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On Completeness of Sliced Spaces under the Alexandrov Topology

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Abstract: We show that in a sliced spacetime (V, g), global hyperbolicity in V is equivalent to T_A-completeness of a slice, if and only if the product topology T_P, on V, is equivalent to T_A, where T_A denotes the usual spacetime Alexandrov "interval" topology.

Keywords: sliced space; Alexandrov interval topology; global hyperbolicity; slice completeness

1. Preliminaries

Sliced spaces have attracted the attention of several authors in studies related to systems of Einstein equations (see [1]), completeness (see [2]), global hyperbolicity (see [3,4]), as well as in problems of a more geometric nature on quantum cosmology (see [5,6]).

Definition 1. *Let $V = M \times \mathbb{R}$, where M is an n-dimensional smooth manifold, such that V is equipped with an $n+1$-dimensional Lorentz metric g, which splits in the following way:*

$$g = -N^2(\theta^0)^2 + g_{ij}\theta^i\theta^j,$$

where $\theta^0 = dt$, $\theta^i = dx^i + \beta^i dt$, $N = N(t, x^i)$ is called lapse function, *$\beta^i(t, x^j)$ is called* shift function *and $M_t = M \times \{t\}$, called spatial slices of V, are spacelike submanifolds equipped with the time-dependent spatial metric $g_t = g_{ij}dx^i dx^j$. Such a product space V is called a* sliced space.

Let (V, g) be a sliced space. A base for the product topology T_P, on V, consists of all sets of the form $A \times B$, where $A \in T_M$ and $B \in T_{\mathbb{R}}$. Here T_M denotes the natural topology of the manifold M where, for an appropriate Riemann metric h, it has a base consisting of open balls $B^h_\epsilon(x)$ and $T_{\mathbb{R}}$ is the usual topology on the real line.

The Alexandrov topology (or "interval topology") T_A on a spacetime V has a base consisting of open sets of the form $< x, y >= I^+(x) \cap I^-(y)$, where $I^+(x) = \{z \in V : x \ll z\}$ and $I^-(y) = \{z \in V : z \ll y\}$, where $\ll$ is the chronological order defined as $x \ll y$ iff there exists a future oriented timelike curve, joining x with y. By $J^+(x)$ one denotes the topological closure of $I^+(x)$ and by $J^-(y)$ that one of $I^-(y)$ (see [7]).

A spacetime V is strongly causal, if and only if it is strongly causal at every point, that is, for every point $p \in V$, p has arbitrarily small causally convex neighbourhoods. We say that V is globally hyperbolic, if and only if V is strongly causal and every set $J^+(x) \cap J^-(y)$ (called a "closed diamond") is compact. Global hyperbolicity is considered the strongest causality condition in the causal hierarchy of spacetimes (see [8]) and is equivalent to the existence of a Cauchy hypersurface S for V (see Section 5, in [7]); this supplies us with the benefit to construct on V well-defined initial-value problems (see [9,10], Theorem 10.2.2). One can also view global hyperbolicity as a property on a spacetime which guarantees the absence of naked singularities in V (for its role in the strong cosmic censorship, see [11]).

In the next section we will show that global hyperbolicity in a sliced spacetime (V, g) is equivalent to completeness with respect to the Alexandrov topology of a slice (M_t, g_t). Although completeness of the Alexandrov topology T_A, by itself, is not a criterion of nonsingularity (in the Schwartzschild space and the Friedmann-Robertson-Walker cosmologies, for example, T_A is complete, but these spaces are singular; see [12]), it is interesting that in the particular case of sliced spacetimes that are equipped with their natural product topology, completeness of a slice with respect to T_A can be considered as a criterion of global hyperbolicity for the entire space.

Thoughout our text, for topological terms like Hausdorff space and completeness, we refer to the seminal book of Engelking, [13].

2. A Topological Condition for the Completeness of a Sliced Space

In [3], sliced spaces are being considered to have uniformly bounded lapse, shift and spatial metric, in order to achieve the equivalence of global hyperbolicity of (V, g) with the completeness of the slice (M_0, γ) (Theorem 2.1). Being motivated by this result, in the Theorem that follows, we consider global topological conditions, for showing the equivalence of global hyperbolicity of (V, g) with a slice (M_t, g_t) being T_A-complete. Our Theorem 1, below, differs from Theorem 2.1 of [3] in that the slices in [3] are complete Riemannian manifolds (with uniformly bounded spatial metric, lapse and shift functions) while in our case the slices are T_A-complete. We discuss this further in Section 3.

Theorem 1. *Let (V, g) be a sliced space, with respect to its natural product topology T_P, where $V = M \times \mathbb{R}$, M is an n-dimensional manifold and g the $n+1$-Lorentz "metric" on V. Let also T_A be the Alexandrov topology on V. Then, the following statements are equivalent:*

1. *(V, g) is globally hyperbolic.*
2. *$T_P \equiv T_A$.*
3. *(M_t, g_t) is complete with respect to T_A.*

Proof. *1.* has been shown to be equivalent to *2.* in [4].

To show that *2.* implies *3.*, we first notice that since (V, g) is globally hyperbolic, it is also strongly causal. Since, also, $T_P \equiv T_A$, we have that for every $t \in \mathbb{R}$, M_t is a subset of a spacetime V, with nondegenerate spacetime metric, with subspace topology T_A inherited from V, such that M_t is strongly causal. Hence, T_A, on M_t, is complete (see [12], Theorem 2).

For proving that *3.* implies *1.*, for each $t \in \mathbb{R}$, we let (M_t, g_t) to be complete with respect to T_A, where each M_t is a spacelike submanifold with time dependent spatial metric $g_t \equiv g_{ij} dx^i dx^j$. But since each M_t is complete, the Alexandrov topology T_A, on M_t, is strongly causal (see, again, [12]). So, each point of M_t is strongly causal, which means that for every point $P \in M_t$ there exists an arbitrarily small convex neighbourhood. But, $V = \bigcup_{t \in \mathbb{R}} M \times \{t\}$, so $P \in V$ if and only if there exists $M_t = M \times \{t\}$, such that $P \in M_t$, and hence V is strongly causal with respect to T_A. That the closed T_A-diamonds, in V, are compact, has been shown in Theorem 3, from [4]. Thus, (V, g) is globally hyperbolic. □

3. Discussion

Question 1: Can our Theorem 1 hold, if one substituted in 3. "(M_t, g_t) is complete with respect to T_A" with the statement "(M_t, g_t) is a complete Riemannian manifold"? The answer is negative, since in a spacetime manifold, T_A is usually a coarser topology than the spacetime topology, and it is equivalent to the manifold topology only if it is Hausdorff (see [7], Theorem 4.24). So, in order for this question to have a positive answer, one would have to add in Theorem 1 the extra condition that T_A, on V, is Hausdorff. As a continuation of this question, we ask whether the spacetimes considered in [3] may well have their Alexandrov topology T_A not being Hausdorff. In such a case, strong causality will fail due to this (see, for example, Remark 4.25 of [7]). Spacetimes where T_A fails to be Hausdorff, according to Penrose, admit a null geodesic along which strong causality fails and this is one aspect of a general result concerning the region of strong causality failure in a spacetime [7]. Given the above argument, we conjecture that for a physically reasonable spacetime, the statements of Theorem 2.1 of [3] and of Theorem 1 here should be equivalent. A rigorous proof showing the equivalence of a uniformly bounded spatial metric, lapse and shift functions with the condition of the topologies T_P and T_A to be equivalent, will be of a great interest.

In [3], there are conditions introduced, so that global hyperbolicity to be equivalent to geodesic completeness. In particular, in Theorem 3.1, the term *trivially sliced space* is introduced, so that a slice is a complete Riemannian manifold, if and only if the space (V, g) is geodesically complete. The "disadvantage" of this condition is that the spatial metric g_{ij} is time-independent.

Question 2: Can one relate slice-completeness and geodesic completeness of (V, g) with a time-dependent spatial metric g_{ij}?

Question 2 does not seem to have a trivial answer. In a possible variation of Theorem 1, towards an answer to this question, one could make use of the classical Hopf-Rinow Theorem (see [14]), which gives that metric completeness, in a spacetime, is equivalent to geodesic completeness. Again, T_A should be Hausdorff.

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