Vector coherent state theory of the non-compact orthosymplectic superalgebras

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Abstract

The vector coherent state and K-matrix combined theory is applied to construct matrix realizations of the positive discrete series irreps of the orthosymplectic superalgebras osp(P/2N,R) (P = 2M or 2M+1) in osp(P/2N,R) $\supset$ so(P) $\oplus$ sp(2N,R) $\supset$ so(P) $\oplus$ u(N) bases. As an example, the case of osp(4/2,R) is treated in detail.

1 Introduction

Vector coherent states (VCS), also called partially coherent states, were independently introduced by Rowe [1], and by Deenen and Quesne [2] as a natural extension of generalized coherent states [3,4]. At the same time, it was noted that coherent states provide a very powerful method for constructing matrix realizations of Lie algebra ladder irreps in bases symmetry-adapted to some maximal rank subalgebra [5,6]. Such a construction is carried out by the so-called K-matrix technique [7,8].

Since then, the VCS and K-matrix combined theory has been applied to a lot of algebra-subalgebra chains (Refs. [7,8] and references quoted therein). Recent extensions have allowed the method to be used for non-semisimple Lie algebras [9] and for Lie superalgebras [10].

In the present communication, we report on a new application to the positive discrete series irreps of the non-compact orthosymplectic superalgebras osp(P/2N,R), where P = 2M or 2M+1. In

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Refs. [11] and [12], a general method is provided for determining the conditions for the existence of star irreps (and of grade star irreps in the osp(2/2N,R) case), the branching rule for their decomposition into a direct sum of so(P) $\oplus$ sp(2N,R) irreps, and the matrix elements of the odd generators in osp(P/2N,R) $\supset$ so(P) $\oplus$ sp(2N,R) $\supset$ so(P) $\oplus$ u(N) bases. The cases explicitely worked out include the most general irreps of osp(1/2N,R), osp(2/2,R), osp(3/2,R), osp(4/2,R), osp(2/4,R), and the most degenerate irreps of osp(2/2N,R). We shall review here the osp(4/2,R) example.

2 The positive discrete series irreps of osp(4/2,R)

The osp(4/2,R) superalgebra is spanned by the so(4) generators $A^{\dagger}_{12}$, A^{12}, C^b_a, a, $b = 1, 2$, the sp(2,R) generators $D^{\dagger}$, D, E, and the odd generators G^a, H^a, I_a, J_a, $a = 1, 2$. We choose to enumerate the weight generators in the order E, C^1_1, C^2_2. Then the lowering generators are A^{12}, C^1_2, D, G^a, and J_a, and the raising ones $A^{\dagger}_{12}$, C^2_1, $D^{\dagger}$, H^a and I_a.

The adjoint operation in so(4) $\oplus$ sp(2,R) can be extended to an adjoint operation in osp(4/2,R) in two ways differing in a sign choice : $(G^a)^{\dagger} = \pm I_a, (J_a)^{\dagger} = \pm H^a$. On the contrary, it cannot be extended to a grade adjoint operation. Hence, osp(4/2,R) may have star, but no grade star irreps [13].

The positive discrete series irreps of osp(4/2,R) can be induced from a lowest-weight so(4) $\oplus$ sp(2,R) irrep $[\Xi_1\Xi_2] \oplus \langle\Omega\rangle$ or, equivalently, from a lowest-weight so(4) $\oplus$ u(1) irrep $[\Xi_1\Xi_2] \oplus \{\Omega\}$. They will be denoted by $[\Xi_1\Xi_2\Omega\rangle$. Here, Ξ_1, Ξ_2, and Ω are some integers subject to the conditions $\Xi_1 \geq |\Xi_2|$, and $\Omega > 1$.

To construct a basis of the $[\Xi_1\Xi_2\Omega\rangle$ carrier space, symmetry-adapted to the chain osp(4/2,R) $\supset$ so(4) $\oplus$ sp(2,R) $\supset$ so(4) $\oplus$ u(1), one may start from a basis $\{|[\Xi_1\Xi_2]\{\Omega\}\alpha\rangle\}$ of the lowest-weight so(4) $\oplus$ u(1) irrep. Since the raising generators H^a and I_a $(D^{\dagger})$ are the components of an so(4) $\oplus$ u(1) irreducible tensor $\mathfrak{K}$ $(D^{\dagger})$, transforming under the irrep $[10] \oplus \{1\}$ ($[00] \oplus \{2\}$), one can construct polynomials in $\mathfrak{K}$ $(D^{\dagger})$, transforming under an so(4) $\oplus$ u(1) irrep $[\lambda_1\lambda_2] \oplus \{\mu\}$ ($[00] \oplus \{\nu\}$). Here, ν runs over all even integers, μ over the set $\{0, 1, ..., 4\}$, and $[\lambda_1\lambda_2]$ over those so(4) irreps contained in the u(4) irrep $\{1^{\mu}\dot{0}\}$. By acting with these two sets of polynomials on the states $|[\Xi_1\Xi_2]\{\Omega\}\alpha\rangle$ and by performing so(4) couplings, one can form the set of states

$$
\begin{aligned}
&|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{h\}\chi\rangle \\
&\quad = \left[P^{[00]\{\nu\}}(D^{\dagger}) \times \left[Q^{[\lambda_1\lambda_2]\{\mu\}}(\mathfrak{K}) \times |[\Xi_1\Xi_2]\{\Omega\}\rangle\right]^{[\xi_1\xi_2]\{\omega\}}\right]^{[\xi_1\xi_2]\{h\}}_{\chi}, \quad (1)
\end{aligned}
$$

characterized by a given so(4) $\oplus$ u(1) irrep $[\xi_1\xi_2] \oplus \{h\}$, and where $\mu = \omega - \Omega$, and $\nu = h - \omega$.

In general, however, the states (1) corresponding to $\{\nu\} = \{0\}$, thence to $\{h\} = \{\omega\}$, do not belong to a definite sp(2,R) irrep. To obtain the lowest-weight state $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi\rangle$ of an sp(2,R) irrep $\langle\omega\rangle$, one has then to combine $|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{\omega\}\chi\rangle$ with some states (1) for which $\{h\} \neq \{\omega\}$. Once this has been done, it still remains to calculate and diagonalize the overlap matrix

$$\langle[\lambda_1'\lambda_2'][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi\rangle = \left(KK^\dagger([\xi_1\xi_2]\{\omega\})\right)_{[\lambda_1'\lambda_2'],[\lambda_1\lambda_2]}, \tag{2}$$

since neither the states (1), nor the states $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi\rangle$ form orthonormal sets.

This painful calculation was actually performed by Schmitt *et al.* [14], who determined in this way the branching rule osp(4/2,R) $\downarrow$ so(4) $\oplus$ sp(2,R). The construction of the corresponding osp(4/2,R) matrix realization would necessitate the computation of some additional complicated scalar products and was not carried out in Ref. [14].

In the next section, we shall see how the VCS and K-matrix combined theory allows the same problems to be solved in a much more elegant and efficient way.

3 VCS and K-matrix combined theory of osp(4/2,R)

The VCS corresponding to the osp(4/2,R) irrep $[\Xi_1\Xi_2\Omega\rangle$ are defined by

$$|z,\sigma,\tau;\alpha\rangle = \exp(Z^\dagger)|[\Xi_1\Xi_2]\{\Omega\}\alpha\rangle, \qquad Z = \frac{1}{2}zD + \sigma_a G^a + \tau^a J_a \, . \tag{3}$$

Here, there is a summation over repeated covariant and contravariant indices, z is a complex variable, and $\sigma_a, \tau^a, a = 1, 2$ are Grassmann variables. The set of variables $\{z, \sigma_a, \tau^a\}$ parametrize the complex extension of the super coset space OSp(4/2,R)/[SO(4) $\otimes$ U(1)]. The VCS (3) differ from standard generalized coherent states [3,4] by the replacememt of a single reference state by a set of such states, spanning the lowest-weight so(4) $\oplus$ u(1) irrep carrier space, which will henceforth be referred to as the intrinsic subspace.

The VCS representation of an arbitrary state $|\Psi\rangle$, belonging to the irrep $[\Xi_1\Xi_2\Omega\rangle$ carrier space, is given by a function $\Psi(z,\sigma,\tau)$ taking vector values in the intrinsic subspace. Its components $\Psi_\alpha(z,\sigma,\tau)$ are holomorphic functions in the variable z, and polynomials in the Grassmann variables σ_a, τ^a. The carrier space of the osp(4/2,R) VCS representation is defined as the graded Hilbert space of all such vector-valued functions which are square integrable with respect to the VCS scalar product $(\Psi'|\Psi)_{VCS}$. K-matrix theory replaces the difficult calculation of the integral form of $(\Psi'|\Psi)_{VCS}$ by an implicit determination through the construction of an orthonormal basis with respect to this scalar product.

The VCS representation $\Gamma(X)$ of an osp(4/2,R) generator X is a differential operator on $\Psi(z,\sigma,\tau)$ depending in addition on the intrinsic representation $\mathbb{A}^{\dagger}_{12}$, $\mathbb{A}^{12}$, $\mathbb{C}^{b}_{a}$, a , b = 1, 2, and $\mathbb{E}$ of so(4) $\oplus$ u(1). Its explicit form can be easily found by using Baker-Campbell-Hausdorff formula.

To apply the K-matrix technique to the osp(4/2,R) irrep $[\Xi_1\Xi_2\Omega\rangle$, we start by considering a vector Bargmann-Berezin (VBB) space. The latter is defined as the space of vector-valued functions $\Psi(z,\sigma,\tau)$ which are square integrable with respect to a Bargmann-Berezin (BB) scalar product $(\Psi'|\Psi)$ [15, 16]. With respect to such a scalar product, the differential operators $2\partial/\partial z$, $\partial/\partial\sigma_a$, and $\partial/\partial\tau^a$ are adjoint to the corresponding variables z, σ_a, and τ^a.

Starting from the intrinsic subspace, the Γ representation of the raising generators generates an irreducible invariant subspace of the VBB space, which is by definition the VCS space. Although the domain of the operators $\Gamma(X)$ is restricted to the latter, one can extend it in a natural way to the whole VBB space. As a result, one obtains the so-called extended Γ representation [10], which may be reducible, and even not fully reducible, although the VCS representation is irreducible.

Since the variables z, σ_a, and τ^a transform under so(4) $\oplus$ u(1) in the same way as the generators $D^\dagger$, I_a, and H^a, the set of states $\{|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{h\}\chi)\}$, obtained by substituting z and $\mathfrak{z} = (\sigma_a, \tau^a)$ for $D^\dagger$ and $\mathfrak{H} = (I_a, H^a)$ in (1), form a VBB basis reducing the subalgebra so(4) $\oplus$ u(1). Contrary to the set (1), the VBB basis is orthonormal (with respect to the BB scalar product).

Let us now introduce a transformation K mapping the VBB basis $\{|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{h\}\chi)\}$ onto a VCS one $\{K|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{h\}\chi)\}$, orthonormal with respect to the unknown VCS scalar product. Instead of using this VCS basis and the VCS representation Γ, which would have to be a star representation with respect to the VCS scalar product, it is much more convenient to keep on working with the VBB basis and transform the VCS representation Γ into an equivalent one γ, defined by $\gamma(X) = K^{-1}\Gamma(X)K$, and satisfying star conditions with respect to the known BB scalar product, *i.e.* $\gamma(X^\dagger) = \gamma^\dagger(X)$.

We may restrict ourselves to the submatrices $\mathcal{K}(|[\xi_1\xi_2]\{\omega\})$ of the full K matrix, defined by

$$\left(\mathcal{K}([\xi_1\xi_2]\{\omega\})\right)_{[\lambda'_1\lambda'_2],[\lambda_1\lambda_2]} = ([\lambda'_1\lambda'_2][\xi_1\xi_2]\{\omega\}\{\omega\}\chi|K|[\lambda_1\lambda_2][\xi_1\xi_2]\{\omega\}\{\omega\}\chi) \ . \qquad (4)$$

By imposing star conditions to $\gamma(\chi)$, it can be shown that the matrix $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\}) \equiv \mathcal{K}([\xi_1\xi_2]\{\omega\})\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ satisfies a recursion relation, whose explicit form can be easily obtained from the Γ representation by using tensor calculus with respect to so(4) $\oplus$ u(1). In addition, it can be proved that $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ is nothing else but the overlap matrix defined in (2). Hence, K-matrix theory provides a simple and systematic method for evaluating the scalar products (2) without having to construct the sp(2,R) lowest-weight states $|[\lambda_1\lambda_2][\xi_1\xi_2]\langle\omega\rangle\{\omega\}\chi\rangle$.

There are at most 15 different irreps $[\xi_1\xi_2] \oplus \{\omega\}$ in the VBB basis. The conditions for their existence, to be referred to as the VBB conditions, can be easily determined from the coupling rules of so(4) irreps. All submatrices $\mathcal{K}([\xi_1\xi_2]\{\omega\})$ are one-dimensional, except for $\mathcal{K}([\Xi_1\Xi_2]\{\Omega+2\})$ which is two-dimensional whenever $\Xi_1 \neq |\Xi_2|$. There are at most 16 matrix elements (2) to be determined from $\mathcal{K}\mathcal{K}^\dagger([\Xi_1\Xi_2]\{\Omega\}) = 1$, corresponding to the intrinsic subspace. The recursion relation provides 40 equations to be satisfied by these 16 unknowns, hence allowing the calculations to be cross-checked.

By definition, the matrices $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ are positive semi-definite. The solutions of the system of 40 equations have such a property if and only if : (i) the plus sign is chosen in the adjoint relations for the odd generators, *i.e.* $(G^a)^\dagger = I_a$, $(J_a)^\dagger = H^a$, and (ii) the irrep labels satisfy the condition $\Omega \geq \Xi_1$.

If $\Omega > \Xi_1$, then all the matrices $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ are positive definite, and all the VBB basis states are mapped onto VCS ones. On the contrary, if $\Omega = \Xi_1$, then not all the matrices $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$ are positive definite, showing that the VCS space is a proper subspace of the VBB one. The linear combinations of VBB basis states, corresponding to vanishing eigenvalues of $\mathcal{K}\mathcal{K}^\dagger([\xi_1\xi_2]\{\omega\})$, have to be eliminated. The conditions for the existence of the remaining linear combinations are referred to as the VCS conditions. The branching rule osp(4/2,R) $\downarrow$ so(4) $\oplus$ sp(2,R), obtained by combining the VBB and VCS conditions, is given in Ref. [12].

The so(4) $\oplus$ u(1) reduced matrix elements of the odd generators between two lowest-weight so(4) $\oplus$ u(1) irrep basis states can be easily determined from those of $\mathfrak{z}$ in the VBB basis, and from the matrix elements of $\mathcal{K}([\xi_1\xi_2]\{\omega\})$ corresponding to non-vanishing eigenvalues. Finally, by applying the Wigner-Eckart theorem with respect to sp(2,R) $\supset$ u(1) [17], the so(4) $\oplus$ sp(2,R) (triple) reduced matrix elements of the odd generators can be calculated and are tabulated in Ref. [12].

4 Conclusion

The VCS and K-matrix combined theory provides a simple and systematic procedure for determining matrix realizations of osp(P/2N,R) by exploiting the full power of tensor calculus with respect to so(P) $\oplus$ u(N). Its only practical limitation lies in the necessity for an explicit knowledge of some so(P) and u(N) Racah coefficients. Note however that in addition to the cases treated in Refs. [11] and [12], many other examples might be worked out. Among them, let us mention the most general irreps of osp(3/4,R) and osp(4/4,R), for which only u(2) Racah coefficients are needed.

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