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A PLASMA WAKE FIELD ACCELERATOR†

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1. INTRODUCTION

One of the main parameters that determines the next generation of high energy accelerators is the acceleration gradient. Recently there has been interest in the use of plasma density waves for obtaining high acceleration gradients. The Plasma Beat-Wave scheme of Tajima and Dawson uses two beating lasers to excite the plasma at its resonant frequency.[1] The driven plasma provides an accelerating field that in principle can be of the order of several GeV/m.[2–5] This scheme has also been modified in the Surfatron,[6–8] although the basic principle is similar.

One of the complications of the Plasma Beat-Wave scheme is the need for high-power, high-quality lasers. Recently it has been suggested that the driving lasers could be replaced by a driving electron beam.[9–10] In the simplest case, a single driving electron bunch enters a plasma and excites a plasma-density wave; a second beam trailing the driving beam is then accelerated provided that it is at the correct phase on the plasma wave. It is also possible to have several appropriately spaced bunches in the driving beam.

This idea is similar to various wake field accelerator schemes[11] with the plasma playing the role of the cavity or accelerator structure. In this paper we analyze the idea suggested in Refs. 9 and 10 in more detail and point out the similarities with wake field schemes suggested in Refs. 11–15. After we review some basic results of wake fields, we calculate the wake field in a plasma and show that the plasma wake obeys the general rules of all wake fields. We then address various other accelerator physics issues associated with acceleration in a plasma density wave. Lastly, we give some numerical design examples.

2. ENERGY TRANSFER IN CO-LINEAR WAKE FIELD ACCELERATORS

In this section, we discuss a general property of energy transfer in wake field accelerators, which is valid whether the source of the wake field is a metallic

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cavity or a plasma. The definition of wake field accelerators used in this paper includes only those accelerators in which the stored energy is zero before the driving beam arrives. Although the analysis below can be extended to a more general case, we will specialize to the co-linear case: the driving beam and the trailing beam take the same straight-line path through the wake field medium. We will then show that one basic limitation of co-linear wake field schemes is that the energy gained by a particle in the accelerated beam is severely limited by the energy per particle of the driving bunch.

Consider first a relativistic bunch of electrons entering a structure with a wake field function $W(y)$. $W(y)$ is a characteristic function of the structure independent of the beam and is defined as the longitudinal decelerating field induced by a unit charge at a distance y behind it. The energy change of the bunch due to its own wake per unit distance traveled is given by

$$\frac{d(N_1E_1)}{dz}=-N_1^2e^2W(0), \tag{1}$$

where N_1 is the number of particles in the bunch and E_1 is the energy per particle. Here we have regarded the bunch as a *rigid* collection of particles which has zero length. As we will see later in this section and in Section 4G, these restrictions can be removed.

If a second bunch is injected at a distance y behind the first bunch, it will experience the wake field left by the first bunch as well as its own wake field. Using linear superposition, the energy change of the trailing bunch is given by

$$\frac{d(N_2E_2)}{dz}=-N_2^2e^2W(0)-N_1N_2e^2W(y), \tag{2}$$

where N_2 and E_2 are the number of particles and energy per particle of the trailing bunch. The second term is the contribution from the wake field of the first bunch. Due to energy conservation, the total energy of the system of two beams must not increase, i.e.

$$(N_1^2+N_2^2)W(0)+N_1N_2W(y)\geq 0. \tag{3}$$

Since this must hold for all N_1 and N_2, the accelerating wake field due to the first bunch at the second bunch $[-W(y)]$ must satisfy

$$[-W(y)]\leq 2W(0). \tag{4}$$

Therefore the acceleration gradient seen by a single particle in the trailing bunch must satisfy

$$G\equiv\frac{dE_2}{dz}\leq(2N_1-N_2)e^2W(0). \tag{5}$$

To calculate the maximum total energy gain by the trailing bunch, let us assume that the leading bunch can transfer all of its energy to the wake field. In this case, the leading bunch stops in a distance L given by

$$L=\frac{E_1}{N_1e^2W(0)}. \tag{6}$$

Note that L is inversely proportional to N_1. An intense driving bunch produces a high accelerating gradient for the trailing bunch, but the acceleration lasts only for a short distance.

Using Eqs. (5) and (6), the energy gain for a particle in the trailing bunch satisfies

$$\Delta E_2 = GL \leq E_1\left(2-\frac{N_2}{N_1}\right). \tag{7}$$

Thus we obtain the well known result: in a co-linear wake field accelerator, the total energy gain per particle of the trailing bunch is less than twice the initial energy per particle of the driving bunch. This is a severe limit on co-linear wake field accelerators. Notice that the only assumptions necessary to derive this result are conservation of energy, linear superposition and a rigid point bunch. The above results also hold for a plasma provided we are in the linear regime.

The inequality (Eq. 7) can be made an equality for a single-mode lossless medium in which the wake field oscillates with a single frequency behind the driving bunch. In this case, the energy-transfer efficiency from the driving bunch to the trailing bunch is given by

$$\eta \equiv \frac{\Delta(N_2 E_2)}{N_1 E_1} = \frac{N_2}{N_1}\left(2-\frac{N_2}{N_1}\right). \tag{8}$$

In such a case, the maximum efficiency is achieved by choosing $N_2 = N_1$. The energy of the leading bunch is then completely transferred to the trailing bunch and no wake field is left after the trailing bunch.

One might ask if the situation would improve if there were multiple driving bunches preceding the bunch to be accelerated. Naively one might expect that if the bunches were spaced by the wavelength of the wake field, the field would grow linearly with the number of bunches. However, the situation is again modified because a bunch sees not only is own retarding wake, but also the wake of all preceding bunches. If the bunches are spaced by one wavelength of the wake field oscillation, the second bunch comes to a stop in 1/3 the distance travelled by the first bunch, the third bunch stops in 1/5 that distance, etc. For the case of M bunches spaced by one wavelength, the energy gain of the trailing bunch to be accelerated is limited by

$$\Delta E_2 \leq E_1\left[\sum_{k=1}^{M} \frac{2}{2k-1} - \frac{N_2}{N_1}\right], \tag{9}$$

where N_1 is the number of particles per driving bunch. Thus the total energy gain increases logarithmically with M, in spite of the fact that the longitudinal electric field at the very beginning of the device grows linearly with M.

Actually, there is a better method to extract energy from the M bunches for a single-frequency system. The object here is to prevent the deceleration of a driving bunch by the field left by the preceding bunches. For example, for $M = 2$ this can be done provided the second bunch follows the first by 1/4 of the wavelength of the wake field. In this case the fields add such that the maximum amplitude increases to $\sqrt{2}$ times the field induced by one bunch, and the phase of the field is shifted by 45 degrees. In addition, both bunches lose all their energy in the same distance L. Generalizing this to M bunches, one must inject the Mth driving bunch at a phase θ_M relative to the first bunch given by

$$\theta_M = \sum_{n=2}^{M} \tan^{-1}\left(\frac{1}{\sqrt{n-2}}\right), \qquad M \geq 2. \tag{10}$$

This yields a resultant phase of the field relative to the field of the first bunch

$$\theta_{\text{field}} = \sum_{n=2}^{M} \tan^{-1}\left(\frac{1}{\sqrt{n-1}}\right). \tag{11}$$

After M bunches of energy E_1 have filled the device, the maximum energy gain for a trailing bunch is

$$\Delta E_2 = E_1\left(2\sqrt{M} - \frac{N_2}{N_1}\right). \tag{12}$$

The energy gain in Eq. 12 is much more favorable than Eq. (9), but it is far from being linear in M.

So far we have assumed rigid driving bunches of zero length. For the case of finite length bunches, we must consider the fact that particles at different longitudinal positions within a given driving bunch experience different decelerating fields. Some of the previous equations must be modified to take this fact into account. To do so, we will divide a bunch into series of slices representing different longitudinal positions. The leading slice sees no induced wake field and hence never comes to a stop. The middle slice comes to a stop in the distance L given by Eq. (6). The slice at the tail of the driving bunch sees twice the average induced wake field and stops in a distance $L/2$. In this distance the maximum energy gain for a particle in the middle of the trailing bunch is one-half that given by Eq. (7), and the efficiency is one-half that given by Eq. (8).

If we assume that by some means the particles in the driving bunch are removed from the plasma just as they come to a halt, then the maximum energy *per particle* that can be gained by a trailing test particle in distance $z_s \geq L/2$ is

$$\Delta E_2 = E_1\left(1 + \ln\frac{2z_s}{L}\right). \tag{13}$$

Thus, for non-rigid bunches, the factor of 2 is replaced by $[1 + \ln(2z_s/L)]$. Although this appears to be better for large z_s, in practice the logarithm increases too slowly to be of use.

If the number of particles in the accelerated bunch is comparable to that in the driving bunch, then the induced field from the accelerated bunch must be subtracted from ΔE_2 in Eq. (13) to obtain the net acceleration. The effect of the finite bunch length will also lead to an energy spread within the accelerated bunch. In analogy to the case of a bunch being accelerated by an rf wave in a conventional accelerator, this beam-loading energy spread can be compensated to some extent by adjusting the position of the bunch with respect to the crest of the plasma wave.

3. THE PLASMA WAKE FIELD

To understand the basic mechanism, we analyze explicitly in this section the response of a cold plasma to a driving bunch by calculating the wake field for three cases: a one-dimensional nonrelativistic plasma, a three-dimensional nonrelativistic plasma, and a one-dimensional relativistic plasma. In all three cases, the plasma is a single-frequency medium and Eq. (7) with an equal sign applies.

Case 1. One-Dimensional Nonrelativistic Plasma

The nonrelativistic fluid equations are

$$\frac{\partial n}{\partial t}+\nabla \cdot (n\mathbf{v})=0$$
$$\frac{\partial \mathbf{v}}{\partial t}+(\mathbf{v}\cdot\nabla)\mathbf{v}=\frac{e}{m}(\mathscr{E}+\mathbf{v}\times\mathscr{B}). \tag{14}$$

These together with Maxwell's equations form the system of equations to be solved.

Consider a plasma with density n_0, an injected bunch with density n_b, and a density perturbation n_1. To linearize we assume

$$n_1 \ll n_0 \tag{15}$$

and that the quantities $\mathbf{v}$, $\mathscr{E}$, and $\mathscr{B}$ are all first-order perturbations. Keeping only linear terms in Eq. (14), we have

$$\frac{\partial n_1}{\partial t}+n_0(\nabla\cdot\mathbf{v})=0$$
$$\frac{\partial \mathbf{v}}{\partial t}=\frac{e\mathscr{E}}{m}. \tag{16}$$

We see that only the electric field appears in first order, and only one of Maxwell's equations is necessary

$$\nabla\cdot\mathscr{E}=4\pi e(n_1+n_b). \tag{17}$$

Combining Eqs. (16) and (17) yields the equation for the density perturbation,

$$\frac{\partial^2 n_1}{\partial t^2}+\omega_p^2 n_1=-\omega_p^2 n_b, \tag{18}$$

where ω_p, the plasma frequency, is given by

$$\omega_p\equiv\left[\frac{4\pi e^2 n_0}{m}\right]^{1/2}.$$

So far we have not restricted the system to be one-dimensional.

For the one-dimensional case, consider an external beam given by

$$n_b = \sigma\delta(z - v_b t) \tag{19}$$

where σ is a uniform surface number density and $\delta(x)$ is the Dirac delta function. The beam is moving with velocity v_b. Although the plasma has been assumed nonrelativistic ($v \ll c$), we have *not* assumed that $v_b \ll c$. Changing variables to

$$y = v_b t - z, \tag{20}$$

Eq. (18) becomes

$$\frac{\partial^2 n_1}{\partial y^2} + k^2 n_1 = -k^2 \sigma \delta(y) \tag{21}$$

where

$$k \equiv \frac{\omega_p}{v_b}.$$

Integrating over $y = 0$, we find

$$\left.\frac{\partial n_1}{\partial y}\right|_{0-}^{0+} = -k^2\sigma. \tag{22}$$

At all other values of y, the density perturbation obeys the homogeneous equation of motion. Thus the density perturbation induced by the injected beam is

$$n_1 = \begin{cases} -k\sigma \sin ky & y > 0 \\ 0 & y < 0. \end{cases} \tag{23}$$

There is no plasma wave ahead of the driving beam. This is due to the fact that the plasma wave has zero group velocity; it does not propagate in space and therefore does not overtake the driving beam even if the driving beam moves nonrelativistically. Mathematically, this is manifested by the absence of spatial derivatives in Eq. (18).

From Eq. (17), the electric field is

$$\mathscr{E} = \begin{cases} -4\pi e\sigma \cos ky & y > 0 \\ -2\pi e\sigma & y = 0 \\ 0 & y < 0. \end{cases} \tag{24}$$

$\mathscr{E}$ is zero in front of the driving beam because the net charge in the plasma obtained by integrating n_1 is equal to $-e\sigma$. Thus ahead of the beam the field from the perturbed plasma charge density exactly cancels the surface-charge field of the driving bunch.

Notice also that in Eq. (24) the electric field at $y = 0$ is 1/2 of the peak value. This factor can be checked by energy conservation as follows. The energy deposited per unit length by the exciting bunch can be calculated from the peak electric field in the wake,

$$\frac{\mathscr{E}^2(\text{peak})}{8\pi} = 2\pi e^2 \sigma^2. \tag{25}$$

On the other hand, the energy lost by the driving beam is

$$-\Delta E_1 = -\mathscr{E}(0)e\sigma = 2\pi e^2\sigma^2. \tag{26}$$

Thus energy conservation is satisfied. Note that this factor of 1/2 is simply the inverse of the factor of 2 discussed in the first section. In addition, note that the energy deposited by the driving beam in the plasma depends only on the surface density of the driving beam, σ and is independent of the plasma density. This does not mean that the plasma density is arbitrary. It must be large enough to satisfy the linearity condition in Eq. (15). This can be conveniently rewritten as

$$\tfrac{1}{2}n_0 m v_b^2 \gg \frac{\mathscr{E}^2(\text{peak})}{8\pi} = \tfrac{1}{2}n_0 m v^2(\text{peak}). \tag{27}$$

The peak plasma electron velocity produced by the field must be much smaller than the beam velocity.

Case 2. Three-Dimensional Nonrelativistic Plasma

We now consider a cylindrically symmetric leading bunch with density given by

$$n_b = \sigma(r)\delta(z - v_b t). \tag{28}$$

Equation (18) can be solved just as in the one dimensional case; the perturbed density is

$$n_1(r) = \begin{cases} -k\sigma(r)\sin ky & y > 0 \\ 0 & y < 0. \end{cases} \tag{29}$$

Note that the r dependence of n_1 is equal to that of the driving beam. This is again a consequence of zero group velocity. Introducing the electrostatic potential ϕ, we must solve

$$\frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial}{\partial r}\phi\right) + \frac{\partial^2\phi}{\partial z^2} = -4\pi e n_1. \tag{30}$$

The electric field of the plasma wave is then given by

$$\mathscr{E} = -\nabla\phi. \tag{31}$$

Effects due to the magnetic field are second order and again are ignored as in the one dimensional case.

To be specific, we will use a parabolic distribution for the surface charge density of the beam;

$$\sigma(r) = \begin{cases} \dfrac{2N}{\pi a^2}(1 - r^2/a^2) & r < a \\ 0 & r > a. \end{cases} \tag{32}$$

It is then straightforward to show that the potential behind the bunch is given by

$$\phi = R(r)\sin(kz - \omega_p t)$$

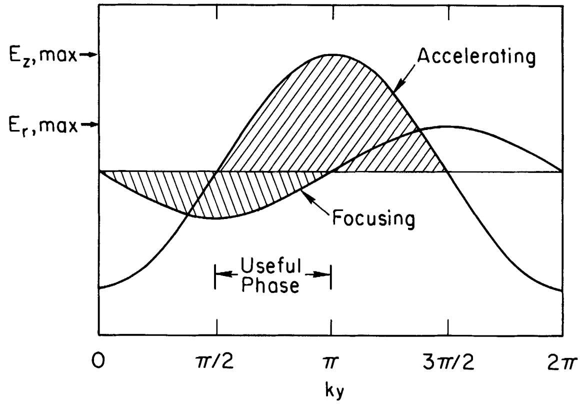

FIGURE 1 The longitudinal and radial electric fields in the plasma wave. The relative size of the two fields depends upon the radial position, *r*.

with

$$R(r)=\frac{16eN}{ka^2}\begin{cases}K_2(ka)I_0(kr)+\dfrac{1}{2}-\dfrac{2}{(ka)^2}-\dfrac{r^2}{2a^2}, & r<a\\ I_2(ka)K_0(kr), & r>a,\end{cases}\tag{33}$$

which yields the electric fields

$$\mathscr{E}_z=-kR(r)\cos(kz-\omega_p t),\qquad r<a$$
$$\mathscr{E}_r=\frac{-16eN}{a^2}\left\{K_2(ka)I_1(kr)-\frac{r}{ka^2}\right\}\sin(kz-\omega_p t),\qquad r<a,\tag{34}$$

where I_n and K_n are modified Bessel functions. Notice that there are both longitudinal (accelerating or decelerating) and radial (focusing or defocusing) electric field components. Figure 1 shows that over 1/4 of the plasma oscillation, the field is both accelerating and focusing.

It is interesting and useful to calculate the fields for $ka\gg 1$ and for $r\ll a$. Since the modified Bessel function K_2 is exponentially small in this case, we find

$$\begin{aligned}\mathscr{E}_z&\simeq\frac{-8eN}{a^2}\left(1-\frac{r^2}{a^2}\right)\cos(kz-\omega_p t)\\ \mathscr{E}_r&\simeq\frac{16eN}{a^2}\left(\frac{r}{ka^2}\right)\sin(kz-\omega_p t).\end{aligned}\tag{35}$$

In this case, the longitudinal field at $r=0$ is identical to the one-dimensional calculation and the radial field is linear in *r*. It is important to note that if $a=\lambda_p$, the plasma wavelength, then $ka=2\pi$, and Eqs. (35) are a good approximation to Eqs. (34) for small values of *r*.

Case 3. The Nonlinear One-Dimensional Relativistic Plasma

It has been shown in Refs. 16 and 17 that it is possible to find one-dimensional analytical solutions to describe a nonlinear free plasma oscillation using the relativistic fluid equations and Maxwell's equations. Since the exciting beam is a delta function, it is also possible to find analytical solutions for the wake function. From Ref. 17 the equations governing a free nonlinear plasma oscillation in one dimension are

$$n=\frac{n_0v_0}{v_0-v}$$
$$\frac{d^2}{dy^2}\left[\frac{c^2-v_0v}{\sqrt{1-v^2/c^2}}\right]=\frac{\omega_p^2v}{v_0-v} \tag{36}$$

where

v_0	is the wave phase velocity	
v	is the velocity of the plasma electrons	
n	is the density of the plasma electrons	(37)
n_0	is the ion density.	

For the case of an exciting beam, the phase velocity v_0 is equal to the velocity of the driving beam v_b. Furthermore, Eq. (36) becomes

$$\frac{d^2}{dy^2}\left[\frac{c^2-v_bv}{\sqrt{1-v^2/c^2}}\right]=\omega_p^2\left[\frac{v}{v_b-v}+\frac{n_b}{n_0}\right]. \tag{38}$$

If we substitute Eq. (19) for the beam density and integrate over $y=0$, we find

$$\frac{d}{dy}\left[\frac{c^2-v_bv}{\sqrt{1-v^2/c^2}}\right]\Bigg|_{0-}^{0+}=\frac{\omega_p^2\sigma}{n_0}. \tag{39}$$

On the other hand, for the homogeneous equation there is an invariant γ_m which satisfies

$$\gamma_m=\frac{1}{\sqrt{1-v^2/c^2}}+\frac{1}{2\omega_p^2}\left\{\frac{d}{dy}\left[\frac{c^2-v_bv}{c\sqrt{1-v^2/c^2}}\right]\right\}^2. \tag{40}$$

γ_m is the maximum energy of the plasma electrons in units of the rest mass.

Initially there is no plasma wave and $\gamma_m=1$. At $y=0+$ we can calculate the invariant using Eqs. (39) and (40); we obtain

$$\gamma_m=1+\frac{\omega_p^2\sigma^2}{2n_0^2c^2}. \tag{41}$$

The electric field behind the exciting bunch written in terms of v and γ_m is, from Refs. 16 and 17

$$\mathscr{E}(y)=\pm\sqrt{2}\,\frac{m\omega_pc}{e}\left[\gamma_m-\frac{1}{\sqrt{1-v(y)^2/c^2}}\right]^{1/2}. \tag{42}$$

Thus the maximum electric field is given by ($v=0$)

$$\mathscr{E}_{\max}=\sqrt{2}\,\frac{m\omega_pc}{e}[\gamma_m-1]^{1/2}=4\pi e\sigma. \tag{43}$$

Comparing this result with that obtained for the one-dimensional linear case we see that the peak electric field is unchanged! This is true even though the oscillation is nonlinear, and the plasma oscillation frequency depends on γ_m.

To understand this result, first note that both the electric field and the kinetic energy of the plasma electrons are zero in front of the driving bunch. As for the linear case, this follows from the fact that

$$\int_0^\infty (n - n_0)\, dy = -\sigma. \tag{44}$$

Immediately behind the driving beam, the plasma kinetic energy has not changed, but the electric field has changed discontinuously to its maximum value $-4\pi e\sigma$. The shape of the longitudinal wave and its frequency are determined by the plasma dynamics; however, the energy density in the plasma wave is completely determined by the field immediately after the exciting bunch. Since this nonlinear plasma wave is periodic, the electric field reaches its maximum periodically.

The result for the maximum electric field in Eq. (43) does not mean that the plasma density is arbitrary. Equations (36) are for wave solutions with phase velocity v_b. However, there are singularities in Eqs. (36) if the plasma electron velocity v is equal to v_b. This indicates that the wave assumption breaks down at this point. If we restrict v to be less than v_b, and write the condition in terms of energy density in the wave, we find that the plasma density must satisfy

$$n_0(\gamma_b - 1) > \left[\frac{\mathscr{E}_z^2(\text{peak})}{8\pi mc^2}\right] = n_0(\gamma_m - 1). \tag{45}$$

Note that this is just the relativistic generalization of Eq. (27) for the nonrelativistic plasma.

4. OTHER ACCELERATOR PHYSICS ISSUES

In this section, we discuss some other accelerator physics issues that are relevant to the plasma wake field accelerator. We will assume that the accelerator is made up of many stages. The energy gain of each stage is assumed to be small compared with the total gain in the entire accelerator. Each stage is driven by one driving bunch. For simplicity we will assume that all the driving bunches are identical in energy, number of particles and transverse size. We will not consider cases in which these quantities vary from stage to stage, although this might be desirable for the optimization of some parameters. Note however that the accelerated-bunch energy changes from stage to stage, and thus its transverse size is adiabatically damped. (See Section B below).

A. *Focusing*

Due to the radial field given in Eq. (34) there is focusing (or defocusing) in the transverse dimensions. The magnitude and sign depend upon the phase at which the accelerated bunch resides in the plasma wave. In cases of interest, the accelerated bunch and the phase velocity of the plasma wave (equal to the velocity of the driving bunch) are so close to c that the phase slippage is acceptable (see Section 4D). Therefore, one can select the desired focusing by the position of the bunch on the wave. Of course, there is a trade off between the accelerating field and the focusing field. It is useful to calculate the 'beta function' of this focusing system, defined in this case to be the wavelength/2π of the transverse oscillation.

The differential equation governing the transverse oscillations of a highly relativistic particle is

$$\frac{d^2x}{dz^2}=\frac{e\mathscr{E}_x}{\gamma mc^2}, \tag{46}$$

where $\mathscr{E}_x$ is the electric field in the transverse dimension x, and z is the length along the linac. The energy is assumed to be constant or varying adiabatically. If we consider small-amplitude oscillations, then from Eq. (35) we find

$$\frac{d^2x}{dz^2}+\left[\frac{16e^2N\sin\phi}{ka^4\gamma mc^2}\right]x=0, \tag{47}$$

where ϕ is the phase along the plasma wave. Identifying the coefficient of x above with β^{-2} yields

$$\beta=\frac{a^2}{4}\left[\frac{\gamma k}{r_e N\sin\phi}\right]^{1/2}. \tag{48}$$

The beta function therefore scales as $\gamma^{1/2}$ if a, k, N, and ϕ are held constant during acceleration.

B. The Maximum Efficiency

It is useful to neglect the effect of the finite bunch length of the trailing bunch and consider it rigid to calculate the maximum possible efficiency. From the equation for the longitudinal accelerating field (Eq. (35)), it is obvious that the trailing bunch should have a size somewhat smaller than the leading bunch. On the other hand, this size does not stay constant during acceleration. If we assume that the initial beam size of the trailing beam is some fraction α of the leading beam, then the beam size at other points along the accelerator is given by

$$b=\alpha a\left(\frac{\gamma_i}{\gamma}\right)^{1/4}. \tag{49}$$

This is true because the beam size is given by $\sqrt{\beta\cdot\text{emittance}}$, while

$$\beta\propto\gamma^{1/2}$$

and the emittance

$$\epsilon\propto\frac{1}{\gamma}. \tag{50}$$

The energy gain per stage of the trailing beam is

$$\Delta E_2 = E_1\left(2-\frac{N_2/b^2}{N_1/a^2}\right)$$
$$= E_1\left[2-\frac{N_2}{\alpha^2 N_1}\left(\frac{\gamma}{\gamma_i}\right)^{1/2}\right]. \tag{51}$$

Integrating this over the accelerator length L_{tot}, we obtain

$$\Delta E_2 = \frac{E_1}{L}\int_0^{L_{\text{tot}}}\left(2-\frac{N_2}{\alpha^2 N_1}\sqrt{1+gs}\right)ds, \tag{52}$$

where L is the length of each stage and

$$\frac{\gamma}{\gamma_i} = 1+gs. \tag{53}$$

Integrating the above expression yields ($\gamma_f \gg \gamma_i$)

$$\Delta E_2 \simeq \frac{E_1 L_{\text{tot}}}{L}\left[2-\frac{N_2}{\alpha^2 N_1}\frac{2}{3}\left(\frac{\gamma_f}{\gamma_i}\right)^{1/2}\right]. \tag{54}$$

The energy-transfer efficiency, which is given by

$$\eta = \frac{N_2}{N_1}\frac{\Delta E_2}{E_1}\frac{L}{L_{\text{tot}}}, \tag{55}$$

is maximum when

$$\frac{N_2}{N_1} = \frac{3}{2}\alpha^2\left(\frac{\gamma_i}{\gamma_f}\right)^{1/2} \tag{56}$$

and

$$\eta_{\max} = \frac{N_2}{N_1} = \frac{3}{2}\alpha^2\left(\frac{\gamma_i}{\gamma_f}\right)^{1/2}. \tag{57}$$

Note that this efficiency might be improved by decreasing the transverse size of the leading bunch from stage to stage. This is not considered here for simplicity and also because it may cause other problems.

C. Radiation

The radiation due to linear acceleration is very small; however, since the focusing fields in the plasma can be large, one must also include the radiation due to the local bending of the focusing fields. The formula for the energy loss per unit length due to a local bending radius ρ is

$$\frac{dE}{dz} = \frac{2}{3}\frac{e^2\gamma^4}{\rho^2}. \tag{58}$$

For motion in a focusing system, this is replaced by

$$\frac{1}{\rho^2} \to \frac{b_{\mathrm{rms}}^2}{2\beta^4} \tag{59}$$

where b_{rms} is the rms radius of the accelerated bunch. This energy loss should be small compared to the acceleration gradient. It is interesting to note that if the integrated loss is greater than the injection energy, the beam will damp transversely.

D. Phase Slippage

Phase slippage can occur in two ways in a plasma wake field accelerator.

1. The driving bunch and the accelerated bunch have different energies and hence slightly different velocities.
2. The transverse motion of the accelerated bunch leads to a path-length change. This effect cannot be compensated at each stage because different particles in the bunch travel on different paths.

In the first case, the relative slip along the plasma wave is given by

$$\Delta L = \frac{1}{c}\int_0^L [v_1(s) - v_2(s)]\, ds, \tag{60}$$

where L is the length of one acceleration stage, and subscripts 1 and 2 refer to the driving and accelerated bunches respectively. Integrating for velocities close to c yields

$$\frac{\Delta L}{L} \simeq \frac{1}{2}\left[\frac{1}{\gamma_{1i}\gamma_{1f}} - \frac{1}{\gamma_{2i}\gamma_{2f}}\right], \tag{61}$$

where subscripts f and i refer to final and initial respectively. To avoid phase slip over a stage of length L, we need ΔL much less than the plasma wavelength λ_p, i.e.,

$$\frac{\lambda_p}{L} \gg \frac{1}{2}\left[\frac{1}{\gamma_{1i}\gamma_{1f}} - \frac{1}{\gamma_{2i}\gamma_{2f}}\right]. \tag{62}$$

In practice, the first term in Eq. (62) dominates. This yields a restriction of the final energy of the driving beam given by

$$\gamma_{1f} \gg \frac{1}{2}\frac{L}{\gamma_{1i}\lambda_p}. \tag{63}$$

In the second case, the change in path length due to the transverse oscillations is given by

$$\Delta L = \int_0^{L_{\mathrm{tot}}} \sqrt{1 + x'^2}\, ds, \tag{64}$$

where $x' = dx/ds$ is the local slope. Note that we must consider the change in path length over the total length, since its effect can not be compensated for all particles at each stage. Using Eq. (47) to find x' and expanding the square root for small x' yields a path-length change

$$\frac{\Delta L}{L_{\text{tot}}} \simeq \frac{\hat{b}_i^2 \gamma_i}{2\beta_i^2 \gamma_f}, \tag{65}$$

where β_i is the beta-function at injection and $\hat{b}_i$ is the peak transverse-oscillation amplitude at injection. The factors of γ_i and γ_f come once again from the transverse damping of the accelerated beam. To avoid phase slip, this path length change should also be small compared with a plasma wavelength, i.e.

$$\frac{\gamma_i}{2\gamma_f} \frac{\hat{b}^2}{\beta_i^2} \ll \frac{\lambda_p}{L_{\text{tot}}}. \tag{66}$$

If conditions (62) and (66) are satisfied, one can neglect phase slippage between the accelerated beam and the plasma wave.

E. Transverse Variation of the Accelerating Field

From Eq. (35) it is evident that for a driving bunch with finite transverse size, the longitudinal field varies transversely. This means that particles performing large oscillations in the focusing field see on the average a lower accelerating gradient than those on axis. This leads to a decrease in the average energy gained by the trailing beam and to a spread in energy. If we average the acceleration gradient over the transverse beam distribution (assumed parabolic) and integrate over the accelerator length, we find a shift in average energy for a trailing beam of maximum initial radius b_i,

$$E_{\text{ave}} = E\left(1 - \frac{2}{3}\left(\frac{b_i}{a}\right)^2 \left(\frac{\gamma_i}{\gamma_f}\right)^{1/2}\right) \tag{67}$$

and a spread in energy given by

$$\left(\frac{\Delta E}{E}\right)_{\text{rms}} = \frac{\sqrt{2}}{3}\left(\frac{b_i}{a}\right)^2 \left(\frac{\gamma_i}{\gamma_f}\right)^{1/2}. \tag{68}$$

F. The Transverse Emittance

Assuming that the injected beam is matched to the focusing properties of the plasma wake, the transverse emittance of the accelerated beam at injection is given by

$$\epsilon_i = \frac{b_i^2}{\beta_i} = 4\alpha^2 \left[\frac{r_e N \sin\phi}{\gamma_i k}\right]^{1/2} \tag{69}$$

while at the end of acceleration, the emittance is

$$\epsilon_f = \epsilon_i \frac{\gamma_i}{\gamma_f}. \tag{70}$$

The emittance should be kept small enough that it can be focused to a sufficiently small spot size by the final focus system of the high-energy collider.

G. Beam Loading

As we have shown previously, a plane of charge moving through a plasma leaves behind a wake field which varies as $4\pi e\sigma \cos ky$, while ahead of the charge the field vanishes. This is similar to the case for a velocity-of-light point charge moving through an accelerating structure with metallic walls. The beam-induced longitudinal wake field vanishes ahead of the charge because of the causality condition, while behind the charge the field varies as $\cos k_n y$ for the nth mode. In this case, if a longitudinal charge distribution is considered rather than a point charge, a convolution of the wake field for a point bunch with the charge density distribution is required to obtain the wake field. For the case of an electron sheet with density distribution $\sigma(y)$ moving through a plasma, the analogous beam-induced field is

$$\mathscr{E}_b(y) = 4\pi e \int_{-\infty}^{y} dy' \sigma(y') \cos k(y - y'). \tag{71}$$

As an example, consider the case of a rectangular charge distribution that has a length small compared with the plasma wavelength and let σ_0 be the total number of particles per unit area in the distribution. The beam-induced field rises linearly from zero at the head of the bunch and reaches $4\pi e\sigma_0$ at the tail. The beam loading as described by Eq. (71) has two effects. First, different particles in the driving bunch experience different rates of energy transfer to the plasma and therefore travel different distances before they come to a stop. The tail particles are stopped earliest while the head particles do not stop at all. This effect was discussed in Section 2.

Another beam-loading effect occurs when the driving bunch length is comparable to or longer than the plasma wavelength. Then the convolution (71) yields a peak wake field that is much reduced as compared with the wake produced by a short bunch of equal intensity. For a rectangular distribution of total length $2l$, the reduction factor for the maximum amplitude reached by the wake behind the bunch is $\sin(kl)/kl$. However, the ratio of the maximum accelerating wake *behind* the bunch to the maximum accelerating wake within the bunch (the transformer ratio) is always less than or equal to 2. This limit on the transformer ratio is valid for any symmetric bunch distribution.

For an *asymmetric* bunch distribution in which the bunch current rises gradually from the front of the bunch toward the peak and then falls off more sharply behind the peak, the transformer ratio as defined above can be *larger* than 2.

However, for a given intensity the peak wake field is again much reduced as compared to the case of a short bunch. This effect could be exploited by incressing the number of particles in the bunch while using longer asymmetric bunches. In this way it should be possible to decrease the energy of the driving bunch while maintaining the same energy gain for the trailing bunch. We will not, however, exploit this possibility in the following section.

In addition to the longitudinal beam-loading effects, there are also transverse effects. The head of the driving bunch experiences no transverse focusing or defocusing fields, while the tail of the bunch sees the transverse focusing wake fields left by all the preceding particles. The beta function describing this focusing action is given by Eq. (48), where for a rectangular bunch with a bunch length $2l$, the factor $\sin\phi$ is replaced by $(\sin kl)^2/kl$. In order to have similar transverse behavior for the head and tail of the driving bunch, we again need to have a bunch length much shorter than the plasma wavelength.

5. A NUMERICAL CONCEPTUAL DESIGN

It is an interesting exercise to imagine a 1 TeV accelerator 1 kilometer long which uses a plasma wake field to generate the longitudinal fields for acceleration. In this case, the acceleration gradient necessary is

$$G = 1\ \text{GeV/m}. \tag{72}$$

From the discussions in the previous sections, it is clear that if the driving bunches all go through the same plasma, the maximum energy gain of the accelerated particles increases only as the square root of the number of bunches. If, however, each driving bunch excites a separate accelerating section, then the energy gain for the accelerated particle is proportional to the number of driving bunches. With this in mind, we consider an accelerator made up of sections of length L with one short driving bunch for each section. If we elect *not* to use the slow logarithmic increase in the energy gain shown in Eq. (13), then each driving bunch must have an energy

$$E_1 > GL. \tag{73}$$

To be specific we let

$$L = 5\ \text{m}. \tag{74}$$

Thus we require 200 driving beams of energy

$$E_1 > 5\ \text{GeV}. \tag{75}$$

In addition we would like the driving beam to lose 5 GeV in the 5 meters to yield the required acceleration gradient. If we use bunches with

$$N_1 = 5 \times 10^{10}, \tag{76}$$

then from Eq. (35) with $r = 0$,

$$G = \frac{8e^2N}{a^2} = \frac{8r_eN}{a^2}\,mc^2. \tag{77}$$

Thus, we require a beam radius of

$$a = 0.76\,\text{mm}. \tag{78}$$

To obtain an approximately one-dimensional plasma it is necessary to restrict the plasma wavelength to

$$\lambda_p \lesssim 0.76\,\text{mm} \tag{79}$$

which implies that

$$n_0 \gtrsim 1.9\times 10^{15}/\text{cm}^3. \tag{80}$$

One could produce a train of driving bunches with only one linac, and then bend the bunches onto the straight line of the wake field accelerator with proper path lengths to give the correct timing. In this regard, it is interesting to note that the necessary precision for bunch placement is much less than a plasma wavelength.

To set the size of the accelerated bunch we restrict the rms spread in energy to 1%. This yields a bunch radius of

$$b \simeq 0.46a\left(\frac{\gamma_i}{\gamma}\right)^{1/4} = 0.35\,\text{mm}\left(\frac{\gamma_i}{\gamma}\right)^{1/4}. \tag{81}$$

The beta function of the transverse focusing from Eq. (48) is

$$\beta \simeq 1.5\,\text{m}\frac{[\gamma/\gamma_f]^{1/2}}{[\sin\phi]^{1/2}}. \tag{82}$$

The radiation from the focusing fields is then

$$\frac{dE}{ds} \simeq 5.7\,\text{MeV/m}[\gamma/\gamma_f]^{3/2}\sin^2\phi. \tag{83}$$

The phase slip in Eq. (65) can now be calculated, yielding

$$\Delta L \simeq 0.03\,\text{mm}[\sin\phi]. \tag{84}$$

Using Eq. (69) the *normalized* emittance is given by

$$\begin{aligned}\epsilon_{\text{normalized}} &= \gamma b^2/\beta \\ &\simeq 8.2\times 10^{-8}\,\text{m}\sqrt{\gamma_i\gamma_f\sin\phi} \\ &\simeq 1.6\times 10^{-2}\,\text{m}\sqrt{\sin\phi},\end{aligned} \tag{85}$$

where we have taken 10 GeV to be the injection energy and 1 TeV to be the final energy. For reference, the normalized emittance for the SLAC Linear Collider is

$$\epsilon_{\text{SLC}} = 3\times 10^{-5}\,\text{m}. \tag{86}$$

Since for colliding beams we need to keep the beam emittance small enough for the beam to be focused to sub-micron size, we should restrict the phase ϕ on the plasma wave;

$$\sin\phi \ll 1. \tag{87}$$

In order to have the same emittance as the SLC, for instance, we need to choose

$$\sin\phi = 3.5 \times 10^{-6}. \tag{88}$$

Such an accurate phase requirement would indeed be difficult. The bunch would have to have a microscopic length. However, we do not think a discussion of detailed optimization is appropriate here; the question of the transverse emittance should be addressed more completely when the design becomes more sophisticated.

Finally we can calculate the maximum efficiency using Eq. (57); we find

$$\eta_{\max} \simeq 0.21\left(\frac{\gamma_i}{\gamma_f}\right)^{1/2} = 0.021. \tag{89}$$

This is a rather low value, but again improvements may be possible with a more sophisticated design.

6. CONCLUSION

In the previous sections, we have addressed many accelerator physics issues associated with the plasma wake field accelerator. There were several key points.

The plasma wake field is subject to the same limitations as the wake field in a metallic structure. In all practical cases with short driving bunches, this limits the energy gain of a trailing particle to about twice the energy of a particle in the leading bunch. This limitation led us to consider a multistage design with one driving bunch per stage. Note that in the wake field accelerator, the leading bunch is used to *obtain* a high field; if there is already a field in the plasma or structure and the bunch is used to *sustain* that high field, then the results would be quite different. This case will be treated in a future paper.

The plasma wake field was calculated for several interesting cases. It is clear from these calculations that, in order to keep radial fields low and longitudinal fields high, it is necessary that the radius of the exciting beam be the order of or greater than the plasma wavelength. In addition, the bunch length of both the driving bunch and the trailing bunch must be much less than the plasma wavelength. Finally, the nonlinear relativistic plasma wake field calculation indicates that for a given driving bunch, the peak longitudinal electric field obtained is *not* improved by nonlinear oscillations.

The finite transverse size of the exciting bunch led to a transverse variation of the longitudinal accelerating field and, more importantly, to radial transverse focusing. With these transverse fields in hand, it was possible to address many issues: synchrotron radiation, phase slippage, energy-transfer efficiency, beam loading and transverse emittance.

Finally, a numerical conceptual design was given. The accelerating fields obtained are impressive, the order of 1 GeV/m, and the driving bunches to obtain these are similar to the SLC bunches in transverse size and number of particles, although the bunch length must be a good deal less. There are also potential difficulties; the bunch length must be very short and the efficiency is rather low.

However, the design given here is certainly not optimum and these problems might have solutions.

The comparison of the plasma wakefield accelerator with the Plasma Beat-Wave scheme should now be quite straightforward. Many of the issues addressed here are independent of the method used to obtain the plasma density wave. At first glance, a comparison of the results here with those obtained in Ref. 3 indicates that the plasma wake field accelerator is at least as interesting as the Plasma Beat–Wave scheme and may have decided advantages.

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