Reducibility of Euclidean Motion Groups

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Introduction

There exist several views of the concept of reducibility and of decomposability in algebra. Reducibility of matrix groups and of groups of linear operators is now about a century old and forms a background for reducibility theory of representations which found its applications in determination of selection rules and classification of spectra. The reducibility in physics is usually the reducibility of linear spaces under the action of some symmetry groups and over the field R of real numbers; the so-called physically irreducible representations are exactly representations irreducible over R. The results of more general reducibility theory which includes reducibility of Z-modules are exposed in the book of Curtis and Reiner [1]; let us only remind that reducibility over fields is equivalent to decomposability which is not true over the rings. The translation subgroups of space groups are exactly ZG-modules and arithmetic classes (G,T) may be either reducible or irreducible; a reducible class may still be either indecomposable at all or its reducibility and decomposability patterns may not coincide. Reducibility of arithmetic classes implies a certain reducibility of space groups. Though reducibility patterns over various fields are given already in the book on four-dimensional space groups [2], this implication has been considered much later [3,4,5,6]. We want to show now that the concept of reducibility and the results of its theory for space groups, especially the factorization and intersection theorem can be applied as well to more general cases of Euclidean and even affine groups.

Extension of the concept of the reducibility of space groups

We denote by $E(n)$ an Euclidean space and by $V(n)$ its difference space, the orthogonal vector space, where n is the dimension. Then every Euclidean motion can be expressed by Seitz symbol $\{g|\mathbf{t}\}_P$ with respect to a certain origin P and every Euclidean motion group $\mathcal{G}$ by a symbol $\{G, T, P, \mathbf{u}_G(g)\}$, where G is a subgroup of the orthogonal group $O(n)$ acting on $V(n)$, T is a G-invariant translation subgroup of $V(n)$ and $\mathbf{u}_G(g)$ the system of nonprimitive translations. By the latter we mean a function $\mathbf{u}_G : G \rightarrow V(n)$ which satisfies Frobenius congruences:

$$\mathbf{w}_G(g,h) = \mathbf{u}_G(g) + g\mathbf{u}_G(h) - \mathbf{u}_G(gh) = \mathbf{0} \bmod T.$$

We can extend the concept of arithmetic class (G,T) to any kind of Euclidean groups and it is also suitable to use the word space group for any group, the translation subgroup T of which spans the whole $V(n)$ over R, while other groups are called subperiodic.

It is easy to realize that the same scheme is valid also on the level of affine groups, where g in the Seitz symbol is from the general linear group $\mathcal{G}L(n)$ acting on the linear space $L(n)$ which turns into orthogonal space $V(n)$ with introduction of an orthogonal scalar product. Every affine group $\mathcal{G}$, the point group G of which is orthogonalizable by a suitable choice of scalar product is affinely equivalent to some Euclidean group and hence all further considerations apply to it as well.

We have defined reducibility of space groups [3,4,5] as a consequence of Q-reducibility of the action of G on T. It would be more appropriate to distinguish this reducibility in a wider context as a *crystallographic* (or *arithmetic*) reducibility. To extend the concept of reducibility to arbitrary Euclidean groups, we have to realize, that the translation subgroup T itself as well as its reducibility under the action of G may have various features. In particular, T may be a direct sum of G-invariant modules or even spaces, each of which spans $V(n)$. Figuratively expressed, the algebraic structure of T does not reflect the geometric meaning of T as a subset of $V(n)$. We should therefore distinguish between *algebraic* reducibility which is simply the reducibility of T as ZG-module and *geometric* reducibility, which is a consequence of the reducibility of the action of G on the space $V(n)$.

The geometric reducibility implies at least partially the algebraic one and both reducibilities imply a certain reducibility of Euclidean groups of the class (G, T). We shall say that the Euclidean group $\mathcal{G}$ is *geometrically reducible*, if the action of its point group G on $V(n)$ is reducible. Subperiodic groups with nontrivial T are naturally always reducible. The reducibility of crystallographic space groups is defined as a consequence of geometric reducibility with an additional requirement of arithmetic reducibility. The latter is automatically fullfilled for orthogonal reductions, while inclined ones may create some problems, which have been considered in more detail in the study of reducible space groups in arbitrary dimensions [5]. To avoid complications, we assume further that we are dealing with orthogonal reducibility.

Consequences of geometric reducibility

Geometric reducibility of Euclidean groups is a reducibility of the action of a group on a point space and the main construction connected with it is the *subdirect product* or *subdirect sum*. This construction is of frequent use but it occurs rarely in textbooks. We can trace it to Goursat [7] and recognize it in many recent constructions. In the book by Huppert [8] it appears under the name *das direkte Produkt von Gruppen mit vereinigter Faktorgruppe*. The subdirect product is also used in consideration of transitivity [9], which is a kind of reducibility of the group action on sets. The importance of this construction is realized by Opechowski [10] who uses it throughout his book. We gave an overview of its use and an analysis for cases of more then two components, when we prefer the term *multiple subdirect product (sum)* [11]. The work with subdirect products is based on a theorem which says that *a subgroup of a direct product of groups O_i is either a direct or subdirect product of subgroups G_i° of groups O_i*. These groups are obtained by homomorphisms σ_i which map G onto its components in O_i and the greatest direct product of subgroups of O_i contained in G is the product of intersections $G_i = G \cap O_i$.

The main theorem on reducible Euclidean groups. The geometric reducibility of (G, T) and hence of all Euclidean groups $\mathcal{G}$ of this class means that $V(n)$ splits into a direct sum of G-invariant subspaces $V_i(k_i)$ of which T is a subgroup. Further, G is a subgroup of a direct product of orthogonal groups O_i which act on spaces $V_i(k_i)$ and $\mathcal{G}$ is a subgroup of a direct product of corresponding Euclidean groups $\mathcal{E}_i$ acting on Euclidean spaces $E(k_i)$. It follows immediately, that T is a subdirect sum of certain groups $T_i^\circ \subseteq V_i(k_i)$, G is a subdirect product of groups $G_i^\circ \subseteq O_i$ and $\mathcal{G}$ is a subdirect product of groups $\mathcal{G}_i^\circ \subseteq \mathcal{E}_i$.

Compare this result with the splitting of reducible representations into a **direct sum** of irreducible components and let us observe that to express it in terms of operator or matrix groups we have to use again the subdirect products [11].

Factorization theorem and projection homomorphisms. If (G,T) is geometrically reducible, then there appear neccessarily G-invariant subgroups $T_i = T \cap V_i(k_i)$. Each such subgroup is normal in every group $\mathcal{G}$ of the class (G,T). The factorization theorem asserts that the factor group $\mathcal{G}/T_i$ is isomorphic to a certain subperiodic group. The group $\mathcal{G}$ is mapped onto so-called *contracted* subperiodic groups by *projection homomorphisms* which are unique for orthogonal reductions. These projections, described in detail in [5] can be applied to any geometrically reducible Euclidean group.

Intersection theorem. This theorem is valid in each case when T splits into a direct sum of G-invariant subgroups $T_i = T \cap V_i(k_i)$ and its meaning is very transparent. The system of nonprimitive translations $\mathbf{u}_G$ can be uniquely expressed as a sum of its components $\mathbf{u}_{Gi}$ in individual subspaces $V_i(k_i)$. Since these subspaces are G-invariant, each of the components $\mathbf{u}_{Gi}$ satisfies Frobenius congruences

$$\mathbf{w}_{Gi}(g,h) = \mathbf{u}_{Gi}(g) + g\mathbf{u}_{Gi}(h) - \mathbf{u}_{Gi}(gh) = \mathbf{0} \bmod T_i.$$

We can introduce now either subperiodic groups $\mathcal{L}_i = \{G, T_i, P, \mathbf{u}_{Gi}\}$ or, since congruences mod T_i imply the same congruences mod T, we can as well introduce groups $\mathcal{G}_i = \{G, T, P, \mathbf{u}_{Gi}\}$ and classify groups of arithmetic class (G,T) into *subperiodic classes* $\mathcal{L}_i$ of which the groups $\mathcal{G}_i$ are the *symmorphic representatives* [4,5]. The essence of the intersection theorem lies in the statement that each space group of the class (G,T) lies on the intersection of subperiodic classes $\mathcal{L}_i$.

An example. The following table shows how intersection and factorization theorem work together in practice. The upper row of this table lists rod groups, the first column

4mmP	p4mm	$p4_2$cm	p4cc	$p4_2$mc
p4mm	P4mm	$P4_2$cm	P4cc	$P4_2$mc
p4bm	P4bm	$P4_2$nm	P4nc	$P4_2$bc

lists the layer groups and on intersections of rows and columns stand the eight space groups of the arithmetic class 4mmP. In these symbols P denotes the translation subgroup $T(\mathbf{a},\mathbf{b},\mathbf{c})$, p stands for $T(\mathbf{a},\mathbf{b})$ and p for $T(\mathbf{c})$. Each layer group of the first column is a common factor group by $T(\mathbf{c})$ for all the space groups of the row and each rod group of the first row is a common factor group by $T(\mathbf{a},\mathbf{b})$ for all space groups of the column.

Views for the future development

We can see already now a few of valuable ramifications and consequences of reducibility theory of Euclidean groups. Its first natural use concerns the crystallography in spaces of arbitrary dimensions. It is easy to see that irreducible space groups and the laws of their composition into reducible ones are of primary interest. We believe, however, that algebraic reducibility is more adequate from the viepoint of applications to incommensurate structures and/or quasicrystals while geometric reducibility is of rather academic interest. The mentioned structures are after all structures of three-dimensional space.

Factorization and intersection theorems are very valuable in three-dimensional crystallography. Both two types of subperiodic groups in three dimensions, the *layer* and the *rod* groups clearly appear as subgroups of space groups. Now we know that they appear also as factor groups of reducible space groups. In this role it is suitable to consider them as *contracted* layer groups acting on $E(\mathbf{a},\mathbf{b}) \times V(\mathbf{c})$, and *contracted* rod groups acting on $V(\mathbf{a},\mathbf{b}) \times E(\mathbf{c})$. The relationship between these contracted subperiodic groups and subperiodic groups acting on the ordinary Euclidean space $E(\mathbf{a},\mathbf{b},\mathbf{c})$ is in all respects analogous to the relationship between point groups G acting on $V(n)$ and site-point groups acting on $E(n)$.

Factorization theorem enables us to classify space groups into *layer and rod classes* [6] and introduce their standards not in an *ad hoc* manner but in correlation with standards of space groups. This is a first step in a solution of an important problem of **bicrystallography** - *scanning* of layer and rod groups through the space for defined space symmetries and plane or line directions, definition and classification of *Wyckoff types* of orbits for planes and lines in a crystal. Such problems can be easily solved for significant directions of reducible space groups and, as we have shown [12], this solution can be extended to arbitrary groups and directions with use of so-called *scanning theorem* and *scanning group*.

LAST BUT NOT LEAST. Since layer and rod groups appear as factor groups of space groups, their representations are connected with certain representations of space groups via the well known process of *engendering*. Every expert in representation theory will probably realize at once the importance of this fact for systemization of our knowledge of representations of space and subperiodic groups. But this is quite a new story.

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