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Tetrad formalism in the solution of spherically symmetric spacetime in general relativity

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Abstract. Spherically symmetric solutions in general relativity are the most fundamental solutions to the Einstein field equation. The first exact solution of the Einstein field equation is the spherically symmetric solution given by the Schwarzschild metric, as easily found in any standard textbook on general relativity. The FLRW (Friedmann-Lemaitre-Robertson-Walkers) metric is another spherically symmetric solution of Einstein's equation describing the standard model in Cosmology. The standard approach to solving Einstein's equations is by considering the metric. However, we can also adopt a tetrad-based method or tetrad formalism. We review these two solutions by the tetrad formalism as an alternative approach. In addition, we give some more cases, including the cosmological constant and the Taub-NUT metric.

1. Introduction

General relativity theory provides the physical framework for studying gravitation and the nature of spacetime. Einstein's Theory of General Relativity describes gravity as a result of a mass warping spacetime, not distances or masses. It provides a more comprehensive understanding of gravity, especially in extreme conditions close to ultra-compact objects and black holes. This theory successfully enlarged the scope of Newtonian gravity elegantly. However, finding the Einstein field equations solutions is a painful task since the equations are non-linear partial differential equations that may depend on the energy-momentum tensor of the matter fields.

The first solution to Einstein's equations is the spacetime metric of the Schwarzschild distribution. This solution applies to the vacuum region outside the region of static symmetric matter because the mass does not have an electric charge or angular momentum and no cosmological constant. The spherically symmetric condition was chosen because it is the simplest solution and accurately describes natural astrophysical bodies.

Indeed, there are many approaches to finding the spherically symmetric solutions. As well as the standard metric formalism, tetrad formalism gives an alternative. Tetrad or *vierbeins* are a set of four linearly free vector fields. Tetrad method could directly accommodate non-uniform stress, has an unambiguous gauge, and is expressed in physical (non-comoving) radial coordinate, resulting in clear and intuitive physical interpretation. Tetrad formalism also has no limitations on its use [1]. In this article, we review the spherically symmetric solutions for Schwarzschild, FLRW, Schwarzschild-de Sitter, and Taub-NUT spacetimes in tetrad formalism. For Schwarzschild, FLRW, and Schwarzschild-de Sitter spacetimes represent static spacetime, as well as the Taub-NUT spacetime which represents non-static spacetime.

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2. Tetrad formalism

2.1. Tetrad field

In Riemannian geometry, every event occurring at every point in space-time is described by a set of x^μ-coordinates (global coordinate) with a set of basis vectors e_μ. Therefore, the metric tensor is given by

$$g_{\mu\nu} = e_\mu \cdot e_\nu \,. \tag{1}$$

Moreover, at every point in the manifold, the Lorentz frames are defined by a set of orthogonal basis vectors $\hat{e}_a$ for the tangent space. Those orthogonal basis vector are connected to the Minkowski metric and are independent from any coordinate system,

$$\eta_{\mu\nu} = \hat{e}_a \cdot \hat{e}_b \,. \tag{2}$$

The relation between the two sets of basis vectors (global basis vector and local basis vector) is described by a bundle of linearly free vector fields called *tetrad* or *vierbeins* e_a^μ and its inverse e_μ^a. Taking into account the invariance of the line elements (which have the same value locally and globally), we have the relation

$$\begin{gathered} ds^2{}_{\text{global}} = ds^2{}_{\text{local}} \\ g = \eta \\ g_{\mu\nu} = e_\mu^a e_\nu^b \eta_{ab} \,, \end{gathered} \tag{3}$$

and the inverse relation

$$\eta_{ab} = e_a^\mu e_b^\nu g_{\mu\nu} \,. \tag{4}$$

Finally, it is clear that tetrad are square root of metric [1][2].

2.2. Spin connection

Analogous to the concept of parallel transport, to compare different vectors in cotangent space introduced another parallel transport which determined by the spin connection. In differential geometry that is not dependent on any coordinate, affine connection coefficient $\Gamma^\alpha_{\beta\gamma}$ is replaced by spin connection coefficient $\omega^a_{\mu b}$, but those two are just the same in principle [3][4].

$$\nabla_\mu X^a = \partial_\mu X^a + \omega^a_{\mu b} X^b \,. \tag{5}$$

The relation between affine connection and spin connection is given by

$$\omega^a_{\mu b} = e^a_\nu e^\lambda_b \Gamma^\nu_{\mu\lambda} - e^\lambda_b \partial_\mu e^a_\lambda \,. \tag{6}$$

2.3. Riemann tensor in tetrad formalism

In the context of Rimannian geometry, the curvature of space-time is described by Riemann curvature or the Riemann tensor. This tensor is built from second order covariant derivative from a vector. In calculus, second order derivative tells the curvature of the given function. In the same sense, Riemann tensor also tells the curvature of the given manifold and often said as the "curvature tensor".

$$R^\mu_{\nu\rho\sigma} = \frac{\partial \Gamma^\mu_{\nu\sigma}}{\partial x^\rho} - \frac{\partial \Gamma^\mu_{\nu\rho}}{\partial x^\sigma} + \Gamma^\lambda_{\nu\sigma}\Gamma^\mu_{\rho\lambda} - \Gamma^\lambda_{\nu\rho}\Gamma^\mu_{\sigma\lambda} \,. \tag{7}$$

The Riemann tensor could also be expressed in the covariant form by contraction with the metric tensor,

$$R_{\mu\nu\rho\sigma} = g_{\mu\lambda} R^\lambda_{\nu\rho\sigma} \tag{8}$$

and if written in the dorm of tetrad terms becomes

$$R_{\mu\nu\rho\sigma} = e^a_\mu e^b_\nu e^c_\rho e^d_\sigma R_{abcd} \;\leftrightarrow\; R_{abcd} = e^\mu_a e^\nu_b e^\rho_c e^\sigma_d R_{\mu\nu\rho\sigma} \,. \tag{9}$$

2.4. Einstein's field equation in tetrad formalism
The Ricci tensor combined with the metric tensor can explain the relationship between the curvature of spacetime and the distribution of matter and energy [5].

$$R_{\mu\nu} - \frac{1}{2}g_{\mu\nu}R = \kappa T_{\mu\nu} \tag{10}$$

In tetrad terms, the Einstein field equation can be written as follows.

$$\begin{gathered} e^a_\mu e^b_\nu R_{ab} - \frac{1}{2}e^a_\mu e^b_\nu \eta_{ab}R = \kappa e^a_\mu e^b_\nu T_{ab} \\ R_{ab} - \frac{1}{2}\eta_{ab}R = \kappa T_{ab} \end{gathered} \tag{11}$$

2.5. Mathematical framework of the tetrad formalism
A spherically symmetric system was chosen by four vector field (tetrad) that was defined in terms of arbitrary functions $f_1(r,t)$, $f_2(r,t)$, $g_1(r,t)$ dan $g_2(r,t)$. From these fucntions, components of the tetrad expressed as follows [2],

$$\begin{aligned} &e^0_0 = f_1, \quad && e^0_1 = f_2, \\ &e^1_0 = g_2, \quad && e^1_1 = g_1, \\ &e^2_2 = 1/r, \quad && e^3_3 = 1/(r\sin\theta). \end{aligned} \tag{12}$$

As such, line element of a spherically symmetric system composed of these tetrad components above has been acquired in this form,

$$ds^2 = \frac{(g_1)^2 - (g_2)^2}{(f_1)^2(g_1)^2}dt^2 + \frac{2g_2}{f_1(g_1)^2}dt\,dr - \frac{1}{(g_1)^2}dr^2 - r^2 d\Omega^2 \tag{13}$$

with $d\Omega$ is a solid angle element, and this system assumes a physical non-moving coordinate for a proper surface, which is a sphere with radius r and surface area $4\pi r^2$.

By the assumption of matter regarded as perfect fluid, the 4-velocity can be constructed from fluid particle (or an observer comoving with the fluid) that has components $[\hat{v}^a] = [1,0,0,0]$ in tetrad frame. Using the fact that $\hat{v}^a\hat{e}_a = v^\mu e_\mu = v^0e_0 + v^1e_1 + v^2e_2 + v^3e_3$, so, the 4-velocity in basis coordinate becomes $[\hat{v}^a] = [f_1, g_2, 0, 0]$ or written in another form $[v^\mu] = \left[\dot{t}, \dot{r}, \dot{\theta}, \dot{\phi}\right] = [f_1, g_2, 0, 0]$. Next, two linear differential operators is given by,

$$L_t \equiv f_1\partial_t + g_2\partial_r \,, \qquad L_r \equiv g_1\partial_r. \tag{14}$$

Then, arbitrary functions $F(r,t)$, $G(r,t)$ and $M(r,t)$ are defined by

$$L_t g_1 \equiv G g_2 \,, \qquad L_r g_2 \equiv F g_1, \tag{15}$$

$$M \equiv \frac{1}{2}r\left((g_2)^2 - (g_1)^2 + 1 - \frac{1}{3}\Lambda r^2\right), \tag{16}$$

with Λ as a constant. From the assumption of perfect fluid with density $\rho(r,t)$ and pressure $p(r,t)$, the Einstein field equation and Bianchi identities are used to find the relations among above arbitrary functions [6]

$$L_r f_1 = -G f_1, \qquad \longrightarrow \qquad f_1 = \exp\left[-\int \frac{G}{g_1}dr\right] \tag{17}$$

$$L_r g_1 = F g_2 + \frac{M}{r^2} - \frac{1}{3}\Lambda r - 4\pi r\rho, \qquad L_t g_2 = G g_1 - \frac{M}{r^2} - \frac{1}{3}\Lambda r - 4\pi r\rho \tag{18}$$

$$L_t M = -4\pi g_2 r^2 p, \qquad L_r M = 4\pi g_1 r^2 p \tag{19}$$

$$L_t\rho = -\left(\frac{2g_2}{r} + F\right)(\rho + p), \qquad L_r p = -G(\rho + p). \tag{20}$$

From the equation 19, it is clear that the function $M(r,t)$ plays a huge role as intrinsic mass (or energy) and in equation 20 shows that the function $G(r,t)$ can be interpreted as radial acceleration, and Λ is the cosmological constant indeed. F and G can also be defined as spin connection coefficients.

Furthermore, by combining the expressions $L_t M$ and the definition of M in 15, we have the following expression as Euler equation in Newtonian fluid dynamics $g_2 \equiv \dot{r}$.

$$(\partial_t + g_2\partial_r)g_2 = -\frac{M}{r^2} + \frac{1}{3}\Lambda \tag{21}$$

Finally, setting the constant $\Lambda = 0$, the definition of M can be rearranged to acquire the infamous Bernoulli equation in zero pressure.

$$\frac{1}{2}(g_2)^2 - \frac{M}{r} = \frac{1}{2}\left((g_1)^2 - 1\right) \tag{22}$$

3. The spherically symmetric spacetimes via tetrad formalism

3.1. Schwarzschild space-time

In this case, a static matter source ($M =$ constant) was given in a concentrated point $r = 0$ and the cosmological constant is not taken into account ($\Lambda = 0$). The solution for this case has zero pressure and density $\rho = p = 0$ everywhere except the centre. By the definition of M in (25), g_1 seemed to correspond to the total energy per unit rest mass of the free falling particle and the gauge condition $g_1 = 1$ could be adopted for particle released from spatial infinity. Finally, the tetrad components are expressed as follows

$$f_1 = 1, \qquad g_1 = 1, \qquad g_2 = -\sqrt{\frac{2M}{r}}, \tag{23}$$

and spin connection coefficients,

$$F = \frac{M}{r^2}\left(\frac{2M}{r}\right)^{-1/2}, \qquad G = 0. \tag{24}$$

This condition ($G = 0$) is consistant with the geodesics of test particle. Next, by subtituting equation (23) to the metric ansatz (13), we found the solution to the field equation for static spherically symmetric or Schwarzschild spacetime in Painleve-Gullstrand coordinate,

$$ds^2 = dt^2 - \left(dr + \sqrt{\frac{2M}{r}}dt\right)^2 - r^2 d\Omega^2, \tag{25}$$

or in standard coordinate,

$$ds^2 = \left(1 - \frac{2M}{r}\right)dt^2 - \left(1 - \frac{2M}{r}\right)^{-1} dr^2 - r^2 d\Omega^2. \tag{26}$$

3.2. FLRW spacetime

FLRW spacetime is homogeneous and isotropic. This makes ρ and p a function of t only, so that $M(r,t) = \frac{4}{3}\pi r^3 \rho$. Therefore, the non-zero tetrad component is given by

$$f_1 = 1\,, \quad {g_1}^2 = 1 - kr^2 \exp\left[-2\int H(t')dt'\right], \quad g_2 = rH(t) \tag{27}$$

where k is an arbitrary constant of integration and the spin connection coefficients F and G become

$$F = H(t) \qquad G = 0. \tag{28}$$

The condition $G = 0$ explains the geodesic motion of the fluid particles because there is no pressure gradient. Using the Hubble parameter equation $H^2(t) \equiv \left(\frac{\partial_t S(t)}{S(t)}\right)^2$, so that $H(t) \equiv \frac{\partial_t S(t)}{S(t)}$ and ${g_1}^2 = 1 - k\frac{r^2}{S^2(t)}$, the line element (4.6) corresponding to the tetrad component (20) becomes

$$ds^2 = dt^2 - \left[1 - \frac{kr^2}{S^2(t)}\right]^{-1} (dr - rH(t)dt)^2 - r^2 d\Omega^2 \tag{29}$$

By redefining the radial coordinates $\tilde{r} \equiv \frac{r}{S(t)}$, the metric (41) will be the general form of the FLRW metric in standard coordinates [7].

$$ds^2 = dt^2 - S^2(t)\left[\frac{d\tilde{r}^2}{1 - k\tilde{r}^2} + \tilde{r}^2 d\Omega^2\right] \tag{30}$$

3.3. Schwarzschild-de Sitter spacetime

Schwarzschild-de Sitter spacetime is an extension of Schwarzschild spacetime that includes a positive cosmological constant (Λ). The existence of the cosmological constant (Λ) represents the expanding universe. In general, Schwarzschild-de Sitter spacetime has similar properties to Schwarzschild spacetime, except that, in addition to the mass (M), the geometry of Schwarzschild-de Sitter spacetime is also affected by the presence of the cosmological constant (Λ). Therefore, the procedure for finding the following spherically symmetric solution of Schwarzschild-de Sitter spacetime will be the same as that of Schwarzschild spacetime, but by maintaining the existence of the cosmological constant [8] [9] [10].

The following is the result of the non-zero tetrad component

$$f_1 = 1, \qquad g_1 = 1, \qquad g_2 = -\sqrt{\frac{2M}{r} + \frac{1}{3}\Lambda r^2}, \tag{31}$$

and spin connection coefficients

$$F = \left(\frac{2M}{r} + \frac{1}{3}\Lambda r^2\right)^{-\frac{1}{2}} \left(\frac{M}{r^2} - \frac{1}{3}\Lambda r\right), \qquad G = 0. \tag{32}$$

Thus, we obtained the Schwarzschild-de Sitter spacetime metric expressed in Painleve-Gullstrand coordinates [11] [12].

$$ds^2 = dt^2 - \left(dr + \sqrt{\frac{2M}{r} + \frac{1}{3}\Lambda r^2}dt\right)^2 - r^2 d\Omega^2 \tag{33}$$

Thus, the standard form of the Schwarzschild-de Sitter metric is as follows.

$$ds^2 = \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right) dt^2 - \left(1 - \frac{2M}{r} - \frac{1}{3}\Lambda r^2\right)^{-1} dr^2 - r^2 d\Omega^2 \tag{34}$$

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3.4. Taub-NUT spacetime

In 1951, Taub proposed a homogeneous, non-isotropic, expanding, empty model of the universe solution later called as Taub space. Amongst homogenous cosmological models, Taub-NUT metric yields the most complex anisotropic behaviour, and belongs to the Bianchi type IX spatial geometry group [13] [14] [15]. This metric can be constructed by determining the tetrad coefficient $g_2 = 0$, then we have,

$$f_1 = \left(1 + \frac{t^2}{4}\right)^{-1}, \qquad g_1 = \left(1 + \frac{t^2}{4}\right), \tag{35}$$

and the spin connection coefficients

$$F = 0, \qquad G = \left(1 + \frac{t^2}{4}\right)^{-1} \frac{4M}{t^2}. \tag{36}$$

After following the same formalism as previous metrics, we have the Taub metric in spherical coordinate,

$$ds^2 = \left(1 + \frac{t^2}{4}\right)^{-2} (dt^2 - dx^2) - \frac{t^2}{4}(d\theta^2 + \sin^2\theta d\phi^2). \tag{37}$$

4. Conclusion

From the work of this article, the metric solutions of spherically symmetric spacetime can be obtained through the review of tetrad formalism. The method provides more intuitive interpretation and less complicated calculations than the standard formalism. By determining the properties of the spacetime to be solved and choosing the appropriate 'gauge' condition, the tetrad formalism can directly construct the metric corresponding to the spacetime.

References

[1] Kim D Y, Lasenby A N and Hobson M P 2018 *General Relativity and Gravitation* **50** 1–37
[2] Nandra R, Lasenby A N and Hobson M P 2012 *Monthly Notices of the Royal Astronomical Society* **422** 2931–2944
[3] Yepez J 2011 *arXiv preprint arXiv:1106.2037*
[4] Zhou T 2022 Lorentz gauge and coulomb gauge for tetrad field of gravity. universe 2022, 8, 659
[5] Bambi C 2018 *Introduction to General Relativity: A Course for Undergraduate Students of Physics* (Springer)
[6] Lasenby A, Doran C and Gull S 1998 *Philosophical Transactions of the Royal Society of London. Series A: Mathematical, Physical and Engineering Sciences* **356** 487–582
[7] Gaur R and Visser M 2022 *Journal of Cosmology and Astroparticle Physics* **2022** 030
[8] Weinberg S 2008 *Cosmology* (OUP Oxford)
[9] Hidayat T 2010 *Bandung: Penerbit ITB*
[10] Gautama S E 2016 Pengantar teori relativitas umum dan kosmologi
[11] Volovik G 2023 *Annals of Physics* 169219
[12] Volovik G 2023 *arXiv preprint arXiv:2304.09847*
[13] Halpern P 2006 *Astrophysics and Space Science* **306** 279–283
[14] Cahill M and Taub A 1971 *Communications in Mathematical Physics* **21** 1–40
[15] Hajicek P 1970 *Communications in Mathematical Physics* **17** 109–126