Quantum techniques for classical black holes

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Abstract

Modern theoretical physics has benefited from a rapid growth in mathematical technology. In particular, technology developed in one field can be quickly adapted for use in another. Two key techniques developed for simplifying calculations of Feynman diagrams are spinor-helicity and the double copy. This thesis will discuss how they can be applied to general relativity.

Spinor-helicity is used in particle physics to simplify expressions. In $D > 4$ this is done by observing that the residual symmetry of the little group is non-trivial. We adapt this technology to classify higher-dimensional spacetimes in the style of the $D = 4$ Petrov classification. Focusing on $D = 5$, our scheme naturally reproduces the full structure previously seen in both the CMPP and de Smet classifications, and resolves long-standing questions concerning their relationship.

We review the exact classical double copy introduced for stationary Kerr-Schild spacetimes. We consider a time-dependent generalisation: the accelerating, radiating point particle. This Kerr-Schild solution has a non-trivial stress-energy tensor which we interpret as the radiative part of the field and find the corresponding single copy. Using Bremsstrahlung as an example, we determine a scattering amplitude describing the radiation which is consistent with the quantum double copy. This indicates a profound connection between exact classical solutions and the double copy.

The double copy relates YM and gravity amplitudes through the observation that numerators of Feynman diagrams can be made to obey a Jacobi relation mirroring the colour charges. This additional structure can be adapted for use in classical perturbative calculations. The double copy maps to $\mathcal{N} = 0$ supergravity requiring careful treatment of the dilaton. Using the Janis-Newman-Winicour family of naked singularities as an example we demonstrate how to construct spacetime metrics through a systematic perturbative expansion.

Lay Summary

This thesis will make steps towards answering two interesting questions about general relativity. Firstly, what kind of solutions to the Einstein equations exist in five dimensions? And secondly, how can we manifest quantum structures in general relativity?

General relativity only has one parameter: the number of dimensions. We experience life in four dimensions – three space and one time – but it is interesting to vary this number to try and understand the theory better. To go higher than four dimensions is difficult because the number of parameters to solve for becomes very high. This thesis develops some mathematical technology to help deal with this using five dimensions as an example.

Normally, distances are calculated so that the distance between two opposite corners of a square is found by adding the distance along each side squared. An alternative is spinor space, defined by the property that the "length" between two points is always zero. Calculations done in spinor space can be made equivalent to normal space and are sometimes more convenient. In particular, when general relativistic calculations are done in spinor space it is possible to classify solutions to the Einstein equations in four dimensions in an intuitive way.

This thesis develops equivalent technology for five dimensions by considering what symmetries remain once a direction is fixed in place. Making this symmetry explicit has the effect of simplifying many expressions.

The second half of the thesis deals with a structure in general relativity called the double copy. In particle physics, it is often necessary to calculate the likelihood that two particles will interact if they are nearby. We will consider particles which have a special kind of charge called colour charge. If all the information that relates to the colour of the particle is replaced with a second copy of the information about its motion, the resulting expression can describe the likelihood

that gravity particles will interact. This is called the double copy.

Gravity particles are a quantum phenomenon, but if we take the classical limit of these calculations, we find that the double copy structure is also present in general relativity. To demonstrate this we will consider a massive accelerating particle. We will find that this can be written as two copies of a charged particle undergoing the same motion, and the radiation from the particles also obeys a double copy relationship.

In the final section of the thesis, we introduce a “transformation function” to explicitly demonstrate how we can move between a quantum-like double copy language and general relativistic expressions.

Declaration

I declare that this thesis was composed by myself, that the work contained herein is my own, except where explicitly stated otherwise in the text, and that this work has not been submitted for any other degree or professional qualification except as specified.

Parts of this work have been published in [1–3].

- R. Monteiro, I. Nicholson and D. O'Connell, *Spinor-helicity and the algebraic classification of higher-dimensional spacetimes*, 1809.03906
- A. Luna, R. Monteiro, I. Nicholson, D. O'Connell and C. D. White, *The double copy: Bremsstrahlung and accelerating black holes*, 1603.05737
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Chapter 1

Introduction

We know a lot about general relativity. The Einstein equations have been studied in huge detail and many exact solutions have been found, filling hundreds of books and articles. The regions, Killing vectors and horizons of these exact solutions have been examined. We understand that they obey the laws of black hole thermodynamics and that they undergo Hawking radiation. We can calculate the trajectories of particles scattering off the background of an exact solution to make astronomical predictions. Still, there are limits to our understanding. The highly non-linear nature of the theory means that it is difficult to gain an intuition for what a solution will look like. Exact solutions must be found through complex extensions and solution-generating algorithms while real-world solutions are found through numerical and perturbative techniques, which is a difficult gap to bridge.

However, there is structure in the theory that is yet to be explored fully and this may help to connect the two approaches. In particular, experience from particle physics has shown that a great deal of perturbative theory can be fixed by proper consideration of the theory's structure and symmetries. It can be hoped that this will be possible for general relativity as well. Using techniques of particle theory to reveal new structures and symmetries in general relativity will be a key theme of this thesis.

Why does this matter? We are fortunate to live in a time when experiments are able to probe gravitational waves to reveal details of black hole binary inspirals. Current and future gravitational wave detectors will provide a spectrum of frequencies through which we can see the universe. Precision calculations may highlight the discrepancies between general relativity and nature - we know that general relativity is non-renormalizable and therefore must emerge as some quantum theory's weak field limit. But precision calculations are difficult and rely on approximations, or use large amounts of computing power. A useful shortcut can be exact solutions, which work as analytic, solvable toy models. This is how black holes were first understood: a mathematical solution to the Einstein equations indicated that physical realisation was possible. If we can understand symmetries and structure in the context of the exact solutions we may be able to improve intuition and calculational ability.

The only parameter that general relativity has is the number of dimensions. This is therefore a good place to start. It turns out that exact solutions change dramatically with the dimension. For example, in two dimensions the Einstein tensor has no algebraically independent degrees of freedom, while in three there are no gravitational waves or asymptotically Minkowskian black holes [6]. We know that general relativity in five dimensions also has fascinating properties - the uniqueness theorem is violated and non-spherical horizon topologies are allowed. Exact solutions such as the black ring and black Saturn have been found, but the huge degree of freedom count makes it hard to find others. In successive dimensions this degree of freedom count gets larger, increasing the complexity of the situation. Therefore this is an important barrier to overcome to understand how general relativity varies with dimensionality.

Many discoveries of exact solutions in four dimensions were made using spinors to reduce the number of components. The removal of redundant gauge degrees of freedom makes it possible to rewrite the equations of general relativity as a much more tractable list of scalar coupled differential equations. This is referred to as the Newman-Penrose formalism. This would be extremely useful in higher dimensions, since there are even more redundant degrees of freedom, but despite several attempts [7, 8] no equivalent formalism was found. However, the particle physics version of this technique, the spinor-helicity formalism, was extended to higher dimensions in [9]. The key observation made was to preserve the residual spacetime symmetry left when a null direction is fixed. This symmetry is called the little group. In the first half of this thesis we will use this observation to extend the Newman-Penrose formalism to five dimensions and sketch the procedure to extend it to an arbitrary number of dimensions.

There is another structure in general relativity called the double copy. More precisely, this is a perturbative duality between supersymmetric Yang-Mills theory and supergravity which was found in [10]. The question of how to relate the double copy to general relativity and to exact solutions will be the second theme of this thesis. In the duality, gauge transformations are used to find a set of kinematic factors in the numerators for Yang-Mills Feynman diagrams which satisfy a Jacobi identity matching the colour factors. This has been proven to be true for arbitrary number of legs at tree level and is a conjecture for loop level. Since it is true at tree level, it is also true classically in most situations (see for example [11]), and therefore the structure must be present in general relativity.

Applying this relationship to exact solutions was first proposed in [12] for stationary Kerr-Schild exact solutions. Using only the degrees of freedom found in the gauge theory, exact solutions satisfying the stationary vacuum gravitational field equations can be constructed. This thesis will demonstrate how this can be extended to non-stationary solutions by considering an accelerating particle. The resulting Bremsstrahlung radiation can be related to the normal quantum double copy.

The Kerr-Schild class of solutions are a choice of coordinates where the graviton is traceless and symmetric. In general, the double copy of an $\mathcal{N}=0$ gauge theory is $\mathcal{N}=0$ supergravity where $\mathcal{N}$ is the number of supercharges present in the theory. $\mathcal{N}=0$ supergravity is composed of a spin-2 graviton, a scalar dilaton field and an antisymmetric axion field, while general relativity only contains the graviton. If the double copy is taken in such a way that the resulting field is symmetric and trace-free (such as in the Kerr-Schild case) then there is an automatic map to general relativity. However, in general it is necessary to remove the axion and dilaton to understand the double copy structure hidden in general relativity. In the final chapter of this thesis, we consider this issue and develop a scheme involving projectors to solve the problem perturbatively.

To fix notation and conventions, we will now introduce some of the concepts that we will use later in this thesis.

1.1 Scattering amplitudes

The methodology of modern scattering amplitudes is based on removing redundancy from the expressions. In the traditional Feynman approach there is a very clear physical interpretation but it is necessary to sum over many different terms. Through a combination of colour-ordering, spinor-helicity and little group symmetry, we can eradicate the vast majority of these terms, often fixing physical information directly. Over the next few paragraphs, we will elaborate on what this terminology mean and show how this is done using the example of the Yang-Mills 3-vertex. For more detail, the review [13] is excellent.

1.1.1 Colour-ordering

Yang-Mills theory is a spin-1 theory with a gauge group which we choose to be $SU(N)$. It has a Lagrangian

$$\mathcal{L}_{YM} = -\frac{1}{4}\mathrm{Tr}\, F_{\mu\nu}F^{\mu\nu}, \quad F_{\mu\nu} = D_\mu A_\nu - D_\nu A_\mu, \quad A_\mu = A_\mu^a T^a \tag{1.1}$$

where the covariant derivative acts on the gauge field in the adjoint representation as

$$D_\mu = \mathbb{1}\partial_\mu - igT^a A_\mu^a. \tag{1.2}$$

The space-time index is given by greek indices $\mu, \nu = 0, 1, .., D-1$ in D dimensions, while the gauge group index is $a, b, c = 1, 2, ..., N^2 - 1$. Without even extracting the Feynman rules, we can see that any expressions we derive will be composed of two different types of data, firstly the $SU(N)$ colour factors constructed from the generators T, and secondly the space-time-dependent data which we will write in momentum space as functions of momenta p_i^μ and polarisation vectors ε_i^μ where i indicates a particle number. The generators are normalised according to [13] and obey

$$[T^a, T^b] = i\tilde{f}^{abc}T^c, \qquad \mathrm{Tr} T^a T^b = \delta^{ab} \tag{1.3}$$

for antisymmetric structure constants $\tilde{f}^{abc}$.

Let us consider a n-leg tree amplitude $\mathcal{A}_n^{\text{tree}}$. We can always write an amplitude solely in terms of the trace of the generators attached to the n external legs:

$$\mathcal{A}_n^{\text{tree}} = g^{n-2} \sum_{\text{perms } \sigma} A_n[\sigma(1, 2, ..., n)]\mathrm{Tr}(T^{\sigma(a_1}T^{a_2}...T^{a_n)}), \tag{1.4}$$

where $A_n[1, 2, ..., n]$ is some function of the kinematics that we will refer to as the colour-ordered amplitude. This can be shown as follows [14]. We proceed diagrammatically by using that the 3-vertex from the gauge fixed Feynman rules is proportional to f^{abc}, while a propagator has colour factor δ^{ab}. Therefore since we can write the structure constants as

$$i\tilde{f}^{abc} = \mathrm{Tr}(T^a T^b T^c) - \mathrm{Tr}(T^b T^a T^c) \tag{1.5}$$

we can always use the Fierz identity

$$(T^a)_i{}^j \, (T^a)_k{}^l = \delta_i{}^l \delta_k{}^j - \frac{1}{N}\delta_i{}^j \delta_k{}^l \tag{1.6}$$

to hook together any repeated colour indices until eventually only traces of generators are left. The colour-ordered amplitudes have a number of nice simplifying properties (it turns out that there are actually only $(n-3)!$ independent ones) but most importantly they are gauge invariant.

1.1.2 Helicities and polarisation vectors

It is now possible to disregard the colour information and focus only on the colour-ordered amplitudes. We stated earlier that they were constructed from momenta p_i and polarisation vectors ϵ_i^μ, which we will define more concretely now. The electromagnetic field strength tensor $F_{\mu\nu}$ is invariant under the gauge transform

$$A_\mu \to A_\mu + \partial_\mu \lambda, \tag{1.7}$$

so it is possible to fix the gauge, for example by the requirement that $\partial \cdot A = 0$. There is still some residual gauge freedom when λ solves the wave equation $\partial^2 \lambda = 0$. Transforming into momentum space as

$$\lambda(p) = \int d^4x e^{ip\cdot x} \lambda(x), \tag{1.8}$$

we fix this by introducing a second reference vector q^μ satisfying $p \cdot q \neq 0$. Now, choosing $q \cdot A = 0$, we have fully fixed the gauge freedom. Since $p \cdot q \neq 0$, we have two remaining independent directions, namely the two polarisation directions. We write these in terms of the two helicities of a circularly polarized wave ε_h^μ where $h = +/-$ such that they complete the basis

$$\sum_h \varepsilon_h^\mu \varepsilon_h^\nu = \eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p \cdot q}. \tag{1.9}$$

Since $p \cdot A = q \cdot A = 0$, we can write the gauge boson as

$$A^\mu(x) = \varepsilon_h^\mu c_h(x). \tag{1.10}$$

So despite initially appearing as a spacetime 4-vector, the gauge boson only contains two physical degrees of freedom. This is because the state that we are describing is a unitary representation of the Poincaré group [15, 16].

1.1.3 Spinor-helicity formalism

A helpful alternative to this is to use spinors. There is an isomorphism between $SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ in four dimensions which allows us to write 4-vectors as two component spinors. To do this, we will need the standard Pauli matrices

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{1.11}$$

to form $\sigma^\mu_{\alpha\dot{\alpha}} = (\sigma^0, \sigma^1, \sigma^2, \sigma^3)$ and conjugate basis $\tilde{\sigma}^{\mu\dot{\alpha}\alpha} = (\sigma^0, -\sigma^1, -\sigma^2, -\sigma^3)$. The spacetime spinor indices are $\alpha, \beta = 1, 2$ and, for the conjugate basis, $\dot{\alpha}, \dot{\beta} = 1, 2$. We move into spin space by constructing the object $p_{\alpha\dot{\alpha}} = p_\mu \sigma^\mu_{\alpha\dot{\alpha}}$ where p_μ is null:

$$p_{\alpha\dot{\alpha}} = \begin{pmatrix} p^0 + p^3 & p^1 - ip^2 \\ p^1 + ip^2 & p^0 - p^3 \end{pmatrix}_{\alpha\dot{\alpha}}. \tag{1.12}$$

Since this has vanishing determinant, it must linearly factorize: $p_{\alpha\dot{\alpha}} = \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. We will use the conventions

$$s^\alpha = \epsilon^{\alpha\beta} s_\beta, \quad s^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} s_{\dot{\beta}} \tag{1.13}$$

to raise and lower spinor indices, and for convenience we will introduce the notation $\langle ij \rangle$ and $[ij]$ to represent

$$\langle ij \rangle = \lambda_i^\alpha \lambda_{j\alpha}, \quad [ij] = \tilde{\lambda}_i^{\dot{\alpha}} \tilde{\lambda}_{j\dot{\alpha}} \tag{1.14}$$

where i and j indicate the ith and jth particle in a Yang-Mills theory.

There is now no need to solve for the on-shell condition $p_i^2 = 0$, since

$$p_{i\mu} p_i^\mu = \left(\sigma^\mu_{\alpha\dot{\alpha}} \lambda_i^\alpha \tilde{\lambda}_i^{\dot{\alpha}}\right) \left(\sigma_{\mu\beta\dot{\beta}} \lambda_i^\beta \tilde{\lambda}_i^{\dot{\beta}}\right) = -2\langle ii \rangle [ii] = 0, \tag{1.15}$$

and in fact it turns out that amplitudes can often be expressed with manifest gauge invariance when spinors are used. A good example of this is the Parke-Taylor formula for the n-gluon tree amplitude which depends only on the n particles' momenta. If the gluons i and j have negative helicity while the remaining $n-2$ gluons have positive helicity then the formula is given by

$$A_n[1^+ ... \, i^- ... \, j^- ... \, n^+] = \frac{\langle ij \rangle^4}{\langle 12 \rangle \langle 23 \rangle ... \langle n1 \rangle} \tag{1.16}$$

which is explicitly independent of the gauge. Since there is no obvious downside, spinors have been wholeheartedly adopted for use both in scattering amplitudes and as we will see in section 2, in general relativity. Using spinors in this way is called the spinor-helicity formalism in the amplitudes community, and the Newman-Penrose formalism by relativistic communities.

1.1.4 Little group symmetry

Now that we have focused on the colour-ordered amplitude and are using spinors, it turns out that calculating scattering amplitudes is a lot less involved than one might anticipate. A big part of the reason for this is the Lorentz transforms that the spinors undergo. When p^μ and q^μ are fixed, there is still an $SO(D-2)$ group of residual Lorentz freedoms. This is often understood by visualising a null vector oriented with the x^{D-1} axis such that it is given by[1]

$$p^\mu = (p, \underbrace{0, ..., 0}_{D-2}, p), \tag{1.18}$$

leaving $\frac{(D-2)(D-3)}{2}$ possible rotations still available without changing p or q. This $SO(D-2)$ group is called the little group and as we shall see, preserving its covariance in Lorentz transforms turns out to be a very powerful tool.

Let us consider what that means for our 2-component spinors in four dimensions.

[1]In fact, the situation is slightly more subtle - the freedom left if you keep the null vector p invariant is the group $SO(D-2) \times T_{D-2}$ where T_n is the n-dimensional group of translations. It is only when you also fix the gauge vector q, for example by the choice

$$q^\mu = (q, 0, ..., 0, -q) \tag{1.17}$$

that only the rotations $SO(D-2)$ are left.

We complete the spinor basis with the orthogonal spinors μ^α and $\tilde{\mu}^{\dot{\alpha}}$ such that

$$\lambda_\alpha \mu^\alpha = \tilde{\lambda}_{\dot{\alpha}} \tilde{\mu}^{\dot{\alpha}} = 1. \tag{1.19}$$

It is then natural to define $q^\mu \equiv \sigma^\mu_{\alpha\dot{\alpha}} \mu^\alpha \tilde{\mu}^{\dot{\alpha}}$, since this satisfies $p \cdot q \neq 0$. The two polarisation vectors can be given by

$$\varepsilon^\mu_+ = \sigma^\mu_{\alpha\dot{\alpha}} \mu^\alpha \tilde{\lambda}^{\dot{\alpha}}, \quad \varepsilon^\mu_- = \sigma^\mu_{\alpha\dot{\alpha}} \lambda^\alpha \tilde{\mu}^{\dot{\alpha}}. \tag{1.20}$$

The only Lorentz transform that leaves p^μ, q^μ and the spinor basis definition $\langle \lambda\mu \rangle = [\lambda\mu] = 1$ invariant is

$$\lambda \to e^{i\theta}\lambda, \quad \tilde{\lambda} \to e^{-i\theta}\tilde{\lambda}, \quad \mu \to e^{-i\theta}\mu, \quad \tilde{\mu} \to e^{i\theta}\tilde{\mu}, \tag{1.21}$$

where θ parametrises the single rotation available in the residual $SO(2)$ Lorentz symmetries. Under this transformation, we find that the positive helicity polarisation transforms as $\varepsilon^\mu_+ \to e^{-2i\theta}\varepsilon^\mu_+$ while the negative helicity polarisation transforms as $\varepsilon^\mu_- \to e^{2i\theta}\varepsilon^\mu_-$.

Surprising as it may seem, even this tiny amount of technology is enough to fix amplitudes. Consider a 3-point amplitude of particles $i = 1, 2, 3$, each with some helicity $h_i = +1$ for positive helicity particles and $h_i = -1$ for negative. As we saw above, the polarisation vectors transform with a scaling factor of $e^{-2h_i\theta}$. The only non-zero scalars we can use to build the (scalar) amplitude are contractions of the three spinors λ^α_i and $\tilde{\lambda}^{\dot{\alpha}}_i$; we do not expect the gauge spinor μ^α_i to appear in the final expression because q^μ is an arbitrary reference vector. One final observation: conservation of momentum for three particles tells us $p^\mu_1 + p^\mu_2 + p^\mu_3 = 0$. So

$$p_3^2 = (p_1 + p_2)^2 = 2p_1 \cdot p_2 = \langle 12 \rangle [12] = 0. \tag{1.22}$$

By repeatedly using momentum conservation, we can show that if $\langle 12 \rangle$ vanishes, then all other angle bracket products also vanish. Conversely, if $[12]$ vanishes, the other square bracket products vanish.

Putting this all together, since the amplitude must transform with the same factors of $e^{-2ih_i\theta}$ as its external particles we can use an angle bracket ansatz

$$A_3(1^{h_1} 2^{h_2} 3^{h_3}) = \langle 12 \rangle^{x_{12}} \langle 23 \rangle^{x_{23}} \langle 31 \rangle^{x_{31}} \tag{1.23}$$

and perform a little group rotation on each particle to find

$$x_{12} = h_3 - h_1 - h_2, \quad x_{23} = h_1 - h_2 - h_3, \quad x_{31} = h_2 - h_3 - h_1. \tag{1.24}$$

For example, let us consider the 3-gluon amplitude with two negative helicity and one positive helicity polarisations $A_3(1^-2^-3^+)$. Implementing this formula, we find

$$A_3(1^-2^-3^+) = \frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}. \tag{1.25}$$

Note that we chose a hypothesis with angle brackets because the kinematics requires square brackets to vanish. We can see this easily: a square bracket ansatz obtains $x_{ij}^{\mathrm{sq}} = -x_{ij}$ such that the 3-point amplitude would look like

$$\frac{[23][31]}{[12]^3}. \tag{1.26}$$

This has a negative mass dimension and therefore cannot come from our local theory. Recursively using the little group symmetry and locality is enough to specify every single tree amplitude in some theories [13].

1.2 General relativity

In contrast, general relativity initially seems very different. The traditional formulation of general relativity is to begin with the spacetime interval $ds^2 = g_{\mu\nu}dx^\mu dx^\nu$, to specify the metric. We can then define the Riemann tensor in the standard way as

$$R^\rho{}_{\sigma\mu\nu} = \partial_\mu \Gamma^\rho{}_{\nu\sigma} - \partial_\nu \Gamma^\rho{}_{\mu\sigma} + \Gamma^\rho{}_{\mu\lambda}\Gamma^\lambda{}_{\nu\sigma} - \Gamma^\rho{}_{\nu\lambda}\Gamma^\lambda{}_{\mu\sigma} \tag{1.27}$$

where the Christoffel symbol $\Gamma^\mu_{\alpha\beta}$ is given by

$$\Gamma^\mu_{\rho\sigma} = \frac{1}{2} g^{\mu\nu}\left(\partial_\rho g_{\nu\sigma} + \partial_\sigma g_{\nu\rho} - \partial_\nu g_{\rho\sigma}\right) \tag{1.28}$$

and ∇_μ is a covariant derivative given by

$$\begin{aligned}\nabla_\mu T^{\rho_1\rho_2\ldots}{}_{\sigma_1\sigma_2\ldots} =& \partial_\mu T^{\rho_1\rho_2\ldots}{}_{\sigma_1\sigma_2\ldots} + \Gamma^{\rho_1}{}_{\mu\nu} T^{\nu\rho_2\ldots}{}_{\sigma_1\sigma_2\ldots} + \Gamma^{\rho_2}{}_{\mu\nu} T^{\rho_1\nu\ldots}{}_{\sigma_1\sigma_2\ldots} + \ldots \\ & - \Gamma^\nu{}_{\mu\sigma_1\ldots} T^{\rho_1\rho_2\ldots}{}_{\nu\sigma_2\ldots} - \Gamma^\nu{}_{\mu\sigma_2\ldots} T^{\rho_1\rho_2\ldots}{}_{\sigma_1\nu\ldots} - \ldots\end{aligned} \tag{1.29}$$

for arbitrary tensor T. Despite this rather dry definition, the Riemann tensor is of essential importance since it encodes the curvature of the spacetime:

$$\nabla_\mu \nabla_\nu v^\rho - \nabla_\nu \nabla_\mu v^\rho = R^\alpha{}_{\sigma\mu\nu} v^\sigma \tag{1.30}$$

for arbitrary vector v^ρ. The trace of the Riemann tensor is defined to be the Ricci tensor $R_{\mu\nu} = R^\alpha{}_{\mu\alpha\nu}$. This encodes the matter present in the spacetime through the Einstein equations

$$R_{\mu\nu} + \frac{1}{2} g_{\mu\nu} R = 8\pi G_N T_{\mu\nu} \tag{1.31}$$

where R is the Ricci scalar $R = g^{\mu\nu} R_{\mu\nu}$ and $T_{\mu\nu}$ is the stress-energy tensor. The tracefree part of the Riemann tensor, the Weyl tensor

$$C_{\mu\nu\rho\sigma} = R_{\mu\nu\rho\sigma} - \frac{2}{D-2}\left(g_{\mu[\rho}R_{\sigma]\nu} - g_{\nu[\rho}R_{\sigma]\mu}\right) + \frac{2}{(D-1)(D-2)} R g_{\mu[\rho}g_{\sigma]\nu} \tag{1.32}$$

encodes non-local matter.

In this formulation, general relativity seems to have very little in common with particle physics. Nevertheless, in this thesis we will show how the two theories can be formulated more similarly and how techniques from amplitudes can be used in general relativity.

The structure of this thesis is as follows. In part I, we will consider the use of spinor helicity techniques in general relativity: in chapter 2 we will review the Newman-Penrose formalism in four dimensions and in chapter 3 we will extend this formalism to five dimensions with a focus on how this naturally implies a Petrov-like classification for all vacuum spacetimes of five dimensions. Then in part II, we will consider how the double copy can be used classically in general relativity. In chapter 4 we will review the details of the quantum double copy, before extending the Kerr-Schild double copy to the accelerating particle case in chapter 5. Finally, we will show a methodology for removing the dilaton to apply the double copy to general relativity in chapter 6, before presenting our conclusions in 7. Chapters 2 and 3 were published in [1], while chapter 5 and chapter 6 were published in [2] and [3] respectively.

We will use the mostly plus metric convention $(-, +, ..., +)$ for the bulk of the thesis with the exception of chapter 5 which is written in the mostly minus metric convention $(+, -, ..., -)$.

Part I

Spinorial techniques in GR

Chapter 2

Exact solutions and spinorial techniques in four dimensions

As we described in chapter 1, using spinors to remove redundant degrees of freedom can be very convenient. In this chapter we will review this process in four dimensions in preparation for the five dimensional formalism developed in chapter 3. We then review the Petrov classification for four-dimensional spacetimes. This classification can be understood from a variety of perspectives; we emphasise the Newman-Penrose (NP) approach [17, 18] because it is closest in spirit to our approach in five dimensions.

This chapter is based on work published in collaboration with Ricardo Monteiro and Donal O'Connell in [1].

2.1 Spinors in four dimensions

In flat Minkowski space, the Clifford algebra is

$$\sigma^{\mu}{}_{\alpha\dot{\alpha}}\,\tilde{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^{\nu}{}_{\alpha\dot{\alpha}}\,\tilde{\sigma}^{\mu\dot{\alpha}\beta} = -2\eta^{\mu\nu}\,\mathbb{1}_{\alpha}{}^{\beta}, \tag{2.1}$$

where $\eta_{\mu\nu}$ is the Minkowski metric.[1] To be explicit, we choose a basis of σ^μ matrices given by

$$\sigma^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{2.2}$$

while the $\tilde{\sigma}^\mu$ matrices are

$$\tilde{\sigma}^0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad \tilde{\sigma}^1 = -\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \tilde{\sigma}^2 = -\begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \tilde{\sigma}^3 = -\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{2.3}$$

For any non-vanishing null vector V, the matrices $V \cdot \sigma$ and $V \cdot \tilde{\sigma}$ have rank 1. Hence we may construct solutions of the (massless) Dirac equations:

$$V \cdot \sigma_{\alpha\dot{\alpha}} \, \tilde{\lambda}^{\dot{\alpha}} = 0, \tag{2.4}$$

$$V \cdot \tilde{\sigma}^{\dot{\alpha}\alpha} \, \lambda_\alpha = 0. \tag{2.5}$$

These spinors can be normalised so that $V \cdot \sigma_{\alpha\dot{\alpha}} = -\sqrt{2}\, \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$. We may raise and lower the indices α and $\dot{\alpha}$ on these spinors with the help of the two-dimensional Levi-Civita tensor. We choose conventions such that $\epsilon^{12} = 1$, $\epsilon_{12} = -1$ and $s^\alpha = \epsilon^{\alpha\beta} s_\beta$ while $\tilde{s}^{\dot{\alpha}} = \epsilon^{\dot{\alpha}\dot{\beta}} \tilde{s}_{\dot{\beta}}$.

In the curved space case, we simply introduce a frame $e^\mu{}_M$, such that

$$g^{\mu\nu} = e^\mu{}_M \, e^\nu{}_N \, \eta^{MN}. \tag{2.6}$$

On the tangent space at each point, the Clifford algebra can be written as before,

$$\sigma^M{}_{\alpha\dot{\alpha}} \, \tilde{\sigma}^{N\dot{\alpha}\beta} + \sigma^N{}_{\alpha\dot{\alpha}} \, \tilde{\sigma}^{M\dot{\alpha}\beta} = -2\eta^{MN} \, \mathbb{1}_\alpha{}^\beta, \tag{2.7}$$

whereas

$$\sigma^\mu{}_{\alpha\dot{\alpha}} \, \tilde{\sigma}^{\nu\dot{\alpha}\beta} + \sigma^\nu{}_{\alpha\dot{\alpha}} \, \tilde{\sigma}^{\mu\dot{\alpha}\beta} = -2g^{\mu\nu} \, \mathbb{1}_\alpha{}^\beta, \tag{2.8}$$

with $\sigma^\mu = e^\mu{}_M \sigma^M$, and a similar definition for $\tilde{\sigma}$. We use the explicit Clifford bases of equations (2.2) and (2.3) in the tangent space.

It may be worth commenting briefly on reality conditions in four dimensions, since the reality conditions in five dimensions will play a more significant role

[1]We work in the mostly-plus signature.

later. The Lorentz group in real Minkowski space is isomorphic to $SL(2,\mathbb{C})/\mathbb{Z}_2$. It is consistent to choose a basis of Hermitian σ matrices – and indeed we have chosen such a basis in equations (2.2) and (2.3). Then, given a real null vector V, we may choose our spinors λ and $\tilde{\lambda}$ such that $\lambda^\dagger = \tilde{\lambda}$. This is consistent with the choice that $V \cdot \sigma_{\alpha\dot{\alpha}} = -\sqrt{2}\,\lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$.

2.2 The four-dimensional Newman-Penrose tetrad

In four dimensions, the NP formalism [17, 18] exploits the correspondence between the Lie algebras $\mathfrak{so}(4)$ and $\mathfrak{su}(2) \times \mathfrak{su}(2)$. A key element of the method is the spinorial construction of a particular basis set of vectors, known as the NP tetrad. We begin by choosing two null vectors k^μ and n^μ which satisfy $k \cdot n \neq 0$, and constructing an associated basis of spinors $\{o_\alpha, \imath_\alpha\}$ by solving the equations

$$k \cdot \tilde{\sigma}_{\dot{\alpha}\alpha}\, o^\alpha = 0, \quad n \cdot \tilde{\sigma}_{\dot{\alpha}\alpha}\, \imath^\alpha = 0. \tag{2.9}$$

Since $k \cdot n \neq 0$, we may normalise the vectors so that $k \cdot n = -1$, and also normalise our spinors so that $o^\alpha\, \imath_\alpha = 1$.

Similarly, we construct a conjugate basis by solving the equations

$$k \cdot \sigma_{\alpha\dot{\alpha}}\, \tilde{o}^{\dot{\alpha}} = 0, \quad n \cdot \sigma_{\alpha\dot{\alpha}}\, \tilde{\imath}^{\dot{\alpha}} = 0, \tag{2.10}$$

to find the dual spinors $\{\tilde{o}_{\dot{\alpha}}, \tilde{\imath}_{\dot{\alpha}}\}$, which we also normalise so that $\tilde{o}^{\dot{\alpha}}\tilde{\imath}_{\dot{\alpha}} = 1$. For real k and n, we may take $\tilde{o} = o^\dagger$ and $\tilde{\imath} = \imath^\dagger$ as discussed in section 2.1.

Let us now complete the construction of the NP tetrad of vectors using our spinor basis. The tetrad includes the vectors k and n, so we must find two more. Since the spinor basis is complete, we can construct the last two elements of the NP tetrad, m and $\widetilde{m}$, from

$$m^\mu = \frac{1}{\sqrt{2}}\, \sigma^\mu_{\alpha\dot{\alpha}}\, \imath^\alpha\, \tilde{o}^{\dot{\alpha}}, \qquad \widetilde{m}^\mu = \frac{1}{\sqrt{2}}\, \sigma^\mu_{\alpha\dot{\alpha}}\, o^\alpha\, \tilde{\imath}^{\dot{\alpha}}. \tag{2.11}$$

Of course, when k and n are real, $\widetilde{m}$ is the conjugate of m. It is then a straightforward exercise to show that all four vectors in the NP tetrad are null, and satisfy $-k\cdot n = m\cdot\widetilde{m} = 1$ with all other dot products vanishing. Furthermore, by use of these properties the spinorial completeness relation transmutes into the

NP metric,

$$g^{\mu\nu} = -k^\mu\, n^\nu - k^\nu\, n^\mu + m^\mu\, \widetilde{m}^\nu + m^\nu\, \widetilde{m}^\mu. \tag{2.12}$$

Thus we can fully describe the spacetime in terms of spinors.

2.3 The Petrov classification for 2-forms and the Weyl spinor

These four-dimensional spinors make it possible to rewrite the field strength 2-form and the Weyl tensor in a convenient form. For an arbitrary 2-form $F_{\mu\nu}$, we can build a complex symmetric spinor

$$\Phi_{\alpha\beta} = F_{\mu\nu}\, \sigma^{\mu\nu}{}_{\alpha\beta}, \tag{2.13}$$

where $\sigma^{\mu\nu}{}_{\alpha\beta} = \frac{1}{2}\left(\sigma^\mu{}_{\alpha\dot\gamma}\, \tilde\sigma^{\nu\dot\gamma}{}_\beta - \sigma^\nu{}_{\alpha\dot\gamma}\, \tilde\sigma^{\mu\dot\gamma}{}_\beta\right)$. The symmetric two-dimensional matrix $\Phi_{\alpha\beta}$ is parametrised by three complex scalars,

$$\phi_0 = \Phi_{\alpha\beta}\, o^\alpha\, o^\beta, \quad \phi_1 = \Phi_{\alpha\beta}\, o^\alpha\, \imath^\beta, \quad \phi_2 = \Phi_{\alpha\beta}\, \imath^\alpha\, \imath^\beta. \tag{2.14}$$

Similarly, we can build a symmetric 4-spinor, known as the Weyl spinor, from the Weyl tensor $C_{\mu\nu\rho\sigma}$

$$\Psi_{\alpha\beta\gamma\delta} = C_{\mu\nu\rho\sigma}\, \sigma^{\mu\nu}{}_{\alpha\beta}\, \sigma^{\rho\sigma}{}_{\gamma\delta}. \tag{2.15}$$

The Weyl spinor can be decomposed into 5 complex scalars defined by:

$$\begin{aligned} \psi_0 &= \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta o^\gamma o^\delta, \quad \psi_1 = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta o^\gamma\, \imath^\delta, \quad \psi_2 = \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, o^\beta\, \imath^\gamma\, \imath^\delta, \\ \psi_3 &= \Psi_{\alpha\beta\gamma\delta}\, o^\alpha\, \imath^\beta\, \imath^\gamma\, \imath^\delta, \quad \psi_4 = \Psi_{\alpha\beta\gamma\delta}\, \imath^\alpha\, \imath^\beta\, \imath^\gamma\, \imath^\delta. \end{aligned} \tag{2.16}$$

The Petrov classification [19] is a way of categorizing Weyl and field strength spinors depending on how "algebraically special" they are. It is well known that a symmetric $SU(2)$ n-spinor will always factorise into the symmetrisation of n basic spinors. The idea of the Petrov classification is that the more of these individual spinors that are the same (up to scale), the more special the original n-spinor is. For example, a field strength spinor $\Phi_{\alpha\beta} = \alpha_{(\alpha}\beta_{\beta)}$ is algebraically special if and only if $\beta \propto \alpha$. This also has an interpretation in terms of the complex scalars ϕ_i (and ψ_i for the Weyl tensor): it is possible to find a tetrad where some of these scalars vanish, depending on how algebraically special the n-spinor is. A summary of the classification for the field strength tensor is given in table 2.1, and for the Weyl tensor in table 2.2. The Petrov scalars have the interesting property that it is always possible to choose a tetrad where ϕ_0 vanishes. This turns out to not always be true for higher dimensions, as originally found by CMPP in [7].

Type	Spinor Alignment	Scalars
Type I	11	$\phi_0 = 0$
Type II	$\underline{11}$	$\phi_0 = \phi_1 = 0$

Table 2.1 *Table showing the Petrov classes of a 2-form. There are two possible classes, only one of which is algebraically special. We denote spinor alignment, i.e., when two spinors are the same (up to scale), by underlining them. Note that the scalars only vanish in certain tetrads.*

Type	Spinor Alignment	Scalars
Type I	1111	$\psi_0 = 0$
Type II	$\underline{11}\,11$	$\psi_0 = \psi_1 = 0$
Type D	$\underline{11}\,\underline{11}$	$\psi_0 = \psi_1 = \psi_3 = \psi_4 = 0$
Type III	$\underline{111}\,1$	$\psi_0 = \psi_1 = \psi_2 = 0$
Type N	$\underline{1111}$	$\psi_0 = \psi_1 = \psi_2 = \psi_3 = 0$

Table 2.2 *Table showing the Petrov classes of a Weyl tensor. There are four different algebraically special classes. The spinor alignment indicates when two or more spinors are the same by underlining them, for example $\underline{11}$ $\underline{11}$ refers to two different pairs of identical spinors. Note that the scalars only vanish in certain tetrads.*

As we mentioned above, the Weyl spinor is a totally symmetric rank-4 spinor and therefore can always be decomposed in terms of four rank-1 spinors as

$$\Psi_{\alpha\beta\gamma\delta} = \alpha_{(\alpha}\beta_{\beta}\gamma_{\gamma}\delta_{\delta)}\,. \tag{2.17}$$

This decomposition allows for an alternative viewpoint on the Petrov classification. The distinct algebraic classes are given by the alignment of the rank-1 spinors, i.e., the equivalence of the rank-1 spinors up to scale. We have represented the aligned spinors in tables 2.1 and 2.2 by underlining them.

The reduction of the four-dimensional formalism reviewed in this section to three dimensions is discussed in [20].

2.4 Field equations

In this chapter, we have reviewed the four dimensional Newman-Penrose formalism and the Petrov classification in preparation for the introduction of a higher dimensional formalism in chapter 3. We have not discussed why this is useful in general relativity further than the insight that the Petrov classification gives. However, they give very concrete computational advantages as well which we will very briefly sketch now.

The final step of the Newman-Penrose process is to introduce the spin connection and rewrite the field equations in terms of four scalar total derivatives and sixteen complex scalars. For example, we might be interested in whether k^μ is geodetic. We would need to study Dk^μ:

$$Dk^\mu = (\epsilon + \tilde{\epsilon})k^\mu - \tilde{\kappa}m^\mu - \kappa\tilde{m}^\mu \tag{2.18}$$

where D is one of the four scalar derivatives defined by

$$D = o^\alpha \tilde{o}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, \quad \Delta = \imath^\alpha \tilde{\imath}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, \quad \delta = o^\alpha \tilde{\imath}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, \quad \tilde{\delta} = \imath^\alpha \tilde{o}^{\dot{\beta}} \nabla_{\alpha\dot{\beta}}, \quad \nabla_{\alpha\dot{\beta}} = \sigma^\mu_{\alpha\dot{\beta}}\partial_\mu \tag{2.19}$$

and where κ and ϵ are two of the 16 complex scalars defined by

$$\kappa = o^\alpha D o_\alpha, \quad \epsilon = o^\alpha D \imath_\alpha. \tag{2.20}$$

We see that κ is a measure of whether or not k^μ is geodetic while a vanishing ϵ indicates affine parametrisation of the geodesic.

The Bianchi identity, Ricci identity and Maxwell equations are then all rewritten in terms of these quantities. The Maxwell equations, for example, are rewritten as four scalar equations, of which the first is

$$D\phi_1 - \tilde{\delta}\phi_0 = (\pi - 2\alpha)\phi_0 + 2\rho\phi_1 - \kappa\phi_2. \tag{2.21}$$

The use of the spinor formalism means that each scalar corresponds to a physical degree of freedom and makes analytic results easy to find. For example, we can see that if we have a type II Maxwell field as described in table 2.1 then a frame can be found where $\phi_0 = \phi_1 = 0$. From equation (2.21) we can immediately read off that in this frame, κ must vanish and therefore k^μ must be geodetic. This kind of insight would be much harder to find without the use of the Newman-

Penrose formalism. As such, we hope that the development of the five dimensional formalism in chapter 3 will lead to similar insight in five dimensions.

Chapter 3

Spinorial techniques in higher dimensions

3.1 Introduction

Representations of the Lorentz group play a prominent role in particle physics. Particle states are famously classified according to irreducible representations, and the requirement of Lorentz invariance strongly constrains their interactions. This constraint is particularly powerful when dealing with massless particles. As we discussed in chapter 2, the isomorphism $SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ allows us to write any massless momentum as a product of two spinors, $k_\mu \mapsto \lambda_\alpha \tilde{\lambda}_{\dot{\alpha}}$ in four spacetime dimensions [15]. For the scattering of massless particles, an S-matrix element is a function of these spinors only, and the helicities h_i of each particle fix the relative homogeneity weight of the function for each type of spinor. This is known as the spinor-helicity formalism, and it has become a major tool in high-energy physics. See for example [13] for a recent review of this formalism and its applications.

As outlined in the previous chapter, general relativity has also seen fruitful applications of this type of idea, starting with Penrose's spinorial approach [21] and its development into the Newman-Penrose formalism [18]. The basic principles are to define a frame $e^\mu{}_M$ that takes us from coordinate space to the tangent space, $\eta_{MN} = g_{\mu\nu}\, e^\mu{}_M\, e^\nu{}_N$, and then to explore the isomorphism $SO(3,1) \cong SL(2,\mathbb{C})/\mathbb{Z}_2$ for the tangent space Lorentz transformations. For instance, the Weyl tensor $C_{\mu\nu\rho\sigma}$ is described in tangent space by a rank 4 spinor $\psi_{\alpha\beta\gamma\delta}$ and its complex conjugate. The algebraic classification of this rank 4 spinor elegantly reproduces the Petrov classification of four-dimensional spacetimes [22], which had a profound impact in the development of general relativity; see for example refs. [23, 24]. In particular, the Kerr solution, which represents a vacuum asymptotically flat stationary black hole, and is perhaps the most important exact solution of astrophysical interest, was originally discovered by imposing a condition of algebraic speciality [25].

There are a variety of motivations for extending these constructions to higher spacetime dimensions. In the case of general relativity, extra dimensions are naturally motivated by string theory, and also by the fact that the number of spacetime dimensions is the natural parameter of the vacuum Einstein equations. Indeed, the catalogue of higher-dimensional vacuum asymptotically flat black hole solutions is incredibly rich, in contrast with the four-dimensional case, where the unique solution is the Kerr black hole; see for example [26–29] for reviews.

In the case of particle physics, analogous motivations apply to developing the spinor-helicity formalism in various dimensions. There is also a more practical application to the computation of S-matrix elements in dimensional regularisation, where the loop momenta cannot be restricted to four dimensions. An elegant extension of the spinor-helicity formalism approach to higher dimensions was presented in [9], where the main focus was on six dimensions. The method was extended to general dimensions in [30, 31]. In this chapter, we will apply this extension to the algebraic classification of solutions in general relativity.

As we mentioned, the space of solutions to the vacuum Einstein equations in higher dimensions is much richer than that in four dimensions, and the question of extending the Petrov classification naturally arose in the past. In fact, different approaches have been taken. Coley, Milson, Pravda and Pravdová (CMPP) defined a classification [7, 32] that has been investigated over many years, for example in [33–40]; see [41] for a review. In analogy to the four-dimensional story, the classification is based on the grouping of Weyl tensor components according to boost weight. Subgroups within the groups of boost-weighted components were found by Coley and Hervik in [37], and in [39] these sub-types were investigated in five dimensions. The CMPP classification has not been studied from a purely spinorial approach.

A different classification had been previously constructed by de Smet [4] for five-dimensional spacetimes, based on the factorisation properties of the Weyl spinor. This spinorial approach can also be considered a natural extension of the four-dimensional story, and yet it takes a very different form to the CMPP construction. An in-depth comparison by Godazgar [5] showed that there was poor agreement in what was considered algebraically special by the de Smet classification versus the CMPP classification. None of two appeared to be the 'finest' classification, since a solution could be special in one classification and general in another.

There are two main goals to this chapter. The first is to apply the higher-dimensional spinor-helicity formalism of ref. [9] to the algebraic classification of solutions of the Einstein equations, in the spirit of the spinorial approach of Penrose. The second is to show the versatility of this spinorial approach, which exhibits manifestly the two relevant types of spinor spaces, by clarifying the relation between the CMPP and the de Smet classifications, and the question of the 'finest' algebraic classification. We will be mostly interested in five-dimensional solutions, where the spinorial formalism is based on the isomorphism $SO(4,1) \cong Sp^*(1,1)/\mathbb{Z}_2$, but we will also briefly discuss the six-dimensional case in order to demonstrate generic features. We will be careful to describe when we consider reality conditions in our spinorial formalism, so that it can be applied both to real spacetimes and to potentially interesting cases of complexified spacetimes.

In addition to the classification of the Weyl tensor, we will study – for illustration and as customary in this context – the classification of its analogue in electromagnetism, the Maxwell field strength. There is a modern motivation to include this. A relation between gravity and gauge theory known as the 'double copy' has emerged from the study of scattering amplitudes in quantum field theories [10, 42]. This relation, which applies in any number of spacetime dimensions, has a counterpart in terms of solutions to the field equations. It can be expressed most clearly for certain algebraically special solutions, namely Kerr-Schild spacetimes [2, 12, 43–46], which we will elaborate on in chapter 5, but it should apply more generally [3, 47–65] as we will demonstrate in chapter 6. It is clear from these developments that there is a close relation between the algebraic properties of spacetimes and those of gauge field configurations. Indeed, it will be obvious from our results that an analogy exists. We hope to address elsewhere how this analogy can be turned into a precise double-copy relationship; progress towards this goal has been begun in [66].

This chapter is organised as follows. Following the review of the four-dimensional spinorial approach to the Petrov classification in the previous chapter, we will introduce the five-dimensional spinorial formalism in section 3.2. The five-dimensional algebraic classification is described in section 3.3 for the field strength tensor, for illustration, and then in section 3.4 for the Weyl tensor. The extension of this spinorial approach to higher dimensions is discussed in section 3.5. We conclude with a discussion of the results and possible future directions in section 3.6. The chapter is based on work done in collaboration with Ricardo Monteiro and Donal O'Connell in [1].

3.2 A Newman-Penrose basis in five dimensions

In the study of scattering amplitudes, it is important to construct a basis of vectors associated with a given particle. As we described in chapter 1, physically, these vectors are the momenta of a particle, a choice of gauge, and a basis of polarisation vectors. A method to construct this basis, known as the spinor-helicity method, is known in any dimension [9, 30, 31]. The method builds on foundational work on amplitudes in four dimensions [67–71].

In four dimensions, the spinor-helicity construction is reminiscent of the Newman-Penrose tetrad, suggesting that the spinor-helicity method can be adapted to craft a higher-dimensional Newman-Penrose basis. We will see below that this turns out to be the case, focusing on five dimensions for concreteness. Apart from some comments on six dimensions in section 3.5, we leave higher dimensions for future work.

We begin with five-dimensional flat space. We will generalise to curved space in section 3.4.1.

3.2.1 Spinors in five dimensions

Our five-dimensional setup is based on the six-dimensional conventions of [9], taking into account simplifications which occur in odd dimensions [31]. Even dimensions always have the property that one can choose a chiral basis of γ matrices, leading to the Clifford algebra[1]. But in odd dimensions no such chiral

[1] In even dimensions, there is always a matrix γ_* with the property that $\{\gamma^\mu, \gamma_*\} = 0$. In four dimensions, this γ_* is usually denoted γ_5. With the help of γ_*, one can define projectors

choice exists. We therefore work with a basis of five γ matrices. One can always raise and lower indices of γ matrices; see e.g. [72] for a useful review. In five dimensions, we may also exploit the accidental isomorphism between $\mathfrak{so}(5)$ and $\mathfrak{sp}(2)$ to choose our γ basis so that the matrices with lower indices are antisymmetric. Since it is convenient to understand the dimensional reduction to four dimensions, we found it useful to pick an explicit basis given by

$$\gamma^{\hat{\mu}}{}_{AB} = \begin{pmatrix} 0 & \sigma^{\hat{\mu}\alpha}{}_{\dot{\beta}} \\ -\tilde{\sigma}^{\hat{\mu}}{}_{\dot{\alpha}}{}^{\beta} & 0 \end{pmatrix}, \quad \hat{\mu} = 0, 1, 2, 3, \tag{3.1}$$

where the matrices σ and $\tilde{\sigma}$ are nothing but the four-dimensional Clifford bases given in equations (2.2) and (2.3) with their spinor indices appropriately raised or lowered. The final component of the basis, $\gamma^4{}_{AB}$, is chosen to be

$$\gamma^4{}_{AB} = -i \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix}. \tag{3.2}$$

This choice of basis is for convenience. The results we derive are independent of basis and hold generally. These results are summarised in section 3.2.5

With this choice of basis, we may build on our understanding of the four-dimensional NP tetrad to lay the foundations of a five-dimensional formalism. To do so, we pick null vectors k and n satisfying $k \cdot n \neq 0$, and choose a coordinate system in which k^μ and n^μ take the form

$$k^\mu = (k^0, k^1, k^2, k^3, 0), \quad n^\mu = (n^0, n^1, n^2, n^3, 0). \tag{3.3}$$

Without loss of generality, we may choose $k \cdot n = -1$. Our first task is to construct a basis of the space of spinors in five dimensions. As in the four-dimensional case described in section 2.2, we will find this basis by solving the massless Dirac equations for the null vectors k and n.

Let us take k^μ as an example. We must find the null space of the matrix

$$k \cdot \gamma_{AB} = \begin{pmatrix} 0 & k \cdot \sigma^{\alpha}{}_{\dot{\beta}} \\ -k \cdot \tilde{\sigma}_{\dot{\alpha}}{}^{\beta} & 0 \end{pmatrix}. \tag{3.4}$$

Since $k \cdot \sigma$ and $k \cdot \tilde{\sigma}$ have rank one, the matrix $k \cdot \gamma$ has rank two and the null

$P_\pm = (1 \pm \gamma_*)/2$. Spinors which are eigenstates of these projectors are called chiral. The Clifford algebra $\sigma^\mu \tilde{\sigma}^\nu + \sigma^\nu \tilde{\sigma}^\mu = -2\eta^{\mu\nu}$ can be obtained from the usual Dirac gamma algebra by defining $\sigma^\mu = P_+ \gamma^\mu P_-$ and $\tilde{\sigma}^\mu = P_- \gamma^\mu P_+$.

space is two-dimensional. We conclude that the null space of $k \cdot \gamma_{AB}$ is spanned by the spinors

$$\mathrm{k}^A{}_1 = \begin{pmatrix} 0 \\ \tilde{o}^{\dot\alpha} \end{pmatrix}, \qquad \mathrm{k}^A{}_2 = \begin{pmatrix} o_\alpha \\ 0 \end{pmatrix}, \tag{3.5}$$

which are evidently linearly independent and lie in the null space by virtue of the definitions, equations (2.9) and (2.10), of o and $\tilde{o}$. It is very convenient to package these spinors up using a Roman two-dimensional index a:

$$\mathrm{k}^A{}_a = \begin{pmatrix} 0 & o_\alpha \\ \tilde{o}^{\dot\alpha} & 0 \end{pmatrix}. \tag{3.6}$$

We will see below that the spinors $\mathrm{k}^A{}_1$ and $\mathrm{k}^A{}_2$ transform into one another under the action of a particular group.

To get a feel for $\mathrm{k}^A{}_a$, it is helpful to understand its relationship with the vector k^μ. The simplest way we can construct a spacetime vector is to hook up the indices as $\mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a$, where we use $\circ$ to denote the contraction of $SO(4,1)$ spinor indices, and have defined $\mathrm{k}^a = \epsilon^{ab}\mathrm{k}_b$. This turns out to be correct: for the first four components $\hat\mu = 0, 1, 2, 3$, we find

$$\begin{aligned} \mathrm{k}_a \circ \gamma^{\hat\mu} \circ \mathrm{k}^a =& \mathrm{Tr}\left[\begin{pmatrix} 0 & \tilde{o}^{\dot\alpha} \\ o_\alpha & 0 \end{pmatrix}\begin{pmatrix} 0 & \sigma^{\hat\mu\alpha}{}_{\dot\beta} \\ -\tilde{\sigma}^{\hat\mu}{}_{\dot\alpha}{}^{\beta} & 0 \end{pmatrix}\begin{pmatrix} o_\beta & 0 \\ 0 & -\tilde{o}^{\dot\beta} \end{pmatrix}\right] \\ =& \sigma^{\hat\mu}{}_{\alpha\dot\beta}\, o^\alpha\, \tilde{o}^{\dot\beta} + \tilde{\sigma}^{\hat\mu}{}_{\dot\alpha\beta}\, \tilde{o}^{\dot\alpha}\, o^\beta \\ =& 2\sqrt{2}\, k^{\hat\mu}, \end{aligned} \tag{3.7}$$

while for the final component we find

$$\mathrm{k}_a \circ \gamma^4 \circ \mathrm{k}^a = \; -i\,\mathrm{Tr}\left[\begin{pmatrix} 0 & \tilde{o}^{\dot\alpha} \\ o_\alpha & 0 \end{pmatrix}\begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot\alpha\dot\beta} \end{pmatrix}\begin{pmatrix} o_\beta & 0 \\ 0 & -\tilde{o}^{\dot\beta} \end{pmatrix}\right] = 0. \tag{3.8}$$

Thus, using only the four-dimensional definitions, we have recovered $k^\mu = (k^{\hat\mu}, 0)$. The complete formula is therefore:

$$k^\mu = \frac{1}{2\sqrt{2}}\, \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a. \tag{3.9}$$

It is worth commenting further on this formula. The spinors k_a for $a = 1, 2$ are a basis of solutions of the equation $k \cdot \gamma_{AB}\, \mathrm{k}^B{}_a = 0$. We may, of course, perform a complex linear change of basis in this space of solutions. The normalisation condition $k^\mu = \frac{1}{2\sqrt{2}}\, \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a$ restricts this change of basis to be an element of

$SL(2,\mathbb{C})$, so we can think of the null space as a two-dimensional representation of $SL(2,\mathbb{C})$. In fact, we will see below in section 3.2.3 that if we choose a real vector k^μ, and impose both our normalisation condition and a reality condition on the spinors k_a, we must further restrict this group to $SU(2)$. The physical role of this group is simply the three-dimensional rotations on the spacetime dimensions orthogonal to both k and n.

Now we construct the other half of the spinor basis $\mathrm{n}^A{}_a$. In view of the normalisation condition $k \cdot n = -1$ satisfied by the vectors, we can choose the spinors $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$ to satisfy $\mathrm{k}_a \circ \mathrm{n}_b \equiv \mathrm{k}^A{}_a \Omega_{AB} \mathrm{n}^B{}_b = \epsilon_{ab}$, where the raising/lowering matrix Ω_{AB} is, explicitly,

$$\Omega_{AB} = \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & -\epsilon_{\dot\alpha\dot\beta} \end{pmatrix}. \tag{3.10}$$

Incidentally, for notational simplicity we define

$$\mathrm{k}_{Aa} = \Omega_{AB}\, \mathrm{k}^B{}_a, \quad \mathrm{n}_{Aa} = \Omega_{AB}\, \mathrm{n}^B{}_a. \tag{3.11}$$

Following the recipe described above we find a basis of spinors in the null space of $n \cdot \sigma_{AB}$. However, a naive application of the method leads to a basis which does not satisfy our normalisation condition $\mathrm{k}_a \circ \mathrm{n}_b = \epsilon_{ab}$. To correct this, we simply perform a change of basis, finding

$$\mathrm{n}^A{}_a = \begin{pmatrix} \imath_\alpha & 0 \\ 0 & -\tilde\imath^{\dot\alpha} \end{pmatrix}. \tag{3.12}$$

The spacetime vector n^μ can be reconstructed from the spinors as before:

$$n^\mu = \frac{1}{2\sqrt{2}}\, \mathrm{n}_a \circ \gamma^\mu \circ \mathrm{n}^a. \tag{3.13}$$

The other two contractions are $\mathrm{k}_a \circ \mathrm{k}_b = \mathrm{n}_a \circ \mathrm{n}_b = 0$, which follows from the antisymmetry of Ω_{AB}.

3.2.2 Polarisation vectors

The spinors $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$ are a complete basis of spinors. As in the four-dimensional case, we can use the spinorial basis to construct vectors which, accompanied by k^μ and n^μ, form a complete basis of vectors in five dimensions –

a pentad. Recall that the vectors k^μ and n^μ are given by

$$k^\mu = \frac{1}{2\sqrt{2}}\,\mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a, \quad n^\mu = \frac{1}{2\sqrt{2}}\,\mathrm{n}_a \circ \gamma^\mu \circ \mathrm{n}^a. \tag{3.14}$$

We define the remaining independent contraction to be

$$\varepsilon^\mu{}_{ab} \equiv \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{n}_b = -\mathrm{n}_b \circ \gamma^\mu \circ \mathrm{k}_a \tag{3.15}$$

where it can be shown that $\varepsilon^\mu{}_{ab} = \varepsilon^\mu{}_{ba}$ by use of gamma matrix algebra. To interpret this object $\varepsilon^\mu{}_{ab}$, we can break it down in terms of its four-dimensional components. Firstly, we will consider $\hat{\mu} = 0, 1, 2, 3$. For these values of $\hat{\mu}$, $\varepsilon^{\hat{\mu}}{}_{ab}$ is given by:

$$\begin{aligned}
\varepsilon^{\hat{\mu}}{}_{ab} =& \begin{pmatrix} 0 & \tilde{o}^{\dot{\alpha}} \\ o_\alpha & 0 \end{pmatrix} \begin{pmatrix} 0 & \sigma^{\hat{\mu}\alpha}{}_{\dot{\beta}} \\ -\tilde{\sigma}^{\hat{\mu}}{}_{\dot{\alpha}}{}^{\beta} & 0 \end{pmatrix} \begin{pmatrix} \imath_\beta & 0 \\ 0 & -\tilde{\imath}^{\dot{\beta}} \end{pmatrix} \\
=& \begin{pmatrix} \tilde{\sigma}^{\hat{\mu}}{}_{\dot{\alpha}\beta}\, \tilde{o}^{\dot{\alpha}}\, \imath^\beta & 0 \\ 0 & \sigma^{\hat{\mu}}{}_{\alpha\dot{\beta}}\, o^\alpha\, \tilde{\imath}^{\dot{\beta}} \end{pmatrix} \\
=&\sqrt{2}\, \begin{pmatrix} m^{\hat{\mu}} & 0 \\ 0 & \widetilde{m}^{\hat{\mu}} \end{pmatrix}.
\end{aligned} \tag{3.16}$$

Thus we can see that as long as $\hat{\mu} = 0, 1, 2, 3$, the diagonal components of $\varepsilon^{\hat{\mu}}{}_{ab}$ are precisely the vectors $m^{\hat{\mu}}$ and $\widetilde{m}^{\hat{\mu}}$ which appeared in the Newman-Penrose tetrad in four dimensions. The final value of μ, $\mu = 4$, is given by

$$\begin{aligned}
\varepsilon^4{}_{ab} =& \,\mathrm{k}_a \circ \gamma^4 \circ \mathrm{n}_b \\
=& -i \begin{pmatrix} 0 & \tilde{o}^{\dot{\alpha}} \\ o_\alpha & 0 \end{pmatrix} \begin{pmatrix} \epsilon^{\alpha\beta} & 0 \\ 0 & \epsilon_{\dot{\alpha}\dot{\beta}} \end{pmatrix} \begin{pmatrix} \imath_\beta & 0 \\ 0 & -\tilde{\imath}^{\dot{\beta}} \end{pmatrix} \\
=& \begin{pmatrix} 0 & i \\ i & 0 \end{pmatrix}.
\end{aligned} \tag{3.17}$$

We therefore find

$$\begin{aligned}
\varepsilon^\mu{}_{11} =& \sqrt{2}\, \left(m^{\hat{\mu}}, 0\right) \\
\varepsilon^\mu{}_{22} =& \sqrt{2}\, \left(\widetilde{m}^{\hat{\mu}}, 0\right) \\
\varepsilon^\mu{}_{12} = \varepsilon^\mu{}_{21} =& \, (0, 0, 0, 0, i)\,.
\end{aligned} \tag{3.18}$$

Finally, we can establish the useful property

$$\varepsilon^\mu{}_{ab}\, \varepsilon_{\mu\, cd} = \epsilon_{ac}\, \epsilon_{bd} + \epsilon_{ad}\, \epsilon_{bc} \tag{3.19}$$

by explicit computation. The spinorial completeness relations imply that

$$\eta^{\mu\nu} = -k^\mu n^\nu - k^\nu n^\mu + \frac{1}{2}\,\epsilon^{ac}\,\epsilon^{cd}\,\varepsilon^\mu{}_{ab}\,\varepsilon^\nu{}_{cd}. \tag{3.20}$$

These properties are characteristic of polarisation vectors, which in part accounts for the utility of this formalism in scattering amplitudes.

3.2.3 Reality conditions

Our γ basis satisfies

$$(\gamma^\mu)^\dagger = -H \circ \gamma^\mu \circ H^T \tag{3.21}$$

where the matrices γ^μ have lower indices and

$$H = \begin{pmatrix} 0 & \epsilon^{\dot\alpha\dot\beta} \\ -\epsilon_{\alpha\beta} & 0 \end{pmatrix}. \tag{3.22}$$

For a real null vector V in five-dimensional Minkowski space, we may impose a reality condition on the associated basis of spinors $\lambda^A{}_a$. Regarding λ as a four-by-two matrix, reality of V implies

$$V \cdot \gamma \circ \lambda = 0 \Rightarrow V \cdot \gamma \circ H^T \circ \lambda^* = 0. \tag{3.23}$$

Thus the spinors $H^T \circ \lambda^*$ are linear combinations of the two basis spinors λ_a, so we may write $H^T \circ \lambda^* = \lambda X$, where X is a two-by-two matrix.

Recall from section 3.2.1 that the two-dimensional space of λ_a furnishes a representation of $SL(2,\mathbb{C})$. The reality condition $H^T \circ \lambda^* = \lambda X$ is not covariant under the full $SL(2,\mathbb{C})$, because the left-hand side transforms under the conjugate representation of the right-hand side. Thus the group is broken to $SU(2)$, which has the well-known property that the conjugate representation is equivalent to the fundamental representation. Requiring that the reality condition is covariant under this $SU(2)$ determines $X \propto \epsilon$. Thus, in our conventions, we arrive at the reality condition in the form [73]

$$H^T \circ \lambda^* = -\lambda\epsilon. \tag{3.24}$$

Using index notation, we may write this as follows. First we define $\bar\lambda^{\dot A a} \equiv (\lambda^A{}_a)^*$;

then the reality condition is

$$\bar{\lambda}^{\dot{A}a} H_{\dot{A}}{}^{A} = \epsilon^{ab} \lambda^{A}{}_{b}. \tag{3.25}$$

Our main focus will be on real spacetimes with Minkowski signature. Therefore we will pick real vectors k^μ and n^μ and impose the reality condition, equation (3.25), on the spinors $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$.

We must now investigate what this means for our pentad, in particular for the "polarisations" $\varepsilon^\mu{}_{ab}$. They are defined by $\varepsilon^\mu{}_{ab} = \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{n}_b$; we define the conjugate of these vectors to be $\overline{\varepsilon}^{\mu ab} \equiv (\varepsilon^\mu{}_{ab})^*$. Using the reality condition we find

$$\begin{aligned}
\overline{\varepsilon}^{\mu ab} &= (\mathrm{k}_a \circ \gamma^\mu \circ \mathrm{n}_b)^* \\
&= (\mathrm{k}_a)^* \circ (\gamma^\mu)^* \circ (\mathrm{n}_b)^* \\
&= \bar{\mathrm{k}}^a \circ (H \circ \gamma^\mu \circ H^T) \circ \bar{\mathrm{n}}^b \\
&= (\epsilon^{ac}\, \mathrm{k}_c) \circ \gamma^\mu \circ (\epsilon^{bd}\, \mathrm{n}_d) \\
&= \epsilon^{ac}\, \epsilon^{bd}\, \varepsilon^\mu{}_{cd} \\
&= \varepsilon^{\mu ab}.
\end{aligned} \tag{3.26}$$

In short, $\varepsilon^{\mu ab} = (\varepsilon^\mu{}_{ab})^*$. So $\varepsilon^\mu{}_{11} = (\varepsilon^\mu{}_{22})^*$, while $\varepsilon^\mu{}_{12} = -(\varepsilon^\mu{}_{12})^*$. This is exactly as we found in section 3.2.2: $\varepsilon^\mu{}_{11}$ and $\varepsilon^\mu{}_{22}$ relate to m^μ and $\widetilde{m}^\mu$ respectively while $\varepsilon^\mu{}_{12}$ is given by $ie^\mu{}_4$, which is indeed imaginary.

3.2.4 Lorentz transformations and the little group

To build some intuition into the objects $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$, it is worth pausing our development to understand how these spinors transform under symmetries, especially (local) Lorentz transformations. Recall that the index A takes values from 1 to 4, spanning the four dimensions of the spinorial representation of $SO(4,1)$, while the index a takes values 1 and 2 and spans the two-dimensional solutions space of, for example, the equation $k_\mu \gamma^\mu_{AB} \mathrm{k}^B_a = 0$. We will see that the $SU(2)$ acting on the two-dimensional solution space is the subgroup of Lorentz transformations which preserve the vector k^μ. This subgroup is the little group of the null vector k^μ.

Boosts and spins

We have defined the spinors $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$ to be solutions of the Dirac equations $k \cdot \gamma_{AB}\mathrm{k}^B_a = 0 = n \cdot \gamma_{AB}\mathrm{n}^B_a$, subject to the normalisation condition $\mathrm{k}_a \cdot \mathrm{n}_b = \epsilon_{ab}$, and obeying a reality condition for real spacetimes. Obviously the rescaling

$$\mathrm{k}^A{}_a \to b\,\mathrm{k}^A{}_a, \qquad \mathrm{n}^A{}_a \to \frac{1}{b}\,\mathrm{n}^A{}_a \tag{3.27}$$

will preserve the definitions, provided that the factor b is real for real spacetimes. We may therefore investigate how this rescaling acts on the pentad we have constructed from the spinors, equations (3.14) and (3.15). It is easy to see that the action is

$$k^\mu \to b^2\,k^\mu, \qquad n^\mu \to \frac{1}{b^2}\,n^\mu, \qquad \varepsilon^\mu{}_{ab} \to \varepsilon^\mu{}_{ab}. \tag{3.28}$$

This simple transformation is nothing but a Lorentz boost in the two-dimensional space spanned by k^μ and n^μ, leaving the remaining three dimensions invariant.

We may also consider a more non-trivial change of basis of the solution space of the Dirac equations:

$$\mathrm{k}^A{}_a \to \mathrm{k}'^A{}_a = M_a{}^b\,\mathrm{k}^A{}_b, \qquad \mathrm{n}^A{}_a \to \mathrm{n}'^A{}_a = N_a{}^b\mathrm{n}^A{}_b. \tag{3.29}$$

This change of basis automatically preserves the conditions that $\mathrm{k}_a \circ \mathrm{k}_b = 0$ and $\mathrm{n}_a \circ \mathrm{n}_b = 0$. We have already seen that M and N are elements of $SL(2, \mathbb{C})$. The normalisation condition is that

$$M_a{}^c\,N_b{}^d\,\epsilon_{cd} = \epsilon_{ab}, \tag{3.30}$$

which implies that $N = M$.

We may now investigate the action of this group of transformations on our spacetime pentad. A straightforward calculation shows that the transformation is

$$k^\mu \to k^\mu, \qquad n^\mu \to n^\mu, \qquad \varepsilon^\mu{}_{ab} \to M_a{}^c\,M_b{}^d\,\varepsilon^\mu{}_{cd}. \tag{3.31}$$

This is a Lorentz transformation preserving k and n.

In the real case, we have already seen that the transformation M is an element of $SU(2)$. This makes sense: in the real case, the subgroup of the Lorentz group which preserves k^μ and n^μ is evidently $SO(3)$. We can see this more concretely by introducing a vectorial basis of the three-dimensional representation of $SU(2)$, which is also the fundamental representation of $SO(3)$. As usual, the symmetric Pauli matrices $\varsigma^{ab}{}_i$, $i = 1, 2, 3$ provide a convenient mapping from the $\underline{2} \otimes \underline{2}$ tensor product of $SU(2)$ representations to the $\underline{3}$. In view of the reality condition, we find it convenient to take

$$\varsigma_1 = \frac{1}{2}\begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix}, \qquad \varsigma_2 = \frac{1}{2}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \qquad \varsigma_3 = \frac{1}{2}\begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \tag{3.32}$$

Notice, for example, that this choice of basis has the property that $(\varsigma^{11}{}_i)^* = \varsigma^{22}{}_i$, consistent with our reality condition. They relate to the usual Pauli matrices as $\sigma_i = \frac{1}{2}\varsigma_i \cdot \epsilon$.

We may then define

$$\varepsilon^\mu{}_i = \varepsilon^\mu{}_{ab}\, \varsigma^{ab}{}_i, \tag{3.33}$$

and

$$m_i = M_{ab}\, \varsigma^{ab}{}_i. \tag{3.34}$$

The antisymmetric degree of freedom in M is defined to be $M_{\text{tr}} = \epsilon^{ab} M_{ab}$. In this language, the condition that M has unit determinant becomes $\frac{1}{4}{M_{\text{tr}}}^2 + \underline{m} \cdot \underline{m} = 1$, and the polarisation vector transformation is

$$\underline{\varepsilon}^\mu \to \left(-\underline{m} \cdot \underline{m} + \frac{1}{4}{M_{\text{tr}}}^2\right) \underline{\varepsilon}^\mu + 2\left(\underline{m} \cdot \underline{\varepsilon}^\mu\right) \underline{m} + M_{\text{tr}}\left(\underline{m} \times \underline{\varepsilon}^\mu\right). \tag{3.35}$$

We can compare this with the standard formula for a rotation by angle θ around an axis $\underline{n}$ in three-dimensional Euclidean space,

$$\underline{x} \to \cos\theta\, \underline{x} + (1 - \cos\theta)\ (\underline{n} \cdot \underline{x})\ \underline{n} + \sin\theta\ (\underline{n} \times \underline{x}), \tag{3.36}$$

to see that the transformation M rotates the polarisation vectors by an angle $\sin\theta = M_{\text{tr}}|\underline{m}|$ around the axis $\underline{m}$ in the Euclidean 3-space of the little group, leaving k^μ and n^μ invariant.

The null rotations

The boost and spin transformations comprise four of the ten Lorentz transformations available in a five-dimensional spacetime. It is interesting to understand the remaining six. To do so, we look to the null rotations of the four-dimensional NP tetrad for inspiration, and construct the ansatz $\mathrm{k}^A{}_a \to \mathrm{k}^A{}_a + T_a{}^b\,\mathrm{n}^A{}_b$, $\mathrm{n}^A{}_a \to\ \mathrm{n}^A{}_a$. To preserve $\mathrm{k}_a \cdot \mathrm{n}_b$, we require that the matrix T is symmetric:

$$\begin{aligned}\mathrm{k}'_a \cdot \mathrm{k}'_b &= (\mathrm{k}_a + T_a{}^c\,\mathrm{n}_c) \cdot \big(\mathrm{k}_b + T_b{}^d\,\mathrm{n}_d\big) \\ &= T_a{}^c\,(\mathrm{n}_c \cdot \mathrm{k}_b) + T_b{}^d\,(\mathrm{k}_a \cdot \mathrm{n}_d) \\ &= T_{ab} - T_{ba} = 0.\end{aligned} \tag{3.37}$$

Similarly the transformation $\mathrm{k}^A{}_a \to \mathrm{k}^A{}_a$, $\mathrm{n}^A{}_a \to \mathrm{n}^A{}_a + S_a{}^b\,\mathrm{k}^A{}_b$ is valid as long as S is symmetric. The symmetric matrices S and T comprise three degrees of freedom each, so combined with the boost and spin, this is a complete parametrisation of the Lorentz group. The action of these transformations on our pentad is:

- Null rotation about n: $\quad \mathrm{k}^A{}_a \to \mathrm{k}^A{}_a + T_a{}^b\,\mathrm{n}^A{}_b, \quad \mathrm{n}^A{}_a \to \mathrm{n}^A{}_a,$

$$k^\mu \to k^\mu + T^{ab}\,\varepsilon^\mu{}_{ab} - \det T\,n^\mu, \quad n^\mu \to n^\mu, \quad \varepsilon^\mu{}_{ab} \to \varepsilon^\mu{}_{ab} + T_{ab}\,n^\mu. \tag{3.38}$$

- Null rotation about k: $\quad \mathrm{k}^A{}_a \to \mathrm{k}^A{}_a, \quad \mathrm{n}^A{}_a \to \mathrm{n}^A{}_a + S_a{}^b\,\mathrm{k}^A{}_b,$

$$k^\mu \to k^\mu, \quad n^\mu \to n^\mu + S^{ab}\,\varepsilon^\mu{}_{ab} - \det S\,k^\mu, \quad \varepsilon^\mu{}_{ab} \to \varepsilon^\mu{}_{ab} + S_{ab}\,k^\mu. \tag{3.39}$$

3.2.5 Summary

We can now summarise the key results. The pentad is constructed from null orthogonal vectors

$$k^2 = n^2 = 0, \qquad k_\mu\,n^\mu = -1, \tag{3.40}$$

with the symmetric polarisation vector $\varepsilon^\mu{}_{ab}$ satisfying

$$k \cdot \varepsilon_{ab} = n \cdot \varepsilon_{ab} = 0, \qquad \varepsilon_{\mu ab}\,\varepsilon^\mu{}_{cd} = \epsilon_{ac}\,\epsilon_{bd} + \epsilon_{ad}\,\epsilon_{bc}. \tag{3.41}$$

This pentad spans the spacetime as

$$\eta^{\mu\nu} = \ - k^\mu n^\nu - k^\nu n^\mu + \frac{1}{2}\,\epsilon^{ac}\,\epsilon^{cd}\,\varepsilon^\mu{}_{ab}\,\varepsilon^\nu{}_{cd}. \tag{3.42}$$

We choose spinors $\mathrm{k}^A{}_a$, $\mathrm{n}^A{}_a$, where $A = 1, ..., 4$ is a spacetime spinor index and $a = 1, 2$ is a little group spinor index, to satisfy

$$\mathrm{k}_a \circ \mathrm{k}_b = \mathrm{n}_a \circ \mathrm{n}_b = 0, \qquad \mathrm{k}_a \circ \mathrm{n}_b = \epsilon_{ab}, \tag{3.43}$$

where "$x \circ y$" indicates a contraction on the spacetime spinor index, i.e., $x_A\, y^A$. The pentad can be defined in terms of the spinors:

$$k^\mu = \frac{1}{2\sqrt{2}}\, \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a, \qquad n^\mu = \frac{1}{2\sqrt{2}}\, \mathrm{n}_a \circ \gamma^\mu \circ \mathrm{n}^a, \qquad \varepsilon^\mu{}_{ab} = \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{n}_b, \tag{3.44}$$

in order to automatically satisfy the properties given above. To restrict to real Minkowski space, the spinors must satisfy reality conditions. In particular, any real objects which transform under the little group indices must obey

$$\left(X_{a_1 \ldots a_n}{}^{b_1 \ldots . b_m}\right)^* = X^{a_1 \ldots a_n}{}_{b_1 \ldots . b_m}. \tag{3.45}$$

Finally, we note that the ten transformations of the standard five-dimensional Lorentz group can be parametrised as a boost b, three spins M_{ab} where $\det M = 1$, and two three-dimensional null transformations T_{ab} and S_{ab} which are both symmetric:

- Boost: $\mathrm{k}_A{}^a \to b\, \mathrm{k}_A{}^a, \quad \mathrm{n}_A{}^a \to \frac{1}{b}\, \mathrm{n}_A{}^a$
- Spin: $\mathrm{k}_A{}^a \to M^a{}_b\, \mathrm{k}_A{}^b, \quad \mathrm{n}_A{}^a \to M^a{}_b \mathrm{n}_A{}^b$
- Null rotation about n: $\quad \mathrm{k}^A{}_a \to \mathrm{k}^A{}_a + T_a{}^b\, \mathrm{n}^A{}_b, \quad \mathrm{n}^A{}_a \to \mathrm{n}^A{}_a$
- Null rotation about k: $\quad \mathrm{k}^A{}_a \to \mathrm{k}^A{}_a, \quad \mathrm{n}^A{}_a \to \mathrm{n}^A{}_a + S_a{}^b\, \mathrm{k}^A{}_b$.

3.3 The field strength tensor

Although our main goal is to apply the results of section 3.2 to gravity, it is helpful to apply them to the simpler field strength tensor $F_{\mu\nu}$ first.

3.3.1 Set up and classifications

To begin, we contract $F_{\mu\nu}$ with the rotation generator

$$\sigma^{\mu\nu}{}_{AB} = \frac{1}{2}\left(\gamma^\mu{}_{AC}\, \gamma^{\nu\, C}{}_B - \gamma^\nu{}_{AC}\, \gamma^{\mu\, C}{}_B\right) \tag{3.46}$$

to find a symmetric bi-spinor,

$$\Phi_{AB} = F_{\mu\nu}\, \sigma^{\mu\nu}{}_{AB}. \tag{3.47}$$

This is analogous to the four-dimensional Newman-Penrose formalism, as described in section 2. Now, however, upon contraction with our basis spinors, we do not obtain scalars but little group bi-spinors:

$$\Phi^{(0)}_{ab} = \Phi_{AB}\, \mathrm{k}^A{}_a\, \mathrm{k}^B{}_b, \quad \Phi^{(1)}_{ab} = \Phi_{AB}\, \mathrm{k}^A{}_a\, \mathrm{n}^B{}_b, \quad \Phi^{(2)}_{ab} = \Phi_{AB}\, \mathrm{n}^A{}_a\, \mathrm{n}^B{}_b, \tag{3.48}$$

where the bracketed numbers label the little group bi-spinors according to the number of $\mathrm{n}^A{}_a$ spinors they are contracted with. To begin with, we will consider complex-valued $F_{\mu\nu}$, and restrict to the real case later on.

In four dimensions, the Petrov classification based on the scalars defined in (2.14) had two classes, type I and type II, the latter of which was considered algebraically special. Type II was defined by the existence of a tetrad where both of the four-dimensional Petrov scalars ϕ_0 and ϕ_1 vanished; see table 2.1. Since the scalars from equation (2.14) and the spinors from (3.48) are clearly analogous, this motivates a Petrov-like classification for five dimensions, which is shown in table 3.1. The guaranteed existence of a tetrad where ϕ_0 vanishes is a special feature of four dimensions, and so we also require an additional "general" class for 2-forms in five dimensions. As we will show in section 3.3.3, this is exactly the original CMPP classification for the 2-form.

Type	Little group spinor characteristic
Type G	$\Phi^{(i)} \neq 0\ \forall\, i$
Type I	$\Phi^{(0)} = 0$
Type II	$\Phi^{(0)} = \Phi^{(1)} = 0$

Table 3.1 *Table showing a proposed Petrov-like classification for a 2-form. There are now three possible classes, two of which are analogous to four dimensions and one of which, Type G, is new to higher dimensions.*

The bi-spinors defined in (3.48) are reducible, and therefore we will refer to this classification as a "coarse" classification. A more fine-grained classification is available if we break the bi-spinors down into their irreducible representations, namely the symmetric bi-spinor and the scalar. To do this, we will use the notation that $\phi^{(i)}$ refers to the symmetrisation of $\Phi^{(i)}$, such that $\phi^{(i)}_{ab} = \Phi^{(i)}_{(ab)}$. Since $\Phi_{AB} = \Phi_{BA}$, we can see that $\Phi^{(0)}$ and $\Phi^{(2)}$ are already symmetric, so

$\phi^{(0)} = \Phi^{(0)}$ and $\phi^{(2)} = \Phi^{(2)}$. The bi-spinor $\Phi^{(1)}$ is not symmetric in general, but it is always possible to write a two-component bi-spinor as the sum of a symmetric bi-spinor and a trace term proportional to the Levi-Civita tensor[2]. We will refer to this trace as ${\Phi^{(1)}}_a{}^a = \Phi^{(1)}_{\text{tr}}$ such that:

$$\Phi^{(1)}_{ab} = \phi^{(1)}_{ab} + \frac{1}{2}\,\Phi^{(1)}_{\text{tr}}\,\epsilon_{ab}. \tag{3.49}$$

This is simply the statement that a **4** decomposes as $\mathbf{4} = \mathbf{3} + \mathbf{1}$ where the symmetric bi-spinor **3** and the scalar **1** are both irreducible representations. The 10 degrees of freedom in the five-dimensional field strength tensor have therefore been split up into 3 symmetric bi-spinors and a single scalar. We can write this as in table 3.2, where the terms have been organised by the dimension of their irreducible representation along the horizontal axis and by the bracketed number in the vertical direction. This fine-grained classification is sensitive to the vanishing of the columns as well as the rows. For example, a 2-form with vanishing $\phi^{(1)}_{ab}$ or $\phi^{(1)}_{\text{tr}}$ is considered more special than one where both are non-zero. We will give some examples in section 3.3.2.

Reducible representation		**3**	**1**
$\Phi^{(0)}_{ab}$		$\phi^{(0)}_{ab}$	
$\Phi^{(1)}_{ab}$	$\Rightarrow$	$\phi^{(1)}_{ab}$	$\Phi^{(1)}_{\text{tr}}$
$\Phi^{(2)}_{ab}$		$\phi^{(2)}_{ab}$	

Table 3.2 *The three little group spinors of the 2-form can be broken up into three symmetric bi-spinors, **3**, and a scalar **1**. This fine-grained structure is able to provide more detail on the nature of the 2-form than the coarse classification. For example, a type I solution with vanishing* $\Phi^{(1)}_{\text{tr}}$ *is more special than one where both* $\Phi^{(1)}_{\text{tr}}$ *and* $\phi^{(1)}$ *are non-zero.*

In the real case, these objects are subject to the conditions $\phi^{(i)}_{ab} = \left(\phi^{(i)\,ab}\right)^*$. We can easily recast them into real vectors acted on by $SO(3)$ using the Pauli matrices

[2]Since a two-dimensional index has only two possible values,

$$\epsilon_{a[b}\epsilon_{cd]} = 0 = \epsilon_{ab}\,\epsilon_{cd} + \epsilon_{ac}\,\epsilon_{db} + \epsilon_{ad}\,\epsilon_{bc}.$$

Contracting this with an arbitrary bi-spinor s^{cd}, we obtain

$$s_{ab} - s_{ba} = \epsilon_{ab}\,{s_c}^c.$$

Spinor notation			Vector notation	
$\phi^{(0)}_{ab}$ $\phi^{(1)}_{ab}$ $\phi^{(2)}_{ab}$	$\Phi^{(1)}_{\rm tr}$	$\leftrightarrow$	$\underline{\phi_0}$ $\underline{\phi_1}$ $\underline{\phi_2}$	$\Phi^{(1)}_{\rm tr}$

Table 3.3 *The little group irreps can be written in terms of spinors or vectors by standard use of the Pauli matrices.*

${\varsigma^i}_{ab}$:

$$\left(\underline{\phi_0}\right)^i = \phi^{(0)}_{ab}\, \varsigma^{iab}, \tag{3.50}$$

where $i = 1, 2, 3$ is an $SO(3)$ index, and of course $\Phi^{(1)}_{\rm tr}$ remains a scalar. The little group irreps therefore change into a combination of 3-vectors and scalars as shown in table 3.3. Vector notation will be useful when making contact with the existing literature.

Finally, it is always possible to factorise a symmetric bi-spinor into two symmetrised spinors

$$\phi_{ab} = \alpha_{(a}\, \beta_{b)}. \tag{3.51}$$

It is natural to ask if there exists some sub-classification where $\alpha = \beta$ as is the case in four dimensions. From the vectorial perspective it is easy to see that this will not be the case if we restrict ourselves to real Minkowski space. If we consider an arbitrary symmetric bi-spinor

$$\left(\underline{\phi}\right)^i = \phi_{ab}\, \varsigma^{iab} = \alpha_a\, \beta_b\, \varsigma^{iab}, \tag{3.52}$$

we can see that the modulus of this vector is given by

$$|\underline{\phi}| = \frac{1}{2}\, |\alpha_a \beta^a|, \tag{3.53}$$

using $\varsigma^{i\,ab}\,\varsigma^{i\,cd} = (\epsilon^{ac}\epsilon^{bd} + \epsilon^{ad}\epsilon^{bc})/4$. Therefore, there is no non-vanishing real vector $\underline{\phi}$ such that $\alpha = \beta$, and the irreps that we describe in table 3.2 cannot be broken down further. In contrast, in the complex case they can, leading to a Russian doll-like structure of nested classifications where each bi-spinor $\phi^{(i)}$ can itself be type I ($\alpha \neq \beta$) or type II ($\alpha = \beta$).

3.3.2 Examples

To be more concrete, we will discuss some simple examples: the plane wave, an electric field and a magnetic field. This will illuminate some details of the fine structure.

A plane wave

The simplest solution is a plane wave which has a field strength tensor of the form

$$F_{\mu\nu} = k_{[\mu}\varepsilon_{\nu]}{}^{ab} P_{ab}\, e^{ik\cdot x}, \tag{3.54}$$

where the symmetric P_{ab} corresponds to an arbitrary choice of polarisation. It is natural to choose k_μ and $\varepsilon_\mu{}^{ab}$ to be elements of our pentad. Using the normalisations in equation (3.44) we have

$$\begin{aligned} \Phi_{AB} &= F_{\mu\nu}\sigma^{\mu\nu}{}_{AB} \\ &= k_{[\mu}\varepsilon_{\nu]}{}^{ab}\gamma^{\mu}{}_{AC}\gamma^{\nu\,C}{}_{B} P_{ab}\, e^{ik\cdot x} \\ &= -2\sqrt{2}\,\mathrm{k}_{(A}{}^{a}\mathrm{k}_{B)}{}^{b} P_{ab}\, e^{ik\cdot x}, \end{aligned} \tag{3.55}$$

and comparison with equation (3.48) tells us that we have

$$\phi^{(0)} = \Phi^{(1)} = 0, \quad \phi^{(2)}_{ab} = -2\sqrt{2}P_{ab}e^{ik\cdot x}. \tag{3.56}$$

A plane wave is therefore a type II solution under the coarse classification. Since $\phi^{(2)}$ is symmetric, it is an irreducible representation of $SU(2)$. However, it is possible that $P_{ab} = \alpha_a\alpha_b$ in the complex case, which describes a circularly polarised electromagnetic field.

A constant electric field

Our second example is a constant electric field $\underline{E}$ in the x direction. Then the Maxwell spinor has the form

$$\Phi_{AB} = 2|\underline{E}|\sigma^{tx}{}_{AB}. \tag{3.57}$$

We choose $k = \frac{1}{\sqrt{2}}(\partial_t + \partial_x)$ and $n = \frac{1}{\sqrt{2}}(\partial_t - \partial_x)$. Taking contractions with $\mathrm{k}^A{}_a$ and $\mathrm{n}^A{}_a$, we find

$$\phi^{(0)} = \phi^{(1)} = \phi^{(2)} = 0, \qquad \Phi^{(1)}_{\mathrm{tr}} = 4|\underline{E}|. \tag{3.58}$$

Hence the electric field has a coarse type I classification, but the fine structure is able to pinpoint that this is more special than a general type I.

A constant magnetic field

Finally, we consider a simple magnetic field B which is trivial in the x direction such that $F^{\mu\nu} = B^{ij}$. We use the same pentad as the previous section, so $k = \frac{1}{\sqrt{2}}(\partial_t + \partial_x)$ and $n = \frac{1}{\sqrt{2}}(\partial_t - \partial_x)$. The Maxwell spinor is

$$\Phi_{AB} = B^{ij}\sigma^{ij}{}_{AB}. \tag{3.59}$$

Taking contractions again and using the Pauli matrices $\varsigma^i{}_{ab}$ to recast $\phi^{(1)}$ as a vector, we find

$$\phi^{(0)} = \phi^{(2)} = \Phi^{(1)}_{\mathrm{tr}} = 0, \qquad \left(\underline{\phi_1}\right)^i = \epsilon^{ijk}B^{jk}. \tag{3.60}$$

Therefore, although this magnetic field and the electric field have the same coarse classification, type I, they can be differentiated by their fine structure.

3.3.3 Relations to the literature: CMPP and de Smet

As we have mentioned earlier, there exist previously proposed classifications for five-dimensional spacetimes. Two of these are the classification derived by CMPP in 2004 [7, 32] and the de Smet classification proposed in 2002 [4]. We will understand both in terms of the spinorial formalism.

The CMPP classification

In their papers [7, 32], CMPP observe that each component of the Weyl tensor in D dimensions has a boost weight when the pentad is rescaled by $\{k,\, n,\, m^{(i)}\} \to \{\rho\, k,\, \rho^{-1}\, n,\, m^{(i)}\}$ for some scalar ρ, where $i = 2, ..., D-1$, and $m^{(i)}$ is any of the remaining space-like directions. This boost weight is simply the power of ρ by which the component of the 2-form transforms. The independent components of the 2-form have the following boost weights:

$$\begin{array}{|c||c|c|c|} \hline \text{Boost weight} & 1 & 0 & -1 \\ \hline \text{Component} & F_{0i} & F_{01}, F_{ij} & F_{1i} \\ \hline \end{array} \tag{3.61}$$

where the index 0 indicates a contraction with k, the index 1 indicates a contraction with n, and a Roman index i corresponds to the space-like direction $m^{(i)}$. The CMPP k and n have an identical role to our own usage, so we will use the same symbols. The relevant choices of k are made by demanding that F_{0i} is set to zero if possible, in which case a choice of n is made to also send F_{01} and F_{ij} to zero if possible. Next, the boost weights are organised into a Petrov-like classification as shown in table 3.4.

Type	Components	CMPP special?
Type G	$F_{0i} \neq 0$	No
Type I	$F_{0i} = 0$	No
Type II	$F_{0i} = F_{01} = F_{1i} = 0$	Yes

Table 3.4 *Table showing the CMPP classes of a 2-form according to which components can be found to vanish. There are three possible classes, only one of which is considered special. The pentad is chosen so that the 2-form is as special as possible.*

In order to compare our formalism with CMPP, we can simply rewrite our little group field strength tensors in terms of $F_{\mu\nu}$. Doing this, we find the simple relationships

$$F_{0i} = \frac{1}{2\sqrt{2}}\,\phi_i^{(0)}, \quad F_{01} = \frac{1}{4}\,\Phi_{\text{tr}}^{(1)}, \quad F_{ij} = \frac{1}{2}\,\epsilon_{ijk}\phi_k^{(1)}, \quad F_{1i} = -\frac{1}{2\sqrt{2}}\,\phi_i^{(2)}. \tag{3.62}$$

Since each boost weight component is exactly identifiable as one of our little group irreps, the coarse classification that we introduced in section 3.3.1 is exactly the CMPP classification as introduced in [7]. Furthermore, the bracketed number (i) of a little group spinor $\Phi^{(i)}$ relates directly to its boost weight, as it would in four dimensions.

The de Smet classification

The de Smet classification [4] has a very different set up to the CMPP classification. It uses a gamma basis such as in equations (3.1) and (3.2) to create a symmetric field strength 2-spinor Φ_{AB}, and considers its factorisation properties to create a classification. There are two cases: in de Smet notation, if the 2-form does not factorise it is a **2**, and if it does it either a **11** or a $\underline{\mathbf{11}}$, with the two factors being equal in the latter case. Let us examine this in more detail. The symmetric 2-spinor is constructed using the rotation generator as usual,

$$\Phi_{AB} = F_{\mu\nu}\, \sigma^{\mu\nu}{}_{AB}. \tag{3.63}$$

Now, the field strength polynomial $\mathcal{F}$ is constructed by contracting in an arbitrary spinor ξ^A, such that

$$\mathcal{F} = \Phi_{AB}\, \xi^A\, \xi^B. \tag{3.64}$$

If the original bi-spinor had the structure $\Phi_{AB} = \alpha_{(A}\, \beta_{B)}$, the polynomial will factorise. Our formalism is based on irreducible representations of $SU(2)$, namely symmetric $SU(2)$ spinors. These have the useful property that they always totally factorise. Therefore, each little group irrep will have its own de Smet structure. We can compute this by studying each of them in turn.

The field strength spinor can be expanded in terms of our little group irreps as

$$\Phi_{AB} = \phi^{(0)}_{ab}\, \mathrm{n}_A{}^a\, \mathrm{n}_B{}^b + 2\, \phi^{(1)}_{ab}\, \mathrm{n}_{(A}{}^a\, \mathrm{k}_{B)}{}^b + \phi^{(1)}_{\mathrm{tr}}\, \mathrm{n}_{(A}{}^a\, \mathrm{k}_{B)a} + \phi^{(2)}_{ab}\, \mathrm{k}_A{}^a\, \mathrm{k}_B{}^b. \tag{3.65}$$

As an example, let us consider a case where only $\phi^{(2)}$ is non-zero, such as the plane wave example given in section 3.3.2. Now, the field strength polynomial is given by

$$\begin{aligned} \mathcal{F} &= \phi^{(2)}_{ab}\, \mathrm{k}_A{}^a\, \mathrm{k}_B{}^b\, \xi^A\, \xi^B \\ &= \alpha_{(a}\, \beta_{b)}\, (\mathrm{k} \circ \xi)^a\, (\mathrm{k} \circ \xi)^b \\ &= \left[\alpha, \mathrm{k} \circ \xi\right] \left[\beta, \mathrm{k} \circ \xi\right], \end{aligned} \tag{3.66}$$

where we have defined the factorisation of $\phi^{(2)}$ to be $\phi^{(2)}_{ab} = \alpha_{(a}\, \beta_{b)}$, and "$\circ$" indicates a contraction on a spacetime spinor index, while "[· , ·]" is a little group spinor contraction. Clearly, this is of de Smet type **11**.

The $\phi^{(0)}$ spinor has the same structure as $\phi^{(2)}$, and therefore a 2-form for which only $\phi^{(0)}$ was non-zero would also be a **11**. However, the k and n structure of the $\phi^{(1)}$ component means that its field strength polynomial behaves differently. Let us consider a 2-form where only $\phi^{(1)}$ is non-zero, for example the magnetic field from section 3.3.2. This would have a field strength polynomial of the form

$$\begin{aligned}\mathcal{F} &= 2\,\phi^{(1)}_{ab}\,\mathrm{n}_{(A}{}^{a}\,\mathrm{k}_{B)}{}^{b}\,\xi^{A}\,\xi^{B} \\ &= [\alpha, \mathrm{n}\circ\xi]\,[\beta, \mathrm{k}\circ\xi] + [\alpha, \mathrm{k}\circ\xi]\,[\beta, \mathrm{n}\circ\xi]\,,\end{aligned} \tag{3.67}$$

and thus it is of de Smet type **2**.

For a solution like the electric field in section 3.3.2, only the $\Phi^{(1)}_{\mathrm{tr}}$ term is non-zero. So the field strength polynomial is

$$\begin{aligned}\mathcal{F} &= \Phi^{(1)}_{\mathrm{tr}}\,\epsilon_{ab}\,\mathrm{n}_{(A}{}^{a}\,\mathrm{k}_{B)}{}^{b}\,\xi^{A}\,\xi^{B} \\ &= \Phi^{(1)}_{\mathrm{tr}}\left([o, \mathrm{n}\circ\xi]\,[\imath, \mathrm{k}\circ\xi] - [o, \mathrm{k}\circ\xi]\,[\imath, \mathrm{n}\circ\xi]\right),\end{aligned} \tag{3.68}$$

where we have used the property $\epsilon_{ab} = o_a\,\imath_b - \imath_a\,o_b$ for some basis spinors o and $\imath$, normalised as $o^a\,\imath_a = 1$. Therefore this is also a de Smet type **2**.

If we organise the little group irreps according to boost weight along the vertical direction and according to irrep dimension along the horizontal direction, we see that each irrep corresponds to a de Smet class, as shown in table 3.5. Any combination of little group irreps will result in a **2**.

Little group spinors				de Smet class	
$\phi^{(0)}_{ab}$				**11**	
$\phi^{(1)}_{ab}$	$\Phi^{(1)}_{\mathrm{tr}}$	$\leftrightarrow$		**2**	**2**
$\phi^{(2)}_{ab}$				**11**	

Table 3.5 *Each little group spinor has a predefined de Smet class.*

As we discussed in section 3.3.1, in the case of complex field strength, there is a Russian doll-like secondary layer of structure, where each $\phi^{(i)}$ can itself be either type I or type II corresponding to $\alpha \neq \beta$ or $\alpha = \beta$, respectively. It is simple to read off from equation (3.66) that these have distinct de Smet types **11** and <u>**11**</u> respectively, in the cases of $\phi^{(0)}$ or $\phi^{(2)}$, while we can see from equation (3.67) that $\phi^{(1)}$ will be **2** and **11** respectively. However, when we restrict to real spacetimes, only the possibilities shown in table 3.5 are possible, since the repeated case $\alpha = \beta$ is not permitted [5].

3.4 General relativity and the Weyl tensor

3.4.1 Spinors in curved space

So far, our analysis has been based on flat spacetime. To generalise our results to curved space, we introduce coordinate indices μ, ν and tangent space indices M, N. We can then pick an arbitrary frame $e^\mu{}_M$ satisfying $g^{\mu\nu} = e^\mu{}_M \, e^\nu{}_N \, \eta^{MN}$. Both $g^{\mu\nu}$ and η^{MN} can be expressed in terms of an NP pentad,

$$\begin{aligned} g^{\mu\nu} &= -\, k^\mu \, n^\nu - k^\nu \, n^\mu + \epsilon^{ac} \, \epsilon^{bd} \, \varepsilon^\mu{}_{ab} \, \varepsilon^\nu{}_{cd} \\ &= e^\mu{}_M \, e^\nu{}_N \left(-k^M \, n^N - k^N \, n^M + \epsilon^{ac} \, \epsilon^{bd} \, \varepsilon^M{}_{ab} \, \varepsilon^N{}_{cd} \right), \end{aligned} \tag{3.69}$$

so we can read off that the curved pentad $\{k^\mu,\, n^\mu,\, \varepsilon^\mu{}_{ab}\}$ is obtained from our flat pentad $\{k^M,\, n^M,\, \varepsilon^M{}_{ab}\}$ by contraction with $e^\mu{}_M$. Similarly, the gamma basis becomes

$$\gamma^\mu{}_{AB} = e^\mu{}_M \, \gamma^M{}_{AB}, \tag{3.70}$$

such that the Clifford algebra is still satisfied, exactly as for the Newman-Penrose construction in four dimensions. Notice that the index μ of previous sections should now be seen as the index M, and μ is henceforth a curved spacetime index.

The results we derived in section 3.2 still apply for the tangent space at each spacetime point. Thus it is possible to choose spinors of the form

$$\mathrm{k}^A{}_a = \begin{pmatrix} 0 & o_\alpha \\ \bar{o}^{\dot\alpha} & 0 \end{pmatrix}, \quad \mathrm{n}^A{}_a = \begin{pmatrix} \imath_\alpha & 0 \\ 0 & -\bar{\imath}^{\dot\alpha} \end{pmatrix}, \tag{3.71}$$

where o and $\imath$ are now curved space spinors of $SU(2) \times SU(2)$. Using the curved space gamma basis, we can construct the same relationships between the spinors and the pentad,

$$k^\mu = \frac{1}{2\sqrt{2}} \, \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{k}^a, \quad n^\mu = \frac{1}{2\sqrt{2}} \, \mathrm{n}_a \circ \gamma^\mu \circ \mathrm{n}^a, \quad \varepsilon^\mu{}_{ab} = \mathrm{k}_a \circ \gamma^\mu \circ \mathrm{n}_b, \tag{3.72}$$

using the properties of the four-dimensional spinors. Similarly, the contraction relation $\mathrm{k}_a \circ \mathrm{n}_b = \epsilon_{ab}$ is upheld, as are the spinor transformations. The reality conditions are also unaffected. We can therefore proceed and use these results for curved spacetime.

3.4.2 The little group spinors

In order to construct the Weyl spinor Ψ_{ABCD}, we simply contract the Weyl tensor $C_{\mu\nu\rho\sigma}$ with the curved space gamma basis to obtain

$$\Psi_{ABCD} = C_{\mu\nu\rho\sigma}\, \sigma^{\mu\nu}{}_{AB}\, \sigma^{\rho\sigma}{}_{CD} \tag{3.73}$$

as in section 3.3.1. The rotation generator $\sigma^{\mu\nu}{}_{AB}$ is constructed from the curved space γ's now but is otherwise defined as in equation (3.46). Given the symmetries of the Weyl tensor, it is easy to show that the Weyl spinor is totally symmetric, and thus comprises the 35 degrees of freedom in the five-dimensional Weyl tensor.

As in section 3.3.1, we would like to break up these 35 degrees of freedom according to their boost weight by contracting in our spinor basis. The little group objects $\Psi^{(i)}_{abcd}$ are defined by

$$\begin{aligned}
\Psi^{(0)}_{abcd} &= \Psi_{ABCD}\, \mathrm{k}^A{}_a\, \mathrm{k}^B{}_b\, \mathrm{k}^C{}_c\, \mathrm{k}^D{}_d \\
\Psi^{(1)}_{abcd} &= \Psi_{ABCD}\, \mathrm{k}^A{}_a\, \mathrm{k}^B{}_b\, \mathrm{k}^C{}_c\, \mathrm{n}^D{}_d \\
\Psi^{(2)}_{abcd} &= \Psi_{ABCD}\, \mathrm{k}^A{}_a\, \mathrm{k}^B{}_b\, \mathrm{n}^C{}_c\, \mathrm{n}^D{}_d \\
\Psi^{(3)}_{abcd} &= \Psi_{ABCD}\, \mathrm{k}^A{}_a\, \mathrm{n}^B{}_b\, \mathrm{n}^C{}_c\, \mathrm{n}^D{}_d \\
\Psi^{(4)}_{abcd} &= \Psi_{ABCD}\, \mathrm{n}^A{}_a\, \mathrm{n}^B{}_b\, \mathrm{n}^C{}_c\, \mathrm{n}^D{}_d,
\end{aligned} \tag{3.74}$$

where the bracketed superscript number (i) indicates the number of $\mathrm{n}^A{}_a$ spinors in the contraction. These definitions are analogous to the field strength objects $\Phi^{(i)}_{ab}$ in equation (3.48) and to the four-dimensional definitions (2.16). Ψ_{ABCD} can equivalently be expressed as the sum of the little group objects:

$$\begin{aligned}
\Psi_{ABCD} = \Psi^{(0)}_{abcd}\, \mathrm{n}_A{}^a\, \mathrm{n}_B{}^b\, \mathrm{n}_C{}^c\, \mathrm{n}_D{}^d &+ 4\, \Psi^{(1)}_{abcd}\, \mathrm{n}_{(A}{}^a\, \mathrm{n}_B{}^b\, \mathrm{n}_C{}^c\, \mathrm{k}_{D)}{}^d \\
+ 6\, \Psi^{(2)}_{abcd}\, \mathrm{n}_{(A}{}^a\, \mathrm{n}_B{}^b\, \mathrm{k}_C{}^c\, \mathrm{k}_{D)}{}^d& \\
+ 4\, \Psi^{(3)}_{abcd}\, \mathrm{n}_{(A}{}^a\, \mathrm{k}_B{}^b\, \mathrm{k}_C{}^c\, \mathrm{k}_{D)}{}^d &+ \Psi^{(4)}_{abcd}\, \mathrm{k}_A{}^a\, \mathrm{k}_B{}^b\, \mathrm{k}_C{}^c\, \mathrm{k}_D{}^d.
\end{aligned} \tag{3.75}$$

We observe from the definitions of the little group objects $\Psi^{(i)}$ that they possess different symmetries. The totally symmetric ones, $\Psi^{(0)}_{abcd}$ and $\Psi^{(4)}_{abcd}$, have 5 degrees of freedom, while $\Psi^{(1)}_{abcd} = \Psi^{(1)}_{(abc)d}$ and $\Psi^{(3)}_{abcd} = \Psi^{(3)}_{a(bcd)}$ each contain 8. $\Psi^{(2)}_{abcd} = \Psi^{(2)}_{(ab)(cd)}$ comprises the final 9 degrees of freedom to reach 35. It is sensible to break these 4-spinors into irreducible representations of $SU(2)$. We will use the notation that a lower case $\psi^{(i)}$ indicates a totally symmetric object, i.e., $\psi^{(i)}_{abcd} =$

$\psi^{(i)}_{(abcd)}$ for any value of i, and we also introduce $\chi^{(i)}$ to indicate a symmetric bi-spinor. Clearly $\Psi^{(0)}$ and $\Psi^{(4)}$ are already irreducible, since they sit in the totally symmetric representation **5**, so $\Psi^{(0)} = \psi^{(0)}$ and $\Psi^{(4)} = \psi^{(4)}$. $\Psi^{(1)}$ and $\Psi^{(3)}$ contain a bi-spinor trace that can be removed to decompose them as $\mathbf{8} = \mathbf{5} + \mathbf{3}$:

$$\begin{aligned} \Psi^{(1)}_{abcd} &= \psi^{(1)}_{abcd} - \frac{1}{4}\left(\epsilon_{ad}\chi^{(1)}_{bc} + \epsilon_{bd}\chi^{(1)}_{ac} + \epsilon_{cd}\chi^{(1)}_{ab}\right) \\ \Psi^{(3)}_{abcd} &= \psi^{(3)}_{abcd} - \frac{1}{4}\left(\epsilon_{ab}\chi^{(3)}_{cd} + \epsilon_{ac}\chi^{(3)}_{bd} + \epsilon_{ad}\chi^{(3)}_{bc}\right), \end{aligned} \tag{3.76}$$

while $\Psi^{(2)}$ splits into a symmetric rank 4 spinor, a symmetric rank 2 spinor and a scalar: $\mathbf{9} = \mathbf{5} + \mathbf{3} + \mathbf{1}$ as

$$\Psi^{(2)}_{abcd} = \psi^{(2)}_{abcd} - \frac{1}{4}\left(\epsilon_{ac}\chi^{(2)}_{bd} + \epsilon_{ad}\chi^{(2)}_{bc} + \epsilon_{bc}\chi^{(2)}_{ad} + \epsilon_{bd}\chi^{(2)}_{ac}\right) + \frac{1}{6}\left(\epsilon_{ac}\epsilon_{bd} + \epsilon_{ad}\epsilon_{bd}\right)\Psi^{(2)}_{\mathrm{tr}}. \tag{3.77}$$

This is summarised in table 3.6.

Reducible little group spinor		**5**	**3**	**1**	Total dof
$\Psi^{(0)}_{abcd} = \Psi^{(0)}_{(abcd)}$		$\psi^{(0)}_{abcd}$			5
$\Psi^{(1)}_{abcd} = \Psi^{(1)}_{(abc)d}$		$\psi^{(1)}_{abcd}$	$\chi^{(1)}_{ab}$		8
$\Psi^{(2)}_{abcd} = \Psi^{(2)}_{(ab)(cd)}$	$\Rightarrow$	$\psi^{(2)}_{abcd}$	$\chi^{(2)}_{ab}$	$\Psi^{(2)}_{\mathrm{tr}}$	9
$\Psi^{(3)}_{abcd} = \Psi^{(3)}_{a(bcd)}$		$\psi^{(3)}_{abcd}$	$\chi^{(3)}_{ab}$		8
$\Psi^{(4)}_{abcd} = \Psi^{(4)}_{(abcd)}$		$\psi^{(4)}_{abcd}$			5

Table 3.6 *The table shows how each little group 4-spinor is decomposed into irreducible representations.* ***5*** *is a totally symmetric 4-spinor,* ***3*** *is a symmetric bi-spinor, and* ***1*** *is a scalar. We write "dof" as a shorthand for degrees of freedom.*

We will also use vectorial language for the little group irreps, translating between the two using the Pauli matrices ${\varsigma^i}_{ab}$ as usual such that, for example,

$$\psi^{(0)}_{ij} = {\varsigma_i}^{ab}\,{\varsigma_j}^{cd}\,\psi^{(0)}_{abcd}. \tag{3.78}$$

Table 3.7 summarises the notation. This is a simple matter of representation, and makes it easier to compare our results with the vectorial techniques used in the literature. In this notation, imposing the reality conditions is equivalent to the requirement that the objects are real.

4-spinor	2-spinor	scalar		3-matrix	3-vector	scalar
$\psi^{(0)}_{abcd}$				$\psi^{(0)}_{ij}$		
$\psi^{(1)}_{abcd}$	$\chi^{(1)}_{ab}$			$\psi^{(1)}_{ij}$	$\underline{\chi}^{(1)}$	
$\psi^{(2)}_{abcd}$	$\chi^{(2)}_{ab}$	$\Psi^{(2)}_{\text{tr}}$	$\leftrightarrow$	$\psi^{(2)}_{ij}$	$\underline{\chi}^{(2)}$	$\Psi^{(2)}_{\text{tr}}$
$\psi^{(3)}_{abcd}$	$\chi^{(3)}_{ab}$			$\psi^{(3)}_{ij}$	$\underline{\chi}^{(3)}$	
$\psi^{(4)}_{abcd}$				$\psi^{(4)}_{ij}$		

Table 3.7 *The irreducible representations of the Weyl spinor can be easily moved between spinor space on the left and vector space on the right by use of the Pauli matrices $\varsigma^i{}_{ab}$. We will use the two notations interchangeably. Note that all spinors are totally symmetric, and that all 3-matrices are symmetric and tracefree.*

Coarse and finely grained classifications

This construction naturally highlights two levels of classification, one coarse-grained which depends only on the little group spinors, and one which is more finely grained which also depends on the irreducible representation. The coarse classification arises due to the similarities in construction between the little group spinors

$$\Psi^{(i)}_{abcd}, \qquad i = 1, ..., 4, \tag{3.79}$$

defined in equation (3.74), and the complex scalars from four dimensions

$$\psi_i \qquad i = 1, ..., 4, \tag{3.80}$$

defined in equation (2.16). Thus the $\Psi^{(i)}$ will obey a classification which is analogous to the four-dimensional Petrov one shown in table 2.2 [3]. This coarse classification is proposed in table 3.8 and as we will show in section 3.4.4, it turns out to be equivalent to the CMPP classification [7, 32].

The fine grained classification notes that the coarse types in table 3.8 referred only to the rows of table 3.6. The columns spreading out into different irreducible representations of the little group shows that a greater level of detail is possible. For example, imagine two type D solutions: then a pentad can be found for each where only $\Psi^{(2)}$ is non-zero. Suppose further that when the fine structure is

[3]There is one caveat, which is that in four dimensions it is always possible to find a tetrad where ψ_0 vanishes. This is not the case in general so we require the additional type G to account for such spacetimes; see [7].

Type	Little group spinor characteristic
Type G	$\Psi^{(0)} \neq 0$
Type I	$\Psi^{(0)} = 0$
Type II	$\Psi^{(0)} = \Psi^{(1)} = 0$
Type D	$\Psi^{(0)} = \Psi^{(1)} = \Psi^{(3)} = \Psi^{(4)} = 0$
Type III	$\Psi^{(0)} = \Psi^{(1)} = \Psi^{(2)} = 0$
Type N	$\Psi^{(0)} = \Psi^{(1)} = \Psi^{(2)} = \Psi^{(3)} = 0$

Table 3.8 *Table showing the coarse grained, Petrov-like classification of a five-dimensional Weyl tensor built in analogy with the four-dimensional Petrov formalism. The classification refers to the vanishing of the reducible little group spinors $\Psi^{(i)}$, which is equivalent to the vanishing of a whole row in table 3.6.*

analysed, it is seen that $\chi^{(2)}$ and $\psi^{(2)}$ vanish for the first spacetime but only $\chi^{(2)}$ vanishes for the second, indicating that the first example is more special. This is exactly the case for the Tangherlini-Schwarzschild black hole and the black string respectively - the details of this example are given in the following section.

We can delve deeper into the irreps themselves to ask whether they also have sub-classifications. First we will consider a complex spacetime. In this case, the structure of the irreducible representations $\psi^{(i)}$ and $\chi^{(i)}$, namely complex symmetric spinors with two-dimensional indices, is exactly that of the four-dimensional Weyl and field strength spinors respectively. Like a Russian doll, hiding inside the Weyl tensor are additional lower-dimensional Weyl tensors. These also have a classification, which can be found in the usual way for four dimensions. For example, a 4-spinor $\psi_{abcd} = \alpha_{(a}\beta_b\gamma_c\delta_{d)}$ could have any of four different specialisations:

- Type II: Two repeated spinors with the other two spinors distinct $\psi_{abcd} = \alpha_{(a}\alpha_b\gamma_c\delta_{d)}$
- Type D: Two pairs of repeated spinors $\psi_{abcd} = \alpha_{(a}\alpha_b\gamma_c\gamma_{d)}$
- Type III: Three repeated spinors $\psi_{abcd} = \alpha_{(a}\alpha_b\alpha_c\delta_{d)}$
- Type N: Four repeated spinors $\psi_{abcd} = \alpha_a\alpha_b\alpha_c\alpha_d$,

whereas for a 2-spinor $\chi_{ab} = \alpha_{(a}\beta_{b)}$ there is only one specialisation

- Type II: Two repeated spinors $\chi_{ab} = \alpha_a \alpha_b$.

In contrast, when we restrict to a real spacetime we find that much of this second layer of hidden lower-dimensional Weyl tensor classification is forbidden. We already know from our analysis of the field strength tensor in section 3.3.1 that a bi-spinor $\chi^{(i)}$ which obeys the reality conditions $\chi = \overline{\chi}$ cannot be written as the outer product of a single spinor, $\chi_{ab} \neq \alpha_a \alpha_b$. A similar analysis can be applied to real symmetric 4-spinor objects ψ_{abcd} which satisfy $\psi = \overline{\psi}$. This will restrict the number of subclasses available, as we will now show.

It is well known from four dimensions (see for example [74]) that if we define $I = \psi^{abcd}\,\psi_{abcd}$ and $J = \psi_{ab}{}^{cd}\,\psi_{cd}{}^{ef}\,\psi_{ef}{}^{ab}$, then the requirements for each class are:

- Type II: $I^3 = 6J^2$
- Type D: $\psi_{pqr(a}\,\psi_{bc}{}^{pq}\,\psi^{r}{}_{def)} = 0$
- Type III: $I = J = 0$
- Type N: $\psi_{(ab}{}^{ef}\,\psi_{cd)ef} = 0$.

Since our ψ's obey the reality condition, they can be rewritten as symmetric tracefree matrices with real entries. In contrast, if we had chosen to consider complex space, or a different signature, the entries would be complex. A real symmetric matrix may always be diagonalised to obtain

$$D = \begin{pmatrix} \lambda_1 & 0 & 0 \\ 0 & -(\lambda_1 + \lambda_2) & 0 \\ 0 & 0 & \lambda_2 \end{pmatrix}, \tag{3.81}$$

and so we can rewrite the conditions in terms of the eigenvalues as

- Type II: $2\lambda_1^3 + 3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 - 2\lambda_2^3 = 0$
- Type D: $2\lambda_1^3 + 3\lambda_1^2\lambda_2 - 3\lambda_1\lambda_2^2 - 2\lambda_2^3 = 0$
- Type III: $\lambda_1^2 + \lambda_1\lambda_2 + \lambda_2^2 = 0$ and $\lambda_1\lambda_2(\lambda_1 + \lambda_2) = 0$
- Type N: $\lambda_1^2 = \lambda_2^2$ and $\lambda_1^2 + 4\lambda_1\lambda_2 + \lambda_2^2 = 0$.

The type II condition has reduced to the more specialised type D condition and is solved only when two of the eigenvalues are equal (or trivially when all the eigenvalues vanish). In contrast, there are no non-trivial solutions for type N and type III, that is, we must have $\lambda_1 = \lambda_2 = 0$. This tells us that under our reality conditions, only type D-like lower-dimensional Weyl tensors are possible.[4] We note that interesting behaviour relating to dimensional reduction also occurs when a single eigenvalue vanishes, which is not reflected by this classification. We hope to explore this property further in future work.

To summarise, we have found three layers of structure naturally embedded in our formalism. The first is a Petrov-like coarse layer in the little group spinors. The second is more fine-grained, breaking the little group spinors into irreducible representations. Finally, the third looks at the irreps themselves and uses their similarity to four-dimensional objects to classify them in a Petrov-like way. This has two possibilities depending on whether or not reality conditions have been imposed as summarised in table 3.9.

Complex	ψ:	I, II, D, III, N
	χ:	I, II
Real	ψ:	I, D
	χ:	I

Table 3.9 *The classification of the lower-dimensional objects hidden within the Weyl tensor depends on whether or not reality conditions have been imposed.*

3.4.3 Examples

To illustrate a few key features of the formalism, we shall give a few very simple examples: the plane wave, a Tangherlini-Schwarzschild black hole and a black string.

A pp-wave

The metric for a pp-wave can be expressed in Brinkmann coordinates

$$ds^2 = -H(u, x, y, z)du^2 - 2du\,dv + dx^2 + dy^2 + dz^2, \tag{3.82}$$

[4] We note that this argument is invalidated when complex entries occur because in general complex symmetric matrices cannot be diagonalised.

such that if we choose the pentad

$$k = \partial_v, \quad n = \partial_u - \frac{1}{2} H(u,x,y,z)\partial_v, \quad \varepsilon_{ab} = \begin{pmatrix} \partial_x + i\partial_y & i\partial_z \\ i\partial_z & \partial_x - i\partial_y \end{pmatrix}, \tag{3.83}$$

then the Weyl tensor is given by

$$C_{\mu\nu\rho\sigma} = 2\,\partial_i\partial_j H(u,x,y,z)\, n_{[\mu}\,\varepsilon^i_{\nu]}\, n_{[\rho}\,\varepsilon^j_{\sigma]}, \tag{3.84}$$

where the index $i = 1,2,3$ runs over the three polarisation directions $\{x, y, z\}$ as usual as in the definition (3.33). Recasting this as a spinor using the curved space gamma basis we find

$$\begin{aligned} \Psi_{ABCD} &= C_{\mu\nu\rho\sigma}\sigma^{\mu\nu}{}_{AB}\sigma^{\rho\sigma}{}_{CD} \\ &= 4\,\partial_i\partial_j H(u,x,y,z)\,\varsigma^i{}_{ab}\,\varsigma^j{}_{cd}\,\mathrm{k}_A{}^a\,\mathrm{k}_B{}^b\,\mathrm{k}_C{}^c\,\mathrm{k}_D{}^d. \end{aligned} \tag{3.85}$$

Therefore the pp-wave is a type N solution with $\psi^{(4)}_{ij} = 4\,\partial_i\partial_j H(u,x,y,z)$. If we were to specify the function $H(u,x,y,z)$ we could classify $\psi^{(4)}_{abcd}$ further since it has all of the properties of a four dimensional Weyl tensor.

The Tangherlini-Schwarzschild black hole

Another simple example is a five-dimensional Schwarzschild black hole, with metric

$$ds^2 = -\Delta(r)du^2 - 2\,du\,dr + r^2\left(d\theta^2 + \sin^2\theta\left(d\phi^2 + \sin^2\,\theta d\chi^2\right)\right), \tag{3.86}$$

where $\Delta(r) = 1 - \frac{r_s^2}{r^2}$. We choose the pentad

$$k = -\partial_u + \frac{1}{2}\,\Delta(r)\,\partial_r, \quad n = \partial_r, \quad \varepsilon_{ab} = \frac{1}{r}\begin{pmatrix} \partial_\theta + i\csc\theta\,\partial_\phi & i\csc\theta\csc\phi\,\partial_\chi \\ i\csc\theta\csc\phi\,\partial_\chi & \partial_\theta - i\csc\theta\,\partial_\phi \end{pmatrix}, \tag{3.87}$$

such that the Weyl tensor is

$$C_{\mu\nu\rho\sigma} = \frac{2r_s^2}{r^4}\left(2k_{[\mu}\,\varepsilon^i_{\nu]}\,n_{[\rho}\,\varepsilon^i_{\sigma]} + 2n_{[\mu}\,\varepsilon^i_{\nu]}\,k_{[\rho}\,\varepsilon^i_{\sigma]} - 6k_{[\mu}\,n_{\nu]}\,k_{[\rho}\,n_{\sigma]} - \varepsilon^i_{[\mu}\,\varepsilon^j_{\nu]}\varepsilon^{[i}_{[\rho}\,\varepsilon^{j]}_{\sigma]}\right). \tag{3.88}$$

The Weyl spinor is

$$\begin{aligned}\Psi_{ABCD} &= C_{\mu\nu\rho\sigma}\sigma^{\mu\nu}{}_{AB}\sigma^{\rho\sigma}{}_{CD} \\ &= -\frac{48r_s^2}{r^4}\left(\epsilon_{ac}\epsilon_{bd}+\epsilon_{ad}\epsilon_{bc}\right)\mathrm{k}_{(A}{}^{a}\,\mathrm{k}_{B}{}^{b}\,\mathrm{n}_{C}{}^{c}\,\mathrm{n}_{D)}{}^{d},\end{aligned} \tag{3.89}$$

and so we can read off that the only non-zero little group irrep for the Tangherlini-Schwarzschild black hole is the scalar $\Psi^{(2)}_{\mathrm{tr}} = -\frac{48r_s^2}{r^4}$. Therefore, it is a very special type D solution, since it only has a single non-zero irrep.

The black string

It is interesting to contrast this with another type D solution, the black string. This is a four-dimensional Schwarzschild black hole trivially extended along the $x^4 = z$ direction with the metric

$$ds^2 = -\Gamma(r)du^2 - 2\,du\,dr + r^2\left(d\theta^2+\sin^2\theta d\phi^2\right) + dz^2 \tag{3.90}$$

where $\Gamma(r) = 1-\frac{r_s}{r}$. We choose a pentad which is similar to the previous example:

$$k=\partial_r, \quad n=\partial_u-\frac{1}{2}\Gamma(r)\,\partial_r, \quad \varepsilon_{ab}=\frac{1}{r}\begin{pmatrix}\partial_\theta+i\csc\theta\,\partial_\phi & i\partial_z \\ i\partial_z & \partial_\theta-i\csc\theta\,\partial_\phi\end{pmatrix}, \tag{3.91}$$

to find that the Weyl tensor is

$$\begin{aligned}C_{\mu\nu\rho\sigma} = 2\frac{r_s}{r^3}\Big(2\,\delta^{ij}_{\mathrm{red}}\left(k_{[\mu}\,\varepsilon^i_{\nu]}\,n_{[\rho}\,\varepsilon^j_{\sigma]}+n_{[\mu}\,\varepsilon^i_{\nu]}\,k_{[\rho}\,\varepsilon^j_{\sigma]}\right) &- 2k_{[\mu}\,n_{\nu]}\,k_{[\rho}\,n_{\sigma]} \\ &+\delta^{ik}_{\mathrm{red}}\,\delta^{jl}_{\mathrm{red}}\,\varepsilon^i_{[\mu}\,\varepsilon^j_{\nu]}\varepsilon^{[k}_{[\rho}\,\varepsilon^{l]}_{\sigma]}\Big),\end{aligned} \tag{3.92}$$

where the reduced identity matrix δ_{red} is trivial in the z direction, $\delta^{ij}_{\mathrm{red}} = \delta^{ij} - e^i_z e^j_z$. Note the similarity to equation (3.88) if δ_{red} is replaced by δ. As usual, we recast as a spinor to find

$$\Psi_{ABCD} = -\frac{96r_s}{r^3}\,\delta^{ij}_{\mathrm{red}}\,\varsigma^i{}_{ab}\,\varsigma^j{}_{cd}\,\mathrm{k}_{(A}{}^{a}\,\mathrm{k}_{B}{}^{b}\,\mathrm{n}_{C}{}^{c}\,\mathrm{n}_{D)}{}^{d}. \tag{3.93}$$

This time there is more than one little group irrep present. The reducible little group spinor $\Psi^{(2)}$ is given by

$$\Psi^{(2)ij} = -\frac{4r_s}{r^3}\delta^{ij}_{\mathrm{red}}, \tag{3.94}$$

which decomposes into a trace term and a traceless symmetric **5**:

$$\psi^{(2)ij} = -\frac{4r_s}{r^3}\left(\frac{1}{3}\delta^{ij} - e_z^i e_z^j\right), \qquad \Psi^{(2)}_{\rm tr} = -\frac{16r_s}{r^3}. \tag{3.95}$$

Therefore the black string is still a type D solution but it has a very different fine structure to the Tangherlini-Schwarzschild black hole.

Finally, we can consider the structure of $\psi^{(2)}$ itself: since it has two equal eigenvalues ($\lambda_x = \lambda_y = -\frac{4r_s}{3r^3}$), the irrep is itself type D.

3.4.4 Relations to the literature: CMPP and de Smet

As we have previously mentioned, there exist previously proposed classifications for five dimensions, notably the CMPP and de Smet classifications [7, 32], [4], which were shown in [5] to disagree on their definition of specialness, since some spacetimes are algebraically special in CMPP but not in de Smet, and vice versa.

The CMPP classification

In the CMPP classification, each component of the Weyl tensor in D dimensions has a boost weight when the pentad is rescaled by $\{k,\, n,\, m^{(i)}\} \to \{\rho\, k,\, \rho^{-1}\, n,\, m^{(i)}\}$ for some scalar ρ, where $i = 2, ..., D-2$, and $m^{(i)}$ is any of the remaining space-like directions. This boost weight is the power of ρ by which the component of the Weyl tensor transforms. The independent components of the Weyl tensor have the following boost weights:

Boost weight	2	1	0	−1	−2
Component	C_{0i0j}	C_{010i}, C_{0ijk}	$C_{0101}, C_{01ij}, C_{0i1j}, C_{ijkl}$	C_{011i}, C_{1ijk}	C_{1i1j}

(3.96)

where the index 0 indicates a contraction with k, the index 1 indicates a contraction with n, and a Roman index i corresponds to the space-like direction $m^{(i)}$. Our usage of k and n is identical, while the CMPP polarisation directions $m^{(i)}$ can be chosen to correspond to our $\varepsilon^\mu{}_i$ as

$$m^{\mu(i)} = \varsigma^{i\;ab}\,\varepsilon^\mu{}_{ab}. \tag{3.97}$$

The Weyl tensor components, combined by boost weight, are then organised into a classification which is shown in table 3.10. This is valid in any dimension, and of course reduces to the Petrov classification in four dimensions.

Type	Characteristic
Type G	$C_{0i0j} \neq 0$
Type I	$C_{0i0j} = 0$
Type II	$C_{0i0j} = C_{010i} = C_{0ijk} = 0$
Type D	$C_{0i0j} = C_{010i} = C_{0ijk} = C_{011i} = C_{1ijk} = C_{1i1j} = 0$
Type III	$C_{0i0j} = C_{010i} = C_{0ijk} = C_{0101} = C_{01ij} = C_{0i1j} = C_{ijkl} = 0$
Type N	$C_{0i0j} = C_{010i} = C_{0ijk} = C_{0101} = C_{01ij} = C_{0i1j} = C_{ijkl}$ $= C_{011i} = C_{1ijk} = 0$

Table 3.10 *The CMPP classification considers the vanishing of the components of the Weyl tensor in some pentad in order to specify a type. The more special the classification, the more components, grouped by boost weight, must vanish.*

The boost transformation is clearly identical to the boost that we have previously defined through spinor space as $\mathrm{k}_A{}^a \to c\,\mathrm{k}_A{}^a$, $\mathrm{n}_A{}^a \to \frac{1}{c}\,\mathrm{n}_A{}^a$. As shown in equation (3.28), the effect on the pentad is identical when we identify $\rho = c^2$. We therefore expect to see a correlation between the components of the Weyl tensor and the little group 4-spinors. This turns out to be exactly the case. We can easily use the equations (3.73), (3.14) and (3.15), which express the Weyl tensor, k, n and $\varepsilon^\mu{}_{ab}$ in terms of spinors, to show that the CMPP components correspond directly to little group irreps:

$$
\begin{aligned}
&C_{0i0j} = \tfrac{1}{8}\psi^{(0)}_{ij} \qquad C_{010i} = -\tfrac{1}{8\sqrt{2}}\chi^{(1)}_i \qquad C_{0ijk} = \tfrac{1}{8\sqrt{2}}\left(2\,\epsilon_{ijl}\,\psi^{(1)}_{lk} - \chi^{(1)}_{[i}\delta_{j]k}\right)\\
&C_{0101} = \tfrac{1}{16}\Psi^{(2)}_{\mathrm{tr}} \qquad C_{01ij} = -\tfrac{1}{8}\,\epsilon_{ijk}\chi^{(2)}_k \qquad C_{0i1j} = -\tfrac{1}{8}\left(\psi^{(2)}_{ij} + \tfrac{1}{2}\epsilon_{ijk}\chi^{(2)}_k + \tfrac{1}{6}\Psi^{(2)}_{\mathrm{tr}}\delta_{ij}\right)\\
&C_{1i1j} = \tfrac{1}{8}\psi^{(4)}_{ij} \qquad C_{011i} = \tfrac{1}{8\sqrt{2}}\chi^{(3)}_i \qquad C_{1ijk} = -\tfrac{1}{8\sqrt{2}}\left(2\epsilon_{ijl}\psi^{(3)}_{lk} + \chi^{(3)}_{[i}\delta_{j]k}\right)\\
&C_{ijkl} = \tfrac{1}{2}\left(\delta_{i[l}\,\psi^{(2)}_{k]j} - \delta_{j[l}\,\psi^{(2)}_{k]i} + \tfrac{1}{12}\Psi^{(2)}_{\mathrm{tr}}\,\delta_{i[l}\,\delta_{k]j}\right).
\end{aligned}
\tag{3.98}
$$

Using this correspondence, it is clear that the classifications shown in tables 3.10 and 3.8 are identical. Thus, the coarse classification inspired by the similarities of our construction with the four-dimensional Petrov classification is exactly the original CMPP classification.

Little group irreps

The irreducible representations $\psi^{(i)}$, $\chi^{(i)}$ and $\Psi_{\rm tr}^{(2)}$ also make an appearance in the literature. It was noted in [37] that there are subgroups of the Weyl components for a given boost weight by noting their grouping under Lorentz transformations. For example, Coley and Hervik define two subclasses of type I by

- Type I(A) $\Leftrightarrow C^{i}{}_{ji0} = 0$
- Type I(B) $\Leftrightarrow C_{ijk0}\, C^{ijk}{}_{0} = \frac{1}{2} C^{ji}{}_{j0}\, C^{k}{}_{ik0}$

in the Weyl-aligned basis for an arbitrary number of dimensions. As before, we can cast this into little group space in five dimensions to find that this corresponds to

- Type I(A) $\Leftrightarrow \chi^{(1)}_{ab} = 0$
- Type I(B) $\Leftrightarrow \psi^{(1)}_{abcd} = 0.$

The other little group irreps are identified in a similar way. In [39], now joined by Ortaggio and Wylleman, Coley and Hervik apply their results to five dimensions and find that the Weyl tensor can be written in terms of 5 symmetric trace-free matrices, three vectors and a scalar, which produce exactly the fine structure that we presented based on spinor-helicity considerations. Thus, the spinorial techniques we have developed are precisely the spinor underpinnings of the refined CMPP classification.

The de Smet classification

As we previously mentioned, another notable higher-dimensional classification is that of de Smet [4]. In this work, de Smet constructs the $SO(4,1)$ 4-spinor Ψ_{ABCD} exactly as we have done, and then constructs a classification based on the factorisation properties of the Weyl polynomial $\mathcal{W}$, defined by

$$\mathcal{W} \equiv \Psi_{ABCD}\, \xi^{A}\, \xi^{B}\, \xi^{C}\, \xi^{D}, \tag{3.99}$$

for an arbitrary ξ^A. Originally containing 12 classes, further work by Godazgar [5] found that consideration of the reality conditions brought the total number of classes down to 8. It was proposed that these can be arranged in order of "specialness" as shown in figure 3.1. We only consider real spacetimes in this section. The de Smet labels work as follows. The numbers indicate the rank of each factorised part of the Weyl polynomial and groups of underlined numbers signify that these are repeated factors. Thus, a **211** indicates a Weyl polynomial with one factor quadratic in ξ and two factors linear in ξ. If the spacetime is a **$\underline{\mathbf{22}}$**, then there are two identical quadratic factors.

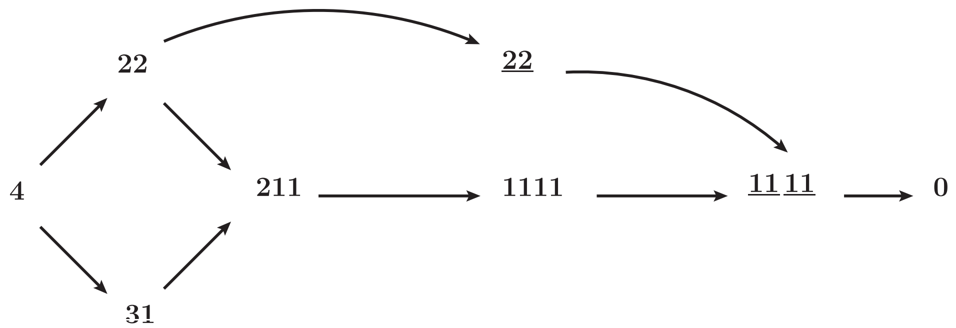


Figure 3.1 *The real de Smet classification proposed by [4] and restricted with reality conditions by [5] contains 8 classes including the flat spacetime class **0**, for which the Weyl tensor vanishes.*

We can interpret the de Smet construction in terms of our formalism by expanding equation (3.75) in terms of its little group irreps. Because our formalism splits the spacetime into totally symmetric little group irreps, the factorisation properties can be easily investigated. To take a simple example, let us consider a spacetime for which only $\psi^{(2)}_{\text{tr}}$ is non-zero (such as the Tangherlini-Schwarzschild solution), so that

$$\begin{aligned} \mathcal{W} &= \psi^{(2)}_{\text{tr}}(\epsilon_{ac}\,\epsilon_{bd} + \epsilon_{ad}\,\epsilon_{bc})\,(\text{n}\circ\xi)^a\,(\text{n}\circ\xi)^b\,(\text{k}\circ\xi)^c\,(\text{k}\circ\xi)^d \\ &= 2\,\psi^{(2)}_{\text{tr}}\,[(\text{n}\circ\xi),(\text{k}\circ\xi)]^2. \end{aligned} \tag{3.100}$$

We have used $[\cdot,\cdot]$ to indicate a contraction on little group spinor indices, distinguishing it from the centre dot " $\circ$ " used to indicate contraction on spacetime spinor indices. Clearly, this factorises beautifully into a de Smet **$\underline{\mathbf{22}}$**, which means that the Weyl polynomial factorises into two identical bi-spinors.

Next, consider a type III solution for which only $\chi^{(3)}$ is non-zero. The Weyl polynomial is

$$\begin{aligned}\mathcal{W} =& \; -\left(\epsilon_{ab}\,\chi^{(3)}_{cd} + \epsilon_{ac}\,\chi^{(3)}_{bd} + \epsilon_{ad}\,\chi^{(3)}_{bc}\right)\,(\mathrm{n}\circ\xi)^a\,(\mathrm{k}\circ\xi)^b\,(\mathrm{k}\circ\xi)^c\,(\mathrm{k}\circ\xi)^d \\ =& \; -3\,[\mathrm{n}\circ\xi,\mathrm{k}\circ\xi]\,\left[\mathrm{k}\circ\xi,\theta^{(3)}\right]\,\left[\mathrm{k}\circ\xi,\kappa^{(3)}\right],\end{aligned} \tag{3.101}$$

where, in the last line, we have used the property that symmetric $SU(2)$ bi-spinors can always be written as the symmetrisation of two spinors to define $\chi^{(3)}_{ab} \equiv \theta^{(3)}_{(a}\,\kappa^{(3)}_{b)}$. This has de Smet type **211**. Using the k $\leftrightarrow$ n symmetry, we can see that $\chi^{(1)}$ must also be a **211**:

$$\mathcal{W} = -3\,[\mathrm{n}\circ\xi,\mathrm{k}\circ\xi]\,\left[\mathrm{n}\circ\xi,\theta^{(1)}\right]\,\left[\mathrm{n}\circ\xi,\kappa^{(1)}\right], \tag{3.102}$$

where again we have defined $\chi^{(1)}_{ab} \equiv \theta^{(1)}_{(a}\,\kappa^{(1)}_{b)}$. By contrast, when $\chi^{(2)}$ gives the sole contribution to Ψ_{ABCD}, the Weyl polynomial has de Smet class **22**:

$$\mathcal{W} = -3\,[\mathrm{n}\circ\xi,k\circ\xi]\,\left\{\left[\mathrm{n}\circ\xi,\theta^{(2)}\right]\left[\mathrm{k}\circ\xi,\kappa^{(2)}\right] + \left[\mathrm{n}\circ\xi,\kappa^{(2)}\right]\left[\mathrm{k}\circ\xi,\theta^{(2)}\right]\right\}. \tag{3.103}$$

The $\psi^{(i)}$'s also have characteristic de Smet types. For example, if only $\psi^{(4)}$ is non-zero as for a type N spacetime, then the Weyl spinor is oriented in the k direction as

$$\Psi_{ABCD} = \psi^{(4)}_{abcd}\,\mathrm{k}_{(A}{}^a\,\mathrm{k}_B{}^b\,\mathrm{k}_C{}^c\,\mathrm{k}_{D)}{}^d. \tag{3.104}$$

The explicit symmetrisation on the little group indices is not required, and thus the Weyl polynomial factorises totally to form a de Smet **1111**:

$$\mathcal{W} = \left[\mathrm{k}\circ\xi,\alpha^{(4)}\right]\left[\mathrm{k}\circ\xi,\beta^{(4)}\right]\left[\mathrm{k}\circ\xi,\gamma^{(4)}\right]\left[\mathrm{k}\circ\xi,\delta^{(4)}\right]. \tag{3.105}$$

Using the invariance of de Smet classes under the interchange $n \leftrightarrow k$, we can see that $\psi^{(0)}$ is also of this type. However, the remaining $\psi^{(i)}$ do require proper symmetrisation over the little group indices, leading to sums over the different permutations which do not factorise at all and are de Smet **4**'s. For example, the Weyl polynomial for $\psi^{(1)}$ is:

$$\mathcal{W} = \sum_{\text{Perms }\{\alpha,\beta,\gamma,\delta\}}\left[\mathrm{k}\circ\xi,\alpha^{(1)}\right]\left[\mathrm{n}\circ\xi,\beta^{(1)}\right]\left[\mathrm{n}\circ\xi,\gamma^{(1)}\right]\left[\mathrm{n}\circ\xi,\delta^{(1)}\right]. \tag{3.106}$$

As usual, $\psi^{(3)}$ can be obtained by $k \leftrightarrow n$ interchange. The expression for $\psi^{(2)}$ is very similar, except that it contains 6 terms due to the symmetrisation over two

k's and two n's.

As we can see, the de Smet classification is highly sensitive to the fine structure of the Weyl tensor. This is summarised in table 3.11. At this point, it is possible to see that the hierarchy between de Smet classes proposed in [4] and shown in figure 3.1 is not actually present. For example, the **211** class does not contain the full **1111** class. A spacetime formed of more than one irrep will generically be a de Smet **4**. Although some special multi-irrep spacetimes exist, which are detailed in appendix A, there are not very many of them and they arise only in highly specialised circumstances. This explains the disagreement between the de Smet and CMPP classifications elucidated by Godazgar in [5]. On the one hand, because the CMPP classification is sensitive to the presence of the reducible little group spinors, it attributes the same Petrov class to a number of different possible de Smet classes[5]. On the other hand, the de Smet classification is most sensitive to the presence of a single irrep, irrespective of its boost weight. The two classifications clearly disagree in the notion of algebraic specialness.

Little group irreps				de Smet class		
$\psi^{(0)}_{abcd}$				**1111**		
$\psi^{(1)}_{abcd}$	$\chi^{(1)}_{ab}$			**4**	**211**	
$\psi^{(2)}_{abcd}$	$\chi^{(2)}_{ab}$	$\Psi^{(2)}_{\text{tr}}$	$\leftrightarrow$	**4**	**22**	<u>**22**</u>
$\psi^{(3)}_{abcd}$	$\chi^{(3)}_{ab}$			**4**	**211**	
$\psi^{(4)}_{abcd}$				**1111**		

Table 3.11 *The de Smet class of each little group irrep. The irreps are arranged by boost weight in the vertical direction and by dimension in the horizontal direction. Note the reflection symmetry in the central horizontal line, indicating invariance under the $k \leftrightarrow n$ interchange.*

3.4.5 Further refinements

The classification we propose is based on identifying representations of the little group: the $\psi^{(i)}_{abcd}$, for $i = 0, \ldots, 4$, $\chi^{(j)}_{ab}$, for $j = 1, 2, 3$, and $\Psi^{(2)}_{\text{tr}}$. An algebraically general spacetime has a full set of these objects, none of which are vanishing, and furthermore satisfying no algebraic relations amongst them.

[5]Although of course the refined CMPP classification in [37, 39] captures the little group irreps in full detail.

Algebraically special cases can occur in a number of ways. We have already observed that it is possible for some of the little group objects to vanish, and a more subtle possibility is that one or more of the $\psi^{(i)}_{abcd}$'s could be type D. In terms of spinors, we can always find two-component spinors α_a, β_b, γ_c and δ_d such that $\psi^{(i)}_{abcd} = \alpha_{(a}\beta_b\gamma_c\delta_{d)}$ for a particular i. In the type D case, there are really only two different spinors up to scaling. In group theoretic terms, this particular $\psi^{(i)}$ is actually a three-dimensional representation rather than a five-dimensional representation.

It is also possible to have situations in which spinors are shared among different little group objects. In the complex case, there are many possibilities, but in the real case we are more limited. It is still possible that $\chi^{(i)} \propto \chi^{(j)}$ for some choices of i and j. Alternatively, it could happen that a particular ψ could be composed of some χ: e.g., $\psi^{(1)}_{abcd} = \chi^{(2)}_{(ab}\chi^{(2)}_{cd)}$. The de Smet classification can be sensitive to such alignments in particular cases, as we discuss in Appendix A.

3.5 Higher dimensions

Although we focused on five dimensions in the previous sections, our approach is quite general. Indeed, our starting point, the spinor-helicity method, is available in any number of dimensions [9, 30, 31]. In this section we will briefly discuss the classification in six dimensions. As this is an even number of dimensions, we choose a chiral basis of spinors, with Clifford algebra

$$\sigma^{\mu}{}_{AB}\tilde{\sigma}^{BC\,\nu} + \sigma^{\nu}{}_{AB}\tilde{\sigma}^{BC\,\mu} = -2\eta^{\mu\nu}\mathbb{1}_A^C. \tag{3.107}$$

It happens that the Lie algebra of the Lorentz group in six dimensions, $\mathfrak{so}(6)$, is isomorphic to $\mathfrak{su}(4)$. This is reflected in the facts that the spinor representation of $\mathfrak{so}(6)$ is the four-dimensional fundamental representation of $\mathfrak{su}(4)$. From the point of view of $\mathfrak{su}(4)$, the six-dimensional vector representation of $\mathfrak{so}(6)$ is the antisymmetric tensor product of two $\mathbf{4}$s. Consequently, we can choose σ^μ and $\tilde{\sigma}^\mu$ to be antisymmetric 4×4 matrices.

In six dimensions, the little group is $SO(4) \cong SU(2) \times SU(2)\,/\,\mathbb{Z}_2$, so our first task is to understand how this product group structure is encoded in the spinors. Let k^μ be a six-dimensional null vector; then we define spinors associated with the vector by

$$k \cdot \sigma_{AB} \mathrm{k}^{Ba} = 0. \tag{3.108}$$

The index a labels linearly independent solutions of this equation. The matrix $k \cdot \sigma_{AB}$ has vanishing determinant and, in fact, has rank 2. Thus the label a takes values 1 and 2.

How can we reconstruct the null vector k from the spinor k^{Aa}? The observation that the **6** is an antisymmetric combination of two **4**s is helpful. There are six linearly independent 4×4 antisymmetric matrices, so if we expand an antisymmetric combination of the two spinors k^{Aa} (for $a = 1, 2$) on the basis ${\sigma^\mu}_{AB}$, the result is guaranteed to transform as a vector. Since k^μ is the only vector available, we simply have to fix the normalisation. Indeed,

$$k^\mu = \frac{1}{2\sqrt{2}} \, {\mathrm{k}^A}_a \, {\sigma^\mu}_{AB} \, \mathrm{k}^{Ba}, \tag{3.109}$$

where ${\mathrm{k}^A}_a = \epsilon_{ab} \mathrm{k}^{Ab}$; from this perspective, the matrix ϵ_{ab} is introduced to antisymmetrise the two possible k^a spinors.

This expression, equation (3.109), is manifestly invariant under an $SU(2)$ transformation $\mathrm{k}^a \to {U^a}_b \mathrm{k}^b$. This is part of the $SO(4)$ little group. The other $SU(2)$ factor acts on the anti-chiral spinors defined via

$$k \cdot \tilde{\sigma}^{AB} {\tilde{\mathrm{k}}_B}{}^{\dot{a}} = 0, \tag{3.110}$$

which implies that we may also write k^μ as

$$k^\mu = \frac{1}{2\sqrt{2}} \, \tilde{\mathrm{k}}_{A\dot{a}} \, \tilde{\sigma}^{\mu AB} \, {\tilde{\mathrm{k}}_B}{}^{\dot{a}}. \tag{3.111}$$

To construct the analogue of the NP tetrad in six dimensions we pick a second null vector n with the property that $k \cdot n = -1$, and introduce spinors $\mathrm{n}^{A\dot{a}}$ and ${\tilde{\mathrm{n}}_A}{}^a$. Then

$$n^\mu = \frac{1}{2\sqrt{2}} \, {\mathrm{n}^A}_{\dot{a}} \, {\sigma^\mu}_{AB} \, \mathrm{n}^{B\dot{a}} \tag{3.112}$$

$$= \frac{1}{2\sqrt{2}} \, \tilde{\mathrm{n}}_{Aa} \, \tilde{\sigma}^{\mu AB} \, {\tilde{\mathrm{n}}_B}{}^a. \tag{3.113}$$

The set of spinors k^{Aa}, $\mathrm{n}^{A\dot{a}}$, ${\tilde{\mathrm{k}}_A}{}^a$, ${\tilde{\mathrm{n}}_A}{}^a$ spans the spinor spaces, so it is a simple matter to break the 15 degrees of freedom of the tracefree 2-form spinor ${F^A}_B$

and the 84 degrees of freedom in the Weyl spinor $C^{AB}{}_{CD}$ into little group irreps. Because this is done in exactly the same way as we did for five dimensions (subject to the details of the spinor spaces), we are guaranteed that the connection to CMPP will continue to be expressed. The representations of the little group spinors are now labelled by two numbers in six dimensions, (i, j), and the boost weight is given by their average. The CMPP classification is simply the statement that each row of tables 3.12 and 3.13 for the 2-form and Weyl tensor, respectively, vanishes appropriately.

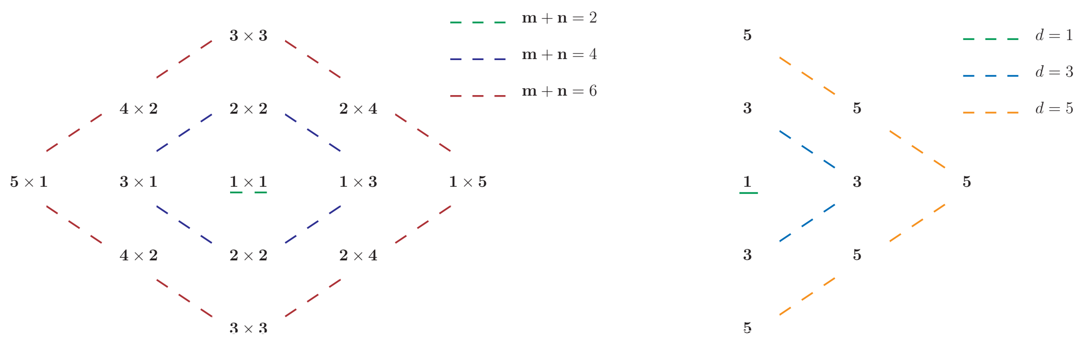
The appearance of a second number in the little group representation labels is due to a second symmetry in the irreps, that of an interchange between the two $SU(2)$ parts of the little group. This corresponds to an interchange $i \leftrightarrow j$ and dotted to undotted indices $a \leftrightarrow \dot{a}$, and manifests itself as a vertical line of symmetry through the centre of tables 3.12 and 3.13. This also explains the shape of the tables: previously, in five dimensions, where there was only a single $SU(2)$ little group, these decompositions had the shape of arrowheads which when reflected through the vertical axis form the characteristic rhombi of six dimensions. The dimensions of the irreps are not as regular as five dimensions, but have the pleasing distribution shown in figure 3.2 for the case of the Weyl spinor, laid next to their five-dimensional equivalent for comparison.

Reducible spinors		Irreducible spinors		Irrep dimensionality
$\Phi^{(0,0)}_{a\dot{b}}$		$\phi^{(0,0)}_{a\dot{b}}$		$\mathbf{2}\times\mathbf{2}$
$\Phi^{(0,2)}_{ab}$ $\Phi^{(2,0)}_{\dot{a}\dot{b}}$	$\Rightarrow$	$\phi^{(0,2)}_{ab}$ $\Phi^{(1,1)}_{\mathrm{tr}}$ $\phi^{(2,0)}_{\dot{a}\dot{b}}$	$\Leftrightarrow$	$\mathbf{1}\times\mathbf{3}$ $\mathbf{1}\times\mathbf{1}$ $\mathbf{3}\times\mathbf{1}$
$\Phi^{(2,2)}_{\dot{a}b}$		$\phi^{(2,2)}_{\dot{a}b}$		$\mathbf{2}\times\mathbf{2}$

Table 3.12 *The six-dimensional 2-form contains 4 reducible little group representations, which can be broken into 5 irreps. The rows are organised by boost weight, equal to the average of the bracketed superscripts. The columns are arranged such that the representations respect the $SU(2)$ interchange symmetry through the central vertical axis, hence the scalar $\Phi^{(1,1)}_{\mathrm{tr}} = \epsilon^{ab}\,\Phi^{(0,2)}_{ab} = \epsilon^{\dot{a}\dot{b}}\,\Phi^{(2,0)}_{\dot{a}\dot{b}}$ sits at the centre of the array.*

Reducible 6D little group spinors		Irreducible 6D little group spinors
$\Psi^{(0,0)}_{(ab)\,(\dot{c}\dot{d})}$		$\psi^{(0,0)}_{ab\,\dot{c}\dot{d}}$
$\Psi^{(0,2)}_{(ab)\,c\dot{d}}$ $\Psi^{(2,0)}_{a\dot{b}\,\dot{c}\dot{d}}$		$\psi^{(0,2)}_{abc\dot{d}}$ $\chi^{(1,1)}_{a\dot{b}}$ $\psi^{(2,0)}_{a\dot{b}\dot{c}\dot{d}}$
$\Psi^{(0,4)}_{(ab)\,(cd)}$ $\Psi^{(2,2)}_{a\dot{b}\,c\dot{d}}$ $\Psi^{(4,0)}_{(\dot{a}\dot{b})\,(\dot{c}\dot{d})}$	$\Rightarrow$	$\psi^{(0,4)}_{abcd}$ $\chi^{(1,3)}_{ab}$ $\Psi^{(2,2)}_{\mathrm{tr}}$ $\chi^{(3,1)}_{\dot{a}\dot{b}}$ $\psi^{(4,0)}_{\dot{a}\dot{b}\dot{c}\dot{d}}$
$\Psi^{(2,4)}_{\dot{a}b\,(cd)}$ $\Psi^{(4,2)}_{(\dot{a}\dot{b})\,\dot{c}d}$		$\psi^{(2,4)}_{\dot{a}bcd}$ $\chi^{(3,3)}_{\dot{a}b}$ $\psi^{(4,2)}_{\dot{a}\dot{b}\dot{c}d}$
$\Psi^{(4,4)}_{(\dot{a}b)\,(cd)}$		$\psi^{(4,4)}_{(\dot{a}b)cd}$

Table 3.13 *Connections between the traces of the reducible six-dimensional little group spinors allow us to break down the components into irreps. The indices of the reducible spinors (left) are organised in symmetrised pairs such that two like indices, for example ab or $\dot{c}\dot{d}$ comprise 3 degrees of freedom each, while pairs such as $a\dot{b}$ and $\dot{c}d$ have no symmetrisation and constitute 4 degrees of freedom. For the table of irreducible representations on the right, all indices of the same $SU(2)$ type (i.e. dotted or undotted) are totally symmetric. The boost weight of each representation (i,j) is given by $(i+j)/2$.*



(a) $D = 6$ (b) $D = 5$

Figure 3.2 *The irreps of the six-dimensional Weyl spinor $\boldsymbol{m} \times \boldsymbol{n}$ form a kite-like pattern (3.2a), while the the irreps of the five-dimensional Weyl spinor can be arranged in an arrowhead with concentric arrows of irrep dimension d (3.2b). As usual, rows correspond to boost weight. Each concentric rhombus corresponds to a different value of $\boldsymbol{m}+\boldsymbol{n}$.*

3.6 Discussion

We have demonstrated that higher-dimensional spinors provide a convenient formalism for the algebraic classification of spacetimes, extending Penrose's spinorial approach to the Petrov classification in four dimensions. The crucial element of the higher-dimensional spinorial construction, first proposed in [9] in the context of particle physics, is the explicit consideration of the little group. We have shown that the formalism not only leads naturally to the CMPP classification and its refinements, but it also allows for a natural connection with the de Smet classification. In particular, we have demonstrated that the de Smet classes mostly correspond to spacetimes where a single little group irrep is present, except for interesting cases where algebraic relations exist between distinct irreps. This analysis completes the work begun by [5].

In this work, we have set up a basic framework but there is much to be done. We have not described in detail the choice of vector basis (pentad in five dimensions) that makes manifest the algebraic properties of a spacetime. We have also only considered a few very simple examples of solutions to the Einstein equations. Further work should provide us with invaluable intuition for the interpretation of the various algebraic classes. Moreover, we have not discussed here the higher-dimensional extension of the Newman-Penrose formalism for the Einstein equations, which has been the subject of much previous work concerning, for instance, problems of existence and stability of solutions [75–85]. Another interesting problem to investigate with our formalism is the use of curvature (and Cartan) invariants to characterise spacetimes; see [86] for a brief introduction and [87–90] for recent work on this topic.

To the obvious possible directions mentioned above, we add one further direction that we already alluded to in the introduction. This is the 'double copy' between gauge theory and gravity, which appeared in the context of scattering amplitudes, and whose application to classical solutions is now under study. The existence of an analogy is, of course, natural from discussions such as the one in this chapter, when comparing the classifications of the field strength tensor and the Weyl tensor. The point is, however, that there is a precise formulation of the double copy in this context. This is the subject of the paper [66], and it was an important motivation for us to revisit the classification problem in this chapter. In the second half of the thesis we will discuss more to do with the double copy and its relations with exact solutions.

Part II

The double copy in GR

Chapter 4

BCJ duality and the double copy

Our most refined understanding of nature is founded on two major theoretical frameworks: general relativity and Yang-Mills theory. There is much in common between these two: local symmetries play an important role in their structure; there are simple action principles for both theories; the geometry of fibre bundles is common to the physical interpretation of the theories. But at the perturbative level, general relativity seems to be a vastly different creature to Yang-Mills theory. Indeed, the Einstein-Hilbert Lagrangian, when expanded in deviations of the spacetime metric from some reference metric (such as the Minkowski metric) contains terms with arbitrarily many powers of the deviations. This is in stark contrast to the Yang-Mills Lagrangian, which contains at most fourth order terms in perturbation theory. Nevertheless, a powerful correspondence between the two exists called the *double copy*. In this chapter, we will review the double copy and take its classical limit. The chapter references work done in collaboration with Andrés Luna, Ricardo Monteiro, Alexander Ochirov, Donal O'Connell, Chris White and Niklas Westerberg in [2, 3].

4.1 GR and perturbation theory

In the first half of this thesis, we focused on exact solutions to the Einstein equations. However, only very limited solutions to general relativity's field equations can be understood in this framework, and for the remainder we must use perturbative field theory techniques. The Einstein-Hilbert action in D dimensions is given by

$$S_{EH} = \frac{2}{\kappa^2} \int d^D x \sqrt{-g} R \tag{4.1}$$

where κ is the gravitational constant $\kappa^2 = 32\pi G_N$ which couples the gravitational field to matter. We can expand around a flat Minkowski background in terms of κ:

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{4.2}$$

to find the perturbations of the gravitational field $h_{\mu\nu}$ around the flat background. However, expanding Einstein-Hilbert action this way will create an expression containing infinitely many terms, since the inverse metric $g^{\mu\nu}$ which enters the definition R multiple times, can only satisfy $g_{\mu\alpha} g^{\alpha\nu} = \delta_\mu{}^\nu$ as an infinite series:

$$g^{\mu\nu} = \eta^{\mu\nu} - \kappa h^{\mu\nu} + \kappa^2 h^\mu{}_\alpha h^{\alpha\nu} - \kappa^3 h^{\mu\alpha} h_{\alpha\beta} h^{\beta\nu} + ... \tag{4.3}$$

where indices on $h_{\mu\nu}$ are raised and lowered using the Minkowski metric. There is also an infinite series of terms from $\sqrt{-g}$. Schematically, the Einstein-Hilbert action will look like

$$S_{EH} = \int d^D x \left(h\partial^2 h + \kappa h^2 \partial^2 h + \kappa^2 h^3 \partial^2 h + ... + \kappa^n h^{n+1} \partial^2 h + ... \right), \tag{4.4}$$

where each term indicates a group of terms with some complicated index structure. There are some gauge fixings and field redefinitions that can be done to simplify matters, such as the field redefinition $\mathfrak{h}_{\mu\nu} = \sqrt{g} g_{\mu\nu} - \eta_{\mu\nu}$ which in the de Donder gauge $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$ simplifies the 3-graviton vertex to

$$\begin{aligned} \frac{\delta S_{3EH}}{\delta h_{\mu_1\nu_1}(k_1) \delta h_{\mu_2\nu_2}(k_2) \delta h_{\mu_3\nu_3}(k_3)} &= -\frac{1}{8} \mathrm{sym} P_6 \big\{ -4\eta_{\mu_3\mu_2}\eta_{\nu_2\mu_1}\eta_{\nu_3\nu_1} k_2 \cdot k_3 \\ &+ 2\eta_{\mu_2\nu_2}\eta_{\mu_3\mu_1}\eta_{\nu_3\nu_1} k_2 \cdot k_3 - \eta_{\mu_2\nu_2}\eta_{\mu_3\nu_3} k_{2\mu_1} k_{3\nu_1} \\ &+ 2\eta_{\mu_3\mu_2}\eta_{\nu_2\nu_3} k_{2\mu_1} k_{3\nu_1} + 4\eta_{\mu_2\mu_1}\eta_{\nu_3\nu_1} k_{2\mu_3} k_{3\nu_2} \big\}. \end{aligned} \tag{4.5}$$

Here "sym" indicates a symmetrization on each index pair $\alpha_i, \beta_i, i = 1, 2, 3$ while P_6 indicates a summation over the six permutations of the particle number i. The 60 terms in this expression when expanded out are merely the beginning of the problem, as each of the infinitely many more n-graviton vertex expressions contain even more terms. A review of the progress made in perturbative gravity calculations is given in [91].

4.2 Squaring relations in amplitudes

Of course, if we wished to calculate the 3-point graviton amplitude M_3 we would actually fix it using little group symmetry and locality as described in section 1 and therefore bypass these complicated expressions. This gives a remarkable result, namely that the 3-point graviton amplitude is the square of the Yang-Mills 3-gluon amplitude found in equation (1.25):

$$M_3(1^-2^-3^+) = \left(\frac{\langle 12\rangle^3}{\langle 23\rangle\langle 31\rangle}\right)^2 = \left(A_3(1^-2^-3^+)\right)^2. \tag{4.6}$$

This fact is part of a much larger equivalence called the KLT relations [92]. These remarkable relations link gravity to gauge theory by considering open and closed strings: a closed string can be expressed as the product of one left-moving open string and one right-moving open string. When the field theory limit is taken, closed string vertex operators become gravity amplitudes while open ones turn into gauge theory amplitudes, giving the relation:

$$\begin{aligned} M_n^{tree}(1,2,...,n) =& i(-1)^{n+1}\Bigg[A_n^{tree}(1,2,...,n)\sum_{perms} f(l_1,...l_i)\tilde{f}(r_1,...,r_j) \\ & \times \tilde{A}_n^{tree}(l_1,...,l_j,1,n-1,r_1,...,r_j,n)\Bigg] + \mathcal{P}(2,...,n-2) \end{aligned} \tag{4.7}$$

where the notation of [93] has been used. The sum over the permutations refers to the permutations of $\{l_1,...,l_i\}$ and also $\{r_1,...,r_j\}$, while the notation $\mathcal{P}(2,...,n-2)$ indicates an additional sum over all permutations of the legs $\{2,...,n-2\}$. The functions f and $\tilde{f}$ are products of Mandelstam invariants given explicitly in the appendix of [93]. Therefore the KLT relations and the Yang-Mills Lagrangian together can be used to reconstruct the Lagrangian of general relativity [94].

4.2.1 BCJ duality

A more convenient expression of this fact was found in [10, 42, 95]. It requires us to express the full Yang-Mills amplitude, including colour factors, in terms of cubic vertices by breaking up quartic interaction terms. Then an m-point tree-level amplitude in non-abelian gauge theory may be written in the general form

$$\mathcal{A}_m^{\text{tree}} = g^{m-2} \sum_{i\in\Gamma} \frac{n_i\, c_i}{\prod_{\alpha_i} p_{\alpha_i}^2}, \tag{4.8}$$

where g is the coupling constant, and the sum is over the set of cubic graphs Γ. The denominator arises from propagators associated with each internal line, and c_i is a colour factor obtained by dressing each vertex with structure constants. Finally, n_i is a kinematic numerator, composed of momenta and polarisation vectors, where i runs over the diagrams. The form is not unique, however, owing to the fact that the numerators $\{n_i\}$ can be modified by gauge transformations and field redefinitions, neither of which affect the amplitude. A compact way to summarise this is that one is free to modify each individual numerator according to the generalised gauge transformation

$$n_i \to n_i + \Delta_i, \qquad \sum_i \frac{\Delta_i c_i}{\prod_{\alpha_i} p_{\alpha_i}^2} = 0, \tag{4.9}$$

where the latter condition expresses the invariance of the amplitude.

As we showed in section 1, the colour factors of the diagrams c_i can always be expressed in terms of traces of the generators T^a and therefore obey Jacobi relations of the form $c_i + c_j + c_k = 0$ for some i, j, k running over the list of diagrams. Other pairs of diagrams are simply related by $c_i = -c_j$. It turns out that it is always possible to find numerators such that the n_i obey identical relationships. For example in a five-point amplitude there are 15 different diagrams, of which only 6 have independent colour factors. The remainder can be expressed using expressions of the form $c_i + c_j + c_k = 0$, or $c_i = -c_j$. Colour-kinematics duality tells us that it is possible to make a choice of kinematic numerators such that the relations obeyed by the n_i are identical. The validity of the BCJ double copy and the existence of colour-dual numerators has been proven at tree-level [47, 95–102] (where it is equivalent to the KLT relations [92]). One very exciting feature of the BCJ procedure is that it admits a simple extension to loop diagrams in the quantum theory [42]. This extension remains conjectural, but it has been verified in highly non-trivial examples at multi-loop level [42, 93, 103–123]. All-order evidence can be obtained in special kinematic regimes [48, 105, 124–126], but a full proof of the correspondence has to date been missing (see, however, refs. [127–141] for related studies).

The BCJ choice of numerators is not unique. Consider the 4-point amplitude which has three diagrams corresponding to the s, t and u channels. The expression can be written as

$$A_4^{\text{tree}} = \frac{c_s n_s}{s} + \frac{c_t n_t}{t} + \frac{c_u n_u}{u} \tag{4.10}$$

for some choice of n_i which obeys colour-kinematics duality. If we make the transformation

$$n_s \to n_s + s\Delta, \quad n_t \to n_t + t\Delta, \quad n_u \to n_u + u\Delta \tag{4.11}$$

for some arbitrary function of the momenta and polarisations Δ, then the Jacobi identity $c_s + c_t + c_u = 0$ ensures that the amplitude is invariant:

$$A_4^{\text{tree}} \to A_4^{\text{tree}} + (c_s + c_t + c_u)\Delta = A_4^{\text{tree}}. \tag{4.12}$$

The duality hints at an intriguing correspondence between colour and kinematic degrees of freedom that is still not fully understood, although progress has been made in the self-dual sector of the theory [47]. More generally, the field-theory limit of superstring theory has been very fruitful for understanding colour-kinematics duality [101, 140, 142] and there has been recent progress on more formal aspects of the duality [143–145].

4.2.2 The double copy

Now the magic of the BCJ basis is that when the gauge theory numerators n_i are given in this form, performing the replacement $c_i \to \tilde{n}_i$ obtains gravity amplitudes:

$$M_n^{\text{tree}} = \sum_i \frac{n_i \tilde{n}_i}{\Pi_{\alpha_i} p_{\alpha_i}^2}. \tag{4.13}$$

As we will see in chapter 6, this gravity theory depends on which gauge theory the kinematic numerators are taken from and is often used in the context of supergravity. The new kinematic numerators $\tilde{n}_i$ do not need to be in a form that respects colour-kinematics duality: as long as they are a valid representation of the same Yang-Mills amplitude as the n_i then as in equation (4.9), the difference $\Delta_i \equiv \tilde{n}_i - n_i$ must satisfy

$$\sum_i \frac{c_i \Delta_i}{\Pi_{\alpha_i} p_{\alpha_i}^2} = 0. \tag{4.14}$$

This argument holds despite knowing nothing about the colour factors except that they come from a non-abelian group whose structure constants obey the Jacobi identities. This is true by design for the kinematic numerators n_i as well, so

$$\sum_i \frac{n_i \Delta_i}{\Pi_{\alpha_i} p^2_{\alpha_i}} = 0 \tag{4.15}$$

is also true, and therefore

$$M_n^{\text{tree}} = \sum_i \frac{n_i \tilde{n}_i}{\Pi_{\alpha_i} p^2_{\alpha_i}} = \sum_i \frac{(n_i)^2}{\Pi_{\alpha_i} p^2_{\alpha_i}}. \tag{4.16}$$

We will not use the quantum form of the double copy much in the following sections. Instead, we will take the classical limit and consider only general relativity. Since colour-kinematics duality has been proven to hold for all n-point tree level diagrams, it is also true classically in most situations, see for example [11]. In order to relate the double copy to general relativity, we need to consider the vacuum expectation value of a field rather than its amplitude - namely LSZ reduction from n-point functions to amplitudes. The process of extracting the metric from Feynman diagrams is demonstrated in [146] for the Schwarzschild solution. Then it is clear that we expect the double copy to live in the metric perturbation as we will demonstrate in the next two chapters.

Chapter 5

Bremsstrahlung and an exact double copy

5.1 Introduction

In the previous section, we reviewed the incredible way that BCJ duality and the double copy relates general relativity and Yang-Mills theory. Motivated by this, a double copy for classical field solutions (which we will refer to as the classical double copy) was proposed [12]. This classical double copy is similar in structure to the BCJ double copy for scattering amplitudes: in both cases, the tensor structure of gravity is constructed from two copies of the vector structure of gauge theory. In addition, scalar propagators are present in both cases; these scalars are exactly the same in gauge and gravitational processes. However, the classical double copy [12] was previously only understood for the special class of Kerr-Schild solutions in general relativity. This reflects the particularly simple structure of Kerr-Schild metrics: the Kerr-Schild ansatz has the remarkable property that the Einstein equations exactly linearise. Therefore we can anticipate that any Yang-Mills solution related to a Kerr-Schild spacetime must be particularly simple. Indeed, the authors of [12] showed that any stationary Kerr-Schild solution has a well-defined single copy that satisfies the Yang-Mills equations, which also take the linearised form. While the structure of the classical double copy is very reminiscent of the BCJ double copy, so far no precise link has been made between the two. One aim of the present chapter is to provide such a link.

Although the classical double copy is only understood for a restricted class of solutions, many of these are familiar. For example, the Schwarzschild and Kerr black holes are members of this class; in higher dimensions, the Myers-Perry black holes are included [12]. The relationship between classical solutions holds for all stationary Kerr-Schild solutions, but other Kerr-Schild solutions are known to have appropriate single copies. A particularly striking example is the shockwave in gravity and gauge theory; the double copy of this pair of solutions was pointed out by Saotome and Akhoury [48]. In further work, the classical double copy has been extended [43] to the Taub-NUT solution [147, 148], which has a double Kerr-Schild form and whose single copy is a dyon in gauge theory.

Despite this success, Kerr-Schild solutions are very special and do not easily describe physical systems which seem very natural from the point of view of the double copy for scattering amplitudes. For example, there is no two-form field or dilaton on the gravity side; there are no non-abelian features on the gauge theory side; the status of the sources must be better understood. In cases where the sources are point particle-like, the classical double copy relates the gauge theory current density to the gravity energy-momentum tensor in a natural way [12, 43]. For extended sources, extra pressure terms on the gravity side are needed to stabilise the matter distribution. Furthermore, reference [44] pointed out that in certain gravity solutions the energy-momentum tensor does not satisfy the weak and/or strong energy conditions of general relativity.

In this chapter, we will extend the classical double copy of [12, 43] by considering one of the simplest situations involving explicit time dependence, namely that of an arbitrarily accelerating, radiating point source. We will see that this situation can indeed be interpreted in the Kerr-Schild language, subject to the introduction of additional source terms for which we provide a clear interpretation. One important fact which will emerge is that these sources themselves have a double copy structure. We will demonstrate that the sources can be related directly to scattering amplitudes, maintaining the double copy throughout. This provides a direct link between the classical double copy and the BCJ procedure for amplitudes, strongly bolstering the argument that these double copies are the same. The gravitational solution of interest to us is a time-dependent generalisation of the Schwarzschild solution; we will see that this gravitational system is a precise double copy of an accelerating point particle. Since there is a double copy of the sources, and these describe the radiation fields, we learn that the gravitational radiation emitted by a black hole which undergoes a short period of acceleration is a precise double copy of electromagnetic Bremsstrahlung.

The structure of the chapter is as follows. In section 5.2, we briefly review the Kerr-Schild double copy. In section 5.3, we present a known Kerr-Schild solution for an accelerating particle, before examining its single copy. We will find that additional source terms appear in the gauge and gravity field equations, and in section 5.4 we relate these to scattering amplitudes describing radiation, by considering the example of Bremsstrahlung. In section 5.5, we examine the well-known energy conditions of GR for the solutions under study. Finally, we discuss our results and conclude in section 5.6. Technical details are contained in appendix B. The chapter is based on work done in collaboration with Andrés Luna, Ricardo Monteiro, Donal O'Connell and Chris White in [2].

5.2 Review of the Kerr-Schild double copy

Let us begin with a brief review of the Kerr-Schild double copy, originally proposed in [12, 43]. We define the graviton field as in chapter 4 via

$$g_{\mu\nu} = \bar{g}_{\mu\nu} + \kappa h_{\mu\nu}, \quad \kappa = \sqrt{32\pi G_{\rm N}} \tag{5.1}$$

where $G_{\rm N}$ is Newton's constant, and $\bar{g}_{\mu\nu}$ is a background metric, which, for the purposes of the present paper, we will take to be the Minkowski metric.[1] There is a special class of *Kerr-Schild* solutions of the Einstein equations, in which the graviton has the form

$$h_{\mu\nu} = -\frac{\kappa}{2}\phi k_\mu k_\nu, \tag{5.2}$$

consisting of a scalar function ϕ multiplying the outer product of a vector k_μ with itself. We have inserted a negative sign in this definition for later convenience. The vector k_μ must be null and geodesic with respect to the background:

$$\bar{g}_{\mu\nu}\, k^\mu\, k^\nu = 0, \quad (k\cdot D)k = 0, \tag{5.3}$$

where D^μ is the covariant derivative with respect to the background metric. It follows that k_μ is also null and geodesic with respect to the metric $g_{\mu\nu}$. These solutions have the remarkable property that the Ricci tensor with mixed

[1]We continue to work with a negative signature metric $\eta = \text{diag}(1, -1, -1, -1)$.

upstairs/downstairs indices is *linear* in the graviton. More specifically, one has

$$R^{\mu}{}_{\nu} = \bar{R}^{\mu}{}_{\nu} - \kappa \left[h^{\mu}{}_{\rho} \bar{R}^{\rho}{}_{\nu} - \frac{1}{2} D_{\rho} \left(D_{\nu} h^{\mu\rho} + D^{\mu} h^{\rho}{}_{\nu} - D^{\rho} h^{\mu}{}_{\nu} \right) \right], \tag{5.4}$$

where $\bar{R}_{\mu\nu}$ is the Ricci tensor associated with $\bar{g}_{\mu\nu}$, and we have used the fact that $h^{\mu}{}_{\mu} = 0$. It follows that the Einstein equations themselves linearise. Furthermore, it was shown in [12] that for every stationary Kerr-Schild solution (i.e. where neither ϕ nor k^{μ} has explicit time dependence), the gauge field

$$A^{a}_{\mu} = c^{a} \phi \, k^{\mu}, \tag{5.5}$$

for a constant colour vector c^{a}, solves the Yang-Mills equations. Analogously to the gravitational case, these equations take a linearised form due to the trivial colour dependence of the solution. We then refer to such a gauge field as the *single copy* of the graviton $h_{\mu\nu}$, since it involves only one factor of the Kerr-Schild vector k_{μ} rather than two. Note that the scalar field ϕ is left untouched by this procedure. This was motivated in [12] by taking the *zeroth copy* of equation (5.5) (i.e. stripping off the remaining k^{μ} factor), which leaves the scalar field itself. The zeroth copy of a Yang-Mills theory is a biadjoint scalar field theory, and the field equation linearises for the scalar field obtained from equation (5.5). The scalar function ϕ then corresponds to a propagator, and is analogous to the untouched denominators (themselves scalar propagators) in the BCJ double copy for scattering amplitudes.

Source terms for the biadjoint, gauge and gravity theories also match up in a natural way in the Kerr-Schild double copy. Point-like sources in a gauge theory map to point particles in gravity, where electric and (monopole) magnetic charge are replaced by mass and NUT charge respectively [43]. Extended source distributions (such as that for the Kerr black hole considered in [12]) lead to additional pressure terms in the gravity theory, which are needed to stabilise the source distribution so as to be consistent with a stationary solution. Conceptual questions relating to extended source distributions have been further considered in [44], regarding the well-known energy conditions of general relativity. In this work, we will consider point-like objects throughout, and therefore issues relating to extended source distributions will not trouble us. Nevertheless we will discuss the energy conditions in section 5.5 below.

Let us emphasise that the Kerr-Schild double copy cannot be the most general relationship between solutions in gauge and gravity theories. Indeed, the field one obtains upon taking the outer product of k^μ with itself is manifestly symmetric. Moreover, the null condition on k^μ means that the trace of the field vanishes. Hence, the Kerr-Schild double copy is unable to describe situations in which a two-form and/or dilaton are active in the gravity theory. This contrasts sharply with the double copy procedure for scattering amplitudes, which easily incorporates these fields. We will demonstrate how these fields can be incorporated in chapter 6. Furthermore, Yang-Mills amplitudes only obey the double copy when written in BCJ dual form, meaning that certain Jacobi relations are satisfied by the kinematic numerator functions [10, 42, 95]. It is not known what the analogue of this property is in the classical double copy procedure. All of these considerations suggest that the Kerr-Schild story forms part of a larger picture, and in order to explore this it is instructive to seek well-defined generalisations of the results of [12, 43].

5.3 Kerr-Schild description of an accelerating point particle

In this chapter, we will go beyond previous work on the Kerr-Schild double copy [12, 43] by considering an accelerating point particle. This is a particularly attractive case, because an accelerating point particle must radiate, so we may hope to make direct contact between the double copy for scattering amplitudes and for Kerr-Schild backgrounds. We first describe a well-known Kerr-Schild spacetime containing an accelerating point particle, before constructing the associated single-copy gauge theoretic solution. We find that the physics of the single copy is particularly clear, allowing a refined understanding of the gravitational system. We will build on this understanding in section 5.4 to construct a double copy pair of scattering amplitudes from our pair of Kerr-Schild solutions in gauge theory and gravity in a manner that preserves the double copy throughout.

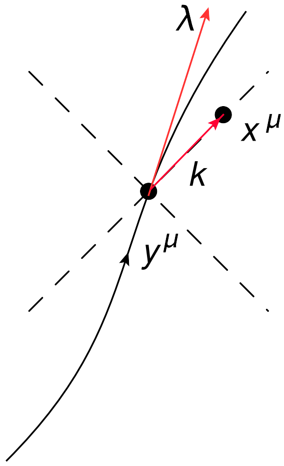


Figure 5.1 *Geometric interpretation of the Kerr-Schild solution for an accelerated particle.*

5.3.1 Gravity solution

Consider a particle of mass M following an arbitrary timelike worldline $y(\tau)$, parametrised by its proper time τ so that the proper velocity of the particle is the tangent to the curve

$$\lambda^\mu = \frac{dy^\mu}{d\tau}. \tag{5.6}$$

An exact Kerr-Schild spacetime containing this massive accelerating particle is known, though the spacetime contains an additional stress-energy tensor; we will understand the physical role of this stress-energy tensor below. A useful geometric interpretation of the null vector k_μ appearing in the solution has been given in [149–151] (see [23] for a review), as follows. Given an arbitrary point $y^\mu(\tau)$ on the particle worldline, one may draw a light cone as shown in figure 5.1. At all points x^μ along the light-cone, one may then define the null vector

$$k^\mu(x) = \left.\frac{(x - y(\tau))^\mu}{r}\right|_{\text{ret}}, \quad r = \lambda \cdot (x - y)|_{\text{ret}}\,, \tag{5.7}$$

where the instruction ret indicates that y and λ should be evaluated at the retarded time τ_{ret}, i.e. the value of τ at which a past light cone from x^μ intersects the worldline. Calculations are facilitated by noting that:

$$\partial_\mu k_\nu = \partial_\nu k_\mu = \frac{1}{r}\left(\eta_{\mu\nu} - \lambda_\mu k_\nu - k_\mu \lambda_\nu - k_\mu k_\nu \left(-1 + rk \cdot \dot{\lambda}\right)\right), \tag{5.8}$$

$$\partial_\mu r = \lambda_\mu + k_\mu(-1 + rk \cdot \dot{\lambda}), \tag{5.9}$$

where dots denote differentiation with respect to the proper time τ.

The Kerr-Schild metric associated with this particle is

$$g_{\mu\nu} = \eta_{\mu\nu} - \frac{\kappa^2}{2}\phi k_\mu k_\nu \tag{5.10}$$

where k_μ is precisely the vector of equation (5.7) and different functional forms for ϕ lead to different solutions. The scalar function corresponding to an accelerating particle is given by [152]

$$\phi = \frac{M}{4\pi r}. \tag{5.11}$$

Plugging this into the Einstein equations, one finds

$$G^\mu{}_\nu \equiv R^\mu{}_\nu - \frac{R}{2}\delta^\mu{}_\nu = \frac{\kappa^2}{2} T_{\text{KS}}{}^\mu{}_\nu, \tag{5.12}$$

where[2]

$$T^{\mu\nu}_{\text{KS}} = \frac{3M}{4\pi}\frac{k\cdot\dot{\lambda}}{r^2}k^\mu k^\nu\bigg|_{\text{ret}}. \tag{5.13}$$

Thus, the use of Kerr-Schild coordinates for the accelerating particle leads to the presence of a non-trivial energy-momentum tensor on the right-hand side of the Einstein equations. We can already see that this extra term vanishes in the stationary case ($\dot{\lambda}^\mu = 0$), consistent with the results of [12]. More generally, this stress-energy tensor $T^{\mu\nu}_{\text{KS}}$ describes a pure radiation field present in the spacetime. The physical interpretation of this source is particularly clear in the electromagnetic "single copy" of this system, to which we now turn.

5.3.2 Single copy

Having examined a point particle in arbitrary motion in a Kerr-Schild spacetime, we may apply the classical single copy of equation (5.5) to construct a corresponding gauge theoretic solution. This procedure is not guaranteed to work, given that the single copy of [12, 43] was only shown to apply in the case of stationary fields. However, we will see that we can indeed make sense of the single copy in the present context. Indeed, the physical interpretation

[2]We note what appears to be a typographical error in [23], where the energy-momentum tensor contains an overall factor of 4 rather than 3. We have explicitly carried out the calculation leading to equation (5.12), and found agreement with [149–151].

of the stress-energy tensor $T^{\mu\nu}_{\text{KS}}$ we encountered in the gravitational situation is illuminated by the single copy.

The essence of the Kerr-Schild double-copy is a relationship between gauge theoretic solutions $A^\mu = k^\mu \phi$ and Kerr-Schild metrics which is simply expressed as $k_\mu \to k_\mu k_\nu$. Thus, the single-copy of

$$h^{\mu\nu} = -\frac{M\kappa}{2}\frac{1}{4\pi r}k^\mu k^\nu \tag{5.14}$$

is[3]

$$A^\mu = g\frac{1}{4\pi r}k^\mu, \tag{5.15}$$

where g is the coupling constant.[4] Inserting this gauge field into the Yang-Mills equations, one finds that non-linear terms vanish, leaving the Maxwell equations

$$\partial^\mu F_{\mu\nu} = j_{\text{KS}\,\nu}, \tag{5.16}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \tag{5.17}$$

is the usual electromagnetic field strength tensor.

A key result is that the current density appearing in the Maxwell equations is given by

$$j_{\text{KS}\,\nu} = 2\frac{g}{4\pi}\frac{k\cdot\dot{\lambda}}{r^2}k_\nu\Bigg|_{\text{ret}}. \tag{5.18}$$

It is important to note that the current density j_{KS} is related to the energy-momentum tensor, equation (5.13), we encountered in the gravitational case. Indeed the relationship between these sources is in accordance with the Kerr-Schild double copy: it involves a single factor of the Kerr-Schild vector k^μ, with similar prefactors, up to numerical constants. We will return to this interesting fact in the following section.

[3]In principle, one should include an arbitrary colour index on the field strength and current density. Given that the field equations are Abelian, however, we ignore this. The resulting solution can be easily embedded in a non-abelian theory, as in [12, 43]. Note that the Abelian character of this theory also implies that we make the replacement $\frac{M\kappa}{2} \to g$ (cf. equation (38) from [12]).

[4]The relative sign between $h_{\mu\nu}$ and A_μ is necessary in our conventions to ensure that positive masses yield attractive gravitational fields while positive scalar potentials A^0 are sources for electric field lines $\boldsymbol{E} = -\boldsymbol{\nabla} A^0$.

The role of the Kerr-Schild current density j_{KS} can be understood by examining our single-copy gauge field, equation (5.15), in more detail. Let us compute the electromagnetic field strength tensor of this system. Using the results (5.8) and (5.9), it is easy to check that

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu = \frac{g}{4\pi r^2}\left(k_\mu \lambda_\nu - \lambda_\mu k_\nu\right). \tag{5.19}$$

A first observation about this field strength tensor is that it falls off as $1/r^2$ and does not depend on the acceleration of the particle. Therefore, it does not describe the radiation field of the accelerated point particle in empty space, since the radiation fields must fall off as $1/r$ and are linear in the acceleration. Secondly, this tensor is manifestly constructed from Lorentz covariant quantities. In the instantaneous rest frame of the particle, $\lambda^\mu = (1,0,0,0)$ and $k^\mu = (1, \hat{\boldsymbol{r}})$, and in this frame it is easy to see that the field strength is simply the Coulomb field of the point charge. Therefore, in a general inertial frame, our field strength tensor describes precisely the boosted Coulomb field of a point charge, omitting the radiation field completely.

The absence of radiation in the electromagnetic field strength makes the interpretation of the current density j_{KS} in the Maxwell equation obvious. This source must describe the radiation field of the point particle. To see this more concretely, let us compare our Kerr-Schild gauge field to the standard Liénard-Wiechert solution $A^\mu_{\text{LW}} = \frac{g}{4\pi r}\lambda^\mu$, which describes a point particle moving in an arbitrary manner in empty space (see e.g. [153]). This comparison is facilitated by defining a "radiative gauge field"

$$A^\mu_{\text{rad}} = \frac{g}{4\pi r}(\lambda^\mu - k^\mu), \tag{5.20}$$

which satisfies

$$F^{\mu\nu}_{\text{rad}} \equiv \partial^\mu A^\nu_{\text{rad}} - \partial^\nu A^\mu_{\text{rad}} = \frac{g}{4\pi r}\left(k^\mu \beta^\nu - \beta^\mu k^\nu\right), \tag{5.21}$$

where $\beta_\mu = \dot{\lambda}_\mu - \lambda_\mu k \cdot \dot{\lambda}$. Thus, $F^{\mu\nu}_{\text{rad}}$ is the radiative field strength of the point particle: it is linear in the particle acceleration, and falls off as $1/r$ at large distances.

Now, since the Liénard-Wiechert field is a solution of the vacuum Maxwell equation, we know that $\partial_\mu \left(F^{\mu\nu} + F^{\mu\nu}_{\rm rad}\right) = 0$ and, consequently,

$$\partial_\mu F^{\mu\nu}_{\rm rad} = -j^\nu_{\rm KS}. \tag{5.22}$$

We interpret $j_{\rm KS}$ as a divergence of the radiative field strength: we have put the radiation part of the gauge field on the right-hand side on the Maxwell equations, rather than the left.

Let us now summarise what has happened. By choosing Kerr-Schild coordinates for the accelerating particle in gravity, an extra energy-momentum tensor $T^{\mu\nu}_{\rm KS}$ appeared on the right-hand side of the Einstein equations. The single copy turns an energy density into a charge density (as in [12, 43, 44]). Thus, the energy-momentum tensor in the gravity theory becomes a charge current $j^\mu_{\rm KS}$ in the gauge theory. We have now seen that this current represents the radiation coming from the accelerating charged particle, and this also allows us to interpret the corresponding energy-momentum tensor on the gravity side: it represents gravitational radiation from an accelerating point mass.

Indeed, our use of Kerr-Schild coordinates forced the radiation to appear in this form. The vector k_μ which is so crucial for our approach is twist-free: $\partial_\mu k_\nu = \partial_\nu k_\mu$. It is known that twist-free, vacuum, Kerr-Schild metrics are of Petrov type D, and therefore there is no gravitational radiation in the metric; see [23] for a review. Correspondingly, the radiation is described by the Kerr-Schild sources.

The radiation fields of the accelerating charge in gauge theory, and the accelerating point mass in gravity, are described in Kerr-Schild coordinates by sources $j^\mu_{\rm KS}$ and $T^{\mu\nu}_{\rm KS}$. The structure of these sources reflects the Kerr-Schild double copy procedure: up to numerical factors, one replaces the vector k_μ by the symmetric trace-free tensor $k_\mu k_\nu$ to pass from gauge theory to gravity. This relationship between the sources, which describe radiation, is highly suggestive. Indeed, it is a standard fact that scattering amplitudes can be obtained from (amputated) currents. We may therefore anticipate that the structural relationship between the Kerr-Schild currents is related to the standard double copy for scattering amplitudes.

Nevertheless, there are still some puzzles regarding the analysis above. What, for example, are we to make of the different numerical factors appearing in the definitions equations (5.13) and (5.18) of the Kerr-Schild stress tensor and current density? If these sources are related to amplitudes, we expect a double copy which is local in momentum space. How can our currents be local in position space? More generally, how can we be sure that the Kerr-Schild double copy is indeed related to the standard BCJ procedure? The answer to these questions is addressed in the following section, in which we interpret the radiative sources directly in terms of scattering amplitudes.

Before proceeding, however, let us comment on the physical interpretation of the particle in the solutions under study. We considered how the particle affects the gauge or gravity fields, but we did not consider the cause of the acceleration of the particle, i.e. its own equation of motion. In the standard Liénard-Wiechert solution, the acceleration is due to a background field. It is therefore required that this background field does not interact with the radiation, otherwise the solution is not valid. This is true in electromagnetism or in its embedding in Yang-Mills theory. However, in the gravity case, one cannot envisage such a situation. Therefore, one should think of this particle merely as a boundary condition, and not as a physical particle subject to forces which would inevitably affect the Einstein equations. What we are describing here is a mathematical map between solutions in gauge theory and gravity, a map which exists irrespective of physical requirements on the solutions. In a similar vein, [44] showed that energy-momentum tensors obtained through the classical double copy do not necessarily obey the positivity of energy conditions in general relativity.

5.4 From Kerr-Schild sources to amplitudes

In the previous section, we saw that the Kerr-Schild double copy can indeed describe radiating particles. The radiation appears as a source term on the right-hand side of the field equations. In this section, we consider a special case of this radiation, namely Bremsstrahlung associated with a sudden rapid change in direction. By Fourier transforming the source terms in the gauge and gravity theory to momentum space, we will see that they directly yield known scattering amplitudes which manifestly double copy. Moreover, the manipulations required to extract the scattering amplitudes in gauge theory and in gravity are precisely parallel. We will preserve the double copy structure at each step, so that the double copy property of the scattering amplitudes emerges from the $k_\mu \to k_\mu k_\nu$

structure of the Kerr-Schild double copy. In this way, we firmly establish a link between the classical double copy and the BCJ double copy of scattering amplitudes.

In order to study Bremsstrahlung, we consider a particle which moves with velocity

$$\lambda^\mu(\tau) = u^\mu + f(\tau)(u'^\mu - u^\mu), \tag{5.23}$$

where

$$f(\tau) = \begin{cases} 0, & \tau < -\epsilon \\ 1, & \tau > \epsilon \end{cases} \tag{5.24}$$

and, in the interval $(-\epsilon, \epsilon)$, $f(\tau)$ is smooth but otherwise arbitrary. This describes a particle which moves with constant velocity $\lambda^\mu = u^\mu$ for $\tau < -\epsilon$, while for $\tau > \epsilon$ the particle moves with a different constant velocity $\lambda^\mu = u'^\mu$. Thus, the particle undergoes a rapid change of direction around $\tau = 0$, assuming ϵ to be small. The form of $f(\tau)$ acts as a regulator needed to avoid pathologies in the calculation that follows. However, dependence on this regulator cancels out, so that an explicit form for $f(\tau)$ will not be needed. Owing to the constant nature of u and u', the acceleration is given by

$$\dot{\lambda}^\mu = \dot{f}(\tau)\left(u'^\mu - u^\mu\right). \tag{5.25}$$

The acceleration vanishes for $\tau < -\epsilon$ and $\tau > \epsilon$, but is potentially large in the interval $(-\epsilon, \epsilon)$. Without loss of generality, we may choose the spatial origin to be the place at which the particle changes direction, so that $y^\mu(0) = 0$.

5.4.1 Gauge theory

We first consider the gauge theory case, and start by using the definitions of equations (5.7) to write the current density of equation (5.18) as

$$j^\nu_{\mathrm{KS}} = \frac{2g}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x - y(\tau))}{[\lambda(\tau)\cdot(x-y(\tau))]^4}(x - y(\tau))^\nu \delta(\tau - \tau_{\mathrm{ret}}), \tag{5.26}$$

where we have introduced a delta function to impose the retarded time constraint. Using the identity

$$\frac{\delta(\tau - \tau_{\mathrm{ret}})}{\lambda\cdot(x - y(\tau))} = 2\theta(x^0 - y^0(\tau))\delta\left((x - y(\tau))^2\right), \tag{5.27}$$

one may rewrite equation (5.26) as

$$j^{\nu}_{\rm KS} = \frac{4g}{4\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x-y(\tau))}{[\lambda(\tau)\cdot(x-y(\tau))]^3}(x-y(\tau))^{\nu}\theta(x^0-y^0(\tau))\delta\left((x-y(\tau))^2\right). \quad (5.28)$$

Any radiation field will be associated with the non-zero acceleration only for $|\tau| < \epsilon$, where $y^{\mu}(\tau)$ is small. We may thus neglect this with respect to x^{μ} in equation (5.28). Substituting equation (5.25) then gives

$$j^{\nu}_{\rm KS} = \frac{4g}{4\pi}x^{\nu}\theta(x^0)\delta(x^2)\int_{-\epsilon}^{\epsilon} d\tau \frac{b\dot{f}(\tau)}{(a+bf(\tau))^3}, \quad (5.29)$$

where

$$a = x\cdot u, \quad b = x\cdot u' - x\cdot u. \quad (5.30)$$

The integral is straightforwardly carried out to give

$$\begin{aligned} j^{\nu}_{\rm KS} &= -\frac{2g}{4\pi}x^{\nu}\theta(x^0)\delta(x^2)\left[\frac{1}{(x\cdot u')^2} - \frac{1}{(x\cdot u)^2}\right] \\ &= \frac{2g}{4\pi}\theta(x^0)\delta(x^2)\left[\frac{\partial}{\partial u'_{\nu}}\left(\frac{1}{x\cdot u'}\right) - (u'\to u)\right]. \end{aligned} \quad (5.31)$$

One may now Fourier transform this expression, obtaining a current depending on a momentum k conjugate to the position x. As our aim is to extract a scattering amplitude from the Fourier space current, $\tilde{j}^{\mu}_{\rm KS}(k)$, we consider only the on-shell limit of the current where $k^2 = 0$; we also drop terms in $\tilde{j}^{\mu}_{\rm KS}(k)$ which are proportional to k^{μ} as these terms are pure gauge. The technical details are presented in appendix B, and the result is

$$\tilde{j}^{\nu}_{\rm KS}(k) = -ig\left(\frac{u'^{\nu}}{u'\cdot k} - \frac{u^{\nu}}{u\cdot k}\right). \quad (5.32)$$

We may now interpret this as follows. First, we note that the current results from acting on the radiative gauge field with an inverse propagator, consistent with the LSZ procedure for truncating Green's functions. It follows that the contraction of $\tilde{j}^{\nu}_{\rm KS}$ with a polarisation vector gives the scattering amplitude for emission of a gluon. Upon doing this, one obtains the standard eikonal scattering amplitude for Bremsstrahlung (see e.g. [154])

$$\mathcal{A}_{\rm gauge} \equiv \epsilon_{\nu}(k)\tilde{j}^{\nu}_{\rm KS} = -ig\left(\frac{\epsilon\cdot u'}{u'\cdot k} - \frac{\epsilon\cdot u}{u\cdot k}\right). \quad (5.33)$$

We thus see directly that the additional current density in the Kerr-Schild approach corresponds to the radiative part of the gauge field.

5.4.2 Gravity

We now turn to the gravitational case. Our goal is to extract the eikonal scattering amplitude for gravitational Bremsstrahlung from the Kerr-Schild stress-energy tensor $T^{\mu\nu}_{\text{KS}}$ for a particle of mass M moving along precisely the same trajectory as our point charge. Thus, the acceleration of the particle is, again,

$$\dot{\lambda}^{\mu} = \dot{f}(\tau)\left(u'^{\mu} - u^{\mu}\right). \tag{5.34}$$

The calculation is a precise parallel to the calculation of the Bremsstrahlung amplitude for the point charge. However, as we will see, the presence of an additional factor of the Kerr-Schild vector k^{ν} in the gravitational case leads to a slightly different integral which we encounter during the calculation. This integral cancels the factor of 3 which appears in $T^{\mu\nu}_{\text{KS}}$, restoring the expected numerical factors in the momentum space current. Let us now turn to the explicit calculation.

We begin by writing the stress tensor as an integral over a delta function which enforces the retardation and causality constraints

$$\begin{aligned} T^{\mu\nu}_{\text{KS}} =& \frac{3M}{2\pi}\int d\tau \frac{\dot{\lambda}(\tau)\cdot(x-y(\tau))}{[\lambda(\tau)\cdot(x-y(\tau))]^4}(x-y(\tau))^{\mu}(x-y(\tau))^{\nu}\theta(x^0-y^0(\tau)) \\ &\times\delta\left((x-y(\tau))^2\right), \end{aligned} \tag{5.35}$$

corresponding to equation (5.28) in the gauge theoretic case. The fourth power in the denominator in the gravitational case arises as a consequence of the additional factor of $k^{\mu} = (x-y(\tau))^{\mu}/[\lambda(\tau)\cdot(x-y(\tau))]$. As before, the integral is strongly peaked around $y^{\mu} = 0$, and we may perform the integral in this region to find that

$$\begin{aligned} T^{\mu\nu}_{\text{KS}} &= -\frac{2M}{4\pi}x^{\mu}x^{\nu}\theta(x^0)\delta(x^2)\left[\frac{1}{(x\cdot u')^3} - \frac{1}{(x\cdot u)^3}\right] \\ &= -\frac{M}{4\pi}\theta(x^0)\delta(x^2)\left[\frac{\partial}{\partial u'_{\mu}}\frac{\partial}{\partial u'_{\nu}}\left(\frac{1}{x\cdot u'}\right) - (u'\to u)\right]. \end{aligned} \tag{5.36}$$

Notice that the factor 3 in the numerator of the stress-energy tensor has cancelled due to the additional factor of $\lambda(\tau)\cdot(x-y(\tau))$ in the denominator of the integrand in the gravitational case. The double copy structure is evidently now captured by a replacement of one derivative $\frac{\partial}{\partial u'_\nu}$ in gauge theory with two derivatives $\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}$ in gravity.

Our next step is to Fourier transform to momentum space. The calculation is extremely similar to the gauge theoretic case (again, see appendix B). As our goal is to compute a scattering amplitude, we work in the on-shell limit $k^2 = 0$ and omit pure gauge terms. After a short calculation, we find

$$\tilde{T}^{\mu\nu}_{\text{KS}}(k) = -iM\left(\frac{u'^\mu u'^\nu}{u'\cdot k} - \frac{u^\mu u^\nu}{u\cdot k}\right). \tag{5.37}$$

To construct the scattering amplitude, we must contract this Fourier-transformed stress-energy tensor with a polarisation tensor, which may be written as an outer product of two gauge theory polarisation vectors:

$$\epsilon^{\mu\nu}(k) = \epsilon^\mu(k)\epsilon^\nu(k). \tag{5.38}$$

The scattering amplitude is then given by

$$\mathcal{A}_{\text{grav}} \equiv \epsilon_\mu(k)\epsilon_\nu(k)\tilde{T}^{\mu\nu}_{\text{KS}}(k) = -iM\left(\frac{\epsilon\cdot u'\,\epsilon\cdot u'}{u'\cdot k} - \frac{\epsilon\cdot u\,\epsilon\cdot u}{u\cdot k}\right), \tag{5.39}$$

corresponding to the known eikonal amplitude for gravitational Bremsstrahlung [155]. Again we see that the additional source term in the Kerr-Schild approach corresponds to the radiative part of the field. Furthermore, in this form the standard double copy for scattering amplitudes is manifest: numerical factors agree between equations (5.32) and (5.37), such that the mass in the gravity theory is replaced with the colour charge in the gauge theory, as expected from the usual operation of the classical single copy [12, 43].

Let us summarise the results of this section. We have examined the particular case of a particle which undergoes a rapid change in direction, and confirmed that the additional source terms appearing in the Kerr-Schild description (in both gauge and gravity theory) are exactly given by known radiative scattering amplitudes. This directly links the classical double copy to the BCJ procedure for amplitudes.

It is interesting to compare the BCJ double copy for scattering amplitudes with the Kerr-Schild double copy, which has been formulated in position space. It is clear that momentum space is the natural home of the double copy. For scattering amplitudes, the amplitudes themselves and the double copy procedure are local in momentum space. In our Bremsstrahlung calculation, the numerical coefficients in the sources are also more natural after the Fourier transform. On the other hand, the currents $T^{\mu\nu}_{\text{KS}}$ and j^{ν}_{KS} are also local in position space. This unusual situation arises because the scattering amplitudes do not conserve momentum: in any Bremsstrahlung process, some momentum must be injected in order to bend the point particle trajectory. Of course, in the case of a static point particle locality in both position space and momentum space is more natural. This is reflected by the structure of the Fourier transform in the present case: as explained in appendix B, the factor $1/x \cdot u$ describing a particle worldline Fourier transforms to an integrated delta function $\int_0^\infty dm\, \delta^4(q - mu)$ (see equation (B.3)).

5.5 Gravitational energy conditions

In this section, we consider the null, weak and strong energy conditions of general relativity. These were recently examined in the context of the Kerr-Schild double copy in [44], where it was shown that extended charge distributions double copy to matter distributions that cannot simultaneously obey the weak and strong energy conditions, if there are no spacetime singularities or horizons. Although the point particle solution of interest to us has both singularities and horizons, it is still interesting to examine the energy conditions.

The null energy condition on a given energy-momentum tensor can be expressed by

$$T_{\mu\nu}\ell^{\mu}\ell^{\nu} \geq 0, \tag{5.40}$$

where ℓ^{μ} is any future-pointing null vector. The weak energy condition is similarly given by

$$T_{\mu\nu}t^{\mu}t^{\nu} \geq 0, \tag{5.41}$$

for any future-pointing timelike vector t^{μ}. The interpretation of this condition is that observers see a non-negative matter density. The null energy condition is implied by the weak energy condition (despite the names, the former is the weakest condition). One may also stipulate that the trace of the tidal tensor

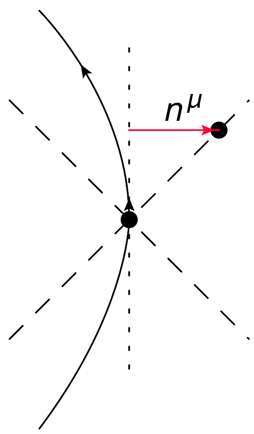


Figure 5.2 *Physical interpretation of $(k \cdot \dot{\lambda})$, where this denotes the component of acceleration in the direction n^μ.*

measured by such an observer is non-negative, which leads to the strong energy condition

$$T_{\mu\nu}t^\mu t^\nu \geq \frac{T}{2} g_{\mu\nu}t^\mu t^\nu, \quad T \equiv T^\alpha_\alpha. \tag{5.42}$$

Let us now examine whether these conditions are satisfied by the Kerr-Schild energy-momentum tensor of equation (5.13). First, the null property of the vector k^μ implies that the trace vanishes, so that the weak and strong energy conditions are equivalent. We may further unify these with the null energy condition, by noting that equation (5.13) implies

$$T^{\mu\nu}_{\text{KS}} V_\mu V_\nu = (k \cdot \dot{\lambda}) \left[\frac{3M(k \cdot V)^2}{4\pi r^2} \right], \tag{5.43}$$

for *any* vector V^μ. The quantity in the square brackets is positive definite, so that whether or not the energy conditions are satisfied is purely determined by the sign of $k \cdot \dot{\lambda}$. This scalar quantity is easily determined in the instantaneous rest-frame of the point particle; it is the negative of the component of acceleration in the direction n^μ of the observer (at the retarded time), see figure 5.2. Thus the energy conditions are not satisfied throughout the spacetime. In particular, any observer which sees the particle accelerating towards (away from) her will measure a negative (positive) energy density.

We remind the reader that the energy-momentum tensor is, in the case under study, an effective way of representing the full vacuum solution. The latter will have no issues with energy conditions. Analogously, the Liénard-Wiechert vacuum solution in gauge theory can be represented, as we have shown in section 5.3, by a boosted Coulomb field, together with a charged current encoding the radiation.

5.6 Discussion

In this chapter, we have extended the classical double copy of [12, 43] to consider accelerating, radiating point sources. This significantly develops previous results, which were based on stationary Kerr-Schild solutions, to a situation involving explicit time dependence. The structure of the double copy we have observed in the radiating case is precisely as one would expect. Passing from the gauge to the gravity theory, the overall scalar function ϕ is left intact; indeed it is the well-known scalar propagator in four dimensions. This is the same as the treatment of scalar propagators in the original BCJ double copy procedure for amplitudes. Similarly, the tensor structure of the gravitational field is obtained from the gauge field by replacing the vector k_μ by the symmetric, trace-free tensor $k_\mu k_\nu$. Finally, our use of Kerr-Schild coordinates in gravity linearised the Einstein tensor (with mixed indices). Reflecting this linearity, the associated single copy satisfies the linearised Yang-Mills equations.

It is worth dwelling a little on the physical implication of our work. The classical double copy is known to relate point sources in gauge theory to point sources in general relativity, in accordance with intuition arising from scattering amplitudes. In this chapter, we have simply considered the case where the point sources move on a specified, arbitrarily accelerated, timelike worldline. On general grounds we expect radiation to be emitted due to the acceleration. Our use of Kerr-Schild coordinates organised the radiation into sources appearing on the right-hand side of the field equations: a current density in gauge theory, and a stress-energy tensor in gravity. Intriguingly, we found that the expressions for these sources also have a double copy structure: one passes from the gauge current to the gravitational stress-energy tensor by replacing k_μ by $k_\mu k_\nu$ while leaving a scalar factor intact, up to numerical factors which are canonical in momentum space. Since these sources encode the complete radiation fields for the accelerating charge and point mass, there is a double copy between the radiation generated by these two systems. This double copy is a property of the exact solution of gauge theory and general relativity.

We further extracted one simple perturbative scattering amplitude from this radiation field, namely the Bremsstrahlung scattering amplitude. The double copy property was maintained as we extracted the scattering amplitude, which firmly establishes a link between the double copy for amplitudes and the double copy for classical solutions.

However, we should emphasise one unphysical aspect of our setup. We mandated a worldline for our point particle in both gauge theory and general relativity. In gauge theory, this is fine: one can imagine that an external force acts on the particle causing its worldline to bend. However, in general relativity such an external force would contribute to the stress-energy tensor in the spacetime. Since we ignored this component of the stress-energy tensor, our calculation is not completely physical. Instead, one should regard the point particle in both cases as a specified boundary condition, rather than as a physical particle. We have therefore seen that the radiation generated by this boundary condition enjoys a precise double copy.

There are a number of possible extensions of our results. One may look at time-dependent extended sources in the Kerr-Schild description, for example, or particles accelerating in non-Minkowski backgrounds (for preliminary work in the stationary case, see [43]). It would also be interesting to examine whether a double copy procedure can be set up in other coordinate systems, such as the more conventional de Donder gauge. One particularly important issue is to understand the generalisation of the colour-dual requirement on kinematic numerators to classical field backgrounds. The Jacobi relations satisfied by colour-dual numerators hint at the existence of a kinematic algebra [47, 143] underlying the connection between gauge theory and gravity; revealing the full detail of this structure would clearly be an important breakthrough. The study of the classical double copy is in its infancy, and many interesting avenues have yet to be explored. In the next chapter we will focus on extending our understanding of the classical double copy to arbitrary gauge. We will use the de Donder gauge, taking into account that the use of a gauge with trace requires the use of projectors to remove the dilaton field.

Chapter 6

Perturbative spacetimes from the double copy

6.1 Introduction

The existence of the double copy hints at a profound relationship between gauge and gravity theories, that should transcend perturbative amplitudes. To this end, the previous chapter discussed work done in [2, 12, 43, 44] which generalised the notion of the double copy to exact classical solutions. That is, a large family of gravitational solutions was found that could be meaningfully associated with a gauge theory solution, such that the relationship between them was consistent with the BCJ double copy. As we described in chapter 5, these solutions all had the special property that they linearised the Einstein and Yang-Mills equations, so that the graviton and gauge field terminate at first order in the coupling constant, with no higher-order corrections. A special choice of coordinates (*Kerr-Schild coordinates*) had to be chosen in the gravity theory, reminiscent of the fact that the amplitude double copy is not manifest in all gauge choices. An alternative approach exists in a wide variety of linearised supersymmetric theories which consists of writing the graviton as a direct convolution of gauge fields [49, 50, 52, 156–158]. This in principle works for general gauge choices, but it is not yet clear how to generalise this prescription to include non-linear effects. One may also consider whether the double copy can be generalised to intrinsically non-perturbative solutions, and first steps have been taken in [159].

As is hopefully clear from the above discussion, it is not yet known how to formulate the double copy for arbitrary field solutions, and in particular for those which are non-linear. However, such a procedure would have highly useful applications. Firstly, the calculation of metric perturbations in classical general relativity is crucial for a plethora of astrophysical applications, but is often cumbersome. A non-linear double copy would allow one to calculate gauge fields relatively simply, before porting the results to gravity. Secondly, hints were provided in [43] that the double copy may work in a non-Minkowski spacetime. This opens up the possibility to obtain new insights (and possible calculational techniques) in cosmology.

The aim of this chapter is to demonstrate explicitly how the BCJ double copy can be used to generate non-linear gravitational solutions order-by-order in perturbation theory[1], from simpler gauge theory counterparts. This is similar in spirit to work done in [160–162], which extracted both classical and quantum gravitational corrections from amplitudes obtained from gauge theory ingredients; and to [146, 163], which used tree-level amplitudes to construct perturbatively the Schwarzschild spacetime. Recently, the double copy procedure has been studied in [51] for classical radiation emitted by multiple point charges. Here we take a more direct approach, namely to calculate the graviton field generated by a given source, rather than extracting this from a scattering amplitude. Another recent work proposes applications to cosmological gravitational waves, pointing out a double copy of radiation memory [164].

As will be explained in detail in what follows, our scheme involves solving the Yang-Mills equations for a given source order-by-order in the coupling constant. We then copy this solution by duplicating kinematic numerators, before identifying a certain product of gauge fields with a two-index field $H_{\mu\nu}$, motivated by [95]. This field contains degrees of freedom associated with a conventional graviton $h_{\mu\nu}$, together with a scalar field ϕ and two-form field $B_{\mu\nu}$. For convenience, we will refer to $H_{\mu\nu}$ as the *fat graviton*, and the physical field $h_{\mu\nu}$ as the *skinny graviton*. As we will see, the skinny fields $h_{\mu\nu}$, $B_{\mu\nu}$ and ϕ can be obtained from knowledge of $H_{\mu\nu}$, though this extraction requires knowledge of a certain gauge transformation and field redefinition in general.

[1] This is the post-Minkowskian expansion, as opposed to the post-Newtonian expansion where the non-relativistic limit is also taken.

The structure of this chapter is as follows. In section 6.1.1, we briefly review the BCJ double copy. In section 6.2, we work at leading order in perturbation theory, and outline our procedure for obtaining gravity solutions from Yang-Mills fields. In section 6.3, we work to first and second subleading order in perturbation theory, thus explicitly demonstrating how non-linear solutions can be generated in our approach. Finally, we discuss our results and conclude in section 6.4. The chapter is based on work done in collaboration with Andrés Luna, Ricardo Monteiro, Alexander Ochirov, Donal O'Connell, Chris White and Niklas Westerberg which was published in [3].

6.1.1 Conventions

As we reviewed in chapter 4, given a gauge theory amplitude in BCJ-dual form, the double copy prescription states that

$$\mathcal{M}_m = i\left(\frac{\kappa}{2}\right)^{m-2}\sum_{i\in\Gamma}\frac{n_i\,\tilde{n}_i}{\prod_{\alpha_i}p_{\alpha_i}^2} \tag{6.1}$$

is an m-point gravity amplitude, where

$$g_{\mu\nu} = \eta_{\mu\nu} + \kappa h_{\mu\nu} \tag{6.2}$$

can be chosen to define the graviton field, and $\kappa = \sqrt{32\pi G}$ is the gravitational coupling constant.[2] This result is obtained from equation (4.8) by replacing the gauge theory coupling constant with its gravitational counterpart, and colour factors with a second set of kinematic numerators $\tilde{n}_i$. Therefore, the procedure modifies the numerators of amplitudes term by term, but leaves the denominators in equations (4.8, 6.1) intact. As we discuss in chapter 5, a similar phenomenon occurs in the double copy for exact classical solutions of [2, 12, 43], in which scalar propagators play a crucial role.

The gravity theory associated with the scattering amplitudes (6.1) depends on the two gauge theories from which the numerators $\{n_i\}$, $\{\tilde{n}_i\}$ are taken. In this chapter, both will be taken from pure Yang-Mills theory, which is mapped by the double copy to "$\mathcal{N} = 0$ supergravity". This theory is defined as Einstein gravity

[2] We work in the mostly plus metric convention.

coupled to a scalar field ϕ (known as the dilaton) and a two-form $B_{\mu\nu}$ (known as the Kalb-Ramond field, which can be replaced by an axion in four spacetime dimensions). The action for these fields is

$$S = \int d^D x \sqrt{-g} \left[\frac{2}{\kappa^2} R - \frac{1}{2(D-2)} \partial^\mu \phi \partial_\mu \phi - \frac{1}{6} e^{-2\kappa\phi/D-2} H^{\lambda\mu\nu} H_{\lambda\mu\nu} \right], \tag{6.3}$$

where $H_{\lambda\mu\nu}$ is the field strength of $B_{\mu\nu}$. In the following, we will study perturbative solutions of this theory around Minkowski space. The starting point is to consider linearised fields, for which the equations of motion are

$$\begin{aligned} &\partial^2 h_{\mu\nu} - \partial_\mu \partial^\rho h_{\rho\nu} - \partial_\nu \partial^\rho h_{\rho\mu} + \partial_\mu \partial_\nu h + \eta_{\mu\nu} \left[\partial^\rho \partial^\sigma h_{\rho\sigma} - \partial^2 h \right] = 0, \\ &\partial^2 B_{\mu\nu} - \partial_\mu \partial^\rho B_{\rho\nu} + \partial_\nu \partial^\rho B_{\rho\mu} = 0, \\ &\quad \partial^2 \phi = 0. \end{aligned} \tag{6.4}$$

Instead of the straightforward graviton field $h_{\mu\nu}$ defined by (6.2), we will often work with the "gothic" metric perturbation $\mathfrak{h}^{\mu\nu}$ such that

$$\sqrt{-g}\, g^{\mu\nu} = \eta^{\mu\nu} - \kappa\, \mathfrak{h}^{\mu\nu}, \tag{6.5}$$

as it is common in perturbation theory [165]. In terms of this gothic graviton field, the de Donder gauge condition is simply $\partial_\mu \mathfrak{h}^{\mu\nu} = 0$ to all orders. At the linear order, the two metric perturbations are simply related:

$$\mathfrak{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h, \tag{6.6}$$

and the linear gauge transformation generated by $x^\mu \to x^\mu - \kappa\, \xi^\mu$ is

$$\mathfrak{h}_{\mu\nu} \;\to\; \mathfrak{h}'_{\mu\nu} = \mathfrak{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi. \tag{6.7}$$

This transformation is more convenient in what follows than the standard gauge transformation for $h_{\mu\nu}$ (where the last term is missing). Finally, the linearised equation of motion is

$$\partial^2 \mathfrak{h}_{\mu\nu} - \partial_\mu \partial^\rho \mathfrak{h}_{\rho\nu} - \partial_\nu \partial^\rho \mathfrak{h}_{\rho\mu} + \eta_{\mu\nu} \partial^\rho \partial^\sigma \mathfrak{h}_{\rho\sigma} = 0. \tag{6.8}$$

In de Donder gauge, we have simply $\partial^2 \mathfrak{h}_{\mu\nu} = 0$.

6.2 Linear gravitons from Yang-Mills fields

Our goal is to rewrite gravitational perturbation theory in terms of the fat graviton $H_{\mu\nu}$, rather than more standard perturbative fields such as $\{\mathfrak{h}_{\mu\nu}, B_{\mu\nu}, \phi\}$. The idea is that the fat graviton is the field whose interactions are directly dictated by the double copy from gauge theory. In this section, we will discuss in some detail the mapping between the skinny fields and the fat graviton at the linearised level. Indeed, we will see that there is an invertible map, so that the fat graviton may be constructed from skinny fields $H_{\mu\nu} = H_{\mu\nu}(\mathfrak{h}_{\alpha\beta}, B_{\alpha\beta}, \phi)$, but also the skinny fields can be determined from the fat field, $\mathfrak{h}_{\mu\nu} = \mathfrak{h}_{\mu\nu}(H_{\alpha\beta}), B_{\mu\nu} = B_{\mu\nu}(H_{\alpha\beta}), \phi = \phi(H_{\alpha\beta})$. We will determine the relations between the fields beginning with the simplest case: linearised waves.

6.2.1 Linear waves

As a prelude to obtaining non-linear gravitational solutions from Yang-Mills theory, we first discuss linear solutions of both theories. The simplest possible solutions are linear waves. These are well-known to double copy between gauge and gravity theories (see e.g. [166]). This property is crucial for the double copy description of scattering amplitudes, whose incoming and outgoing states are plane waves. Here, we use linear waves to motivate a prescribed relationship between fat and skinny fields, which will be generalised in later sections.

Let us start by considering a gravitational plane wave in the de Donder gauge. The free equation of motion for the graviton is simply $\partial^2 \mathfrak{h}_{\mu\nu} = 0$. Plane wave solutions take the form

$$\mathfrak{h}_{\mu\nu} = a_{\mu\nu} e^{ip\cdot x}, \qquad p^\mu a_{\mu\nu} = 0, \qquad p^2 = 0, \tag{6.9}$$

where $a_{\mu\nu}$ is a constant tensor, and the last condition follows from the equation of motion. Symmetry of the graviton implies $a_{\mu\nu} = a_{\nu\mu}$, and one may also fix a residual gauge freedom by setting $a \equiv a^\mu_\mu = 0$, so that $\mathfrak{h}_{\mu\nu}$ becomes a traceless, symmetric matrix. It is useful to further characterise the matrix $a_{\mu\nu}$ as we did in chapter 1 by introducing a set of $(D-2)$ polarisation vectors ε^i_μ satisfying the orthogonality conditions

$$p \cdot \varepsilon^i = 0, \qquad q \cdot \varepsilon^i = 0, \tag{6.10}$$

where q^μ ($q^2 = 0$, $p \cdot q \neq 0$) is an auxiliary null vector used to project out physical degrees of freedom for an on-shell massless vector boson. These polarisation vectors are a complete set, so they satisfy a completeness relation

$$\varepsilon^i_\mu \varepsilon^i_\nu = \eta_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{p \cdot q}. \tag{6.11}$$

Then the equation of motion for $\mathfrak{h}_{\mu\nu}$, together with the symmetry and gauge conditions on $a_{\mu\nu}$, imply that one may write

$$a_{\mu\nu} = f^{t\!\!/}_{ij} \varepsilon^i_\mu \varepsilon^j_\nu, \tag{6.12}$$

where $f^{t\!\!/}_{ij}$ is a traceless symmetric matrix. Thus, the linearised gravitational waves have polarisation states which can be constructed from outer products of vector waves, times traceless symmetric matrices.

Similarly, one may consider linear plane wave solutions for a two-form and ϕ field. Imposing Lorenz gauge $\partial^\mu B_{\mu\nu} = 0$ for the antisymmetric tensor, its free equation of motion becomes simply $\partial^2 B_{\mu\nu} = 0$. Thus plane wave solutions are

$$B_{\mu\nu} = \tilde{f}_{ij} \varepsilon^i_\mu \varepsilon^j_\nu e^{ip\cdot x}, \tag{6.13}$$

where $\tilde{f}_{ij}$ is a constant antisymmetric matrix. Meanwhile the free equation of motion for the scalar field is $\partial^2 \phi = 0$, with plane wave solution

$$\phi = f_\phi e^{ip\cdot x}. \tag{6.14}$$

The double copy associates these skinny waves with a single fat graviton field $H_{\mu\nu}$ satisfying the field equation $\partial^2 H_{\mu\nu} = 0$,

$$H_{\mu\nu} = f_{ij} \varepsilon^i_\mu \varepsilon^j_\nu e^{ip\cdot x}, \tag{6.15}$$

where now f_{ij} is a general $D-2$ matrix and we have chosen a gauge condition

$\partial^\mu H_{\mu\nu} = 0 = \partial^\mu H_{\nu\mu}$. One may write this decomposition as

$$H_{\mu\nu} = \left(f^{t}_{ij} + \tilde{f}_{ij} + \delta_{ij} \frac{f_\phi}{D-2} \right) \varepsilon^i_\mu \varepsilon^j_\nu e^{ip\cdot x}, \tag{6.16}$$

$$= \mathfrak{h}_{\mu\nu} + B_{\mu\nu} + \left(\eta_{\mu\nu} - \frac{p_\mu q_\nu + p_\nu q_\mu}{p \cdot q} \right) \frac{\phi}{D-2}, \tag{6.17}$$

which explicitly constructs the fat graviton from skinny fields. Working in position space for constant q, this becomes

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}\phi, \tag{6.18}$$

where we have defined the projection operator

$$P^q_{\mu\nu} = \frac{1}{D-2} \left(\eta_{\mu\nu} - \frac{q_\mu \partial_\nu + q_\nu \partial_\mu}{q \cdot \partial} \right), \tag{6.19}$$

which will be important throughout this article.[3]

Our goal in this work is not to construct fat gravitons from skinny fields, but on the contrary to determine skinny fields using a perturbative expansion based on the double copy and the fat graviton. Therefore it is important that we can determine the skinny fields given knowledge of the fat graviton. To that end, recall that we have been able to choose a gauge so that the trace, $\mathfrak{h}$, of the metric perturbation vanishes. Therefore the trace of the fat graviton determines the dilaton:

$$\phi = H^\mu{}_\mu \equiv H. \tag{6.20}$$

We may now use symmetry to determine the skinny graviton and antisymmetric tensor from the fat graviton:

$$B_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} - H_{\nu\mu} \right), \tag{6.21}$$

$$\mathfrak{h}_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu} \right) - P^q_{\mu\nu} H. \tag{6.22}$$

The basic strategy of this construction is simple: we have decomposed the matrix field $H_{\mu\nu}$ into its antisymmetric, traceless symmetric, and trace parts.

[3]Notice that $\hat{P}^q_{\mu\nu} = (D-2)P^q_{\mu\nu}$ is the properly normalised projection operator, such that $\hat{P}^{q}_{\mu}{}^{\lambda} \hat{P}^{q}_{\lambda}{}^{\nu} = \hat{P}^{q}_{\mu}{}^{\nu}$, and $\hat{P}^{q}_{\mu}{}^{\mu} = D-2$.

It is worth dwelling on the decomposition of the fat graviton into skinny fields a little further. Having constructed $\mathfrak{h}_{\mu\nu}$ from the fat graviton, we are free to consider a gauge transformation of the skinny graviton:

$$\begin{aligned} \mathfrak{h}'_{\mu\nu} &= \mathfrak{h}_{\mu\nu} + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi & (6.23) \\ &= \frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right) - \frac{1}{D-2}\left(\eta_{\mu\nu} - \frac{q_\mu \partial_\nu + q_\nu \partial_\mu}{q \cdot \partial}\right) H + \partial_\mu \xi_\nu + \partial_\nu \xi_\mu - \eta_{\mu\nu} \partial \cdot \xi. & (6.24) \end{aligned}$$

If we choose

$$\xi_\mu = -\frac{1}{D-2}\left(\frac{q_\mu}{q \cdot \partial}\right) H, \tag{6.25}$$

then we find that the expression for the $\mathfrak{h}'_{\mu\nu}$ simplifies to

$$\mathfrak{h}'_{\mu\nu} = \frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right). \tag{6.26}$$

Thus, up to a gauge transformation, the skinny graviton is the symmetric part of the fat graviton. It may be worth emphasising that ϕ and $B_{\mu\nu}$ also transform under this gauge transformation, which is, of course, a particular diffeomorphism. However, the transformation of ϕ and $B_{\mu\nu}$ is suppressed by a power of κ, and so we may take them to be gauge invariant for diffeomorphisms at this order.

We will see below that the perturbative expansion for fat gravitons is much simpler than the perturbative expansion for the individual skinny fields. But before we embark on that story, it is important to expand our understanding of the relationship between the fat graviton and the skinny fields beyond the sole case of plane waves.

6.2.2 General linearised vacuum solutions

For plane waves, the fat graviton is given in terms of skinny fields in equation (6.18), and at first glance this equation is not surprising: one may always choose to decompose an arbitrary rank two tensor into its symmetric traceless, antisymmetric and trace parts. However, equation (6.18) contains non-trivial physical content, namely that the various terms on the RHS are the genuine propagating degrees of freedom associated with each of the skinny fields. The auxiliary vector q_μ plays a crucial role here: it is associated in the gauge theory with the definition of physical polarisation vectors, and thus can be used to project out physical degrees of freedom in the gravity theory. One may then ask whether equation (6.18) generalises for arbitrary solutions of the linearised equations of motion. There is potentially a problem in that the relationship becomes ambiguous: the trace of the skinny graviton may be non-zero (as is indeed the case in general gauges), and one must then resolve how the trace degree of freedom in $H^{\mu\nu}$ enters the trace of the skinny graviton, and the scalar field ϕ. Furthermore, it is not immediately clear that equation (6.18) (derived for plane waves) will work when non-zero sources are present in the field equations. In order to use the double copy in physically relevant applications, we must consider this possibility.

Here we will restrict ourselves to skinny gravitons that are in de Donder gauge. However, we will relax the traceless condition on the skinny graviton which was natural in the previous section. To account for the trace, we postulate that equation (6.18) should be replaced by

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}(\phi - \mathfrak{h}). \tag{6.27}$$

To be useful, this definition of the fat graviton must be invertible. First, note that the trace of $H_{\mu\nu}$ determines ϕ as before, while the antisymmetric part of $H_{\mu\nu}$ determines $B_{\mu\nu}$. Finally, the traceless symmetric part of the fat graviton is

$$\frac{1}{2}\left(H_{\mu\nu} + H_{\nu\mu}\right) - P^q_{\mu\nu}H = \mathfrak{h}_{\mu\nu}(x) - P^q_{\mu\nu}\mathfrak{h} = \mathfrak{h}'_{\mu\nu}(x), \tag{6.28}$$

where $\mathfrak{h}'_{\mu\nu}(x)$ is a gauge transformation of $\mathfrak{h}_{\mu\nu}(x)$. In practice, we find it useful to work with $\mathfrak{h}_{\mu\nu}(x)$ rather than $\mathfrak{h}'_{\mu\nu}(x)$, because at higher orders the gauge transformation to $\mathfrak{h}'_{\mu\nu}(x)$ leads to more cumbersome formulae. It is also worth noticing that both $\mathfrak{h}_{\mu\nu}$ and $\mathfrak{h}'_{\mu\nu}$ are in de Donder gauge, since

$$\partial^\mu P^q_{\mu\nu}\mathfrak{h} = \frac{1}{D-2}\left(\partial_\nu - \frac{q_\nu \partial^2 + q\cdot\partial\,\partial_\nu}{q\cdot\partial}\right)\mathfrak{h} = -\frac{1}{D-2}\frac{q_\nu}{q\cdot\partial}\partial^2\mathfrak{h} = 0. \tag{6.29}$$

Our relationship between skinny and fat fields still holds only for linearised fields; we will explicitly find corrections to equation (6.27) at higher orders in perturbation theory in section 6.3. Before doing so, however, it is instructive to illustrate the above general discussion with some specific solutions of the linear field equations, showing how the fat and skinny fields are mutually related.

6.2.3 The linear fat graviton for Schwarzschild

One aim of our programme is to be able to describe scattering processes involving black holes. To this end, let us see how to extend the above results in the presence of point-like masses. It is easy to construct a fat graviton for the linearised Schwarzschild metric: we begin by noticing that, in the case of Schwarzschild ($D = 4$), we have

$$\mathfrak{h}_{\mu\nu}(r) = \frac{\kappa}{2}\frac{M}{4\pi r}u_\mu u_\nu + \mathcal{O}(\kappa^2), \quad B_{\mu\nu}(x) = 0, \quad \phi(x) = 0, \quad u_\mu = (1,0,0,0). \tag{6.30}$$

The fat graviton depends on an arbitrary constant null vector q^μ. In this section, for illustration, we will make an explicit choice of $q^\mu = (1,0,0,1)$ and evaluate the action of the projector (6.19) in position space in full. A computation gives

$$H_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}u_\mu u_\mu + P^q_{\mu\nu}\left(\frac{\kappa}{2}\frac{M}{4\pi r}\right) \tag{6.31}$$

$$= \frac{\kappa}{2}\frac{M}{4\pi r}\left(u_\mu u_\nu + \frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right), \tag{6.32}$$

where $l_\mu = -(0,x,y,r+z)/(r+z)$, such that $q \cdot l = 1$. It is easy to check that $\partial^\mu H_{\mu\nu} = 0$, $\partial^2 H_{\mu\nu} = 0$.

Going in the other direction, it is easy to compute the skinny fields given this fat graviton. Since $H_{\mu\nu}$ is traceless, the dilaton vanishes. Similarly $H_{\mu\nu}$ is symmetric, and therefore $B_{\mu\nu} = 0$. The skinny graviton can therefore be taken to be equal to the fat graviton. While this result seems to be at odds with (6.30), recall that they differ only by a gauge transformation (which leaves ϕ and $B_{\mu\nu}$ unaffected at this order) and that the skinny graviton we recover is traceless, as we would expect from equation (6.28).

It may not seem that we have gained much by passing to equation (6.32) from equation (6.30). However, it is our contention that it is simpler to compute perturbative corrections to metrics using the formalism of the fat graviton than with the traditional approach. We will illustrate this in a specific example later in this chapter.

6.2.4 Solutions with linearised dilatons

The linearised Schwarzschild metric corresponds to a somewhat complicated fat graviton. Since the fat graviton's equation of motion is simply $\partial^2 H_{\mu\nu} = 0$, it is natural to consider the solution

$$H_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r} u_\mu u_\nu, \qquad \text{with } \; u_\mu = (1,0,0,0), \tag{6.33}$$

which corresponds to inserting a singularity at the origin. We will see that this solution has the physical interpretation of a point mass which is also a source for the scalar dilaton. Indeed, the dilaton contained in the fat graviton is given by its trace:

$$\phi = -\frac{\kappa}{2}\frac{M}{4\pi r}. \tag{6.34}$$

Since the fat graviton is symmetric, $B_{\mu\nu} = 0$. Meanwhile the skinny graviton is

$$\mathfrak{h}_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}\left(u_\mu u_\nu + \frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right). \tag{6.35}$$

Again, a linearised diffeomorphism can give the skinny graviton the same form as the fat graviton.

It is natural to ask what is the non-perturbative static spherically-symmetric solution for which we are finding the linearised fields. Exact solutions of the Einstein equations minimally coupled to a scalar field of this form were discussed by Janis, Newman and Winicour (JNW) [167] and have been extensively studied in the literature [167–173]. The complete solution is, in fact, a naked singularity, consistent with the no-hair theorem. The general JNW metric and dilaton can be expressed as

$$ds^2 = -\left(1 - \frac{\rho_0}{\rho}\right)^{\gamma} dt^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{-\gamma} d\rho^2 + \left(1 - \frac{\rho_0}{\rho}\right)^{1-\gamma} \rho^2 d\Omega^2, \tag{6.36}$$

$$\phi = \frac{\kappa}{2}\frac{Y}{4\pi\rho_0}\log\left(1 - \frac{\rho_0}{\rho}\right), \tag{6.37}$$

where the two parameters ρ_0 and γ can be given in terms of the mass M and the scalar coupling Y as

$$\rho_0 = 2G\sqrt{M^2 + Y^2} = \left(\frac{\kappa}{2}\right)^2 \frac{\sqrt{M^2+Y^2}}{4\pi}, \qquad \gamma = \frac{M}{\sqrt{M^2+Y^2}}. \tag{6.38}$$

For $Y = 0$ and $M > 0$, we recover the Schwarzschild black hole, with the event horizon at $\rho = \rho_0$. For $|Y| > 0$ and $M > 0$, the solution also decays for large ρ, but there is a naked singularity at $\rho = \rho_0$, which now corresponds to zero radius (since the metric factor in front of $d\Omega^2$ vanishes) [167]. We can write the JNW solution in de Donder gauge by applying the coordinate transformation $\rho = r + \rho_0/2$, where r is the Cartesian radius in the de Donder coordinates. Expanding in κ, the result is

$$\mathfrak{h}_{\mu\nu} = \frac{\kappa}{2}\frac{M}{4\pi r}u_\mu u_\nu + \left(\frac{\kappa}{2}\right)^3 \frac{1}{8(4\pi r)^2}\big((7M^2 - Y^2)u_\mu u_\nu + (M^2 + Y^2)\hat{r}_\mu \hat{r}_\nu\big) + \mathcal{O}(\kappa^5), \tag{6.39}$$

$$\phi = -\frac{\kappa}{2}\frac{Y}{4\pi r} + \mathcal{O}(\kappa^5), \tag{6.40}$$

with $\hat{r}^\mu = (0, \mathbf{x}/r)$. Despite its somewhat esoteric nature, this naked singularity is a particularly natural object from the point of view of the perturbative double copy. At large distances from the singularity, both the metric perturbation and the scalar field fall off as $1/r$, and for $Y = M$ this leading part reproduces the skinny fields obtained above, up to a linearised diffeomorphism in $\mathfrak{h}_{\mu\nu}$. In Section 6.3, we will discuss the first two non-linear corrections to the JNW metric using fat gravitons, and, in the case of the first correction, we will match the expansion above. We conclude that the JNW solution with $Y = M$ is the exact solution associated to the linearised fat graviton (6.33).

We can also ask what fat graviton would be associated to the general JNW family of solutions, with M and Y generic. Since we are dealing with linearised fields, we can superpose contributions, and so we arrive at

$$H_{\mu\nu} = \frac{\kappa}{2}\frac{1}{4\pi r}\left(M\, u_\mu u_\nu + (M - Y)\,\frac{1}{2}(\eta_{\mu\nu} - q_\mu l_\nu - q_\nu l_\mu)\right). \tag{6.41}$$

The gauge theory "single copy" associated to this field is simply the Coulomb solution, which presents an apparent puzzle: it is argued in [12] that the double copy of the Coulomb solution is a pure Schwarzschild black hole, with no dilaton field. Above, however, the double copy produces a JNW solution. The latter was also found in [51], which thus concluded that the Schwarzschild solution is not obtained by the double copy, but can only be true in certain limits (such as the limit of an infinite number of dimensions). The resolution of this apparent contradiction is that one can choose whether or not the dilaton is sourced upon taking the double copy. It is well-known in amplitude calculations, for example, that gluon amplitudes can double copy to arbitrary combinations of amplitudes for gravitons, dilatons and/or B-fields. A simple example is amplitudes for linearly polarised gauge bosons: the double copied "amplitude" involves mixed waves of gravitons and dilatons. Thus, the result in the gravity theory depends on the linear combinations of the pairs of gluon polarisations involved in the double copy. Here, we may say that the Schwarzschild solution is a double copy of the Coulomb potential, as given by the Kerr-Schild double copy [12], just as one may say that appropriate combinations of amplitudes of gluons lead to amplitudes of pure gravitons. The analogue of more general gravity amplitudes with both gravitons and dilatons, obtained via the double copy, is the JNW solution. Therefore the double copy of the Coulomb solution is somewhat ambiguous: in fact, it is any member of the JNW family of singularities, including the Schwarzschild metric. Note that the Kerr-Schild double copy is applicable only in the Schwarzschild special case since the other members of the JNW family of spacetimes do not admit Kerr-Schild coordinates.

For the vacuum Kerr-Schild solutions studied in [12], in particular for the Schwarzschild black hole, it was possible to give an exact map between the gauge theory solution and the exact graviton field, making use of Kerr-Schild coordinates (as opposed to the de Donder gauge used here). For the general JNW solution, the double copy correspondence was inferred above from the symmetries of the problem and from the perturbative results. A more general double copy map would also be able to deal with the exact JNW solution. This remains an important goal, and is addressed in appendix C.

6.3 Perturbative Corrections

Now that we have understood how to construct fat gravitons in several cases, let us finally put them to use. In this section, we will construct non-linear perturbative correction to spacetime metrics and/or dilatons using the double copy. Thus, we will map the problem of finding perturbative corrections to a simple calculation in gauge theory.

6.3.1 Perturbative metrics from gauge theory

Since the basis of our calculations is the perturbative expansion of gauge theory, we begin with the vacuum Yang-Mills equation

$$\partial^\mu F^a_{\mu\nu} + g f^{abc} A^{b\mu} F^c_{\mu\nu} = 0, \tag{6.42}$$

where g is the coupling constant, while the field strength tensor is

$$F^a_{\mu\nu} = \partial_\mu A^a_\nu - \partial_\nu A^a_\mu + g f^{abc} A^b_\mu A^c_\nu. \tag{6.43}$$

We are interested in a perturbative solution of these equations, so that the gauge field A^a_μ can be written as a power series in the coupling:

$$A^a_\mu = A^{(0)a}_\mu + g A^{(1)a}_\mu + g^2 A^{(2)a}_\mu + \cdots . \tag{6.44}$$

In this expansion, the perturbative coefficients $A^{(i)a}_\mu$ are assumed to have no dependence on the coupling g. We use a similar notation for the perturbation series for the skinny and fat gravitons:

$$\mathfrak{h}^{\mu\nu} = \mathfrak{h}^{(0)\mu\nu} + \frac{\kappa}{2}\mathfrak{h}^{(1)\mu\nu} + \left(\frac{\kappa}{2}\right)^2 \mathfrak{h}^{(2)\mu\nu} + \cdots , \tag{6.45}$$

$$H^{\mu\nu} = H^{(0)\mu\nu} + \frac{\kappa}{2}H^{(1)\mu\nu} + \left(\frac{\kappa}{2}\right)^2 H^{(2)\mu\nu} + \cdots . \tag{6.46}$$

We can construct solutions in perturbation theory in a straightforward manner. To zeroth order in the coupling, the Yang-Mills equation in Lorenz gauge $\partial^\mu A^a_\mu = 0$ is simply

$$\partial^2 A^{(0)a}_\mu = 0. \tag{6.47}$$

For our present purposes, two basic solutions of this equation will be of interest: wave solutions, and Coulomb-like solutions with isolated singularities.

Given a solution $A^{(0)a}_\mu$ of the linearised Yang-Mills equation, it is easy to write down an expression for the first order correction $A^{(1)a}_\mu$ by expanding the Yang-Mills equation to first order in g:

$$\partial^2 A^{(1)a}_\nu = -2 f^{abc} A^{(0)b\mu} \partial_\mu A^{(0)c}_\nu + f^{abc} A^{(0)b\mu} \partial_\nu A^{(0)c}_\mu. \tag{6.48}$$

The double copy is most easily understood in Fourier (momentum) space. To simplify our notation, we define

$$\int đ^D p F(p) \equiv \int \frac{d^D p}{(2\pi)^D} F(p), \qquad \delta\!\!\!{}^-{}^D(p) \equiv (2\pi)^D \delta^{(D)}(p). \tag{6.49}$$

Using this notation, we may write the solution for the first perturbative correction in Fourier space in the familiar form

$$\begin{aligned} A^{(1)a\mu}(-p_1) = \frac{i}{2p_1^2} f^{abc} \int đ^D p_2 đ^D p_3 \delta\!\!\!{}^-{}^D(p_1+p_2+p_3) \\ \times \left[(p_1-p_2)^\gamma \eta^{\mu\beta} + (p_2-p_3)^\mu \eta^{\beta\gamma} + (p_3-p_1)^\beta \eta^{\gamma\mu}\right] A^{(0)b}_\beta(p_2) A^{(0)c}_\gamma(p_3). \end{aligned} \tag{6.50}$$

Notice that the factor in square brackets in this equation obeys the same algebraic symmetries as the colour factor, f^{abc}, appearing in the equation. This is a requirement of colour-kinematics duality. Before using the double copy, it is necessary to ensure that this duality holds.

The power of the double copy is that it is now completely trivial to compute the perturbative correction $H^{(1)}_{\mu\nu}$ to a linearised fat graviton $H^{(0)}_{\mu\nu}$. All we need to do, following [10, 42, 95], is to square the numerator in equation (6.50), ignore the colour structure, and assemble fat gravitons by the rule that $A^{(0)a}_\mu(p) A^{(0)b}_\nu(p) \to H^{(0)}_{\mu\nu}(p)$. This straightforward procedure leads to

$$\begin{aligned} H^{(1)\mu\mu'}(-p_1) = \frac{1}{4p_1^2} \int đ^D p_2 đ^D p_3 \delta\!\!\!{}^-{}^D(p_1+p_2+p_3) \\ \times \left[(p_1-p_2)^\gamma \eta^{\mu\beta} + (p_2-p_3)^\mu \eta^{\beta\gamma} + (p_3-p_1)^\beta \eta^{\gamma\mu}\right] \\ \times \left[(p_1-p_2)^{\gamma'} \eta^{\mu'\beta'} + (p_2-p_3)^{\mu'} \eta^{\beta'\gamma'} + (p_3-p_1)^{\beta'} \eta^{\gamma'\mu'}\right] H^{(0)}_{\beta\beta'}(p_2) H^{(0)}_{\gamma\gamma'}(p_3). \end{aligned} \tag{6.51}$$

Notice that the basic structure of the perturbative calculation is that of gauge theory. The double copy upgrades the gauge-theoretic perturbation into a calculation appropriate for gravity, coupled to a dilaton and an antisymmetric tensor.

As a simple example of this formalism at work, let us compute the first order correction to the simple fat graviton equation (6.33) corresponding to a metric and scalar field. To begin, we need to write $H^{(0)}_{\mu\nu}(p)$ in momentum space; it is simply

$$H^{(0)\mu\nu}(p) = \frac{\kappa}{2} M u^\mu u^\nu \frac{\delta\!\!\!{}^-{}^1(p^0)}{p^2}. \tag{6.52}$$

Inserting this into our expression for $H^{(1)}$, equation (6.51), we quickly find

$$H^{(1)\mu\mu'}(-p_1) = \left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4p_1^2} \int d\!\!{}^-{}^3 p_2 d\!\!{}^-{}^3 p_3 \delta\!\!{}^-{}^4(p_1+p_2+p_3) \frac{(p_2-p_3)^\mu (p_2-p_3)^{\mu'}}{p_2^2\, p_3^2}, \tag{6.53}$$

where $p_2^0 = 0 = p_3^0$, and consequently $p_1^0 = 0$. For future use, we note that $p_{1\mu} H^{(1)\mu\mu'}(-p_1) = 0$. Since all of the components of $H^{(1)}$ in the time direction vanish, we need only calculate the spatial components $H^{(1)ij}$. To do so, it is convenient to Fourier transform back to position space and compute firstly the Laplacian of $\nabla^2 H^{(1)ij}(x)$; we find

$$\begin{aligned}
\nabla^2 H^{(1)ij} &= -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4} \int d\!\!{}^-{}^3 p_2 d\!\!{}^-{}^3 p_3 \frac{e^{-i\mathbf{p}_2\cdot\mathbf{x}} e^{-i\mathbf{p}_3\cdot\mathbf{x}}}{\mathbf{p}_2^2 \mathbf{p}_3^2} (\mathbf{p}_2 - \mathbf{p}_3)^i (\mathbf{p}_2 - \mathbf{p}_3)^j \\
&= \left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4} \int d^3 y\, \delta^{(3)}(\mathbf{x}-\mathbf{y}) (\nabla^i_{\mathbf{x}} - \nabla^i_{\mathbf{y}})(\nabla^j_{\mathbf{x}} - \nabla^j_{\mathbf{y}}) \frac{1}{4\pi|\mathbf{x}|} \frac{1}{4\pi|\mathbf{y}|} \\
&= -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi)^2} \left(\frac{2\delta^{ij}}{r^4} - \frac{4x^i x^j}{r^6}\right).
\end{aligned} \tag{6.54}$$

It is now straightforward to integrate this expression using spherical symmetry and the known boundary conditions to find

$$H^{(1)}_{\mu\nu}(x) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2} \hat{r}_\mu \hat{r}_\nu, \tag{6.55}$$

where $\hat{r}_\mu = (0, \mathbf{x}/r)$.

It is interesting to pause for a moment to contrast this calculation with its analogue in Yang-Mills theory. The simplest gauge counterpart of the JNW linearised fat graviton is

$$A^{(0)a}_\mu(x) = g c^a u_\mu \frac{1}{4\pi r} \qquad \Rightarrow \qquad A^{(0)a}_\mu(p) = g c^a u_\mu \frac{\delta\!\!{}^-{}^1(p^0)}{p^2}. \tag{6.56}$$

To what extent is the first non-linear correction to the Yang-Mills equation similar to the equivalent in our double-copy theory? The answer to this question is clear: they are distinctly different. Indeed, the colour structure of $A^{(1)a}_\mu$ is $f^{abc} c^b c^c = 0$, so $A^{(1)a}_\mu = 0$. However, the kinematic numerator of $A^{(1)a}_\mu$ identified by colour-kinematics duality is non-zero, so there is no reason for $H^{(1)}_{\mu\nu}$ to vanish. How the double copy propagates physical information from one theory to the other is unclear, but as a mathematical statement there is no issue with using the double copy to simplify gravitational calculations.

Given our expression, equation (6.55), for the fat graviton, it is now straightforward to extract the trace and the symmetric fields:

$$\tilde{\phi}^{(1)} \equiv H^{(1)} = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2}, \tag{6.57}$$

$$\tilde{\mathfrak{h}}^{(1)}_{\mu\nu} \equiv \frac{1}{2}\left(H^{(1)}_{\mu\nu} + H^{(1)}_{\nu\mu}\right) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{4(4\pi r)^2}\hat{r}_\mu \hat{r}_\nu. \tag{6.58}$$

However, we cannot directly deduce that this $\tilde{\phi}^{(1)}$ is the usual dilaton and that $\tilde{\mathfrak{h}}^{(1)}_{\mu\nu}$ is the first order correction to the metric in some well-known gauge. The double copy is only guaranteed to compute quantities which are field redefinitions or gauge transformations of the graviton and dilaton. This suggests structuring calculations to compute only quantities which are invariant under field redefinitions and gauge transformations [51, 160–162, 174, 175]. However, if desired, it is nevertheless possible to determine explicitly the relevant field redefinitions and gauge transformations. This is the topic of the next section.

6.3.2 Relating fat and skinny fields: gauge transformations and field redefinitions

In section 6.2, we argued that the relationship between the fat and skinny fields in linear theory is

$$H^{(0)}_{\mu\nu}(x) = \mathfrak{h}^{(0)}_{\mu\nu}(x) + B^{(0)}_{\mu\nu}(x) + P^q_{\mu\nu}(\phi^{(0)}(x) - \mathfrak{h}^{(0)}(x)). \tag{6.59}$$

Beyond linear theory, we can expect perturbative corrections to this formula, so that

$$H_{\mu\nu}(x) = \mathfrak{h}_{\mu\nu}(x) + B_{\mu\nu}(x) + P^q_{\mu\nu}(\phi(x) - \mathfrak{h}(x)) + \mathcal{O}(\kappa). \tag{6.60}$$

We define a quantity $\mathcal{T}_{\mu\nu}$, which we call the *transformation function* to make this equation exact:

$$H^{(1)}_{\mu\nu}(x) = \mathfrak{h}^{(1)}_{\mu\nu}(x) + B^{(1)}_{\mu\nu}(x) + P^q_{\mu\nu}(\phi^{(1)}(x) - \mathfrak{h}^{(1)}(x)) + \mathcal{T}^{(1)}_{\mu\nu}. \tag{6.61}$$

We can require that $\mathcal{T}^{(1)}_{\mu\nu}$ is only constructed from linearised fields, so that $\mathcal{T}^{(1)}_{\mu\nu} = \mathcal{T}^{(1)}_{\mu\nu}(\mathfrak{h}^{(0)}_{\alpha\beta}, B^{(0)}_{\alpha\beta}, \phi^{(0)})$. More generally, at the nth order of perturbation theory

$$H^{(n)}_{\mu\nu}(x) = \mathfrak{h}^{(n)}_{\mu\nu}(x) + B^{(n)}_{\mu\nu}(x) + P^{q}_{\mu\nu}(\phi^{(n)}(x) - \mathfrak{h}^{(n)}(x)) + \mathcal{T}^{(n)}_{\mu\nu}(\mathfrak{h}^{(m)}_{\alpha\beta}, B^{(m)}_{\alpha\beta}, \phi^{(m)}), \tag{6.62}$$

where $m < n$. We can therefore determine $\mathcal{T}^{(n)}_{\mu\nu}$ iteratively in perturbation theory.

Before we compute $\mathcal{T}^{(1)}_{\mu\nu}$ explicitly, let us pause for a moment to discuss its physical significance. Our understanding of $\mathcal{T}^{(n)}_{\mu\nu}$ rests on two facts. Firstly, the double copy is known to work to all orders in perturbation theory for tree amplitudes. Secondly, the classical background field which we have been discussing is a generating function for tree scattering amplitudes. Therefore it must be the case that scattering amplitudes computed from the classical fat graviton background fields equal their known expressions. So consider computing $H^{(n)}_{\mu\nu}$ via the double copy, and computing $\mathfrak{h}^{(n)}_{\mu\nu}, B^{(n)}_{\mu\nu}$ and $\phi^{(n)}$ using a standard perturbative solution of their coupled equations of motion. Then the difference $H^{(n)}_{\mu\nu} - \mathfrak{h}^{(n)}_{\mu\nu} - B^{(n)}_{\mu\nu}(x) - P^{q}_{\mu\nu}(\phi^{(n)}(x) - \mathfrak{h}^{(n)}(x)) \equiv \mathcal{T}^{(n)}_{\mu\nu}$ must vanish upon use of the LSZ procedure. We conclude that $\mathcal{T}_{\mu\nu}$ parametrises redundancies of the physical fields which are irrelevant for computing scattering amplitudes: gauge transformations and field redefinitions. Indeed, the very definition of $\mathcal{T}_{\mu\nu}$ requires choices of gauge: for example, the choice of de Donder gauge for the skinny graviton.

Since $\mathcal{T}_{\mu\nu}$ parametrises choices which can be made during a calculation, such as the choice of gauge, we do not expect a particularly simple form for it. Nevertheless, to compare explicit skinny gravitons computed via the double copy with standard metrics, it may be useful to have an explicit form of $\mathcal{T}^{(1)}_{\mu\nu}$. It is always possible to compute $\mathcal{T}^{(n)}_{\mu\nu}$ directly through its definition, at the expense of perturbatively solving the coupled Einstein, scalar and antisymmetric tensor equations of motion. For example, consider the fat graviton $H^{(1)}_{\mu\nu}(x)$, equation (6.55), we computed in the previous section. Since there is no antisymmetric tensor in this system, we may compute $\mathcal{T}^{(1)}_{\mu\nu}$ under the simplifying assumption that $B_{\mu\nu} = 0$ so that $H_{\mu\nu}$ is symmetric. We find that when $\partial_\mu \mathfrak{h}^{(0)\mu\nu} = \partial_\mu H^{(0)\mu\nu} = 0$, then the transformation function is

$$\mathcal{T}^{(1)\mu\nu}(-p_1) = \int \text{đ}^D p_2 \text{đ}^D p_3 \bar{\delta}^D(p_1+p_2+p_3)\frac{1}{4p_1^2}\Big\{ H^{(0)}_{2\alpha\beta} H_3^{(0)\alpha\beta} p_1^\mu p_1^\nu$$
$$+ 8p_2^\alpha H^{(0)}_{3\alpha\beta} H_2^{(0)\beta(\mu} p_1^{\nu)} + 8p_2\cdot p_3\, H_2^{(0)\mu\alpha} H_3^{(0)\nu}{}_\alpha - 2\eta^{\mu\nu} p_2\cdot p_3\, H^{(0)}_{2\alpha\beta} H_3^{(0)\alpha\beta}$$
$$+ 4\eta^{\mu\nu} p_2^\alpha H^{(0)}_{3\alpha\beta} H_2^{(0)\beta\gamma} p_{3\gamma} + P_q^{\mu\nu}\big[2(D-6)p_2\cdot p_3\, H^{(0)}_{2\alpha\beta} H_3^{(0)\alpha\beta}$$
$$- 4(D-2) p_2^\alpha H^{(0)}_{3\alpha\beta} H_2^{(0)\beta\gamma} p_{3\gamma}\big]\Big\}, \tag{6.63}$$

where we have used a convenient short-hand notation

$$H_i^{\mu\nu} \equiv H^{\mu\nu}(p_i), \qquad p^{(\mu} q^{\nu)} \equiv \frac{1}{2}\left(p^\mu q^\nu + p^\nu q^\mu\right). \tag{6.64}$$

This expression is valid for any symmetric $H^{(0)}_{\mu\nu}$, and the extension to general $H^{(0)}_{\mu\nu}$ is straightforward.

While the information in the transformation function contains little content of physical interest, it may be of some interest from the point of view of the mathematics of colour-kinematics duality. Indeed, in the special case of the self-dual theory, it is known how to choose an explicit parametrisation of the metric perturbation so that the double copy is manifest [47]. Choosing these variables therefore sets $\mathcal{T}_{\mu\nu} = 0$ to all orders, for self-dual spacetimes. Once the relevant variables have been chosen, then the kinematic algebra in the self-dual case was manifest at the level of the equation of motion of self-dual gravity: the algebra is one of area-preserving diffeomorphisms. Perhaps it is the case that an understanding of the transformation function in the general case will open the way towards a simple understanding of the full kinematic algebra.

6.3.3 The perturbative corrections to the JNW fields

We are now in a position to convert our fat graviton $H^{(1)}_{\mu\nu}(x)$, equation (6.55) into skinny fields. The simple form of the $H^{(0)}_{\mu\nu}(x)$ leads to a simplification in the transformation function, since $p\cdot u = 0$ for a stationary source. Thus $\mathcal{T}^{(1)\mu\nu}$ is simply

$$\mathcal{T}^{(1)\mu\nu}(-p_1) = -\left(\frac{\kappa}{2}\right)^2 M^2 \int \hat{d}^4 p_2 \hat{d}^4 p_3 \hat{\delta}^4(p_1+p_2+p_3) \frac{1}{4p_1^2} \frac{\hat{\delta}^1(p_2^0)}{p_2^2} \frac{\hat{\delta}^1(p_3^0)}{p_3^2} \\ \times \left\{ 8p_2 \cdot p_3 u^\mu u^\nu - p_1^\mu p_1^\nu + 2\eta^{\mu\nu} p_2 \cdot p_3 + P_q^{\mu\nu} \left[4p_2 \cdot p_3\right] \right\}, \tag{6.65}$$

in $D = 4$. Performing the Fourier transform, we find

$$\mathcal{T}^{(1)}_{\mu\nu}(x) = -\left(\frac{\kappa}{2}\right)^2 \left[3u_\mu u_\nu + 2\hat{r}_\mu \hat{r}_\nu + 2P^q_{\mu\nu}\right] \frac{M^2}{4(4\pi r)^2}. \tag{6.66}$$

Let us now extract the skinny fields in de Donder gauge from our fat graviton, equation (6.55). The relation between the fat and skinny fields is now given by

$$\begin{aligned} \mathfrak{h}^{(1)}_{\mu\nu}(x) + P^q_{\mu\nu}\left[\phi^{(1)}(x) - \mathfrak{h}^{(1)}(x)\right] &= H^{(1)}_{\mu\nu}(x) - \mathcal{T}^{(1)}_{\mu\nu}(x) \\ &= -\left(\frac{\kappa}{2}\right)^2 \hat{r}_\mu \hat{r}_\nu \frac{M^2}{4(4\pi r)^2} + \left(\frac{\kappa}{2}\right)^2 \left[3u_\mu u_\nu + 2\hat{r}_\mu \hat{r}_\nu + 2P^q_{\mu\nu}\right] \frac{M^2}{4(4\pi r)^2}. \end{aligned} \tag{6.67}$$

Thus, the dilaton vanishes as anticipated in section 6.2.4, since

$$\phi^{(1)}(x) = H^{(1)}(x) - \mathcal{T}^{(1)}(x) = 0. \tag{6.68}$$

Consequently, the negative of the trace of the metric is the only term acted upon by $P_q^{\mu\nu}$, so we find

$$\mathfrak{h}^{(1)}(x) = -\left(\frac{\kappa}{2}\right)^2 \frac{M^2}{2(4\pi r)^2}, \tag{6.69}$$

The metric is easily seen to be

$$\mathfrak{h}^{(1)}_{\mu\nu}(x) = \left(\frac{\kappa}{2}\right)^2 (3u_\mu u_\nu + \hat{r}_\mu \hat{r}_\nu) \frac{M^2}{4(4\pi r)^2}, \tag{6.70}$$

consistent with the anticipated trace, and in agreement with the known result for the JNW metric, equation (6.39), when $M = Y$.

6.3.4 Higher orders

In section 6.3.1, we saw how fat graviton fields can be obtained straightforwardly from perturbative solutions of the Yang-Mills equations. These can then be translated to skinny fields, if necessary, after obtaining the relevant transformation functions $\mathcal{T}^{\mu\nu}$. Now let us briefly describe how this procedure generalises to higher orders.

As we explained in section 6.1.1, the validity of the double copy relies on writing Yang-Mills diagrams such that colour-kinematics duality is satisfied. But, in general, a perturbative solution of the conventional Yang-Mills equations will not satisfy this property. So before using the double copy, one must reorganise the perturbative solution of the theory so that, firstly, only three-point interaction vertices between fields occur, and secondly, the numerators of these three-point diagrams satisfy the same algebraic identities (Jacobi relations and antisymmetry properties) as the colour factors. The Jacobi identities can be enforced by using an explicit Yang-Mills Lagrangian designed for this purpose [95, 128]. It is known how to construct this Lagrangian to arbitrary order in perturbation theory. This Lagrangian is non-local and contains Feynman vertices with an infinite number of fields. If desired, it is possible to obtain a local Lagrangian containing only three point vertices at the expense of introducing auxiliary fields. For now, we will restrict ourselves to four-point order. At this order Bern, Dennen, Huang and Kiermaier (BDHK) introduced [95] an auxiliary field $B^a_{\mu\nu\rho}$ so as to write a cubic version of the Yang-Mills Lagrangian,

$$\mathcal{L}_{\text{BDHK}} = \frac{1}{2}A^{a\mu}\partial^2 A^a_\mu + B^{a\mu\nu\rho}\partial^2 B^a_{\mu\nu\rho} - gf^{abc}\left(\partial_\mu A^a_\nu - \partial^\rho B^a_{\rho\mu\nu}\right)A^{b\mu}A^{c\nu}. \tag{6.71}$$

Since the role of the field $B^a_{\mu\nu\rho}$ is essentially to be a Lagrange multiplier, it is understood that no sources for $B^a_{\mu\nu\rho}$ should be introduced.

To illustrate the procedure in a non-trivial example, let us compute the second order correction to the JNW fat graviton, $H^{(2)}_{\mu\nu}(x)$. In fact, a number of simplifications make this calculation remarkably straightforward. Firstly, the momentum space equation of motion for the auxiliary field appearing in the BDHK Lagrangian, equation (6.71), is

$$\begin{aligned} p_1^2 B^{(1)a}_{\mu\nu\rho}(-p_1) =& \frac{i}{4}f^{abc}\int đ^4p_2 đ^4p_3 \delta\!\!\!^{-}{}^4(p_1+p_2+p_3)p_{1\mu} \\ & \times\left[\eta_{\nu\beta}\eta_{\rho\gamma} - \eta_{\nu\gamma}\eta_{\rho\beta}\right]A^{(0)b\beta}(p_2)A^{(0)c\gamma}(p_3). \end{aligned} \tag{6.72}$$

Notice that the term in square brackets is antisymmetric under interchange of β and γ; imposing this symmetry is a requirement of colour-kinematics duality because the associated colour structure is antisymmetric under interchange of b

and c. A consequence of this simple fact is that, in the double copy, the auxiliary field vanishes in the JNW case (to this order of perturbation theory). In fact, two auxiliary fields appear in the double copy: one can take two copies of the field B, or one copy of B times one copy of the gauge boson A. In either case, the expression for an auxiliary field in the double copy in momentum space will contain a factor

$$\begin{aligned}
&p_{1\mu}\left[\eta_{\nu\beta}\eta_{\rho\gamma}-\eta_{\nu\gamma}\eta_{\rho\beta}\right]H^{(0)\beta\beta'}(p_2)H^{(0)\gamma\gamma'}(p_3)\\
&\quad=p_{1\mu}\left[\eta_{\nu\beta}\eta_{\rho\gamma}-\eta_{\nu\gamma}\eta_{\rho\beta}\right]\frac{\delta^1(p_2^0)}{p_2^2}\frac{\delta^1(p_3^0)}{p_3^2}u^\beta u^{\beta'}u^\gamma u^{\gamma'}=0,
\end{aligned} \tag{6.73}$$

because of the antisymmetry of the vertex in square brackets, and the factorisability of the tensor structure of the zeroth order JNW expression.

Consequently, the Yang-Mills four-point vertex plays no role in the the double copy for JNW at second order. Thus the Yang-Mills equation to be solved is simply

$$\begin{aligned}
p_1^2A^{(2)a\mu}(-p_1)&=if^{abc}\int đ^4p_2đp_3\delta\!\!\!^-{}^4(p_1+p_2+p_3)\\
&\times\left[(p_1-p_2)^\gamma\eta^{\mu\beta}+(p_2-p_3)^\mu\eta^{\beta\gamma}+(p_3-p_1)^\beta\eta^{\gamma\mu}\right]A_\beta^{(0)b}(p_2)A_\gamma^{(1)c}(p_3),
\end{aligned} \tag{6.74}$$

using the symmetry of the expression under interchange of p_2 and p_3. Thus, $H^{(2)}$ is the solution of

$$\begin{aligned}
p_1^2H^{(2)\mu\mu'}(-p_1)&=\frac{1}{2}\int đ^4p_2đ^4p_3\delta\!\!\!^-{}^4(p_1+p_2+p_3)\\
&\times\left[(p_1-p_2)^\gamma\eta^{\mu\beta}+(p_2-p_3)^\mu\eta^{\beta\gamma}+(p_3-p_1)^\beta\eta^{\gamma\mu}\right]\\
&\times\left[(p_1-p_2)^{\gamma'}\eta^{\mu'\beta'}+(p_2-p_3)^{\mu'}\eta^{\beta'\gamma'}+(p_3-p_1)^{\beta'}\eta^{\gamma'\mu'}\right]H^{(0)}_{\beta\beta'}(p_2)H^{(1)}_{\gamma\gamma'}(p_3).
\end{aligned} \tag{6.75}$$

This expression simplifies dramatically when we recall that $H^{(0)}_{\beta\beta'}(p_2)$ and $H^{(1)}_{\gamma\gamma'}(p_3)$ both have vanishing components of momentum in the time direction, so that $p_2^0=0=p_3^0=p_1^0$. Meanwhile $H^{(0)}_{\beta\beta'}(p_2)\propto u_\beta u_{\beta'}$. Thus,

$$p_1^2H^{(2)}_{\mu\mu'}(-p_1)=2\int đ^4p_2đ^4p_3\delta\!\!\!^-{}^4(p_1+p_2+p_3)H^{(0)}_{\mu\mu'}(p_2)\,p_2^\alpha H^{(1)}_{\alpha\beta}(p_3)p_2^\beta. \tag{6.76}$$

We find it convenient to Fourier transform back to position space, where we must solve the simple differential equation

$$\partial^2 H^{(2)}_{\mu\mu'}(x) = 2H^{(1)}_{\alpha\alpha'}\partial^\alpha\partial^{\alpha'} H^{(0)}_{\mu\mu'}. \tag{6.77}$$

Inserting explicit expressions for $H^{(0)}$, equation (6.33) and $H^{(1)}$, equation (6.55), and bearing in mind that the situation is static, the differential equation simplifies to

$$\nabla^2 H^{(2)}_{\mu\mu'}(x) = -\left(\frac{\kappa}{2}\right)^3 \frac{M^3}{(4\pi r)^3}\frac{u_\mu u_{\mu'}}{r^2}, \tag{6.78}$$

with solution

$$H^{(2)}_{\mu\mu'}(x) = -\left(\frac{\kappa}{2}\right)^3 \frac{M^3}{6(4\pi r)^3} u_\mu u_{\mu'}. \tag{6.79}$$

We could now, if we wished, extract the metric perturbation and scalar field corresponding to this expression. Indeed, it is always possible to convert fat gravitons into ordinary metric perturbations in a specified gauge.

It is possible to continue to continue this calculation to higher orders. In that case, more work is required in order to satisfy the requirement of colour-kinematics duality. It is possible to supplement the BDHK Lagrangian by higher-order effective operators involving the gluon field, constructed order-by-order in perturbation theory, which act to enforce colour-kinematics duality. Furthermore, one may introduce further auxiliary fields so that only cubic interaction terms appear in the Lagrangian. This procedure is explained in detail in [95, 128], and can be carried out to arbitrary perturbative order. The fat graviton equation of motion is constructed as a term-by-term double copy of the fields in the colour-kinematics satisfying Yang-Mills Lagrangian. In this way, it is possible to calculate perturbative fat gravitons to any order using Yang-Mills theory and the double copy.

6.4 Discussion

In this chapter, we have addressed how classical solutions of gravitational theories can be obtained by double-copying Yang-Mills solutions. These results go beyond the classical double copies of [2, 12, 43, 44, 49, 50, 52, 156–158] in that the solutions are non-linear. However, the price one pays is that they are no longer exact, but must be constructed order-by-order in perturbation theory. We have concentrated on solutions obtained from two copies of pure (nonsupersymmetric) Yang-Mills theory, for which the corresponding gravity theory is $\mathcal{N} = 0$ supergravity. The double copy then relates the Yang-Mills fields to a single field $H_{\mu\nu}$, that we call the *fat graviton*, and which in principle can be decomposed into its constituent *skinny fields*, which we take to be the graviton $\mathfrak{h}^{\mu\nu}$ (defined according to equation (6.5)), the dilaton ϕ, and the two-form $B^{\mu\nu}$.

Our procedure for calculating gravity solutions is as follows:

1. For a given distribution of charges, one may perturbatively solve the Yang-Mills equations for the gauge field $A^{\mu\, a}$, given in terms of integrals of interaction vertices and propagators.
2. The solution for the fat graviton is given by double copying the gauge theory solution expression according to the rules of [10, 42, 95] once colour-kinematics duality is satisfied. That is, one strips off all colour information, and duplicates the interaction vertices, leaving propagators intact.
3. The fat graviton can in principle be translated into skinny fields using the transformation law of equation (6.62), which iteratively defines the *transformation function* $\mathcal{T}^{\mu\nu}$. This function can be obtained from matching the fat graviton solution to a perturbative solution of the conventional $\mathcal{N} = 0$ supergravity equations. Once found, however, it can be used for arbitrary source distributions.

The presence of the transformation function $\mathcal{T}^{\mu\nu}$ is at first glance surprising. One may always decompose the fat graviton in terms of its symmetric traceless, anti-symmetric and trace degrees of freedom. Then one could simply define that these correspond to the physical graviton, two-form and dilaton. However, one has the freedom to perform further field redefinitions and gauge transformations of the skinny fields, in order to put these into a more conventional gauge choice (e.g. de Donder). The role of $\mathcal{T}^{\mu\nu}$ is then to perform this redefinition. It follows that it carries no physical degrees of freedom itself, and indeed is irrelevant for any physical observable.

We have given explicit examples of fat gravitons, and their relation to de Donder gauge skinny fields, up to the first subleading order in perturbation theory. We took a stationary point charge as our source, finding that one can construct either the Schwarzschild metric (as in the Kerr-Schild double copy of [12]), or the JNW solution [167] for a black hole with non-zero scalar field ϕ. Which solution one obtains on the gravity side amounts to the choice of whether or not to source the dilaton upon performing the double copy. This mirrors the well-known situation for amplitudes, namely that the choice of polarisation states in gauge theory amplitudes determines whether or not a dilaton or two-form is obtained in the corresponding gravity amplitudes at tree level. This clarifies the apparent puzzle presented in [51], regarding whether it is possible for the same gauge theory solution to produce different gravity solutions.

Underlying the simplicity of the double copy is the mystery of the kinematic algebra. While it is known that one can always find kinematic numerators for gauge theory diagrams so that colour-kinematics duality is satisfied, it is not known whether an off-shell algebraic structure exists in the general case which can compute these numerators. If this algebra exists, it may further simplify the calculations we have described in this chapter. The kinematic algebra should allow for a more algebraic computation of the numerators of appropriate gauge-theoretic diagrams, perhaps without the need for auxiliary fields. Similarly, it seems possible that a detailed understanding of the kinematic algebra will go hand-in-hand with deeper insight into the transformation function $\mathcal{T}_{\mu\nu}$ which parametrises the choice of gauge and field redefinition picked out by the double copy.

Our ultimate aim is to use the procedure outlined in this chapter in astrophysical applications, namely to calculate gravitational observables for relevant physical sources (a motivation shared by [51]). To this end, our fat graviton calculations must be extended to include different sources, and also higher orders in perturbation theory. In order to translate the fat graviton to more conventional skinny fields, one would then need to calculate the relevant transformation functions $\mathcal{T}^{(n)}_{\mu\nu}$. An alternative possibility exists, namely to calculate physical observables, which must be manifestly invariant under gauge transformations and field redefinitions, directly from fat graviton fields, without referring to skinny fields at all. Work on these issues is ongoing.

Chapter 7

Conclusions and outlook

In this thesis we have discussed a range of mathematical techniques from particle physics that can be applied to general relativity. In the first half, we began by discussing the spinor-helicity formalism and how the elimination of redundant degrees of freedom and preservation of little group covariance could be used to uplift the Newman-Penrose procedure from four dimensions to five. In the process of doing this, we focused on the irreducible representations of the Maxwell and Weyl spinors to find that they had a non-trivial little group structure. This led to a better understanding of the relation between the CMPP and de Smet classifications building on [75]. The lack of overlap between the two classifications is because the de Smet classification is highly sensitive to which irreducible little group spinors are non-zero while the CMPP classification is sensitive to boost weight. Our spinorial formalism ascribes physical degrees of freedom to components directly and as such in the future it is hoped that it will lead to better understanding of higher dimensional solutions. In particular, the interesting solutions of five dimensions such as the black ring, which defies the uniqueness theorem of four dimensions may be understood and extended.

Also in chapter 3, we sketched a six-dimensional outline of the spinorial formalism and emphasized that our approach can be generalised to any number of dimensions. The rich geometries which have already been found in five and six dimensions are an excellent reason to be interested in higher dimensions, but there are many formal and phenomenological motivations as well. It can further be hoped that an excellent understanding of general relativity in an arbitrary dimension may lead to deeper insight into the theory itself and the theory of quantum gravity underlying it.

Turning away from higher dimensions for the second half of the thesis, we discussed how the structure of the double copy originally found in quantum scattering amplitudes can also be found in general relativity. We began by outlining the Kerr-Schild double copy, which had previously been used to relate stationary exact Kerr-Schild spacetimes to electromagnetism. By considering the Kinnersley photon rocket, a non-vacuum Kerr-Schild particle which describes an accelerating mass, we extended the Kerr-Schild double copy to include time-dependent solutions. Having rewritten the Liénard-Wiechert solution to put the radiation content as a Maxwell current, it was possible to write the gravity solution in the form of a double copy of the gauge theory solution, namely an accelerating charged particle. Taking careful account of the factors of the stress-energy tensor as compared to the Maxwell current, it was possible to express the radiation part of the respective sources in terms of scattering amplitudes for the case of Bremsstrahlung. The resulting expressions were explicitly a double and single copy of each other. This result is interesting because the nature of the "null fluid" in the Kinnersley photon rocket has been debated numerous times over the years and this is an excellent example of how particle physics techniques can help to bring new light to the discussion.

Part of the reason for the Kerr-Schild double copy's success is that the Kerr-Schild choice of coordinates ensures a graviton that is both symmetric and trace-free, meaning that when the double copy is taken, both the axion and the dilaton are automatically excluded. To obtain general relativity from the double copy in general it is necessary to introduce projectors to handle the extra degrees of freedom. This formalism was introduced in chapter 6 and used the example of linearized waves to motivate the double copy at zeroth order. Using a simple stationary massive particle as an example, the formalism was used to generate the JNW solution for a black hole charged under a dilatonic field to third order.

Making the double copy structure hidden in general relativity explicit is important because of its potential application to gravitational wave physics: the high precision of calculations needed for comparison with new gravitational wave experiments will require all computational tools available. However it is also very interesting to speculate on the geometric implications of the double copy and considering exact solutions is an intriguing first step towards this.

A next logical step from this work is to draw these two exciting themes together.

The classical double copy has been extended to all four dimensional type D vacuum solutions in the Weyl double copy in [66]. This work shows that the solutions can be expressed in a double copy-like form through their Weyl and Maxwell spinors:

$$\Psi_{ABCD} = \frac{1}{S}\Phi_{(\alpha\beta}\Phi_{\gamma\delta)} \tag{7.1}$$

where S is a scalar field which plays the role of the 'propagator'. This relationship holds for all vacuum four dimensional spacetimes of the type D form and therefore to test its applicability further it is necessary to find a slightly but not too much more complicated test bed. Five dimensions is ideal for this purpose. In particular, the C-metric is a well known non-stationary exact solution, which it is found in [66] is the double copy of the Liénard-Weichert solution. This is the vacuum generalisation of the accelerating Kerr-Schild double copy considered in chapter 5. However, the five-dimensional analogue of the C-metric is unknown. Using the double copy to generate new five-dimensional black hole solutions would be an excellent demonstration of its utility.

Appendix A

Multi-irrep spacetimes in the de Smet classification

In section 3.4.4, it was shown that the de Smet classification is highly sensitive to the presence of a single little group irrep. What about when more than one irrep contributes to the Weyl tensor? Generically, this will lead to a **4**. For example, it can be seen from the discussion in section 3.4.4 that combining a **22** or a $\underline{\mathbf{22}}$ with a **1111** will always produce a **4**. Similarly, while de Smet classes are invariant under the interchange $k \leftrightarrow n$, combining any irrep with its $k \leftrightarrow n$ pair creates a **4**, if the two irreps are distinct. However, there are two cases when more than one irrep is present and the spacetime is still special in the de Smet classification:

- Absence of any $\psi^{(i)}$

  The Weyl polynomials of all four irreps of dimension 3 or less contain a factor $[n \cdot \xi, k \cdot \xi]$. This means that when only irreps of dimension 3 or less are present in the spacetime, they will in general form a **22**. However, if $\chi^{(1)}$, $\chi^{(2)}$ and $\chi^{(3)}$ are present and all directly proportional to each other, they can form into a **211**. This works as follows. Let the $\chi^{(i)}$ factorise as

$$\chi^{(1)}_{ab} = X\,\theta_{(a}\kappa_{b)}, \quad \chi^{(2)}_{ab} = Y\,\theta_{(a}\kappa_{b)}, \quad \chi^{(3)}_{ab} = Z\,\theta_{(a}\kappa_{b)}. \tag{A.1}$$

  Now the Weyl polynomial is of the form

$$\begin{aligned}\mathcal{W} = \; -3\,[\mathrm{n}\circ\xi, \mathrm{k}\circ\xi]\big\{ & X\;[\mathrm{n}\circ\xi,\theta]\,[\mathrm{n}\circ\xi,\kappa] + Y\;[\mathrm{n}\circ\xi,\theta]\,[\mathrm{k}\circ\xi,\kappa] \\ & + Y\;[\mathrm{n}\circ\xi,\kappa]\,[\mathrm{k}\circ\xi,\theta] + Z\,[\mathrm{k}\circ\xi,\theta]\,[\mathrm{k}\circ\xi,\kappa]\,\big\},\end{aligned} \tag{A.2}$$

which factorizes into a **211** if $X\,Z = Y^2$:

$$\mathcal{W} = -3\,[\mathrm{n}\circ\xi, \mathrm{k}\circ\xi]\,(X\,\,[\mathrm{n}\circ\xi,\theta] + Y\,\,[\mathrm{k}\circ\xi,\theta])\left([\mathrm{n}\circ\xi,\kappa] + \frac{Y}{X}\,\,[\mathrm{k}\circ\xi,\kappa]\right). \tag{A.3}$$

In other words, if the three vectors $\underline{\chi_1}$, $\underline{\chi_2}$ and $\underline{\chi_3}$ all point in the same direction with relative magnitudes satisfying $|\underline{\chi_1}|\,|\underline{\chi_3}| = |\underline{\chi_2}|^2$ then a special **211** composite spacetime is formed.

- **211 + 1111**

If the Weyl tensor contains only non-zero $\psi^{(4)}$ and $\chi^{(3)}$ terms (or $\psi^{(0)}$ and $\chi^{(1)}$), it is possible for these to form a de Smet **31** or **211**. Let us define

$$\psi^{(4)}_{abcd} = \alpha^{(4)}_{(a}\,\beta^{(4)}_{b}\,\gamma^{(4)}_{c}\,\delta^{(4)}_{d)}, \qquad \chi^{(3)}_{ab} = \theta^{(3)}_{(a}\,\kappa^{(3)}_{b)}. \tag{A.4}$$

Now, if one direction is the same, for example $\theta^{(3)} \propto \alpha^{(4)}$, then the Weyl polynomial forms a **31**,

$$\begin{aligned}\mathcal{W} = \left[\mathrm{k}\circ\xi, \alpha^{(4)}\right]\Big\{ &\left[\mathrm{k}\circ\xi, \beta^{(4)}\right]\left[\mathrm{k}\circ\xi, \gamma^{(4)}\right]\left[\mathrm{k}\circ\xi, \delta^{(4)}\right] \\ &+ \frac{|\theta^{(3)}|}{|\alpha^{(4)}|}\left[\mathrm{k}\circ\xi, \kappa^{(3)}\right][n\circ\xi, \mathrm{k}\circ\xi]\Big\},\end{aligned} \tag{A.5}$$

while if two directions are shared such that $\theta^{(3)} \propto \alpha^{(4)}$ and $\kappa^{(3)} \propto \beta^{(4)}$ then the Weyl polynomial remains a **211**,

$$\begin{aligned}\mathcal{W} = \left[\mathrm{k}\circ\xi, \alpha^{(4)}\right]\left[\mathrm{k}\circ\xi, \beta^{(4)}\right]\Big\{ &\left[\mathrm{k}\circ\xi, \gamma^{(4)}\right]\left[\mathrm{k}\circ\xi, \delta^{(4)}\right] \\ &+ \frac{|\theta^{(3)}|}{|\alpha^{(4)}|}\frac{|\kappa^{(3)}|}{|\beta^{(4)}|}\,[\mathrm{n}\circ\xi, \mathrm{k}\circ\xi]\Big\}.\end{aligned} \tag{A.6}$$

In contrast, if $\psi^{(4)}$ is of the special de Smet form **<u>11</u> <u>11</u>** and shares a direction with $\chi^{(3)}$, then the spacetime is always a **211**: the reality conditions prevent us from constructing a **31**. This is because the reality conditions on a $\psi^{(4)}$ of the form

$$\psi^{(4)}_{abcd} = \alpha_{(a}\,\beta_b\,\alpha_c\,\beta_{d)} \tag{A.7}$$

are

$$\alpha_1\,\beta_1 = \pm(\alpha_2\,\beta_2)^*, \quad \alpha_1\,\beta_2 + \alpha_2\,\beta_1 = \mp\,(\alpha_1\,\beta_2 + \alpha_2\,\beta_1)^*, \tag{A.8}$$

requiring a β that looks like

$$\beta = \begin{pmatrix} 1 \\ -\alpha_1^* \,/\, \alpha_2^* \end{pmatrix} \beta_1, \quad \beta_1^* = \mp \frac{\alpha_2}{\alpha_2^*} \beta_1. \tag{A.9}$$

The reality conditions for χ_3 of the form $\chi_{ab}^{(3)} = \theta_{(a}\,\kappa_{d)}$ are very similar:

$$\theta_1\,\kappa_1 = (\theta_2\,\kappa_2)^*, \quad \theta_1\,\kappa_2 + \theta_2\,\kappa_1 = -\left(\theta_1\,\kappa_2 + \theta_2\,\kappa_1\right)^*, \tag{A.10}$$

with solution

$$\kappa = \begin{pmatrix} 1 \\ -\theta_1^* \,/\, \theta_2^* \end{pmatrix} \kappa_1, \quad \kappa_1^* = -\frac{\theta_2}{\theta_2^*} \kappa_1. \tag{A.11}$$

Therefore, if $\psi^{(4)}$ and χ_3 share a direction such that $\alpha \propto \theta$, then it can be read off from equations (A.9) and (A.11) that β and κ are proportional.

These are the only ways that a de Smet class can be built - every other combination results in a **4**. Figure 3.1 is therefore misleading, since it implies that each class can be reduced to another wholly contained within it. For example, figure 3.1 implies that de Smet **1111**s are a subset of **211**s. This is not always the case: a spacetime with only $\chi^{(3)}$ non-zero has no overlap with a spacetime which has only $\psi^{(0)}$ non-zero. An attempt to depict this limited specialisation of de Smet classes more accurately has been made in figure A.1 as a contrast to figure 3.1.

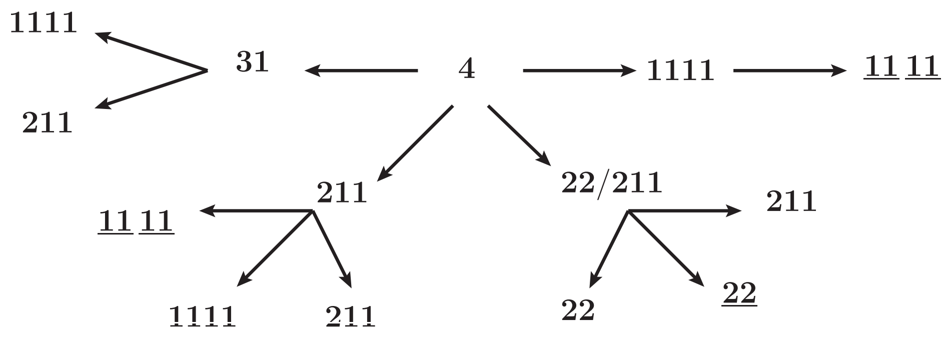


Figure A.1 *There are 4 ways that the de Smet classes can become more specialised. Going clockwise from the top: a type N solution can become more special when its eigenvalues are equal. A spacetime containing more than one irrep of dimension 1 or 3 can be a* ***22*** *or a* ***211*** *if the dimension 3 irreps form a perfect square. A* ***211*** *spacetime can also be formed using the irreps dimension 5 irreps, and a* ***31*** *spacetime always is.*

Appendix B

Fourier transform of source terms

In this appendix, we describe how to carry out the Fourier transform of eqs. (5.31, 5.36), to get the momentum-space expressions of eqs. (5.32, 5.37).

One may first consider the transform of $(u \cdot x)^{-1}$, where we work explicitly in four spacetime dimensions:

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \int d^4x \frac{e^{iq\cdot x}}{u\cdot x} \\ &= \frac{1}{u^0}\int d^3x e^{-i\boldsymbol{q}\cdot\boldsymbol{x}} \int dx^0 \frac{e^{iq^0x^0}}{x^0 - \frac{\boldsymbol{x}\cdot\boldsymbol{u}}{u^0}}. \end{aligned} \tag{B.1}$$

Closing the x^0 contour in the upper half plane gives a positive frequency solution $q^0 > 0$:

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \frac{2\pi i}{u^0}\int d^3x\, e^{-i\boldsymbol{x}\cdot\left[\boldsymbol{q}-\frac{q^0}{u^0}\boldsymbol{u}\right]} \\ &= \frac{i(2\pi)^4}{u^0}\delta^{(3)}\left(\boldsymbol{q}-\frac{q^0}{u^0}\boldsymbol{u}\right). \end{aligned} \tag{B.2}$$

It is possible to regain a covariant form for this expression by introducing a mass variable m, such that

$$\begin{aligned}\mathcal{F}\left\{\frac{1}{u\cdot x}\right\} &= \frac{i(2\pi)^4}{u^0}\int_0^\infty dm\,\delta\left(m-\frac{q^0}{u^0}\right)\delta^{(3)}(\boldsymbol{q}-m\boldsymbol{u}) \\ &= i(2\pi)^4\int_0^\infty dm\,\delta^{(4)}(q-mu), \end{aligned} \tag{B.3}$$

where the integral is over non-negative values of m only, given that $q^0 > 0$. Given that $\theta(x^0)\delta(x^2)$ is a retarded propagator[1], one may also note the transform

$$\mathcal{F}\left\{\theta(x^0)\delta(x^2)\right\} = -\frac{2\pi}{q^2}. \tag{B.4}$$

We then use the convolution theorem to obtain the Fourier transform of the current from eq. (5.31). The theorem states that the Fourier transform of a product is equal to the convolution of the transforms of each term. That is,

$$\mathcal{F}\{f \cdot g\} = \mathcal{F}\{f\} * \mathcal{F}\{g\}, \tag{B.5}$$

where the convolution operation in four dimensions takes the form

$$(F * G)(k) = \frac{1}{(2\pi)^4} \int d^4q F(q)G(k-q). \tag{B.6}$$

Then, we can compute the Fourier transform of the current

$$\begin{aligned} \tilde{j}^\nu(k) &= \mathcal{F}\{j^\nu_{\text{KS}}(x)\} \\ &= \frac{2g}{4\pi}\frac{\partial}{\partial u'_\nu}\left[\mathcal{F}\{\theta(x^0)\delta(x^2)\} * \mathcal{F}\left\{\frac{1}{x \cdot u'}\right\}\right] - (u \leftrightarrow u'), \end{aligned} \tag{B.7}$$

so inserting eqs. (B.4) and (B.3), and using the convolution definition eq. (B.6) we obtain the expression

$$\begin{aligned} \tilde{j}^\nu(k) =& \frac{2g}{4\pi}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(2\pi)^4}\int d^4q \left(-\frac{2\pi}{q^2}\right)\left(i(2\pi)^4\int_0^\infty dm\, \delta^{(4)}(k-q-mu')\right)\right] \\ &- (u \leftrightarrow u') && \text{(B.8)} \\ =& -ig\int_0^\infty dm \left(\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-mu')^2}\right] - (u \leftrightarrow u')\right), && \text{(B.9)} \end{aligned}$$

where we have carried out the integral over q in the last line. The derivative in the m integral can be carried out to give

$$\int_0^\infty dm \frac{2m(k-mu')^\nu}{(k-mu')^4} = -\int_0^\infty dm \frac{2m^2 u'^\nu}{(m^2 - 2mu' \cdot k)^2}, \tag{B.10}$$

where, on the right-hand side, we have used the onshellness condition $k^2 = 0$, and also neglected terms $\sim k^\mu$, which vanish upon contraction of the current with a physical polarisation vector. The remaining integral over m is easily carried out,

[1]The retarded nature of the propagator is implemented by the prescription $\frac{1}{(p^0+i\varepsilon)^2 - \boldsymbol{p}^2}$, where ε ensures convergence of the integrals in what follows.

and leads directly to the result of eq. (5.32).

Similar steps to those leading to eq. (B.9) can be used to rewrite eq. (5.36) in the form

$$T^{\mu\nu}_{\text{KS}} = \frac{iM}{2}\int_0^\infty dm\left(\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-mu')^2}\right] - (u \leftrightarrow u')\right). \tag{B.11}$$

Carrying out the double derivative gives

$$\begin{aligned}\frac{\partial}{\partial u'_\mu}\frac{\partial}{\partial u'_\nu}\left[\frac{1}{(k-mu')^2}\right] &= -\frac{2m^2\eta^{\mu\nu}}{(m^2-2mu'\cdot k)^4} + \frac{8m^2(k-mu')^\mu(k-mu')^\nu}{(m^2-2mu'\cdot k)^3}\\ &\simeq \frac{8m^4u'^\mu u'^\nu}{(m^2-2mu'\cdot k)^3},\end{aligned} \tag{B.12}$$

where in the second line we have again used onshellness ($k^2 = 0$), and ignored terms which vanish when contracted with the graviton polarisation tensor. Substituting eq. (B.12) into eq. (B.11), the m integral is straightforward, and one obtains the result of eq. (5.37).

Appendix C

Obtaining JNW from the fat graviton machinery

In chapter 6, the JNW solution was obtained for a scalar charge equal to the black hole mass, $M = Y$. Using the machinery developed in chapter 6 for the fat graviton, we can extend this to the full family of solutions for arbitrary dilaton Y.

We start by evaluating the zeroth order fat graviton from $\mathfrak{h}^{(0)\mu\nu} = M\frac{u^\mu u^\mu}{r}$ and $\phi^{(0)} = Y\frac{u^2}{r}$:

$$\begin{aligned} H^{(0)\mu\nu} =& \mathfrak{h}^{(0)\mu\nu} - P_q^{\mu\nu}\mathfrak{h}^{(0)} + P_q^{\mu\nu}\phi^{(0)} \\ =& M\frac{u^\mu u^\mu}{r} + (Y - M)P_q^{\mu\nu}\frac{u^2}{r}. \end{aligned} \tag{C.1}$$

We will leave explicit computation of $H^{(1)\mu\nu}$ and $T^{\mu\nu}$ for section C.4 and move straight to finding $\mathfrak{h}^{(1)\mu\nu}$ and $\phi^{(1)}$. We can do this by combining the general expressions as $H^{(1)\mu\nu} - T^{\mu\nu}$ as given in chapter 6 and then manipulating them as:

$$\begin{aligned} \phi^{(1)} =& H^{(1)} - X \\ =& \int \hat{d}^4p_2\hat{d}^4p_3\hat{\delta}^{(4)}(p_1 + p_2 + p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{-1}{8}\right)\left(p_2 \cdot H_3^{(0)} \cdot p_2 H_2^{(0)}\right) \end{aligned} \tag{C.2}$$

and

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu} - P^{\mu\nu}_{q'}\mathfrak{h}^{(1)} =& H^{(1)\mu\nu} - X^{\mu\nu} - P^{\mu\nu}_{q'}(H^{(1)} - X) \\ =& \int \bar{d}^4 p_2 \bar{d}^4 p_3 \bar{\delta}^{(4)}(p_1 + p_2 + p_3) \left(\frac{1}{p_1}\right)^2 \left(\frac{1}{8}\right) \Big\{ 8 p_2 \cdot p_3 H_2^{(0)\mu} \cdot H_3^{(0)\nu} \\ & + 4 H_2^{(0)} \cdot H_3^{(0)} p_2^\mu p_3^\nu - 8 H_3^{(0)\mu\nu} p_3 \cdot H_2^{(0)} \cdot p_3 + 8 p_2 \cdot H_3^{(0)\mu} p_3 \cdot H_2^{(0)\nu} \\ & - 16 p_2 \cdot H_3^{(0)} \cdot H_2^{(0)(\mu} p_3^{\nu)} - 2\eta^{\mu\nu} H_2^{(0)} \cdot H_3^{(0)} p_2 \cdot p_3 \\ & + 4\eta^{\mu\nu} p_2 \cdot H_3^{(0)} \cdot H_2^{(0)} \cdot p_3 P^{\mu\nu}_{q'} \big[- 8 p_2 \cdot H_3^{(0)} \cdot p_2 H_2^{(0)} \\ & - 2(D-6) p_2 \cdot p_3 H_2^{(0)} \cdot H_3^{(0)} + 4(D-2) p_2 \cdot H_3^{(0)} \cdot H_2^{(0)} \cdot p_3 \big] \Big\}. \end{aligned} \tag{C.3}$$

As usual we note that since the q' dependence is arbitrary[1] we can read off:

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu} =& \int \bar{d}^4 p_2 \bar{d}^4 p_3 \bar{\delta}^{(4)}(p_1 + p_2 + p_3) \left(\frac{1}{p_1}\right)^2 \left(\frac{1}{8}\right) \Big\{ 8 p_2 \cdot p_3 H_2^{(0)\mu} \cdot H_3^{(0)\nu} \\ & + 4 H_2^{(0)} \cdot H_3^{(0)} p_2^\mu p_3^\nu - 8 H_3^{(0)\mu\nu} p_3 \cdot H_2^{(0)} \cdot p_3 + 8 p_2 \cdot H_3^{(0)\mu} p_3 \cdot H_2^{(0)\nu} \\ & - 16 p_2 \cdot H_3^{(0)} \cdot H_2^{(0)(\mu} p_3^{\nu)} - 2\eta^{\mu\nu} H_2^{(0)} \cdot H_3^{(0)} p_2 \cdot p_3 \\ & + 4\eta^{\mu\nu} p_2 \cdot H_3^{(0)} \cdot H_2^{(0)} \cdot p_3 \Big\}. \end{aligned} \tag{C.4}$$

C.1 Evaluating the graviton

When we plug any fat graviton $H^{(0)\mu\nu}$ that has a contribution from the $P^{\mu\nu}_q$ terms into this formula, for example

$$H^{(0)\mu\nu} = H^{(0)\mu\nu}_{P_q\to 0} - P^{\mu\nu}_q (H^{(0)}_{P_q\to 0} - \phi^{(0)}), \tag{C.5}$$

[1] Furthermore, note that q' need not be the same as our zeroth order gauge parameter q.

we will get a messy result, containing four times as many terms, three-quarters of which will contain P_{q2} or P_{q3}. We can write this as

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu} = &\int \hat{d}^4p_2\hat{d}^4p_3\hat{\delta}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\Big\{8p_2\cdot p_3 H^{(0)\mu}_{2P_q\to 0}\cdot H^{(0)\nu}_{3P_q\to 0}\\
&+4H^{(0)}_{2P_q\to 0}\cdot H^{(0)}_{3P_q\to 0}p_2^\mu p_3^\nu - 8H^{(0)\mu\nu}_{3P_q\to 0}p_3\cdot H^{(0)}_{2P_q\to 0}\cdot p_3\\
&+8p_2\cdot H^{(0)\mu}_{3P_q\to 0}p_3\cdot H^{(0)\nu}_{2P_q\to 0} - 16p_2\cdot H^{(0)}_{3P_q\to 0}\cdot H^{(0)(\mu}_{2P_q\to 0}p_3^{\nu)}\\
&-2\eta^{\mu\nu}H^{(0)}_{2P_q\to 0}\cdot H^{(0)}_{3P_q\to 0}p_2\cdot p_3 + 4\eta^{\mu\nu}p_2\cdot H^{(0)}_{3P_q\to 0}\cdot H^{(0)}_{2P_q\to 0}\cdot p_3\\
&+F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right)\Big\}\end{aligned} \tag{C.6}$$

where F is a long expression containing every term with at least one P_q in. One route now might be to expand out the projector terms in F and attempt to simplify. However, we can gain insight more directly by expressing the $\mathfrak{h}^{(1)\mu\nu}$ as the result of a gauge transform, $\mathfrak{h}^{(1)\mu\nu}_{final} = \mathfrak{h}^{(1)\mu\nu}_{initial} + \delta\mathfrak{h}^{(1)\mu\nu}$. The initial frame is the "GR-like" frame, where

$$\mathfrak{h}^{(0)\mu\nu}_{initial} = H^{(0)\mu\nu}_{P_q\to 0}, \tag{C.7}$$

and the final frame is obtained by a gauge transform

$$\xi^\nu(p_i) = \frac{1}{D-2}\frac{q^\nu}{p_i\cdot q}(\mathfrak{h}^{(0)}_{initial} - \phi^{(0)}) \tag{C.8}$$

such that $\mathfrak{h}^{(1)\mu\nu}_{final}$, which corresponds to the expression given in equation (C.6), is

$$\begin{aligned}\mathfrak{h}^{(0)\mu\nu}_{final} =&\mathfrak{h}^{(0)\mu\nu}_{initial} + p^\mu\xi^\nu + p^\nu\xi^\mu - \eta^{\mu\nu}\partial\cdot\xi\\
=&\mathfrak{h}^{(0)\mu\nu}_{initial} - \left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}\right)\frac{\mathfrak{h}^{(0)}_{initial} - \phi^{(0)}}{D-2}\\
=&H^{(0)\mu\nu}_{P_q\to 0} - P_q^{\mu\nu}(H^{(0)}_{P_q\to 0} - \phi^{(0)}).\end{aligned} \tag{C.9}$$

The corresponding change in $\mathfrak{h}^{(1)\mu\nu}$ can be calculated using

$$\delta\mathfrak{h}^{(1)\mu\nu} = \mathfrak{h}^{(1)\mu\nu}(\mathfrak{h}^{(0)}_{final}) - \mathfrak{h}^{(1)\mu\nu}(\mathfrak{h}^{(0)}_{intial}). \tag{C.10}$$

Naively, this would lead us to say that the gauge transform is simply

$$\delta\mathfrak{h}^{(1)\mu\nu} = F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right). \tag{C.11}$$

However, comparison with equation (C.6) shows that if this were true, only terms containing $H^{(0)\mu}_{P_q\to 0}$ would remain, removing all the dependences on the dilaton field $\phi^{(0)}$. For the example of JNW it would mean that every possible JNW solution collapsed to the $M = Q$ case.

C.2 Gauge subtleties

The resolution can be found by carefully studying the origin of the terms in the general expression for $\mathfrak{h}^{(1)\mu\nu}$. These come from the graviton 3-vertex and the dilaton/graviton vertex and look like:

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu}_{GR} = &\int\!\! \text{đ}^4p_2\text{đ}^4p_3\text{δ̄}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\Big\{8p_2\cdot p_3\mathfrak{h}^{(0)\mu}_2\cdot\mathfrak{h}^{(0)\nu}_3 \\ &+4\mathfrak{h}^{(0)}_2\cdot\mathfrak{h}^{(0)}_3p^\mu_2p^\nu_3-8\mathfrak{h}^{(0)\mu\nu}_3p_3\cdot\mathfrak{h}^{(0)}_2\cdot p_3+8p_2\cdot\mathfrak{h}^{(0)\mu}_3p_3\cdot\mathfrak{h}^{(0)\nu}_2 \\ &-16p_2\cdot\mathfrak{h}^{(0)}_3\cdot\mathfrak{h}^{(0)(\mu}_2p^{\nu)}_3-2\eta^{\mu\nu}\mathfrak{h}^{(0)}_2\cdot\mathfrak{h}^{(0)}_3p_2\cdot p_3 \\ &+4\eta^{\mu\nu}p_2\cdot\mathfrak{h}^{(0)}_3\cdot\mathfrak{h}^{(0)}_2\cdot p_3-2\mathfrak{h}^{(0)}_2\mathfrak{h}^{(0)}_3p^\mu_2p^\nu_3+\eta^{\mu\nu}p_2\cdot p_3\mathfrak{h}^{(0)}_2\mathfrak{h}^{(0)}_3\Big\},\end{aligned}\tag{C.12}$$

and

$$\mathfrak{h}^{(1)\mu\nu}_{dilaton} = \int\!\! \text{đ}^4p_2\text{đ}^4p_3\text{δ̄}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\left(2p^\mu_2p^\nu_3-\eta^{\mu\nu}p_2\cdot p_3\right)\phi^{(0)}_2\phi^{(0)}_3.\tag{C.13}$$

We used the definitions

$$\begin{aligned}\phi^{(0)} =& H^{(0)}\\ \mathfrak{h}^{(0)\mu\nu} =& H^{(0)\mu\nu}\end{aligned}\tag{C.14}$$

to convert these expressions into "fat" fields, and thus the $\mathfrak{h}^{(1)\mu\nu}_{dilaton}$ terms cancelled exactly with the final two terms of $\mathfrak{h}^{(1)\mu\nu}_{GR}$. However, their gauge transforms do not cancel. The projector-filled part of the gauge transform, $F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right)$, is generated as usual by all the terms of $\mathfrak{h}^{(1)\mu\nu}_{GR}$ except the last two[2], such that the

[2]If in doubt, compare this with equation (C.4)

total transform is:

$$\begin{aligned}\delta\mathfrak{h}^{(1)\mu\nu} =&\mathfrak{h}^{(1)\mu\nu}_{GR}(\mathfrak{h}^{(0)}_{initial}+\delta\mathfrak{h}^{(0)})+\mathfrak{h}^{(1)\mu\nu}_{dilaton}(\mathfrak{h}^{(0)}_{initial}+\delta\mathfrak{h}^{(0)})\\ &-\mathfrak{h}^{(1)\mu\nu}_{GR}(\mathfrak{h}^{(0)}_{initial})-\mathfrak{h}^{(1)\mu\nu}_{dilaton}(\mathfrak{h}^{(0)}_{initial})\\ =&F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right)+\int đ^4p_2đ^4p_3\delta^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\\ &\times\left(-2p_2^\mu p_3^\nu+\eta^{\mu\nu}p_2\cdot p_3\right)\left(2\delta\mathfrak{h}^{(0)}_2\mathfrak{h}^{(0)}_{3initial}+\delta\mathfrak{h}^{(0)}_2\delta\mathfrak{h}^{(0)}_3\right)\\ =&F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right)+\int đ^4p_2đ^4p_3\delta^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\\ &\times\left(-2p_2^\mu p_3^\nu+\eta^{\mu\nu}p_2\cdot p_3\right)\big(-2(H^{(0)}_{2P_q\to0}-\phi^{(0)}_2)H^{(0)}_{3P_q\to0}\\ &\qquad+(H^{(0)}_{2P_q\to0}-\phi^{(0)}_2)(H^{(0)}_{3P_q\to0}-\phi^{(0)}_3)\big)\\ =&F\left(P_{q2},P_{q3},P_{q2}P_{q3}\right)+\int đ^4p_2đ^4p_3\delta^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\\ &\times\left(-2p_2^\mu p_3^\nu+\eta^{\mu\nu}p_2\cdot p_3\right)\left(-H^{(0)}_{2P_q\to0}H^{(0)}_{3P_q\to0}+\phi^{(0)}_2\phi^{(0)}_3\right),\end{aligned} \tag{C.15}$$

where we have used the symmetry on $2\leftrightarrow3$ and recalled from the previous section that

$$\begin{aligned}\delta\mathfrak{h}^{(0)} =&\mathfrak{h}^{(0)}_{final}-\mathfrak{h}^{(0)}_{intial}\\ =&-(H^{(0)}_{P_q\to0}-\phi^{(0)}),\end{aligned} \tag{C.16}$$

while $\mathfrak{h}^{(0)\mu\nu}_{initial}$ is $H^{(0)}_{P_q\to0}$. It is therefore possible to undo the gauge transform on $\mathfrak{h}^{(1)\mu\nu}_{final}$ to find

$$\begin{aligned}\mathfrak{h}^{(1)\mu\nu}_{initial} =&\mathfrak{h}^{(1)\mu\nu}_{final}-\delta\mathfrak{h}^{(0)\mu\nu}\\ =&\int đ^4p_2đ^4p_3\delta^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\Big\{8p_2\cdot p_3H^{(0)\mu}_{2P_q\to0}\cdot H^{(0)\nu}_{3P_q\to0}\\ &+4H^{(0)}_{2P_q\to0}\cdot H^{(0)}_{3P_q\to0}p_2^\mu p_3^\nu-8H^{(0)\mu\nu}_{3P_q\to0}p_3\cdot H^{(0)}_{2P_q\to0}\cdot p_3\\ &+8p_2\cdot H^{(0)\mu}_{3P_q\to0}p_3\cdot H^{(0)\nu}_{2P_q\to0}-16p_2\cdot H^{(0)}_{3P_q\to0}\cdot H^{(0)(\mu}_{2P_q\to0}p_3^{\nu)}\\ &-2\eta^{\mu\nu}H^{(0)}_{2P_q\to0}\cdot H^{(0)}_{3P_q\to0}p_2\cdot p_3+4\eta^{\mu\nu}p_2\cdot H^{(0)}_{3P_q\to0}\cdot H^{(0)}_{2P_q\to0}\cdot p_3\\ &-\left(-2p_2^\mu p_3^\nu+\eta^{\mu\nu}p_2\cdot p_3\right)\left(-H^{(0)}_{2P_q\to0}H^{(0)}_{3P_q\to0}+\phi^{(0)}_2\phi^{(0)}_3\right)\Big\}.\end{aligned} \tag{C.17}$$

This will always return a graviton in the "GR-like" frame, with a trace consistent with those used in the GR community in de Donder gauge. For example, we shall see in the next section that the formula produces the exact form of JNW that we see in the literature for harmonic coordinates.

C.3 Application to JNW

For JNW, the parts of the fat graviton are:

$$\begin{aligned} H^{(0)\mu\nu}_{P_q\to 0} =& M\frac{u^\mu u^\mu}{r} \\ \phi^{(0)} =& Y\frac{u^2}{r}, \end{aligned} \tag{C.18}$$

and therefore using that $p_i \cdot u = 0$, we have

$$\begin{aligned} \mathfrak{h}^{(1)\mu\nu}_{initial} =& \int đ^4p_2 đ^4p_3 \delta\!\!\!^{-(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{1}{8}\right)\Big\{8p_2\cdot p_3 H^{(0)\mu}_{2P_q\to 0}\cdot H^{(0)\nu}_{3P_q\to 0} \\ &+ 4H^{(0)}_{2P_q\to 0}\cdot H^{(0)}_{3P_q\to 0}p_2^\mu p_3^\nu - 2\eta^{\mu\nu}H^{(0)}_{2P_q\to 0}\cdot H^{(0)}_{3P_q\to 0}p_2\cdot p_3 \\ &- (-2p_2^\mu p_3^\nu + \eta^{\mu\nu}p_2\cdot p_3)\left(-H^{(0)}_{2P_q\to 0}H^{(0)}_{3P_q\to 0} + \phi_2^{(0)}\phi_3^{(0)}\right)\Big\} \\ =& \int đ^4p_2 đ^4p_3 \delta\!\!\!^{-(4)}(p_1+p_2+p_3)\left(\frac{1}{8}\right)\frac{\delta(p_2^0)}{p_2^2}\frac{\delta(p_3^0)}{p_3^2}\Big\{4M^2u^4\frac{p_2^\mu p_3^\nu}{p_1^2} \\ &+ 4M^2u^2u^\mu u^\nu - \eta^{\mu\nu}M^2u^4 - u^4\left(-2\frac{p_2^\mu p_3^\nu}{p_1^2} + \frac{1}{2}\eta^{\mu\nu}\right)\left(-M^2+Y^2\right)\Big\}. \end{aligned} \tag{C.19}$$

This is easy to invert back into position space using the identity

$$\int đ^4p_2 đ^4p_3 \delta\!\!\!^{-(4)}(p_1+p_2+p_3)\frac{\delta\!\!\!^{-}(p_2^0)\delta\!\!\!^{-}(p_3^0)}{p_2^2p_3^2}\frac{p_2^\mu p_3^\nu}{p_1^2} = \frac{-1}{4}\left[\frac{x^ix^j}{r^4} - \frac{\delta^{ij}}{r^2}\right] \tag{C.20}$$

to obtain

$$\begin{aligned} \mathfrak{h}^{(1)\mu\nu}_{inital} =& \frac{1}{16(4\pi)^2}\Big\{ -2M^2u^4\left[\frac{x^ix^j}{r^4} - \frac{\delta^{ij}}{r^2}\right] + 8M^2\frac{u^2u^\mu u^\nu}{r^2} \\ &- 2M^2u^4\frac{\eta^{\mu\nu}}{r^2} + u^4\left(\frac{x^ix^j}{r^4} - \frac{\delta^{ij}}{r^2} + \frac{\eta^{\mu\nu}}{r^2}\right)\left(M^2 - Y^2\right)\Big\} \\ =& \frac{-1}{16(4\pi)^2}\Big\{(7M^2 - Y^2)\frac{u^\mu u^\nu}{r^2} + (M^2+Y^2)\frac{x^ix^j}{r^4}\Big\} \end{aligned} \tag{C.21}$$

which is exactly the JNW solution for arbitrary mass M and dilaton charge Y. Note that we did not need to undo the gauge transform from the initial to the final frame when we calculated JNW for $M = Y$ because $\delta\mathfrak{h}^{(0)} \sim M - Y$: the two frames coincide for this special case.

C.3.1 Dilaton correction

Of course, it will also be necessary to reverse the gauge transform on $\phi^{(1)}$. By inputting

$$H^{(0)\mu\nu} = H^{(0)\mu\nu}_{P_q\to 0} - P_q^{\mu\nu}(H^{(0)}_{P_q\to 0} - \phi^{(0)}) \tag{C.22}$$

into the formula for $\phi^{(1)}$ we find that

$$\begin{aligned}\phi^{(1)}_{final} &= \int đ^4 p_2 đ^4 p_3 \bar{\delta}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{-1}{8}\right)\left(p_2\cdot H^{(0)}_3\cdot p_2 H^{(0)}_2\right)\\ &= \int đ^4 p_2 đ^4 p_3 \bar{\delta}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{-1}{8}\right)\Big(p_2\cdot H^{(0)}_{3P_q\to 0}\cdot p_2\phi^{(0)}_2\\ &\quad - p_2\cdot P_{q3}\cdot p_2\left(H^{(0)}_{3P_q\to 0}-\phi^{(0)}_3\right)\phi^{(0)}_2\Big)\end{aligned} \tag{C.23}$$

where $H^{(0)}$ is given by $H^{(0)}_{P_q\to 0} + \Delta^{(0)}$. As before, the second term is easily interpreted if $P_q^{\mu\nu}$ is the gauge transform:

$$P_q^{\mu\nu}(p) = \frac{1}{D-2}\left(\eta^{\mu\nu} - \frac{p^\mu q^\nu + p^\nu q^\mu}{p\cdot q}\right) \tag{C.24}$$

and so $\phi^{(1)}$ transforms as

$$\begin{aligned}\phi^{(1)}_{final} =&\phi^{(1)}(\mathfrak{h}^{(0)}_{final})\\ =&\phi^{(1)}(\mathfrak{h}^{(0)}_{inital} + \delta\mathfrak{h}^{(0)})\\ =&\phi^{(1)}_{initial} + \int đ^4 p_2 đ^4 p_3 \bar{\delta}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{8}\right)\left(\frac{\left(H^{(0)}_{3P_q\to 0}-\phi^{(0)}_3\right)}{D-2}\frac{p_2\cdot q}{p_3\cdot q}\phi^{(0)}_2\right).\end{aligned} \tag{C.25}$$

This fits the standard form

$$\phi^{(1)} \to \phi^{(1)} + \xi^{(0)}\cdot\partial\phi^{(0)} \tag{C.26}$$

for $\xi^\mu(p) = \frac{1}{D-2}\frac{q^\mu}{p\cdot q}\left(H^{(0)}_{P_q\to 0} - \phi^{(0)}\right)$. Since $\phi^{(0)}$ has no gauge transform, the "GR-frame" dilaton is therefore simply

$$\phi^{(1)}_{initial} = \int \text{đ}^4 p_2 \text{đ}^4 p_3 \bar{\delta}^{(4)}(p_1+p_2+p_3)\left(\frac{1}{p_1}\right)^2\left(\frac{-1}{8}\right)\left(p_2\cdot H^{(0)}_{3P_q\to 0}\cdot p_2 H^{(0)}_2\right). \tag{C.27}$$

This is zero for all cases in JNW, as $H^{(0)\mu\nu}_{P_q\to 0} \sim u^\mu u^\nu$ such that $p_i \cdot H^{(0)}_{P_q\to 0}\cdot p_i$ vanishes.

C.4 Fat graviton and transformation function

Previously, $H^{(1)\mu\nu}$ and $T^{(1)\mu\nu}$ were not expanded in terms of the projectors P_q and instead were combined to get general expressions for $\mathfrak{h}^{(1)\mu\nu}$ and $\phi^{(1)}$. Explicit expressions for H and T will always contain a mess of $P_q^{\mu\nu}$ terms. This is because the gauge transform of the fat graviton is given by

$$\begin{aligned}
H^{(0)\mu\nu}_{final} =& H^{(0)\mu\nu}(\mathfrak{h}^{(0)}_{final}) \\
=& \mathfrak{h}^{(0)\mu\nu}_{final} - P_q^{\mu\nu}(\mathfrak{h}^{(0)}_{final} - \phi^{(0)}) \\
=& H^{(0)\mu\nu}(\mathfrak{h}^{(0)}_{initial} + \delta\mathfrak{h}^{(0)}) \\
=& \mathfrak{h}^{(0)\mu\nu}_{intial} + \delta\mathfrak{h}^{(0)\mu\nu} - P_q^{\mu\nu}(\mathfrak{h}^{(0)}_{intial} + \delta\mathfrak{h}^{(0)} - \phi^{(0)}).
\end{aligned} \tag{C.28}$$

Recalling that $\delta\mathfrak{h}^{(0)\mu\nu}$ for the transform out of the "GR-like" frame is given by

$$\delta\mathfrak{h}^{(0)\mu\nu} = -P_q^{\mu\nu}(\mathfrak{h}^{(0)}_{initial} - \phi^{(0)}), \tag{C.29}$$

we see that

$$\begin{aligned}
\delta H^{(0)\mu\nu} =& \delta\mathfrak{h}^{(0)\mu\nu} - P_q^{\mu\nu}\delta\mathfrak{h}^{(0)} \\
=& 0.
\end{aligned} \tag{C.30}$$

This is exactly as we should expect, since the change from initial to final frame consists only of playing around with the trace of the zeroth order skinny graviton. This is a gauge dependent object in GR, but for the fat field it is not, since we must never be able to gauge away the dilaton. However, this does mean that in general, $H^{(1)\mu\nu}$ and $T^{(1)\mu\nu}$ are quite ugly objects whose mass of P_q terms will laboriously combine to produce $F(P_{q2}, P_{q3}, P_{q2}P_{q3})$. It is easier to treat $F(P_{q2}, P_{q3}, P_{q2}P_{q3})$ as an analytic object. Therefore, we bypass a transformation into position space and write

$$H^{(1)\mu\nu} = H^{(1)\mu\nu}(H^{(0)}_{P_q\to 0}) + F_1\left(P_{q2}, P_{q3}, P_{q2}P_{q3}\right) \tag{C.31}$$

$$T^{(1)\mu\nu} = T^{(1)\mu\nu}(H^{(0)}_{P_q\to 0}) + F_2\left(P_{q2}, P_{q3}, P_{q2}P_{q3}\right) \tag{C.32}$$

where F_1 and F_2 satsify

$$F_1\left(P_{q2}, P_{q3}, P_{q2}P_{q3}\right) + F_1\left(P_{q2}, P_{q3}, P_{q2}P_{q3}\right) = F\left(P_{q2}, P_{q3}, P_{q2}P_{q3}\right). \tag{C.33}$$

Note that $H^{(1)\mu\nu}(H^{(0)}_{P_q\to 0})$ is simply $H^{(1)\mu\nu}_{M=Y}$, the JNW result when $M = Y$, and similarly $T^{(1)\mu\nu}(H^{(0)}_{P_q\to 0}) = T^{(1)\mu\nu}_{M=Y}$.

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