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Rough Estimates of Solar System Gravitomagnetic Effects in Post-Newtonian Gravity

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Abstract: In order to properly describe the gravity interactions, including the mass currents, in gravitomagnetism, we construct four Maxwell-type gravitational equations that are shown to be analogs of the Maxwell equations in electromagnetism. Next, exploiting the Maxwell-type gravitational equations, we explicitly predict the mass magnetic fields for both the isolated system of the spinning Moon orbiting the spinning Earth and that of the Sun and solar system planets orbiting the spinning Sun, whose phenomenological values have not been evaluated in the preceding Newtonian gravity formalisms. In gravitomagnetism, we also phenomenologically investigate the mass magnetic general relativity (GR) forces associated with the mass magnetic fields, finding that they are extremely small but non-vanishing compared to the corresponding mass electric Newtonian forces. Moreover, the directions of the mass magnetic GR forces for the solar system planets, except Venus and Uranus, are shown to be anti-parallel to those of their mass electric Newtonian forces. Next, we investigate the mass magnetic dipole moment related to the B ring of Saturn to evaluate $\vec{m}_M(Ring) = -1.141 \times 10^4\ \mathrm{m}^3\ \mathrm{s}^{-1}\ \hat{\omega}$, with $\hat{\omega}$ being the unit vector along the axis direction of the spinning B ring. The predicted value of $\vec{m}_M(Ring)$ is shown to be directly related to the Cassini data on the total mass of the rings of Saturn.

Keywords: gravitomagnetism; mass magnetic field; Saturn ring; mass magnetic GR force; mass magnetic dipole moment; post-Newtonian gravity

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1. Introduction

In the weak field approximation, the Einstein field equations, in general, are relatively similar to the Maxwell equations of electromagnetism (EM). For a brief formalism of EM, see Appendix A. Next in the post-Newtonian corrections to the gravitational law, there have been discussions on frame-dragging effects [1,2] and Lense–Thirring effects [3,4]. For more details on these effects, see Refs. [1–4] and references therein. In investigating phenomenology related to spinning compact objects such as the Sun, the solar system planets and the Moon, for example, one can effectively exploit the rotational degrees of freedom (DOF) of gravitomagnetism. For more details on gravitomagnetism, see Refs. [5–9] and references therein. Recently, we proposed [10] a new formalism of modified linear general relativity (MLGR) that uses a vectorial scheme for the mass scalar and mass vector potentials of the gravitational wave (GW). To do this, in the MLGR, we employed the Wald approximation [9–11]. To be specific, we found the GW radiation intensity profile to possess a prolate ellipsoid geometry due to the merging source of binary compact objects. Here, the radiation lobe is prolate, with a long axis perpendicular to the orbital plane [10,12]. To be more specific, at a given radial distance from the binary compact objects, the GW radiation intensity on the revolution axis of the binary compact objects has been shown to be twice that on the equatorial plane [10].

On the other hand, the obliquity of Saturn has been known to be too large to have risen during Saturn's formation from a protoplanetary disk or from a large impact [13]. The rings of Saturn are known to have appeared about 100 million years ago, based on the estimated strength of satellite ring torques [14] and the estimated rate of darkening of the ice-rich materials [15,16]. However, it is uncertain how the young rings of Saturn could have formed so recently. The Cassini data have recently been exploited to refine estimates of Saturn's moment of inertia. Moreover, it has been proposed [17] that Saturn previously possessed an additional satellite, called Chrysalis, which caused the obliquity of Saturn to increase through the Neptune resonance. Therefore, destabilization of Chrysalis's orbit about 100 million years ago can explain the proximity of the system to the resonance and the formation of the young rings through a grazing encounter of Chrysalis with Saturn. Moreover the simulated Chrysalis has been shown to have had multiple encounters with Titan, which is the largest satellite of Saturn [17]. The other Cassini data also have explicitly shown the total mass of the rings of Saturn [16].

In this paper, we theoretically investigate the precision astrophysics phenomenology with gravitomagnetism related to mass (M), which is the analog of EM [10,11,18,19] associated with charge (Q). Next, we study the mass magnetic fields for the spinning compact objects in gravitomagnetism. To be more specific, we evaluate the mass magnetic fields for both the isolated system of the Moon orbiting the spinning Earth and that of the solar system planets orbiting the spinning Sun. Making use of the mass magnetic fields, we proceed to predict the ensuing Lorentz-type mass magnetic general relativity (GR) forces for these systems. Next, we construct the mass magnetic dipole moment in gravitomagnetism to investigate the phenomenology of the B ring of Saturn.

In Section 2, we first construct the formalism of gravitomagnetism by exploiting MLGR. Next, in this gravitomagnetism, we formulate the mass electric and mass magnetic fields, as well as the corresponding mass electric Newtonian force and the mass magnetic force. In Section 3, we investigate the phenomenology associated with the mass magnetic fields of both the spinning Moon orbiting the spinning Earth and the spinning solar system planets orbiting the spinning Sun. Moreover, we evaluate the mass electric Newtonian forces and mass magnetic forces for these systems. Next, we evaluate the mass magnetic dipole moment associated with the B ring of Saturn. Section 4 includes conclusions.

2. Formalism of Astrophysical Phenomenology

2.1. Gravitomagnetism Originating from MLGR

In this subsection, we construct the formalism of gravitomagnetism by using MLGR [10]. To do this, we briefly recapitulate MLGR originating from Einstein GR. We start with the Einstein–Hilbert action,

$$S_{EH} = \int d^4x \sqrt{-g} R, \tag{1}$$

from which we construct the Einstein GR field equation, [5,11,20]

$$R_{\alpha\beta} - \frac{1}{2} g_{\alpha\beta} R = \frac{8\pi G}{c^4} T_{\alpha\beta}, \tag{2}$$

where $R_{\alpha\beta}$ and R are the Riemann tensor and scalar curvature, respectively, and $T_{\alpha\beta}$ is the energy stress tensor. Next, we assume that the deviation ($h_{\alpha\beta}$) of the space-time metric ($g_{\alpha\beta}$) from a flat metric ($\eta_{\alpha\beta}$) is small.

$$g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}. \tag{3}$$

Exploiting the Einstein GR field equation in (2) and $g_{\alpha\beta}$ in (3), we construct the linearized Einstein equation [5,10,11,20]:

$$-\frac{1}{2}\Box\bar{h}_{\alpha\beta} + \partial^{\gamma}\partial_{(\alpha}\bar{h}_{\beta)\gamma} - \frac{1}{2}\eta_{\alpha\beta}\partial^{\gamma}\partial^{\delta}\bar{h}_{\gamma\delta} = \frac{8\pi G}{c^4}T_{\alpha\beta}, \tag{4}$$

where $\Box = \partial^{\beta}\partial_{\beta}$, with a metric of $\eta_{\alpha\beta} = \mathrm{diag}(-1, +1, +1, +1)$. Here, the trace reversed perturbation $\bar{h}_{\alpha\beta}$ is given by $\bar{h}_{\alpha\beta} = h_{\alpha\beta} - \frac{1}{2}h$, with $h = \eta^{\alpha\beta}\bar{h}_{\alpha\beta}$. Next, exploiting (4) and the Lorentz-type gauge condition in gravity,

$$\partial^{\beta}\bar{h}_{\alpha\beta} = 0, \tag{5}$$

we arrive at the Maxwell-type equation in MLGR.

$$\Box\bar{h}_{\alpha\beta} = -\frac{16\pi G}{c^4}T_{\alpha\beta} \tag{6}$$

Now, we consider the non-vanishing $\bar{h}_{00}$ and $\bar{h}_{0i}$ $(i = 1, 2, 3)$, together with $\bar{h}_{ij} = 0$ [9–11]. We define $\bar{h}_{00}$ and T_{00} as

$$\bar{h}_{00} = -\frac{4}{c^2}\phi_M, \quad T_{00} = c^2\rho_M, \tag{7}$$

where $\phi_M \equiv cA^0_M$ is the linearized physical quantity in MLGR. From now on, subscripts M an Q denote the physical mass and charge quantities, respectively. Inserting (7) into the Maxwell-type equation in (6) yields

$$\Box\phi_M = 4\pi G\rho_M. \tag{8}$$

Next, we define $\bar{h}_{0i}$ and T_{0i} as

$$\bar{h}_{0i} = \frac{4}{c}A^i_M, \quad T_{0i} = -c\rho_M v^i = -cJ^i_M, \tag{9}$$

where the energy stress tensor is approximated to linear order in velocity in the Wald approximation [9–11]. In MLGR, inserting (9) into the Maxwell-type equation in (6) produces

$$\Box A^i_M = \frac{4\pi G}{c^2}J^i_M. \tag{10}$$

Note that in the Wald approximation associated with T_{00} in (7) and T_{0i} in (9), we construct $T_{\alpha\beta}$ in terms of $J^i_M = \rho_M v^i$ [9–11], i.e.,

$$T_{\alpha\beta} = \begin{pmatrix} c^2\rho_M & -cJ^j_M \\ -cJ^i_M & 0 \end{pmatrix}, \tag{11}$$

where the space–space components (T_{ij}) vanish.

In vacuum, inserting $T_{\alpha\beta} = 0$ into (6) and exploiting the definitions of $\bar{h}_{00}$ and $\bar{h}_{0i}$ in (7) and (9), we obtain the wave equations for the non-vanishing fields ($\bar{h}_{00}$ and $\bar{h}_{0i}$),

$$\Box\bar{h}_{00} = \Box A^0_M = 0, \quad \Box\bar{h}_{0i} = \Box A^i_M = 0, \tag{12}$$

from which we arrive at the wave equations in MLGR, i.e.,

$$\Box A^{\alpha}_M = 0, \quad \alpha = 0, 1, 2, 3 \tag{13}$$

which describe a massless spin-one graviton propagating in flat space time. This phenomenology in gravitomagnetism is analogous to that in EM. Note also that we have the Lorentz-type gauge condition for A^{α}_{M}.

$$\partial_{\alpha} A^{\alpha}_{M} = 0 \tag{14}$$

Next, we investigate the formalism of gravitomagnetism. To this end, we define the *mass electric and mass magnetic fields* (E^i_M and B^i_M) in terms of $\bar{h}_{00}$ in (7) and $\bar{h}_{0i}$ in (9), constructed in relativistic MLGR as

$$E^i_M = -\partial_i \phi_M - \frac{\partial A^i_M}{\partial t} = \frac{c^2}{4}(\partial_i \bar{h}_{00} - \partial_0 \bar{h}_{0i}), \quad B^i_M = \epsilon^{ijk}\partial_j A^k_M = \frac{c}{4}\epsilon^{ijk}\partial_j \bar{h}_{0k}. \tag{15}$$

Note that E^i_M contains the DOF related to the $\frac{\partial A^i_M}{\partial t}$ term. After some algebra exploiting $\vec{E}_M$ and $\vec{B}_M$ in (15) in MLGR, we construct the Maxwell-type equations in gravitomagnetism [6] as

$$\begin{aligned} \nabla \cdot \vec{E}_M &= -4\pi G \rho_M, \quad \nabla \times \vec{E}_M + \frac{\partial \vec{B}_M}{\partial t} = 0, \\ \nabla \cdot \vec{B}_M &= 0, \qquad \nabla \times \vec{B}_M - \frac{1}{c^2}\frac{\partial \vec{E}_M}{\partial t} = -\frac{4\pi G}{c^2}\rho_M \vec{v}. \end{aligned} \tag{16}$$

Note that the Maxwell-type gravitational equations in (16) in gravitomagnetism are analogs of the Maxwell equations in EM [18,19]. Note also that in gravitomagnetism, the equations in (16) include the time-derivative terms related to the dynamics of gravitomagnetism interactions.

Now, it seems appropriate to address comments on gravitomagnetism in Refs. [5,7–9], where the gravitational Maxwell equations are given by

$$\begin{aligned} \nabla \cdot \vec{E}_g &= -4\pi G \rho_M, \quad \nabla \times \vec{E}_g = 0, \\ \nabla \cdot \vec{B}_g &= 0, \qquad \nabla \times \vec{B}_g = -\frac{16\pi G}{c^2}\rho_M \vec{v}. \end{aligned} \tag{17}$$

First, in gravitomagnetism in (17), we find curl equations that do not possess the dynamic DOF associated with the time-derivative terms in (16). Note that the equations for $\vec{E}_g$ denote the gravitational field produced by a static mass configuration. The equations for $\vec{B}_g$ yield a notationally means of determining the extra gravitational field produced by moving masses associated with $\rho_M \vec{v}$ [5].

Second, even in the static case, the last equation in (17) has a *coefficient different* from that in gravitomagnetism in (16). This feature in gravitomagnetism in Refs. [5,7–9] originates from the following identification [5]:

$$\epsilon_0 \leftrightarrow -\frac{1}{4\pi G}, \quad \mu_0 \leftrightarrow -\frac{16\pi G}{c^2}. \tag{18}$$

In contrast, in gravitomagnetism associated with (16), we construct the last equation possessing $-\frac{4\pi G}{c^2}$ by using the treatment of the Einstein GR field equation in (2) and $g_{\alpha\beta} = \eta_{\alpha\beta} + h_{\alpha\beta}$ in (3) used in MLGR. Especially in order to predict astrophysical quantities, which are *numerically* evaluated in Section 3, for instance, we need to exploit the equations in (16). This feature in MLGR and the ensuing gravitomagnetism associated with (16) are among the main points of this paper.

Next, we have comments on the gauge invariance of gravitomagnetism and that of EM described in Appendix A. First, in MLGR [10], we do not exploit the traceless, transverse (TT) gauge [5,10,11,20], which is a second-rank tensorial formalism in linearized general relativity (LGR). In the TT gauge, we choose $\bar{h}^{0i}_{TT} \equiv 0$. Next, for MLGR, exploiting the Wald approximation [9–11], we construct a vectorial formalism that is similar to that of EM [10]. Note that in MLGR, we have a vanishing $\bar{h}^{ij} = 0$ associated with the Wald approximation, while in LGR, we possess a nonzero $\bar{h}^{ij}_{TT} \neq 0$ in the TT gauge [5,10,11,20].

Second, the equations of motion (EOMs) for the ϕ_M and $\vec{A}_M$ fields in (8) and (10) produce the covariant EOM of $A^\alpha_M \equiv (\frac{\phi_M}{c}, A^i_M)$:

$$\Box A^\alpha_M = \frac{4\pi G}{c^2} J^\alpha_M, \tag{19}$$

where $J^\alpha_M \equiv (c\rho_M, J^i_M)$. Next, (19) has the same form as the EOM for the fields ($A^\alpha_Q \equiv (\frac{\phi_Q}{c}, A^i_Q)$) that are obtainable from (A3) and $J^\alpha_Q \equiv (c\rho_Q, J^i_Q)$, i.e.,

$$\Box A^\alpha_Q = -\frac{1}{c^2\epsilon_0} J^\alpha_Q, \tag{20}$$

representing the covariant EOM. Note that in MLGR, the GW radiation intensity profile possesses a prolate ellipsoid geometry, with the angle depending on the source of the merging binary compact objects [10].

Thirdly, motivated by the mathematical similarity of the vectorial forms of the EOM for $A^\alpha_M (= -\frac{c}{4}\bar{h}^{0\alpha})$ ($\alpha = 0, 1, 2, 3$) in (19) and A^α_Q in (20), we construct the vectorial formalism for the spin-one graviton in MLGR [10]. Now, we investigate the gauge invariance and U(1) transformation in gravitomagnetism. To do this, we start with the Dirac equation for the electron wave function (ψ) and the mass-scalar and mass-vector potentials (A^μ_M) in units of $\hbar = c = 1$:

$$i\gamma^\mu \partial_\mu \psi - m_e \psi - e\gamma^\mu A_{M,\mu}\psi = 0, \tag{21}$$

with m_e and e being the electron mass and charge, respectively. Next, keeping in mind that in MLGR, we have the spin-one graviton, we study the quantum gravitomagnetic dynamics (QGD) that describes the interactions between electrons and gravitons. Now, we introduce the QGD Lagrangian, which is analogous to that of quantum electrodynamics (QED) in (A7) [21], i.e.,

$$\mathcal{L}_{QGD} = -\frac{1}{4} F_{M,\mu\nu} F^{\mu\nu}_M + \bar{\psi}(i\gamma^\mu - m_e)\psi - e\bar{\psi}\gamma^\mu A_{M,\mu}\psi, \tag{22}$$

where $F^{\mu\nu}_M = \partial^\mu A^\nu_M - \partial^\nu A^\mu_M$ and the third term denotes the interaction between the electron and the gravitomagnetic wave (or spin-one graviton). Now, we find that the QGD Lagrangian in (22) is invariant under the gauge transformations, i.e.,

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{-ie\Theta(x)}\psi(x), \\ A^\mu_M(x) &\rightarrow A'^\mu_M(x) = A^\mu_M(x) + \partial^\mu \Theta(x), \end{aligned} \tag{23}$$

where the first relation is the U(1) transformation.

2.2. Mass Magnetic GR Force

Now, we construct the mass magnetic GR force by exploiting gravitomagnetism as formulated in the previous subsection. For the cases of *mass electrostatics and mass magnetostatics*, which are practically applicable to the phenomenological predictions in the next section, the mass electric and mass magnetic fields are given by

$$\vec{E}_M = -G \int \frac{\rho_M \hat{R}}{R^2} d^3x', \quad \vec{B}_M = -\frac{G}{c^2} \int \frac{\vec{J}_M \times \hat{R}}{R^2} d^3x', \tag{24}$$

where $\vec{R} = \vec{x} - \vec{x}'$ is the vector from d^3x' to field point $\vec{x}$ and $\hat{R} = \vec{R}/R$ with $R = |\vec{x} - \vec{x}'|$. Using (24), we obtain the corresponding force acting on a test mass (M_0) moving with a velocity of v as follows:[1]

$$\vec{F}_M = M_0(\vec{E}_M + \vec{v} \times \vec{B}_M) \equiv \vec{F}_M^E + \vec{F}_M^B. \tag{25}$$

Now, in mass magnetostatics, we construct the mass-vector potential ($\vec{A}_M$) as

$$\vec{A}_M = -\frac{G}{c^2} \int \frac{\vec{J}_M}{R} d^3x'. \tag{26}$$

Note that, exploiting $\vec{B}_M (= \nabla \times \vec{A}_M)$, we find that $\vec{A}_M$ in (26) reproduces $\vec{B}_M$ in (24).

Now, it seems appropriate to address some comments on the applications of formulas (24) and (26). First, exploiting (24), we construct mass magnetic field $\vec{B}_M$ at the center of the loop of the orbital radius (R), which consists of the total mass (M) and orbits its center with an orbital period of T.

$$\vec{B}_M = -\frac{2\pi GM}{c^2 RT} \hat{\omega}. \tag{27}$$

Here, $\hat{\omega}$ is the unit vector along the axis of the loop orbiting its center. Note that inside (outside) the loop, the direction of $\vec{B}_M$ is anti-parallel (parallel) to $\hat{\omega}$.

Secondly, we consider a solid sphere of mass M and radius a spinning with angular velocity ($\vec{\omega} = \omega\hat{\omega}$), as shown in Figure 1. Note that $\vec{J}_M = \rho_M \vec{v} = \rho_M \vec{\omega} \times \vec{r}'$ produces

$$\vec{J}_M = \frac{3M}{4\pi a^3}(-\hat{y}\, \omega r' \cos\theta' + \hat{z}\, \omega r' \sin\theta' \sin\phi'). \tag{28}$$

Inserting (28) into (26), we find the mass vector potential ($\vec{A}_M$) as follows:

$$\vec{A}_M = -\frac{3GM\omega}{2c^2 a^3} \int_0^a dr' r'^3 \int_0^\pi \frac{d\theta' \sin\theta' \cos\theta'}{R} \hat{\omega} \times \hat{r}, \tag{29}$$

where we have use the relation expressed as $\hat{y} = -\hat{\omega} \times \hat{z}$. Next, on the equatorial plane of the solid sphere, we arrive at $\vec{A}_M$ at $\vec{r} = r\hat{r}$ with r $(r \geq a)$[2]

$$\vec{A}_M = -\frac{GMa^2\omega}{5c^2 r^2} \hat{\omega} \times \hat{r}. \tag{30}$$

Exploiting (30), we finally formulate the mass magnetic field ($\vec{B}_M (= \nabla \times \vec{A}_M)$) in the following form:

$$\vec{B}_M = \frac{4\pi GMa^2}{5c^2 r^3 t} \hat{\omega}, \tag{31}$$

where t is the rotational period of the spinning solid sphere. Note that outside the spinning solid sphere, the direction of $\vec{B}_M$ is parallel to $\hat{\omega}$ on the equatorial plane. Note also that in EM, the direction of $\vec{B}_Q$ circles around the charge current, following the right-hand thumb rule, since the charges interact repulsively [18,19]. In contrast, in gravitomagnetism, the direction of $\vec{B}_M$ circles around the mass current, following the left-hand thumb rule due to the fact that the masses interact attractively. Explicitly these features can be explained by the comparison of (26) and (A11).

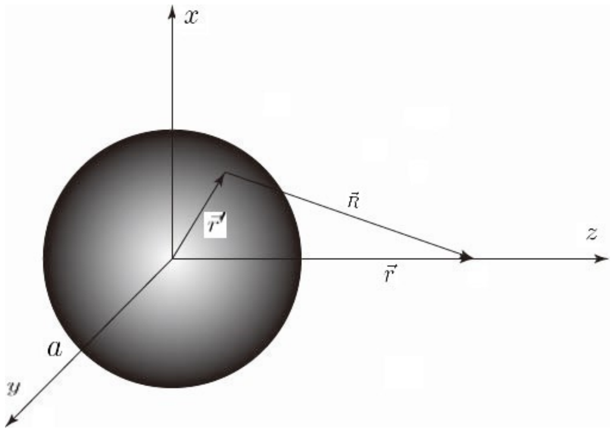

Figure 1. The geometry of a solid sphere of mass (M) and radius (a) spinning with angular velocity ($\vec{\omega} = \omega\hat{\omega}$, where $\hat{\omega}$ is the unit vector along $\hat{x}$). Here, $\vec{r}' = r'(\hat{x}\sin\theta'\cos\phi' + \hat{y}\sin\theta'\sin\phi' + \hat{z}\cos\theta')$ and $R = (r^2 + r'^2 - 2rr'\cos\theta')^{1/2}$, with θ' being the polar angle between $\vec{r}$ and $\vec{r}'$.

3. Phenomenology of Astrophysical Systems in Gravitomagnetism

3.1. Earth and Moon

Now, in gravitomagnetism, we consider the *isolated two-body system* of the Earth of mass $M_\oplus$ and the spinning Moon of mass M_m orbiting the spinning Earth to find the phenomenological prediction for the mass magnetic field ($\vec{B}_M(M_\oplus) = B_M(M_\oplus)\hat{\omega}$) on the surface of the Earth, where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Earth. Note that $\hat{\omega}$ is also the unit vector along both the axis direction of the spinning Moon and the orbital-axis direction of the Moon orbiting the Earth. The mass magnetic field on the surface of the Moon is considered later. Exploiting (27) and (31), we construct the expression of the mass magnetic field ($B_M(M_\oplus)$) for the isolated system of the Earth and Moon, both of which possess rotational DOFs, for an observer located on the surface of the Earth[3]

$$\begin{aligned} B_M(M_\oplus) &= \frac{4\pi G M_\oplus}{5c^2 r_\oplus t_\oplus} - \frac{2\pi G M_m}{c^2 R_m T_m} + \frac{4\pi G M_m r_m^2}{5c^2 R_m^3 t_m} \\ &= (2.029 \times 10^{-14} - 3.777 \times 10^{-19} + 3.085 \times 10^{-24})\ \mathrm{s}^{-1}, \end{aligned} \quad (32)$$

where the first, second and third terms originate from the contributions of the spinning effect of the Earth and the orbital and spinning effects of the Moon, respectively. Here, subscript m denotes the physical quantities of the Moon. $r_\oplus$ and $t_\oplus$ are the radius and rotational period of the spinning Earth, respectively, while r_m, R_m, t_m and T_m are the radius, average orbital distance, rotational period and average orbital period of the spinning Moon orbiting the Earth, respectively.[4] Note that the contributions from the orbital and spinning effects of the Moon are negligible. The contributions of the other solar system planets to $B_M(M_\oplus)$ are discussed in (39).

Next, we evaluate the mass magnetic field ($\vec{B}_M(M_m) = B_M(M_m)\hat{\omega}$) for an observer located on the surface of the Moon using (31). To do this, we obtain the expression of the mass magnetic field ($B_M(M_m)$) for the isolated system of the Earth and Moon as

$$B_M(M_m) = \frac{4\pi G M_m}{5c^2 r_m t_m} + \frac{4\pi G M_\oplus r_\oplus^2}{5c^2 R_m^3 t_\oplus} = (3.343 \times 10^{-17} + 9.267 \times 10^{-20})\ \mathrm{s}^{-1}, \quad (33)$$

where the first and second terms originate from the contributions of the spinning Moon and the spinning Earth, respectively.

Now, we investigate the force acting on the spinning Moon of mass M_m due to the spinning Earth of mass $M_{\oplus}$. Exploiting $\vec{F}^B_M$ in (25), we are left with $\vec{F}^B_M(M_m - M_{\oplus}) = F^B_M(M_m - M_{\oplus})\hat{r}$, with $\hat{r}$ being the radial direction from the Earth[5]

$$F^B_M(M_m - M_{\oplus}) = (2.513 \times 10^9 + 6.966 \times 10^6)\ \mathrm{N}. \tag{34}$$

Next, making use of $\vec{E}_M$ in (24) and $\vec{F}^E_M$ in (25), one can readily construct the mass electric Newtonian force acting on the Moon ($\vec{F}^E_M(M_m - M_{\oplus}) = F^E_M(M_m - M_{\oplus})\hat{r}$) for the isolated system of the Earth and Moon for an observer located on the surface of the Moon as follows[6]:

$$F^E_M(M_m - M_{\oplus}) = -1.983 \times 10^{20}\ \mathrm{N}, \tag{35}$$

from which we arrive at the characteristic ratio ($\chi(M_m - M_{\oplus})$) in the following form:

$$\chi(M_m - M_{\oplus}) = |\vec{F}^B_M(M_m - M_{\oplus})| / |\vec{F}^E_M(M_m - M_{\oplus})| = 1.271 \times 10^{-11}. \tag{36}$$

The ratio of $\chi(M_m - M_{\oplus})$ in (36) implies that gravitomagnetic correction, namely $|\vec{F}^B_M(M_m - M_{\oplus})|$ in gravitomagnetism, to the Newtonian classical prediction ($|\vec{F}^E_M(M_m - M_{\oplus})|$) is extremely small but non-vanishing.

3.2. Sun and Solar System Planets

In gravitomagnetism, we now investigate the isolated system of the Sun and the solar system planets orbiting the spinning Sun to evaluate the mass magnetic field ($\vec{B}_M(M_{\odot}) = B_M(M_{\odot})\hat{\omega}$) for an observer located on the surface of the Sun, where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Sun. Here, for simplicity, we assume that the rotation axes of the solar system planets, except those of Venus and Uranus, are approximately parallel to that of the Sun to produce the vanishing obliquities, and that these axes reside on the equatorial plane of the Sun. We also assume that on the equatorial plane of the Sun, the rotation axes of Venus and Uranus are anti-parallel to and perpendicular to that of the Sun, respectively.

Similar to (32), exploiting (27) and (31), we find the formula of the mass magnetic field ($B_M(M_{\odot})$) for the isolated system of the Sun and solar system planets for an observer located on the surface of the Sun as follows:

$$\begin{aligned} B_M(M_{\odot}) &= \frac{4\pi G M_{\odot}}{5c^2 r_{\odot} t_{\odot}} - \sum_{i=1}^{8} \frac{2\pi G M_i}{c^2 R_i T_i} + \sum_{i=1}^{8} \frac{4\pi G M_i r_i^2}{5c^2 R_i^3 t_i} \\ &= (2.468 \times 10^{-12} - 5.266 \times 10^{-20} + 3.769 \times 10^{-25})\ \mathrm{s}^{-1}, \end{aligned} \tag{37}$$

where the first, second and third terms come from contributions from the spinning effect of the Sun, orbital effects of the solar system planets and spinning effects of the solar system planets, respectively. Here, $M_{\odot}$, $r_{\odot}$ and $t_{\odot}$ are the mass, radius and rotational period of the spinning Sun, while M_i, r_i, R_i, t_i and T_i ($i = 1, 2, 3, \ldots 8$) are the mass, radius, average orbital distance, rotational period and average orbital period of the i-th solar system planet orbiting the Sun, respectively. In treating $B_M(M_{\odot})$ in (37), we ignore the effect of the Sun's radius ($r_{\odot}$) which is negligible compared to $R_{Mercury}$ due to the factor of $r_{\odot}/R_{Mercury} = 1.202 \times 10^{-2}$ [20], for instance, in calculating the second term in (37). From now on, we ignore the contributions from the spinning effects of the solar system planets in the third term, since these effects are extremely small, as shown in (32) for the case of the system of the Earth and Moon.

Now, we investigate the mass magnetic fields ($\vec{B}_M(M_i) = B_M(M_i)\hat{\omega}$ ($i = 1, 2, 3, \ldots, 8$)) for the solar system planets with M_i for an observer located on the surface of the planets, where $\hat{\omega}$ is the unit vector along the axis direction of the spinning Sun. To do this, using (31), we formulate $B_M(M_i)$ as follows:

$$B_M(M_i) = \frac{4\pi G M_i}{5c^2 r_i t_i} + \frac{4\pi G M_\odot r_\odot^2}{5c^2 R_i^3 t_\odot} + \sum_{j\neq i} \frac{4\pi G M_j r_j^2}{5c^2 |R_j - R_i|^3 t_j}, \quad (38)$$

where the first, second and third terms originate from the contributions from the spinning effect of the i-th planet, the spinning effect of the Sun and the spinning effect of the j-th other planet, respectively. Here, we assume that all the solar system planets are approximately aligned on a straight line in planetary order for simplicity, and that for the case of Uranus, $\vec{B}_M(M_7)$ is along the centripetal direction $(-\hat{r})$.

Making use of (38), for the case of the Earth, we arrive at

$$B_M(M_\oplus) = (2.029 \times 10^{-14} + 2.486 \times 10^{-19} + 2.078 \times 10^{-24})\ \mathrm{s}^{-1}, \quad (39)$$

where, for simplicity, we do not include the contribution of $\vec{B}_M(M_7)$ originating from Uranus, since $\vec{B}_M(M_7)$ is perpendicular to the directions of $\vec{B}_M(M_i)$ for the other solar system planets. Note that in (39), we can ignore the negligible contributions of the spinning effect of the Sun and spinning effects of the other solar system planets. In (39), we also exclude the negligible contributions of the orbital effect of the Moon and the spinning effect of Moon, which are listed in $B_M(M_\oplus)$ in (32). Moreover, since the contributions of the spinning effects of the other solar system planets with M_j $(j = 1, 2, 4, \ldots, 8)$ are negligible, as shown in (39), in evaluating $B_M(M_i)$ in (38), we effectively do not need the constraint that all the solar system planets are aligned on a straight line in planetary order.

Next, similar to (34) for the case of the Moon, we evaluate the mass magnetic GR force acting on the Earth ($\vec{F}_M^B(M_\oplus - M_\odot) = F_M^B(M_\oplus - M_\odot)\hat{r}$, with $\hat{r}$ being the unit radial vector from the Sun). Exploiting (25), one can readily evaluate $F_M^B(M_\oplus - M_\odot)$

$$F_M^B(M_\oplus - M_\odot) = (3.612 \times 10^{15} + 4.426 \times 10^{10} + 3.529 \times 10^5)\ \mathrm{N} \cong 1.019 \times 10^{-7} \times |F_M^E(M_\oplus - M_\odot)|, \quad (40)$$

where $|F_M^E(M_\oplus - M_\odot)|$ is given below. Now, we readily formulate the mass electric Newtonian force acting on the Earth as $\vec{F}_M^E(M_\oplus - M_\odot) = F_M^E(M_\oplus - M_\odot)\hat{r}$, where $\hat{r}$ is, again, the unit radial vector from the Sun and $F_M^E(M_\oplus - M_\odot)$ is given by

$$\begin{aligned} F_M^E(M_\oplus - M_\odot) &= -\frac{G M_\oplus M_\odot}{R_\oplus^2} - \sum_{j=1,2} \frac{G M_\oplus M_j}{|R_j - R_\oplus|^2} + \sum_{j>3} \frac{G M_\oplus M_j}{|R_j - R_\oplus|^2}, \\ &= (-3.543 \times 10^{22} - 1.149 \times 10^{18} + 2.103 \times 10^{18})\ \mathrm{N}, \end{aligned} \quad (41)$$

where the first, second and third terms come from the contributions from the Sun, the solar system planets $(i = 1, 2)$ inside the Earth's orbital radius and the solar system planets $(i = 4, 5, \ldots, 8)$ outside the Earth's orbital radius, respectively. Here, for simplicity, we, again, assume that all the solar system planets are approximately aligned on a straight line in planetary order and that, in evaluating $F_M^E(M_\oplus - M_\odot)$, the Sun and the solar system planets do not have rotational DOFs. Note that the leading-order prediction in $F_M^E(M_\oplus - M_\odot)$ in (41) is dominant compared to the corresponding sub-leading-order predictions. Since the contributions of the spinning effects of the other solar system planets with M_j $(j = 1, 2, 4, \ldots, 8)$ are negligible, as shown in (41), as in the case of the evaluation of $B_M(M_\oplus)$ in (39), for the leading-order prediction of $F_M^E(M_\oplus - M_\odot)$ in (41), we can, again, effectively remove the constraint that all the solar system planets are aligned on a straight line in planetary order.

Next, even the sub-leading-order contributions of an order of about 10^{18} N in $F_M^E(M_\oplus - M_\odot)$ in (41) are greater than the total contribution in $F_M^B(M_\oplus - M_\odot)$ in (40), implying that $F_M^B(M_\oplus - M_\odot)$ is hidden in Newtonian gravity. From (40) and (41), we arrive at the characteristic ratio ($\chi(M_\oplus - M_\odot)$) in the following form:

$$\chi(M_\oplus - M_\odot) = |\vec{F}_M^B(M_\oplus - M_\odot)|/|\vec{F}_M^E(M_\oplus - M_\odot)| = 1.019 \times 10^{-7}, \quad (42)$$

implying that the correction of $|\vec{F}_M^B(M_\oplus - M_\odot)|$ to the Newtonian classical prediction ($|\vec{F}_M^E(M_\oplus - M_\odot)|$) is extremely small but non-vanishing, as in the case of (36) for the isolated system of the Earth and the spinning Moon orbiting the spinning Earth.

Now, we investigate the general cases of the solar system planets with masses of M_i ($i = 1, 2, 3, \ldots, 8$), including the Earth, with a mass of $M_3 = M_\oplus$. The predictions of $B_M(M_i)$ in (38) for the solar system planets with masses of M_i are given in Table 1. Exploiting (25), we can readily construct $\vec{F}_M^B(M_i - M_\odot) = F_M^B(M_i - M_\odot)\hat{r}$, where $F_M^B(M_i - M_\odot)$ is listed in Table 1.[7] Next, in order to find the characteristic ratios, we construct the mass electric Newtonian forces acting on the solar system planets as $\vec{F}_M^E(M_i - M_\odot) = F_M^E(M_i - M_\odot)\hat{r}$, where $F_M^E(M_i - M_\odot)$ is given by

$$F_M^E(M_i - M_\odot) = -\frac{GM_iM_\odot}{R_i^2} - \sum_{j<i} \frac{GM_iM_j}{|R_j - R_i|^2} + \sum_{j>i} \frac{GM_iM_j}{|R_j - R_i|^2}. \quad (43)$$

Table 1. $\vec{B}_M(M_i) = B_M(M_i)\hat{\omega}$, $\vec{F}_M^B(M_i - M_\odot) = F_M^B(M_i - M_\odot)\hat{r}$, $\vec{F}_M^E(M_i - M_\odot) = F_M^E(M_i - M_\odot)\hat{r}$ and $\chi(M_i - M_\odot) = |\vec{F}_M^B(M_i - M_\odot)|/|\vec{F}_M^E(M_i - M_\odot)|$ are listed for the i-th planet ($i = 1, 2, 3, \ldots, 8$). Here, $\hat{\omega}$ and $\hat{r}$ are the unit vector along the axis direction of the spinning Sun and that along the radial vector from the Sun, respectively. The magnitudes of $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ are given in units of $|F_M^E(M_\oplus - M_\odot)|$. For the case of Venus, the direction of $\vec{B}_M(M_2)$ is along $-\hat{\omega}$ to produce the centripetal force ($\vec{F}_M^B(M_2 - M_\odot)$), while for the case of Uranus, the direction of $\vec{B}_M(M_7)$ is along $-\hat{r}$ to yield the force ($\vec{F}_M^B(M_7 - M_\odot)$) along $\hat{\omega}$. The predictions for the exceptional cases of Venus and Uranus are indicated by $*$.

i-th Planet	$B_M(M_i)$ (s^{-1})	$F_M^B(M_i - M_\odot)$	$F_M^E(M_i - M_\odot)$	$\chi(M_i - M_\odot)$
Mercury	5.423×10^{-17}	2.422×10^{-11}	-3.691×10^{-1}	6.563×10^{-11}
Venus	-7.081×10^{-17} *	-3.409×10^{-10} *	-1.558	2.188×10^{-10}
Earth	2.029×10^{-14}	1.019×10^{-7}	-1.000	1.019×10^{-7}
Mars	3.973×10^{-15}	1.737×10^{-9}	-4.622×10^{-2}	3.758×10^{-8}
Jupiter	1.388×10^{-12}	9.723×10^{-4}	-1.174×10	8.280×10^{-5}
Saturn	4.583×10^{-13}	7.097×10^{-5}	-1.046	6.784×10^{-5}
Uranus	1.021×10^{-13} *	1.703×10^{-6} *	-3.948×10^{-2}	4.314×10^{-5}
Neptune	1.355×10^{-13}	2.126×10^{-6}	-1.895×10^{-2}	1.122×10^{-4}

Furthermore, the predictions for the mass magnetic GR forces ($F_M^B(M_i - M_\odot)$) associated with $B_M(M_i)$ in (38), the mass electric Newtonian forces $F_M^E(M_i - M_\odot)$ in (43), and the corresponding characteristic ratios, i.e.,

$$\chi(M_i - M_\odot) = |\vec{F}_M^B(M_i - M_\odot)|/|\vec{F}_M^E(M_i - M_\odot)| \sim 10^{-11} - 10^{-4}, \quad (44)$$

are also listed in Table 1. Here, we observe that $F_M^B(M_i - M_\odot)$ and $F_M^E(M_i - M_\odot)$ acting on the solar system planets with M_i show magnitudes of orders of about $(10^{11} - 10^{19})$ N and $(10^{20} - 10^{23})$ N, respectively. Note that for the case of Jupiter ($B_M(M_5)$), $F_M^B(M_5 - M_\odot)$ and $F_M^E(M_5 - M_\odot)$ have the largest predicted values among those of the solar system planets because Jupiter has the heaviest mass.

Note that the smallness of these characteristic ratios ($\chi(M_m - M_\oplus)$ and $\chi(M_i - M_\odot)$) originate from the $\frac{G}{c^2}$ factor involved in $B_M(M_m)$ in (33) and $B_M(M_i)$ in (38), with which $F_M^B(M_m - M_\oplus)$ in (34) and $F_M^B(M_i - M_\odot)$ are constructed. In contrast, $F_M^E(M_m - M_\oplus)$ and $F_M^E(M_i - M_\odot)$ possess the G factor only as shown in (35) and (43). Note also that, for the case of Venus, the direction of $\vec{B}_M(M_2)$ is along $-\hat{\omega}$ to produce the centripetal force ($\vec{F}^B_M(M_2 - M_\odot)$). Next, for the case of Uranus, the direction of $\vec{B}_M(M_7)$ is along $-\hat{r}$ to yield the force ($\vec{F}^B_M(M_7 - M_\odot)$) along $\hat{\omega}$.

3.3. B Ring of Saturn

Now, it seems appropriate to address some comments on the rings of Saturn to investigate the corresponding mass magnetic dipole moments. It is well known that the B ring of Saturn is the brightest, most opaque and—most likely—most massive of Saturn's rings [24]. Motivated by this, we study the isolated system of the B ring of Saturn for simplicity. Making use of a multipole expansion valid at a distant observation point, we find $\vec{A}_M$ in terms of the dominant mass magnetic dipole moment contribution[8]

$$\vec{A}_M = \frac{\vec{m}_M \times \vec{x}}{r^3}, \tag{45}$$

where $r = |\vec{x}|$ and the mass magnetic dipole moment $\vec{m}_M$ is given by

$$\vec{m}_M = -\frac{G}{2c^2} \int \vec{x}' \times \vec{J}_M d^3x'. \tag{46}$$

Note that the mass magnetic dipole moment ($\vec{m}_M$) in (46) is the analog of the charge magnetic dipole moment ($\vec{m}_Q$) [18,19], and the direction of $\vec{m}_M$ is opposite to that of $\vec{m}_Q$ because of the difference in the interaction patterns between repulsive EM and attractive mass gravitational forces.

Now, we construct the mass magnetic dipole moment of the B ring orbiting Saturn. To do this, we first find the period (t_B) of the B ring orbiting Saturn. In the absence of non-gravitational fields, the B ring of Saturn moves in a circular orbit according to the Kepler's third law to a high degree of approximation [25], where the period (t_B) is given by [26]

$$t_B^2 = \frac{4\pi^2 R_B^3}{GM_6}. \tag{47}$$

Here, R_B is the orbital radius of the B ring. Note that in (47), we neglect the effect of the B-ring mass (M_B) which is extremely small compared to the mass (M_6). Combining the formulas for $\vec{m}_M$ in (46) and t_B in (47), for the B ring of Saturn, we finally arrive at the mass magnetic dipole moment ($\vec{m}_M(Ring) = m_M(Ring)\hat{\omega}$), where[9]

$$m_M(Ring) = -\frac{G^{3/2} M_6^{1/2} M_B R_B^{1/2}}{2c^2} = -1.141 \times 10^4 \text{ m}^3 \text{ s}^{-1}. \tag{48}$$

Note that the direction of this mass magnetic dipole moment is opposite to that of the charge magnetic dipole moment in the electromagnetic interaction because of the differences in the interaction patterns between charge-repulsive electromagnetic and mass-attractive gravitational forces. Now, the predicted value for $m_M(Ring)$ in (48) is intrinsically associated with the Cassini data [16] concerning the total mass of the rings of Saturn via the explicit formula possessing M_B in (48), which is the approximated total mass of Saturn's rings [24].

Next, we investigate the recent Cassini data [17] on the formation of the young rings through a grazing encounter of an additional satellite, named Chrysalis, which is assumed to be predominantly composed of water ice. This encounter of Chrysalis with Saturn would have caused Chrysalis to break apart to yield debris, which could have developed into Saturn's young rings [17,27]. The loss of the hypothetical satellite Chrysalis can explain the obliquity of Saturn and the young age of its rings. Moreover the value of $m_M(Ring)$ in

(48), which could come from the debris mass of Chrysalis via (48), could be related to the formation mechanism [17] of Saturn's rings.

4. Conclusions

In summary, we have constructed a formalism of gravitomagnetism by making use of MLGR. Exploiting gravitomagnetism, we have *numerically* evaluated astrophysical quantities such as the mass magnetic GR force and the mass magnetic dipole moment of Saturn's B ring, for instance. To be more specific, in order to identify the effectiveness of gravitomagnetism, we treated the astrophysical observables such as the mass magnetic fields for both the isolated system of the Moon orbiting the spinning Earth and that of the solar system planets orbiting the spinning Sun. Exploiting the mass magnetic fields, we predicted the ensuing Lorentz-type mass magnetic GR forces for these systems. Note that in investigating gravitomagnetism, we used MLGR associated with the Wald approximation, in contrast to LGR possessing the TT gauge. Note also that in MLGR related to gravitomagnetism, dependence of the angle of GW radiation can be explained by the vectorial formalism [10]. In contrast, LGR is mathematically defined differently in the second-rank tensorial formalism from in the vectorial formalism in MLGR, and LGR cannot present the angle dependence of GW radiation [10]. In this paper, gravitomagnetism predicted the existence and ensuing numerical estimates of physical observables such as the mass magnetic GR force hidden in the Newtonian gravity. Note that LGR cannot predict astrophysical physical quantities such as the mass magnetic fields and the mass magnetic GR forces, for instance.

Next, exploiting the Maxwell-type gravitational equations in gravitomagnetism we obtained the mass electric and mass magnetic fields ($\vec{E}_M$ and $\vec{B}_M$, respectively), with which we constructed the corresponding mass electric Newtonian and Lorentz-type mass magnetic GR forces ($\vec{F}_M^E$ and $\vec{F}_M^B$, respectively). One of the advantages of gravitomagnetism is to evaluate the mass magnetic field ($\vec{B}_M$), whose phenomenological value is hidden in Newtonian gravity, as newly predicted in this paper. For instance, in the isolated Earth of mass $M_\oplus$ and the Moon of mass M_m, we found $|\vec{B}_M(M_\oplus)|$ acting on the Earth, while in the isolated system of the Sun of mass $M_\odot$ and solar system planets of masses M_i $(i = 1, 2, 3, \ldots, 8)$, we obtained $|\vec{B}_M(M_\odot)|$ acting on the Sun, which is about $10^2 \times |\vec{B}_M(M_\oplus)|$. Moreover, we noticed that the mass magnetic GR force ($\vec{F}_M^B$) is relatively small compared with $\vec{F}_M^E$. To be specific, considering the isolated system of the Moon and Earth, in gravitomagnetism we estimated $|\vec{F}_M^B(M_m - M_\oplus)|$ acting on the Moon, which is approximately 10^{-11} of the mass of electric Newtonian force ($|\vec{F}_M^E(M_m - M_\oplus)|$). For the case of the Earth and the Sun, we found that $|\vec{F}_M^B(M_\oplus - M_\odot)|$ is about 10^{-7} of the corresponding mass electric Newtonian force ($|\vec{F}_M^E(M_\oplus - M_\odot)|$). Next, for the isolated Sun and solar system planets, in gravitomagnetism, we have evaluated $|\vec{F}_M^B(M_i - M_\odot)|$ acting on the solar system planets, which are of an order of about $10^{-11} - 10^{-4}$ of the corresponding mass electric Newtonian forces ($|\vec{F}_M^E(M_i - M_\odot)|$), as shown in Table 1. This feature implies that gravitomagnetic correction, namely $|\vec{F}_M^B(M_i - M_\odot)|$, to the Newtonian classical prediction ($|\vec{F}_M^E(M_i - M_\odot)|$) is extremely small but non-vanishing. To be specific, this gravitomagnetic correction has been hidden in astrophysical phenomenology until now. The theoretical prediction of the mass magnetic GR force in gravitomagnetism is one of the main points of this paper. Note that $|\vec{F}_M^B(M_i - M_\odot)|$ possesses the leading-order correction, which originates from the spinning effects of the solar system planets of M_i, and this correction is dominant compared to the other next-order effects. Note also that the directions of $\vec{F}_M^B(M_i - M_\odot)$ for the solar system planets (except Venus of mass M_2 and Uranus of mass M_7) are anti-parallel to those of $\vec{F}_M^E(M_i - M_\odot)$, while the directions of $\vec{F}_M^B(M_2 - M_\odot)$ and $\vec{F}_M^B(M_7 - M_\odot)$ for Venus and Uranus are parallel to and perpendicular to those of $\vec{F}_M^E(M_i - M_\odot)$ $(i = 2, 7)$, respectively.

Now, as a typical example of the mass magnetic dipole moment ($\vec{m}_B$) in gravitomagnetism, we investigated Saturn's B ring to predict $\vec{m}_M(Ring)$. Note that the prediction of $\vec{m}_M(Ring)$ has been shown to be closely related to the Cassini data [16] on the total mass of the rings of Saturn. Next, $\vec{m}_M(Ring)$, which could originate from the debris mass of hypothetical satellite Chrysalis [17], also could be associated with the formation of the young rings of Saturn. Note that the mass magnetic GR force ($\vec{F}^B_M(Ring)$) acting on the B ring of Saturn contributes, in small amounts, to the formation of the width of the B ring.

Next in this paper, we phenomenologically predicted physical quantities such as the mass magnetic fields and their ensuing *mass magnetic GR forces*, which exist in the Universe according to gravitomagnetism (or MLGR) proposal. It will be interesting to search for observational evidence for these physical quantities in the Universe. Once this is done, the theoretical formalism developed in this paper could have some progressive impacts on *precision astrophysics phenomenology.* Note that research on GR effects such as the relativistic precision and frame dragging conducted saliently by the Gravity Probe B satellite has been conducted, as is well documented in Refs. [2,3,28] and references therein.

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Appendix A. Sketch of EM

In order to construct the gravitomagnetism formalism discussed in Section 2, we digress to pedagogically recapitulate the EM associated with charge (Q) [18,19]. To do this, we start with the charge electric and charge magnetic fields in EM, where we find the Maxwell equations in EM:

$$\begin{aligned} \nabla \cdot \vec{E}_Q &= \frac{1}{\epsilon_0}\rho_Q, \quad \nabla \times \vec{E}_Q + \frac{\partial \vec{B}_Q}{\partial t} = 0, \\ \nabla \cdot \vec{B}_Q &= 0, \qquad \nabla \times \vec{B}_Q - \frac{1}{c^2}\frac{\partial \vec{E}_Q}{\partial t} = \frac{1}{\epsilon_0 c^2}\vec{J}_Q, \end{aligned} \tag{A1}$$

where ρ_Q and $\vec{J}_Q$ are the volume charge density and volume charge current, respectively. Here, we use $\epsilon_0\mu_0 = c^{-2}$, where ϵ_0, μ_0 and c are the permittivity, permeability and speed of light of free space, respectively. Note that the charge electric and mass magnetic fields (E^i_Q and B^i_Q, respectively) are given in terms of the four charge vector potentials ($A^\alpha_Q \equiv \left(\frac{\phi_Q}{c}, \vec{A}_Q\right)$) as

$$\vec{E}_Q = -\nabla\phi_Q - \frac{\partial \vec{A}_Q}{\partial t}, \quad \vec{B}_Q = \nabla \times \vec{A}_Q. \tag{A2}$$

We also find that the charge vector potentials (A^α_Q) fulfill

$$\Box\phi_Q = -\frac{1}{\epsilon_0}\rho_Q, \quad \Box\vec{A}_Q = -\frac{1}{\epsilon_0 c^2}\vec{J}_Q, \tag{A3}$$

where we exploit the Lorentz gauge condition:

$$\partial_\alpha A^\alpha_Q = 0. \tag{A4}$$

In vacuum, inserting $\rho_Q = \vec{J}_Q = 0$ into (A3), we find the wave equation for A_Q^α:

$$\Box A_Q^\alpha = 0, \quad \alpha = 0, 1, 2, 3 \tag{A5}$$

which implies a massless spin-one photon propagating in flat space time.

Next, we investigate the gauge invariance and U(1) transformation in EM. To do this, we start with the Dirac equation for the electron wave function (ψ) and the charge-vector potentials (A_Q^μ) in units of $\hbar = c = 1$, i.e.,

$$i\gamma^\mu \partial_\mu \psi - m_e \psi - e\gamma^\mu A_{Q,\mu} \psi = 0, \tag{A6}$$

where m_e and e are the electron mass and charge, respectively. Next, the QED describes the interactions between electrons and photons. Now, we introduce the QED Lagrangian [21]:

$$\mathcal{L}_{QED} = -\frac{1}{4} F_{Q,\mu\nu} F_Q^{\mu\nu} + \bar{\psi}(i\gamma^\mu - m_e)\psi - e\bar{\psi}\gamma^\mu A_{Q,\mu}\psi, \tag{A7}$$

where $F_Q^{\mu\nu} = \partial^\mu A_Q^\nu - \partial^\nu A_Q^\mu$ and the third term denotes the interaction between the electron and the electromagnetic wave (or spin-one photon). Note that the QED Lagrangian is invariant under the gauge transformations:

$$\begin{aligned} \psi(x) &\rightarrow \psi'(x) = e^{-ie\Gamma(x)}\psi(x), \\ A_Q^\mu(x) &\rightarrow A_Q'^\mu(x) = A_Q^\mu(x) + \partial^\mu \Gamma(x), \end{aligned} \tag{A8}$$

where the first relation is the U(1) transformation.

Next, for the cases of charge electrostatics and charge magnetostatics, corresponding to ρ_Q and $\vec{J}_Q$, respectively, inside a given volume, we obtain the charge electric and charge magnetic fields as

$$\vec{E}_Q = \frac{1}{4\pi\epsilon_0} \int \frac{\rho_Q \hat{R}}{R^2} d^3x', \quad \vec{B}_Q = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\vec{J}_Q \times \hat{R}}{R^2} d^3x', \tag{A9}$$

where $\vec{R} = \vec{x} - \vec{x}'$ is the vector from d^3x' to the field point ($\vec{x}$) and $\hat{R} = \vec{R}/R$ with $R = |\vec{x} - \vec{x}'|$. Exploiting the charge electric and magnetic fields ($\vec{E}_Q$ and $\vec{B}_Q$, respectively) in (A9), we end up with the corresponding force acting on a test charge (Q_0) moving with a velocity of v:

$$\vec{F}_Q = Q_0(\vec{E}_Q + \vec{v} \times \vec{B}_Q). \tag{A10}$$

Moreover, in charge magnetostatics, we construct the charge-vector potential ($\vec{A}_Q$) as follows:

$$\vec{A}_Q = \frac{1}{4\pi\epsilon_0 c^2} \int \frac{\vec{J}_Q}{R} d^3x'. \tag{A11}$$

Making use of a multipole expansion valid at a distant observation point, we reformulate $\vec{A}_Q$ in terms of the dominant charge magnetic dipole moment contribution as

$$\vec{A}_Q = \frac{\vec{m}_Q \times \vec{x}}{r^3}, \tag{A12}$$

where $r = |\vec{x}|$ and the charge magnetic dipole moment ($\vec{m}_Q$) is given by

$$\vec{m}_Q = \frac{1}{8\pi\epsilon_0 c^2} \int \vec{x}' \times \vec{J}_Q d^3x'. \tag{A13}$$

Notes

[1] Superscripts E and B stand for the mass *electric* Newtonian force and the mass *magnetic* force, respectively.

[2] Here, we have exploited the identity $\int_0^\pi \frac{d\theta' \sin\theta' \cos\theta'}{R} = \frac{2r'}{3r^2}$.

[3] Here, we assume that this observer resides on a space-fixed frame and does not co-rotate together with the Earth so that we can include the spinning effect of the Earth. The same statement is applied to the observers who are located on the surfaces of the Moon, the Sun and the solar system planets.

[4] For the observational data for M_I, r_I, R_I, t_I and T_I $(I = m, \oplus)$, we exploit [20,22,23]. Similarly, we use [20,22,23] for the observational data for M_i, r_i, R_i, t_i and T_i $(i = 1, 2, \ldots 8)$ for the i-th planet in the solar system, as discussed in the next subsections.

[5] From now on, $\vec{F}^B_M(M_m - M_\oplus)$ denotes the force ($\vec{F}^B_M$) acting on the position at M_m under the field at $M_\oplus$, for instance. Here, the argument expressed as $M_m - M_\oplus$ indicates the vector of the direction from $M_\oplus$ to M_m with a magnitude of $|M_m - M_\oplus|$.

[6] From now on, the minus sign in $F^E_M(M_m - M_\oplus)$ indicates the attractive Newtonian force between M_m and $M_\oplus$, for example.

[7] In Table 1, the magnitudes of $F^B_M(M_i - M_\odot)$ and $F^E_M(M_i - M_\odot)$ are given in units of $|F^E_M(M_\oplus - M_\odot)|$, and these values are dimensionless. The minus signs in $F^E_M(M_i - M_\odot)$, again, denote the attractive Newtonian forces between M_i and $M_\odot$.

[8] Here, we exploit the approximation at a distant observation point of $\vec{A}_M = -\frac{G}{c^2}\left(\frac{1}{r}\oint \vec{J}_M d^3x' + \frac{x_i}{r^3}\int \vec{J}_M x'_i d^3x' + \cdots\right)$, where the ellipsis stands for the higher-order terms that can be ignored for the mass-vector potential of a localized mass-current distribution. Note that the first term denotes the mass magnetic monopole term, which vanishes for the closed-loop integral, as expected. Thus, in this paper, we exclude the possibility of a mass magnetic monopole, in addition to that of a charge magnetic monopole.

[9] The recent Cassini data show that the total mass of the rings of Saturn is 1.54×10^{19} kg [16]. In evaluating (48), we exploit this value for the B-ring mass (M_B) for simplicity, since the B ring is known to probably be the most massive of Saturn's rings [24]. Next the orbital radius of the B ring is given by $R_B = (9.232 \times 10^7 - 1.177 \times 10^8)$ m [20]; thus, we use the average value of $R_B = 1.050 \times 10^8$ m. Moreover, we ignore the effect of the width of the B ring of Saturn in the evaluation in (48) for simplicity.

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