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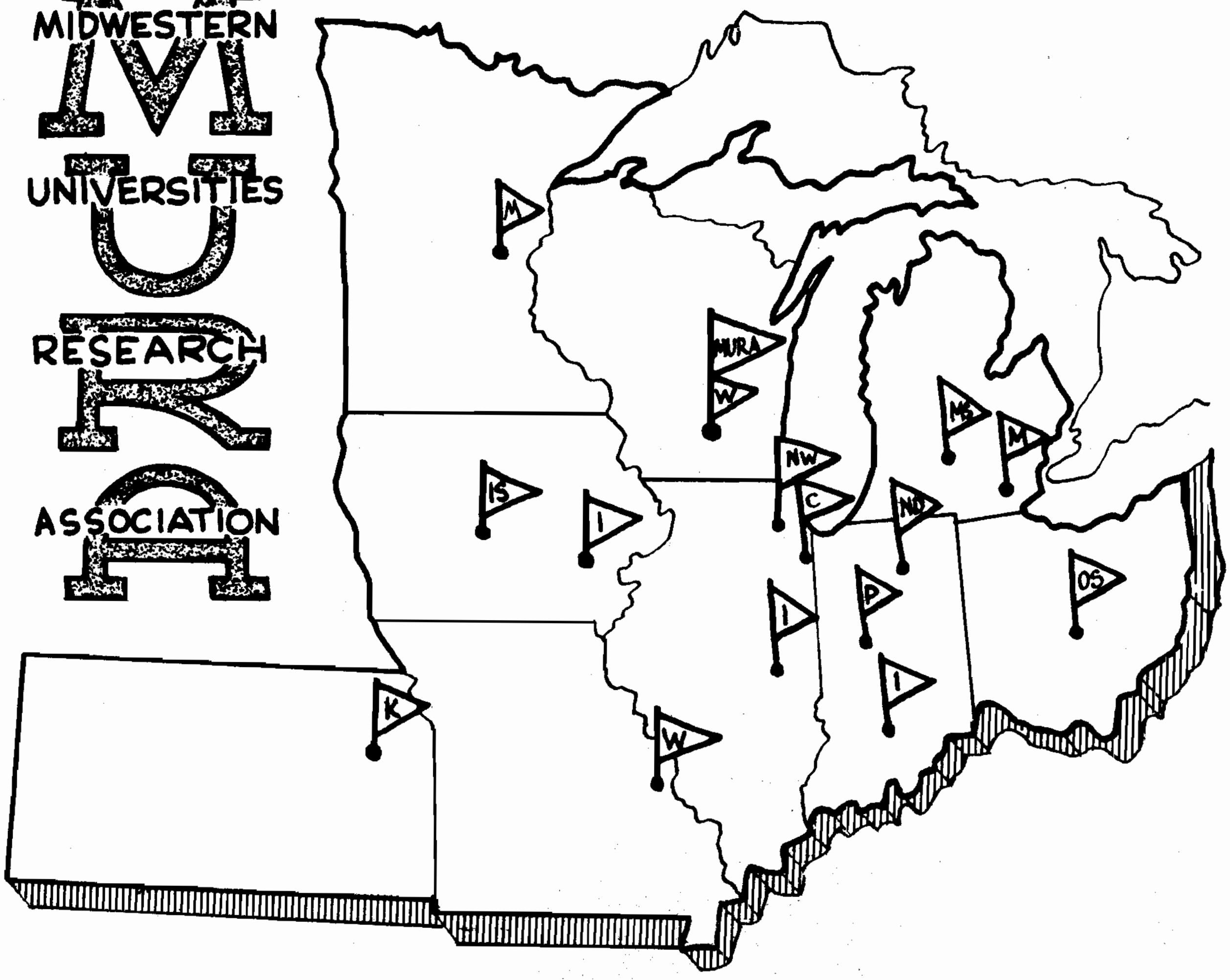

CONCERNING THE $\nu/N \longrightarrow 1/3$ RESONANCE, II

APPLICATION OF A VARIATIONAL PROCEDURE AND OF

THE MOSER METHOD TO THE EQUATION

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2}\left[\sum_{m=1} b_m \sin 2mt\right] v^2 = 0$$

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L. Jackson Laslett**

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ABSTRACT

As a continuation of an earlier report pertaining to the $\nu/N \longrightarrow 1/3$ resonance, the stability boundary for the equation

$$\frac{d^2v}{dt^2} + \left(\frac{2\nu}{N}\right)^2 v + \frac{1}{2}\left[\sum_{m=1} b_m \sin 2mt\right] v^2 = 0$$

has been studied analytically and (for $b_1 = 1$, $b_3 = 3/4$, $b_5 = 1/2$) by digital computation. A relatively simple trial function,

$$v = \sum_{m=1}\left[A_m \sin(2m - 4/3)t + B_m \sin 2mt + C_m \sin(2m + 4/3)t\right]$$

is employed in a variational procedure or with harmonic balance to obtain an estimate of the unstable equilibrium (periodic) solution and associated fixed points. Application of the Moser method of solution is also carried through, to include terms of order $(\nu/N - 1/3)^2$. The results are compared with computational data for $\nu/N = 0.3267$, 0.33, 0.3367, and 0.34.

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A. MOTIVATION

In a previous report,[1*] hereinafter designated as I, a study was made of the differential equation

$$\frac{d^2v}{dt^2} + (2\nu/N)^2 v + (1/2)(\sin 2t)\, v^2 = 0, \tag{1}$$

with particular attention to the limiting-amplitude solution governed by the one-third resonance ($\nu/N \rightarrow 1/3$). As was pointed out in I, if the coefficient of the linear term in (1) had not been constant but involved a periodic function of the independent variable t, it would be possible[2] to remove this t-dependence by a suitable transformation. Such a transformation, however, has the effect that the quadratic term becomes more complicated than in eqn. (1). As an extension of the results of I, we therefore consider in the present report the equation

$$\frac{d^2v}{dt^2} + (2\nu/N)^2 v + (1/2)\left[\sum_{m=1} b_m \sin 2mt\right] v^2 = 0, \tag{2}$$

with $b_1 \neq 0$.

As before,[1] results of a variational solution and of application of the Moser procedure[3] will be presented and compared with computational results. In particular we shall be concerned with the limiting-amplitude solution governed by the one-third resonance, and undertake to carry the analysis consistently through terms of order $(\nu/N - 1/3)^2$.

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*References are given in Section E.

B. THE VARIATIONAL METHOD

The unstable equilibrium orbit, or the associated "fixed points" characterizing the limiting-amplitude solution of eqn. (2),

$$\frac{d^2v}{dt^2} + (2\,\nu/N)^2\, v + (1/2)\left[\sum_{m=1} b_m \sin 2\,m\,t\right] v^2 = 0,$$

may be sought by insertion of a suitable trial function into the variational statement

$$\delta\left\{\langle (dv/dt)^2\rangle - (2\,\nu/N)^2\langle v^2\rangle - (1/3)\sum_{m=1} b_m \langle v^3 \sin 2\,m\,t\rangle\right\} = 0. \quad (3)$$

We shall employ here the trial function

$$v = A_1 \sin 2\,t/3 + B_1 \sin 2\,t + C_1 \sin 10\,t/3$$
$$+ \sum_{m=2}\left[A_m \sin (2\,m - 4/3)\,t + B_m \sin 2\,m\,t + C_m \sin (2\,m + 4/3)\,t\right], \quad (4)$$

in which the first term is the dominant one and the remaining terms are then of a form suggested by considerations of harmonic balance.

In the substitution of the trial function (4) into the variational statement (3), only those terms need be retained which will contribute terms of order no higher than $(\nu/N - 1/3)^2$ to the solution--to this accuracy it is then sufficient to retain (cubic) terms in $\langle v^3 \sin 2\,m\,t\rangle$ which involve A_1 squared or cubed. With this approximation the variational statement (3) then becomes (on multiplication of (3) by 72):

$$16\left[1 - 9\,(\nu/N)^2\right]A_1^2 + 16\left[9 - 9\,(\nu/N)^2\right]B_1^2 + 16\left[25 - 9\,(\nu/N)^2\right]C_1^2$$
$$+ 16\sum_{m=2}\Big\{\left[(3\,m-2)^2 - 9\,(\nu/N)^2\right]A_m^2 + \left[(3\,m)^2 - 9\,(\nu/N)^2\right]B_m^2 + \left[(3\,m+2)^2 - 9\,(\nu/N)^2\right]C_m^2\Big\}$$
$$+ 9\,b_1\left[A_1^3/3 - 2\,A_1^2 B_1 + A_1^2 C_1\right]$$
$$+ 9\sum_{m=2} b_m\left[A_1^2\,(A_m - 2\,B_m + C_m)\right] \text{ to be stationary.} \quad (5)$$

By performing the appropriate differentiations of the algebraic form (5) the simultaneous algebraic equations for the coefficients of the trial function are then obtained directly:

$$32\left[1 - 9\,(\nu/N)^2\right] A_1 + 9\, b_1 \left[A_1^2 - 4\, A_1\, B_1 + 2\, A_1\, C_1\right] + 18 \sum_{m=2} b_m\, A_1\, (A_m - 2\, B_m + C_m) = 0 \qquad (6a)$$

$$32\left[9 - 9\,(\nu/N)^2\right] B_1 - 18\, b_1\, A_1^2 = 0 \qquad (6b)$$

$$32\left[25 - 9\,(\nu/N)^2\right] C_1 + 9\, b_1\, A_1^2 = 0 \qquad (6c)$$

$$\left.\begin{aligned} &32\left[(3\,m - 2)^2 - 9\,(\nu/N)^2\right] A_m + 9\, b_m\, A_1^2 = 0 \qquad (6d)\\ &32\left[(3\,m)^2 - 9\,(\nu/N)^2\right] B_m - 18\, b_m\, A_1^2 = 0 \qquad (6e)\\ &32\left[(3\,m + 2)^2 - 9\,(\nu/N)^2\right] C_m + 9\, b_m\, A_1^2 = 0\,. \qquad (6f) \end{aligned}\right\} m \geqslant 2$$

In solution of eqns. (6a-f), one may first express B_1, C_1, A_m, ... in terms of A_1 by means of eqns. (6b-f) and substitute the results into eqn. (6a) to obtain an equation involving the unknown A_1 alone. An approximate solution of this last-named equation, valid through terms of order $(\nu/N - 1/3)^2$, may then be obtained and the remaining coefficients (B_1, C_1, A_m, ...) determined [Appendix A]. We thus find

$$A_1 = -\frac{64}{3\, b_1}\,(1/3 - \nu/N)\left\{1 - 8\left[1 + \sum_{m=2}\left(\frac{b_m}{b_1}\right)^2 \frac{9\, m^2 - 5}{(m^2 - 1)(9\, m^2 - 1)}\right](1/3 - \nu/N)\right\} \qquad (7a)$$

$$B_1 = \frac{32}{b_1}\,(1/3 - \nu/N)^2 \qquad (7b)$$

$$C_1 = -\frac{16}{3\, b_1}\,(1/3 - \nu/N)^2 \qquad (7c)$$

$$\left.\begin{aligned} A_m &= -\frac{128}{3\, b_1}\,\frac{b_m/b_1}{(m - 1)\,(3\,m - 1)}\,(1/3 - \nu/N)^2 \qquad (7d)\\ B_m &= \frac{256}{b_1}\,\frac{b_m/b_1}{9\, m^2 - 1}\,(1/3 - \nu/N)^2 \qquad (7e)\\ C_m &= -\frac{128}{3\, b_1}\,\frac{b_m/b_1}{(m + 1)\,(3\,m + 1)}\,(1/3 - \nu/N)^2 \qquad (7f) \end{aligned}\right\} m \geqslant 2$$

These coefficients, when employed in the trial function (4), provide us with an approximate representation of the unstable equilibrium orbit in the form of a trigonometric series.

From the foregoing results for the unstable equilibrium orbit, the coordinates of the fixed points may be obtained, as desired. Thus, at $t = 0$, one finds

$$v = 0 \tag{8a}$$

$$\begin{aligned} p_v \equiv \frac{dv}{dt} &= \frac{2}{3} A_1 + 2 B_1 + \frac{10}{3} C_1 \\ &\quad + \sum_{m=2} \left[\frac{2}{3} (3m - 2) A_m + 2 m B_m + \frac{2}{3} (3m + 2) C_m \right] \\ &= - \frac{128}{9 b_1} \left(\frac{1}{3} - \frac{\nu}{N} \right) \left\{ 1 - \left[\frac{45}{4} - 8 \sum_{m=2} \frac{2m (b_m/b_1) - (9m^2 - 5)(b_m/b_1)^2}{(m^2 - 1)(9m^2 - 1)} \right] \left(\frac{1}{3} - \frac{\nu}{N} \right) \right\}. \end{aligned} \tag{8b}$$

From the experience reported previously in I (Section C of reference 1) it may be expected that the accuracy of these results, being carried only through second order terms, will be somewhat limited unless $\left| \frac{1}{3} - \frac{\nu}{N} \right|$ is small; reasonable accuracy might be expected, however, if $\left| \frac{1}{3} - \frac{\nu}{N} \right|$ were, say, as small as 0.01. A comparison of the analytic results with digital computations will be presented later in this report (Sect. D). We turn next to the applications of the analytic method of Moser to eqn. (2).

C. THE MOSER PROCEDURE

1. The Forward Transformations

In this section we undertake to treat eqn. (2) by the Moser procedure,[3] in a manner paralleling that presented in Sect. D 3 of I.[1] Our basic equation, eqn. (2), follows from the Hamiltonian

$$H = (1/2)\, p^2 + (1/2)\, (2\,\nu/N)^2\, v^2 + (1/6) \left[\sum_{m=1} b_m \sin 2\,m\,t \right] v^3, \tag{9}$$

which we now subject to a series of canonical transformations designed to eliminate the t-dependence from the cubic term in (9).

We commence by employing the generating function

$$G_o\,(v,\ \gamma_o) = (\nu/N)\, v^2 \operatorname{ctn}\ \gamma_o\ , \tag{10}$$

so that

$$p = \partial G_o / \partial v = (2\,\nu/N)\, v \operatorname{ctn}\ \gamma_o \tag{11a}$$

$$J_o = -\ \partial G_o / \partial \gamma_o = (\nu/N)\, v^2 \csc^2\ \gamma_o\ ; \tag{11b}$$

thus

$$\operatorname{ctn}\ \gamma_o = \frac{N}{2\nu}\ \frac{p}{v} \tag{12a}$$

$$J_o = \frac{1}{2} \left(\frac{N}{2\nu}\right) p^2 + \frac{1}{2} \left(\frac{2\,\nu}{N}\right) v^2\ , \tag{12b}$$

$$v = (N/\nu)^{1/2}\ J_o^{1/2}\ \sin\ \gamma_o \tag{12c}$$

$$p = 2\,(\nu/N)^{1/2}\ J_o^{1/2}\ \cos\ \gamma_o\ , \tag{12d}$$

and the new Hamiltonian is

$$\begin{aligned} K_o &= H + \partial G_o / \partial t \\ &= H \\ &= 2\,(\nu/N)\, J_o + (1/6)(N/\nu)^{3/2}\, J_o^{3/2} \sin^3 \gamma_o \sum_{m=1} b_m \sin 2\,m\,t \\ &= 2\,(\nu/N)\, J_o \\ &\quad + (1/48)(N/\nu)^{3/2} J_o^{3/2} \sum_{m=1} b_m \left[\begin{array}{l} 3 \cos(\gamma_o - 2\,m\,t) - 3 \cos(\gamma_o + 2\,m\,t) \\ + \cos(3\,\gamma_o + 2\,m\,t) - \cos(3\gamma_o - 2\,m\,t) \end{array} \right], \end{aligned} \tag{13}$$

with γ_o and J_o constituting respectively the new coordinate and momentum.

We now select as a second generating function

$$G_1(\gamma_o, J_1) = J_1 \cdot \gamma_o + \frac{1}{96}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left\{ b_1 \left[3\frac{\sin(\gamma_o - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_o + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_o + 2t)}{1 + 3\nu/N} \right] + \sum_{m=2} b_m \left[\begin{array}{l} 3\frac{\sin(\gamma_o - 2mt)}{m - \nu/N} + 3\frac{\sin(\gamma_o + 2mt)}{m + \nu/N} \\ - \frac{\sin(3\gamma_o - 2mt)}{m - 3\nu/N} - \frac{\sin(3\gamma_o + 2mt)}{m + 3\nu/N} \end{array} \right] \right\}, \tag{14}$$

so that

$$\begin{aligned} J_o &= \partial G_1/\partial \gamma_o \\ &= J_1 + \frac{1}{32}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left\{ b_1 \left[\frac{\cos(\gamma_o - 2t)}{1 - \nu/N} + \frac{\cos(\gamma_o + 2t)}{1 + \nu/N} - \frac{\cos(3\gamma_o + 2t)}{1 + 3\nu/N} \right] + \sum_{m=2} b_m \left[\begin{array}{l} \frac{\cos(\gamma_o - 2mt)}{m - \nu/N} + \frac{\cos(\gamma_o + 2mt)}{m + \nu/N} \\ - \frac{\cos(3\gamma_o - 2mt)}{m - 3\nu/N} - \frac{\cos(3\gamma_o + 2mt)}{m + 3\nu/N} \end{array} \right] \right\} \end{aligned} \tag{15a}$$

$$\begin{aligned} \gamma_1 &= \partial G_1/\partial J_1 \\ &= \gamma_o + \frac{1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2} \left\{ b_1 \left[3\frac{\sin(\gamma_o - 2t)}{1 - \nu/N} + 3\frac{\sin(\gamma_o + 2t)}{1 + \nu/N} - \frac{\sin(3\gamma_o + 2t)}{1 + 3\nu/N} \right] + \sum_{m=2} b_m \left[\begin{array}{l} 3\frac{\sin(\gamma_o - 2mt)}{m - \nu/N} + 3\frac{\sin(\gamma_o + 2mt)}{m + \nu/N} \\ - \frac{\sin(3\gamma_o - 2mt)}{m - 3\nu/N} - \frac{\sin(3\gamma_o + 2mt)}{m + 3\nu/N} \end{array} \right] \right\}, \end{aligned} \tag{15b}$$

and

$$\begin{aligned} K_1 &= K_o + \partial G_1/\partial t \\ &= K_o + \frac{1}{48}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \left\{ b_1 \left[-3\frac{\cos(\gamma_o - 2t)}{1 - \nu/N} + 3\frac{\cos(\gamma_o + 2t)}{1 + \nu/N} - \frac{\cos(3\gamma_o + 2t)}{1 + 3\nu/N} \right] + \sum_{m=2} b_m \left[\begin{array}{l} -3\frac{\cos(\gamma_o - 2mt)}{1 - \nu/mN} + 3\frac{\cos(\gamma_o + 2mt)}{1 + \nu/mN} \\ + \frac{\cos(3\gamma_o - 2mt)}{1 - 3\nu/mN} - \frac{\cos(3\gamma_o + 2mt)}{1 + 3\nu/mN} \end{array} \right] \right\}. \end{aligned} \tag{16}$$

It is now in order, of course, to express the new Hamiltonian, K_1, explicitly in terms of γ_1 and J_1. As a first step, substitution of J_o, as given by eqn. (15a), into K_o, as given by eqn. (13), results (after considerable simplification) in eqn. (16) assuming the following form, through terms of order J_1^2:

$$K_1 = 2\,(\nu/N)\,J_1 - \frac{b_1}{48}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2} \cos(3\gamma_o - 2t)$$
$$+ \frac{b_1^2}{2048}\left(\frac{N}{\nu}\right)^3 J_1^2 \left\{ \frac{6\,\nu/N}{1-\nu^2/N^2} - \frac{1}{1+3\,\nu/N} + \frac{6\nu}{N}\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\left[\frac{1}{m^2-\nu^2/N^2} + \frac{1}{m^2-9\nu^2/N^2}\right] + \sum_{m=1}\frac{b_m b_{m+2}}{b_1^2}\left[\frac{1}{m+3\,\nu/N} - \frac{1}{m+2-3\,\nu/N}\right]\cos 2(3\gamma_o - 2t) \right.$$
$$\left. + \text{ terms which are neither constant, nor involve circular functions of an argument which is a multiple of } 3\gamma_o - 2t \right\} \qquad (17)$$

It can be seen that the introduction of γ_1 in place of γ_o in eqn. (17) need not change the form of this result, since the substitution, based on eqn. (15b), which is involved in expressing $\cos(3\gamma_o - 2t)$ in terms of γ_1 does not introduce into the J_1^2 term any terms of the form which we have elected to retain. It may moreover be noted that there is little point to retaining the last term in eqn. (17), involving the cross products $b_m b_{m+2}$, since, to this order, $3\,\nu/N$ may here be set equal to unity with the result that the term in question vanishes. In this spirit, and in the interest of simplicity, we therefore write

$$K_1 = 2\,(\nu/N)\,J_1 - \frac{b_1}{48}\left(\frac{N}{\nu}\right)^{3/2} J_1^{3/2}\cos(3\gamma_1 - 2t) + \alpha\,\frac{b_1^2}{2048}\left(\frac{N}{\nu}\right)^3 J_1^2\,, \qquad (18)$$

where

$$\alpha \equiv \frac{6\,\nu/N}{1 - \nu^2/N^2} - \frac{1}{1 + 3\,\nu/N}$$

$$+ 6\frac{\nu}{N} \sum_{m=2} \left(\frac{b_m}{b_1}\right)^2 \left[\frac{1}{m^2 - \nu^2/N^2} + \frac{1}{m^2 - 9\,\nu^2/N^2}\right] \qquad (19)$$

[cf. eqn. (25) of I] and in which t-dependent terms have deliberately been omitted from the J_1^2 term of K_1.

For the final transformation we now, as in I, introduce the third generating function

$$G_2(\gamma_1, J_2) = J_2 \cdot (\gamma_1 - \tfrac{2}{3} t), \qquad (20)$$

which effects the transformation

$$J_1 = \partial G_2 / \partial \gamma_1 = J_2 \qquad (21a)$$

$$\gamma_2 = \partial G_2 / \partial J_2 = \gamma_1 - \frac{2}{3} t \qquad (21b)$$

with

$$\begin{aligned} K_2 &= K_1 + \partial G_2 / \partial t \\ &= K_1 - \frac{2}{3} J_2 \\ &= -2\left(\frac{1}{3} - \frac{\nu}{N}\right) J_2 - \frac{b_1}{48}\left(\frac{N}{\nu}\right)^{3/2} J_2^{3/2} \cos 3\gamma_2 + \alpha \frac{b_1^2}{2048}\left(\frac{N}{\nu}\right)^3 J_2^2 \end{aligned} \qquad (22)$$

and in which α is given by eqn. (19). K_2, which, as written, is independent of t, is now to be regarded as substantially a constant of the motion.

2. The Separatrix and Fixed Points

The expression (22) for K_2, which we take to be a constant of the motion, is virtually identical in form to eqn. (57) of I [Section D 3 of reference 1] and the succeeding step thus will parallel the corresponding work in I, save that the values of J_2 $(= J_1)$ will contain a factor $1/b_1^2$ and α is to be interpreted in the manner of eqn. (19).

The fixed points, corresponding to the unstable equilibrium orbit, are characterized by K_2 being stationary; i.e., by

$$\cos 3\gamma_2 = -1 \tag{23a}$$

$$\gamma_2 = \pm \pi/3, \pi \tag{23b}$$

$$\gamma_1 = \pm \pi/3 + 2t/3, \quad \pi + 2t/3 \tag{23c}$$

and

$$J_1^{1/2} = J_2^{1/2} = \frac{64}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^{3/2} \eta_1 , \tag{24}$$

where

$$\eta_1 = \frac{\sqrt{1 + 8\alpha\,(1/3 - \nu/N)} - 1}{4\alpha\,(1/3 - \nu/N)} \tag{25a}$$

$$= 1 - 2\alpha\,(1/3 - \nu/N) + \cdots . \tag{25b}$$

Other points on the separatrix are determined by eqn. (22), with K_2 given the value [implied by eqns. (23a) and (24)]

$$K_2 = -\frac{8192}{3 b_1^2}\left(\frac{\nu}{N}\right)^3\left(\frac{1}{3} - \frac{\nu}{N}\right)^3 \frac{\eta_1^2\,(3 - \eta_1)}{2} . \tag{26}$$

3. The Inverse Transformation

To obtain an expression for the unstable equilibrium orbit in terms of the original dependent variable, v, we perform the inverse transformation from γ_1, J_1, making use of eqn. (24) and (say) setting $\gamma_1 = \pi + 2t/3$ [cf. eqn. (23c)]. We thus write

$$J_0^{1/2} \doteq J_1^{1/2}\left[1 - \frac{b_1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2} \cdot R\right] \tag{27a}$$

$$\sin\gamma_0 \doteq \sin\gamma_1 - (\cos\gamma_1)(\gamma_1 - \gamma_0)$$

$$\doteq \sin\gamma_1 + \frac{b_1 \cos\gamma_1}{64}\left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2} \cdot S \tag{27b}$$

and

$$\cos \gamma_o \doteq \cos \gamma_1 + (\sin \gamma_1)(\gamma_1 - \gamma_o)$$
$$\doteq \cos \gamma_1 - \frac{b_1 \sin \gamma_1}{64} \left(\frac{N}{\nu}\right)^{3/2} J_1^{1/2} \cdot S\,, \qquad (27c)$$

where

$$R \equiv \frac{\cos 4\,t/3}{1 - \nu/N} + \frac{\cos 8\,t/3}{1 + \nu/N} - \frac{\cos 4\,t}{1 + 3\nu/N} + \sum_{m=2} \frac{b_m}{b_1} \left[\frac{\cos (2/3)(3\,m - 1)\,t}{m - \nu/N} + \frac{\cos (2/3)(3\,m + 1)\,t}{m + \nu/N} - \frac{\cos 2\,(m - 1)\,t}{m - 3\,\nu/N} - \frac{\cos 2\,(m + 1)\,t}{m + \nu/N} \right] \qquad (27d)$$

and

$$S \equiv -3\,\frac{\sin 4\,t/3}{1 - \nu/N} + 3\,\frac{\sin 8\,t/3}{1 + \nu/N} - \frac{\sin 4\,t}{1 + 3\,\nu/N} + \sum_{m=2} \frac{b_m}{b_1} \left[-3\,\frac{\sin (2/3)(3\,m - 1)\,t}{m - \nu/N} + 3\,\frac{\sin (2/3)(3\,m + 1)\,t}{m + \nu/N} + \frac{\sin 2\,(m - 1)\,t}{m - 3\,\nu/N} - \frac{\sin 2\,(m + 1)\,t}{m + 3\,\nu/N} \right]. \qquad (27e)$$

Accordingly [cf. eqn. (12c)]

$$v = (N/\nu)^{1/2} J_o^{1/2} \sin \gamma_o$$
$$= -(N/\nu)^{1/2} J_1^{1/2} \left[1 - \left(\frac{1}{3} - \frac{\nu}{N}\right) \eta_1 \cdot R \right] \left[\sin 2\,t/3 + \left(\frac{1}{3} - \frac{\nu}{N}\right) \eta_1 (\cos 2\,t/3)\, S \right]$$

$$= -\frac{64}{b_1} \left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right) \eta_1 \left\{ \sin 2\,t/3 - \left[\frac{\sin 2\,t/3}{1 - \nu/N} + \frac{4(\nu/N)\sin 2\,t}{1 - \nu^2/N^2} - \left(\frac{1}{1 + \nu/N} - \frac{1}{1 + 3\nu/N} \right) \sin 10t/3 + \sum_{m=2} \frac{b_m}{b_1} \left\{ \left(\frac{1}{m - \nu/N} - \frac{1}{m - 3\,\nu/N} \right) \sin (2/3)(3\,m - 2)\,t + \frac{4\,(\nu/N) \sin 2\,m\,t}{m^2 - \nu^2/N^2} - \left(\frac{1}{m + \nu/N} - \frac{1}{m + 3\,\nu/N} \right) \sin (2/3)(3\,m + 2)\,t \right\} \right] \left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \right\}; \qquad (28a)$$

similarly [cf. eqn. (12d)]

$$p = 2\,(\nu/N)^{1/2} J_o^{1/2} \cos \gamma_o$$

$$= -2\,(\nu/N)^{1/2} J_1^{1/2}\left[1 - \left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \cdot R\right]\left[\cos 2\,t/3 - \left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \left(\sin\frac{2\,t}{3}\right) S\right]$$

$$= -\frac{128}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2 \eta_1 \left\{ \cos 2\,t/3 + \left[\left(\frac{\cos 2t/3}{1 - \nu/N} - \frac{4\cos 2t}{1 - \nu^2/N^2}\right) + \left(\frac{1}{1 + \nu/N} + \frac{1}{1 + 3\nu/N}\right)\cos 10t/3 + \sum_{m=2} \frac{b_m}{b_1}\left\{ \begin{array}{l} \left(\frac{1}{m - \nu/N} + \frac{1}{m - 3\,\nu/N}\right)\cos\,(2/3)\,(3\,m - 2)t \\ -\ \frac{4\,m\cos 2\,m\,t}{m^2 - \nu^2/N^2} \\ +\left(\frac{1}{m + \nu/N} + \frac{1}{m + 3\,\nu/N}\right)\cos\,(2/3)\,(3\,m + 2)t \end{array} \right\} \right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \right\}. \qquad (28b)$$

For comparison with the results of Section B, we may first examine the coefficient of $\sin 2\,t/3$ in the expression for v shown in eqn. (28a), making certain simplifications consistent with retention of terms through those of order $\left(\frac{1}{3} - \frac{\nu}{N}\right)^2$. This coefficient is

$$A_1 = -\frac{64}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1 \left[1 - \frac{1/3 - \nu/N}{1 - \nu/N}\,\eta_1\right] \qquad (29a)$$

$$\doteq -\frac{64}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\left[1 - \left(2\,\alpha + \frac{1}{1 - \nu/N}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\right] \qquad (29b)$$

$$\doteq -\frac{64}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\left[1 - \left(2\,\alpha + \frac{3}{2}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\right] \qquad (29c)$$

$$\doteq -\frac{64}{3\,b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left[1 - \left(2\,\alpha + \frac{9}{2}\right)\left(\frac{1}{3} - \frac{\nu}{N}\right)\right] \qquad (29d)$$

and, with

$$\alpha \doteq 7/4 + 4\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2 \frac{9\,m^2 - 5}{(9\,m^2 - 1)\,(m^2 - 1)} \qquad \left[\text{cf. eqn. (19)}\right], \qquad (30)$$

$$A_1 \doteq -\frac{64}{3\,b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left\{1-8\left[1+\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\frac{9\,m^2-5}{(m^2-1)(9\,m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}, \tag{29e}$$

in agreement with the expression given as eqn. (7a). A similar reduction of the coefficient of $\cos 2\,t/3$ in the expression (28b) for p leads to a quantity which is 2/3 of formula (29e) for A_1, as it of course should since $p = dv/dt$.

Similar reductions of the remaining (second order) terms in the trigonometric series for v and p, as given by eqns. (28a, b), leads to the coefficients listed below in Table I.

TABLE I

COEFFICIENTS OF SECOND ORDER TERMS IN THE TRIGONOMETRIC SERIES FOR v AND p, FROM EQUATIONS 28a AND 28b.

Argument	Sine Coefficient in v	Cosine Coefficient in p
$2\,t$	$+\frac{32}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$	$+\frac{64}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$
$10\,t/3$	$-\frac{16}{3\,b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$	$-\frac{160}{9\,b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$
$(2/3)(3m-2)t$	$-\frac{128\,b_m}{3\,b_1^2}\frac{1}{(m-1)(3\,m-1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$	$-\frac{256\,b_m}{9\,b_1^2}\frac{3\,m-2}{(m-1)(3\,m-1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$
$2\,m\,t$	$+\frac{256\,b_m}{b_1^2}\frac{1}{9\,m^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$	$+\frac{512\,b_m}{b_1^2}\frac{m}{9\,m^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$
$(2/3)(3m+2)t$	$-\frac{128\,b_m}{3\,b_1^2}\frac{1}{(m+1)(3\,m+1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$	$-\frac{256\,b_m}{9\,b_1^2}\frac{3\,m+2}{(m+1)(3\,m+1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$.

The coefficients listed here for the terms appearing in eqn. (28a) for v are immediately seen to be concordant with the coefficients of the trial function of Section B, as listed in eqns. (7b-f). Similarly the coefficients listed for p are seen to be related to those given r v in a way consistent with $p = dv/dt$.

Coordinates of fixed points may of course be obtained directly from eqns. (28a, b). Thus, for one of the fixed points at $t = 0$ one finds

$$v = 0 \tag{31a}$$

$$p = -\frac{128}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2 \eta_1 \left\{ 1 - \left[\begin{array}{l} \dfrac{2}{1 - \nu^2/N^2} - \dfrac{1}{1 + 3\,\nu/N} \\ -2\sum\limits_{m=2} m\,\dfrac{b_m}{b_1}\left(\dfrac{1}{m^2 - 9\nu^2/N^2} - \dfrac{1}{m^2 - \nu^2/N^2}\right) \end{array} \right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \right\}. \tag{31b}$$

This expression (31b) for p may be somewhat simplified if various reductions are made by aid of $\eta_1 \cong 1 - 2\alpha\left(\frac{1}{3} - \frac{\nu}{N}\right)$, use of eqn. (30), and the approximation $(\nu/N)^2 \cong \frac{1}{9}\left[1 - 6\left(\frac{1}{3} - \frac{\nu}{N}\right)\right]$:

$$p \doteq -\frac{128}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2 \eta_1 \left\{ 1 - \left[\frac{7}{4} - 16\sum_{m=2}\frac{m\,(b_m/b_1)}{(m^2-1)(9\,m^2-1)}\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\right\}$$

$$\doteq -\frac{128}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2 \left\{ 1 - \left[\frac{21}{4} - 8\sum_{m=2}\frac{2m(b_m/b_1) - (9m^2-5)(b_m/b_1)^2}{(m^2-1)\,(9\,m^2-1)}\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\right\}$$

$$\doteq -\frac{128}{9\,b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left\{ 1 - \left[\frac{45}{4} - 8\sum_{m=2}\frac{2\,m\,(b_m/b_1) - (9\,m^2-5)\,(b_m/b_1)^2}{(m^2-1)\,(9\,m^2-1)}\right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\right\}, \tag{31b'}$$

which is in agreement with the result (8b) found in Section B. The other unstable fixed points associated with this value of t likewise may be obtained, by the substitution of $t = \pm\pi$ in eqns. (28a, b):

$$v = \mp\frac{32\sqrt{3}}{b_1}\left(\frac{1}{3} - \frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1 \left\{ 1 - \left[\begin{array}{l} \dfrac{2}{1 - \nu^2/N^2} - \dfrac{1}{1 + 3\,\nu/N} \\ -2\sum\limits_{m=2} m\,\dfrac{b_m}{b_1}\left(\dfrac{1}{m^2 - 9\nu^2/N^2} - \dfrac{1}{m^2 - \nu^2/N^2}\right) \end{array} \right]\left(\frac{1}{3} - \frac{\nu}{N}\right)\eta_1 \right\} \tag{32a}$$

$$\doteq \mp \frac{32\sqrt{3}}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\eta_1\left\{1-\left[\frac{7}{4}-16\sum_{m=2}\frac{b_m}{b_1}\frac{m}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}$$

$$\doteq \mp \frac{32\sqrt{3}}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)\left\{1-\left[\frac{21}{4}-8\sum_{m=2}\frac{2m(b_m/b_1)-(9m^2-5)(b_m/b_1)^2}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}$$

$$\doteq \mp \frac{32\sqrt{3}}{3b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left\{1-\left[\frac{33}{4}-8\sum_{m=2}\frac{2m(b_m/b_1)-(9m^2-5)(b_m/b_1)^2}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}, \quad (32a')$$

$$p = \frac{64}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left\{1+\left[\frac{10}{1-\nu^2/N^2}+\frac{1}{1+3\nu/N} + 2\sum_{m=2} m\frac{b_m}{b_1}\left(\frac{5}{m^2-\nu^2/N^2}+\frac{1}{m^2-9\nu^2/N^2}\right)\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\eta_1\right\} \quad (32b)$$

$$\doteq \frac{64}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\eta_1\left\{1+\left[\frac{47}{4}+4\sum_{m=2} m\frac{b_m}{b_1}\frac{27m^2-23}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}$$

$$\doteq \frac{64}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left(\frac{\nu}{N}\right)^2\left\{1+\left[\frac{33}{4}+4\sum_{m=2}\frac{m(27m^2-23)(b_m/b_1)-2(9m^2-5)(b_m/b_1)^2}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}$$

$$\doteq \frac{64}{9b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left\{1+\left[\frac{9}{4}+4\sum_{m=2}\frac{m(27m^2-23)(b_m/b_1)-2(9m^2-5)(b_m/b_1)^2}{(m^2-1)(9m^2-1)}\right]\left(\frac{1}{3}-\frac{\nu}{N}\right)\right\}. \quad (32b')$$

The reduced forms (32a') and (32b') agree with the value of the trial function of Section B and its derivative at $t = \pm\pi$, namely $v = \pm(\sqrt{3}/2)\sum_{m=1}(A_m - C_m)$ and $dv/dt = -(1/3)\sum_{m=1}\left[(3m-2)A_m - 6mB_m + (3m+2)C_m\right]$, when the coefficients are taken as given by eqns. (7a-f).

The coefficients of the trigonometric development of the unstable equilibrium orbit, and particular fixed-point coordinates, are thus seen to agree, through terms in $\left(\frac{1}{3}-\frac{\nu}{N}\right)^2$, when obtained by the variational method or by the Moser procedure. In the following Section we present some computational checks of these results.

D. COMPUTATIONAL CHECKS

The analytic results of Sections B and C for the limiting-amplitude solution of eqn. (2), for which the solution was carried through terms of order $(\nu/N - 1/3)^2$, have been subjected to computational checks[4] for a series of examples in which

$$b_1 = 1, \quad b_3 = 3/4, \quad \text{and} \quad b_5 = 1/2, \tag{33}$$

and in which ν/N successively assumed the values

0.3267,

0.33,

0.3367, and

0.34.

The computational results for the trigonometric representation of the unstable equilibrium orbit, and for the coordinates (v, p) of the fixed points corresponding to t = 0, were compared with the results of the analytic work, both in the form obtained directly from application of the Moser method and in the simplified, or "reduced", forms in which the results also could be expressed. A particularly decisive test of the results might be afforded by examining explicitly the coefficient of $(\nu/N - 1/3)^2$ in the results--thus by forming

$$\frac{1 - \frac{9\, b_1}{128} \frac{(-p)}{\frac{1}{3} - \frac{\nu}{N}}}{\frac{1}{3} - \frac{\nu}{N}}$$

one might expect to obtain a result which would approach

$$\frac{45}{4} - 8 \sum_{m=2} \frac{2\, m\, (b_m/b_1) - (9\, m^2 - 5)\, (b_m/b_1)^2}{(m^2 - 1)\,(9\, m^2 - 1)} \doteq 11.80$$

as $\nu/N \rightarrow 1/3$ [cf. eqn. (31b')]. From such tests it appeared that the coefficients of interest were approximately of the size expected but assumed limiting values which depended appreciably on the Runge-Kutta interval employed in the computations--thus with $N_{RK} = 64$ (requiring runs of length $N_E = 960$ Runge-Kutta steps), the limiting value of

$$\frac{1 - \frac{9\, b_1}{128} \frac{(-\, p)}{\frac{1}{3} - \frac{\nu}{N}}}{\frac{1}{3} - \frac{\nu}{N}}$$

appeared to be about 11.7. In the results reported below, the computational results are taken primarily from runs made with $N_{RK} = 64$.

In Table II we list the Fourier coefficients of the unstable equilibrium orbit for the cases studied. For each argument listed, the first line gives the value of the coefficient expected from the results of the Moser theory [eqns. (28a, b)]; the second line gives the value obtained from the reduced forms [see eqn. (29e) and Table I]; and the third line gives the coefficients obtained computationally.

In Table III we similarly list the fixed-point coordinates, for $t = 0$. The agreement between the analytic and computational results, as illustrated by Table II and Table III, is felt to be completely satisfactory.

TABLE II

FOURIER COEFFICIENTS IN UNSTABLE EQUILIBRIUM ORBIT

$b_1 = 1$ $b_3 =$ [illegible]4 $b_5 = 1/2$

Argument	Sine Coefficient in v				Cosine Coefficient in p			
	ν/N				ν/N			
	0.3267	0.3300	0.3367	0.3400	0.3267	0.3300	0.3367	0.3400
2 t/3	-.133 9152(a) -.133 4186(b) -.134 1$_{351}$(c)	-.069 1337 -.069 0676 -.069 1$_{799}$	+.073 9787 +.073 9068 +.073 9$_{996}$	+.150 9863 +.150 3963 +.151 3$_{083}$	-.089 1973(a) -.088 9457(b) -.089 4$_{23}$ (c)	-.046 0785 -.046 0451 -.046 1$_{20}$	+.049 3068 +.049 2712 +.049 3$_{33}$	+.100 5561 +.100 2642 +.100 8$_{72}$
2 t	+.001 2792 +.001 4080 +.001 25$_{94}$	+.000 3385 +.000 3556 +.000 33$_{57}$	+.000 3818 +.000 3627 +.000 38$_{59}$	+.001 5780 +.001 4222 +.001 61$_{75}$	+.002 5584 +.002 8161 +.002 51$_{9}$	+.000 6771 +.000 7111 +.000 67$_{1}$	+.000 7637 +.000 7254 +.000 77$_{2}$	+.003 1561 +.002 8444 +.003 23$_{5}$
10 t/3	-.000 2175 -.000 2347 -.000 210$_{1}$	-.000 0570 -.000 0593 -.000 056$_{0}$	-.000 0630 -.000 0605 -.000 064$_{3}$	-.000 2578 -.000 2370 -.000 269$_{3}$	-.000 7192 -.000 7822 -.000 70$_{0}$	-.000 1892 -.000 1975 -.000 18$_{7}$	-.000 2109 -.000 2015 -.000 21$_{4}$	-.000 8662 -.000 7901 -.000 89$_{8}$
14 t/3	-.000 0794 -.000 0880 -.000 078$_{5}$	-.000 0211 -.000 0222 -.000 020$_{9}$	-.000 0240 -.000 0227 -.000 024$_{1}$	-.000 0994 -.000 0889 -.000 101$_{3}$	-.000 3724 -.000 4107 -.000 36$_{7}$	-.000 0987 -.000 1037 -.000 09$_{8}$	-.000 1115 -.000 1058 -.000 11$_{3}$	-.000 4611 -.000 4148 -.000 47$_{3}$
6 t	+.000 0964 +.000 1056 +.000 094$_{7}$	+.000 0254 +.000 0267 +.000 025$_{2}$	+.000 0286 +.000 0272 +.000 028$_{9}$	+.000 1178 +.000 1067 +.000 121$_{0}$	+.000 5782 +.000 6336 +.000 56$_{8}$	+.000 1527 +.000 1600 +.000 15$_{1}$	+.000 1714 +.000 1632 +.000 17$_{3}$	+.000 7069 +.000 6400 +.000 72$_{6}$
22 t/3	-.000 0324 -.000 0352 -.000 031$_{8}$	-.000 0085 -.000 0089 -.000 008$_{4}$	-.000 0095 -.000 0091 -.000 009$_{6}$	-.000 0390 -.000 0356 -.000 039$_{9}$	-.000 2365 -.000 2581 -.000 23$_{4}$	-.000 0623 -.000 0652 -.000 06$_{2}$	-.000 0697 -.000 0665 -.000 07$_{0}$	-.000 2869 -.000 2607 -.000 29$_{3}$
26 t/3	-.000 0152 -.000 0168 -.000 014$_{8}$	-.000 0040 -.000 0042 -.000 003$_{9}$	-.000 0045 -.000 0043 -.000 004$_{6}$	-.000 0188 -.000 0169 -.000 019$_{5}$	-.000 1322 -.000 1453 -.000 12$_{8}$	-.000 0350 -.000 0367 -.000 03$_{4}$	-.000 0394 -.000 0374 -.000 04$_{0}$	-.000 1625 -.000 1467 -.000 16$_{9}$
10 t	+.000 0230 +.000 0251 +.000 022$_{5}$	+.000 0061 +.000 0063 +.000 006$_{0}$	+.000 0068 +.000 0065 +.000 006$_{9}$	+.000 0280 +.000 0254 +.000 028$_{8}$	+.000 2295 +.000 2514 +.000 22$_{5}$	+.000 0606 +.000 0635 +.000 06$_{0}$	+.000 0680 +.000 0648 +.000 06$_{9}$	+.000 2804 +.000 2540 +.000 28$_{8}$
34 t/3	-.000 0090 -.000 0098 -.000 009$_{0}$	-.000 0024 -.000 0025 -.000 002$_{3}$	-.000 0026 -.000 0025 -.000 002$_{6}$	-.000 0109 -.000 0099 -.000 010$_{9}$	-.000 1014 -.000 1108 -.000 10$_{3}$	-.000 0267 -.000 0280 -.000 02$_{7}$	-.000 0299 -.000 0285 -.000 03$_{0}$	-.000 1233 -.000 1119 -.000 12$_{3}$

(a)Eqn. (28a)
(b)Reduced forms(29e), et seq.
(c)Computational

(a)Eqn. (28a)
(b)Reduced forms
(c)Computational

TABLE III

FIXED POINT COORDINATES

($t = 0$, mod. 2π)

ν/N	On Symmetry Axis ($b_1 = 1$)	To Right and Left of Symmetry Axis ($b_3 = 3/4$)	($b_5 = 1/2$)
	p	v	p
0.3267	-.087 393[a]	∓.115 832[a]	+.048 746[a]
	-.086 955[b]	∓.115 396[b]	+.049 029[b]
	-.087 64[c]	∓.116 0$_{40}$[c]	+.048 7$_{94}$[c]
0.33	-.045 600	∓.059 834	+.024 136
	-.045 542	∓.059 777	+.024 173
	-.045 65	∓.059 8$_{92}$	+.024 1$_{53}$
0.3367	+.049 849	±.064 108	-.023 420
	+.049 784	±.064 043	-.023 462
	+.049 87	±.064 1$_{12}$	-.023 4$_{13}$
0.34	+.102 799	±.130 922	-.045 185
	+.102 275	±.130 396	-.045 530
	+.103 16	±.131 2$_{00}$	-.045 2$_{04}$

b_1	b_3	b_5
(a) Eqn. (31b)	(a) Eqn. (32a)	(a) Eqn. (32b)
(b) Eqn. (31b')	(b) Eqn. (32a')	(b) Eqn. (32b')
(c) Computed	(c) Computed	(c) Computed

E. REFERENCES

1. L. Jackson Laslett, MURA-452 (April 13, 1959), hereinafter designated as I.

2. E. D. Courant and H. S. Snyder, Annals of Physics 3, No. 1, 1-48 (January, 1958)--Section 4a, esp. eqns. (4.4) and (4.5), p. 18.

3. Jürgen Moser, Nach. Gött. Akad. (Math.-Phys. Kl.) Nr. 6, 87-120 (1955).

4. The computational work was performed with the MURA IBM 704, by means of the DUCK-ANSWER program [J. N. Snyder, (IBM Program 75), MURA-237 (1957)], with the independent variable, τ, of the program identified as $\tau = 5\,t$ and with the dependent variable (ρ or ψ) usually identified as 10 times the dependent variable (v) of eqn. (2). Accordingly, dv/dt is then represented by $0.5\, d\rho/d\tau$ or $0.5\, d\psi/d\tau$. The coefficients of the program are then taken to be

$$S_1 = S_2 = -0.016\,(\nu/N)^2,$$
$$A_3 = A_{15} = 0.001,$$
$$B_2 = B_{15} = 0.002,$$
$$C_3 = C_{15} = 0.0015,$$

with $N_1 = 10$, $N_2 = 5$, and $\alpha_3 = \alpha_{15} = \beta_3 = \beta_{15} = \gamma_3 = \gamma_{15} = 0.5$. If one selects $N_{RK} = 64$, a computational run through an interval $\Delta t = 3\pi$ requires a total of $N_E = 960$ Runge-Kutta integration steps. For Fourier analysis of the results of a DUCK-ANSWER computation, the DUCKNALL program was employed [John McNall, (IBM Program 219), MURA-438 (1958)], this program constituting basically an incorporation into the DUCK-ANSWER program of the FORANAL program [J. N. Snyder, (IBM Program 52), MURA-228 (1957)].

APPENDIX A

SOLUTION OF EQNS. 6a-f FOR THE COEFFICIENTS OF THE TRIAL FUNCTION

From eqns. (6b-f) we immediately obtain

$$B_1 = (1/16)\, b_1 A_1^2 \left[1 - (\nu/N)^2\right]^{-1} \tag{A-1a}$$

$$C_1 = -(9/32)\, b_1 A_1^2 \left[25 - 9\,(\nu/N)^2\right]^{-1} \tag{A-1b}$$

$$\left.\begin{aligned} A_m &= -(9/32)\, b_m A_1^2 \left[(3m-2)^2 - 9\,(\nu/N)^2\right]^{-1} && \text{(A-1c)} \\ B_m &= (9/16)\, b_m A_1^2 \left[(3m)^2 - 9\,(\nu/N)^2\right]^{-1} && \text{(A-1d)} \\ C_m &= -(9/32)\, b_m A_1^2 \left[(3m+2)^2 - 9\,(\nu/N)^2\right]^{-1}. && \text{(A-1e)} \end{aligned}\right\} \quad m \geq 2$$

By insertion of the expressions (A-1a-e) into eqn. (6a), and rejection of the trivial root $A_1 = 0$, the quadratic equation for A_1 is obtained:

$$2\left[1 - 9\,(\nu/N)^2\right] + 9\, b_1 A_1 - 9\, b_1^2 A_1^2 \left[\frac{1/4}{1-(\nu/N)^2} + \frac{9/16}{25 - 9\,(\nu/N)^2}\right]$$
$$- \frac{81}{16} A_1^2 \sum_{m=2} b_m^2 \left[\frac{1}{(3m-2)^2 - 9\,(\nu/N)^2} + \frac{4}{(3m)^2 - 9\,(\nu/N)^2} + \frac{1}{(3m+2)^2 - 9\,(\nu/N)^2}\right] = 0. \tag{A-2}$$

An approximate solution of eqn. (A-2) then gives

$$A_1 = -\frac{32}{9b_1}\left[1 - 9(\nu/N)^2\right]\left[1 - \frac{32}{81}\left\{9\left[\frac{1/4}{1-(\nu/N)^2} + \frac{9/16}{25 - 9\,(\nu/N)^2}\right] + \frac{81}{16}\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\left[\frac{1}{(3m-2)^2 - 9(\nu/N)^2} + \frac{4}{(3m)^2 - 9(\nu/N)^2} + \frac{1}{(3m+2)^2 - 9(\nu/N)^2}\right]\right\}\left[1 - 9\,(\nu/N)^2\right]\right]$$

$$= -\frac{32}{b_1}\left[\frac{1}{9} - \left(\frac{\nu}{N}\right)^2\right]\left[1 - 9\left\{\frac{8/9}{1-(\nu/N)^2} + \frac{2}{25 - 9\,(\nu/N)^2} + 2\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\left[\frac{1}{(3m-2)^2 - 9(\nu/N)^2} + \frac{4}{(3m)^2 - 9(\nu/N)^2} + \frac{1}{(3m+2) - 9(\nu/N)^2}\right]\right\}\left[\frac{1}{9} - \left(\frac{\nu}{N}\right)^2\right]\right]$$

$$\cong -\frac{32}{b_1}\left[\frac{1}{9}-\left(\frac{\nu}{N}\right)^2\right]\left[1-9\left\{\frac{13}{12}+2\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\left[\frac{1}{(3m-2)^2-1}+\frac{4}{(3m)^2-1}+\frac{1}{(3m+2)^2-1}\right]\right\}\left[\frac{1}{9}-\left(\frac{\nu}{N}\right)^2\right]\right]$$

$$\cong -\frac{64}{3b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left[1-\frac{3}{2}\left(\frac{1}{3}-\frac{\nu}{N}\right)\right]\left[1-6\left\{\frac{13}{12}+\frac{4}{3}\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\frac{9m^2-5}{(m^2-1)(9m^2-1)}\right\}\left(\frac{1}{3}-\frac{\nu}{N}\right)\right]$$

$$\cong -\frac{64}{3b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)\left[1-8\left\{1+\sum_{m=2}\left(\frac{b_m}{b_1}\right)^2\frac{9m^2-5}{(m^2-1)(9m^2-1)}\right\}\left(\frac{1}{3}-\frac{\nu}{N}\right)\right], \quad \text{(A-3a)}$$

in which ν/N has been replaced by 1/3 in terms such that a simplification could thereby be achieved consistent with the objective of retaining accuracy through order $(1/3-\nu/N)^2$. To this same order we also obtain, by substitution of $A_1 \cong -\frac{64}{3b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)$ into eqns. (A-1a-e) in turn,

$$B_1 = \frac{32}{b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 \quad \text{(A-3b)}$$

$$C_1 = -\frac{16}{3b_1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 \quad \text{(A-3c)}$$

$$A_m = -\frac{128}{b_1}\frac{b_m/b_1}{(3m-2)^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 = -\frac{128}{3b_1}\frac{b_m/b_1}{(m-1)(3m-1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 \quad \text{(A-3d)}$$

$$B_m = \frac{256}{b_1}\frac{b_m/b_1}{(3m)^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 = \frac{256}{b_1}\frac{b_m/b_1}{9m^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 \qquad m \geq 2 \quad \text{(A-3e)}$$

$$C_m = -\frac{128}{b_1}\frac{b_m/b_1}{(3m+2)^2-1}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2 = -\frac{128}{3b_1}\frac{b_m/b_1}{(m+1)(3m+1)}\left(\frac{1}{3}-\frac{\nu}{N}\right)^2. \quad \text{(A-3f)}$$

It is these equations which have been taken as eqns. (7a-f) in the main body of the text. The results for the special case $b_m = 0$ ($m \geq 2$) can be seen to be consistent, through order ϵ^2, with equations (10a-c) of I [Section C 1 of refereence 1].