = k_1$ and $\varepsilon_2 = k_2$. The amplitude becomes then

$$M = [(1) + \cdots + (6)]_{1q,2q} + [(7) + \cdots + (12)]_{1p,2p} +$$

$$+ [(13) + \cdots + (16)]_{1p,2q} + [(17) + \cdots + (20)]_{1q,2p}, \tag{2.3}$$

where the subscript $1q,2q$ means that $\varepsilon_1 = \varepsilon_{1q}$, $\varepsilon_2 = \varepsilon_{2q}$, etc., and $\varepsilon_q\ (\varepsilon_p)$ is referred to the momentum q_+ or q_- (p_+ or p_-) depending on the helicities of the particles.

For the amplitude $M(-,+,+,-,+,+)$ where the signs represent the helicities of $e^+, e^-, \mu^+, \mu^-, \gamma_1$ and γ_2, respectively, and the photons have identical helicities, the choice of the polarization vectors

$$\not{\varepsilon}^+_{iq} = \not{\varepsilon}^+(k_i, q_-) = \frac{\sqrt{2}}{\langle q_- | k_i + \rangle}[\,|k_i -\rangle\langle q_- -| + |q_- +\rangle\langle k_i +|\,] \tag{2.4}$$

and

$$\not{\varepsilon}^+_{ip} = \not{\varepsilon}^+(k_i, p_-) = \frac{\sqrt{2}}{\langle p_- | k_i + \rangle}[\,|k_i -\rangle\langle p_- -| + |p_- +\rangle\langle k_i +|\,] \tag{2.5}$$

(i=1.2) leaves six amplitudes non-vanishing, corresponding to diagrams (1), (2), (7), (8), (13) and (17), due to the helicity match rule [8, 9]. The resulting amplitude is

$$M(-,+,+,-,+,+) = 4ie^4 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (A_{1p} - A_{1q})(A_{2p} - A_{2q}), \tag{2.6}$$

where

$$A_{ip} = \frac{\langle p_+ p_- \rangle}{\langle p_+ k_i \rangle \langle p_- k_i \rangle} \quad , \quad A_{iq} = \frac{\langle q_+ q_- \rangle}{\langle q_+ k_i \rangle \langle q_- k_i \rangle} \quad . \quad (i = 1, 2) \tag{2.7}$$

In comparison with the single Bremsstrahlung amplitude [9]

$$M(-,+,+,-,+) = 2\sqrt{2}\, ie^3 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (A_p - A_q) , \tag{2.8}$$

and the elastic scattering $(e^+e^- \to \mu^+\mu^-)$ amplitude

$$M(-,+,+,-) = 2ie^2 \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} , \tag{2.9}$$

an interesting rule follows: starting with the elastic scattering amplitude (2.9) for each additional bremsstrahlung photon the amplitude is multiplied by a factor

$$T = \sqrt{2}\, e (A_{ip} - A_{iq}), \tag{2.10}$$

in which A_{ip} and A_{iq} (2.7) are associated with the momentum k_i of the added photon. Therefore, the helicity amplitude for $e^+e^- \to \mu^+\mu^- + n\gamma$ where all photons have the same helicity is

$$M(-,+,+,-;+,+,\cdots,+) = i(\sqrt{2}\, e)^{n+2} \frac{\langle p_- q_- \rangle^2}{\langle p_+ p_- \rangle \langle q_+ q_- \rangle} (A_{1p} - A_{1q}) \cdots (A_{np} - A_{nq}). \tag{2.11}$$

This rule is also discussed in Ref.[8] in some different form.

As to the amplitudes with different helicities of the photons the calculation is more complicated, the result turns out to be:

$$\begin{aligned} M(-,+,+,-,+,-) &= -4ie^4 \cdot \\ &\cdot \Big\{ \frac{1}{s} \alpha_{1q} \alpha_{2q}^* \Big[\Big(A - \frac{C}{(q_- + k_1 + k_2)^2}\Big)\Big(B - \frac{D}{(q_+ + k_1 + k_2)^2}\Big) - \\ &\quad - \frac{4(s-s')(q_+ k_1)(q_- k_2)}{(q_+ + k_1 + k_2)^2 (q_- + k_1 + k_2)^2} \langle p_- q_- \rangle \langle p_+ q_+ \rangle^* \Big] + \\ &+ \frac{1}{s'} \alpha_{1p} \alpha_{2p}^* \Big[\Big(A - \frac{C'}{(p_+ - k_1 - k_2)^2}\Big)\Big(B - \frac{D'}{(p_- - k_1 - k_2)^2}\Big) - \\ &\quad - \frac{4(s-s')(p_+ k_1)(p_- k_2)}{(p_+ - k_1 - k_2)^2 (p_- - k_1 - k_2)^2} \langle p_- q_- \rangle \langle p_+ q_+ \rangle^* \Big] + \\ &+ \frac{1}{s_2} \alpha_{1q} \alpha_{2p}^* A^2 + \frac{1}{s_1} \alpha_{1p} \alpha_{2q}^* B^2 \Big\}, \end{aligned} \tag{2.12}$$

in which

$$\alpha_{ip} = \frac{1}{\langle p_+ k_i \rangle \langle p_- k_i \rangle}\,, \qquad \alpha_{iq} = \frac{1}{\langle q_+ k_i \rangle \langle q_- k_i \rangle}\,, \qquad (i = 1, 2) \tag{2.13}$$

$$\left.\begin{aligned}
A &= \langle p_+ + | (\not{q}_+ + \not{k}_1) | q_- + \rangle = \langle q_- - | (\not{p}_- - k_2) | p_+ - \rangle, \\
B &= \langle q_+ + | (\not{q}_- + \not{k}_2) | p_- + \rangle = \langle p_- - | (\not{p}_+ - \not{k}_1) | q_+ - \rangle, \\
C &= \langle p_+ + | \not{q}_+ \not{k}_1 \not{k}_2 | q_- + \rangle, \\
D &= \langle q_+ + | \not{k}_1 \not{k}_2 \not{q}_- | p_- + \rangle, \\
C' &= \langle p_+ + | \not{k}_1 \not{k}_2 \not{p}_- | q_- + \rangle, \\
D' &= \langle q_+ + | \not{p}_+ \not{k}_1 \not{k}_2 | p_- + \rangle,
\end{aligned}\right\} \tag{2.14}$$

$$\left.\begin{aligned}
&s = (p_+ + p_-)^2, \qquad\qquad s' = (q_+ + q_-)^2, \\
&s_1 = (p_+ + p_- - k_1)^2 = (q_+ + q_- + k_2)^2, \quad s_2 = (p_+ + p_- - k_2)^2 = (q_+ + q_- + k_1)^2.
\end{aligned}\right\} \tag{2.15}$$

They are consistent with those given in [8], if the phase difference in the polarization vectors used here from those in [8] is taken into account. But numerical computation shows that the norms of amplitudes with different photon helicities are about one fifth or less of that with identical photon helicities for most of the final angular configurations. As approximation we ignore their contributions and get for the average norm squared a compact expression [8]:

$$\overline{|M(e^+e^- \to \mu^+\mu^-\gamma\gamma)|^2} = \frac{1}{2} e^8 \, \frac{t^2 + u^2 + t'^2 + u'^2}{s s'} \, (v_{1p} - v_{1q})^2 \, (v_{2p} - v_{2q})^2, \tag{2.16}$$

where

$$v_{ip} = \frac{p_+}{(p_+ k_i)} - \frac{p_-}{(p_- k_i)}\,, \qquad v_{iq} = \frac{q_+}{(q_+ k_i)} - \frac{q_-}{(q_- k_i)}\,. \qquad (i = 1, 2) \tag{2.17}$$

3. Double Bremsstrahlung in QCD

Among the thirty-six Feynman diagrams contributing to the QCD process

$$\bar{q}(p_+) + q(p_-) \longrightarrow \bar{q}'(q_+) + q'(q_-) + g(k_1) + g(k_2), \tag{3.1}$$

there are twenty-four QED-like diagrams shown in Fig.4, and eight diagrams with one gluon emitted from the internal gluon line shown in Fig.5, and four with both gluons emitted from the internal gluon line shown in Fig.6. The diagrams in Fig.4 through Fig.6 are referred to as the diagrams with naught, single and double internal radiations, and denoted by $M^{(0)}$, $M^{(1)}$ and $M^{(2)}$, respectively.

Just as in QED we can seperate $M^{(0)}$ into four subsets: $(0)_{qq}$, $(0)_{pp}$, $(0)_{pq}$ and $(0)_{qp}$, as shown in Fig 4(1) through (4), in which $(0)_{qq}$ is the abbreviation of $M^{(0)}_{1q,2q}$ calculated with $\varepsilon_1=\varepsilon_{1q}$, $\varepsilon_2=\varepsilon_{2q}$, etc. But each set is no longer gauge-invariant:

$$[(0)_{qq}]_{\varepsilon=k} \neq 0, \qquad \text{etc.} \tag{3.2}$$

Contributions from single and double internal radiation diagrams must be included in order to form gauge-invariant subsets.

We apply the decomposition rule for single Bremsstrahlung process (eq.(1.18) and (1.19)) to the single internal radiation diagrams, that is, for each diagram with a piece shown in Fig.7 we decompose the amplitude into two terms,

$$R = R_q + R_p , \tag{3.3}$$

where

$$R_q = -\frac{Q'^2}{P'^2 - Q'^2} R , \tag{3.4}$$

$$R_p = \frac{P'^2}{P'^2 - Q'^2} R, \tag{3.5}$$

and calculate R_q by putting $\varepsilon=\varepsilon_q$ and R_p by putting $\varepsilon=\varepsilon_p$. Here P' and Q' are not necessarily $p_+ + p_-$ and $q_+ + q_-$. So the single internal radiation amplitude $M^{(1)}$ are also decomposed into four subsets: $(1)_{qq}$, $(1)_{pp}$, $(1)_{pq}$ and $(1)_{qp}$. The sum of the corresponding subsets of $M^{(0)}$ and $M^{(1)}$ is not yet gauge-invariant:

$$[(0)_{qq}+(1)_{qq}]_{\varepsilon=k}\neq 0, \qquad \text{etc.} \tag{3.6}$$

The crucial step here is to decompose the amplitude $M^{(2)}$ into four parts:

$$(2)=(2)_{qq}+(2)_{pp}+(2)_{pq}+(2)_{qp} \tag{3.7}$$

such that

$$[(0)_{qq}+(1)_{qq}+(2)_{qq}]_{\varepsilon=k}=0, \qquad \text{etc.} \tag{3.8}$$

The decomposition of $M^{(2)}$ is to cancel the following factors respectively when ε is put being k :

$$[(0)_{qq}+(1)_{qq}]_{\varepsilon=k}=-\frac{1}{s}\left[\frac{1}{2}F_{12}G+\frac{1}{2}F_{21}G-\frac{1}{2(Pk_1)}F_{12}K_{12}-\frac{1}{2(Pk_2)}F_{21}K_{21}\right], \tag{3.9}$$

$$[(0)_{pp}+(1)_{pp}]_{\varepsilon=k}=-\frac{1}{s'}\left[\frac{1}{2}F_{12}G+\frac{1}{2}F_{21}G+\frac{1}{2(Qk_2)}F_{12}K_{12}+\frac{1}{2(Qk_1)}F_{21}K_{21}\right], \tag{3.10}$$

$$[(0)_{pq}+(1)_{pq}]_{\varepsilon=k}=\frac{1}{s_1}\left[F_{12}G-\left(\frac{1}{2(Pk_1)}-\frac{1}{2(Qk_2)}\right)F_{12}K_{12}\right], \tag{3.11}$$

$$[(0)_{qp}+(1)_{qp}]_{\varepsilon=k}=\frac{1}{s_2}\left[F_{21}G-\left(\frac{1}{2(Pk_2)}-\frac{1}{2(Qk_1)}\right)F_{21}K_{21}\right], \tag{3.12}$$

where

$$F_{12}=f_{c_1ae}f_{c_2be}T_a(q)T_b(q'), \tag{3.13}$$

$$F_{21}=f_{c_2ae}f_{c_1be}T_a(q)T_b(q'), \tag{3.14}$$

$$G=g^{\mu\nu}W_{\mu\nu}, \tag{3.15}$$

$$K_{12}=k_1^{\mu}k_2^{\nu}W_{\mu\nu}, \tag{3.16}$$

$$K_{21}=k_2^{\mu}k_1^{\nu}W_{\mu\nu}, \tag{3.17}$$

$$W_{\mu\nu}=ig^4\,\bar{v}(p_+)\gamma_\mu u(p_-)\bar{u}(q_-)\gamma_\nu v(q_+), \tag{3.18}$$

and

$$P=p_++p_-, \qquad Q=q_++q_-, \tag{1.15}$$

$$\left.\begin{aligned} &s=P^2, \qquad s'=Q^2, \\ &s_1=(P-k_1)^2=(Q+k_2)^2, \qquad s_2=(P-k_2)^2=(Q+k_1)^2. \end{aligned}\right\} \tag{2.15}$$

There are four double internal radiation diagrams, as shown in Fig.6 (a) —— (d):

$$M_{(a)} = -\frac{1}{ss's_1} F_{12} W_{\mu\nu}\cdot[\ (s+s'-s_1-2s_2)\,\varepsilon_1^\mu\varepsilon_2^\nu - 4\,(\varepsilon_2 k_1)\,\varepsilon_1^\mu k_2^\nu - 4\,(\varepsilon_1 k_2)\,k_1^\mu\varepsilon_2^\nu +$$

$$+\ 4\,(\varepsilon_2 Q)(\varepsilon_1^\mu k_1^\nu - k_1^\mu\varepsilon_1^\nu) + 4\,(\varepsilon_1 P)(\varepsilon_2^\mu k_2^\nu - k_2^\mu\varepsilon_2^\nu) +$$

$$+\ 4\,(\varepsilon_1\varepsilon_2)\,k_1^\mu k_2^\nu - 4\,(\varepsilon_1 P)(\varepsilon_2 Q)\,g^{\mu\nu}\ ]\,, \tag{3.19}$$

$$M_{(b)} = -\frac{1}{ss's_2} F_{21} W_{\mu\nu}\cdot[\ (s+s'-s_2-2s_1)\,\varepsilon_2^\mu\varepsilon_1^\nu - 4\,(\varepsilon_2 k_1)\,k_2^\mu\varepsilon_1^\nu - 4\,(\varepsilon_1 k_2)\,\varepsilon_2^\mu k_1^\nu +$$

$$+\ 4\,(\varepsilon_1 Q)(\varepsilon_2^\mu k_2^\nu - k_2^\mu\varepsilon_2^\nu) + 4\,(\varepsilon_2 P)(\varepsilon_1^\mu k_1^\nu - k_1^\mu\varepsilon_1^\nu) +$$

$$+\ 4\,(\varepsilon_1\varepsilon_2)\,k_2^\mu k_1^\nu - 4\,(\varepsilon_1 Q)(\varepsilon_2 P)\,g^{\mu\nu}\ ]\,, \tag{3.20}$$

$$M_{(c)} = -\frac{1}{ss'(k_1+k_2)^2}\,(F_{12}-F_{21})\,W_{\mu\nu}\cdot$$

$$\cdot\{\ 4\,(\varepsilon_2 k_1)\,[\,(k_1+k_2)^\mu\varepsilon_1^\nu - \varepsilon_1^\mu(k_1+k_2)^\nu\,] +$$

$$+\,4\,(\varepsilon_1 k_2)\,[\,\varepsilon_2^\mu(k_1+k_2)^\nu - (k_1+k_2)^\mu\varepsilon_2^\nu\,] + 4\,(\varepsilon_1\varepsilon_2)(k_1^\mu k_2^\nu - k_2^\mu k_1^\nu) +$$

$$+\,[\,4(\varepsilon_1 Q)(\varepsilon_2 P) - 4\,(\varepsilon_1 P)(\varepsilon_2 Q) - 2\,(\varepsilon_1\varepsilon_2)\,((Pk_1)-(Pk_2))\,]\,g^{\mu\nu}\}, \tag{3.21}$$

$$M_{(d)} = -\frac{1}{ss'}\,W_{\mu\nu}\cdot[\ F_{12}\,(\varepsilon_1^\mu\varepsilon_2^\nu - 2\,\varepsilon_2^\mu\varepsilon_1^\nu + (\varepsilon_1\varepsilon_2)\,g^{\mu\nu}) -$$

$$-\,F_{21}\,(2\,\varepsilon_1^\mu\varepsilon_2^\nu - \varepsilon_2^\mu\varepsilon_1^\nu - (\varepsilon_1\varepsilon_2)\,g^{\mu\nu})\ ]\,, \tag{3.22}$$

where we have made use of the Bianchi identity

$$f_{c_1c_2e}\,f_{abe} = f_{c_1ae}\,f_{c_2be} - f_{c_2ae}\,f_{c_1be} \tag{3.23}$$

and the energy-momentum conservation

$$P = Q + k_1 + k_2 \tag{3.24}$$

to simplify the expressions. It is convenient to define $\widetilde{M}_{(c)}$ and $\widetilde{M}_{(d)}$ in terms of $M_{(c)}$ and $M_{(d)}$ such that

$$\widetilde{M}_{(c)} + \widetilde{M}_{(d)} = M_{(c)} + M_{(d)}\,, \tag{3.25}$$

where

$$\widetilde{M}_{(c)} = -\frac{1}{ss'} F_{12} W_{\mu\nu} \cdot \Big\{ \varepsilon_1^\mu \varepsilon_2^\nu - 2\,\varepsilon_2^\mu \varepsilon_1^\nu + \frac{2(\varepsilon_2 k_1)}{(k_1 k_2)} \left[(k_1+k_2)^\mu \varepsilon_1^\nu - \varepsilon_1^\mu (k_1+k_2)^\nu \right] +$$
$$+ \frac{2(\varepsilon_1 k_2)}{(k_1 k_2)} \left[\varepsilon_2^\mu (k_1+k_2)^\nu - (k_1+k_2)^\mu \varepsilon_2^\nu \right] + \frac{2(\varepsilon_1 \varepsilon_2)}{(k_1 k_2)} (k_1^\mu k_2^\nu - k_2^\mu k_1^\nu) +$$
$$+ \frac{1}{(k_1 k_2)} \left[2(\varepsilon_1 Q)(\varepsilon_2 P) - 2(\varepsilon_1 P)(\varepsilon_2 Q) - (\varepsilon_1 \varepsilon_2)(Q k_1) + (\varepsilon_1 \varepsilon_2)(P k_2) \right] g^{\mu\nu} \Big\}, \quad (3.26)$$

$$\widetilde{M}_{(d)} = \frac{1}{ss'} F_{21} W_{\mu\nu} \cdot \Big\{ 2\varepsilon_1^\mu \varepsilon_2^\nu - \varepsilon_2^\mu \varepsilon_1^\nu + \frac{2(\varepsilon_2 k_1)}{(k_1 k_2)} \left[(k_1+k_2)^\mu \varepsilon_1^\nu - \varepsilon_1^\mu (k_1+k_2)^\nu \right] +$$
$$+ \frac{2(\varepsilon_1 k_2)}{(k_1 k_2)} \left[\varepsilon_2^\mu (k_1+k_2)^\nu - (k_1+k_2)^\mu \varepsilon_2^\nu \right] + \frac{2(\varepsilon_1 \varepsilon_2)}{(k_1 k_2)} (k_1^\mu k_2^\nu - k_2^\mu k_1^\nu) +$$
$$+ \frac{1}{(k_1 k_2)} \left[2(\varepsilon_1 Q)(\varepsilon_2 P) - 2(\varepsilon_1 P)(\varepsilon_2 Q) + (\varepsilon_1 \varepsilon_2)(Q k_2) - (\varepsilon_1 \varepsilon_2)(P k_1) \right] g^{\mu\nu} \Big\}. \quad (3.27)$$

The new amplitudes with twiddles appear more symmetrical.

Now we decompose these diagrams in the following way:

$$\left.\begin{aligned} M_{(a)} &= a_1 M_{(a)} + a_2 M_{(a)} + a_3 M_{(a)} , \\ M_{(b)} &= b_1 M_{(b)} + b_2 M_{(b)} + b_4 M_{(b)} , \\ \widetilde{M}_{(c)} &= c_1 \widetilde{M}_{(c)} + c_2 \widetilde{M}_{(c)} + c_3 \widetilde{M}_{(c)} , \\ \widetilde{M}_{(d)} &= d_1 \widetilde{M}_{(d)} + d_2 \widetilde{M}_{(d)} + d_4 \widetilde{M}_{(d)} , \end{aligned}\right\} \quad (3.28)$$

with

$$a_1 + a_2 + a_3 = b_1 + b_2 + b_4 = c_1 + c_2 + c_3 = d_1 + d_2 + d_4 = 1 , \quad (3.29)$$

and put

$$\left.\begin{aligned} (2)_{gg} &= a_1 M_{(a)} + b_1 M_{(b)} + c_1 \widetilde{M}_{(c)} + d_1 \widetilde{M}_{(d)} , \\ (2)_{pp} &= a_2 M_{(a)} + b_2 M_{(b)} + c_2 \widetilde{M}_{(c)} + d_2 \widetilde{M}_{(d)} , \\ (2)_{pg} &= a_3 M_{(a)} + c_3 \widetilde{M}_{(c)} , \\ (2)_{gp} &= b_4 M_{(b)} + d_4 \widetilde{M}_{(d)} . \end{aligned}\right\} \quad (3.30)$$

Taking into account that

$$[M_{(a)}]_{\varepsilon=k} = -\frac{1}{ss's_1}[-4(Pk_1)(Qk_2)F_{12}G + (s_1-s-s')F_{12}K_{12}], \tag{3.31}$$

$$[M_{(b)}]_{\varepsilon=k} = -\frac{1}{ss's_2}[-4(Pk_2)(Qk_1)F_{21}G + (s_2-s-s')F_{21}K_{21}], \tag{3.32}$$

$$[\widetilde{M}_{(c)}]_{\varepsilon=k} = \frac{1}{ss'}[(-(Pk_1)+(Qk_2))F_{12}G + F_{12}K_{12}], \tag{3.33}$$

$$[\widetilde{M}_{(d)}]_{\varepsilon=k} = \frac{1}{ss'}[(-(Pk_2)+(Qk_1))F_{21}G + F_{21}K_{21}], \tag{3.34}$$

we set the coefficients of $F_{12}G$, $F_{21}G$, $F_{12}K_{12}$ and $F_{21}K_{21}$ in (3.8) equal to zero, respectively, in order to meet the requirement of separate gauge invariance of the subsets of diagrams, and obtain

$$\left.\begin{aligned}
&a_1 = \frac{1}{\alpha_1}\frac{s's_1}{2}\frac{(Qk_2)}{(Pk_1)}, \quad a_2 = \frac{1}{\alpha_1}\frac{ss_1}{2}\frac{(Pk_1)}{(Qk_2)}, \quad a_3 = \frac{1}{\alpha_1}\left(-\frac{ss'}{2}\right)\left[\frac{(Qk_2)}{(Pk_1)} + \frac{(Pk_1)}{(Qk_2)}\right],\\
&b_1 = \frac{1}{\alpha_2}\frac{s's_2}{2}\frac{(Qk_1)}{(Pk_2)}, \quad b_2 = \frac{1}{\alpha_2}\frac{ss_2}{2}\frac{(Pk_2)}{(Qk_1)}, \quad b_4 = \frac{1}{\alpha_2}\left(-\frac{ss'}{2}\right)\left[\frac{(Qk_1)}{(Pk_2)} + \frac{(Pk_2)}{(Qk_1)}\right],\\
&c_1 = \frac{1}{\alpha_1}\left(-\frac{s'}{2}\right)[s+2(Qk_2)], \quad c_2 = \frac{1}{\alpha_1}\left(-\frac{s}{2}\right)[s'-2(Pk_1)], \quad c_3 = \frac{1}{\alpha_1}ss',\\
&d_1 = \frac{1}{\alpha_2}\left(-\frac{s'}{2}\right)[s+2(Qk_1)], \quad d_2 = \frac{1}{\alpha_2}\left(-\frac{s}{2}\right)[s'-2(Pk_2)], \quad d_4 = \frac{1}{\alpha_2}ss',
\end{aligned}\right\} \tag{3.35}$$

in which

$$\alpha_1 = s(Pk_1) - s'(Qk_2), \tag{3.36}$$

$$\alpha_2 = s(Pk_2) - s'(Qk_1). \tag{3.37}$$

Thus the decomposition is accomplished.

We calculate the helicity amplitude with identical gluon helicities $M(-,+,+,-,+,+)$. There are twelve different colour factors in the amplitude and the notations (1.29) and (1.30) are to be used.

$$M(-,+,+,-,+,+) = Z\sum_{i=1}^{12} c_i D_i\,, \tag{3.38}$$

where

$$Z = 4ig^4\frac{\langle p_- q_-\rangle^2}{\langle p_+ p_-\rangle\langle q_+ q_-\rangle}\,, \tag{3.39}$$

and

$$c_1=\frac{1}{2}T_a(q)(T_a\{T_{c_1},T_{c_2}\})(q'), \qquad D_1=A(q_+,q_-,k_1)A(q_+,q_-,k_2),$$

$$c_2=\frac{1}{2}(\{T_{c_2},T_{c_1}\}T_a)(q)T_a(q'), \qquad D_2=A(p_+,p_-,k_1)A(p_+,p_-,k_2),$$

$$c_3=\frac{1}{2}T_a(q)(T_a[T_{c_1},T_{c_2}])(q'), \qquad D_3=B(q_+,q_-,k_1,k_2)-B(q_+,q_-,k_2,k_1),$$

$$c_4=\frac{1}{2}([T_{c_2},T_{c_1}]T_a)(q)T_a(q'), \qquad D_4=B(p_+,p_-,k_1,k_2)-B(p_+,p_-,k_2,k_1),$$

$$c_5=(T_{c_1}T_a)(q)(T_aT_{c_2})(q'), \qquad D_5=-A(p_+,p_-,k_1)A(q_+,q_-,k_2),$$

$$c_6=(T_{c_2}T_a)(q)(T_aT_{c_1})(q'), \qquad D_6=-A(q_+,q_-,k_1)A(p_+,p_-,k_2),$$

$$c_7=([T_{c_1},T_a])(q)(T_aT_{c_2})(q'), \qquad D_7=-A(p_-,q_-,k_1)A(q_+,q_-,k_2),$$

$$c_8=(T_{c_2}T_a)(q)([T_a,T_{c_1}])(q'), \qquad D_8=A(p_-,q_-,k_1)A(p_+,p_-,k_2),$$

$$c_9=(T_{c_1}T_a)(q)([T_a,T_{c_2}])(q'), \qquad D_9=A(p_+,p_-,k_1)A(p_-,q_-,k_2),$$

$$c_{10}=([T_{c_2},T_a])(q)(T_aT_{c_1})(q'), \qquad D_{10}=-A(q_+,q_-,k_1)A(p_-,q_-,k_2),$$

$$c_{11}=([T_{c_1},T_a])(q)([T_a,T_{c_2}])(q'),$$

$$\begin{aligned}D_{11}=\frac{1}{\alpha_1}\{&-2(Qk_2)[(q_+k_1)B(q_+,q_-,k_1,k_2)-(q_+k_2)B(q_+,q_-,k_2,k_1)]+\\&+2(Pk_1)[(p_+k_1)B(p_+,p_-,k_1,k_2)-(p_+k_2)B(p_+,p_-,k_2,k_1)]+\\&+(2(Qk_2)-s)(q_+k_2)A(p_-,q_-,k_1)A(q_+,q_-,k_2)+\\&+(-2(Pk_1)-s')(p_+k_1)A(p_+,p_-,k_1)A(p_-,q_-,k_2)+\\&-4(p_+k_1)(q_+k_2)A(p_+,p_-,k_1)A(q_+,q_-,k_2)\},\end{aligned}$$

$$c_{12}=([T_{c_2},T_a])(q)([T_a,T_{c_1}])(q'),$$

$$\begin{aligned}D_{12}=\frac{1}{\alpha_2}\{&+2(Qk_1)[(q_+k_1)B(q_+,q_-,k_1,k_2)-(q_+k_2)B(q_+,q_-,k_2,k_1)]+\\&-2(Pk_2)[(p_+k_1)B(p_+,p_-,k_1,k_2)-(p_+k_2)B(p_+,p_-,k_2,k_1)]+\\&+(2(Qk_1)-s)(q_+k_1)A(q_+,q_-,k_1)A(p_-,q_-,k_2)+\\&+(-2(Pk_2)-s')(p_+k_2)A(p_-,q_-,k_1)A(p_+,p_-,k_2)+\\&-4(q_+k_1)(p_+k_2)A(q_+,q_-,k_1)A(p_+,p_-,k_2)\},\end{aligned}$$

and

$$A\,(p_1, p_2, k) = \frac{\langle p_1 p_2 \rangle}{\langle p_1 k \rangle \langle p_2 k \rangle}\ , \tag{3.41}$$

$$B\,(p_1, p_2, k_1, k_2) = \frac{\langle p_1 p_2 \rangle}{\langle p_1 k_1 \rangle \langle p_2 k_2 \rangle \langle k_1 k_2 \rangle}\ . \tag{3.42}$$

4. Evaluation of the amplitudes and their norms

The helicity amplitude (3.38) with (3.39) —— (3.42) are expressed in terms of the basic spinor product $\langle pq \rangle$ and colour factors. For easy numerical evaluation of $\langle pq \rangle$ we introduce Weyl representation of γ-matrices and the notations $k_\pm$ and $k_\perp$ for momentum k^μ [5,7,8]:

$$\gamma^0 = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \qquad \gamma^i = \begin{pmatrix} 0 & -\sigma^i \\ \sigma^i & 0 \end{pmatrix}, \qquad (i=1,2,3) \tag{4.1}$$

and

$$k_\pm = k_0 \pm k_3\ , \qquad k_\perp = k_x + i k_y = |k_\perp|\, e^{i\phi_k} = \sqrt{k_+ k_-}\, e^{i\phi_k}\ , \tag{4.2}$$

where $\mathbb{1}$ is the 2×2 unit matrix and σ^i (i=1.2.3) the Pauli matrices. By suitable choice of the phase we get [5]

$$|k+\rangle = \begin{pmatrix} \sqrt{k_+} \\ \sqrt{k_-}\, e^{i\phi_k} \\ 0 \\ 0 \end{pmatrix}, \qquad |k-\rangle = \begin{pmatrix} 0 \\ 0 \\ \sqrt{k_-}\, e^{-i\phi_k} \\ -\sqrt{k_+} \end{pmatrix}, \tag{4.3}$$

and

$$\langle k_1 k_2 \rangle \equiv \langle k_1 - | k_2 + \rangle = \sqrt{k_{1-} k_{2+}}\, e^{i\phi_1} - \sqrt{k_{1+} k_{2-}}\, e^{i\phi_2}\ . \tag{4.4}$$

It satisfies the relation

$$\langle k_1 k_2 \rangle = -\langle k_2 k_1 \rangle . \tag{4.5}$$

Therefore the D_i's in (3.38) can be computed. The square of the amplitude (3.38) is then

$$|M(-,+,+,-,+,+)|^2 = 16\, g^8 \frac{t'^2}{ss'}\, \mathscr{D}^\dagger \mathscr{C} \mathscr{D}, \tag{4.6}$$

in which

$$\mathscr{D} = \begin{pmatrix} D_1 \\ D_2 \\ \cdot \\ \cdot \\ \cdot \\ D_{12} \end{pmatrix} , \qquad \mathscr{D}^\dagger = (D_1^*, D_2^*, \cdots, D_{12}^*), \tag{4.7}$$

and

$$\mathscr{C} = (C_{ij}) = (c_i^* c_j) . \qquad (i,j = 1, \cdots, 12) \tag{4.8}$$

Since

$$(T_a \cdots T_b)(\xi) [(T_c \cdots T_d)(\xi)]^* = Tr (T_a \cdots T_b T_d \cdots T_c), \tag{4.9}$$

the element $c_i^* c_j$ is a product of two spurs of product of several generators T_a, and by repeatedly making use of the identities

$$T_a T_b = \frac{1}{2N} \delta_{ab}\, I + \frac{1}{2} (d_{abc} + i f_{abc}) T_c \tag{1.20}$$

and

$$Tr(I) = N \quad , \qquad Tr(T_a) = 0 , \tag{4.10}$$

they are reduced to contractions of the tensors δ_{ab} , d_{abc} , and f_{abc}, which can be calculated by the method given in the Appendix. We find that $c_i^* c_j$ is a polynomial in (N^2-1) up to the third power divided by N^2, and we can retain only the highest term in the numerator since $(N^2-1) \gg 1$. So the matrix $\mathscr{C}$ becomes

$$\mathscr{C} = \frac{(N^2-1)^3}{32 N^2}\, \mathscr{C}' , \tag{4.11}$$

$$\mathcal{C}' = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 & 1 & 1 \\ 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & -1 & 1 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & -1 & 1 & 0 & 1 & -1 \\ 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 & 0 \\ 0 & 0 & 0 & 0 & 0 & 2 & 0 & 2 & 0 & 2 & 0 & 2 \\ 1 & 1 & -1 & 0 & 2 & 0 & 4 & 2 & 2 & 0 & 4 & 2 \\ 1 & 1 & 0 & -1 & 0 & 2 & 2 & 4 & 0 & 2 & 2 & 4 \\ 1 & 1 & 0 & 1 & 2 & 0 & 2 & 0 & 4 & 2 & 4 & 2 \\ 1 & 1 & 1 & 0 & 0 & 2 & 0 & 2 & 2 & 4 & 2 & 4 \\ 1 & 1 & -1 & 1 & 2 & 0 & 4 & 2 & 4 & 2 & 8 & 4 \\ 1 & 1 & 1 & -1 & 0 & 2 & 2 & 4 & 2 & 4 & 4 & 8 \end{pmatrix} \qquad (4.12)$$

The amplitudes with different helicities of gluons are much more complicated, but can be ignored according to the reason explained in Section 2. Therefore the unpolarized, colour-averaged norm squared of the process (3.1) is

$$\overline{|M|^2} = g^8 \frac{(N^2-1)^3}{4N^4} \frac{t^2+u^2+t'^2+u'^2}{s s'} \mathcal{D}^{\dagger} \mathcal{C}' \mathcal{D}. \qquad (4.13)$$

5. Conclusion

In the conventional formalism of QCD it is very difficult to find compact expressions of the amplitudes for multiple Bremsstrahlung processes. The method proposed by CALKUL collaboration promotes the solution of this problem. By means of the new polarization vectors and the decomposition of gauge-invariant subsets we achieve satisfactory results for single and double Bremsstrahlung in QCD. The method of decomposition of the amplitudes into gauge-invariant subsets is systematic and can be generalized to higher Bremsstrahlung processes. More features concerning higher internal radiation diagrams will be discussed in a subsequent paper [10].

Appendix

SU(N) as Gauge Group

Suppose that T_a ($a=1,\cdots,N^2-1$) is the hermitian generator of the matrix group SU(N) satisfying Lie relation

$$[T_a, T_b] = i f_{abc} T_c , \qquad (a, b, c = 1, \cdots, N^2-1) \tag{A.1}$$

in which f_{abc} is the antisymmetric structure constant. T_a is traceless:

$$Tr(T_a) = 0 \tag{A.2}$$

and normalized by

$$Tr(T_a T_b) = \frac{1}{2}\delta_{ab} . \tag{A.3}$$

We have another relation

$$\{T_a, T_b\} = \frac{1}{N}\delta_{ab} I + d_{abc} T_c , \tag{A.4}$$

in which d_{abc} is symmetric, thus

$$T_a T_b = \frac{1}{2N}\delta_{ab} I + \frac{1}{2}(d_{abc} + i f_{abc}) T_c , \tag{A.5}$$

$$T_a T_a = \frac{N^2-1}{2N} I . \tag{A.6}$$

From the following identities

$$\left.\begin{aligned}
&[[A, B], C] + [[B, C], A] + [[C, A], B] = 0 ,\\
&\{\{A, B\}, C\} - \{\{B, C\}, A\} - [[C, A], B] = 0 ,\\
&\{[A, B], C\} - \{[B, C], A\} - [\{C, A\}, B] = 0 ,\\
&[\{A, B\}, C] + [\{B, C\}, A] + [\{C, A\}, B] = 0 ,
\end{aligned}\right\} \tag{A.7}$$

we get

$$\left.\begin{aligned}
&f_{abe} f_{ecd} + f_{ade} f_{ebc} + f_{ace} f_{edb} = 0 , \qquad \text{(Bianchi)}\\
&\frac{2}{N}(\delta_{ab}\delta_{cd} - \delta_{ad}\delta_{bc}) + d_{abe} d_{ecd} - d_{ade} d_{ebc} + f_{ace} f_{edb} = 0 ,\\
&f_{abe} d_{ecd} + f_{ade} d_{ebc} + f_{ace} d_{edb} = 0 ,\\
&d_{abe} f_{ecd} - d_{ade} f_{ebc} - f_{ace} d_{edb} = 0 .
\end{aligned}\right\} \tag{A.8}$$

What we need is contractions of the tensors δ_{ab}, f_{abc} and d_{abc}. It is known that

$$\delta_{aa} = N^2 - 1, \tag{A.9}$$

$$f_{abc} f_{abc'} = N \delta_{cc'}, \tag{A.10}$$

and by making use of the above relations it is easy to get all the contractions of up to four tensors f and d.

A schematic symbol is helpful here. The tensor f_{abc} or d_{abc} can be represented by a circle with three legs, see Fig. A(1). In the case of antisymmetric f the circle is directed and we put an arrow to indicate the order of its indices. The legs can rotate freely, but exchanging of two legs or reversing of the arrow direction leads to a minus sign in f. More circles appearing in the same scheme represent the product of the tensors. When two indices are contracted the corresponding legs are connected together, see Fig. A(2) for some examples. If all the indices are contracted out there will be no free leg in the scheme and it is a scalar.

In this formulation the contractions we need in calculating the norms of amplitudes for single and double Bremsstrahlung processes are shown in Fig. A(3). Note that for scalar the tensors f (d) must appear in pairs, otherwise the contraction is zero.

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Figure Captions

Fig.1 Feynman diagrams of $e^+e^- \to \mu^+\mu^-\gamma$.

Fig.2 Feynman diagrams of $\bar{q}q \to \bar{q}'q'g$.

Fig.3 Feynman diagrams of $e^+e^- \to \mu^+\mu^-\gamma\gamma$.

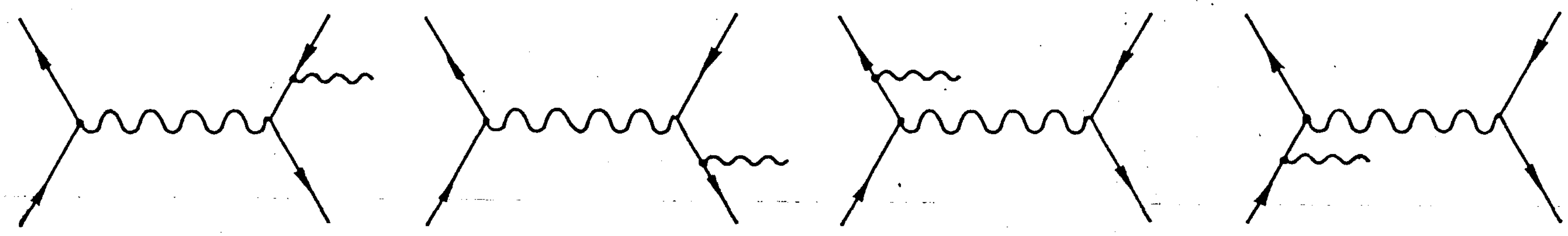
Fig.4 Naught internal radiation (QED-like) diagrams of $\bar{q}q \to \bar{q}'q'gg$. (1) qq-subset, (2) pp-subset, (3) pq-subset, (4) qp-subset.

Fig.5 Single internal radiation diagrams of $\bar{q}q \to \bar{q}'q'gg$.

Fig.6 Double internal radiation diagrams of $\bar{q}q \to \bar{q}'q'gg$.

Fig.7 Decomposition of single internal radiation diagram.

Fig.A Schematic representation of colour tensors and their contractions. (1) Tensors f and d. (2) Contractions of tensors. (3) Values of contractions needed in single and double Bremsstrahlung processes.



(1) (2) (3) (4)

FIG. 1

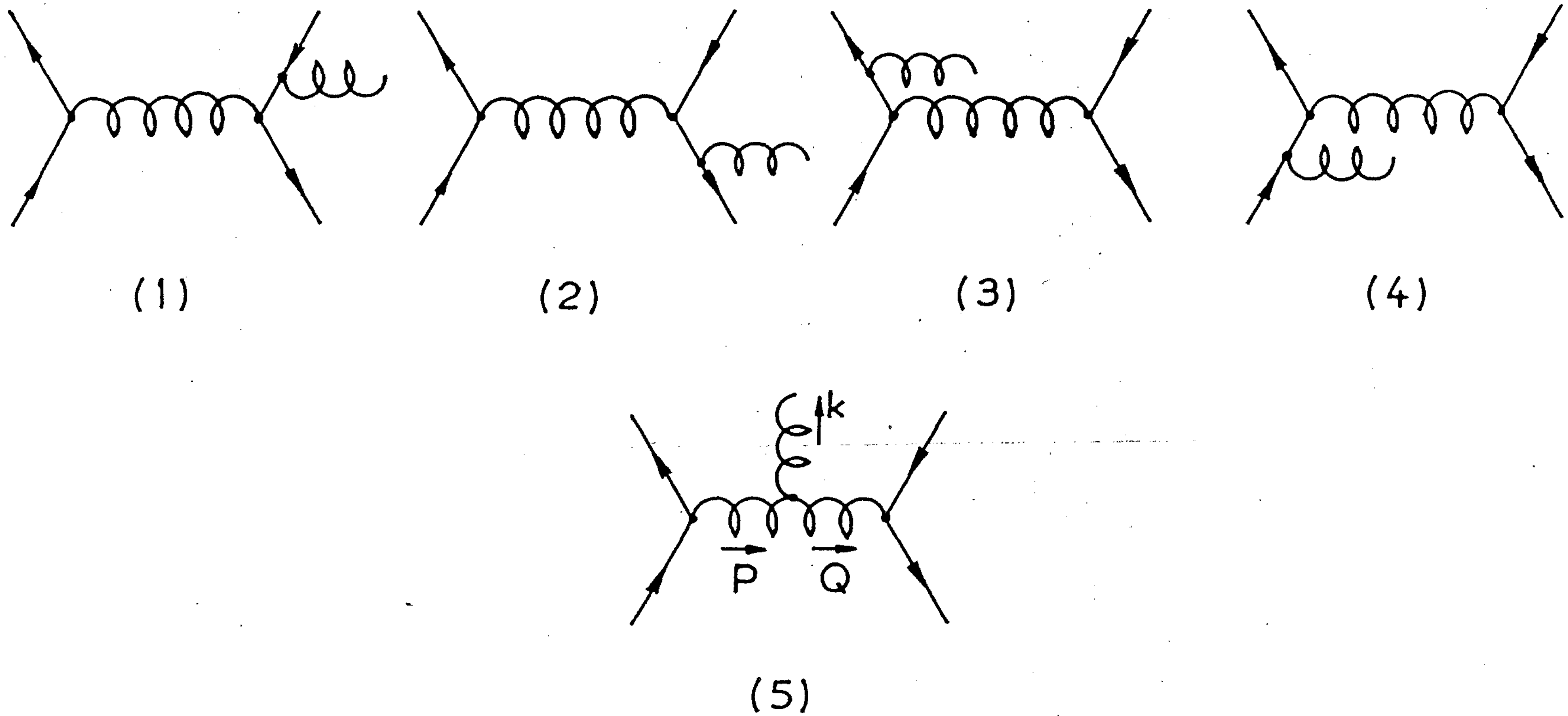


FIG. 2

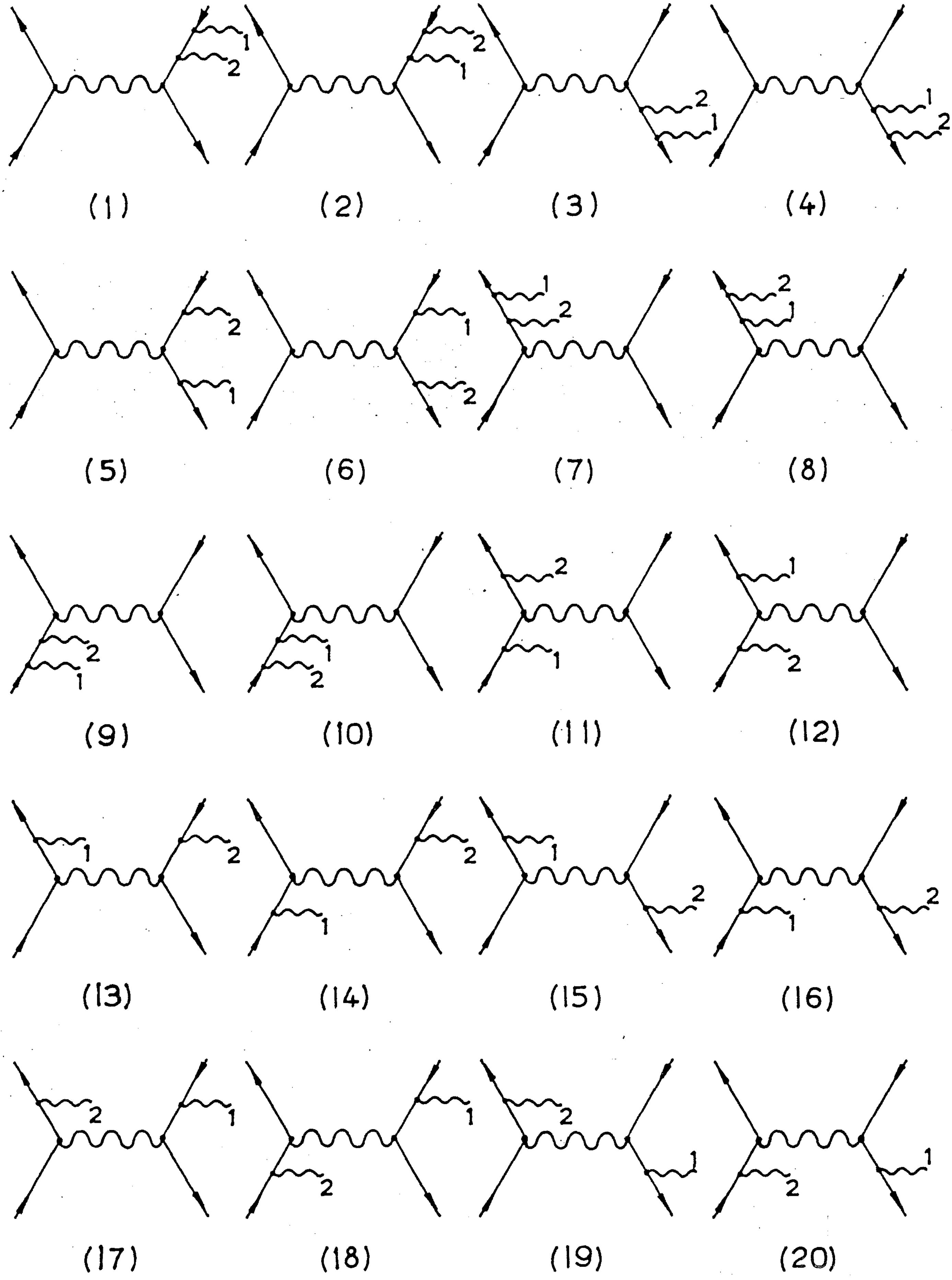
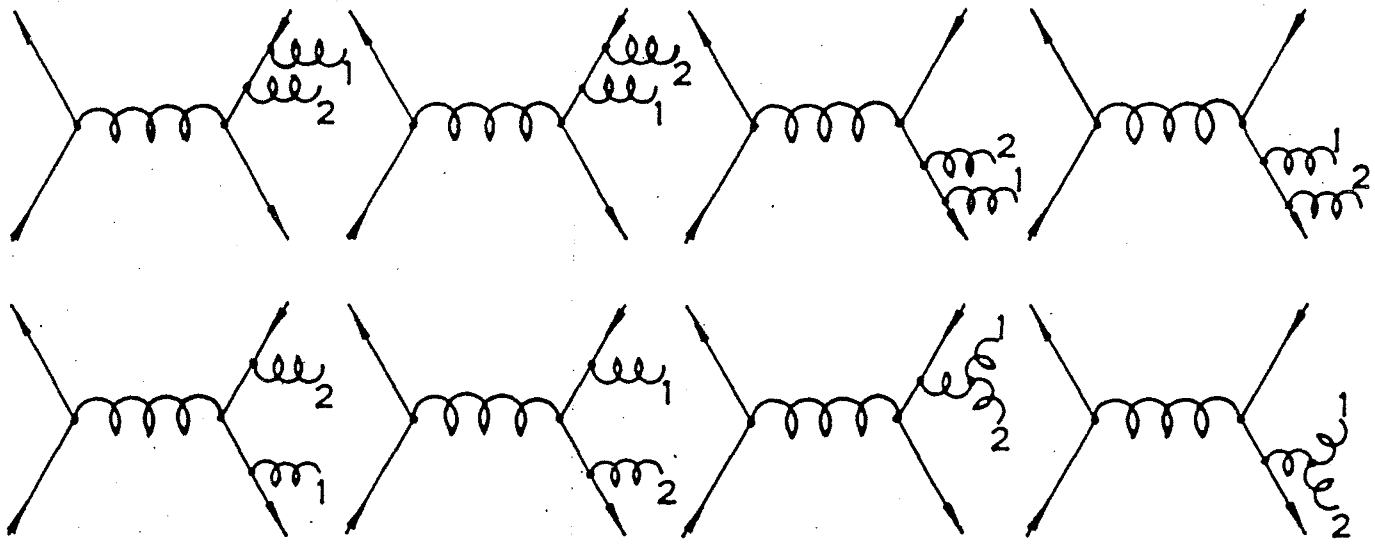
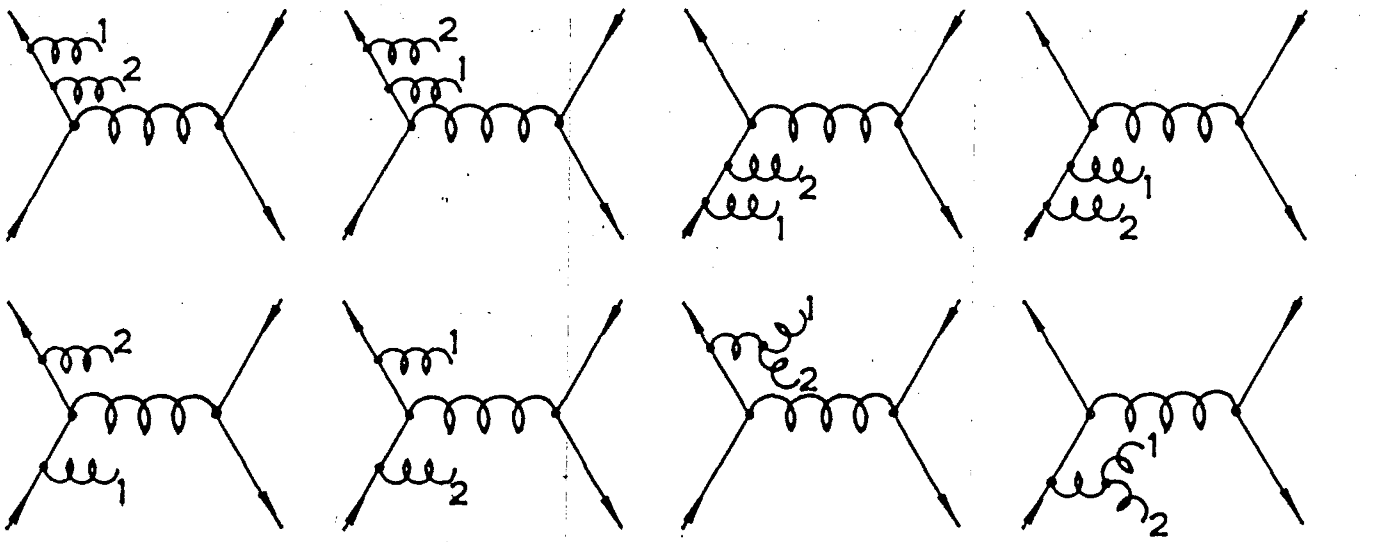


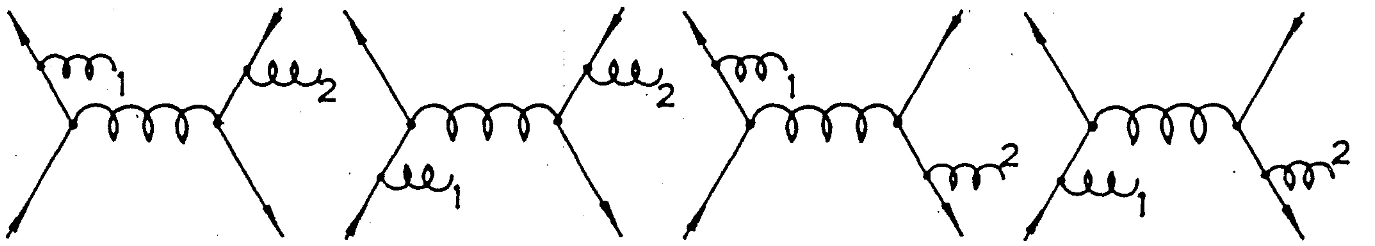
FIG. 3



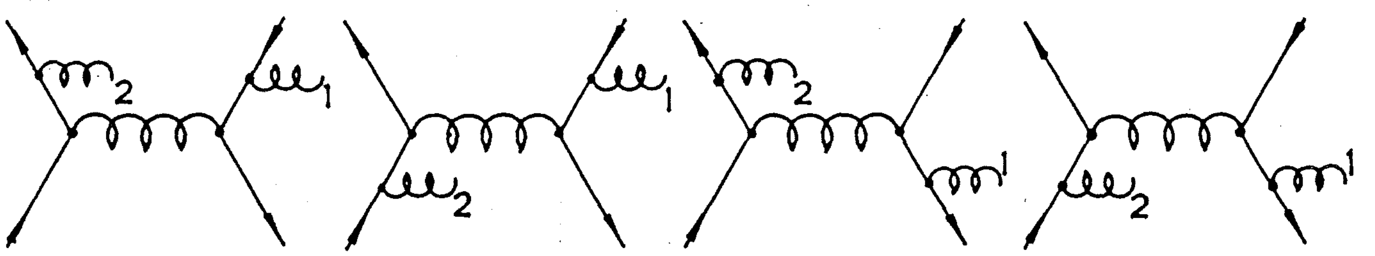
(1)



(2)



(3)



(4)

FIG. 4

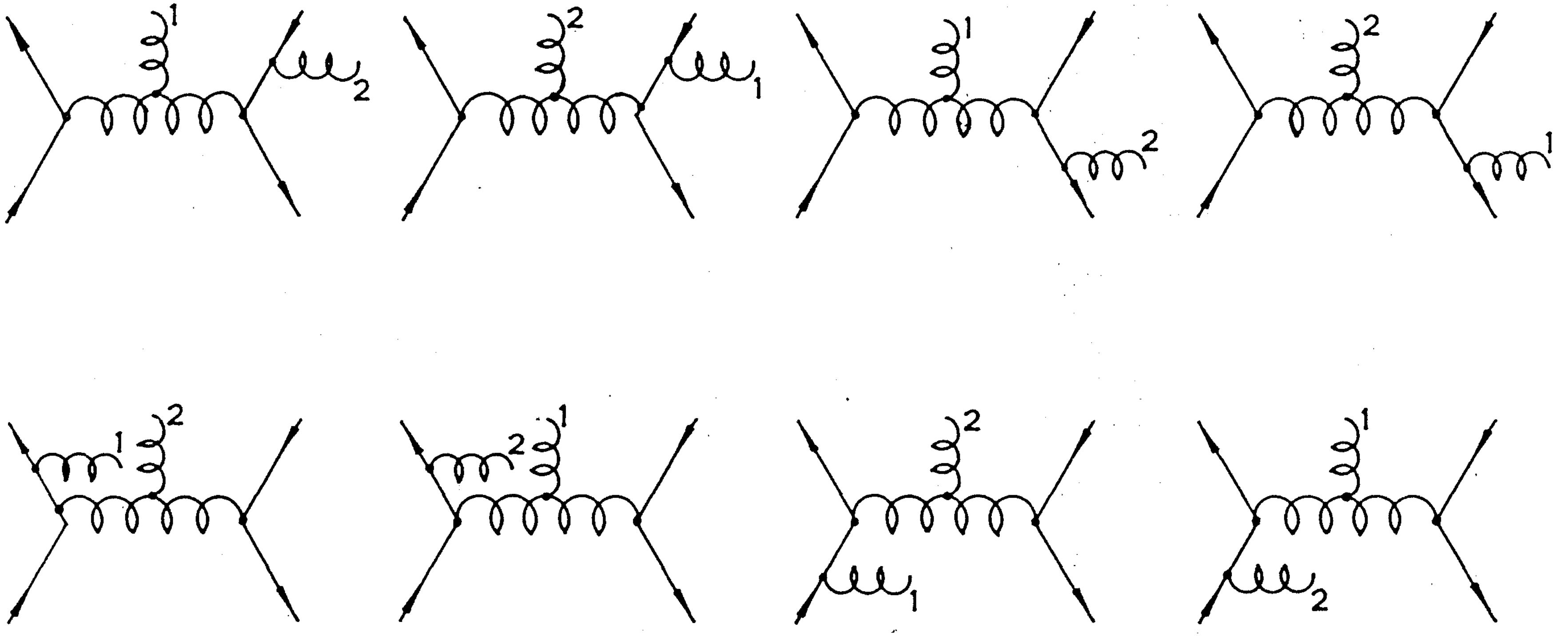
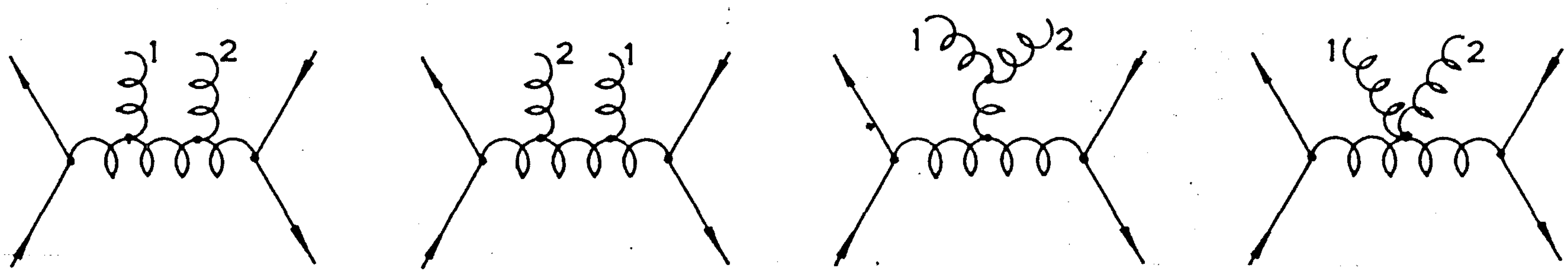
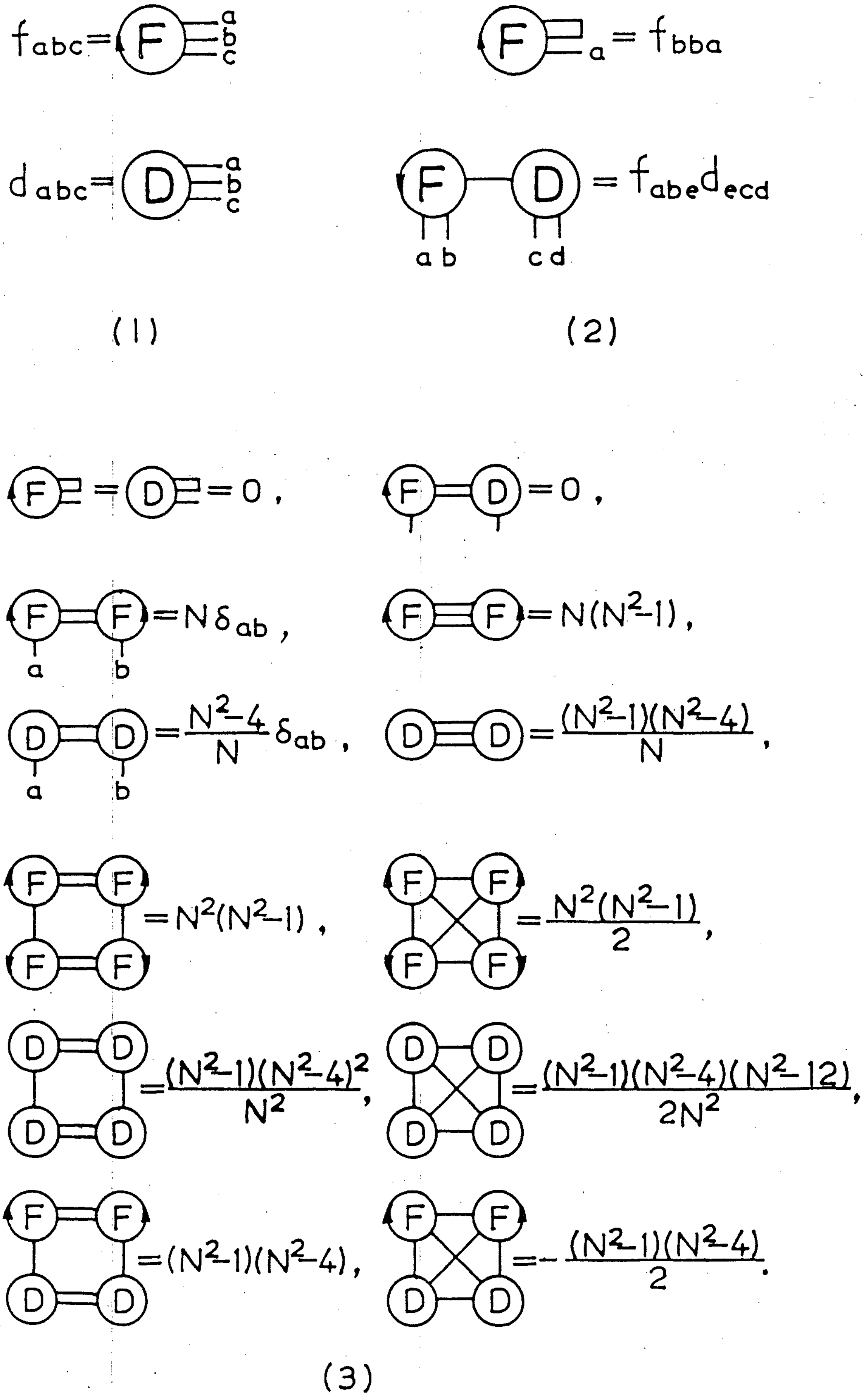


FIG. 5



(a) (b) (c) (d)

FIG. 6


f abc =
F
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d abc =
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= 0 ,
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b
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= N(N2-1) ,
D
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= (N2-4)/N δ ab ,
a
b
D
D
= (N2-1)(N2-4)/N ,
= N2(N2-1) ,
= N2(N2-1)/2 ,
= (N2-1)(N2-4)2/N2 ,
= (N2-1)(N2-4)(N2-12)/2N2 ,
= (N2-1)(N2-4) ,
= - (N2-1)(N2-4)/2 .
(3)

FIG. A

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Jan. 1985

HELICITY AMPLITUDES FOR MULTIPLE BREMSSTRAHLUNG IN MASSLESS NON-ABELIAN GAUGE THEORY(III): AMPLITUDES OF MULTIPLE BREMSSTRAHLUNG PROCESSES INVOLVING GLUON SELF-COUPLING VERTICES*)

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1. Introduction

In order to apply the helicity amplitude formalism developed by the CALKUL collaboration [1] to QCD processes with multiple gluon Bremsstrahlung, we adopt the polarization vectors defined and the method developed in [2] . Two problems are yet to be solved before real simplification for the amplitudes can be achieved. These are: (1) treatment of the gluon self-coupling vertices, or more generally non-Abelian gauge boson self-coupling vertices, and (2) decomposition of the ampli-

ABSTRACT

The method developed on the basis of helicity amplitude formalism proposed by CALKUL collaboration is applied to QCD processes involving gluon self-coupling vertices. A general reduction formula is obtained in which a tree diagram involving gluon self-coupling vertices is related to a corresponding diagram with gluons attached to a fermion line in the case of same helicity of the gluons. The case of different gluon helicities is more complicated just as in QED. We demonstrate in an example that the formalism again leads to significant simplification. When combined with the decomposition of amplitudes into gauge invariant subsets, the present formalism provides a general framework of calculating amplitudes of QCD processes involving tree diagrams with multiple gluon Bremsstrahlung.

tudes into gauge invariant subsets of diagrams in the case of more than one fermion line present in the diagrams.

The formalism has been proved very efficient in QED as well as in QCD with gluon-quark coupling [1]. It remains to be studied how gluon self-coupling vertices behave in the formalism. Attempts to relate amplitudes with gluon self-coupling with those with quark-gluon conpling only were proved to be successful, which ensures that the simplifying program works for amplitudes with gluon self coupling too. In Sec. 3 and Sec. 4 we discuss the reductions of amplitudes of processes involving tree diagrams with gluons of the same and diffrent helicities respectively. In Sec. 5 we illustrate the method in the example of $W^+ \to u + \bar{d} + \gamma + \gamma$ in which the non-Abelian couplings $W\gamma\gamma$ and $WW\gamma\gamma$ are operative. In Sec. 2 the definition of polarization vector for a massive boson (or for an off-shell massless boson) is given. The problem of decomposition of amplitudes into gauge invariant subsets of diagrams with gluon self-coupling is disentangled in [3].

2. Polarization Vector for a Massive Boson

The polarization vector of a gluon is defined to be [2]

$$\epsilon^{\pm\mu}(k,r) = \frac{\pm\langle r\mp|\gamma^\mu|k\mp\rangle}{\sqrt{2}\langle r\mp|k\pm\rangle} = \frac{\pm\langle k\pm|\gamma^\mu|r\pm\rangle}{\sqrt{2}\langle r\mp|k\pm\rangle},$$

or

$$\not{\epsilon}^{\pm}(k,r) = \frac{\pm\sqrt{2}\left(|r\pm\rangle\langle k\pm| + |k\mp\rangle\langle r\mp|\right)}{\langle r\mp|k\pm\rangle}. \tag{2.1}$$

This definition is essentially to fix the gauge choice for the polarization vector of the gluon of momentum k with the help of a reference momentum r on the light cone. Vectors k and r, both on the light cone, enter into the definition (2.1) in a symmetrical way. It follows from (2.1) that

$$r\cdot\epsilon^{\pm}(k,r) = k\cdot\epsilon^{\pm}(k,r) = 0\,, \tag{2.2}$$

and consequently $(\alpha r+\beta k)\cdot\epsilon^{\pm}(k,r)=0$ for any constants α and β. For different choice of r, ϵ differs only by a vector proportional to k.

The 4-momentum K of a massive vector boson can be written as

$$K = k+\alpha r\,, \tag{2.3}$$

with

$$K^2 = 2\alpha\, k\cdot r > 0\,. \tag{2.4}$$

Actually, given any time-like K and fixed reference momentum $r\ (r^2=0)$, there always exists a scalar $\frac{K^2}{2K\cdot r}\equiv\alpha$ such that $K-\alpha r\equiv k$ is a 4-momentum on the light cone

$$K-\frac{K^2}{2K\cdot r}\,r = k, \qquad k^2=0\,. \tag{2.5}$$

The correspondence between the vectors $K\ (K^2\neq 0)$ and $k\ (k^2=0)$ through the reference momentum $r\ (r^2=0)$ given by (2.5) will be repeatedly used where momenta on the light cone are denoted by small letters. The transverse polarization vectors for K are defined as

$$\epsilon^{\pm}(K,r) = \epsilon^{\pm}(k,r), \qquad (2.6)$$

where k is defined in (2.5) and $\epsilon^{\pm}(k,r)$ given by (2.1). The longitudinal polarization vector $\epsilon^{0\mu}$ is defined as

$$\epsilon^{0}(K,r) = \frac{k-\alpha r}{\sqrt{K^2}} \qquad (2.7)$$

for time-like K. For space-like K (momentum of a virtual boson),

$$\epsilon^{0}(K,r) = \frac{k+\alpha r}{\sqrt{|K^2|}} = \frac{K}{\sqrt{|K^2|}} \qquad (2.8)$$

a unit space-like vector in the direction of K. It follows that $\epsilon^{0}\cdot\epsilon^{\pm}=0$ and $\epsilon^{0^2}=-1$ for both (2-7) and (2.8). For a time-like K, we have from (2.6) and (2.7)

$$K\cdot\epsilon^{\pm}(K,r) = K\cdot\epsilon^{0}(K,r) = 0. \qquad (2.9)$$

$\epsilon^{\pm}$ and ϵ^{0} form a triad of polarization vectors for K. The polarization vectors for K are defined in terms of the vectors k and r both on the light cone. This is necessary since all calculations are being carried out in terms of products $\langle r+|k-\rangle$ etc. [2]. For $K^2=0$ we have $\alpha=0$ and only $\epsilon^{\pm}$ survive. The definition (2.1) is a special case of (2.6) and (2.7).

3. Reduction of Amplitudes of Processes Involving Tree Diagrams with Gluons of the Same Helicity

Let two polarization vectors $\epsilon^{\pm}(K,r)$ and $\epsilon^{\pm}(S,r)$ of the same helicity be coupled through a 3-gluon vertex (Fig. 1)

$$g f_{abc} T^{\mu\nu\sigma}(P,-S,-K) = g f_{abc}\left[(P+S)^{\sigma} g^{\mu\nu} + (-S+K)^{\mu} g^{\nu\sigma} + (-K-P)^{\nu} g^{\sigma\mu} \right]. \qquad (3.1)$$

By (2.1) and (2.6) the polarization vectors are

$$\epsilon^{\pm}_{\sigma}(K,r) = \frac{\pm\langle r\mp|\gamma_{\sigma}|k\mp\rangle}{\sqrt{2}\langle r\mp|k\pm\rangle}$$

and (3.2)

$$\epsilon^{\pm}_{\nu}(S,r) = \frac{\pm\langle r\mp|\gamma_{\nu}|s\mp\rangle}{\sqrt{2}\langle r\mp|s\pm\rangle},$$

where

$$k = K - \frac{K^2}{2K\cdot r}\, r \qquad (3.3)$$

and

$$s = S - \frac{S^2}{2S\cdot r}\, r,$$

with $k^2 = s^2 = 0$. Focusing our attention on the spacetime structure, ~~we have~~

$$T^{\mu\nu\sigma}(P,-S,-K)\,\epsilon^{\pm}_{\nu}(S,r)\,\epsilon^{\pm}_{\sigma}(K,r)$$

$$= \epsilon^{\pm}(K,r)\cdot(P+S)\,\epsilon^{\pm\mu}(S,r) \qquad (3.4)$$

$$+ (K-S)^{\mu}\epsilon^{\pm}(S,r)\cdot\epsilon^{\pm}(K,r) - \epsilon^{\pm}(S,r)\cdot(K+P)\,\epsilon^{\pm\mu}(K,r).$$

Substituting (3.2) into (3.4) and making use of 4-momentum conservation $P = K + S$, Dirac equation $\not{r}|r\pm\rangle = 0$, and the following relations proved in [2],

$$\langle r\mp|s\pm\rangle = -\langle s\mp|r\pm\rangle \,, \tag{3.5a}$$

$$\langle r\mp|r\pm\rangle = 0 \,, \tag{3.5b}$$

$$|s\pm\rangle\langle s\pm| = \tfrac{1}{2}(1\pm\gamma_5)\not{s} \,, \tag{3.5c}$$

$$\langle r\pm|\not{K}|s\pm\rangle = \langle r\pm|\tfrac{1}{2}(1\mp\gamma_5)\not{K}|s\pm\rangle = \langle r\pm|k\mp\rangle\langle k\mp|s\pm\rangle \,, \tag{3.5d}$$

$$\langle r\mp|\gamma_\mu|k\mp\rangle\langle t\mp|\gamma^\mu|s\mp\rangle = 2\langle r\mp|t\pm\rangle\langle s\pm|k\mp\rangle \,, \tag{3.5e}$$

$$\langle r\mp|\gamma_\mu|k\mp\rangle\langle r\mp|\gamma^\mu|s\mp\rangle = 0 \,, \tag{3.5f}$$

we rewrite (3.4) in the following way

$$T^{\mu\nu\sigma}(P,-S,-K)\ \epsilon^\pm_\nu(S,r)\ \epsilon^\pm_\sigma(K,r)$$

$$= \frac{\langle r\mp|2\not{s}|k\mp\rangle\langle r\mp|\gamma^\mu|s\mp\rangle - \langle r\mp|2\not{K}|s\mp\rangle\langle r\mp|\gamma^\mu|k\mp\rangle}{2\langle r\mp|k\pm\rangle\langle r\mp|s\pm\rangle}$$

$$= \frac{\langle r\mp|\gamma^\mu\not{s}|r\pm\rangle\langle k\pm|s\mp\rangle + \langle r\mp|\gamma^\mu\not{K}|r\pm\rangle\langle k\pm|s\mp\rangle}{\langle r\mp|k\pm\rangle\ \langle r\mp|s\pm\rangle}$$

$$= \frac{\langle r\mp|p\pm\rangle}{\langle r\mp|s\pm\rangle}\cdot\frac{\pm\sqrt{2}\langle p\mp|r\pm\rangle\langle k\pm|s\mp\rangle}{\langle r\mp|k\pm\rangle}\cdot\frac{\pm\langle r\mp|\gamma^\mu|p\mp\rangle}{\sqrt{2}\langle r\mp|p\pm\rangle}$$

$$= \frac{\langle r\mp|p\pm\rangle}{\langle r\mp|s\pm\rangle}\ \langle p\mp|\not{\epsilon}^\pm(K,r)|s\mp\rangle\ \epsilon^{\pm\mu}(P,r) \,, \tag{3.6}$$

where

$$p = P - \frac{P^2}{2P\cdot r}\,r \,, \qquad p^2 = 0 .$$

We have therefore: Two gluon polarization vectors of the same helicity $\epsilon^\pm_\nu(S,r)$ and $\epsilon^\pm_\sigma(K,r)$ coupled through the 3-gluon vertex $T^{\mu\nu\sigma}(P,-S,-K)$ give rise to a vector proportional to $\epsilon^{\pm}_{\mu}(P,r)$ of the same helicity.

The vector resulting from the coupling of three gluon polarization vectors $\epsilon^{\pm}(K,r)$, $\epsilon^{\pm}(Q,r)$ and $\epsilon^{\pm}(S,r)$ of the same helicity through a 4-gluon vertex is (Fig.2)

$$
\begin{aligned}
V^{\pm\mu} &= ig^2\left[f_{abe}f_{ade}\left(g^{\mu\nu}g^{\lambda\sigma}-g^{\nu\lambda}g^{\mu\sigma}\right)+f_{bce}f_{ade}\left(g^{\nu\lambda}g^{\mu\sigma}-g^{\lambda\mu}g^{\nu\sigma}\right)\right. \\
&\quad \left.+f_{cae}f_{bde}\left(g^{\lambda\mu}g^{\nu\sigma}-g^{\mu\nu}g^{\lambda\sigma}\right)\right]\cdot\epsilon^{\pm}_{\lambda}(K,r)\,\epsilon^{\pm}_{\nu}(Q,r)\,\epsilon^{\pm}_{\sigma}(S,r) \\
&= ig^2\Big[f_{abe}f_{cde}\Big(\epsilon^{\pm}(K,r)\cdot\epsilon^{\pm}(S,r)\;\epsilon^{\pm\mu}(Q,r) \\
&\qquad -\epsilon^{\pm}(K,r)\cdot\epsilon^{\pm}(Q,r)\,\epsilon^{\pm\mu}(S,r)\Big)+\text{cycl. perm.}(a,c,d)(K,Q,S)\Big] \\
&= 0,
\end{aligned}
\tag{3.7}
$$

Since $\epsilon^{\pm}(K,r)\cdot\epsilon^{\pm}(S,r)=0$ etc., because of (3.5f). Consequently 4-gluon vertices do not contribute in the case of identical gluon helicity. This enables us to conclude that any number of polarization vectors of gluons of the same helicity defined by (2.1) with same r couple in a tree diagram to yield a vector proportional to $\epsilon^{\pm}(K,r)$ of the same helicity, where K is the resultant momentum.

We consider the coupling of gluon line to a fermion current. Substituting $\epsilon^{\pm}_{\nu}(S,r)$ and $\epsilon^{\pm\mu}(P,r)$ in (3.6) by respective expressions (2.1) we obtain

$$
\begin{aligned}
&T^{\mu\nu\sigma}(P,-S,-K)\,\epsilon^{\pm}_{\sigma}(K,r)\langle r\mp|\gamma_{\nu}|s\mp\rangle \\
&= \langle r\mp|\gamma^{\mu}|p\mp\rangle\langle p\mp|\not{\epsilon}^{\pm}(K,r)|s\mp\rangle \\
&= \langle r\mp|\gamma^{\mu}\not{p}\,\not{\epsilon}^{\pm}(K,r)|s\mp\rangle
\end{aligned}
\tag{3.8}
$$

Disregarding the vertex factors (coupling constant and color factors) and the scalar factors in the propagators (including denominators and $+i$ or $-i$), the left hand side of (3.8) represents the coupling of a gluon polarization vector $\epsilon^{\pm}(K,r)$ and a fermion current $\langle r\mp|\gamma_\nu|s\mp\rangle$ shown in Fig.3a, and the right hand side represents the amplitude for the diagram Fig.3b since

$$\not{p}\,\not{\epsilon}^{\pm}(K,r)\,|s\mp\rangle = (\not{P}-\not{r})\,\not{\epsilon}^{\pm}(K,r)\,|s\mp\rangle ,$$

where $P-r$ is the momentum of the virtual fermion line appearing in the propagator, which differs from p by a vector proportional to r, owing to (2.5). The sign $\stackrel{\Omega}{=}$ in Fig.3 means equality modulo vertex factors and scalar factors in propagators.

If instead of $\langle r\mp|\gamma_\nu|s\mp\rangle$ we use $\langle s\pm|\gamma_\nu|r\pm\rangle$ for $\epsilon^{\pm}_{\nu}(S,r)$ (2.1) then a relation similar to (3.8) follows:

$$\begin{aligned} &T^{\mu\nu\sigma}(P,-S,-K)\,\epsilon^{\pm}_{\sigma}(K,r)\langle s\pm|\gamma_\nu|r\pm\rangle \\ &= \langle p\pm|\gamma^\mu|r\pm\rangle\langle s\pm|\not{\epsilon}^{\pm}(K,r)|p\pm\rangle \\ &= \langle s\pm|\,\not{\epsilon}^{\pm}(K,r)\not{p}\,\gamma^\mu\,|r\pm\rangle . \end{aligned} \tag{3.8a}$$

This can be interpreted as shown in Fig. 3c and Fig. 3d. What we learn from the foregoing arguments is that when a gluon is coupled through a 3-gluon vertex to a fermion current and the reference for the polarization vector is to be chosen as the momentum of the incoming (outgoing) fermion line when the helicity of the gluon is the same as (different from) the helicity of the fermion line, corresponding to the match rule in [2]. the equivalence between the diagrams shown in Fig. 3c and

Fig 3d (the diagrams Fig. 3a and Fig. 3b) is established: the gluon can be transplanted on the outgoing (incoming) line.

To go a step further we start from the following equality

$$\langle r\mp|\gamma_\nu|q\mp\rangle\langle q\mp|\not\epsilon^{\pm}(K',r)|s\mp\rangle = \langle r\mp|\gamma_\nu \not q\, \not\epsilon^{\pm}(K',r)|s\mp\rangle$$

Applying to both side $T^{\mu\nu\sigma}(P,-Q,-K)\epsilon_\sigma^{\pm}(K,r)$ and using (3.8) we obtain for the left-hand side

$$T^{\mu\nu\sigma}(P,-Q,-K)\,\epsilon_\sigma^{\pm}(K,r)\,\langle r\mp|\gamma_\nu|q\mp\rangle\langle q\mp|\not\epsilon^{\pm}(K',r)|s\mp\rangle$$

$$= \langle r\mp|\gamma^\mu|p\mp\rangle\langle p\mp|\not\epsilon^{\pm}(K,r)|q\mp\rangle\langle q\mp|\not\epsilon^{\pm}(K',r)|s\mp\rangle$$

$$= \langle r\mp|\gamma^\mu \not p\,\not\epsilon^{\pm}(K,r)\,\not q\,\not\epsilon^{\pm}(K',r)|s\mp\rangle$$

$$= \langle r\mp|\gamma^\mu(\not P-\not r)\,\not\epsilon^{\pm}(K,r)\,(\not Q-\not r)\,\not\epsilon^{\pm}(K',r)|s\mp\rangle.$$

Equating this to the right-hand side, we obtain

$$T^{\mu\nu\sigma}(P,-Q,-K)\,\epsilon_\sigma^{\pm}(K,r)\,\langle r\mp|\gamma_\nu(\not Q-\not r)\,\not\epsilon^{\pm}(K',r)|s\mp\rangle$$

$$= \langle r\mp|\gamma^\mu(\not P-\not r)\,\not\epsilon^{\pm}(K,r)\,(\not Q-\not r)\,\not\epsilon^{\pm}(K',r)|s\mp\rangle. \quad (3.9)$$

The left hand side of (3.9) can be interpreted as the ampltude for the diagram Fig. 4a while the right hand side that for Fig. 4b, modulo vertex factors and scalar factors in propagators.

Combining (3.8) and (3.9) (i.e. Fig. 3 and Fig. 4) we see that step-by-step reduction holds when the gluon helicities are the same (Fig. 5). The procedure of generalization from (3.8) to (3.9) can be repeated any number of times, and therefore we conclude:

When a number of gluons of the same helicity are attached to a gluon line coupled with a fermion current and the reference momenta for the polarization vectors of all the gluons are taken to be the momentum of the incoming (outgoing) fermion line in case the helicity of the fermion line is the same as (different from) that of the gluons, then the step-by-step reduction holds, in which the gluons can be transplanted one by one to the outgoing (incoming) fermion line leaving the amplitude unchanged modulo vertex factors and scalar factors in propagators. (Fig. 6)

When a 4-gluon vertex is coupled to a fermion current (Fig.7), in which the two gluons have the same helicity and the same reference momentum r, it is appearent that the fermion curent is equivalent to $\epsilon^{\pm}(S, r)$ when the reference momentum is properly chosen as shown in Fig. 7 such that the diagrams give no contribution to the amplitude by (3.7). This argument persists when the fermion line is loaded with any number of gluons with helicities identical to those involved in the 4-gluon vertex. Actually all gluons loaded on the fermion line can be transplanted on the virtual gluon line according to Fig. 5, leaving the current with unloaded fermion lines $\langle r\mp|\gamma^{\mu}|s\mp\rangle = \langle s\pm|\gamma^{\mu}|r\pm\rangle$ proportional to $\epsilon^{\pm\mu}(S,r)$. Applying our previous knowledge on the coupling of two polarization vectors of the same helicity through 3-gluon vertex and the coupling of three polarization vectors of the same helicity through 4-gluon vertex, we conclude that the above assertion is true.

In the foregoing derivations no restriction was imposed on K^2. $\epsilon^{\pm}(K,r)$ can designate a real gluon, or a "gluon tree" composed of any number of gluons of the same helicity coupled through 3-gluon vertices.

4. Reduction of Amplitudes of Processes with Gluons of Different Helicities

In QED amplituds of multiple Bremsstrahlung processes with photons of the same helicity are much simpler than the amplituds of the respective processes when photons of different helicities are involved. Exactly the same situation occurs with QED-like diagrams in QCD. Furthermore, it is much more difficult to reduce the amplitudes of diagrams with gluons of different helicities attached to the virtual gluon line. We try to relate the amplitude with gluons attached to the virtual gluon line to the corresponding amplitudes with these gluons transplanted to the fermion line. We consider the general case in which the fermion line is already loaded with gluons (Fig.8). We use polarization vector ϵ_k for a gluon of momentum k without referring to the definition (2.1) to obtain a relation of general validity, and go over to (2.1) in specific applications. The amplitude for the diagram Fig. 8a is proportional to

$$T^{\mu\nu\sigma}(P,-S,-k)\epsilon_{\sigma k}\langle s\pm|\cdots\not\epsilon_r \not Q \gamma_\nu \not Q' \not\epsilon_{r'}\cdots|s'\pm\rangle \tag{4.1}$$

where $S = Q + Q'$. We rewrite (4.1) as

$$M_a^\mu = T^{\mu\nu\sigma}(P,-S,-k)\,\epsilon_{\sigma k}\, J_\nu^\pm(S), \tag{4.2a}$$

using the notation

$$J_\nu^\pm(S) = \langle s\pm|\cdots\not\epsilon_r \not Q \gamma_\nu \not Q' \not\epsilon_{r'}\cdots|s'\pm\rangle. \tag{4.2b}$$

It is necessary to reduce

$$T^{\mu\nu\sigma}(P,-S,-k)\epsilon_{\sigma k}\gamma_\nu = -(\not{p}+\not{k})\epsilon_k^\mu + (-S+k)^\mu \not{\epsilon}_k + (P+S)\cdot\epsilon_k\gamma^\mu ,$$

(4.3)

relating it to $\gamma^\mu(\not{p}'+\not{k})\not{\epsilon}_k$ and $\not{\epsilon}_k(\not{q}+\not{k})\gamma^\mu$ corresponding to diagrams Fig. 8b and Fig. 8c. This can be achieved by considering, for instance,

$$(P+S)\cdot\epsilon_k\gamma^\mu = (2P-k)\cdot\epsilon_k\gamma^\mu = 2P\cdot\epsilon_k\gamma^\mu = (\not{P}\not{\epsilon}_k+\not{\epsilon}_k\not{P})\gamma^\mu .$$

By permuting γ^μ through, we have finally

$$\begin{aligned} T^{\mu\nu\sigma}(P,-S,-k)\,\epsilon_{\sigma k}\gamma_\nu = {} & \gamma^\mu(\not{q}'+\not{k})\not{\epsilon}_k + \not{\epsilon}_k(\not{q}+\not{k})\gamma^\mu \\ & - \not{q}\gamma^\mu\not{\epsilon}_k - \not{\epsilon}_k\gamma^\mu\not{q}' + \not{S}\epsilon_k^\mu - P^\mu\not{\epsilon}_k \end{aligned}$$

Substituting this equality back into (4.1), we have

$$\begin{aligned} M_a^\mu = {} & \langle s\pm|\cdots\not{\epsilon}_r\not{q}\,\gamma^\mu(\not{q}'+\not{k})\not{\epsilon}_k\not{q}'\not{\epsilon}_{r'}\cdots|s'\pm\rangle \\ & + \langle s\pm|\cdots\not{\epsilon}_r\not{q}\,\not{\epsilon}_k(\not{q}+\not{k})\gamma^\mu\not{q}'\not{\epsilon}_{r'}\cdots|s'\pm\rangle \\ & - Q^2\langle s\pm|\cdots\not{\epsilon}_r\gamma^\mu\not{\epsilon}_k\not{q}'\not{\epsilon}_{r'}\cdots|s'\pm\rangle \\ & - Q'^2\langle s\pm|\cdots\not{\epsilon}_r\not{q}\not{\epsilon}_k\gamma^\mu\not{\epsilon}_{r'}\cdots|s'\pm\rangle \\ & + \epsilon_k^\mu\, S\cdot J^\pm(S) - P^\mu\epsilon_k\cdot J^\pm(S). \end{aligned}$$

(4.4)

The first two terms are equal to M_b^μ and M_c^μ (Figs. 8b and 8c) modulo vertex factors and propagator scalar factors. Therefore, in the general case a gluon from 3-gluon vertex coupled to a fermion line can be transplanted to this fermion line with 4 additional terms.

In the derivation of (4.4) the relation $k\cdot\epsilon_k = 0$ has been used.

When all the gluons have the same helicity which is, say, opposite to (the same as) that of the fermion line and the polarization vectors assume the form (2.1) with reference momentum s (s'), the outgoing (incoming) momentum of the fermion line, all terms except the first (second) one in (4.4) vanish. This is just what we obtained before. In more general case, the equivalence takes a more complicated form, equation (4.4). We demonstrate the application of (4.4) to the process $W^+ \to u + \bar{d} + \gamma + \gamma$ in the next section, and we will find that many terms in (4.4) do not contribute.

5. Application to the Process $W^+ \to u + \bar{d} + \gamma + \gamma$

We consider the process

$$W^+(P) \to u(q) + \bar{d}(q') + \gamma(k_1) + \gamma(k_2). \tag{5.1}$$

Altogether 13 Feynman diagrams contribute to the amplitude to the lowest order of couplings (Fig.9). This is a process in which only electromagnetic and weak interactions are involved. In the Glashow-Weinberg-Salam theory of $SU(2)_L \times U(1)$ gauge fields, the electromagnetic interaction of $W^\pm$ originates from the non-Abelian coupling

$$-\frac{1}{2} g \epsilon_{ijk} (\partial^\mu W_i^\nu - \partial^\nu W_i^\mu) W_\mu^j W_\nu^k$$

and

$$-\frac{1}{4} g^2 \epsilon_{ijl} \epsilon_{kml} W_\mu^i W_\nu^j W^{k\mu} W^{m\nu} ,$$

which gives among others the effective coupling

$$-ie(W_\nu^{+}\overleftrightarrow{\partial}_\mu W^\nu)A^\mu$$

and
$$2e^2(W_\mu^{*}W^\mu A_\nu A^\nu - W_\mu^{*}W_\nu A^\mu A^\nu)\,, \tag{5.2}$$

where $W = \frac{1}{\sqrt{2}}(W_1 + iW_2)$. Therefore the arguments on the reduction of non-Abelian coupling are applicable and at the same time field theory rules for vector particles can be used. The W boson has a finite mass, and the numerator of its propagator is $-g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}$, where P is the momentum. The additional term $P_\mu P_\nu/M^2$ has no contribution to the amplitude, because in all diagrams of Fig. 9 the fermion current to which a W boson propagator is coupled is either unloaded with photons or loaded with only one photon; such that the current is proportional to $\epsilon(P,r)$ and $P_\mu \epsilon^\mu(P,r) = 0$ where r can be either q or q'. In addition, the left handedness of the charged weak current demands the negative helicity of the fermion line in this process.

First consider the case in which the photons have the same, say positive, helicity. The amplitude of process (5.1) is then

$$M^a(++) = \epsilon_P^{a\mu}\mathcal{M}_\mu(++)\,, \tag{5.3}$$

where $\epsilon_P^{a\mu}$ is the polarization vector of the W^+ particle and a takes three possible states +, 0 and −. Since the helicity of photons is different from that of the fermion line, the reference momentum for the polarization vectors of photons is chosen to be q, the outgoing momentum, and consequently

$$\not{\epsilon}^{+}(k_i, q) = \frac{\sqrt{2}\,(|q+\rangle\langle k_i+| + |k_i-\rangle\langle q-|)}{\langle q-|k_i+\rangle}\,. \tag{5.4}$$

It immediately follows that diagrams b, b', c, c', e, e' do not contribute, because in these diagrams a photon of positive helicity hangs on the outgoing end of fermion line of negative helicity. Among the nonvanishing contributions, we have

$$\mathcal{M}_a \triangleq \mathcal{M}_d \triangleq \mathcal{M}_f ,$$
$$\mathcal{M}_{a'} \triangleq \mathcal{M}_{d'} \triangleq \mathcal{M}_{f'} . \tag{5.5}$$

The primed diagrams differ from the corresponding unprimed ones only by the exchange of $\epsilon(k_1)$ and $\epsilon(k_2)$. Therefore it is sufficient to evaluate one diagram to get $\mathcal{M}^\mu$. For instance

$$\mathcal{M}_a^\mu(++) = i\frac{g}{\sqrt{2}} e^2 Q_d^2 \frac{1}{(P-q)^2\, 2k_1\cdot q'}\, 2\langle q-|\gamma^\mu(\not{P}-\not{q})|q+\rangle$$
$$\cdot\langle k_2+|(\not{q}'+\not{k}_1)|q+\rangle\langle k_1+|q'-\rangle \frac{1}{\langle q-|k_2+\rangle\langle q-|k_1+\rangle} , \tag{5.6}$$

where Q_d is the d quark charge in unit of e. We define the momentum p on the light cone, connected with P by the following relation (2.5):

$$p = P - \frac{P^2}{2P\cdot q}\, q . \tag{5.7}$$

Then it follows that

$$(\not{P}-\not{q})|q+\rangle = \not{P}|q+\rangle = \not{p}|q+\rangle$$

and

$$\mathcal{M}_a^\mu(++) = i\frac{g}{\sqrt{2}} e^2 \frac{Q_d^2}{(P-q)^2\, 2k_1\cdot q'} \frac{2\langle q-|\gamma^\mu|p-\rangle\langle p-|q+\rangle}{\langle q-|k_2+\rangle\langle q-|k_1+\rangle}\langle k_2+|(\not{q}'+\not{k}_1)|q+\rangle\langle k_1+|q'-\rangle \tag{5.8}$$

Introducing short-hand notations which appear in amplitudes of various diagrams:

$$X_a = \frac{Q_d^2}{(P-q)^2\, 2k_1\cdot q'},$$

$$X_{a'} = \frac{Q_d^2}{(P-q)^2\, 2k_2\cdot q'}, \qquad (5.9a)$$

characterizing the vertex factors and propagator scalar factors of diagrams (a) and (a'), respectively, and

$$G_{21} = \langle k_2+|(\not q' + \not K_1)|q+\rangle\langle k_1+|q'-\rangle,$$

$$G_{12} = \langle k_1+|(\not q' + \not K_2)|q+\rangle\langle k_2+|q'-\rangle, \qquad (5.9b)$$

characterizing the emission of photons (k_2, k_1) and (k_1, k_2) and the propagator spinor factor between the emission of two photons. The X factors for diagrams (d), (d'), (f) and (f') are respectively

$$X_d = \frac{-Q_d}{[(P-k_2)^2 - M^2 + iM\Gamma]\, 2q'\cdot k_1},$$

$$X_{d'} = \frac{-Q_d}{[(P-k_1)^2 - M^2 + iM\Gamma]\, 2q'\cdot k_2}, \qquad (5.10)$$

$$X_f = \frac{1}{[(P-k_2)^2 - M^2 + iM\Gamma][2qq' - M^2 + iM\Gamma]},$$

and

$$X_{f'} = \frac{1}{[(P-k_1)^2 - M^2 + iM\Gamma][2q\cdot q' - M^2 + iM\Gamma]},$$

where W width is taken into account by using Breit-Wigner expression in the propagator denominator. Finally we have

$$\mathcal{M}^\mu(++) = i\frac{g}{\sqrt{2}} e^2 \frac{2\langle q-|\gamma^\mu|p-\rangle\langle p-|q+\rangle}{\langle q-|k_1+\rangle\langle q-|k_2+\rangle}\cdot$$

$$\cdot\left[(X_a + X_d + X_f)G_{21} + (X_{a'} + X_{d'} + X_{f'})G_{12}\right]. \qquad (5.11)$$

To find $\sum_a |M^a(++)|^2$ we make use of the relations

$$\sum_a \epsilon^a_{P\mu}(\epsilon^a_{P\nu})^* = -g_{\mu\nu} + \frac{P_\mu P_\nu}{M^2}$$

and

$$P_\mu\langle q-|\gamma^\mu|p-\rangle = \langle q-|(\not p + \frac{P^2}{2P\cdot q}\not q)|p-\rangle = 0 .$$

The result is

$$\sum_a |M^a(++)|^2 = -\mathcal{M}^{\mu *}(++)\mathcal{M}_\mu(++)$$

$$= g^2 e^4 \frac{4(g\cdot P)^2}{g\cdot k_1\ g\cdot k_2}\left|(X_a+X_d+X_f)G_{21}+(X_{a'}+X_{d'}+X_{f'})G_{12}\right|^2. \tag{5.12}$$

The amplitude $M^a(--)$ is connected with $M^a(++)$ in a straightforward way. When the helicity is changed, a photon attached to the u quark line is now switched over to the $\bar{d}$ antiquark line and vice versa, due to the helicity match rule [2]. At the same time the reference momentum for the polarization vectors changes from q to q', and vice versa [2]. The helicities in the basic product $\langle a+|b-\rangle$ undergo sign changes everywhere. Therefore finally we have

$$M^a(--) = M^a(++)\Big|_{g\pm \leftrightarrow g'\mp;\ Q_u\leftrightarrow -Q_d;\ k_i\mp\to k_i\pm}, \tag{5.13}$$

and

$$\sum_a |M^a(--)|^2 = g^2 e^4 \frac{4(g'\cdot P)^2}{(g'\cdot k_1)(g'\cdot k_2)}\left|(X_c+X_e+X_f)G_{12}'+(X_{c'}+X_{e'}+X_{f'})G_{21}'\right|^2, \tag{5.14}$$

in which

$$X_c = X_a\,(g\leftrightarrow g',\ Q_u\leftrightarrow -Q_d),$$
$$X_{c'} = X_{a'}\,(g\leftrightarrow g',\ Q_u\leftrightarrow -Q_d),$$
$$X_e = X_d\,(g\leftrightarrow g',\ Q_u\leftrightarrow -Q_d), \tag{5.16}$$
and
$$X_{e'} = X_{d'}\,(g\leftrightarrow g',\ Q_u\leftrightarrow -Q_d)$$

are just the X factors of Figs. 8(c) (c') (e) (e') respectively, and

$$G_{21}' = G_{21}(g\leftrightarrow g'),$$
$$G_{12}' = G_{12}(g\leftrightarrow g'). \tag{5.17}$$

Certainly (5.13) can be confirmed by direct evaluation of the amplitude $M^a(- -)$.

Next we consider the case of different photon helicities. Assuming the helicities of k_1 and k_2 to be positive and negative respectively, we then choose q and q' to be the reference momenta for $\epsilon^+(k_1)$ and $\epsilon^-(k_2)$:

$$\epsilon^{+\mu}(k_1,q) = \frac{\langle q-|\gamma^\mu|k_1-\rangle}{\sqrt{2}\langle q-|k_1+\rangle}, \tag{5.18}$$

$$\epsilon^{-\mu}(k_2,q') = \frac{\langle k_2-|\gamma^\mu|q'-\rangle}{\sqrt{2}\langle k_2+|q'-\rangle}.$$

Consequently $\not\epsilon^-(k_2,q')|q'-\rangle = \langle q-|\not\epsilon^+(k_1,q) = 0$, i.e. $\epsilon^+(k_1,q)$ and $\epsilon^-(k_2,q')$ can only be attached to $|q'-\rangle$ and $\langle q-|$ respectively for a non-vanishing amplitude. The non-vanishing amplitudes are therefore

$\mathcal{M}_a,\ \mathcal{M}_b,\ \mathcal{M}_{c'}$ —— QED like diagrams.

$\mathcal{M}_d,\ \mathcal{M}_{e'},\ \mathcal{M}_f,\ \mathcal{M}_{f'},\ \mathcal{M}_g$ —— diagrams with non-Abelian coupling

We concentrate on the reduction of amplitudes with non-Abelian coupling, to which (4.4) can be applied directly.

From (3.8) we have

$$M_d \overset{\Omega}{=} M_f, \qquad M_{e'} \overset{\Omega}{=} M_{f'}.$$

Consider first M_d. The corresponding quantities in (4.4) are

$$J^\mu = \langle q-|\gamma^\mu(\not q'+\not k_1)\not\epsilon^+(k_1,q)|q'-\rangle$$

$$Q = q, \quad Q' = q'+k_1, \quad S = q+q'+k_1.$$

Taking into account the following observations:

$$\epsilon_{p\mu}P^\mu = 0$$

$$\begin{aligned} S\cdot J^-(S) &= \langle q-|\not S(\not q'+\not k_1)\not\epsilon^+(k_1,q)|q'-\rangle \\ &= 2q'\cdot k_1\langle q-|\not\epsilon^+(k_1,q)|q'-\rangle \\ &= 0, \end{aligned}$$

and factorizing an amplitude M_i into the product of the X factor and a reduced amplitude m_i:

$$M_i = m_i X_i, \tag{5.19a}$$

we find according to (4.4)

$$m_f = m_d = m_a + m_b - 2q'\cdot k_1 Z_0, \tag{5.19b}$$

where

$$Z_0 = +\epsilon_{p\mu} \langle q-|k_2+\rangle\langle k_1+|q'-\rangle \frac{i\sqrt{2}ge^2}{\langle q'+|k_2-\rangle\langle q-|k_1+\rangle} \langle q-|\gamma^\mu|q'-\rangle; \tag{5.20a}$$

as well as

$$m_{f'} = m_{e'} = m_b + m_c - 2q\cdot k_2 Z_0. \tag{5.19c}$$

We also find that the amplitude of diagram 8g is

$$M_g = X_g Z_0, \tag{5.21}$$

where

$$X_g = \frac{2}{2q\cdot q' - M^2 + iM\Gamma}. \tag{5.22}$$

Consequently, everything is reduced to the calculation of QED-like amplitudes and an additional term Z_0. To obtain the final result, we write down the explicit expressions for m_a, m_b and m_c:

$$m_a = \epsilon_{p\mu} \frac{i\sqrt{2}\,ge^2}{\langle q'+|k_2-\rangle\langle q-|k_1+\rangle} \langle q-|\gamma^\mu(\not{p}-\not{q})|k_2+\rangle\langle q'+|\not{k}_1|q+\rangle\langle k_1+|q'-\rangle,$$

$$m_b = \epsilon_{p\mu} \frac{i\sqrt{2}ge^2}{\langle q'+|k_2-\rangle\langle q-|k_1+\rangle} \langle q-|k_2+\rangle\langle q'+|(\not{q}'+\not{k}_2)\gamma^\mu(\not{q}'+\not{k}_1)|q+\rangle\langle k_1+|q'-\rangle,$$

$$m_c = \epsilon_{p\mu} \frac{i\sqrt{2}ge^2}{\langle q'+|k_2-\rangle\langle q-|k_1+\rangle} \langle q-|k_2+\rangle\langle q'+|\not{k}_2|q+\rangle\langle k_1+|(\not{p}-\not{q}')\gamma^\mu|q'-\rangle.$$

(5.20b)

Then we have finally

$$M(+-)=\Big\{ m_a (X_a+X_d+X_f)+m_b(X_b+X_d+X_f+X_{e'}+X_{f'})$$
$$+m_c(X_c+X_{e'}+X_{f'})+Z_0\big[2\rho'\cdot k_1(X_d+X_f)$$
$$+2\rho\cdot k_2(X_{e'}+X_{f'})+X_g\big]\Big\}, \qquad (5.23)$$

expressed in terms of QED-like amplitudes and the additional factor Z_0. All X factors are given by (5.10) (5.14) and (5.22a) except for $X_{e'}$, X_b and X_c . For these factors we have

$$X_{e'}=X_e\,(k_1\leftrightarrow k_2)\,,$$
$$X_b=\frac{-Q_uQ_d}{4\rho'\cdot k_1\,\rho\cdot k_2}\,,$$
and
$$X_c=Q_u^2/(P-\rho')^2\,2\rho\cdot k_1\,. \qquad (5.22b)$$

For $M(-+)$, we have

$$M(-+)=M(+-)\Big|_{k_1\leftrightarrow k_2} \qquad (5.24)$$

The differential cross-sections of $W^+\to u+\bar{d}+\gamma+\gamma$ will be calculated numerically and given in a forthcoming paper.

Concluding Remarks

The formalism developed provides a framework for calculating amplitudes of multiple Bremsstrahlung processes in non-Abelian gauge theory so far as tree diagrams are concerned. The calculation is reduced to the evaluation of amplitudes of a few QED-like diagrams. When all the gauge vector bosons have the same helicity, the reduction is clear-cut and complete. When the bosons have both kinds of helicities we have to make case study, but the reduction is also systematic, leading to the evaluations of amplitudes of QED-like diagrams plus some simple terms. When supplemented by the decomposition of amplitudes into gauge invariant subsets [3], the formalism is efficient in calculating tree diagram contributions to amplifudes in non-Abelian gauge theory. The simplification achieved in rather remarkable.

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Figure Captions

Fig.1 Two polarization vectors of same helicity coupled through a 3-gluon vertex.

Fig.2 Three polarization vectors of same helicity conpled through a 4-gluon vertex.

Fig.3 Equivalence of 3-gluon coupling to the coupling of a gluon to a fermion line

Fig.4 Equivalence of amplitudes.

Fig.5 Step-by-step reduction of amplituds with two gluons having same helicity

Fig.6 Step-by-step reduction for n gluons of the same helicity.

Fig.7 4-gluon vertex coupled to a fermion current

Fig.8 Reduction of amphtude in the general case. Dots in diagrams represent any number of gluon insertions.

Fig.9 Feynman diagrams contributimg to $W^{+} \rightarrow u + \bar{d} + \gamma + \gamma$

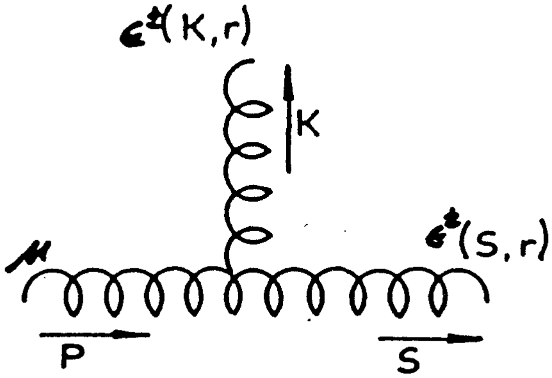


FIG.1

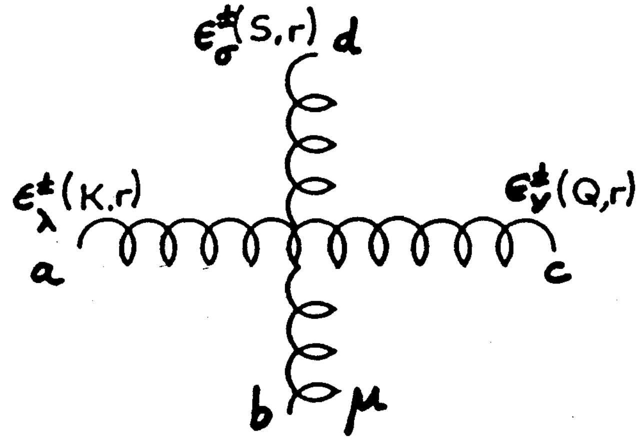


FIG.2

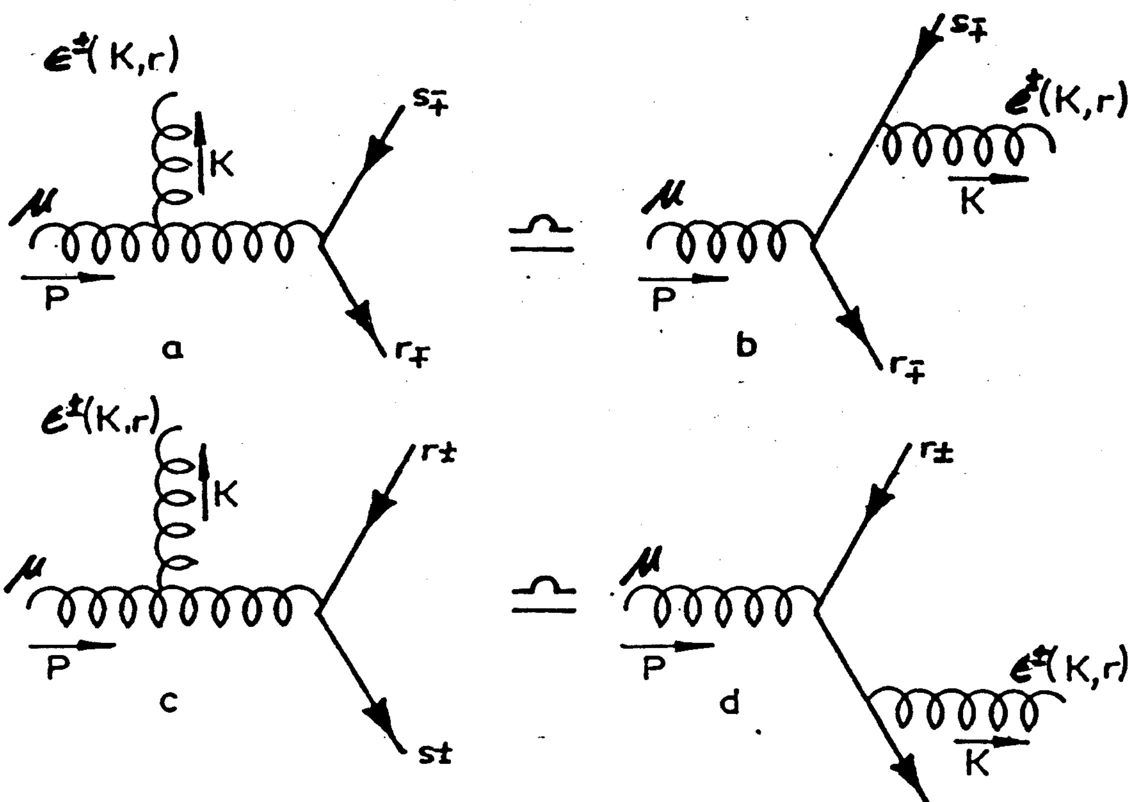
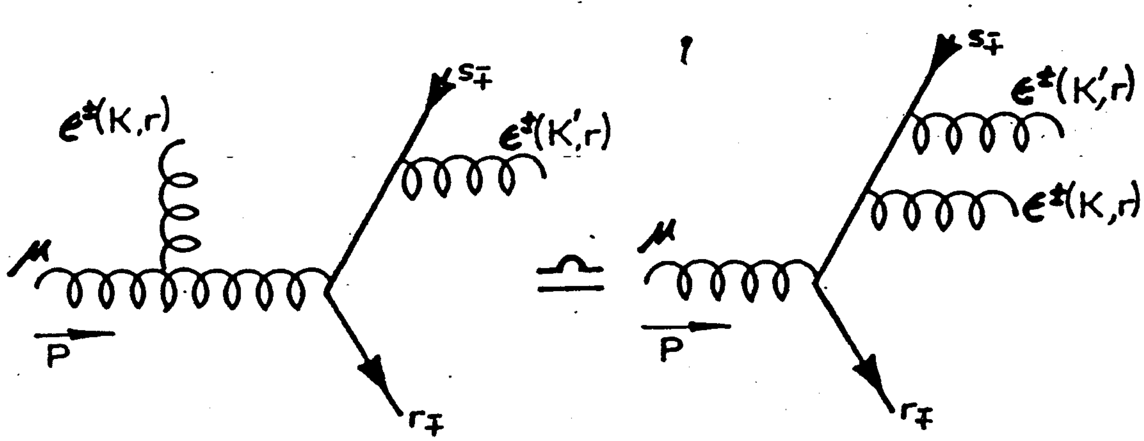
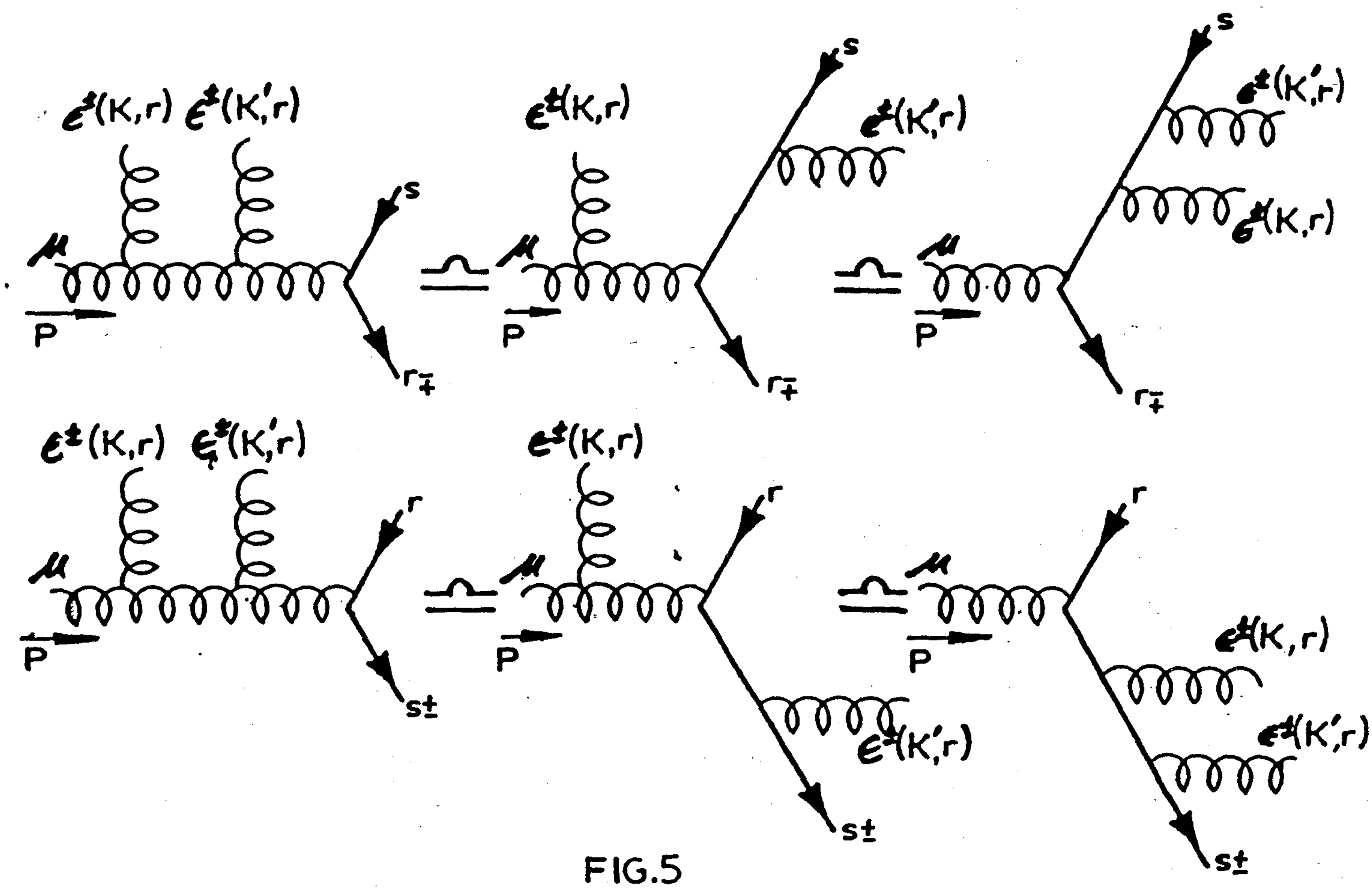


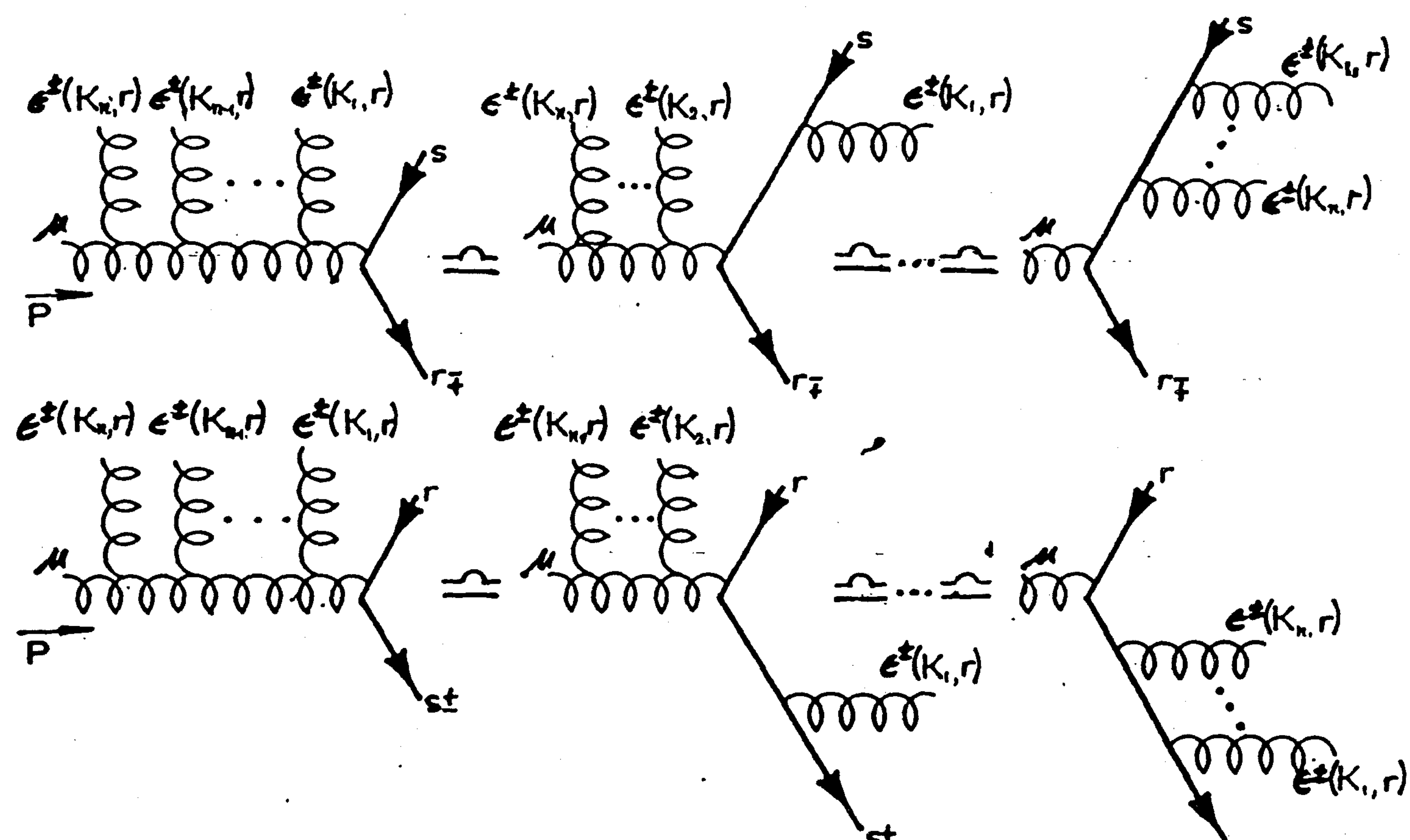
FIG.3





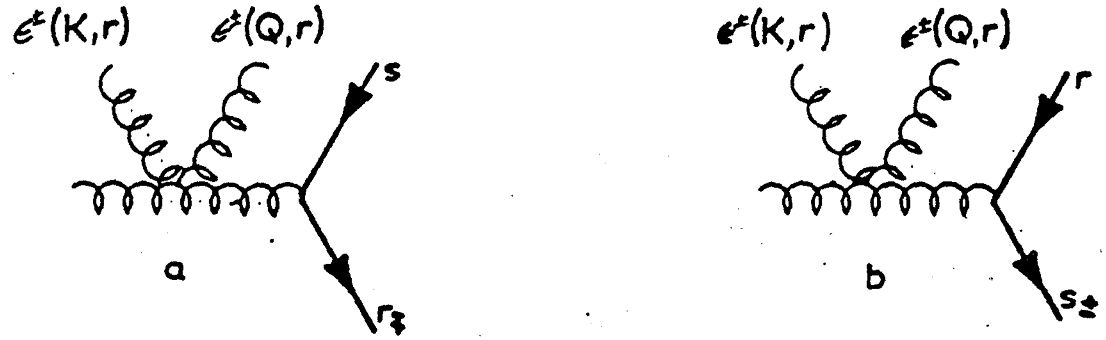
$\epsilon^{\pm}(K,r)$
$\epsilon^{\pm}(K',r)$
μ
P
s
$r\mp$
s±
r

FIG.5



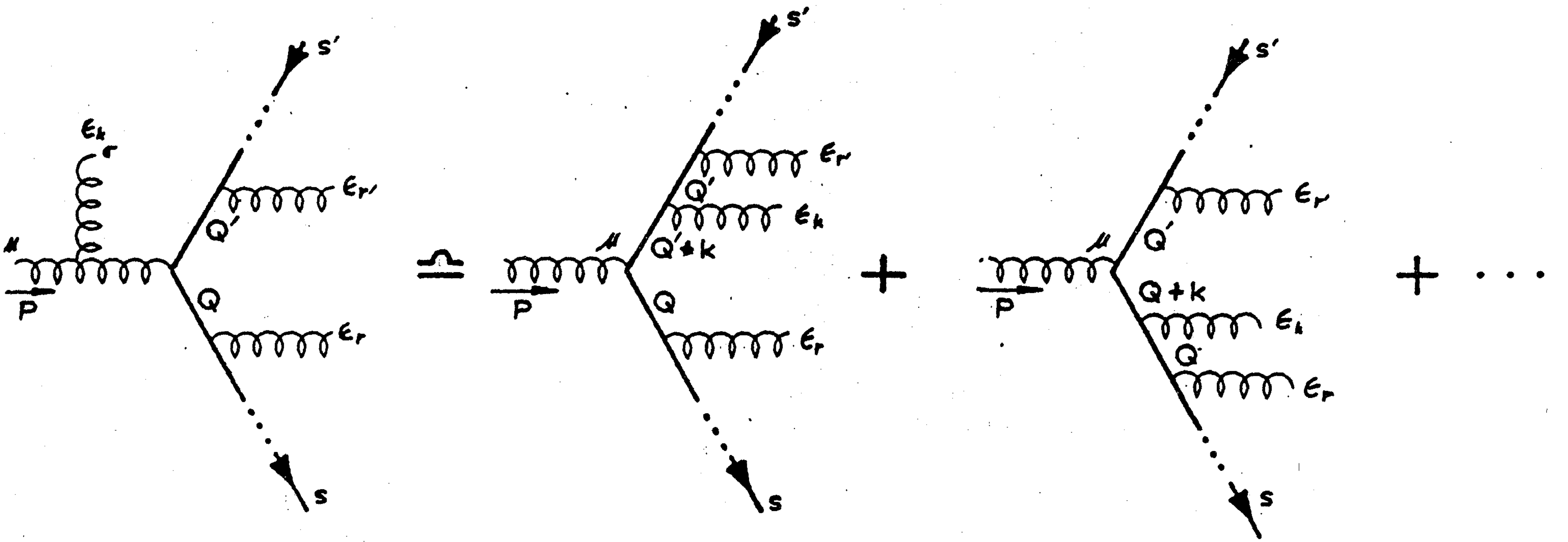
$\epsilon^{\pm}(K_n,r)$
$\epsilon^{\pm}(K_{n-1},r)$
$\epsilon^{\pm}(K_1,r)$
$\epsilon^{\pm}(K_2,r)$
μ
P
s
$r\mp$
s±
r

FIG. 6



ε±(K,r)
ε±(Q,r)
s
r∓
a
ε±(K,r)
ε±(Q,r)
r
s±
b

FIG.7



s'
ε_k
ε_r'
Q'
μ
P
Q
ε_r
s
≙
s'
ε_r'
Q'
ε_k
μ
Q'+k
P
Q
ε_r
s
+
s'
ε_r'
Q'
μ
P
Q+k
ε_k
Q
ε_r
s
+ ....

FIG.8

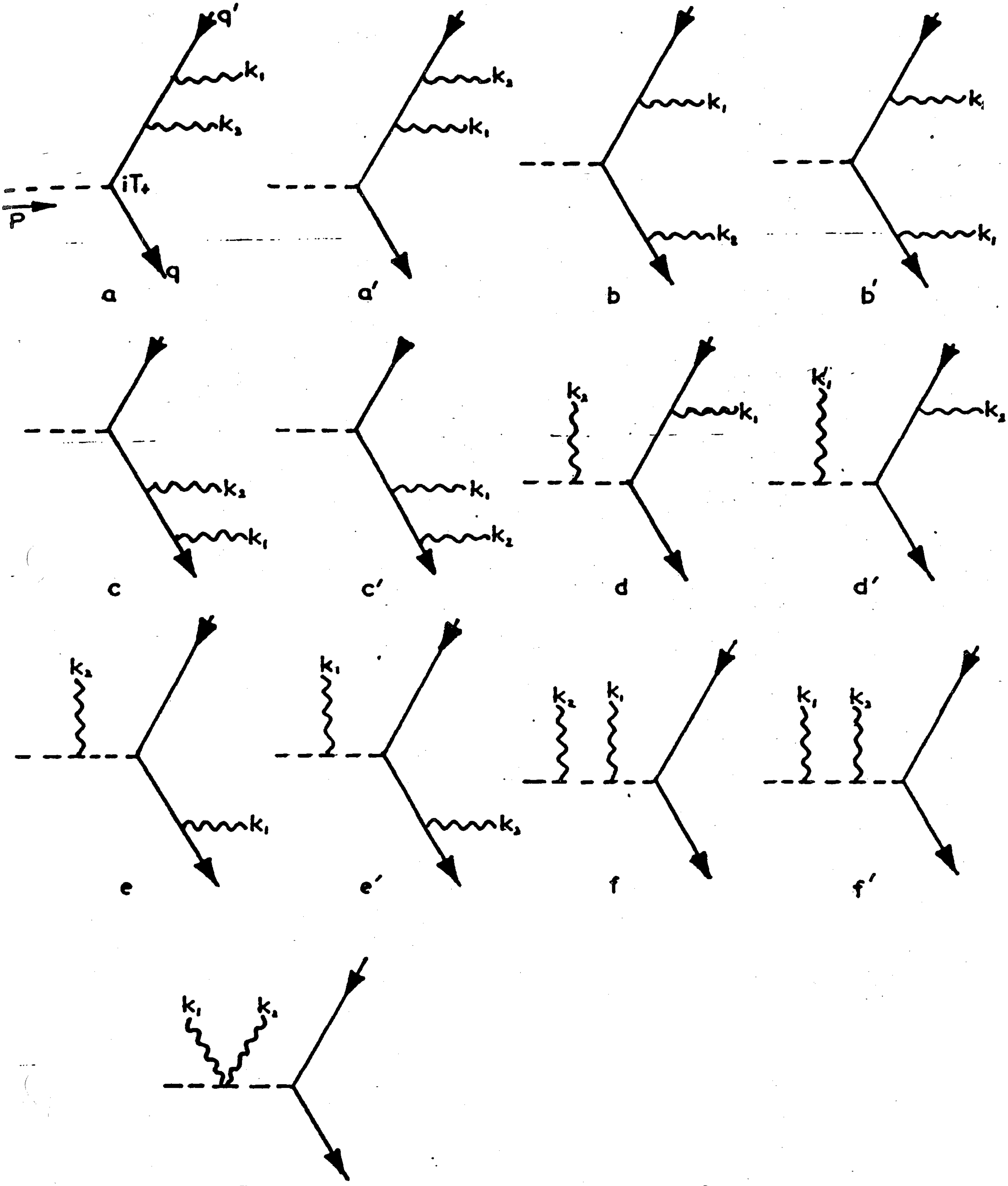


FIG.9