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IMPROVEMENTS IN THE SPHERICAL COLLAPSE MODEL AND DARK ENERGY COSMOLOGIES

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Abstract. In the present paper, we study how the effects of deviations from spherical symmetry of a system, produced by angular momentum, and shear stress, influence typical parameters of the spherical collapse model, like the linear density threshold for collapse of the non-relativistic component (δ_c) and its virial overdensity (Δ_V). The study is performed in the framework of the Einstein-de Sitter and ΛCDM models, and assuming that the vacuum component is not clustering within the homogeneous non-spherical overdensities. We start from the standard spherical top hat model (SCM) which does not take account the non-spherical effects, and we add to this model the shear term and angular momentum term, which are finally expressed in terms of the density contrast, δ. We find that the non-spherical terms change the non-linear evolution of the system and that the collapse stops "naturally" at the virial radius, differently from the standard spherical collapse model. Moreover, shear and rotation gives rise to higher values of the linear overdensity parameter and different values of Δ_V with respect to the standard spherical collapse model.

Key words: cosmology: theory – large-scale structure of Universe – dark energy – galaxies: formation

1. INTRODUCTION

Current analyses of high quality cosmological data coming from Supernovae Type Ia (Riess et al. 1998; Perlmutter et al. 1999; Kowalski et al. 2008; Amanullah et al. 2010), CMB (Spergel et al. 2003; 2007), and clusters (Allen et al. 2002; Lima et al. 2003; Eisenstein et al. 2005; Allen et al. 2008) are suggesting a cosmic expansion history involving some sort of dark energy and a flat spatial geometry in order to explain the recent accelerating expansion of the Universe. Among a number of possibilities to describe the dark energy (DE) component, the simplest one based on a cosmological constant Λ (see Padmanabhan 2003; Peebles & Ratra 2003; Lima & Braz 2004; Frieman et al. 2008; Li et al. 2011 for reviews), usually interpreted as the vacuum energy density (ρ_v) which acts on the Friedmann's equations as a perfect fluid with negative pressure ($p_{\rm v} = -\rho_{\rm v}$). In the present cosmic concordance ΛCDM model, the overall cosmic fluid contains non-relativistic matter (baryons + cold dark matter, $\Omega_{\rm nr} = 0.274$) plus a vacuum energy density ($\Omega_\Lambda = 0.726$) that fits accurately the current observational data and thus it provides an excellent scenario to describe the present observed Universe (Komatsu et al. 2011; Allen et al. 2011; Del Popolo 2014). Nowadays, one of the most challenging problems in the so-called ΛCDM cosmology (see Del Popolo 2007, 2013, for a review) is to understand the role played by the different cosmic components during the non-linear regime of gravitational clustering and how the many possible physical effects contribute to determine the total mass of virialized halos (Del Popolo & Gambera 1996; Del Popolo 2002) (galaxy and galaxy clusters)[1].

Most of the field of structure formation and galaxy formation concerns understanding the non-linear regime and collapsed objects in the density field. This can be done through N-body simulations or in some cases, using analytical models.

A popular analytical approach to study the non-linear evolution of perturbations of dark matter (in the presence of a non-clustered dark energy (DE)) is the standard spherical collapse model (SSCM) proposed in the seminal paper of Gunn & Gott (1972) and extended in subsequent papers (Ryden & Gunn 1987; Gurevich & Zybin 1988a,b; White & Zaritsky 1992; Sikivie et al. 1997; Le Delliou & Henriksen 2003; Williams et al. 2004; Basilakos et al. 2010; Del Popolo & Kroupa 2009; Cardone et al. 2011a,b; Del Popolo 2012a,b.).

In the simplest case, termed "top-hat" model, one considers a uniform, spherical perturbation characterized by an overdensity $\delta = \rho(t)/\rho_b - 1$, where ρ_b is the smooth background density of matter. In the quoted model, a proto-structure is considered as formed by concentric shells, expanding with the Hubble flow. The equation of motion for a shell in the perturbation is given by

$$\ddot{R} = -GM/R^2 \tag{1}$$

For a positively curved matter dominated universe the previous equation has the parametric form solution

$$\begin{aligned} R &\propto (1-\cos(\theta)) \\ t &\propto (1-\sin(\theta)) \end{aligned} \tag{2}$$

Starting from an initial comoving radius x_i, each shell expands until $\theta = \pi$ ($t = t_{\rm max}$) then turn around and collapse at $\theta = 2\pi$ ($t = 2t_{\rm max}$), formally reaching an infinite density[2].

[1] Note that the ΛCDM model suffers from other problems, like the small-scale problems (e.g., the cusp/core problem (Cardone & Del Popolo 2012; Del Popolo et al. 2013a; Del Popolo & Hiotelis 2014), and the missing satellite problem, the too-big-to-fail problem (Del Popolo & Gambera 1997; Del Popolo et al. 2014), and other problems like the cosmological constant problem (Weinberg 1989; Astashenok & Del Popolo 2012), and the cosmic coincidence problem. Moreover, is still debated the universal nature of dark matter density profiles forming in the ΛCDM cosmology (Navarro et al. 2010; Del Popolo 2010, 2011).

[2] In this simple-minded model matter has no internal pressure, so there is nothing stopping the spherical blob to collapse to the infinite density. In real life collapse will, of course, stop before the infinite density is reached, giving rise to a "virialized" structure, when non-linear processes in the collapse phase convert kinetic energy into random motions. The final result will be a system which satisfies the virial theorem, and $r_{\rm vir} = 1/2 r_{\rm max}$.

At that time the density of the spherical region compared to the Einsten de Sitter (EdS) background is

$$\rho/\rho_b = \frac{9\pi^2}{16} = 5.55 \tag{3}$$

So the spherical perturbation starts to collapse when its density has reached 5.55 times of the background density.

When the spherical region starts to collapse, the linear perturbation theory predicts that $\delta_{\rm lin} = 3(6\pi)^{2/3}/20 \simeq 1.06$. Assuming virialization to occur at $t = 2t_{\rm max}$, the linear density contrast has at this point increased to

$$\delta_{\rm lin} \equiv \delta_c = 3/20(6\pi\frac{2t_{\rm max}}{t_{\rm max}})^{2/3} \simeq 1.686. \tag{4}$$

The model describes how a spherical symmetric overdensity decouples from the Hubble flow, slows down, turns around and collapse. In the last decade, the SSCM has been applied to study density perturbation evolution and structure formation in the presence of DE. However, when solving the density contrast (δ) in the SSCM, the local shear (σ) and the rotation (ω) parameters are usually not taken into account. While the first assumption is correct, since for a sphere the shear tensor vanishes, the rotation term, or angular momentum is not negligible. A simple approach preserving spherical symmetry is to assume that the particles are described by a random distribution of angular momenta such that the mean angular momentum at any point in space is zero (White & Zaritsky 1992). Nevertheless, in any proper extension of the SSCM both effects need to be considered (Engineer et al. 2000, hereafter E00; Del Popolo et al. 2013b,c,d) since shear induces contraction while vorticity induces expansion as expected from a centrifugal effect.

In this paper, we study the net physical effect of shear and rotation in the framework of an extended spherical collapse model (ESCM). We restrict our analysis to the Einstein-de Sitter (EdS) and the flat ΛCDM background cosmologies. For the ΛCDM model we assume the following cosmological parameters: $\Omega_{\rm m} = 0.274$, $\Omega_\Lambda = 0.726$ and $h = 0.7$. In particular, we discuss how the linear density threshold for collapsing non-relativistic component ($\delta_{\rm c}$) and its virial overdensity ($\Delta_{\rm V}$) change. We recall that the change of these two parameters has a strong effect on the mass function and other fundamental cosmological quantities. As a general result, it is also found that the extra terms appearing in the ESCM are responsible for higher values of the linear overdensity parameter at galactic scales as compared to the case without shear and rotation. We also show that the non-spherical terms give rise to a collapse that "naturally" stops at the virial radius, differently from the SSCM, in which the collapse has a singular behavior predicting infinite density contrasts for all collapsed objects. In real systems, the collapse to a point will never occurr in practice. Dissipative physics and the process of violent relaxation will eventually intervene and convert the kinetic energy of collapse into random motions leading the system to the virial equilibrium. The virial radius can be easily computed to be half the maximum radius reached by the system ($r_{\rm vir} = 1/2r_{\rm max}$).

Even if the virialization argument is physically well motivated in real systems, in the SSCM no mechanism exists leading the system to virialization. Usually, one introduces by hand the assumption that in the collapse the shells constituting the system stop at a fixed radius (e.g., $1/2r_{\rm max}$). Taking into account angular momentum and shear in the SSCM, the system will not collapse to a point but the shells of the system will smoothly evolve towards $r_{\rm vir} = 1/2r_{\rm max}$.

2. $\delta_{\rm c}$ AND $\Delta_{\rm V}$

To begin with, let us now consider that the only clustering component in the cosmic medium is the cold dark matter. Following standard lines, the evolution of the overdensity δ in the SSCM, under the assumption that only DM can form clumps and that DE is present as a background fluid (Fosalba et al. 1998; Ohta et al. 2003; Mota & van de Bruck 2004; Abramo et al. 2007), is given by a second order non-linear differential equation (Padmanabhan 1996; Ohta et al. 2003; Pace et al. 2010), namely

$$\ddot{\delta} + 2H\dot{\delta} - \frac{4}{3}\frac{\dot{\delta}^2}{1+\delta} - 4\pi G\bar{\rho}\delta(1+\delta) - (1+\delta)(\sigma^2 - \omega^2) = 0\ , \tag{5}$$

where the shear term $\sigma^2 = \sigma_{ij}\sigma^{ij}$ and the rotation term $\omega^2 = \omega_{ij}\omega^{ij}$ are connected to the shear tensor, which is a symmetric traceless tensor, while the rotation is antisymmetric.

Recalling that $\delta = \rho/\overline{\rho} - 1 = (a/R)^3 - 1$ (a is the scale factor and R is the radius of the perturbation), and inserting it into Eq. 5, it is easy to check that the evolution equation for δ reduces to the SSCM (Fosalba et al. 1998; E00; Ohta et al. 2003)

$$\frac{d^2R}{dt^2} = 4\pi G\rho R - 1/3(\sigma^2 - \omega^2)R = -\frac{GM}{R^2} - 1/3(\sigma^2 - \omega^2)R\ , \tag{6}$$

comparable with the usual expression for the SSCM with angular momentum (Peebles 1993; Nusser 2001; Zukin & Bertschinger 2010):

$$\frac{d^2R}{dt^2} = -\frac{GM}{R^2} + \frac{L^2}{M^2R^3} = -\frac{GM}{R^2} + \frac{4}{25}\Omega^2 R, \tag{7}$$

where in the last expression we have used the momentum of inertia of a sphere, $I = 2/5MR^2$.

The previous argument shows that vorticity, ω, is strictly connected to angular velocity, Ω. In the simple case of a uniform rotation with angular velocity $\mathbf{\Omega} = \Omega_z\mathbf{e}_z$, we have that $\mathbf{\Omega} = \omega/2$ (see also Chernin 1993, for a more complex and complete treatment of the interrelation of vorticity and angular momentum in galaxies).

One assumption generally used when solving the SSCM equations for the density contrast δ (Eq. 5) is to neglect the shear, σ, and the rotation ω. While the first assumption is correct, since for a sphere the shear tensor vanishes, the rotation term, or angular momentum is not negligible. In fact, if we consider the ratio of the rotational term and the gravitational one in Eq. 7, we get $\frac{L^2}{M^3RG}$ that for a spiral galaxy like the Milky Way, with $L \simeq 2.5\times10^{74} gcm^2/s$ (Ryden & Gunn 1987; Catelan & Theuns 1996) and the radius 15 kpc, is of the order of 0.4, showing, as well known, that the rotation is not negligible in the case of galaxy sized perturbations. The quoted ratio is larger for smaller size perturbations (of the dwarf galaxies size) and smaller for larger size perturbations (for clusters of galaxies the ratio is of the order of 10^{-6}). The value of angular momentum, L, or similarly Ω, can be obtained and added to the SCM as described in Del Popolo (2009) or as described previously, assigning an angular momentum $\propto \sqrt{GM(< r_*)r_*}$ at turn-around (White & Zaritsky 1992; Sikivie et al. 1997; Nusser 2001)[3].

E00 studied the effect of the term $\sigma^2 - \omega^2$ on the SSCM model just for the Einstein-de Sitter (EdS) model.

In the present paper, we will study how the typical parameters of the SSCM (in Universes dominated by DE), namely δ_c and Δ_V, are changed by a non-zero σ and ω terms. We also will show how the non-sphericity can "naturally" lead the system to virialization without the need to introduce it by hand, as usually is done in the SSCM.

To this aim, we notice that Equation (5) can be written in terms of the scale factor $a(t)$ as (Pace et al. 2010)

$$\delta''+\left(\frac{3}{a}+\frac{E'}{E}\right)\delta'-\frac{4}{3}\frac{\delta'^2}{1+\delta}-\frac{3}{2}\frac{\Omega_{\rm m}}{a^5E^2(a)}\delta(1+\delta)-\frac{1}{a^2H^2(a)}(\sigma^2-\omega^2)(1+\delta)=0\ . \quad (8)$$

The quantity $\Omega_{\rm m}$ is the present-day value of the density parameter of the DM component while the quantity $E(a)$ is defined by:

$$E(a)=\sqrt{\frac{\Omega_{\rm m}}{a^3}+\Omega_\Lambda}\ , \quad (9)$$

where Ω_Λ is the present-day value of the vacuum density parameter (at $a = 1$).

In order to calculate the shear and vorticity terms in Eq. (8) it is convenient to define the dimensionless α-number as the ratio between the rotational and the gravitational term in Eq. (7):

$$\alpha=\frac{L^2}{M^3RG}\ . \quad (10)$$

As already stressed, the above quoted ratio, α, is of the order of 0.4, for a spiral galaxy like the Milky Way ($L \simeq 2.5\times10^{74} g\ cm^2/s$; $R \simeq 15$ kpc (Ryden & Gunn 1987; Catelan & Theuns 1996a), larger for smaller size perturbations (dwarf galaxies size perturbations) and smaller for larger size perturbations (for galaxy clusters the ratio is of the order of 10^{-6}).

Based on the above outlined argument for rotation one may calculate the same ratio between the gravitational and the extra term appearing in Eq. (8) thereby obtaining

$$(\sigma^2-\omega^2)H_0^{-2}=-\frac{3}{2}\frac{\alpha\Omega_{\rm m,0}}{a^3}\delta. \quad (11)$$

Note that this result is the same of that assumed by E00 (their Eq. 24) in the limit $\delta >> 1$. However, while the approximation given by Eq. (11) is good in the

[3] As previously stressed, the non-trivial role of angular momentum in the SSCM has been pointed out in a noteworthy number of papers studying structure formation in DM dominated universes (see also Del Popolo 2009; Zukin & Bertschinger 2010, Cupani et al. 2011).

non-linear regime, as noticed by E00, it is insufficient to cover the larger range of density contrast (especially the quasi-linear regime) which is of interest to us. As done by E00, we expand $(\sigma^2 - \omega^2)H_0^{-2}$ in Taylor series and retaining the first two terms, we have

$$(\sigma^2 - \omega^2)H_0^{-2} = -\frac{3}{2}\frac{\alpha\Omega_{\mathrm{m},0}}{a^3}\delta - A/\delta + B/\delta^2 \tag{12}$$

In particular, the values of A and B are calculated numerically (Spedicato et al. 2003) as in E00 by constraining the solution of the spherical collapse by means of the two-point correlation function or, as we did, comparing the threshold of collapse, δ_{c} obtained by Sheth & Tormen (2002) which furnish the base to obtain the Sheth-Tormen mass function (see also Sheth & Tormen 2001; Del Popolo & Gambera 2000; Hiotelis & Del Popolo 2006, 2013) with the δ_{c} parameter which is obtained from Eq. (8). Note that the Sheth-Tormen mass function and the δ_c from whom it is obtained using the excursion set theory and is in very good agreement with simulations (Yahagi et al. 2004; Del Popolo 2006).

3. BASIC RESULTS

In this section we discuss some physical consequences of the ESCM discussed in this paper. In particular, we show how the non-spherical terms introduce physically virialization without the need to add it by hand (as previously stressed).

Then we obtain the linear overdensity parameter δ_{c} and the virial overdensity Δ_{V}. In Fig. 1a we show the result of the integration of Eq. (8), using Eq. (12) for $\omega^2 - \sigma^2$. We used the same units as in E00: $y(x) = R/r_{\mathrm{vir}}$ and $x = t/\beta$, with $\beta = \sqrt{(8/3^5)A/(GM)r_{\mathrm{vir}}^3}$.

As expected on physical grounds, the function has a maximum and slowly decreases to unity for large values of x. $r_{\mathrm{vir}}/R_{\mathrm{max}} \simeq 0.6$ and for $t \to \infty$, $R \to r_{\mathrm{vir}}$. The value is slightly different from that obtained by E00 since in our model we have also the Λ term.

In Fig. 1b we compare, similarly to E00, the non-linear density contrast in the modified SCM (dashed line) with that in the SSCM (solid line). Both quantities are plotted against the linearly extrapolated density contrast, δ_L. The plot shows that at the epoch corresponding to $\delta_L \simeq 1.686$ the SSCM has, as known, a singular behavior while our model has a smooth behavior, with $\delta \simeq 120$ (value slightly different from that obtained by E00). When $R = R_{\mathrm{max}}/2$ $\delta_{\mathrm{ESCM}} \simeq 88$.

Since the deviations from spherical symmetry are smaller at early epochs and grow as the system evolves, the two curves should converge going towards smaller values of δ, as observed. However, since we used a Taylor expansion in $1/\delta$, we cannot compare the two curves at $\delta << 1$.

In the next plots, we show how δ_c and Δ_V are changed by the non-spherical terms.

In Fig. 2 (4 plots), we show the evolution of the linear overdensity parameter δ_{c} (upper panels) and of the virial overdensity Δ_{V} (lower panels) for the same EdS and ΛCDM cosmologies. In the left panels, the analyses based on the ESCM are restricted to a halo of 10^{11} $M_\odot/h$ since for galactic masses the effect will be enhanced, while on the right panels we consider also the effect of distinct masses. As before, we concentrate our analyses to three different mass scales: galactic

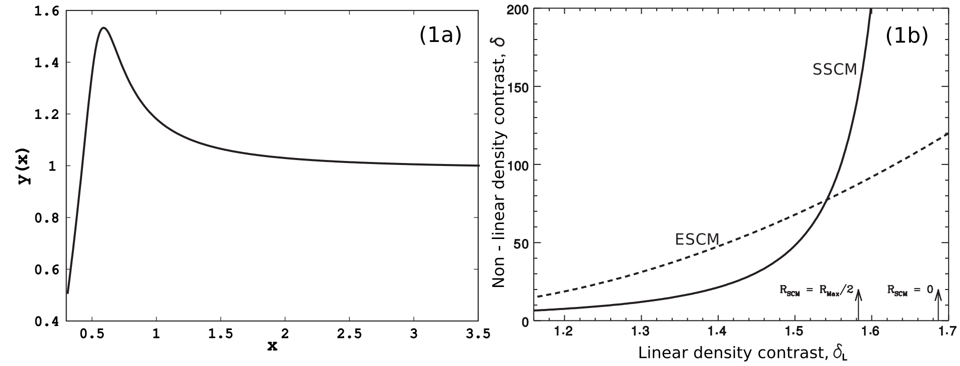

Fig. 1. Left panel: plot of the scaled radius of the shell $y(x)$ as a function of scaled time x (solid line). Right panel: non-linear density contrast in the SSCM (solid line) and in the extended ESCM (dashed line), plotted against the linearly extrapolated density contrast δ_L

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($\approx 10^{11}\ M_{\odot}/h$), groups ($\approx 10^{13}\ M_{\odot}/h$) and clusters ($\approx 10^{15}\ M_{\odot}/h$). As expected from the analysis of Fig. 2, with the growth of the mass the effect of the extra term in the ESCM becomes negligible, and we recover the same values of the SSCM case. It is also worth to notice that the results for the ΛCDM model reduce to the ones of the EdS model for sufficiently high redshifts, since the influence of the cosmological constant becomes rapidly negligible. We will therefore concentrate only on the analysis of the left panels. For the different line colors and styles, we direct to the caption of the figure.

As expected, the $\delta_{\rm c}$ for the ESCM is $\sim 40\%$ higher than for the SSCM case and it decreases towards high redshifts, since the effect of the extra term becomes smaller. For the EdS model, $\delta_{\rm c}$ decreases from a value of ≈ 2.3 at $z = 0$ to ≈ 2.1 at $z = 10$. As expected, the linear overdensity parameter for the ΛCDM model is smaller than the EdS one. This is understood by taking into account that, if we want to have the same number of structures now, we need to have a faster growth of structures to overcome the influence of the cosmological constant. This translates into a lower $\delta_{\rm c}$.

In the lower panels we compare the behavior of $\Delta_{\rm V}$ in the SSCM approach with the one predicted by the ESCM description. The red dashed (blue short dashed) curve show the standard and the extended results for an EdS model, while the green dotted curve represents a ΛCDM model. It is clear that the ESCM description affects also the virial overdensity parameter. In particular, we see that $\Delta_{\rm V}$ is always constant in time for the EdS model. However, with the extra term its value increases reaching $\Delta_{\rm V} \approx 185$, about 4% higher than the standard result. The curve for the ΛCDM model approximates the EdS at high redshifts, as expected. Once again, higher masses are less affected by the ESCM correction term (lower right panel).

4. CONCLUSIONS

In this paper we have discussed how shear and rotation affect the standard spherical collapse model. The net effect of such quantities which is $\propto (\sigma^2 - \omega^2)$

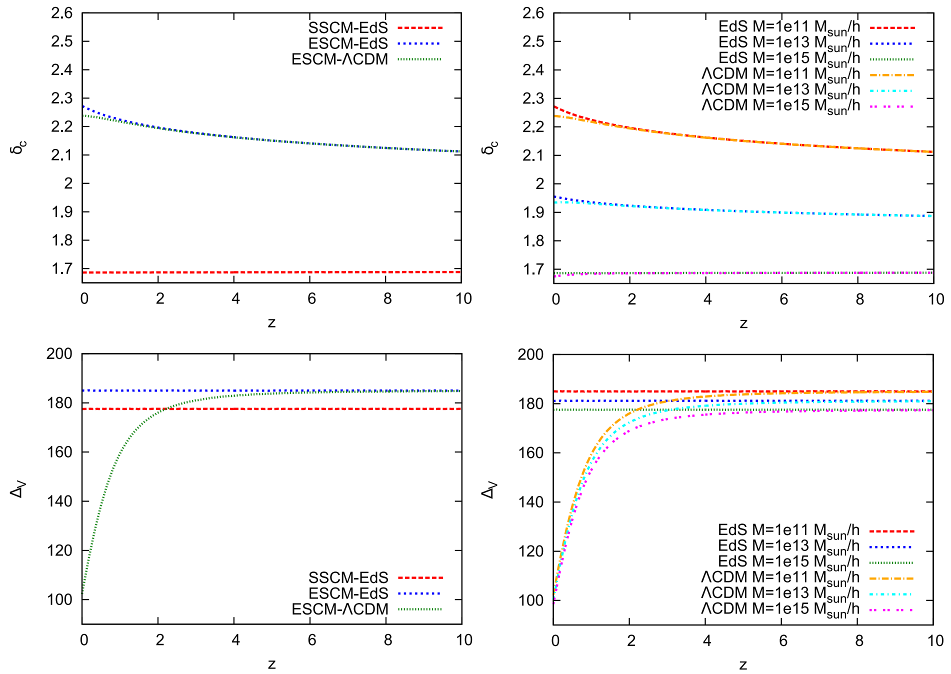

Fig. 2. Upper (lower) panels: evolution of δ_c (Δ_V) with respect to the redshift z for the EdS and the ΛCDM models. The left panels show the time evolution for both parameters at galactic scale (10^{11} $M_\odot/h$). The red curve represents the solution for the EdS model in the non-rotating case while the blue and the green curves represent the EdS and the ΛCDM model when rotation is included. On the right panels we compare the time evolution for three different masses, 10^{11}, 10^{13} and 10^{15} $M_\odot/h$. Different colors and line-styles correspond to different masses and different cosmological models: red dashed (orange dot-dashed) curve represents a halo of 10^{11} $M_\odot/h$ in a EdS (ΛCDM) cosmological background, blue short-dashed (dot-short-dashed cyan) curve represents a halo of 10^{13} $M_\odot/h$ for an EdS (ΛCDM) model, while the green dotted (magenta) curve stands for an object of 10^{15} $M_\odot/h$ in an EdS (ΛCDM) model.

has been phenomenologically described by a power law on the density contrast depending on two parameters (A and B), fixed by comparing the threshold of collapse, δ_c as discussed in Sheth & Tormen (2002), with the δ_c value which is directly obtained from Eq. (8). We have focused our discussion on the influence of such an extra term on (a) the spherical collapse parameters δ_c and Δ_V, and (b) the natural introduction of virialization in the SSCM.

The last point (b) was shown in Fig. 1 , showing that the collapse does not go to a singularity, but reaches a maximum and goes softly down to the virialization radius (Fig. 1, left panel). Similarly the right panel of Fig. 1 compares the non-linear contrast in the SSCM and that in the modified one, ESCM, in terms of the linear density profile. The plot shows the different behavior of the SSCM and the ESCM. Concerning δ_c ($\approx 40\%$), and Δ_V, as it should be expected, the extra term slows down the collapse, and, as such, higher values for the initial perturbations are required in order to have a collapse at the same time of a spherical collapsing sphere. It is also found that the extra term contribution is more important for galactic scales so that its contribution becomes negligible at high masses (galaxy clusters). In Fig. 2 we have numerically evaluated and compared the evolutionary behavior of both the ESCM and SSCM approaches. We have seen that both the linear and the non-linear virial overdensity in the extended spherical collapse model are enhanced with respect to the standard spherical case. Enhancements are more pronounced for δ_c ($\approx 40\%$), while for Δ_V are only of the order of few percent.

These results reinforce the importance of a more complete and rigorous treatment involving the effects of shear and rotation at the late stages of the collapsing halo history mainly for the galactic scales.

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