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Canonical noncommutativity in special and general relativity

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Abstract. There are two main points that concern us in this short contribution. The first one is the conceptual distinction between a intrinsically noncommuting spacetime, *i.e.*, one where the coordinate functions fail to commute among themselves, on the one hand, and the proposal of noncommuting position operators, on the other. The second point concerns a particular form of position operator noncommutativity, involving the spin of the particle, to which several approaches seem to converge. We also suggest an analysis of the effects of spacetime curvature on position operator noncommutativity.

1. Introduction

The traditional approach to spacetime noncommutativity has, up to now, followed invariably the paradigm of a noncommuting "manifold", where the coordinate functions of standard differential geometry are replaced by elements of a noncommutative algebra. Then an appropriate noncommutative differential calculus is developed, and once the geometric substratum is firmly in place, physics enters the picture and predictions are sought. On the other hand, spacetime noncommutativity has long been conjectured invoking *gedanken* experiments, in which, *e.g.*, ultra-energetic probe particles distort spacetime sufficiently in their vicinity so as to render it effectively noncommuting. There is a clear conceptual divergence between these two points of view, the latter being, in our view, closer to an "experimental", or operational, treatment of geometry, in which it is only the combined system of spacetime *and* the particles used to probe it from which geometrical data should be extracted. Such a point of view may be better served by invoking noncommuting position operators for the various particles of the theory. Then, the noncommutativity may well involve other properties of the particles, such as their mass or spin, giving rise to different geometries being perceived by different particles, and avoiding, at the same time, a host of conceptual and technical difficulties so far encountered in the "quantum manifold" approaches, in particular the apparent lack of spacetime homogeneity and isotropy.

The structure of this note is as follows: section 2 offers arguments against using position operators as generators of a Lie algebra — their natural replacement is shown to be the boost operators [2]. Then Jordan y Mukunda's classic work [3] is briefly presented. Section 3 shows that both Dirac's theory, appropriately restricted, and the relativistic center of mass definitions that seem to make the most sense physically point to a particular form of noncommutativity. Section 4 outlines a study of the possible effects of curvature to this problem.

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2. Positions and boosts

The position/momentum duality, captured in Heisenberg's commutation relations,

$$P_i X_j = -i\delta_{ij} + X_j P_i \tag{1}$$

allows the interpretation of momenta as generators of translations in position space, or the mathematically equivalent interpretation of positions as generators of translations in momentum space. The apparent symmetry between these two alternatives, is, however misleading. When a composite object is translated in position space by a certain amount, each of its constituent parts is translated by that same amount. In momentum space this is not so — for example, translating each of an apple's two halves by a momentum $\vec{k}$, results in the apple itself being translated by $2\vec{k}$. At the origin of this asymetry lies the fact that momentum is additive under system composition (*extensive*, in thermodynamical parlance), while position is not. For this latter statement to be precise, we need to define, in some reasonable way, the position of a composite system, the obvious candidate being a suitably defined "center-of-mass". Then what is said above is that the position of the center of mass of, say, two particles, is not the sum of the positions of the individual particles. Is there any position-related quantity that adds up under system composition? A glance at the Newtonian formula for the center of mass position,

$$\vec{X}_{12} = \frac{M_1 \vec{X}_1 + M_2 \vec{X}_2}{M_1 + M_2}, \tag{2}$$

shows that MX is indeed additive, in the above sense. In this Newtonian limit, $K_i \equiv MX_i$ are the generators of galilean boosts, and it can be shown that their relativistic counterparts are also additive. But, why should we look for extensive quantities in the first place? The answer has to do with the Leibniz rule that Lie algebra generators should obey, when acting on composite systems (a property that algebraists express in terms of a *primitive coproduct* structure). A finite version of this (as opposed to the preceeding differential one) is that, given a particular symmetry transformation, its application to a composite system should consist in applying the same symmetry transformation to each of the constituent parts (think of rotating a Lego construction). We will not delve too deeply into this territory, but suffice to say that there are good arguments supporting the statement that *physical quantities that correspond to Lie algebra generators must be extensive.* This rules out the appearance of position operators as generators of a Lie algebra, and points to their replacement by the boosts, which carry equivalent information. We observe this replacement in, *e.g.*, the Poincaré algebra,

$$\begin{array}{lll} [J_i, J_j] = \epsilon_{ijk} J_k & [J_i, K_j] = \epsilon_{ijk} K_k & [J_i, P_j] = \epsilon_{ijk} P_k \\ [K_i, K_j] = -\epsilon_{ijk} J_k & [K_i, P_j] = \delta_{ij} H & [K_i, H] = P_i \end{array}$$

where J_i, K_j, P_r and H generate rotations, boosts, space and time translations, respectively (square brackets are to be interpreted as commutators or Poisson brackets, as the need arises).

What we plan to do next is to look for representations of this algebra, appropriate for spinless particles, in terms of variables $\{q_i, p_j\}$, $i, j = 1, 2, 3$, with canonical brackets $[q_i, p_j] = \delta_{ij}$, $[q_i, q_j] = 0 = [p_i, p_j]$ (we adopt the Poisson bracket interpretation from now on). In this, we follow the work of Jordan y Mukunda [3], where it is shown that the answer, unique up to canonical transformations, is

$$J_i = \epsilon_{ijk} q_j p_k \qquad K_i = q_i \sqrt{\vec{p}^2 + m^2} \qquad P_i = p_i \qquad H = \sqrt{\vec{p}^2 + m^2}\,. \tag{3}$$

Our next aim is to introduce a set of position operators X_i, $i = 1, 2, 3$, and express their physical properties in terms of their commutators with the Poincaré algebra generators. Thus we impose, still following [3],

$$[X_i, P_j] = \delta_{ij} \qquad [J_i, X_j] = \epsilon_{ijk} X_k \qquad [X_j, K_k] = X_k [X_j, H]\,, \tag{4}$$

the last one guaranteeing that simultaneous position measurements, along the three cartesian axes, form part of a four-vector, the zeroth component of which is the time at which they were made. Returning to the representation problem, the solution $X_i = q_i$ is forced upon us, up to canonical transformations, making the X_i commutative.

Notice that $K_i = X_i H$ in this representation. It is instructive to see how the boost commutator produces rotations,

$$\begin{aligned}[K_i, K_j] &= [X_i H, X_j H] \\ &= [X_i, X_j]H^2 + [X_i, H]X_j H + [H, X_j]X_i H \\ &= X_j P_i - X_i P_j \\ &= -\epsilon_{ijk} J_k \,.\end{aligned}$$

An obvious question then is, how does the above calculation work out in the presence of spin? Could it be that the first term in the second line above, involving the X-X commutator, makes the difference? *Very* naively then, one would need something like

$$[X_i, X_j] = -\epsilon_{ijk} S_k / H^2 \,, \tag{5}$$

a formula that, despite dubious origins, seems to capture something fundamental.

3. Hints

3.1. Noncommuting position operators in Dirac's theory

Eq. (5) makes little sense on first sight, as it predicts noncommutative position operators for, *e.g.*, a spin 1/2 particle, while in Dirac's theory position operators are known to commute among themselves. However, in the above discussion, we are dealing with a single particle theory, while Dirac's negative energy states are known to transmute eventually to positron states. Thus, to properly compare (5) and Dirac's theory, the latter should be truncated to positive, say, energies. This is accomplished by writing every operator A in the theory as the sum of its even and odd part, $A = \tilde{A} + \hat{A}$, the former respecting the sign of the energy of the states it acts on, while the latter flipping it. To truncate the theory to single-sign energies, one keeps only the even part $\tilde{A}$. Doing this to the position operators x_i, which act multiplicatively on wavefunctions, one finds $\tilde{x}_i = x_i + i(\alpha_i - p_i/H)/2H$, with $\tilde{x}_i$ satisfying exactly (5), up to factors of i *etc.*, stemming from different conventions [1].

3.2. The relativistic "center of mass"

It is a rather remarkable fact that, two years after special relativity's centennary, a fully satisfactory definition of the relativistic analogue of the newtonian center-of-mass concept does not exist. And while it might well be that no such generalization exists, the deeper lesson to be learned from this circumstance should be within our grasp, and it seems it is not (by "our grasp" we mean at least the authors'). It is also remarkable, and relatively little known, that the center-of-mass definitions that make most sense physically in the relativistic realm, seem to point to (5)-esque noncommutativity. Pryce [4], back in 1948, gives an excellent overview of the various proposals that had seen the light up to his time, and systematically explores and classifies their properties. What he considers desirable among them are

a) The three coordinates of the "center-of-mass" should be part of a four vector, the zeroth component being the time at which they are measured.

b) The "center-of-mass" should be at rest in the center-of-momentum frame.

c) When no external forces act on the system of particles, its "center-of-mass" ought to move with constant velocity.

d) The three coordinates of the "center-of-mass" should commute among themselves (in the sense of Poisson brackets).

The last requirement was included by Pryce with hamiltonian mechanics in mind — we, of course, are willing to drop it. The four main center-of-mass definitions mentioned in Pryce are

1) Average of positions, weighted by rest masses. Obvious weakness: not part of a four-vector.

2) Apply **1** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation.

3) Average of positions, weighted by total energies. Obvious weakness: not part of a four-vector.

4) Apply **3** in the center-of-momentum frame, and obtain the coordinates in any other frame by Lorentz transformation [5].

The table below summarizes the score

	a	**b**	**c**	**d**
1	-	-	-	x
2	x	-	-	x
3	-	x	x	-
4	x	x	x	-

Interestingly, the otherwise sound **3** and **4** fail to satisfy **d**, and they do so according to (5) (for **4**, H is replaced by total mass M).

There are further instances where (5)-like coordinate algebras show up (*e.g.*, [6, 7]). The persistence of these relations, and the diversity of the approaches that converge to them, suggest to us that they deserve a more detailed analysis.

3.3. Atomic versus molecular algebras

Another point worth further consideration is the following: in considering a system of particles, one takes their coordinates and dual momenta to satisfy the canonical Poisson bracket relations and yet, when the system is considered as one composite particle, its coordinates and dual momenta no longer satisfy the canonical relations (*e.g.*, the coordinates do not commute among themselves). The situation is rather unsatisfactory, from a physical point of view, as it necessitates knowledge of (ultimate!) "compositeness" before the appropriate Poisson bracket algebra is selected ("atoms" pick one algebra, "molecules" another). The problem can be further formulated in Hopf algebraic terms ("lack of coproduct") and its resolution points again to (5) as the appropriate starting point.

3.4. Why $[X, X] \sim S/H^2$*?*

We want to ponder here on what does (5) mean physically, and what it does not. To begin with, it is clear there is nothing quantum gravitational about it. Rather, the noncommutativity of the X's seems to stem from the fact that the system is not point like, but extends in space. This statement needs some refinement though. Suppose the system is observed in the center-of-momentum frame, and it is found to possess angular momentum along z (S_z *is* angular momentum in the center-of-momentum frame). Then the uncertainty relation implied by (5) specifies that the system's *center-of-mass* position in the x-y plane cannot be located exactly in a quantum theory. On the other hand, Newtonian systems of particles can be extended too, and yet, their center-of-mass coordinates commute and, hence, their center-of-mass can be located exactly, even in a quantized theory. This should be compared with the fact that, in the Newtonian limit, an extended system behaves *exactly*, as far as Newton's law is concerned, like a point mass located at its center-of-mass — a property not shared by relativistic systems. The uncertainty in the relativistic "center-of-mass" position then reflects exactly this absence of a sharp "effective position", and is therefore a purely relativistic effect. Its relevance for quantum gravitational considerations stems from the rather basic aspiration to find a quantum analogue to the classical geodesic motion of point particles. A quantum particle is inherently spread out, and one would like to assign to it some sort of mean position, in the hope that the latter might follow a (suitably defined) "geodesic". The most one can hope for, in view of (5), is that the effective point particle, located at the "mean position", will feel some sort of average of the metric over a region whose area is of the order of the r.h.s. of (5) — this averaging process might contribute further, quantum gravitational terms in the r.h.s. of (5).

4. The role of curvature

As mentioned already, Eqs. (5) do not capture gravitational effects, as they emerge assuming a Minkowski background. Could it be that the presence of curvature affects them? A straightforward way to address the question is to extend Jordan y Mukunda's work to de Sitter spacetime. We propose the following program:

- Write down the de Sitter algebra and introduce canonical coordinates q_i and momenta p_j, as in section 2. First, representations of the spinless case are to be sought, in the form of deformations of the ones found in [3].
- Spin variables s_i should be admitted, satisfying the rotation algebra, and representations with spin should be determined.
- Antiparticles should be introduced and the previously found representations should be appropriately extended.
- Finally, position operators X_i should be introduced and their algebraic properties imposed. This is not a trivial step, as the de Sitter momenta do not commute, and it is not clear what the appropriate Heisenberg-type relations ought to be.
- Determine a representation for the X_i and the ensuing X-X algebra.

It would suffice, for our purposes, that the computation were done perturbatively in the inverse de Sitter radius. The results would then suggest the effects expected in general spacetimes, provided the curvature changes were over an appropriate length scale.

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References

[1] Thaller B 1992 *The Dirac Equation* (Berlin: Springer)
[2] Chryssomalakos C and Okon E 2004 *Int. J. Mod. Phys.* D **13** 2003
[3] Jordan T F and Mukunda N 1963 *Physical Review* **132** 1842
[4] Pryce M H L 1948 *Proc. Royal Soc. Lon.* **195** 62
[5] Fokker A D 1929 *Relativitätstheorie* (Groningen: P. Noordhoff)
[6] Casalbuoni R 1976 *Nuovo Cimento* A **33** 389
[7] Bette A 1984 *J. Math. Phys.* **25** 2456