Dynamical Instability of Ultra-spinning Myers-Perry Black Holes

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Abstract

We study the dynamical stability of Myers-Perry black holes with single rotation parameter. We derived the gravitational perturbation equation of the most symmetric mode, which is given by a system of partial differential equations. We solve the partial differential equations by the relaxation method and find the marginally stable modes in $d = 6, 7$ dimensions. This result indicates not only the instability of Myers-Perry black holes with a large angular momentum, but also the existence of a sequence of new black hole solutions in $d \geq 6$ dimensions.

1 Introduction

Recently, various higher dimensional black holes have been found motivated by the string theory. A generalization of Kerr black holes to higher dimensions was obtained by Myers and Perry many years ago. There are also many exotic black holes in higher dimension such as black ring, etc. It is important to study the stability of these black holes in order to reveal the nature of higher dimensional gravity.

The stability of higher dimensional Schwarzschild black holes has been shown in [1]. For Myers-Perry black holes with equal angular momenta, the stability was shown for partial modes [2, 3]. On the other hand, it was predicted that Myers-Perry black holes with single rotation parameter can be unstable for $d \geq 6$ [4]. In $d \geq 6$ dimensions, Myers-Perry black holes with single rotation parameter approach black brane solutions for a sufficiently large angular momentum because of the centrifugal force. (In $d = 4, 5$ dimensions, the angular momentum of the Myers-Perry black hole has upper bound and we cannot take this limit.) It is known that the black brane solutions are unstable and, therefore, we can expect that Myers-Perry black holes with single rotation parameter are unstable for a large angular momentum. The purpose of this paper is to show the instability. The spacetime of the Myers-Perry black hole with single rotation parameter has two inhomogeneous directions. Thus, the gravitational perturbation equation is given by the system of partial differential equations in general. We will solve the partial differential equations numerically and find the instability of ultra-spinning Myers-Perry black holes.

2 Myers-Perry black holes with single rotation parameter

In this paper, we will concentrate on the Myers-Perry black holes with single angular momentum. The metric is given by

$$ds^2 = -\frac{\rho^2\Delta}{\Sigma^2}dt^2 + \frac{\Sigma^2\sin^2\theta}{\rho^2}\left(d\phi - \frac{2Ma}{\Sigma^2 r^{d-5}}dt\right)^2 + \frac{\rho^2}{\Delta}dr^2 + \rho^2 d\theta^2 + r^2\cos^2\theta d\Omega_{d-4}^2 \ , \tag{1}$$

where

$$\Delta = r^2 + a^2 - \frac{2M}{r^{d-5}} \ , \quad \rho^2 = r^2 + a^2\cos^2\theta \ , \quad \Sigma^2 = (r^2+a^2)^2 - a^2\Delta\sin^2\theta \ . \tag{2}$$

The horizon of this black holes is located at $\Delta(r_+) = 0$. The symmetry of this spacetime is $R_t \times U(1) \times SO(d-3)$, where R_t is the time translation symmetry, $U(1)$ is the rotational symmetry generated by ∂_ϕ and $SO(d-3)$ is the symmetry of $d\Omega_{d-4}^2$ part. The r and θ directions are inhomogeneous. This

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symmetry is not enough to separate gravitational perturbation equations and the perturbation equations are given by partial differential equations of r and θ coordinates. Hereafter, we will use the another radial coordinate defined by $x = \int_{r_+}^{r} \frac{dr}{\Delta^{1/2}}$. In the x coordinate, the horizon is located at $x = 0$.

3 Perturbation equations

In [4], it is predicted that the Gregory-Laflamme type instability appears in the ultra-spinning Myers-Perry spacetime. Because the original Gregory-Laflamme instability appears in s-wave of the spherical part of the metric, we can expect that the instability of Myers-Perry black holes also appear in the most symmetric mode. Thus, we concentrate on the gravitational perturbation retaining the back ground symmetry, $U(1) \times SO(d-3)$. Moreover, to find the onset of the instability, we can also restrict the perturbation to the static perturbation. The metric which have the symmetry $R_t \times U(1) \times SO(d-3)$ is given by

$$ds^2 = e^{2\chi}(dx^2 + d\theta^2) - e^{\alpha-\beta+2(d-4)\gamma}dt^2 + e^{\alpha+\beta+2(d-4)\gamma}(d\phi + Adt)^2 + e^{-2\gamma}d\Omega_{d-4}^2 \ , \tag{3}$$

where functions α, β, γ, A and χ are depend on x and θ. Now, we consider the infinitesimal variation of these functions as $\alpha \to \alpha + \tilde{\alpha}$, $\beta \to \beta + \tilde{\beta}$, $\gamma \to \gamma + \tilde{\gamma}$, $A \to A + \tilde{A}$ and $\chi \to \chi + \tilde{\chi}$, where variables with tilde represent perturbed variables and background variables are given by

$$\begin{aligned} &\alpha = \frac{1}{2}\ln\left[\Delta\sin^2\theta(r^2\cos^2\theta)^{2(d-4)}\right] \ , \quad \beta = \frac{1}{2}\ln\left[\frac{\Sigma^4\sin^2\theta}{\rho^4\Delta}\right] \ , \\ &\gamma = -\frac{1}{2}\ln\left[r^2\cos^2\theta\right] \ , \quad A = -\frac{2Ma}{\Sigma^2 r^{d-5}} \ , \quad \chi = \ln\rho \ , \quad . \end{aligned} \tag{4}$$

The perturbation equations are given by

$$\partial^2\tilde{\alpha} + \partial(\Phi+\alpha)\cdot\partial\tilde{\alpha} + (d-4)\partial\alpha\cdot\partial\tilde{\gamma} - 4(d-4)(d-5)e^{2\gamma+2\chi}(\tilde{\gamma}+\tilde{\chi}) = 0 \ , \tag{5}$$

$$\partial^2\tilde{\beta} + \partial\Phi\cdot\partial\tilde{\beta} + (d-4)\partial\beta\cdot\partial\tilde{\gamma} + \partial\beta\cdot\partial\tilde{\alpha} + 2e^{2\beta}\partial A\cdot(\partial A\,\tilde{\beta} + \partial\tilde{A}) = 0 \ , \tag{6}$$

$$\partial^2\tilde{\gamma} + \partial\{\Phi + (d-4)\gamma\}\cdot\partial\tilde{\gamma} + \partial\gamma\cdot\partial\tilde{\alpha} + 2(d-5)e^{2\gamma+2\chi}(\tilde{\gamma}+\tilde{\chi}) = 0 \ , \tag{7}$$

$$\partial^2\tilde{A} + \partial(\Phi+2\beta)\cdot\partial\tilde{A} + (d-4)\partial A\cdot\partial\tilde{\gamma} + \partial A\cdot\partial\tilde{\alpha} + 2\partial A\cdot\partial\tilde{\beta} = 0 \ , \tag{8}$$

$$\begin{aligned} \partial^2\tilde{\chi} - \Big[&\frac{1}{2}\partial\alpha\cdot\partial\tilde{\alpha} - \frac{1}{2}\partial\beta\cdot\partial\tilde{\beta} + \frac{1}{2}e^{2\beta}(\partial A)^2\tilde{\beta} + \frac{1}{2}e^{2\beta}\partial A\cdot\partial\tilde{A} \\ &- (d-3)(d-4)\partial\gamma\cdot\partial\tilde{\gamma} - (d-4)(d-5)e^{2\gamma+2\chi}(\tilde{\gamma}+\tilde{\chi})\Big] = 0 \ . \end{aligned} \tag{9}$$

where $\Phi \equiv (d-4)\gamma + \alpha$. There are also two constraint equations,

$$\begin{aligned} C_1 \equiv &-\partial_x\partial_\theta\tilde{\Phi} + \partial_\theta\chi\partial_x\tilde{\Phi} + \partial_x\chi\partial_\theta\tilde{\Phi} - \partial_x\Phi\partial_\theta\tilde{\Phi} - \partial_\theta\Phi\partial_x\tilde{\Phi} + \partial_\theta\tilde{\chi}\partial_x\Phi + \partial_x\tilde{\chi}\partial_\theta\Phi + \tilde{\beta}e^{2\beta}\partial_x A\partial_\theta A \\ &+ \frac{1}{2}\left[\partial_x\alpha\partial_\theta\tilde{\alpha} - \partial_x\beta\partial_\theta\tilde{\beta} + e^{2\beta}\partial_x A\partial_\theta\tilde{A} - 2(d-3)(d-4)\partial_x\gamma\partial_\theta\tilde{\gamma}\right] \\ &+ \frac{1}{2}\left[\partial_\theta\alpha\partial_x\tilde{\alpha} - \partial_\theta\beta\partial_x\tilde{\beta} + e^{2\beta}\partial_\theta A\partial_x\tilde{A} - 2(d-3)(d-4)\partial_\theta\gamma\partial_x\tilde{\gamma}\right] = 0 \ , \end{aligned} \tag{10}$$

$$\begin{aligned} C_2 \equiv &-\partial_\theta^2\tilde{\Phi} - \partial_x\chi\partial_x\tilde{\Phi} + \partial_\theta\chi\partial_\theta\tilde{\Phi} - 2\partial_\theta\Phi\partial_\theta\tilde{\Phi} - \partial_x\tilde{\chi}\partial_x\Phi + \partial_\theta\tilde{\chi}\partial_\theta\Phi - \frac{1}{2}\tilde{\beta}e^{2\beta}\{(\partial_x A)^2 - (\partial_\theta A)^2\} \\ &+ \frac{1}{2}\left[\partial_\theta\alpha\partial_\theta\tilde{\alpha} - \partial_\theta\beta\partial_\theta\tilde{\beta} + e^{2\beta}\partial_\theta A\partial_\theta\tilde{A} - 2(d-3)(d-4)\partial_\theta\gamma\partial_\theta\tilde{\gamma}\right] \\ &- \frac{1}{2}\left[\partial_x\alpha\partial_x\tilde{\alpha} - \partial_x\beta\partial_x\tilde{\beta} + e^{2\beta}\partial_x A\partial_x\tilde{A} - 2(d-3)(d-4)\partial_x\gamma\partial_x\tilde{\gamma}\right] \\ &+ (d-4)(d-5)e^{2\gamma+2\chi}(\tilde{\gamma}+\tilde{\chi}) = 0 \ , \end{aligned} \tag{11}$$

The left hand sides of constraint equations C_1 and C_2 satisfy

$$\partial_\theta(e^\Phi \tilde{C}_1) - \partial_x(e^\Phi \tilde{C}_2) = 0 \ , \quad \partial_x(e^\Phi \tilde{C}_1) + \partial_\theta(e^\Phi \tilde{C}_2) = 0 \ , \qquad (12)$$

where we used equations (5-9). These equations are nothing but Cauchy-Riemann equations. Therefore, if the constraint equations are satisfied at the boundary, the Eqs.(5-9) guarantee the constraint equations at whole region.

4 A method to find the instability

In this section, we explain how to solve the partial differential equations (5-11). For numerical calculation, the variables $(\tilde{\alpha}, \tilde{\beta}, \tilde{\gamma}, \tilde{A}, \tilde{\chi})$ are not good variables because some coefficients of perturbation variables in development equations (5-9) diverge at axes $\theta = 0, \pi/2$. Now, we try to eliminate the singularities at axes by changing variables and using the constraint equation (11). We define new variables $(\tilde{p}, \tilde{q}, \tilde{s})$ as

$$\tilde{p}(x,\theta) \equiv \frac{\tilde{\chi} + \tilde{\gamma}}{1+y} \ , \quad \tilde{q}(x,\theta) \equiv \frac{\tilde{\chi} - (\tilde{\alpha} + \tilde{\beta})/2 - (d-4)\tilde{\gamma}}{1-y} \ , \quad \tilde{s}(x,\theta) \equiv \frac{1}{2}\left(\tilde{\alpha} - \tilde{\beta}\right) + (d-4)\tilde{\gamma} \ , \qquad (13)$$

where $y = \cos 2\theta$. In term of the new set of variables $(\tilde{p}, \tilde{q}, \tilde{s}, \tilde{A}, \tilde{\chi})$, after some appropriate linear combinations of the development equations (5-9), we obtain

$$\left[\partial_x^2 + \mathcal{M}_1 \partial_y^2 + \mathcal{M}_2 \partial_x + \mathcal{M}_3 \partial_y + \mathcal{M}_4\right] \boldsymbol{V} = 0 \ , \qquad (14)$$

where we define $\boldsymbol{V} = (\tilde{p}, \tilde{q}, \tilde{s}, \tilde{A}, \tilde{\chi})^T$ and $\mathcal{M}_i$ $(i = 1,2,3,4)$ are 5×5 matrices whose components are functions of x and $y = \cos 2\theta$. Some components of $\mathcal{M}_i$ still diverge at $\theta = 0, \pi/2$. To eliminate these singularities, we use constraint equation (11). We add a vector $\boldsymbol{V}' = (-\tilde{C}_2/(1+y), -\tilde{C}_2/(1-y), -(d-3)\tilde{C}_2, 0, 0)^T$ to the left hand side of (14). Then, functional matrices are changed as $\mathcal{M}_i \to M_i$ and new matrices M_i are regularized at $\theta = 0, \pi/2$. To find stationary perturbation, we calculate the eigen value of the operator $\mathcal{O} \equiv \partial_x^2 + M_1 \partial_y^2 + M_2 \partial_x + M_3 \partial_y + M_4$. We study the distribution of eigen values of the operator $\mathcal{O}$ with various rotation parameter a and trace the variation of eigen values. If a eigen value cross $\lambda = 0$, it means the existence of the zero mode.

5 Instability of ultra-spinning Myers-Perry black holes

We solve the eigen value of the operator $\mathcal{O}$ by the relaxation method. The domain of our calculation is $\{(x,y) | 0 \le x \le x_{\max}, -1 \le y \le 1\}$, where $x_{\max}$ is given by $x_{\max}(r = r_{\max})$ where $r_{\max} = 5.0 r_+$. The number of grids for x and y directions are $N_x = 30$ and $N_y = 40$, respectively.

Firstly, we consider the case of $d = 7$. The eigen value for various rotation parameter a is depicted in Figure.1. We find that the eigen value crosses $\lambda = 0$ at $a/r_+ = 1.41$ and $a/r_+ = 3.09$. The zero mode at $a/r_+ = 1.41$ is not a onset of the instability. The reason is as follows. Thermodynamical parameters of Myers-Perry black holes are given by

$$M = \frac{1}{2} r_+^{d-5}(r_+^2 + a^2) \ , \quad J = \frac{1}{d-2} a r_+^{d-5}(r_+^2 + a^2) \ , \quad T = \frac{(d-3)r_+^2 + (d-5)a^2}{4\pi r_+ (r_+^2 + a^2)} \ , \quad \Omega = \frac{a}{r_+^2 + a^2} \ , \ (15)$$

Then, the Jacobian of (T, Ω) and (M, J) is given by $\partial(T,\Omega)/\partial(M,J) \propto (d-5)(a/r_+)^2 - (d-3)$. Thus, for $a/r_+ = \sqrt{(d-3)/(d-5)}$, the transformation of (T, Ω) to (M, J) is not one to one correspondence. In our calculation, we impose conditions at the horizon that temperature and angular velocity of the stationary perturbation do not change to exclude mass and angular momentum perturbations of Myers-Perry black holes which always exist. However, these conditions are not enough to exclude trivial perturbations at $a/r_+ = \sqrt{(d-3)/(d-5)}$. Hence, the zero mode at $a/r_+ = 1.41$ is just a trivial perturbation. Thus, for $d = 7$, the onset of instability is given by $a/r_+ = 3.09$.

By the similar way, we can study the eigen value in $d = 6$ dimensions and find zero eigen values. We summarize the onset of instability as

$$a/r_+ = 4.06 \ (d = 6) \ , \quad a/r_+ = 3.09 \ (d = 7) \ . \qquad (16)$$

We checked that these results do not depend on N_x, N_y and $x_{\max}$.

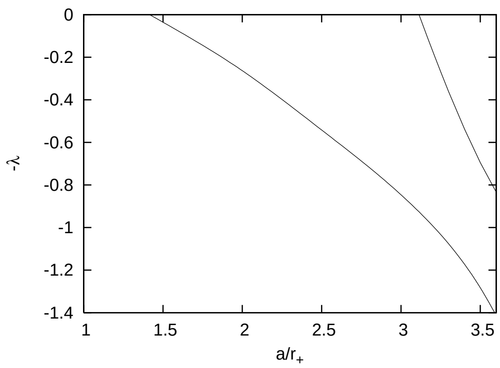

Figure 1: This is the variation of the eigen value with respect to rotation parameter a. We see the eigen value cross the $\lambda = 0$ at $a/r_+ = 1.41$ and $a/r_+ = 3.09$.

6 Summary

We studied the stability of Myers-Perry black holes with single rotation parameter. We derive the perturbation equations for the most symmetric mode. The perturbation equations are given by partial deferential equations. We solved the equations by relaxation method and find the instability in $d = 6, 7$ dimensions. We found the static perturbation of the ultra-spinning Myers-Perry black hole to see the onset of the instability. The existence of the static perturbation indicates that there exist a sequence of new black hole solutions even in non-linear regime.

Recently, the stability analysis of higher dimensional black holes becomes important beyond the gravity theory itself, because of the Gauge/Gravity duality. Since the instabilities of the AdS black holes corresponds to the phase transitions of the dual theories, the stability analysis of the AdS black holes may be the useful tool to understand the phase structure of the dual theory. The Myers-Perry black holes can be generalized to those with the cosmological constant, the so-called Kerr-AdS spacetimes. In Kerr-AdS spacetimes, an instability is also expected, which is similar to the instability found in this paper. The instability of Kerr-AdS black holes may be found using the method to show the instability of Myers-Perry black holes.

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