Superalgebras of Symmetries in Superquantum Mechanics*

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ABSTRACT

We discuss two supersymmetrization procedures -the so-called standard and spin-orbit coupling ones- when applied to the harmonic oscillator in the n = 1, 2, 3, arbitrary-dimensional cases as well as to other examples directly and simply connected with the harmonic context. We study their dynamical and kinematical (super)symmetries and their inclusions are mentioned.

1. The standard and spin-orbit coupling procedures of supersymmetrization

Let us consider the supersymmetric N = 2-system corresponding to a spin-$\frac{1}{2}$ particle moving on a line [1] characterized by two supercharges

$$Q_\pm = \frac{1}{\sqrt{2}} \left(p \mp i \frac{dW}{dx}\right)\xi_\pm \tag{1.1}$$

where $p = -i \frac{d}{dx}$, $[p,x] = -i$, $W(x)$ being the superpotential and $\xi_\pm$ the fermionic variables satisfying

$$\{\xi_+,\xi_-\} = 1 \quad , \quad \{\xi_\pm,\xi_\pm\} = 0 \tag{1.2}$$

and leading to a 2-dimensional Clifford algebra

$$\{\varphi^a,\varphi^b\} = 2\,\delta^{ab} \quad , \quad \varphi^1 = \xi_+ + \xi_- \quad , \quad \varphi^2 = i(\xi_- - \xi_+) \quad . \tag{1.3}$$

We then get the well known hamiltonian

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$$H = \{Q_+,Q_-\} = \tfrac{1}{2}[p^2 + (\frac{dW}{dx})^2 + \frac{d^2W}{dx^2}[\xi_+,\xi_-]] \quad . \tag{1.4}$$

The extension to the n-dimensional case suggests to consider two particular kinds of supersymmetrization procedures. The first one -the so-called standard procedure [1]- consists in a generalization of the relations (1.2) or (1.3) on the form

$$\{\xi_{+,k},\xi_{-,\ell}\} = \delta_{k\ell},\ \{\xi_{\pm,k},\xi_{\pm,\ell}\} = 0,\ \{\varphi_k^a,\varphi_\ell^b\} = 2\ \delta^{ab}\delta_{k\ell} \tag{1.5}$$

between the fermionic variables $\xi_{\pm,k}$ or φ_k^a $(k = 1,\ldots,n\ ;\ a = 1,2)$ corresponding to the description of a 2n-dimensional Clifford algebra. In such a procedure we deal with a description admitting the same number (2n) of bosonic and fermionic degrees of freedom.

The type-Q supercharges generalizing (1.1) are then given by

$$Q_\pm = \frac{1}{\sqrt{2}}\,(p_k \mp i\,\frac{\partial W}{\partial x_k})\xi_{\pm,k} \tag{1.6}$$

and lead with (1.5) to the supersymmetric hamiltonian

$$H = \tfrac{1}{2}[p_k p_k + (\frac{\partial W}{\partial x_k})^2] + \tfrac{1}{2}\,\frac{\partial^2 W}{\partial x_i \partial x_j}[\xi_+^i,\xi_-^j] = H_0 + H_1 \quad , \tag{1.7}$$

where H_0 corresponds to the usual bosonic hamiltonian in the presence of an external potential $U = \tfrac{1}{2}(\frac{\partial W}{\partial x_k})^2$.

The second procedure of supersymmetrization -the so-called spin-orbit coupling one [2,3]- can be characterized [3] by $(k,\ell = 1,\ldots,n)$

$$\{\xi_{+,k}\xi_{-,\ell}\} = \delta_{k\ell}-i\Xi_{k\ell},\ \Xi_{k\ell} = -\Xi_{\ell k}\ (\Xi^+=\Xi),\ \{\xi_{\pm,k},\xi_{\pm,\ell}\} = 0 \tag{1.8}$$

or

$$\{\varphi_k^a,\varphi_\ell^b\} = 2(\delta^{ab}\delta_{k\ell}+\epsilon^{ab}\Xi_{k\ell}),\ \epsilon^{12} = -\epsilon^{21} = 1,\ a,b = 1,2 \quad . \tag{1.9}$$

These relations essentially differ from (1.5) for the standard procedure. We deal here with a smaller number of fermionic degrees of freedom than the bosonic ones.

Now the supersymmetric hamiltonian is obtained from the supercharges (1.6) but with the relations (1.8). We get

$$H' = \tfrac{1}{2}[p_k^2 + (\frac{\partial W}{\partial x_k})^2] + \tfrac{1}{2}\,\frac{\partial^2 W}{\partial x_i \partial x_j}\,[\xi_+^i,\xi_-^j] - \tfrac{1}{2}[(\partial_i W)p_j-(\partial_j W)p_i)\Xi^{ij} \quad . \tag{1.10}$$

2. Examples of supersymmetric systems with the standard procedure

2.a The n-dimensional harmonic oscillator. The supersymmetric harmonic oscillator system in the n-dimensional case is described by the hamiltonian H = (1.4) with the superpotential $W = \frac{1}{2}\,\omega x_k x_k$. We then have

$$H_0^{SS} = \tfrac{1}{2}(p_k^2 + \omega^2 x_k^2) + \frac{\omega}{2}\,[\xi_{+,k},\xi_{-,k}] = H_0 + H_1 \tag{2.1}$$

with the corresponding type Q-supercharges (cf. (1.6))

$$Q_\pm = \frac{1}{\sqrt{2}}\,(p_k \mp i\omega x_k)\xi_{\pm,k} = \mp\, i\sqrt{\omega}\; a_{\mp,k}\xi_{\pm,k} \quad , \tag{2.2}$$

$a_\pm$ being the well known creation and annihilation operators. It is interesting to notice that in such a context the hamiltonian admits distinct bosonic and fermionic parts without coupling term between bosonic and fermionic variables.

Let us now give the largest maximal invariance superalgebras for such a system. Starting from the dynamical point of view [4], let us recall that the maximal dynamical invariance (MDI) algebra for H_0 is the algebra sp(2n) $\oplus\!\!\!\!\supset$ h(2n) generated by

$$T_{k\ell} = \frac{\omega}{2}\,\{a_{-,k},a_{+,\ell}\},\; C_{\pm,k\ell} = \pm\,\frac{i\omega}{2}e^{\mp 2i\omega t}\{a_{\pm,k},a_{\pm,\ell}\} \tag{2.3}$$

for sp(2n) and by

$$P_{\pm,k} = (\pm\, i\, e^{\mp i\omega t}(\omega x_k \mp i p_k)) = \pm\, i\,\sqrt{2\omega}\; e^{\mp i\omega t}a_{\pm,k} \tag{2.4}$$

together with the identity operator for h(2n), the so-called Heisenberg algebra. Now the MDI superalgebra [5,6] is osp(2n/2n) $\oplus\!\!\!\!\supset$

sh(2n/2n) where the generators of osp(2n/2n) are associated with the bosonic symmetries corresponding to sp(2n) but also with the fermionic ones described by the algebra so(2n) with the generators

$$Y_{k\ell} = \frac{\omega}{2}\,[\xi_{+,k},\xi_{-,\ell}],\quad Z_{\pm,k\ell} = \pm\,\frac{i\omega}{2}\,e^{\mp 2i\omega t}[\xi_{\pm,k},\xi_{\pm,\ell}] \tag{2.5}$$

and the supersymmetries of type Q and S [7]

$$Q_{\pm,k\ell} = \mp\, i\sqrt{\omega}\, a_{\mp,k}\xi_{\pm,\ell},\quad S_{\pm,k\ell} = \pm i\sqrt{\omega}\, e^{\mp 2i\omega t} a_{\pm,k}\xi_{\pm,\ell}\ . \tag{2.6}$$

Finally, the generators of sh(2n/2n) , the so-called Heisenberg superalgebra, are the bosonic ones associated with h(2n) and their fermionic analogous

$$T_{\pm,k} = e^{\mp i\omega t}\xi_{\pm,k}\ . \tag{2.7}$$

Now from the kinematical point of view [8] we notice that inside the MDI superalgebra we recognize [6] the generators associated with the maximal kinematical invariance (MKI) superalgebra [osp(2/2) ⊕ so(n)] $\ni\!\!\!\!+$ sh(2n/2n). They are the ones associated with invariances under coordinate transformations only. Let us write

$$\mathrm{osp}(2n/2n) \ni\!\!\!\!+ \mathrm{sh}(2n/2n) \supset [\mathrm{osp}(2/2) \oplus \mathrm{so}(n)] \ni\!\!\!\!+ \mathrm{sh}(2n/2n). \tag{2.8}$$

The 8-dimensional superalgebra osp(2/2) thus contains the so(2,1)-algebra corresponding to bosonic symmetries such that

$$H_0^{SS} = T_{kk}\ ,\ C_\pm = C_{\pm,kk}\ , \tag{2.9}$$

the so(2)-algebra corresponding to the fermionic symmetries with

$$Y = Y_{kk} = \frac{\omega}{2}\,[\xi_{+,k},\xi_{-,k}] \tag{2.10}$$

and finally the four supercharges

$$Q_\pm = Q_{\pm,kk} = (2.2),\ S_\pm = S_{\pm,kk} = \pm i\sqrt{\omega}\ e^{\mp 2i\omega t} a_{\pm,k}\xi_{\pm,k} \quad . \tag{2.11}$$

The so(n)-algebra corresponds to the generators of the total angular momentum

$$J_{k\ell} = L_{k\ell} + \Sigma_{k\ell} = \frac{i}{\omega}(T_{k\ell} - T_{\ell k}) - i(\xi_{-,k}\xi_{+,\ell} - \xi_{-,\ell}\xi_{+,k}) \tag{2.12}$$

and the superalgebra sh(2n/2n) is the same as above. Finally let us insist on the fact that in the dynamical and kinematical contexts, the Heisenberg superalgebra sh(2n/2n) appears as fundamental since all the osp(2n/2n) generators can be written as second order products of the sh(2n/2n)-operators (cf. (2.4) and (2.7)).

2.b The free case. The supersymmetry of the free system is easily deduced from the harmonic oscillator one through a one-to-one correspondence existing between both systems. This corresponds to a change of variables found by Niederer [8] and further exploited [3] to show the isomorphism between the corresponding MKI algebras and superalgebras.

2.c The constant magnetic field system. A charged particle in interaction with a constant magnetic field $\vec{B} = (0,0,B)$ is described by the hamiltonian (subscript M)

$$H_M = \tfrac{1}{2}(\underline{p} - e\underline{A})^2 = \tfrac{1}{2}\,\underline{\Pi}^2,\ \underline{p} \equiv (p_x, p_y) \quad , \tag{2.13}$$

where we refer to the 2-dimensional context in the (x,y)-plane. Using the gauge symmetric potential $\vec{A}^S = -\frac{1}{2}\,\vec{r}\times\vec{B}$, we get

$$H_M = H_0 - \omega L \tag{2.14}$$

when $eB = 2\omega$ and $L = xp_y - yp_x$. This relation clearly shows the connection between the magnetic and the harmonic oscillator context.

Now using the supersymmetric harmonic oscillator hamiltonian (2.1) restricted to the 2-dimensional case with a specific 4 by 4 representation for the fermionic variables $\xi_{\pm,k}$ or φ_k^a (a = 1,2 ;

k = 1,2) satisfying (1.5), we get

$$H_0^{SS} = H_0 + Y = H_0 - \omega\begin{pmatrix}\sigma_3 & 0\\ 0 & 0\end{pmatrix} . \qquad (2.15)$$

The total angular momentum operator (2.12) which corresponds to a symmetry of H_0^{SS} , is given by

$$J = L + \Sigma = L + \begin{pmatrix}0 & 0\\ 0 & \sigma_3\end{pmatrix} .$$

So, in correspondence with (2.14), we propose a supersymmetric magnetic hamiltonian

$$H_M^{SS} = H_0^{SS} - \omega J = H_0 - \omega L - \omega\begin{pmatrix}\sigma_3 & 0\\ 0 & \sigma_3\end{pmatrix} = (H_M - \omega\sigma_3) \otimes \mathbf{1} = H_P \otimes \mathbf{1} \qquad (2.16)$$

which is nothing else than an amplification of the well known Pauli hamiltonian H_P .

Due to the structure of Eq. (2.16) and the results obtained for the harmonic oscillator (cf. § 2.a), we immediately deduce that the supersymmetric magnetic hamiltonian admits the same invariance superalgebra. The Heisenberg superalgebra sh(4/4) has to be generated by new operators which correspond to constants of motion for H_0^{SS} = (2.16). Indeed with the definitions

$$P_{+,\pm} = P_{+,1} \pm iP_{+,2} \;,\; P_{-,\pm} = P_{-,1} \pm iP_{-,2} \qquad (2.17a)$$

and

$$\xi_{+,\pm} = \xi_{+,1} \pm i\xi_{+,2} \;,\; \xi_{-,\pm} = \xi_{-,1} \pm i\xi_{-,2} \qquad (2.17b)$$

we get the four <u>bosonic</u> operators ($\underline{\pi} = \underline{p} + e\underline{A}$)

$$\begin{aligned}\pi_+ &= \pi_x + i\pi_y = e^{i\omega t}P_{+,+} \;,\; \pi_- = \pi_x - i\;{}_y = e^{-i\omega t}P_{-,-} \;,\\ P_+ &= e^{-2i\omega t}(\Pi_x - i\Pi_y) = e^{-i\omega t}P_{+,-} \;,\; P_- = e^{2i\omega t}(\Pi_x + i\Pi_y) = e^{i\omega t}P_{-,+}\end{aligned} \qquad (2.18)$$

and the four <u>fermionic</u> ones

$$\xi_{+,+} = e^{i\omega t}T_{+,+} \ , \ \xi_{-,-} = e^{-i\omega t}T_{-,-}, \ \eta_{+,-} = e^{-i\omega t}T_{+,-}, \ \eta_{-,+} = e^{i\omega t}T_{-,+}. \tag{2.19}$$

The dynamical superalgebra osp(4/4) is then immediately constructed through all the second order products of the sh(4/4)-generators while the kinematical superalgebra osp(2/2) ⊕ so(2) admits unchanged generators (with respect to the harmonic oscillator case). Let us finally mention that the isomorphism between the MDI as well as MKI superalgebras for the magnetic and harmonic oscillator cases can also be shown by another way. Indeed we have proposed [9] a very simple change of variables enlightening the connection between the 2-dimensional harmonic oscillator and the motion in a constant magnetic field. It corresponds to a time-dependent rotation $R(\omega t)$ in the (x,y)-plane acting on the bosonic as well as fermionic coordinates and it leads to the correspondences

$$P^0_{\pm,\pm} \leftrightarrow \pi_\pm, \ P^0_{\pm,\mp} \leftrightarrow P_\pm, \ T^0_{\pm,\pm} \leftrightarrow \xi_{\pm,\pm}, \ T^0_{\pm,\mp} \leftrightarrow \eta_{\pm,\mp} \ . \tag{2.20}$$

This makes clear the isomorphism between the Heisenberg superalgebras and then between the whole invariance superalgebras [6].

3. Examples of supersymmetric system with the spin-orbit coupling procedure

3.a The harmonic oscillator and the constant magnetic field system.

In fact let us consider the supersymmetric harmonic oscillator in the $n = 2$-case with the spin-orbit coupling procedure. Then the hamiltonian obtained from (1.10) with a specific 2 by 2 representation of the fermionic variables is nothing else than the Pauli hamiltonian for the constant magnetic field. Indeed we have

$$(H_0^{SS})' = \tfrac{1}{2}(\underline{p}^2+\omega^2\underline{x}^2) - \omega L - \omega\sigma_3 = H_p \ . \tag{3.1}$$

The associated MDI superalgebra determined by Durand [10] is called here osp(2/4) ⊕ sh(2/4). Its contents is then known. The Heisenberg superalgebra sh(2/4) admits the four bosonic charges (2.18) but only <u>two</u> fermionic ones

$$T_\pm = e^{\mp 2i\omega t}\sigma_\pm \quad . \tag{3.2}$$

The dynamical superalgebra osp(2/4) is once again easily constructed through all the second order products between the sh(2/4)-generators. Let us insist on the fact that in osp(2/4) , there are only <u>one</u> fermionic charge associated with so(2) , <u>ten</u> bosonic charges associated with sp(4) and finally <u>eight</u> supersymmetric charges.

The corresponding MKI superalgebra is $[osp(2/2) \oplus so(2)] \,\supset\!\!\!\!+\, sh(2/4)$. We again have the "dynamical $\supset$ kinematical" inclusion, i.e.

$$osp(2/4) \,\supset\!\!\!\!+\, sh(2/4) \supset [osp(2/2) \oplus so(2)] \,\supset\!\!\!\!+\, sh(2/4) \quad .$$

3.b <u>The harmonic oscillator and the $1/r^2$-potential</u>. We limit here our considerations to the n = 3-case. Besides the examples studied by Balantekin [2] and subtended by the superalgebra osp(2/2) ⊕ so(3) , let us consider another supersymmetric hamiltonian combining a harmonic oscillator and a $1/r^2$-potentials [11]. It admits again an accidental degeneracy [4] in the energy levels. In fact with the fermionic variables realized by 4 by 4 matrices satisfying (1.8) and with the superpotential

$$W(r) = \tfrac{1}{2}\,\omega r^2 + \lambda\, \ell n\, r \;, \; (r = \sqrt{x^2+y^2+z^2}) \quad , \tag{3.3}$$

we get the hamiltonian

$$H = \tfrac{1}{2}(p^2+\omega^2 r^2 + \frac{\lambda^2}{r^2} + 2\omega\lambda) + \omega(\vec{\sigma}.\vec{L} + \tfrac{3}{2}\,\mathbb{1}) \otimes \sigma_3 + \frac{\lambda}{r^2}\,(\vec{\sigma}.\vec{L} + \tfrac{1}{2}\,\mathbb{1}) \otimes \sigma_3 . \tag{3.4}$$

Let us notice that such a hamiltonian clearly contains the usual bosonic part associated with the presence of an external potential $U(r) = \frac{1}{2}\,\omega r^2 + \frac{\lambda^2}{2r^2} + \omega\lambda$ and another part containing coupling terms between bosonic and fermionic variables.

Now the MKI superalgebra associated with (3.4) is once again obtained as the superalgebra osp(2/2) ⊕ so(3). The superalgebra osp(2/2) is generated by the four operators

$$C_{\pm} = \pm \tfrac{i}{2}\, e^{\mp 2i\omega t}[(\vec{p}\pm i\omega\vec{r})^2 \mathbb{1} + \frac{\lambda^2}{r^2}\,\mathbb{1} + \frac{2\lambda}{r^2}\,(\vec{\sigma}.\vec{L} + \tfrac{1}{2}\,\mathbb{1}) \otimes \sigma_3] \quad ,$$

$$H_1 = \tfrac{1}{2}\,(p^2+\omega^2 r^2)\mathbb{1} + \frac{\lambda^2}{2r^2}\,\mathbb{1} + \frac{\lambda}{r^2}\,(\vec{\sigma}.\vec{L} + \tfrac{1}{2}\,\mathbb{1}) \otimes \sigma_3 \quad , \tag{3.5}$$

$$H_2 = \omega(\vec{\sigma}.\vec{L} + \frac{3}{2}\,\mathbb{1}) \otimes \sigma_3 + \omega\lambda\,\mathbb{1} \quad ,$$

associated with so(2,1) ⊕ so(2) and by four supercharges

$$Q_{\pm} = \frac{1}{\sqrt{2}}\,[\vec{p}\mp i(\omega + \frac{\lambda}{r^2})\vec{r}].\vec{\xi}_{\pm} \ , \ S_{\pm} = \frac{1}{\sqrt{2}}\,e^{\mp 2i\omega t}[\vec{p}\pm i(\omega - \frac{\lambda}{r^2})\vec{r}].\vec{\xi}_{\pm} \ . \tag{3.6}$$

The rotational invariance leads to the conservation of the total angular momentum operators $\vec{J} = \vec{L} + \frac{1}{2}\,\vec{\sigma} \otimes 1$.

By taking into account the eigenvalues $\varepsilon = \pm 1$ of σ_3, Eq. (3.4) becomes a direct sum of the two hamiltonians

$$H_{\varepsilon} = \tfrac{1}{2}(p^2+\omega^2 r^2)\mathbb{1} + \frac{\lambda(\lambda+\varepsilon)}{2r^2} + \varepsilon(\omega + \frac{\lambda}{r^2})\vec{\sigma}.\vec{L} + \omega(\lambda + \frac{3}{2}\,\varepsilon)\mathbb{1} \quad . \tag{3.7}$$

Concerning the energy levels, it is easy to show that H_{ε} admits the eigenvalues $E^{\varepsilon}_{n\ell j}$ where n, ℓ and j refer to the usual quantum numbers :

$$\begin{aligned} E^{\varepsilon}_{n\ell j} &= \omega[2n + j + \lambda + 1 + \varepsilon\,(j+\lambda+1)] \quad \text{if} \quad j = \ell + \tfrac{1}{2} \ , \\ E^{\varepsilon}_{n\ell j} &= \omega[2n + j + \lambda + 2 - \varepsilon\,(j+\lambda)] \quad \text{if} \quad j = \ell - \tfrac{1}{2} \ . \end{aligned} \tag{3.8}$$

Such results do contain accidental degeneracies. Indeed we have

$$E^{1}_{n\ell\ (=j-\frac{1}{2})j} = 2\omega(n+j+\lambda+1) = E^{-1}_{n\ell\ (=j+\frac{1}{2})j} \quad . \tag{3.9}$$

Finally, through the subalgebra so(2,1) ⊕ so(2) and its associated Casimir operators, we are led to a clear labelling of the states in an energy basis and the accidental degeneracy is completely eliminated due to the supersymmetric context.

References

1) Witten, E., Nucl.Phys. **B188**,513(1981).

2) Balantekin, A.B., Ann.Phys. (N.Y.) **164**,277(1985).

3) Beckers, J., Dehin, D. and Hussin, V., J.Phys. **A20**,1137(1987).

4) Wybourne, B., "Classical Groups for Physicists", Wiley, N.Y. (1974).

5) de Crombrugghe, M. and Rittenberg, V., Ann.Phys. (N.Y.) **151**,99(1983).

6) Beckers, J., Dehin, D. and Hussin, V., "Dynamical and Kinematical Supersymmetries of the Quantum Harmonic Oscillator and the Motion in a Constant Magnetic Field", to be published J.Phys. A (1988).

7) Fubini, S. and Rabinovici, E., Nucl.Phys. **B245**,17(1984).

8) Niederer, U., Helv.Phys.Acta **45**,802(1972) ; **46**,191(1973).

9) Dehin, D. and Hussin, V., Helv.Phys.Acta **60**,552(1987).

10) Durand, S., "Supersymétries des systèmes mécaniques non relativistes en une et deux dimensions", Maîtrise es Sciences, Université de Montréal, (1985).

11) Celka, P. and Hussin, V., Mod.Phys.Lett. **A2**,391(1987).