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Wick Theorem and Hopf Algebra Structure in Causal Perturbative Quantum Field Theory

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Abstract: We consider the general framework of perturbative quantum field theory for the pure Yang–Mills model. We give a more precise version of the Wick theorem using Hopf algebra notations for chronological products and not for Feynman graphs. Next, we prove that the Wick expansion property can be preserved for all cases in order $n = 2$. However, gauge invariance is broken for chronological products of Wick submonomials.

Keywords: perturbative quantum field theory; causal approach; pure Yang–Mills fields

1. Introduction

The most natural way to arrive at the Bogoliubov axioms of perturbative quantum field theory (pQFT) is by analogy with non-relativistic quantum mechanics [1–4]: in this way, one arrives naturally at Bogoliubov axioms [5–11]. We prefer the formulation from [2] and as presented in [12]: for every set of monomials $A_1(x_1), \dots, A_n(x_n)$ in some jet variables (associated to some classical field theory), one associates the operator-valued distributions $T^{A_1, \dots, A_n}(x_1, \dots, x_n)$ called chronological products. It is convenient to use another notation: $T(A_1(x_1), \dots, A_n(x_n))$.

The Bogoliubov axioms, presented in Section 2.2 express essentially some properties of the scattering matrix understood as a formal perturbation series; the “coefficients” of this formal series are the chronological products and they verify:

- (Skew)symmetry properties in the entries $A_1(x_1), \dots, A_n(x_n)$;
- Poincaré invariance;
- Causality;
- Unitarity;
- The “initial condition”, which says that $T(A(x))$ is a Wick monomial.

So, we need some basic notions on free fields and Wick monomials. One can supplement these axioms by requiring the following:

- Power counting;
- The Wick expansion property.

It is a highly non-trivial problem to find solutions for the Bogoliubov axioms, even in the simplest case of a real scalar field.

There are, at least to our knowledge, three rigorous ways to do that. For completeness, we reiterate them as follows [13]:

- *Hepp axioms* [4];
- *Polchinski flow equations* [14,15];
- *The causal approach* due to Epstein and Glaser [3,6], which we prefer.

The procedure of Epstein and Glaser is a recursive construction for the basic objects $T(A_1(x_1), \dots, A_n(x_n))$ and reduces the induction procedure to a distribution splitting of some distributions with causal support. In an equivalent way, one can reduce the induction procedure to the process of the extension of distributions [16].



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An equivalent point of view uses retarded products [17] instead of chronological products. For gauge models, one has to deal with non-physical fields (the so-called ghost fields) and impose a supplementary axiom, namely, gauge invariance, which guarantees that the physical states are left invariant by the chronological products.

In the next section, we give the main prerequisites we need, following essentially [12]. Then, we present a more precise version of the Wick theorem and emphasize the Hopf structure in Section 3. This Hopf structure differs from the Hopf structure introduced in [18], which is based on the forest formula of Zimmermann. Our Hopf structure is based only on the basic chronological products, and it first appeared in [19]. A similar Hopf structure appeared in [20,21].

We will apply these results to the pure YM model in Section 4. The Kreimer version of the Hopf algebra for gauge models was developed in [22–25].

Next, we discuss in detail the simplest case of $n = 2$ for the pure YM model. We study the basic chronological products of the type $T(T(x_1), T(x_2))$ where $T(x)$ is the interaction Lagrangian, but also chronological products of the type $T(A_1(x_1), A_2(x_2))$ where A_1, A_2 are Wick submonomials.

All these chronological products can be split in the loop and tree contributions. First, we prove in Section 5 that the loop contributions do not produce anomalies. This point can be cleared by direct computations. Then, in Section 6, we investigate the tree contributions. As it is known, they produce anomalies; for the basic chronological products of the type $T(T(x_1), T(x_2))$ these anomalies can be removed by finite renormalizations, but we will prove that this is not true for the associated chronological products of the type $T(A_1(x_1), A_2(x_2))$. However, it is important to establish if there is a clever finite renormalization saving gauge invariance for the basic chronological products of the type $T(T(x_1), T(x_2))$ and Wick expansion property also. This point can be proved in a very compact way, starting from a redefinition of the chronological product $T(v_{a\mu,\nu}(x_1), v_{b\rho,\sigma}(x_1))$ where $v_{a\mu}$ are some jet variables for the vector fields describing the gluons and $v_{a\mu,\nu}$ the associated derivative jet variables. This idea appeared in [26,27].

2. Perturbative Quantum Field Theory

There are two main ingredients in the construction of a perturbative quantum field theory (pQFT): the construction of the Wick monomials and the Bogoliubov axioms. For a pQFT of Yang–Mills theories, one needs one more ingredient, namely, the introduction of ghost fields and gauge charge.

2.1. Wick Products

We follow the formalism from [12]. We consider a classical field theory on the Minkowski space $\mathcal{M} \simeq \mathbb{R}^4$ (with variables $x^\mu, \mu = 0, \dots, 3$ and the metric η with $diag(\eta) = (1, -1, -1, -1)$) described by the Grassmann manifold Ξ_0 with variables $\zeta_a, a \in \mathcal{A}$ (here \mathcal{A} is some index set) and the associated jet extension $J^r(\mathcal{M}, \Xi_0), r \geq 1$ with variables $x^\mu, \zeta_{a;\mu_1, \dots, \mu_n}, n = 0, \dots, r$; we denote generically by $\zeta_p, p \in P$ the variables corresponding to classical fields and their formal derivatives and by Ξ_r the linear space generated by them. The variables from Ξ_r generate the algebra $Alg(\Xi_r)$ of polynomials.

To illustrate this, let us consider a real scalar field in Minkowski space \mathcal{M} . The first jet-bundle extension is

$$J^1(\mathcal{M}, \mathbb{R}) \simeq \mathcal{M} \times \mathbb{R} \times \mathbb{R}^4$$

with coordinates $(x^\mu, \phi, \phi_\mu), \mu = 0, \dots, 3$.

If $\varphi : \mathcal{M} \rightarrow \mathbb{R}$ is a smooth function, we can associate a new smooth function $j^1\varphi : \mathcal{M} \rightarrow J^1(\mathcal{M}, \mathbb{R})$ according to $j^1\varphi(x) = (x^\mu, \varphi(x), \partial_\mu\varphi(x))$.

For higher order jet-bundle extensions, we have to add new real variables $\phi_{\{\mu_1, \dots, \mu_r\}}$ considered completely symmetric in the indexes. For more complicated fields, one needs to add supplementary indexes to the field i.e., $\phi \rightarrow \phi_a$ and similarly for the derivatives. The index a carries some finite dimensional representation of $SL(2, \mathbb{C})$ (Poincaré invariance) and maybe a representation of other symmetry groups. In classical field theory, the jet-

bundle extensions $j^r \varphi(x)$ do verify Euler–Lagrange equations. To write them, we need the formal derivatives defined by

$$d_\nu \phi_{\{\mu_1, \dots, \mu_r\}} \equiv \phi_{\{\nu, \mu_1, \dots, \mu_r\}}. \tag{1}$$

We suppose that in the algebra $\text{Alg}(\Xi_r)$ generated by the variables ξ_p there is a natural conjugation $A \rightarrow A^\dagger$. If A is some monomial in these variables, there is a canonical way to associate to A a Wick monomial: we associate to every classical field $\xi_a, a \in \mathcal{A}$ a quantum free field denoted by $\tilde{\zeta}_a^{\text{quant}}(x), a \in \mathcal{A}$ and determined by the 2-point function

$$\langle \Omega, \tilde{\zeta}_a^{\text{quant}}(x), \tilde{\zeta}_b^{\text{quant}}(y) \Omega \rangle = -i D_{ab}^{(+)}(x - y) \times \mathbf{1}. \tag{2}$$

Here

$$D_{ab}(x) = D_{ab}^{(+)}(x) + D_{ab}^{(-)}(x) \tag{3}$$

is the causal Pauli–Jordan distribution associated to the two fields; it is (up to some numerical factors) a polynomial in the derivatives applied to the Pauli–Jordan distribution. We understand by $D_{ab}^{(\pm)}(x)$ the positive and negative parts of $D_{ab}(x)$. From (2), we have

$$[\tilde{\zeta}_a(x), \tilde{\zeta}_b(y)] = -i D_{ab}(x - y) \times \mathbf{1} \tag{4}$$

where $[\cdot, \cdot]$ means the graded commutator.

The n -point functions for $n \geq 3$ are obtained assuming that the truncated Wightman functions are null: see [28] and Relations (8.74) and (8.75) and Proposition 8.8 therein. The definition of these truncated Wightman functions involves the Fermi parities $|\xi_p|$ of the fields $\xi_p, p \in P$.

Afterward, we define

$$\tilde{\zeta}_{\tilde{a}; \mu_1, \dots, \mu_n}^{\text{quant}}(x) \equiv \partial_{\mu_1} \dots \partial_{\mu_n} \tilde{\zeta}_a^{\text{quant}}(x), a \in \mathcal{A}$$

which amounts to

$$[\tilde{\zeta}_{a; \mu_1 \dots \mu_m}(x), \tilde{\zeta}_{b; \nu_1 \dots \nu_n}(y)] = (-1)^n i \partial_{\mu_1} \dots \partial_{\mu_m} \partial_{\nu_1} \dots \partial_{\nu_n} D_{ab}(x - y) \times \mathbf{1}. \tag{5}$$

More sophisticated ways to define the free fields involve the GNS construction.

The free quantum fields generate a Fock space \mathcal{F} in the sense of the Borchers algebra: formally, it is generated by states of the form $\tilde{\zeta}_{a_1}^{\text{quant}}(x_1) \dots \tilde{\zeta}_{a_n}^{\text{quant}}(x_n) \Omega$ where Ω is the vacuum state. The scalar product in this Fock space is constructed using the n -point distributions, and we denote by $\mathcal{F}_0 \subset \mathcal{F}$ the algebraic Fock space.

One can prove that the quantum fields are free, i.e., they verify some free field equation; in particular, every field must verify the Klein–Gordon equation for some mass m

$$(\square + m^2) \tilde{\zeta}_a^{\text{quant}}(x) = 0 \tag{6}$$

and it follows that in momentum space, they must have the support on the hyperboloid of mass m . This means that they can be split in two parts $\tilde{\zeta}_a^{\text{quant}(\pm)}$ with support on the upper (respectively, lower) hyperboloid of mass m . We convene that $\tilde{\zeta}_a^{\text{quant}(+)}$ respectively, $\tilde{\zeta}_a^{\text{quant}(-)}$ correspond to the creation (respectively, annihilation) part of the quantum field. The expressions $\tilde{\zeta}_p^{\text{quant}(+)}$ respectively, $\tilde{\zeta}_p^{\text{quant}(-)}$ for a generic $\xi_p, p \in P$ are obtained in a natural way, applying partial derivatives. For a general discussion of this method of constructing free fields, see ref. [28]—especially Proposition 8.8. The Wick monomials leave the algebraic Fock space invariant. The definition for the Wick monomials is contained in the following proposition.

Proposition 1. *The operator-valued distributions $N(\tilde{\zeta}_{q_1}(x_1), \dots, \tilde{\zeta}_{q_n}(x_n))$ are uniquely defined by*

$$N(\xi_{q_1}(x_1), \dots, \xi_{q_n}(x_n))\Omega = \xi_{q_1}^{(+)}(x_1) \dots \xi_{q_n}^{(+)}(x_n)\Omega \tag{7}$$

$$\begin{aligned} & [\xi_p(y), N(\xi_{q_1}(x_1), \dots, \xi_{q_n}(x_n))] \\ &= \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||\xi_{q_l}|} [\xi_p(y), \xi_{q_m}(x_m)] N(\xi_{q_1}(x_1), \dots, \hat{m}, \dots, \xi_{q_n}(x_n)) \\ &= -i \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||\xi_{q_l}|} D_{pq_m}(y - x_m) N(\xi_{q_1}(x_1), \dots, \hat{m}, \dots, \xi_{q_n}(x_n)) \end{aligned} \tag{8}$$

$$N(\emptyset) = I. \tag{9}$$

The expression $N(\xi_{q_1}(x_1), \dots, \xi_{q_n}(x_n))$ is (graded) symmetrical in the arguments.

The expressions $N(\xi_{q_1}(x_1), \dots, \xi_{q_n}(x_n))$ are called Wick monomials. There is an alternative definition based on the splitting of the fields into the creation and annihilation part for which we refer to [12].

It is a non-trivial result of Wightman and Gårding [29] that in $N(\xi_{q_1}(x_1), \dots, \xi_{q_n}(x_n))$ one can collapse all variables into a single one and still obtain a well-defined expression.

Proposition 2. The expressions

$$W_{q_1, \dots, q_n}(x) \equiv N(\xi_{q_1}(x), \dots, \xi_{q_n}(x)) \tag{10}$$

are well defined. They verify the following:

$$W_{q_1, \dots, q_n}(x)\Omega = \xi_{q_1}^{(+)}(x) \dots \xi_{q_n}^{(+)}(x)\Omega \tag{11}$$

$$[\xi_p^{(\epsilon)}(y), W_{q_1, \dots, q_n}(x)] = -i \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||\xi_{q_l}|} D_{pq_m}^{(-\epsilon)}(y - x_m) W_{q_1, \dots, \hat{m}, \dots, q_n}(x) \tag{12}$$

$$W(\emptyset) = I. \tag{13}$$

We call expressions of the type $W_{q_1, \dots, q_n}(x)$ Wick monomials. By

$$|W| \equiv \sum_{l=1}^n |\xi_{q_l}| \tag{14}$$

we mean the Fermi number of W . We define the derivative

$$\frac{\partial}{\partial \xi_p} W_{q_1, \dots, q_n}(x) \equiv \sum_{s=1}^n \prod_{l < s} (-1)^{|\xi_p||\xi_{q_l}|} \delta_{pq_s} W_{q_1, \dots, \hat{q}_s, \dots, q_n}(x) \tag{15}$$

and we have a generalization of the preceding proposition.

Proposition 3. Let $W_j = W_{q_1^{(j)}, \dots, q_{r_j}^{(j)}}$, $j = 1, \dots, n$ be Wick monomials. Then the expression $N(W_1(x_1), \dots, W_n(x_n))$ is well defined through

$$N(W_1(x_1), \dots, W_n(x_n))\Omega = \prod_{j=1}^n \prod_{l=1}^{r_j} \xi_{q_l^{(j)}}^{(+)}(x_j)\Omega \tag{16}$$

$$[\xi_p(y), N(W_1(x_1), \dots, W_n(x_n))] = -i \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||W_l|} \sum_q D_{pq}(y - x_m) N(W_1(x_1), \dots, \frac{\partial}{\partial \xi_q} W_m(x_m), \dots, W_n(x_n)) \quad (17)$$

$$N(W_1(x_1), \dots, W_n(x_n), \mathbf{1}) = N(W_1(x_1), \dots, W_n(x_n)) \quad (18)$$

$$N(W(x)) = W(x). \quad (19)$$

The expression $N(W_1(x_1), \dots, W_n(x_n))$ is symmetric (in the Grassmann sense) in the entries $W_1(x_1), \dots, W_n(x_n)$.

One can prove that

$$[N(A(x)), N(B(y))] = 0, \quad (x - y)^2 < 0 \quad (20)$$

where by $[\cdot, \cdot]$ we mean the graded commutator. This is the most simple case of the causal support property. Now, we are ready for the most general setting. We define for any monomial $A \in \text{Alg}(\Xi_r)$ the derivation

$$\xi \cdot A \equiv (-1)^{|\xi||A|} \frac{\partial}{\partial \xi} A \quad (21)$$

for all $\xi \in \Xi_r$. Here, $|A|$ is the Fermi parity of A , and we consider the left derivative in the Grassmann sense. So, the product \cdot is defined as a map $\Xi_r \times \text{Alg}(\Xi_r) \rightarrow \text{Alg}(\Xi_r)$. An expression $E(A_1(x_1), \dots, A_n(x_n))$ is considered to be of the Wick type if and only if it verifies

$$\begin{aligned} & [\xi_p(y), E(A_1(x_1), \dots, A_n(x_n))] \\ &= \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q [\xi_p(y), \xi_q(x_m)] E(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \\ &= -i \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}(y - x_m) E(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \end{aligned} \quad (22)$$

$$E(A_1(x_1), \dots, A_n(x_n), \mathbf{1}) = E(A_1(x_1), \dots, A_n(x_n)) \quad (23)$$

$$E(\mathbf{1}) = \mathbf{1}. \quad (24)$$

The expression $N(W_1(x_1), \dots, W_n(x_n))$ from Proposition 3 is of the Wick type. Then, we easily have the following.

Proposition 4. *If $E(A_1(x_1), \dots, A_k(x_k))$ and $F(A_{k+1}(x_1), \dots, A_n(x_n))$ are expressions of the Wick type, then $E(A_1(x_1), \dots, A_k(x_k)) F(A_{k+1}(x_1), \dots, A_n(x_n))$ is also an expression of the Wick type.*

We can also prove the following.

Theorem 1. *The following formula is true:*

$$\begin{aligned} & N(\xi_p(y), A_1(x_1), \dots, A_n(x_n)) \\ &= \xi_p^{(+)}(y) N(A_1(x_1), \dots, A_n(x_n)) + \prod_{l \leq n} (-1)^{|\xi_p||A_l|} N(A_1(x_1), \dots, A_n(x_n)) \xi_p^{(-)}(y) \\ &+ i \sum_{m=1}^n \prod_{l < m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}^{(+)}(y - x_m) N(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \end{aligned} \quad (25)$$

2.2. Bogoliubov Axioms

Suppose the monomials $A_1, \dots, A_n \in \text{Alg}(\Xi_r)$ are self-adjoint: $A_j^\dagger = A_j, \forall j = 1, \dots, n$ and of Fermi number f_j .

The chronological products

$$T(A_1(x_1), \dots, A_n(x_n)) \equiv T^{A_1, \dots, A_n}(x_1, \dots, x_n) \quad n = 1, 2, \dots$$

are some distribution-valued operators, leaving invariant the algebraic Fock space and verifying the following set of axioms:

- **Skew symmetry** in all arguments:

$$T(\dots, A_i(x_i), A_{i+1}(x_{i+1}), \dots) = (-1)^{f_i f_{i+1}} T(\dots, A_{i+1}(x_{i+1}), A_i(x_i), \dots) \quad (26)$$

- **Poincaré invariance**: we have a natural action of the Poincaré group in the space of Wick monomials and we impose that for all $g \in \text{in}SL(2, \mathbb{C})$ we have

$$U_g T(A_1(x_1), \dots, A_n(x_n)) U_g^{-1} = T(g \cdot A_1(x_1), \dots, g \cdot A_n(x_n)) \quad (27)$$

where, on the right-hand side, we have the natural action of the Poincaré group on Ξ . Sometimes, it is possible to supplement this axiom by other invariance properties: space and/or time inversion, charge conjugation invariance, global symmetry invariance with respect to some internal symmetry group, supersymmetry, etc.

- **Causality**: if $y \cap (x + \tilde{V}^+) = \emptyset$ then we denote this relation by $x \succeq y$. Suppose that we have $x_i \succeq x_j, \forall i \leq k, j \geq k + 1$; then we have the factorization property:

$$T(A_1(x_1), \dots, A_n(x_n)) = T(A_1(x_1), \dots, A_k(x_k)) T(A_{k+1}(x_{k+1}), \dots, A_n(x_n)); \quad (28)$$

- **Unitarity**: We define the *anti-chronological products* using a convenient notation introduced by Epstein–Glaser, adapted to the Grassmann context. If $X = \{j_1, \dots, j_s\} \subset N \equiv \{1, \dots, n\}$ is an ordered subset, we define

$$T(X) \equiv T(A_{j_1}(x_{j_1}), \dots, A_{j_s}(x_{j_s})). \quad (29)$$

Let us consider some Grassmann variables θ_j , of parity $f_j, j = 1, \dots, n$ and let us define

$$\theta_X \equiv \theta_{j_1} \cdots \theta_{j_s}. \quad (30)$$

Now let (X_1, \dots, X_r) be a partition of $N = \{1, \dots, n\}$ where X_1, \dots, X_r are ordered sets. Then we define the (Koszul) sign $\epsilon(X_1, \dots, X_r)$ through the relation

$$\theta_1 \cdots \theta_n = \epsilon(X_1, \dots, X_r) \theta_{X_1} \cdots \theta_{X_r} \quad (31)$$

and the antichronological products are defined according to

$$(-1)^n \tilde{T}(N) \equiv \sum_{r=1}^n (-1)^r \sum_{I_1, \dots, I_r \in \text{Part}(N)} \epsilon(X_1, \dots, X_r) T(X_1) \cdots T(X_r) \quad (32)$$

Then the unitarity axiom is

$$\tilde{T}(N) = T(N)^\dagger. \quad (33)$$

- **The “initial condition”**:

$$T(A(x)) = N(A(x)). \quad (34)$$

- **Power counting:** We can also include in the induction hypothesis a limitation on the order of singularity of the vacuum averages of the chronological products associated to arbitrary Wick monomials A_1, \dots, A_n ; explicitly:

$$\omega(\langle \Omega, T^{A_1, \dots, A_n}(X)\Omega \rangle) \leq \sum_{l=1}^n \omega(A_l) - 4(n-1) \tag{35}$$

where by $\omega(d)$ we mean the order of singularity of the (numerical) distribution d and by $\omega(A)$ we mean the canonical dimension of the Wick monomial W .

- **Wick expansion property:** In analogy to (22), we require

$$\begin{aligned} & [\xi_p(y), T(A_1(x_1), \dots, A_n(x_n))] \\ = & \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q [\xi_p(y), \xi_q(x_m)] T(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \\ = & -i \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}(y-x_m) T(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \end{aligned} \tag{36}$$

In fact, we can impose a sharper form:

$$\begin{aligned} & [\xi_p^{(\epsilon)}(y), T(A_1(x_1), \dots, A_n(x_n))] \\ = & -i \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}^{(-\epsilon)}(y-x_m) T(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \end{aligned} \tag{37}$$

Up to now, we defined the chronological products only for self-adjoint Wick monomials W_1, \dots, W_n but we can extend the definition for Wick polynomials by linearity.

The chronological products $T(A_1(x_1), \dots, A_n(x_n))$ are not uniquely defined by the axioms presented above. They can be modified with quasi-local expressions i.e., expressions localized on the big diagonal $x_1 = \dots = x_n$; such expressions are of the type

$$N(A_1(x_1), \dots, A_n(x_n)) = P_j(\partial)\delta(X) W_j(X) \tag{38}$$

where $\delta(X) \equiv \delta(x_1 - x_n) \dots \delta(x_{n-1} - x_n)$, the expressions $P_j(\partial)$ are polynomials in the partial derivatives and $W_j(X)$ are Wick polynomials. There are some restrictions on these quasi-local expressions such that the Bogoliubov axioms remain true. One such consistency relation comes from the Wick expansion property. Such redefinitions of the chronological products are extremely important for Yang–Mills models because gauge invariance can be preserved only if some redefinitions of this type are made.

For simplicity, let us consider that the variables ξ_a are commutative and the finite renormalizations are of the type

$$N(A_1(x_1), \dots, A_n(x_n)) = \delta(X) N(A_1, \dots, A_n)(x_n) \tag{39}$$

where $N(A_1, \dots, A_n)$ are polynomials in the jet variables. Then the relation (36) from above is preserved if we require

$$\xi_p \cdot N(A_1, \dots, A_n) = \sum_{m=1}^n N(A_1, \dots, \xi_p \cdot A_m, \dots, A_n). \tag{40}$$

In the general case, some combinatorial complications, such as, for instance, Fermi signs, do appear. This consistency check seems to be absent in the literature. We investigate in the context of Yang–Mills models.

The construction of Epstein–Glaser is based on a recursive procedure [6]. We can derive from these axioms the following result [30].

Theorem 2. One can fix the causal products such that the following formula is true:

$$\begin{aligned}
 & T(\xi_p(y), A_1(x_1), \dots, A_n(x_n)) \\
 = & -i \sum_{m=1}^n \prod_{l \leq m} (-1)^{|\xi_p||A_l|} \sum_q D_{pq}^F(y - x_m) T(A_1(x_1), \dots, \xi_q \cdot A_m(x_m), \dots, A_n(x_n)) \\
 & + \xi_p^{(+)}(y) T(A_1(x_1), \dots, A_n(x_n)) + \prod_{l \leq n} (-1)^{|\xi_p||f_l|} T(A_1(x_1), \dots, A_n(x_n)) \xi_p^{(-)}(y) \quad (41)
 \end{aligned}$$

where D_{pq}^F is a Feynman propagator associated to the causal distribution D_{pq} .

Sometimes, (41)—or variants of it—is called the **equation of motion** axiom [2].

2.3. Yang–Mills Fields

First, we can generalize the preceding formalism to the case when some of the scalar fields are odd Grassmann variables. One simply insert everywhere the Fermi sign. The next generalization is to the arbitrary vector and spinorial fields. If we consider, for instance, the Yang–Mills interaction Lagrangian corresponding to pure QCD [12], then the jet variables $\xi_a, a \in \Xi$ are $(v_a^\mu, u_a, \tilde{u}_a), a = 1, \dots, r$ where v_a^μ are Grassmann-even and u_a, \tilde{u}_a are Grassmann-odd variables.

The interaction Lagrangian is determined by gauge invariance. Namely, we define the *gauge charge* operator by

$$d_Q v_a^\mu = i d^\mu u_a, \quad d_Q u_a = 0, \quad d_Q \tilde{u}_a = -i d_\mu v_a^\mu, \quad a = 1, \dots, r \quad (42)$$

where d^μ is the formal derivative. The gauge-charge operator squares to zero:

$$d_Q^2 \simeq 0 \quad (43)$$

where by \simeq we mean the modulo equation of motion. Now we can define the interaction Lagrangian by the relative cohomology relation:

$$d_Q T(x) \simeq \text{total divergence.} \quad (44)$$

If we eliminate the corresponding coboundaries, then a tri-linear Lorentz covariant expression is uniquely given by

$$T = f_{abc} \left(\frac{1}{2} v_{a\mu} v_{b\nu} F_c^{\nu\mu} + u_a v_b^\mu d_\mu \tilde{u}_c \right) \quad (45)$$

where

$$F_a^{\mu\nu} \equiv d^\mu v_a^\nu - d^\nu v_a^\mu, \quad \forall a = 1, \dots, r \quad (46)$$

and f_{abc} are real and completely anti-symmetric. (This is the tri-linear part of the usual QCD interaction Lagrangian from classical field theory.)

Then, we define the associated Fock space by the non-zero 2-point distributions as

$$\begin{aligned}
 \langle \Omega, v_a^\mu(x_1) v_b^\nu(x_2) \Omega \rangle &= i \eta^{\mu\nu} \delta_{ab} D_0^{(+)}(x_1 - x_2), \\
 \langle \Omega, u_a(x_1) \tilde{u}_b(x_2) \Omega \rangle &= -i \delta_{ab} D_0^{(+)}(x_1 - x_2), \\
 \langle \Omega, \tilde{u}_a(x_1) u_b(x_2) \Omega \rangle &= i \delta_{ab} D_0^{(+)}(x_1 - x_2). \quad (47)
 \end{aligned}$$

and construct the associated Wick monomials. Then, the expression (45) gives a Wick polynomial T^{quant} that is formally the same, but (a) the jet variables must be replaced by the associated quantum fields, (b) the formal derivative d^μ goes in the true derivative in

the coordinate space, and (c) Wick ordering should be performed to obtain well-defined operators. We also have an associated gauge-charge operator in the Fock space given by

$$[Q, v_a^\mu] = i \partial^\mu u_a, \quad \{Q, u_a\} = 0, \quad \{Q, \tilde{u}_a\} = -i \partial_\mu v_a^\mu$$

$$Q\Omega = 0. \tag{48}$$

Then, it can be proved that $Q^2 = 0$ and

$$[Q, T^{\text{quant}}(x)] = \text{total divergence} \tag{49}$$

where the equations of motion are automatically used because the quantum fields are on shell. From now on we abandon the super-script *quant* because it will be obvious from the context whether we are referring to the classical expression (45) or to its quantum counterpart.

In (47), we use the Pauli–Jordan distribution

$$D_m(x) = D_m^{(+)}(x) + D_m^{(-)}(x) \tag{50}$$

where

$$D_m^{(\pm)}(x) = \pm \frac{i}{(2\pi)^3} \int dp e^{-ip \cdot x} \theta(\pm p_0) \delta(p^2 - m^2) \tag{51}$$

and

$$D_m^{(-)}(x) = -D_m^{(+)}(-x). \tag{52}$$

We conclude our presentation with a generalization of (49). In fact, it can be proved that (49) implies the existence of Wick polynomials T^I and $T^{\mu\nu}$ such that we have

$$[Q, T^I] = i \partial_\mu T^{I\mu} \tag{53}$$

for any multi-index I with the convention $T^\emptyset \equiv T$. Explicitly,

$$T^I = f_{abc} \left(u_a v_{bv} F_c^{v\mu} - \frac{1}{2} u_a u_b d^\mu \tilde{u}_c \right) \tag{54}$$

and

$$T^{\mu\nu} = \frac{1}{2} f_{abc} u_a u_b F_c^{\mu\nu}. \tag{55}$$

Finally, we give the relation expressing gauge invariance in order n of the perturbation theory. We define the operator δ on chronological products by

$$\delta T(T^{I_1}(x_1), \dots, T^{I_n}(x_n)) \equiv \sum_{m=1}^n (-1)^{s_m} \partial_\mu^m T(T^{I_1}(x_1), \dots, T^{I_{m-1}}(x_{m-1}), \dots, T^{I_m}(x_m), \dots, T^{I_n}(x_n)) \tag{56}$$

with

$$s_m \equiv \sum_{p=1}^{m-1} |I_p|, \tag{57}$$

then we define the operator

$$s \equiv d_Q - i\delta. \tag{58}$$

Gauge invariance in an arbitrary order is then expressed by

$$sT(T^{I_1}(x_1), \dots, T^{I_n}(x_n)) = 0. \tag{59}$$

This concludes the necessary prerequisites.

3. A More Precise Version of Wick Theorem

For simplicity, we assume in this section that the variables ζ_a (see Section 2.1) are commutative. We also use the summation convention over the dummy indices. We begin with the following elementary result.

Theorem 3. *Let us first consider the chronological products of the form*

$$T(\zeta_p(x_1), A_2(x_1), \dots, A_n(x_n)).$$

We define

$$T_1(\zeta_p(x_1), A_2(x_1), \dots, A_n(x_n)) \equiv \zeta_p^{(+)}(x_1) T(A_2(x_1), \dots, A_n(x_n)) + T(A_2(x_1), \dots, A_n(x_n)) \zeta_p^{(-)}(x_1) \tag{60}$$

and

$$T_0 \equiv T - T_1. \tag{61}$$

Then T_0 is of the Wick type only in the entries A_2, \dots, A_n i.e., it verifies

$$\begin{aligned} & [\zeta_q(y), T_0(\zeta_p(x_1), A_2(x_1), \dots, A_n(x_n))] \\ &= \sum_{m=2}^n [\zeta_q(y), \zeta_r(x_m)] T_0(\zeta_p(x_1), A_2(x_1), \dots, \zeta_r \cdot A_m(x_m), \dots, A_n(x_n)). \end{aligned} \tag{62}$$

The proof of (62) is elementary by direct computation. In the general case of arbitrary Grassmann variables, we have to consider a Grassmann sign, i.e.,

$$T_1(\zeta_p(x_1), A_2(x_1), \dots, A_n(x_n)) \equiv: \zeta_p(x_1) T(A_2(x_1), \dots, A_n(x_n)) : \zeta_p^{(+)}(x_1) T(A_2(x_1), \dots, A_n(x_n)) + (-1)^f T(A_2(x_1), \dots, A_n(x_n)) \zeta_p^{(-)}(x_1) \tag{63}$$

where

$$f \equiv |\zeta_p| \sum_{m=2}^n |A_m|. \tag{64}$$

We use another notation emphasizing that we “get out” of the chronological products one factor:

$$T(\zeta_p^{(1)}(x_1), A_2(x_1), \dots, A_n(x_n)) \equiv T_1(\zeta_p(x_1), A_2(x_1), \dots, A_n(x_n)). \tag{65}$$

This new notation will prove useful when we will consider more general cases. We remark that from (61), we trivially have a partial Wick theorem, only in the first variable A_1 :

$$T \equiv T_0 + T_1. \tag{66}$$

Now we consider a more complicated case.

Theorem 4. *Let us first consider that*

$$A_1 = \frac{1}{2} f_{pq} \zeta_p \zeta_q, \quad f_{pq} = f_{qp} \tag{67}$$

and A_2, \dots, A_n are arbitrary. We define

$$f_{pq} [\zeta_p^{(+)}(x_1) T_0(\zeta_q(x_1), A_2(x_2), \dots, A_n(x_n)) + T_0(\zeta_q(x_1), A_2(x_2), \dots, A_n(x_n)) \zeta_p^{(-)}(x_1)] \tag{68}$$

where T_0 was defined above (61); we also define

$$T_2(A_1(x_1), \dots, A_n(x_n)) \equiv f_{pq} \left[\frac{1}{2} \tilde{\xi}_p^{(+)}(x_1) \tilde{\xi}_q^{(+)}(x_1) T(A_2(x_2), \dots, A_n(x_n)) + \tilde{\xi}_p^{(+)}(x_1) T(A_2(x_2), \dots, A_n(x_n)) \tilde{\xi}_q^{(-)}(x_1) + \frac{1}{2} T(A_2(x_2), \dots, A_n(x_n)) \tilde{\xi}_p^{(-)}(x_1) \tilde{\xi}_q^{(-)}(x_1) \right] \tag{69}$$

and

$$T_0 \equiv T - T_1 - T_2. \tag{70}$$

Then T_0 is of the Wick type only in the entries A_2, \dots, A_n i.e., it verifies

$$\begin{aligned} & [\tilde{\xi}_q(y), T_0(A_1(x_1), A_2(x_1), \dots, A_n(x_n))] \\ &= \sum_{m=2}^n [\tilde{\xi}_q(y), \tilde{\xi}_r(x_m)] T_0(A_1(x_1), A_2(x_1), \dots, \tilde{\xi}_r \cdot A_m(x_m), \dots, A_n(x_n)). \end{aligned} \tag{71}$$

The proof is also elementary but more tedious. We also notice that (70) is a partial Wick theorem in the first variable A_1 :

$$T \equiv T_0 + T_1 + T_2. \tag{72}$$

We can use the more compact notations:

$$\begin{aligned} T(A_1^{(1)}(x_1), \dots, A_n(x_n)) &\equiv T_1(A_1(x_1), \dots, A_n(x_n)) = \\ &f_{pq} : \tilde{\xi}_p(x_1) T_0(\tilde{\xi}_q(x_1), A_2(x_2), \dots, A_n(x_n)) : \end{aligned} \tag{73}$$

and

$$\begin{aligned} T(A_1^{(2)}(x_1), \dots, A_n(x_n)) &\equiv T_2(A_1(x_1), \dots, A_n(x_n)) = \\ &\frac{1}{2} f_{pq} : \tilde{\xi}_p(x_1) \tilde{\xi}_q(x_1) T(A_2(x_2), \dots, A_n(x_n)) : \end{aligned} \tag{74}$$

Finally, we consider the more interesting case when the monomial A_1 is tri-linear.

Theorem 5. *Let us first consider that*

$$A_1 = \frac{1}{3!} f_{pqr} \tilde{\xi}_p \tilde{\xi}_q \tilde{\xi}_r, \quad f_{pqr} = \text{is completely symmetric} \tag{75}$$

and A_2, \dots, A_n are arbitrary. We define

$$\begin{aligned} T_1(A_1(x_1), \dots, A_n(x_n)) &\equiv \\ &\frac{1}{2} f_{pqr} [\tilde{\xi}_p^{(+)}(x_1) T_0(\tilde{\xi}_q \tilde{\xi}_r(x_1), A_2(x_2), \dots, A_n(x_n)) \\ &+ T_0(\tilde{\xi}_q \tilde{\xi}_r(x_1), A_2(x_2), \dots, A_n(x_n)) \tilde{\xi}_p^{(-)}(x_1)] \end{aligned} \tag{76}$$

where T_0 was defined above (70); we also define

$$\begin{aligned} T_2(A_1(x_1), \dots, A_n(x_n)) &\equiv f_{pqr} \left[\frac{1}{2} \tilde{\xi}_p^{(+)}(x_1) \tilde{\xi}_q^{(+)}(x_1) T_0(\tilde{\xi}_r(x_1), A_2(x_2), \dots, A_n(x_n)) \right. \\ &\quad \left. + \tilde{\xi}_p^{(+)}(x_1) T_0(\tilde{\xi}_r(x_1), A_2(x_2), \dots, A_n(x_n)) \tilde{\xi}_q^{(-)}(x_1) \right. \\ &\quad \left. + \frac{1}{2} T_0(\tilde{\xi}_r(x_1), A_2(x_2), \dots, A_n(x_n)) \tilde{\xi}_p^{(-)}(x_1) \tilde{\xi}_q^{(-)}(x_1) \right] \end{aligned} \tag{77}$$

where T_0 was defined above (61). Finally,

$$\begin{aligned}
 T_3(A_1(x_1), \dots, A_n(x_n)) \equiv & f_{pqr} \left[\frac{1}{3!} \zeta_p^{(+)}(x_1) \zeta_q^{(+)}(x_1) \zeta_r^{(+)}(x_1) T(A_2(x_2), \dots, A_n(x_n)) \right. \\
 & + \frac{1}{2} \zeta_p^{(+)}(x_1) \zeta_q^{(+)}(x_1) T(A_2(x_2), \dots, A_n(x_n)) \zeta_r^{(-)}(x_1) \\
 & + \frac{1}{2} \zeta_p^{(+)}(x_1) T(A_2(x_2), \dots, A_n(x_n)) \zeta_q^{(-)}(x_1) \zeta_r^{(-)}(x_1) \\
 & \left. + \frac{1}{3!} T(A_2(x_2), \dots, A_n(x_n)) \zeta_p^{(-)}(x_1) \zeta_q^{(-)}(x_1) \zeta_r^{(-)}(x_1) \right] \quad (78)
 \end{aligned}$$

and

$$T_0 \equiv T - T_1 - T_2 - T_3. \quad (79)$$

Then, T_0 is of the Wick type only in the entries A_2, \dots, A_n i.e., it verifies (71).

The computations are long but straightforward. We have more compact notations:

$$\begin{aligned}
 T(A_1^{(1)}(x_1), \dots, A_n(x_n)) \equiv & T_1(A_1(x_1), \dots, A_n(x_n)) = \\
 & \frac{1}{2} f_{pqr} : \zeta_p(x_1) T_0(\zeta_q(x_1) \zeta_r(x_1), A_2(x_2), \dots, A_n(x_n)) : \quad (80)
 \end{aligned}$$

$$\begin{aligned}
 T(A_1^{(2)}(x_1), \dots, A_n(x_n)) \equiv & T_2(A_1(x_1), \dots, A_n(x_n)) = \\
 & \frac{1}{2} f_{pqr} : \zeta_p(x_1) \zeta_q(x_1) T(\zeta_r(x_1), A_2(x_2), \dots, A_n(x_n)) : \quad (81)
 \end{aligned}$$

and

$$\begin{aligned}
 T(A_1^{(3)}(x_1), \dots, A_n(x_n)) \equiv & T_3(A_1(x_1), \dots, A_n(x_n)) = \\
 & \frac{1}{3!} f_{pqr} : \zeta_p(x_1) \zeta_q(x_1) \zeta_r(x_1) T(A_2(x_2), \dots, A_n(x_n)) : \quad (82)
 \end{aligned}$$

From (79), we have a Wick theorem in the first entry:

$$T \equiv T_0 + T_1 + T_2 + T_3. \quad (83)$$

We can iterate the arguments above in the entries A_2, \dots, A_n and obtain the following version of the Wick theorem:

$$T(A_1(x_1), \dots, A_n(x_n)) = \sum T(A_1^{(k_1)}(x_1), \dots, A_n^{(k_n)}(x_n)) \quad (84)$$

where the sum runs over $k_1, \dots, k_n = 0, \dots, 3$ for A_1, \dots, A_n tri-linear. This formula can be written in a more transparent way if we use Hopf algebra notions. We define

$$C_p \equiv \zeta_p \cdot A \quad (85)$$

for an arbitrary Wick polynomial; then, we define the co-product

$$\Delta A \equiv 1 \otimes A + A \otimes 1 + \zeta_p \otimes C_p + C_p \otimes \zeta_p. \quad (86)$$

Now, if we use Sweedler notation

$$\Delta A_j = \sum A_j^{(1)} \otimes A_j^{(2)} \quad (87)$$

we can rewrite (83) as

$$T(A_1(x_1), \dots, A_n(x_n)) = \sum : A_1^{(1)}(x_1) T(A_1^{(2)}(x_1)^{(0)}, A_2(x_2) \dots, A_n(x_n)) : \quad (88)$$

and if we use induction, we arrive at

$$T(A_1(x_1), \dots, A_n(x_n)) = \sum T_0(A_1^{(2)}(x_1), \dots, A_n^{(2)}(x_n)) : A_1^{(1)}(x_1) \dots, A_n^{(1)}(x_n) : \quad (89)$$

where

$$T_0(A_1(x_1), \dots, A_n(x_n)) \equiv T(A_1(x_1)^{(0)}, \dots, A_n(x_n)^{(0)}) \quad (90)$$

has no Wick property in any of the variables, i.e., it commutes with all variables, so it must be the vacuum average:

$$T_0(B_1(x_1), \dots, B_n(x_n)) \equiv \langle \Omega, T(B_1(x_1), \dots, B_n(x_n)) \Omega \rangle . \quad (91)$$

In the general case of arbitrary Grassmann variables, we have to include the appropriate Fermi signs.

We note in the end that the Hopf structure appearing in the preceding form of the Wick theorem does not involve Feynman graphs. The Hopf structure is valid for the chronological product, which is the sum of (many) Feynman contributions but with better smoothness properties: the vacuum averages are well-behaved tempered distributions.

4. Wick Submonomials

4.1. The Case of Pure Yang–Mills Theories

We notice that in (48) and in (53), we have a pattern of the type

$$d_Q A = \text{total divergence.} \quad (92)$$

This pattern remains true for Wick submonomials if we use the definition (21). We consider the expressions (45), (54) and (55) from the pure Yang–Mills case and define

$$\begin{aligned} B_{a\mu} &\equiv \tilde{u}_{a,\mu} \cdot T = -f_{abc} u_b v_{c\mu} \\ C_{a\mu} &\equiv v_{a\mu} \cdot T = f_{abc} (v_b^\nu F_{c\nu\mu} - u_b \tilde{u}_{c,\mu}) \\ D_a &\equiv u_a \cdot T = f_{abc} v_b^\mu \tilde{u}_{c,\mu} \\ E_{a\mu\nu} &\equiv v_{a\mu,\nu} \cdot T = f_{abc} v_{b\mu} v_{c\nu} \\ C_{a\nu\mu} &\equiv v_{a\nu} \cdot T_\mu = -f_{abc} u_b F_{c\nu\mu}. \end{aligned} \quad (93)$$

We also have

$$\begin{aligned} u_a \cdot T &= -C_{a\mu} \\ v_{a\rho,\sigma} \cdot T_\mu &= \eta_{\mu\sigma} B_{a\rho} - \eta_{\mu\rho} B_{a\sigma} \\ u_a \cdot T_{\mu\nu} &= -C_{a\mu\nu} \end{aligned} \quad (94)$$

If we define

$$B_a \equiv \frac{1}{2} f_{abc} u_b u_c \quad (95)$$

we also have

$$\begin{aligned} \tilde{u}_{a,\nu} \cdot T_\mu &= \eta_{\mu\nu} B_a \\ v_{a\rho,\sigma} \cdot T_{\mu\nu} &= (\eta_{\mu\sigma} \eta_{\nu\rho} - \eta_{\nu\sigma} \eta_{\mu\rho}) B_a. \end{aligned} \quad (96)$$

Then we try to extend the structure (92) to the Wick submonomials defined above. We have

$$\begin{aligned}
 d_Q B_a^\mu &= id^\mu B_a \\
 d_Q C_a^\mu &= id_\nu C_a^{\mu\nu} \\
 d_Q D_a &= -id_\mu C_a^\mu \\
 d_Q E_a^{\mu\nu} &= i(d^\nu B_a^\mu - d^\mu B_a^\nu + C_a^{\mu\nu}) \\
 d_Q B_a &= 0 \\
 d_Q C_a^{\mu\nu} &= 0.
 \end{aligned}
 \tag{97}$$

So, we see that the pattern (92) is broken only for $E_a^{\mu\nu}$. We fix this in the following way. We have the formal derivative

$$\delta A \equiv d_\mu A^\mu \tag{98}$$

used in the definition of gauge invariance (56) + (58); we also define the derivative δ' by

$$\delta' E_a^{\mu\nu} = C_a^{\mu\nu}. \tag{99}$$

and 0 for the other Wick submonomials (93) and (95). Finally,

$$s \equiv d_Q - i\delta, \quad s' \equiv s - i\delta' = d_Q - i(\delta + \delta'). \tag{100}$$

Then we have the structure

$$s' A = 0 \tag{101}$$

for all expressions $A = T^I, B_{a\mu}, C_{a\mu},$ etc., and also for the basic jet variables $v_{a\mu}, u_a, \tilde{u}_a.$

4.2. Hopf Structure of the Yang–Mills pQFT

We implement the Wick theorem in the form (89) for the pure YM model. First, we use Theorem 5 for the expressions T^I defined in Section 2.3 and compute the expressions $T(T^I(x_1)^{(k)}, A_2, \dots, A_n).$ Proceeding as in Section 3, we derive by direct computations the following expressions, where the Wick submonomials play an important role.

For the case $k = 1$ i.e., when we “pull out” one factor from the first entry T^I we have

$$\begin{aligned}
 &T(T(x_1)^{(1)}, A_2(x_2), \dots, A_n(x_n)) = \\
 &\quad : v_{a\mu}(x_1) T(C_a^\mu(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &+ \frac{1}{2} : F_{a\nu\mu}(x_1) T(E_a^{\mu\nu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &\quad + : u_a(x_1) T(D_a(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &\quad + : \partial_\mu \tilde{u}_a(x_1) T(B_a^\mu(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned}
 \tag{102}$$

$$\begin{aligned}
 &T(T^\mu(x_1)^{(1)}, A_2(x_2), \dots, A_n(x_n)) = \\
 &\quad : u_a(x_1) T(C_a^\mu(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &+ : v_{a\nu}(x_1) T(C_a^{\nu\mu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &+ : F_a^{\mu\nu}(x_1) T(B_{a\nu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 &\quad - : \partial^\mu \tilde{u}_a(x_1) T(B_a(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned}
 \tag{103}$$

and

$$\begin{aligned}
 & T(T^{\mu\nu}(x_1)^{(1)}, A_2(x_2), \dots, A_n(x_n)) = \\
 & - : u_a(x_1) T(C_a^{\mu\nu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & + : F_a^{\mu\nu}(x_1) T(B_a(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned} \tag{104}$$

For the case $k = 2$ i.e., when we “pull out” two factors from the first entry T^I we have

$$\begin{aligned}
 & T(T(x_1)^{(2)}, A_2(x_2), \dots, A_n(x_n)) = \\
 & : C_a^\mu(x_1) T(v_{a\mu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & + \frac{1}{2} : E_a^{\mu\nu}(x_1) T(F_{a\nu\mu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & - : D_a(x_1) T(u_a(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & - : B_a^\mu(x_1) T(\tilde{u}_{a,\mu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned} \tag{105}$$

$$\begin{aligned}
 & T(T^\mu(x_1)^{(2)}, A_2(x_2), \dots, A_n(x_n)) = \\
 & : C_a^\mu(x_1) T(u_a(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & + : C_a^{\nu\mu}(x_1) T(v_{a\nu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & - : B_{a\nu}(x_1) T(F_a^{\nu\mu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & - : B_a(x_1) T(\tilde{u}_a{}^\mu(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned} \tag{106}$$

and

$$\begin{aligned}
 & T(T^{\mu\nu}(x_1)^{(1)}, A_2(x_2), \dots, A_n(x_n)) = \\
 & : B_a(x_1) T(F_a^{\mu\nu}(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) : \\
 & + : C_a^{\mu\nu}(x_1) T(u(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n)) :
 \end{aligned} \tag{107}$$

For the case $k = 3$ we trivially have

$$T(T^I(x_1)^{(3)}, A_2(x_2), \dots, A_n(x_n)) =: T^I(x_1)T(A_2(x_2), \dots, A_n(x_n)) : \tag{108}$$

Here, the expressions $T(A_1(x_1)^{(0)}, A_2(x_2), \dots, A_n(x_n))$ are of the Wick type only in A_2, \dots, A_n as explained in Section 3. We also note that there are various signs—because we have a true Grassmann structure: the variables u_a, \tilde{u}_a and the Wick submonomials $B_a^\mu, D_a, C_a^{\mu\nu}$ are odd. We can write completely the preceding relations using the Hopf structure. We define the co-product:

$$\begin{aligned}
 \Delta T = 1 \otimes T + T \otimes 1 + v_{a\mu} \otimes C_a^\mu + C_a^\mu \otimes v_{a\mu} + \frac{1}{2} F_{a\nu\mu} \otimes E_a^{\mu\nu} + \frac{1}{2} E_a^{\mu\nu} \otimes F_{a\nu\mu} \\
 + u_a \otimes D_a - D_a \otimes u_a + \tilde{u}_{a,\mu} \otimes B_a^\mu - B_a^\mu \otimes \tilde{u}_{a,\mu}
 \end{aligned} \tag{109}$$

$$\begin{aligned}
 \Delta T^\mu = 1 \otimes T^\mu + T^\mu \otimes 1 + u_a \otimes C_a^\mu + C_a^\mu \otimes u_a + v_{a\nu} \otimes C_a^{\nu\mu} + C_a^{\nu\mu} \otimes v_{a\nu} \\
 + F_a^{\mu\nu} \otimes B_{a\nu} + B_{a\nu} \otimes F_a^{\mu\nu} - \tilde{u}_a{}^\mu \otimes B_a - B_a \otimes \tilde{u}_a{}^\mu
 \end{aligned} \tag{110}$$

$$\Delta T^{\mu\nu} = 1 \otimes T^{\mu\nu} + T^{\mu\nu} \otimes 1 - u_a \otimes C_a^{\mu\nu} + C_a^{\mu\nu} \otimes u_a + F_a^{\mu\nu} \otimes B_a + B_a \otimes F_a^{\mu\nu} \tag{111}$$

Then the relations (102)–(108) can be written in the compact form

$$T(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) = \sum : A_1^{(1)}(x_1) T_0(A_1^{(2)}(x_1), A_2(x_2), \dots, A_n(x_n)) : \tag{112}$$

where $A_1 = T, T^\mu, T^{\mu\nu}$, the expressions $T_0(A_1^{(1)}(x_1), A_2(x_2), \dots, A_n(x_n))$ are of the Wick type only in A_2, \dots, A_n and we use Sweedler notation (87). Now, we can use induction and iterate the procedure in the entries A_2, \dots, A_n . The final result is

$$T(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) = \sum \epsilon_n T_0(A_1^{(2)}(x_1), \dots, A_n^{(2)}(x_n)) : A_1^{(1)}(x_1) \dots A_n^{(1)}(x_n) : \tag{113}$$

where $A_1, \dots, A_n = T, T^\mu, T^{\mu\nu}$, the expressions

$$T_0(A_1^{(2)}(x_1), \dots, A_n^{(2)}(x_n)) \equiv \langle \Omega, T(A_1^{(2)}(x_1), \dots, A_n^{(2)}(x_n)) \Omega \rangle \tag{114}$$

are vacuum averages and we use Sweedler notations for all entries. The sign is given by

$$\epsilon_n = (-1)^{s_n}, \quad s_n = \sum_{p=1}^{n-1} |A_{p+1}^{(1)}| \sum_{q=1}^p |A_q^{(2)}| \tag{115}$$

i.e., the Fermi sign associated with the permutations of the odd factors from the preceding relation.

We can understand the previous sign better if we use the notion of co-multiplication on the product of the algebras. Suppose that W_1, W_2 are two graded algebras, and for $a_j \in W_j, j = 1, 2$ we have

$$\Delta a_j = \sum a_j^{(1)} \otimes a_j^{(2)} \tag{116}$$

(in Sweedler notation) with $a_j^{(1)}, a_j^{(2)}$ of fixed grading. Then, we can define the co-multiplication on $W_1 \otimes W_2$ according to

$$\Delta a_1 \otimes a_2 = \sum (-1)^{|a_2^{(1)}| |a_1^{(2)}|} a_1^{(1)} \otimes a_2^{(1)} \otimes a_1^{(2)} \otimes a_2^{(2)} \tag{117}$$

We can iterate this definition to a product on $W_1 \otimes \dots \otimes W_n$. Suppose that all these algebras are identical $W_1 = \dots = W_n = \mathcal{W}$ with the polynomial algebra in the jet variable of the pure YM model. If $A_1, \dots, A_n \in \mathcal{W}$ and we have

$$\Delta A_1 \otimes \dots \otimes A_n = \sum \mathcal{A}^{(1)} \otimes \mathcal{A}^{(2)} \tag{118}$$

with $\mathcal{A}^{(1)}, \mathcal{A}^{(2)} \in \mathcal{W} \otimes \dots \otimes \mathcal{W}$, then the formula (113) has the simple form

$$T(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) = \sum T_0(\mathcal{A}^{(2)}) : \mathcal{A}^{(1)} : \tag{119}$$

where we use the following compact notations: if $\mathcal{A} = a_1 \otimes \dots \otimes a_n$ then

$$T_0(\mathcal{A}) = T_0(a_1(x_1), \dots, a_n(x_n)) \tag{120}$$

and

$$: \mathcal{A} := a_1(x_1) \dots a_n(x_n) : \tag{121}$$

We try to define now expressions of the type

$$s' T(T^{I_1}(x_1)^{(k)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) \tag{122}$$

in term of similar expressions involving Wick submonomials. We generalize first the operator δ by imposing, as suggested by (97)

$$\begin{aligned} \delta T(B_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) &= \partial_1^\mu T(B_a(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) \\ &- \sum_{m=2}^n (-1)^{s_1(m)} \partial_\mu^m T(B_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_{m-1}}(x_{m-1}), \dots, T^{I_n}(x_n)) \end{aligned} \tag{123}$$

$$\delta T(C_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = \partial_\nu^1 T(C_a^{\mu\nu}(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) + \sum_{m=2}^n \partial_\mu^m T(C_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_m\mu}(x_m), \dots, T^{I_n}(x_n)) \tag{124}$$

$$\delta T(D_a(x), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = -\partial_\mu^1 T(C_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) - \sum_{m=2}^n (-1)^{s_1(m)} \partial_\mu^m T(D_a(x_1), T^{I_2}(x_2), \dots, T^{I_m\mu}(x_m), \dots, T^{I_n}(x_n)) \tag{125}$$

$$\delta T(E_a^{\mu\nu}(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = \partial_1^\nu T(B_a^\mu(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) - (\mu \longleftrightarrow \nu) + \sum_{m=2}^n (-1)^{s_1(m)} \partial_\rho^m T(E_a^{\mu\nu}(x_1), T^{I_2}(x_2), \dots, T^{I_m\rho}(x_m), \dots, T^{I_n}(x_n)) \tag{126}$$

$$\delta T(B_a(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = \sum_{m=2}^n (-1)^{s_1(m)} \partial_\rho^m T(B_a(x_1), T^{I_2}(x_2), \dots, T^{I_m\rho}(x_m), \dots, T^{I_n}(x_n)) \tag{127}$$

$$\delta T(C_a^{\mu\nu}(x_1), T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = \sum_{m=2}^n (-1)^{s_1(m)} \partial_\rho^m T(C_a^{\mu\nu}(x_1), T^{I_2}(x_2), \dots, T^{I_m\rho}(x_m), \dots, T^{I_n}(x_n)) \tag{128}$$

with

$$s_1(m) \equiv \sum_{p=2}^m |I_p| \tag{129}$$

Then, we generalize naturally the operator δ' given by (99) to chronological products; the formula is

$$\delta' T(A_1(x_1), \dots, A_n(x_n)) = \sum_{m=1}^n (-1)^{s_m} T(A_1(x_1), \dots, \delta' A_m(x_m), \dots, A_n(x_n)) \tag{130}$$

Finally, we can define

$$s' = s - i\delta' = d_Q - i(\delta + \delta') \tag{131}$$

on any chronological product of the form $T(A_1(x_1), \dots, A_n(x_n))$ with A_1, \dots, A_n of the type T^I or submonomials as B_a^μ , etc.

In the case $k = 1$ we have directly from (102)–(104):

$$sT(T(x_1)^{(1)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = : v_{a\mu}(x_1) s' T(C_a^\mu(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : + \frac{1}{2} : F_{a\nu\mu}(x_1) s' T(E_a^{\mu\nu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : - : u_a(x_1) s' T(D_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : - : \partial_\mu \tilde{u}_a(x_1) s' T(B_a^\mu(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \tag{132}$$

$$sT(T^\mu(x_1)^{(1)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = - : u_a(x_1) s' T(C_a^\mu(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : + : v_{a\mu}(x_1) s' T(C_a^{\nu\mu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : + : F_a^{\mu\nu}(x_1) s' T(B_{a\nu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : + : \partial^\mu \tilde{u}_a(x_1) s' T(B_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \tag{133}$$

$$\begin{aligned}
 sT(T^{\mu\nu}(x_1)^{(1)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) &=: u_a(x_1) s'T(C_a^{\mu\nu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ : F_a^{\mu\nu}(x_1) T(B_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) :
 \end{aligned}
 \tag{134}$$

Taking into account that we can replace s by s' in the left-hand sides of the preceding three equations, it results that we have a graded “commutativity” between s' and the projection $T^I(x_1) \rightarrow T^I(x_1)^{(1)}$. This “commutativity” is lost if we go to the case $k = 2$. From (105)–(107), we have

$$\begin{aligned}
 s'T(T(x_1)^{(2)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) &=: C_a^\mu(x_1) s'T(v_{a\mu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ \frac{1}{2} : E_a^{\nu\mu}(x_1) s'T(F_{a\nu\mu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ : D_a(x_1) s'T(u_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ : B_a^\mu(x_1) \partial_\mu^1 s'T(\tilde{u}_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &- i : B_{a\mu}(x_1) \square_1 T(v_a^\mu(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ i : B_a(x_1) \square_1 T(\tilde{u}_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) :
 \end{aligned}
 \tag{135}$$

$$\begin{aligned}
 s'T(T^\mu(x_1)^{(2)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) &=: C_a^\mu(x_1) s'T(u_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &- : C_a^{\nu\mu}(x_1) s'T(v_{a\nu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ : B_{a\nu}(x_1) s'T(F_a^{\nu\mu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &- : B_a(x_1) \partial_1^\mu s'T(\tilde{u}_a(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &+ i : B_a(x_1) \square_1 T(v_a^\mu(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) :
 \end{aligned}
 \tag{136}$$

and

$$\begin{aligned}
 s'T(T^{\mu\nu}(x_1)^{(1)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) &=: B_a(x_1) s'T(F_a^{\mu\nu}(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) : \\
 &- : C_a^{\mu\nu}(x_1) s'T(u(x_1)^{(0)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) :
 \end{aligned}
 \tag{137}$$

so in the first two relations, there are supplementary δ terms because if we apply (41) to the extra terms, the d' Alembert operator \square_1 acts on the Feynman distributions $D_0^F(x_1 - x_m)$ and produces $\delta(x_1 - x_m)$.

We also have trivially

$$sT(T^{I_1}(x_1)^{(3)}, T^{I_2}(x_2), \dots, T^{I_n}(x_n)) = T^{I_1}(x_1) sT(T^{I_2}(x_2), \dots, T^{I_n}(x_n)).
 \tag{138}$$

The final step is to express all relations (132)–(138) in a compact way, using Hopf algebra notations. First, we extend the operations d_Q and ∂_μ to the product of algebras through

$$d_Q(a \otimes b) \equiv d_Q a \otimes b + (-1)^{|a|} a \otimes d_Q b
 \tag{139}$$

(for a of a fixed grading number) and

$$\partial_\mu(a \otimes b) \equiv \partial_\mu a \otimes b + a \otimes \partial_\mu b
 \tag{140}$$

respectively. Next, we define

$$s\Delta T^I \equiv d_Q \Delta T^I - i \partial_\mu \Delta T^{I\mu}
 \tag{141}$$

and we can obtain by elementary computations

$$\begin{aligned}
 s\Delta T &= -i (\square v_a^\mu \otimes B_{a\mu} + B_{a\mu} \otimes \square v_a^\mu) + i (\square \tilde{u}_a \otimes B_a + B_a \otimes \square \tilde{u}_a) \\
 s\Delta T^\mu &= i (\square v_a^\mu \otimes B_a + B_a \otimes \square v_a^\mu) \\
 s\Delta T^{\mu\nu} &= 0
 \end{aligned}
 \tag{142}$$

so we have on-shell

$$s\Delta T^I \cong 0. \tag{143}$$

Suppose now that $A_1, \dots, A_n = T, T^\mu, T^{\mu\nu}$ and, according to the previous relations, we have

$$s\Delta A_l = i (B_l^{(1)} \otimes \square B_l^{(2)} + \square B_l^{(2)} \otimes B_l^{(1)}) \tag{144}$$

With these notations, we can write the relations (132)–(138) in a compact way as follows:

$$\begin{aligned}
 s'T(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) &= \sum \tau_1 : A_1^{(1)}(x_1) s'T_0(A_1^{(2)}(x_1), A_2(x_2), \dots, A_n(x_n)) : \\
 &+ i \sum : B_1^{(1)}(x_1) \square T_0(B_1^{(2)}(x_1), A_2(x_2), \dots, A_n(x_n))
 \end{aligned}
 \tag{145}$$

where the expressions $T_0(A_1^{(1)}(x_1), A_2(x_2), \dots, A_n(x_n))$ are of the Wick type only in A_2, \dots, A_n and we have defined the sign

$$\tau_1 = (-1)^{|A_1^{(1)}|}. \tag{146}$$

Now we can use induction and iterate the procedure in the entries A_2, \dots, A_n . The final result is

$$\begin{aligned}
 &s'T(A_1(x_1), A_2(x_2), \dots, A_n(x_n)) \\
 &= \sum \tau_n \epsilon_n s'T_0(A_1^{(2)}(x_1), \dots, A_n^{(2)}(x_n)) : A_1^{(1)}(x_1) \dots A_n^{(1)}(x_n) : \\
 &+ i \sum_{l=1}^n \tau_l' \epsilon_{l,n}' \square_l T_0(A_1^{(2)}(x_1), \dots, B_l^{(2)}(x_l), \dots, A_n(x_n)) : A_1^{(1)}(x_1) \dots B_l^{(1)}(x_l) \dots A_n^{(1)}(x_n) :
 \end{aligned}
 \tag{147}$$

where the sign ϵ_n was defined before by (115)

$$\tau_n = (-1)^{\sum_{l=1}^n |A_l^{(1)}|} \tag{148}$$

$$\tau_l' = (-1)^{\sum_{p=1}^{l-1} |A_p|} \tag{149}$$

and $\epsilon_{l,n}'$ is obtained from ϵ_n making $A_l^{(1)} \rightarrow B_l^{(1)}, A_l^{(2)} \rightarrow B_l^{(2)}$.

We notice that in the right hand sides of the formula (147) the operator s' acts on numerical distributions so we have $s'T_0 = -i (\delta + \delta')T_0$.

It follows that we can reduce the gauge invariance condition (59) to numerical relations of the type

$$(\delta + \delta')T_0(A_1(x_1), \dots, A_n(x_n)) = \text{delta terms}. \tag{150}$$

However, it is not trivial to determine the explicit form of the right-hand side.

5. Second-Order Gauge Invariance—Loop Contributions

To illustrate the advantage of our approach, we study in detail the second-order case of the pure Yang–Mills model. We can iterate Formulas (105)–(107) or use directly (112) to obtain

$$\begin{aligned}
 T(T(x_1)^{(1)}, T(x_2)^{(1)}) = & v_{a\mu}(x_1) v_{bv}(x_2) : T(C_a^\mu(x_1)^{(0)}, C_b^v(x_2)^{(0)}) \\
 & + \frac{1}{4} : F_{a\nu\mu}(x_1) F_{b\sigma\rho}(x_2) : T(E_a^{\mu\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) \\
 & + \frac{1}{2} : v_{a\mu}(x_1) F_{b\sigma\rho}(x_2) : T(C_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) + (1 \longleftrightarrow 2) \\
 & - [: u_a(x_1) \partial_\mu \tilde{u}_b(x_2) : T(D_a^\mu(x_1)^{(0)}, B_b^\mu(x_2)^{(0)}) + (1 \longleftrightarrow 2)] \tag{151}
 \end{aligned}$$

$$\begin{aligned}
 T(T^\mu(x_1)^{(1)}, T(x_2)^{(1)}) = & u_a(x_1) v_{bv}(x_2) : T(C_a^\mu(x_1)^{(0)}, C_b^v(x_2)^{(0)}) \\
 & + \frac{1}{2} : u_{a\mu}(x_1) F_{b\sigma\rho}(x_2) : T(C_a^{\nu\mu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) \\
 & - : v_{av}(x_1) u_b(x_2) : T(C_a^{\nu\mu}(x_1)^{(0)}, D_b^\mu(x_2)^{(0)}) \\
 & - : F_{a\nu\mu}(x_1) u_b(x_2) : T(B_{av}(x_1)^{(0)}, D_b^\mu(x_2)^{(0)}) \tag{152}
 \end{aligned}$$

$$T(T^\mu(x_1)^{(1)}, T^v(x_2)^{(1)}) = u_a(x_1) u_b(x_2) : T(C_a^{\mu\nu}(x_1)^{(0)}, C_b^v(x_2)^{(0)}) \tag{153}$$

and

$$T(T^{\mu\nu}(x_1)^{(1)}, T(x_2)^{(1)}) = u_a(x_1) u_b(x_2) : T(C_a^{\mu\nu}(x_1)^{(0)}, D_b(x_2)^{(0)}) \tag{154}$$

$$T(T^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)^{(1)}) = 0. \tag{155}$$

The expressions of the type $T(C_a^\mu(x_1)^{(0)}, C_b^v(x_2)^{(0)})$, etc., are associated with 2-loop contributions in the second order of the perturbation theory. We have now the following result.

Theorem 6. *The following relations are true:*

$$\begin{aligned}
 sT(T(x_1)^{(1)}, T(x_2)^{(1)}) = & \\
 -[: u_a(x_1) v_{bv}(x_2) : s'T(D_a(x_1)^{(0)}, C_b^v(x_2)^{(0)}) + (1 \longleftrightarrow 2)] & \\
 -\frac{1}{2} : [u_a(x_1) F_{b\sigma\rho}(x_2) : s'T(D_a(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) + (1 \longleftrightarrow 2)] & \tag{156}
 \end{aligned}$$

$$\begin{aligned}
 sT(T^\mu(x_1)^{(1)}, T(x_2)^{(1)}) = & : u_a(x_1) u_b(x_2) : s'T(C_a^\mu(x_1)^{(0)}, D_b(x_2)^{(0)}) \\
 sT(T^\mu(x_1)^{(1)}, T^v(x_2)^{(1)}) = & 0 \tag{157}
 \end{aligned}$$

$$sT(T^{\mu\nu}(x_1)^{(1)}, T(x_2)^{(1)}) = 0 \tag{158}$$

$$sT(T^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)^{(1)}) = 0. \tag{159}$$

Proof. We sketch to prove the first relation. We have to compute the following expressions: $A = d_Q T(T(x_1)^{(1)}, T(x_2)^{(1)})$, $B = -i \partial_\mu^1 T(T^\mu(x_1)^{(1)}, T(x_2)^{(1)})$, $C = -i \partial_\mu^2 T(T(x_1)^{(1)}, T^\mu(x_2)^{(1)})$ and we use for this the relations (151) and (152). In the sum $A + B + C$ (which is the left-hand side of (156)), there are a lot of cancellations; we are left with

$$\begin{aligned}
 a = -i : u_a(x_1) v_{bv}(x_2) : [\partial_\mu^1 T(C_a^\mu(x_1)^{(0)}, C_b^v(x_2)^{(0)}) + \partial_\mu^2 T(D_a(x_1)^{(0)}, C_b^{\nu\mu}(x_2)^{(0)})] & \\
 = i : u_a(x_1) v_{bv}(x_2) : \delta T(D_a(x_1)^{(0)}, C_b^v(x_2)^{(0)}) & \\
 = - : u_a(x_1) v_{bv}(x_2) : s'T(D_a(x_1)^{(0)}, C_b^v(x_2)^{(0)}) & \tag{160}
 \end{aligned}$$

(which is the first term from the right-hand side of (156)) and two more similar terms. \square

All the expressions on the right-hand sides from the preceding theorem are in fact zero.

Theorem 7. *The following relations are true:*

$$\begin{aligned} s' T(D_a(x_1)^{(0)}, C_b^\nu(x_2)^{(0)}) &= 0 \\ s' T(D_a(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) &= 0 \end{aligned} \tag{161}$$

Proof. The basic distributions used for the computations are

$$d_{0,0} \equiv \frac{1}{2} [(D_0^{(+)})^2 - (D_0^{(-)})^2] \tag{162}$$

and associated ones:

$$d_{\mu\nu} \equiv D_0^{(+)} \partial_\mu \partial_\nu D_0^{(+)} - D_0^{(-)} \partial_\mu \partial_\nu D_0^{(-)}, \quad f_{\mu\nu} \equiv \partial_\mu D_0^{(+)} \partial_\nu D_0^{(+)} - \partial_\mu D_0^{(-)} \partial_\nu D_0^{(-)} \tag{163}$$

All these distributions have causal support; in fact, one can derive that

$$d_{\mu\nu} = \frac{2}{3} \left(\partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \square \right) d_{0,0}, \quad f_{\mu\nu} = \frac{1}{3} \left(\partial_\mu \partial_\nu + \frac{1}{2} \eta_{\mu\nu} \square \right) d_{0,0}. \tag{164}$$

The causal splitting of the distribution $d_{0,0}$ gives a Feynman propagator $d_{0,0}^F$ which is not unique because of the degree of singularity of $d_{0,0}$ in 0; any choice will be good. More important, we can obtain the corresponding Feynman propagators by

$$d_{\mu\nu}^F = \frac{2}{3} \left(\partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \square \right) d_{0,0}^F, \quad f_{\mu\nu}^F = \frac{1}{3} \left(\partial_\mu \partial_\nu + \frac{1}{2} \eta_{\mu\nu} \square \right) d_{0,0}^F. \tag{165}$$

With these conventions, we can obtain

$$\begin{aligned} T(C_a^\mu(x_1)^{(0)}, C_b^\nu(x_2)^{(0)}) &= \frac{2}{3} g_{ab} (\partial^\mu \partial^\nu - \eta^{\mu\nu} \square) d_{0,0}^F \\ T(C_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) &= \frac{1}{2} g_{ab} (\eta^{\mu\sigma} \partial^\rho - \eta^{\mu\rho} \partial^\sigma) d_{0,0}^F \\ T(D_a(x_1)^{(0)}, B_b^\mu(x_2)^{(0)}) &= -\frac{1}{2} g_{ab} \partial^\mu d_{0,0}^F \\ T(D_a(x_1)^{(0)}, C_b^{\mu\nu}(x_2)^{(0)}) &= 0 \end{aligned} \tag{166}$$

where

$$g_{ab} \equiv f_{acd} f_{bcd} \tag{167}$$

and from here,

$$\begin{aligned} &\delta T(D_a(x_1)^{(0)}, C_b^\nu(x_2)^{(0)}) \\ &\equiv -[\partial_\mu^1 T(C_a^\mu(x_1)^{(0)}, C_b^\nu(x_2)^{(0)}) - \partial_\mu^2 T(D_a(x_1)^{(0)}, C_b^{\nu\mu}(x_2)^{(0)})] = 0 \\ &\quad (\delta + \delta') T(D_a(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) = \\ &\equiv -\partial_\mu^1 T(C_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) - [\partial_2^\sigma T(D_a(x_1)^{(0)}, B_b^\rho(x_2)^{(0)}) - (\rho \leftrightarrow \sigma)] \\ &\quad - T(D_a(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)^{(0)}) = 0 \end{aligned} \tag{168}$$

and this proves that there are no anomalies from the loop contributions in the second order of the perturbation \square

We can extend the previous arguments to loop contributions associated to chronological products of Wick submonomials. The following formulas are true:

$$\begin{aligned}
 T(B_a^\mu(x_1)^{(0)}, T(x_2)) &= -u_b(x_2) T(B_a^\mu(x_1)^{(0)}, D_b^{(0)}(x_2)) \\
 T(B_a^\mu(x_1)^{(0)}, T^I(x_2)) &= 0, \quad \text{for } |I| = 1, 2 \\
 T(C_a^\mu(x_1)^{(0)}, T(x_2)) &= v_{bv}(x_2) T(C_a^\mu(x_1)^{(0)}, C_b^\nu(x_1)(x_2)^{(0)}) \\
 &\quad + \frac{1}{2} F_{b\sigma\rho}(x_2) T(C_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) \\
 T(C_a^\mu(x_1)^{(0)}, T^\nu(x_2)) &= u_b(x_2) T(C_a^\mu(x_1)^{(0)}, C_b^\nu(x_1)(x_2)^{(0)}) \\
 T(C_a^\mu(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= 0 \\
 T(D_a(x_1)^{(0)}, T(x_2)) &= -\partial_\mu \tilde{u}_b^\mu(x_2) T(D_a(x_1)^{(0)}, B_b^\mu(x_2)^{(0)}) \\
 T(D_a(x_1)^{(0)}, T^\mu(x_2)) &= v_{bv}(x_2) T(D_a(x_1)^{(0)}, C_b^{\nu\mu}(x_1)(x_2)^{(0)}) \\
 &\quad + : F_b^{\mu\nu}(x_2) T(D_a(x_1)^{(0)}, B_{bv}(x_2)^{(0)}) : \\
 T(D_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= u_b(x_2) T(D_a(x_1)^{(0)}, C_b^{\mu\nu}(x_1)(x_2)^{(0)}) \\
 T(E_a^{\mu\nu}(x_1)^{(0)}, T(x_2)) &= v_{b\rho}(x_2) T(E_a^{\mu\nu}(x_1)^{(0)}, C_b^\rho(x_2)^{(0)}) \\
 &\quad + \frac{1}{2} : F_{b\sigma\rho}(x_2) T(E_a^{\mu\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) \\
 T(E_a^{\mu\nu}(x_1)^{(0)}, T^\rho(x_2)) &= u_b(x_2) T(E_a^{\mu\nu}(x_1)^{(0)}, C_b^\rho(x_2)^{(0)}) \\
 T(E_a^{\mu\nu}(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= 0 \\
 T(B_a(x_1)^{(0)}, T^I(x_2)) &= 0, \quad \forall I \\
 T(C_a^{\mu\nu}(x_1)^{(0)}, T(x_2)) &= -u_b(x_2) T(C_a^{\mu\nu}(x_1)^{(0)}, D_b(x_2)^{(0)}) \\
 T(C_a^{\mu\nu}(x_1)^{(0)}, T^I(x_2)) &= 0, \quad \text{for } |I| = 1, 2.
 \end{aligned} \tag{169}$$

Then, similar to Theorem 6, we have

Theorem 8.

$$\begin{aligned}
 s'T(B_a^\mu(x_1)^{(0)}, T^I(x_2)) &= 0, \quad \forall I \\
 s'T(C_a^\mu(x_1)^{(0)}, T(x_2)) &= iu_b(x_2) \delta T(C_a^\mu(x_1)^{(0)}, D_b(x_1)(x_2)^{(0)}) \\
 s'T(C_a^\mu(x_1)^{(0)}, T^I(x_2)) &= 0 \quad \text{for } |I| = 1, 2 \\
 s'T(D_a(x_1)^{(0)}, T(x_2)) &= -iv_{bv}(x_2) \delta T(D_a(x_1)^{(0)}, C_b^\nu(x_2)^{(0)}) \\
 &\quad - \frac{i}{2} F_{b\sigma\rho}(x_2) (\delta + \delta') T(D_a(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)^{(0)}) \\
 s'T(D_a(x_1)^{(0)}, T^\mu(x_2)) &= -iu_b(x_2) \delta T(D_a(x_1)^{(0)}, C_b^\mu(x_1)(x_2)^{(0)}) \\
 s'T(D_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= 0 \\
 s'T(E_a^{\mu\nu}(x_1)^{(1)}, T(x_2)) &= iu_b(x_2) (\delta + \delta') T(E_a^{\mu\nu}(x_1)^{(0)}, D_b(x_2)^{(0)}) \\
 s'T(E_a^{\mu\nu}(x_1)^{(1)}, T^I(x_2)) &= 0 \quad \text{for } |I| = 1, 2 \\
 s'T(B_a(x_1)^{(0)}, T^I(x_2)) &= 0, \quad \forall I \\
 s'T(C_a^{\mu\nu}(x_1)^{(1)}, T^I(x_2)) &= 0, \quad \forall I. \tag{170}
 \end{aligned}$$

Now, using Theorem 7, we have

Theorem 9.

$$\begin{aligned}
 s'T(C_a^\mu(x_1)^{(0)}, T^I(x_2)) &= 0 \quad \forall I \\
 s'T(D_a(x_1)^{(0)}, T^I(x_2)) &= 0 \quad \forall I \\
 s'T(E_a^{\mu\nu}(x_1)^{(0)}, T^I(x_2)) &= 0 \quad \forall I. \tag{171}
 \end{aligned}$$

Finally we have the following.

Theorem 10. *The following relation is true:*

$$s'T(\xi_a \cdot T^I(x_1)^{(0)}, \xi_b \cdot T^J(x_2)^{(0)}) = -i (\delta + \delta') T(\xi_a \cdot T^I(x_1)^{(0)}, \xi_b \cdot T^J(x_2)^{(0)}) = 0. \tag{172}$$

Proof. We use the relations (166). □

6. Second-Order Gauge Invariance—Tree Contributions

Now we study tree contributions. We can iterate (102)–(104) or use directly (112) to obtain

$$\begin{aligned}
 T(T(x_1)^{(2)}, T(x_2)^{(2)}) = & C_a^\mu(x_1) C_b^\nu(x_2) : T(v_{a\mu}(x_1)^{(0)}, v_{b\nu}(x_2)^{(0)}) \\
 & + \frac{1}{4} : E_a^{\nu\mu}(x_1) E_b^{\sigma\rho}(x_2) : T(F_{a\mu\nu}(x_1)^{(0)}, F_{b\rho\sigma}(x_2)^{(0)}) \\
 & + \frac{1}{2} [: E_a^{\mu\nu}(x_1) C_b^\rho(x_2) : T(F_{a\nu\mu}(x_1)^{(0)}, v_{b\rho}(x_2)^{(0)}) + (1 \longleftrightarrow 2)] \\
 & - [: D_a(x_1) B_b^\mu(x_2) : T(u_a(x_1)^{(0)}, \tilde{u}_{b,\mu}(x_2)^{(0)}) + (1 \longleftrightarrow 2)] \tag{173}
 \end{aligned}$$

$$\begin{aligned}
 T(T^\mu(x_1)^{(2)}, T(x_2)^{(2)}) = & C_a^\mu(x_1) B_b^\nu(x_2) : T(u_{a,\nu}(x_1)^{(0)}, \tilde{u}_b(x_2)^{(0)}) \\
 & + : C_a^{\nu\mu}(x_1) C_b^\rho(x_2) : T(v_{a\nu}(x_1)^{(0)}, v_{b\rho}(x_2)^{(0)}) \\
 & + \frac{1}{2} : C_a^{\nu\mu}(x_1) E_b^{\rho\sigma}(x_2) : T(v_{a\nu}(x_1)^{(0)}, F_{b\sigma\rho}(x_2)^{(0)}) \\
 & - : B_{a\nu}(x_1) C_b^\rho(x_2) : T(F_a^{\nu\mu}(x_1)^{(0)}, v_{b\rho}(x_2)^{(0)}) \\
 & - \frac{1}{2} : B_{a\nu}(x_1) E_b^{\rho\sigma}(x_2) : T(F_a^{\nu\mu}(x_1)^{(0)}, F_{b\sigma\rho}(x_2)^{(0)}) \\
 & - : B_a(x_1) D_b(x_2) : T(\tilde{u}_a,^\mu(x_1)^{(0)}, u_b(x_2)^{(0)}) \tag{174}
 \end{aligned}$$

$$\begin{aligned}
 T(T^\mu(x_1)^{(2)}, T^\nu(x_2)^{(2)}) = & - : C_a^\mu(x_1) B_b(x_2) : T(u_a(x_1)^{(0)}, \tilde{u}_b^\nu(x_2)^{(0)}) \\
 & + : C_a^{\rho\mu}(x_1) C_b^{\sigma\nu}(x_2) : T(v_{a\rho}(x_1)^{(0)}, v_{b\sigma}(x_2)^{(0)}) \\
 & - : C_a^{\rho\mu}(x_1) B_{b\sigma}(x_2) : T(v_{a\rho}(x_1)^{(0)}, F_b^{\sigma\nu}(x_2)^{(0)}) \\
 & - : B_{a\rho}(x_1) C_b^{\sigma\nu}(x_2) : T(F_a^{\rho\mu}(x_1)^{(0)}, v_{b\sigma}(x_2)^{(0)}) \\
 & + : B_{a\rho}(x_1) B_{b\sigma}(x_2) : T(F_a^{\rho\mu}(x_1)^{(0)}, F_b^{\sigma\nu}(x_2)^{(0)}) \\
 & - : B_a(x_1) C_b^\nu(x_2) : T(\tilde{u}_a,^\mu(x_1)^{(0)}, u_b(x_2)^{(0)}) \tag{175}
 \end{aligned}$$

$$\begin{aligned}
 T(T^{\mu\nu}(x_1)^{(2)}, T(x_2)^{(2)}) = & B_a(x_1) C_b^\rho(x_2) : T(F_a^{\mu\nu}(x_1)^{(0)}, v_{b\rho}(x_2)^{(0)}) \\
 & + \frac{1}{2} : B_a(x_1) E_b^{\rho\sigma}(x_2) : T(F_a^{\mu\nu}(x_1)^{(0)}, F_{b\sigma\rho}(x_2)^{(0)}) \\
 & + : C_a^{\mu\nu}(x_1) B_b^\rho(x_2) : T(u_a(x_1)^{(0)}, \tilde{u}_{b,\rho}(x_2)^{(0)}) \tag{176}
 \end{aligned}$$

$$\begin{aligned}
 T(T^{\mu\nu}(x_1)^{(2)}, T^\rho(x_2)^{(2)}) = & B_a(x_1) C_b^{\sigma\rho}(x_2) : T(F_a^{\mu\nu}(x_1)^{(0)}, v_{b\sigma}(x_2)^{(0)}) \\
 & - : B_a(x_1) B_{b\sigma}(x_2) : T(F_a^{\mu\nu}(x_1)^{(0)}, F_b^{\sigma\rho}(x_2)^{(0)}) \\
 & - : C_a^{\mu\nu}(x_1) B_b(x_2) : T(u_a(x_1)^{(0)}, \tilde{u}_b,^\rho(x_2)^{(0)}) \tag{177}
 \end{aligned}$$

$$T(T^{\mu\nu}(x_1)^{(2)}, T^{\rho\sigma}(x_2)^{(2)}) = B_a(x_1) B_b(x_2) : T(F_a^{\mu\nu}(x_1)^{(0)}, F_b^{\sigma\rho}(x_2)^{(0)}) \tag{178}$$

We must give the values of the chronological products $T(\xi_p^{(0)}(x_1), \xi_q^{(0)}(x_2))$ which are not unique. For the pure Yang–Mills model, the causal commutators of the basic fields are

$$\begin{aligned}
 D(v_a^\mu(x_1), v_b^\nu(x_2)) & \equiv [v_a^\mu(x_1), v_b^\nu(x_2)] = i \eta^{\mu\nu} \delta_{ab} D_0(x_1 - x_2), \\
 D(u_a(x_1), \tilde{u}_b(x_2)) & \equiv [u_a(x_1), \tilde{u}_b(x_2)] = -i \delta_{ab} D_0(x_1 - x_2), \\
 D(\tilde{u}_a(x_1), u_b(x_2)) & \equiv [\tilde{u}_a(x_1), u_b(x_2)] = i \delta_{ab} D_0(x_1 - x_2). \tag{179}
 \end{aligned}$$

where, on the left-hand side, we have the graded commutator. The causal splitting $D = D^{\text{adv}} - D^{\text{ret}}$ is unique because the degree of singularity of D_0 is $\omega = -2$ and we obtain

$$\begin{aligned} T(v_a^\mu(x_1)^{(0)}, v_b^\nu(x_2)^{(0)}) &= i \eta^{\mu\nu} \delta_{ab} D_0^F(x_1 - x_2), \\ T(u_a(x_1)^{(0)}, \tilde{u}_b(x_2)^{(0)}) &= -i \delta_{ab} D_0^F(x_1 - x_2), \\ T(\tilde{u}_a(x_1)^{(0)}, u_b(x_2)^{(0)}) &= i \delta_{ab} D_0^F(x_1 - x_2). \end{aligned} \tag{180}$$

From the previous relations, we also have uniquely

$$\begin{aligned} T(\xi_{a,\mu}^{(0)}(x_1), \xi_b^{(0)}(x_2)) &= \partial_\mu^1 T(\xi_a(x_1), \xi_b(x_2)) \\ T(\xi_a^{(0)}(x_1), \xi_{b,\nu}^{(0)}(x_2)) &= \partial_\nu^2 D^F(\xi_a(x_1), \xi_b(x_2)). \end{aligned} \tag{181}$$

However, the causal splitting of $T(\xi_{a,\mu}(x_1)^{(0)}, \xi_{b,\nu}(x_2)^{(0)})$ is not unique because the distribution has the degree of singularity $\omega = 0$. This was noticed for the first time in [26,27]. A possible choice is the *canonical* splitting, following from (5):

$$T(\xi_{a,\mu}^{(0)}(x_1), \xi_{b,\nu}^{(0)}(x_2)) = i \partial_\mu^1 \partial_\nu^2 T(\xi_a(x_1)^{(0)}, \xi_b(x_2)^{(0)}) \tag{182}$$

and this gives from (173)–(178)

$$\begin{aligned} T(T(x_1)^{(2)}, T(x_2)^{(2)}) &= i D_0^F(x_1 - x_2) : C_a^\mu(x_1) C_{a\mu}(x_2) : \\ &+ i \partial_\mu D_0^F(x_1 - x_2) [: C_{a\nu}(x_1) E_a^{\mu\nu}(x_2) : - : D_a(x_1) B_a^\mu(x_2) :] + (1 \leftrightarrow 2) \\ &- i \partial_\mu \partial_\nu D_0^F(x_1 - x_2) : E_a^{\mu\rho}(x_1) E_{a\rho}^\nu(x_2) : \end{aligned} \tag{183}$$

$$\begin{aligned} T(T^\mu(x_1)^{(2)}, T(x_2)^{(2)}) &= -i D_0^F(x_1 - x_2) : C_a^{\mu\nu}(x_1) C_{b\nu}(x_2) : \\ &+ i \partial_\nu D_0^F(x_1 - x_2) [: C_a^\mu(x_1) B_a^\nu(x_2) : - : C_a^{\mu\rho}(x_1) E_{a\rho}^\nu(x_2) : - : B_a^\nu(x_1) C_a^\mu(x_2) :] \\ &+ i \partial^\mu D_0^F(x_1 - x_2) [: B_{a\nu}(x_1) C_a^\nu(x_2) : - : B_a(x_1) D_a(x_2) :] \\ &+ i \partial_\nu \partial_\rho D_0^F(x_1 - x_2) : B_a^\nu(x_1) E_a^{\mu\rho}(x_2) : + i \partial^\mu \partial_\nu D_0^F(x_1 - x_2) : B_{a\rho}(x_1) E_a^{\nu\rho}(x_2) : \end{aligned} \tag{184}$$

$$\begin{aligned} T(T^\mu(x_1)^{(2)}, T^\nu(x_2)^{(2)}) &= i D_0^F(x_1 - x_2) : C_a^{\mu\rho}(x_1) C_{a\rho}^\nu(x_2) : \\ &+ i \partial^\mu D_0^F(x_1 - x_2) [: B_{a\rho}(x_1) C_a^{\rho\nu}(x_2) : - : B_a(x_1) C_a^\nu(x_2) :] \\ &- i \partial^\nu D_0^F(x_1 - x_2) [: C_a^\mu(x_1) B_a(x_2) : + : C_a^{\mu\nu}(x_1) B_{a\rho}(x_2) :] \\ &- i \partial_\rho D_0^F(x_1 - x_2) [: C_a^{\mu\nu}(x_1) B_a^\rho(x_2) : + : B_a^\rho(x_1) C_a^{\mu\nu}(x_2) :] \\ &- i [\eta^{\mu\nu} \partial_\rho \partial_\sigma D_0^F(x_1 - x_2) : B_a^\rho(x_1) B_a^\sigma(x_2) : + \partial^\mu \partial^\nu D_0^F(x_1 - x_2) : B_{a\rho}(x_1) B_a^\rho(x_2) :] \\ &+ i [\partial^\mu \partial_\rho D_0^F(x_1 - x_2) : B_a^\nu(x_1) B_a^\rho(x_2) : + \partial^\nu \partial_\rho D_0^F(x_1 - x_2) : B_a^\rho(x_1) B_a^\mu(x_2) :] \end{aligned} \tag{185}$$

$$\begin{aligned} T(T^{\mu\nu}(x_1)^{(2)}, T(x_2)^{(2)}) &= i [\partial^\mu D_0^F(x_1 - x_2) : B_a(x_1) C_a^\nu(x_2) : - (\mu \leftrightarrow \nu)] \\ &+ i \partial_\rho D_0^F(x_1 - x_2) : C_a^{\mu\nu}(x_1) B_a^\rho(x_2) : \\ &- i [\partial^\mu \partial_\rho D_0^F(x_1 - x_2) : B_a(x_1) E_a^{\nu\rho}(x_2) : - (\mu \leftrightarrow \nu)] \end{aligned} \tag{186}$$

$$\begin{aligned} T(T^{\mu\nu}(x_1)^{(2)}, T^\rho(x_2)^{(2)}) &= i [\partial^\mu D_0^F(x_1 - x_2) : B_a(x_1) C_a^{\nu\rho}(x_2) : - (\mu \leftrightarrow \nu)] \\ &- i \partial^\rho D_0^F(x_1 - x_2) : C_a^{\mu\nu}(x_1) B_a(x_2) : \\ &- i [\partial_\mu \partial_\rho D_0^F(x_1 - x_2) : B_a(x_1) B_a^\nu(x_2) : - (\mu \leftrightarrow \nu)] \\ &- i [\eta^{\mu\rho} \partial^\nu \partial_\sigma D_0^F(x_1 - x_2) : B_a(x_1) B_a^\sigma(x_2) : - (\mu \leftrightarrow \nu)] \end{aligned} \tag{187}$$

$$T(T^{\mu\nu}(x_1)^{(2)}, T^{\rho\sigma}(x_2)^{(2)}) = -i (\eta^{\nu\sigma}\partial^\mu\partial^\rho - \eta^{\nu\rho}\partial^\mu\partial^\sigma + \eta^{\mu\rho}\partial^\nu\partial^\sigma - \eta^{\mu\sigma}\partial^\nu\partial^\rho) D_0^F(x_1 - x_2) : B_a(x_1) B_a(x_2) : \tag{188}$$

From these formulas, we can determine now if gauge invariance is true; in fact, we have anomalies, as it is well known.

Theorem 11. *The following formulas are true:*

$$s'T(T(x_1)^{(2)}, T(x_2)^{(2)}) = 2 \delta(x_1 - x_2) (: B_{a\mu} C_a^\mu : - : B_a D_a :)(x_2) + \partial_\mu \delta(x_1 - x_2) [: B_{a\nu}(x_1) E_a^{\mu\nu}(x_2) : - : E_a^{\mu\nu}(x_1) B_a^\mu(x_2) :] \tag{189}$$

$$s'T(T^\mu(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) (-2 : B_a C_a^\mu : + : C_a^{\mu\nu} B_{a\nu} :)(x_2) + \partial_\nu \delta(x_1 - x_2) [: B_a(x_1) E_a^{\mu\nu}(x_2) : + : B_a^\nu(x_1) B_a^\mu(x_2) :] - \partial^\mu \delta(x_1 - x_2) : B_{a\nu}(x_1) B_a^\nu(x_2) : \tag{190}$$

$$s'T(T^\mu(x_1)^{(2)}, T^\nu(x_2)^{(2)}) = -2\delta(x_1 - x_2) : B_a C_a^{\mu\nu} : (x_2) + \partial^\mu \delta(x_1 - x_2) : B_a^\nu(x_1) B_a(x_2) : + \partial^\nu \delta(x_1 - x_2) : B_a(x_1) B_a^\mu(x_2) : - \eta^{\mu\nu} \partial_\rho \delta(x_1 - x_2) [: B_a(x_1) B_a^\rho(x_2) : + : B_a^\rho(x_1) B_a(x_2) :] \tag{191}$$

$$s'T(T^{\mu\nu}(x_1)^{(2)}, T(x_2)^{(2)}) = \delta(x_1 - x_2) : C_a^{\mu\nu} B_a : (x_2) + \partial^\mu \delta(x_1 - x_2) : B_a(x_1) B_a^\nu(x_2) : - (\mu \leftrightarrow \nu) \tag{192}$$

$$s'T(T^{\mu\nu}(x_1)^{(2)}, T^\rho(x_2)^{(2)}) = \eta^{\mu\rho} \partial^\nu \delta(x_1 - x_2) : B_a(x_1) B_a(x_2) : - (\mu \leftrightarrow \nu) \tag{193}$$

$$s'T(T^{\mu\nu}(x_1)^{(2)}, T^{\rho\sigma}(x_2)^{(2)}) = 0 \tag{194}$$

Proof. We consider for illustration only the first relation. We have to compute the expressions $A = d_Q T(T(x_1)^{(2)}, T(x_2)^{(2)})$, $B = -i\partial_\mu^1 T(T^\mu(x_1)^{(2)}, T(x_2)^{(2)})$, $C = -i\partial_\mu^2 T(T(x_1)^{(2)}, T^\mu(x_2)^{(2)})$ using the relations (183) and (184). The expression on the left-hand side of (189) is $A + B + C$. There are a lot of cancellations, and only the terms with \square acting on $D_0^F(x_1 - x_2)$, $\partial_\rho D_0^F(x_1 - x_2)$ survive. They give the right-hand side of (189). \square

The final result is as follows.

Theorem 12. *We have*

$$sT(T^{I_1}(x_1), T^{I_2}(x_2)) = s'T(T^{I_1}(x_1)^{(2)}, T^{I_2}(x_2)^{(2)}) \tag{195}$$

and the anomalies given in the previous theorem can be eliminated if and only if the constants f_{abc} verify the Jacobi identity

$$f_{eab} f_{ecd} + f_{ebc} f_{ead} + f_{eca} f_{ebd} = 0 \tag{196}$$

using the finite renormalizations:

$$T(A_1(x_1), A_2(x_2)) \rightarrow T^{\text{ren}}(A_1(x_1), A_2(x_2)) = T(A_1(x_1), A_2(x_2)) + \delta(x_1 - x_2) N(A_1, A_2)(x_2) \tag{197}$$

where

$$\begin{aligned}
 N(T, T) &\equiv \frac{i}{2} E_a^{\mu\nu} E_{a\mu\nu} \\
 N(T^\mu, T) = N(T, T^\mu) &\equiv -i B_{a\nu} E_a^{\mu\nu} \\
 N(T^\mu, T^\nu) &\equiv i B_a^\mu B_a^\nu \\
 N(T^{\mu\nu}, T) = N(T, T^{\mu\nu}) &\equiv -i B_a E_a^{\mu\nu} \\
 N(T^{\mu\nu}, T^\rho) = N(T^\rho, T^{\mu\nu}) &= 0 \\
 N(T^{\mu\nu}, T^{\rho\sigma}) &= 0
 \end{aligned}
 \tag{198}$$

The previous finite renormalizations can be obtained from (173)–(178) performing the finite renormalization

$$N(v_{a\mu,\nu}, v_{b\rho,\sigma}) = \frac{i}{2} \eta_{\mu\rho} \eta_{\nu\sigma} \delta_{ab}.
 \tag{199}$$

Proof. From (199), we obtain

$$N(F_a^{\mu\nu}, F_b^{\rho\sigma}) = i (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma}) \delta_{ab}.
 \tag{200}$$

We substitute this in (173)–(178) and obtain the new contributions (198). It is useful to prove

$$N(T^{\mu\nu}, T) = -N(T^\mu, T^\nu).
 \tag{201}$$

Then, we compute the supplementary terms in $sT(T^{I_1}(x_1), T^{I_2}(x_2))$.

For instance, we have

$$\begin{aligned}
 sT^{\text{ren}}(T(x_1), T(x_2)) &= sT(T(x_1), T(x_2)) + d_Q[\delta(x_1 - x_2)N(T, T)(x_2)] \\
 &\quad - i \partial_\mu^1[\delta(x_1 - x_2)N(T^\mu, T)(x_2)] - i \partial_\mu^2[\delta(x_1 - x_2)N(T, T^\mu)(x_2)]
 \end{aligned}
 \tag{202}$$

Using (198), we arrive at

$$sT^{\text{ren}}(T(x_1), T(x_2)) = sT(T(x_1), T(x_2)) + \delta(x_1 - x_2)R(T, T)(x_2)
 \tag{203}$$

where

$$R(T, T) \equiv d_Q N(T, T) - i \partial_\mu N(T^\mu, T).
 \tag{204}$$

If we substitute (189), we obtain, after some simplifications,

$$sT^{\text{ren}}(T(x_1), T(x_2)) = \delta(x_1 - x_2)\mathcal{A}(T, T)(x_2)
 \tag{205}$$

where

$$\mathcal{A}(T, T) \equiv 2(: B_{a\mu} C_a^\mu : - : B_a D_a :) - : E_{a\mu\nu} C_a^{\mu\nu} :
 \tag{206}$$

If we use Jacobi identity, we obtain $\mathcal{A} = 0$. One can prove the converse of this statement; suppose that $\delta(x_1 - x_2)\mathcal{A}(T, T)(x_2)$ is a coboundary, i.e.,

$$\begin{aligned}
 \delta(x_1 - x_2)\mathcal{A}(T, T)(x_2) &= d_Q[\delta(x_1 - x_2)N(x_2)] \\
 - i \partial_\mu^1[\delta(x_1 - x_2)N^\mu(x_1)] &- i \partial_\mu^2[\delta(x_1 - x_2)N^\mu(x_2)]
 \end{aligned}
 \tag{207}$$

with N and N^μ constrained only by Lorentz covariance and canonical dimension $\omega \leq 4$. We obtain from here

$$\mathcal{A}(T, T) = d_Q N - i \partial_\mu N^\mu.
 \tag{208}$$

If we write arbitrary expressions for N and N^μ we can prove from here that $\mathcal{A}(T, T) = 0$ and this leads to the Jacobi identity. \square

We now consider chronological products of the type $T(\xi_a \cdot T^I, T^J)$.

The following formulas are true:

$$\begin{aligned}
 T(B_a^\mu(x_1)^{(1)}, A(x_2)) &= -f_{abc}[: u_b(x_1) T(v_c^\mu(x_1)^{(0)}, A(x_2)) : \\
 &\quad - : v_b^\mu(x_1)^{(0)} T(u_c(x_1)^{(0)}, A(x_2)) :] \\
 T(C_{a\mu}(x_1)^{(1)}, A(x_2)) &= f_{abc}[: v_b^\nu(x_1) T(F_{c\nu\mu}(x_1)^{(0)}, A(x_2)) : \\
 &\quad - : F_{b\nu\mu}(x_1)^{(0)} T(v_c^\nu(x_1)^{(0)}, A(x_2)) : \\
 - : u_b(x_1) T(\tilde{u}_{c,\mu}(x_1)^{(0)}, A(x_2)) : &- : \partial_\mu \tilde{u}_b(x_1) T(u_c(x_1)^{(0)}, A(x_2)) :] \\
 T(D_a(x_1)^{(1)}, A(x_2)) &= f_{abc}[: v_b^\mu(x_1) T(\tilde{u}_{c,\mu}(x_1)^{(0)}, A(x_2)) : \\
 &\quad - : \partial_\mu \tilde{u}_b(x_1) T(v_c^\mu(x_1)^{(0)}, A(x_2)) :] \\
 T(E_a^{\mu\nu}(x_1)^{(1)}, A(x_2)) &= f_{abc}[: v_b^\mu(x_1) T(v_c^\nu(x_1)^{(0)}, A(x_2)) : - (\mu \leftrightarrow \nu)] \\
 T(B_a(x_1)^{(1)}, A(x_2)) &= f_{abc} : u_b(x_1) T(u_c(x_1)^{(0)}, A(x_2)) : \\
 T(C_a^{\mu\nu}(x_1)^{(1)}, A(x_2)) &= -f_{abc}[: u_b(x_1) T(F_c^{\mu\nu}(x_1)^{(0)}, A(x_2)) : \\
 &\quad - : F_b^{\mu\nu}(x_1)^{(0)} T(u_c(x_1)^{(0)}, A(x_2)) :] \tag{209}
 \end{aligned}$$

where $A = T, T^\mu, T^{\mu\nu}$.

To compute the right-hand sides above, we need the expressions $T(\xi_a(x_1)^{(0)}, A(x_2))$. From our more precise form of the Wick theorem, again we derive

$$\begin{aligned}
 T(v_a^\mu(x_1)^{(0)}, T(x_2)) &= C_{bv}(x_2) T(v_a^\mu(x_1)^{(0)}, v_b^\nu(x_2)^{(0)}) \\
 &\quad + \frac{1}{2} E_{b\rho\sigma}(x_2) T(v_a^\mu(x_1)^{(0)}, F_b^{\sigma\rho}(x_2)^{(0)}) \\
 T(v_a^\mu(x_1)^{(0)}, T^\nu(x_2)) &= C_b^{\rho\nu}(x_2) T(v_a^\mu(x_1)^{(0)}, v_{b\rho}(x_2)^{(0)}) \\
 &\quad - B_{b\rho}(x_2) T(v_a^\mu(x_1)^{(0)}, F_b^{\rho\nu}(x_2)^{(0)}) \\
 T(v_a^\mu(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= B_b(x_2) T(v_a^\mu(x_1)^{(0)}, F_b^{\rho\sigma}(x_2)^{(0)}) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T(x_2)) &= C_{b\rho}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_b^\rho(x_2)^{(0)}) \\
 &\quad + \frac{1}{2} E_{b\rho\sigma}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, F_b^{\sigma\rho}(x_2)^{(0)}) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T^\rho(x_2)) &= C_b^{\sigma\rho}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_{b\sigma}(x_2)^{(0)}) \\
 &\quad - B_{b\sigma}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, F_b^{\rho\sigma}(x_2)^{(0)}) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= B_b(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, F_b^{\rho\sigma}(x_2)^{(0)}) \\
 T(u_a(x_1)^{(0)}, T(x_2)) &= B_b^\mu(x_2) T(u_a(x_1)^{(0)}, \tilde{u}_{b,\mu}(x_2)^{(0)}) \\
 T(u_a(x_1)^{(0)}, T^\mu(x_2)) &= -B_b(x_2) T(u_a(x_1)^{(0)}, \tilde{u}_b{}^\mu(x_2)^{(0)})
 \end{aligned}$$

$$\begin{aligned}
 T(u_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= 0 \\
 T(\tilde{u}_a(x_1)^{(0)}, T(x_2)) &= D_b(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_b(x_2)^{(0)}) \\
 T(\tilde{u}_a(x_1)^{(0)}, T^\mu(x_2)) &= C_b^\mu(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_b(x_2)^{(0)}) \\
 T(\tilde{u}_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= -C_b^{\mu\nu}(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_b(x_2)^{(0)})
 \end{aligned} \tag{210}$$

The canonical splitting view (180)–(182) leads to more precise forms:

$$\begin{aligned}
 T(v_a^\mu(x_1)^{(0)}, T(x_2)) &= i D_0^F(x_1 - x_2) C_a^\mu(x_2) - i \partial_\nu D_0^F(x_1 - x_2) E_a^{\mu\nu}(x_2) \\
 T(v_a^\mu(x_1)^{(0)}, T^\nu(x_2)) &= i D_0^F(x_1 - x_2) C_a^{\mu\nu}(x_2) \\
 &\quad - i \partial^\nu D_0^F(x_1 - x_2) B_a^\mu(x_2) + i \eta^{\mu\nu} \partial_\rho D_0^F(x_1 - x_2) B_a^\rho(x_2) \\
 T(v_a^\mu(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= -i (\eta^{\mu\sigma} \partial^\rho - \eta^{\mu\rho} \partial^\sigma) D_0^F(x_1 - x_2) B_a(x_2) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T(x_2)) &= -i [\partial^\nu D_0^F(x_1 - x_2) C_a^\mu(x_2) - (\mu \leftrightarrow \nu)] \\
 &\quad + i [\partial^\mu \partial_\rho D_0^F(x_1 - x_2) E_a^{\nu\rho}(x_2) - (\mu \leftrightarrow \nu)] \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T^\rho(x_2)) &= -i [\partial^\nu D_0^F(x_1 - x_2) C_a^{\mu\rho}(x_2) - (\mu \leftrightarrow \nu)] \\
 &\quad - i [\eta^{\mu\rho} \partial^\nu \partial_\sigma D_0^F(x_1 - x_2) B_a^\sigma(x_2) - (\mu \leftrightarrow \nu)] - i [\partial^\mu \partial^\rho D_0^F(x_1 - x_2) B_a^\nu(x_2) - (\mu \leftrightarrow \nu)] \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, T^{\rho\sigma}(x_2)) &= \\
 &\quad i (\eta^{\nu\sigma} \partial^\mu \partial^\rho - \eta^{\mu\sigma} \partial^\nu \partial^\rho + \eta^{\mu\rho} \partial^\nu \partial^\sigma - \eta^{\nu\rho} \partial^\mu \partial^\sigma) D_0^F(x_1 - x_2) B_a(x_2) \\
 T(u_a(x_1)^{(0)}, T(x_2)) &= i \partial_\mu D_0^F(x_1 - x_2) B_a^\mu(x_2) \\
 T(u_a(x_1)^{(0)}, T^\mu(x_2)) &= -i \partial_\mu D_0^F(x_1 - x_2) B_a(x_2) \\
 T(u_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= 0 \\
 T(\tilde{u}_a(x_1)^{(0)}, T(x_2)) &= i D_0^F(x_1 - x_2) D_a(x_2) \\
 T(\tilde{u}_a(x_1)^{(0)}, T^\mu(x_2)) &= i D_0^F(x_1 - x_2) C_a^\mu(x_2) \\
 T(\tilde{u}_a(x_1)^{(0)}, T^{\mu\nu}(x_2)) &= -i D_0^F(x_1 - x_2) C_a^{\mu\nu}(x_2)
 \end{aligned} \tag{211}$$

If we substitute the previous relations in (209), we can obtain the canonical expressions:

$$\begin{aligned}
 T(B_a^\mu(x_1)^{(1)}, T(x_2)) &= -i f_{abc} \{ D_0^F(x_1 - x_2) : u_b(x_1) C_c^\mu(x_2) : \\
 &\quad - \partial_\nu D_0^F(x_1 - x_2) [: u_b(x_1) E_c^{\mu\nu}(x_2) : + : v_b^\mu(x_1) B_c^\nu(x_2) :] \} \\
 T(B_a^\mu(x_1)^{(1)}, T^\nu(x_2)) &= -i f_{abc} \{ D_0^F(x_1 - x_2) : u_b(x_1) C_c^{\mu\nu}(x_2) : \\
 + \partial_\rho D_0^F(x_1 - x_2) [\eta^{\mu\nu} : u_b(x_1) B_c^\rho(x_2) : - \eta^{\nu\rho} : u_b(x_1) B_c^\mu(x_2) : + \eta^{\nu\rho} : v_b^\mu(x_1) B_c(x_2) :] \} \\
 T(B_a^\mu(x_1)^{(1)}, T^{\rho\sigma}(x_2)) &= -i f_{abc} (\eta^{\mu\rho} \partial^\sigma - \eta^{\mu\sigma} \partial^\rho) D_0^F(x_1 - x_2) : u_b(x_1) B_c(x_2) : \tag{212}
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, T(x_2)) &= -i f_{abc} \{ D_0^F(x_1 - x_2) : F_b^{\nu\mu}(x_1) C_{c\nu}(x_2) : \\
 &\quad + \partial^\mu D_0^F(x_1 - x_2) [: v_{b\nu}(x_1) C_c^\nu(x_2) : + : u_b(x_1) D_c(x_2) :] \\
 + \partial_\nu D_0^F(x_1 - x_2) [- : v_b^\nu(x_1) C_c^\mu(x_2) : + : F_b^\mu{}_\rho(x_1) C_c^{\rho\nu}(x_2) : + : \partial^\mu \tilde{u}_b(x_1) B_c^\nu(x_2) :] \\
 + \partial_\nu \partial_\rho D_0^F(x_1 - x_2) : v_b^\nu(x_1) E_c^{\mu\rho}(x_2) : - \partial^\mu \partial_\rho D_0^F(x_1 - x_2) : v_{b\nu}(x_1) E_c^{\nu\rho}(x_2) : \}
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, T^\nu(x_2)) &= -i f_{abc} \{ D_0^F(x_1 - x_2) : F_b^{\mu\rho}(x_1) C_{c\rho}^\nu(x_2) : \\
 &\quad + \partial^\mu D_0^F(x_1 - x_2) [: v_{b\rho}(x_1) C_c^{\rho\nu}(x_2) : + : u_b(x_1) C_c^\nu(x_2) :] \\
 &\quad - \partial^\nu D_0^F(x_1 - x_2) [: F_b^{\rho\mu}(x_1) B_{c\rho}(x_2) : + : \partial^\mu \tilde{u}_b(x_1) B_c(x_2) :] \\
 &\quad + \partial_\rho D_0^F(x_1 - x_2) [- : v_b^\rho(x_1) C_c^{\mu\nu}(x_2) : + : F_b^{\nu\mu}(x_1) B_c^\rho(x_2) :] \\
 + \partial^\mu \partial_\rho D_0^F(x_1 - x_2) : v_b^\nu(x_1) B_c^\rho(x_2) : + \partial^\nu \partial_\rho D_0^F(x_1 - x_2) : v_b^\rho(x_1) B_c^\mu(x_2) : \\
 - \eta^{\mu\nu} \partial_\rho \partial_\sigma D_0^F(x_1 - x_2) : v_b^\rho(x_1) B_c^\sigma(x_2) : - \partial^\mu \partial^\nu D_0^F(x_1 - x_2) : v_b^\rho(x_1) B_{c\rho}(x_2) : \}
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, T^{\rho\sigma}(x_2)) &= -i f_{abc} [\partial^\sigma D_0^F(x_1 - x_2) : F_b^{\rho\mu}(x_1) B_c(x_2) : \\
 &\quad - \partial^\rho D_0^F(x_1 - x_2) : F_b^{\sigma\mu}(x_1) B_c(x_2) : \\
 &\quad - \partial^\mu D_0^F(x_1 - x_2) : u_b(x_1) C_c^{\rho\sigma}(x_2) : \\
 &\quad + (\eta^{\mu\sigma} \partial^\rho \partial_\nu - \eta^{\mu\rho} \partial^\sigma \partial_\nu) D_0^F(x_1 - x_2) : v_b^\nu(x_1) B_c(x_2) : \\
 + \partial^\mu \partial^\sigma D_0^F(x_1 - x_2) : v_b^\rho(x_1) B_c(x_2) : - \partial^\mu \partial^\rho D_0^F(x_1 - x_2) : v_b^\sigma(x_1) B_c(x_2) :] \tag{213}
 \end{aligned}$$

$$\begin{aligned}
 T(D_a(x_1)^{(1)}, T(x_2)) &= -i f_{abc} \{ D_0^F(x_1 - x_2) : \partial_\mu \tilde{u}_b(x_1) C_c^\mu(x_2) : \\
 &\quad + \partial_\mu D_0^F(x_1 - x_2) [- : v_b^\mu(x_1) D_c(x_2) : + : \partial_\nu \tilde{u}_b^\mu(x_1) E_c^{\mu\nu}(x_2) :] \}
 \end{aligned}$$

$$\begin{aligned}
 T(D_a(x_1)^{(1)}, T^\mu(x_2)) &= -i f_{abc} \{ -D_0^F(x_1 - x_2) : \partial_\nu \tilde{u}_b(x_1) C_c^{\mu\nu}(x_2) : \\
 + \partial_\nu D_0^F(x_1 - x_2) [- : v_b^\nu(x_1) C_c^\mu(x_2) : + : \partial^\mu \tilde{u}_b^\nu(x_1) B_c^\nu(x_2) : - \eta^{\mu\nu} : \partial_\rho \tilde{u}_b^\mu(x_1) B_c^\rho(x_2) :] \}
 \end{aligned}$$

$$\begin{aligned}
 T(D_a(x_1)^{(1)}, T^{\rho\sigma}(x_2)) &= -i f_{abc} \{ -\partial_\mu D_0^F(x_1 - x_2) : v_b^\mu(x_1) C_c^{\rho\sigma}(x_2) : \\
 + \partial^\sigma D_0^F(x_1 - x_2) : \partial^\rho \tilde{u}_b(x_1) B_c^\mu(x_2) : - \partial^\rho D_0^F(x_1 - x_2) : \partial^\sigma \tilde{u}_b(x_1) B_c^\mu(x_2) : \} \tag{214}
 \end{aligned}$$

$$\begin{aligned}
 & T(E_a^{\mu\nu}(x_1)^{(1)}, T(x_2)) = \\
 & -i f_{abc} \{ D_0^F(x_1 - x_2) [: v_b^\nu(x_1) C_c^\mu(x_2) : - : v_b^\mu(x_1) C_c^\nu(x_2) :] \\
 & \quad + \partial_\rho D_0^F(x_1 - x_2) [: v_b^\mu(x_1) E_c^{\nu\rho}(x_2) : - v_b^\nu(x_1) E_c^{\mu\rho}(x_2) :] \} \\
 & T(E_a^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)) = -i f_{abc} \{ D_0^F(x_1 - x_2) : v_b^\nu(x_1) C_c^{\mu\rho}(x_2) : \\
 & + \partial_\sigma D_0^F(x_1 - x_2) [-\eta^{\nu\rho} : v_b^\mu(x_1) B_c^\sigma(x_2) : + \eta^{\rho\sigma} v_b^\mu(x_1) B_c^\nu(x_2) :] \} - (\mu \leftrightarrow \nu) \\
 & T(E_a^{\mu\nu}(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = \\
 & -i f_{abc} [(\eta^{\nu\sigma} \partial^\rho - \eta^{\nu\rho} \partial^\sigma) D_0^F(x_1 - x_2) : v_b^\mu(x_1) B_c(x_2) : - (\mu \leftrightarrow \nu)] \tag{215}
 \end{aligned}$$

$$\begin{aligned}
 & T(B_a(x_1)^{(1)}, T(x_2)) = i f_{abc} \partial_\mu D_0^F(x_1 - x_2) : u_b(x_1) B_c^\mu(x_2) : \\
 & T(B_a(x_1)^{(1)}, T^\mu(x_2)) = -i f_{abc} \partial^\mu D_0^F(x_1 - x_2) : u_b(x_1) B_c(x_2) : \\
 & T(B_a(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = 0 \tag{216}
 \end{aligned}$$

$$\begin{aligned}
 & T(C_a^{\mu\nu}(x_1)^{(1)}, T(x_2)) = -i f_{abc} [\partial^\mu D_0^F(x_1 - x_2) : u_b(x_1) C_c^\nu(x_2) : \\
 & - \partial^\nu D_0^F(x_1 - x_2) : u_b(x_1) C_c^\mu(x_2) : - \partial_\rho D_0^F(x_1 - x_2) : F_b^{\mu\nu}(x_1) B_c^\rho(x_2) : \\
 & - \partial^\mu \partial_\rho D_0^F(x_1 - x_2) : u_b(x_1) E_c^{\nu\rho}(x_2) : + \partial^\nu \partial_\rho D_0^F(x_1 - x_2) : u_b(x_1) E_c^{\mu\rho}(x_2) :] \\
 & T(C_a^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)) = -i f_{abc} [\partial^\mu D_0^F(x_1 - x_2) : u_b(x_1) C_c^{\nu\rho}(x_2) : \\
 & \quad - \partial_\nu D_0^F(x_1 - x_2) : u_b C_c^{\mu\rho}(x_2) : \\
 & \quad + \partial^\rho D_0^F(x_1 - x_2) : F_b^{\mu\nu}(x_1) B_c(x_2) : \\
 & \quad + (\eta^{\nu\rho} \partial^\mu \partial_\sigma - \eta^{\mu\rho} \partial^\nu \partial_\sigma) D_0^F(x_1 - x_2) : u_b(x_1) B_c^\sigma(x_2) : \\
 & - \partial^\mu \partial^\rho D_0^F(x_1 - x_2) : u_b(x_1) B_c^\nu(x_2) : + \partial^\nu \partial^\rho D_0^F(x_1 - x_2) : u_b(x_1) B_c^\mu(x_2) :] \\
 & T(C_a^{\mu\nu}(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = -i f_{abc} (\eta^{\mu\sigma} \partial^\nu \partial^\rho - \eta^{\nu\sigma} \partial^\mu \partial^\rho \\
 & \quad + \eta^{\nu\rho} \partial^\mu \partial^\sigma - \eta^{\mu\rho} \partial^\nu \partial^\sigma) D_0^F(x_1 - x_2) : u_b(x_1) B_c(x_2) : \tag{217}
 \end{aligned}$$

We can obtain in a similar way to Theorem 11 the following result.

Theorem 13. *The following formulas are true:*

$$s' T(B_a^\mu(x_1)^{(1)}, T^I(x_2)) = 0 \tag{218}$$

$$\begin{aligned}
 s' T(C_a^\mu(x_1)^{(1)}, T(x_2)) = & \delta(x_1 - x_2) f_{abc} (: u_b C_c^\mu : + : F_b^{\mu\nu} B_{c\nu} : - : \partial^\mu \tilde{u}_b B_c :) (x_2) \\
 & - \partial_\nu \delta(x_1 - x_2) f_{abc} [: u_b(x_1) E_c^{\mu\nu}(x_2) : - : v_b^\nu(x_1) B_c^\mu(x_2) :] \\
 & - \partial^\mu \delta(x_1 - x_2) f_{abc} : v_b^\nu(x_1) B_{c\nu}(x_2) : \tag{219}
 \end{aligned}$$

$$\begin{aligned}
 s' T(C_a^\mu(x_1)^{(1)}, T^\nu(x_2)) = & \delta(x_1 - x_2) f_{abc} (: u_a C_c^{\mu\nu} : - : F_b^{\mu\nu} B_c :) (x_2) \\
 & + \partial_\rho \delta(x_1 - x_2) \eta^{\mu\nu} f_{abc} [: u_b(x_1) B_c^\rho(x_2) : - : v_b^\rho(x_1) B_c(x_2) :] \\
 & - \partial^\nu \delta(x_1 - x_2) f_{abc} : u_b(x_1) B_c^\mu(x_2) : + \partial^\mu \delta(x_1 - x_2) f_{abc} : v_b^\nu(x_1) B_c(x_2) : \tag{220}
 \end{aligned}$$

$$s' T(C^\mu(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = (\eta^{\mu\rho} \partial^\sigma - \eta^{\mu\sigma} \partial^\rho) \delta(x_1 - x_2) f_{abc} : u_b(x_1) B_c(x_2) : \tag{221}$$

$$s'T(D(x_1)^{(1)}, T(x_2)) = \delta(x_1 - x_2) f_{abc} (: v_{b\mu} C_c^\mu : + : u_b D_c : + : \partial_\mu \tilde{u}_b B_c^\mu :)(x_2) + \partial_\mu \delta(x_1 - x_2) f_{abc} : v_{bv}(x_1) E_c^{\mu\nu}(x_2) : \quad (222)$$

$$s'T(D_a(x_1)^{(1)}, T^\mu(x_2)) = \delta(x_1 - x_2) f_{abc} (: v_{bv} C_c^{\nu\mu} : + : u_b C_c^\mu : - : \partial^\mu \tilde{u}_b B_c :)(x_2) + \partial_\nu \delta(x_1 - x_2) f_{abc} : v_b^\mu(x_1) B_c^\nu(x_2) : - \partial^\mu \delta(x_1 - x_2) f_{abc} : v_b^\nu(x_1) B_{c\nu}(x_2) : \quad (223)$$

$$s'T(D(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = -\delta(x_1 - x_2) f_{abc} : u_b C_c^{\rho\sigma} : (x_2) + \partial^\sigma \delta(x_1 - x_2) f_{abc} : v_b^\rho(x_1) B_c(x_2) : - (\rho \leftrightarrow \sigma) \quad (224)$$

$$s'T(E_a^{\mu\nu}(x_1)^{(1)}, T(x_2)) = \delta(x_1 - x_2) f_{abc} : v_b^\mu B_c^\nu : (x_2) - (\mu \leftrightarrow \nu) \quad (225)$$

$$s'T(E_a^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)) = \delta(x_1 - x_2) f_{abc} \eta^{\mu\rho} : v_b^\nu B_c : (x_2) - (\mu \leftrightarrow \nu) \quad (226)$$

$$s'T(E_a^{\mu\nu}(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = 0 \quad (227)$$

$$s'T(B_a(x_1)^{(1)}, T^I(x_2)) = 0 \quad (228)$$

$$s'T(C_a^{\mu\nu}(x_1)^{(1)}, T(x_2)) = -\delta(x_1 - x_2) f_{abc} : F_b^{\mu\nu} B_c : (x_2) + \partial^\mu \delta(x_1 - x_2) f_{abc} : u_b(x_1) B_c^\nu(x_2) : - (\mu \leftrightarrow \nu) \quad (229)$$

$$s'T(C_a^{\mu\nu}(x_1)^{(1)}, T^\rho(x_2)) = (\eta^{\mu\rho} \partial^\nu - \eta^{\nu\rho} \partial^\mu) \delta(x_1 - x_2) f_{abc} : u_b(x_1) B_c(x_2) : \quad (230)$$

$$s'T(C_a^{\mu\nu}(x_1)^{(1)}, T^{\rho\sigma}(x_2)) = 0 \quad (231)$$

Next, we investigate if we can remove the anomalies from the previous theorem with finite renormalizations, as in Theorem 12.

Theorem 14. *We have*

$$sT(\xi_a \cdot T^{I_1}(x_1), T^{I_2}(x_2)) = s'T(\xi_a \cdot T^{I_1}(x_1)^{(1)}, T^{I_2}(x_2)^{(2)}) \quad (232)$$

and the anomalies can be removed using finite renormalizations of the type (197):

$$T^{\text{ren}}(A_1(x_1), A_2(x_2)) = T(A_1(x_1), A_2(x_2)) + \delta(x_1 - x_2) N(A_1, A_2)(x_2)$$

where the non-trivial expressions N are

$$\begin{aligned} N(C_a^\mu, T) &\equiv i f_{abc} : v_{bv} E_c^{\mu\nu} : \\ N(C_a^{\mu\nu}, T) &\equiv i f_{abc} : u_b E_c^{\mu\nu} : \\ N(C_a^\mu, T^\nu) &\equiv i f_{abc} (: v_b^\nu B_c^\mu : - \eta^{\mu\nu} : v_{b\rho} B_c^\rho :) \\ N(C_a^{\mu\nu}, T^\rho) &\equiv -i f_{abc} (\eta^{\mu\rho} : u_b B_c^\nu : - \eta^{\nu\rho} : u_b B_c^\mu :) \\ N(C_a^\mu, T^{\rho\sigma}) &\equiv -i f_{abc} (\eta^{\mu\rho} : v_b^\sigma B_c : - \eta^{\nu\rho} : v_b^\sigma B_c :) \end{aligned} \quad (233)$$

These finite renormalizations can be obtained using the finite renormalization (199) in the expressions (210) and are unique.

Proof. Is quite similar to the proof Theorem 12. First, we obtain (233) by substituting (199) in (209) + (210). Next, we compute the supplementary terms coming from the finite renormalizations. For instance,

$$s' T^{\text{ren}}(C_a^\mu(x_1), T(x_2)) = s' T(C_a^\mu(x_1), T(x_2)) + \delta(x_1 - x_2) R(C_a^\mu, T)(x_2) + i\partial_\nu \delta(x_1 - x_2) R^\nu(C_a^\mu, T)(x_2) \tag{234}$$

where

$$R(C_a^\mu, T) \equiv d_Q N(C_a^\mu, T) - i\partial_\nu N(C_a^\mu, T^\nu) \\ R^\nu(C_a^\mu, T) \equiv i [-N(C_a^{\mu\nu}, T) + N(C_a^\mu, T^\nu)] \tag{235}$$

These expressions can be computed using the formulas from the statement. If we substitute in (219), one obtains after some computations

$$s' T^{\text{ren}}(C_a^\mu(x_1), T(x_2)) = \delta(x_1 - x_2) \mathcal{A}(C_a^\mu, T)(x_2) \tag{236}$$

where

$$\mathcal{A}(C_a^\mu, T) \equiv f_{abc} (- : v_{bv} C_c^{\mu\nu} : + : u_b C_c^\mu : + : F_b^{\mu\nu} B_{c\nu} : - : \partial^\mu \tilde{u}_b B_c :). \tag{237}$$

However, using the Jacobi identity, the equality $\mathcal{A}(C_a^\mu, T) = 0$ follows easily. The same line of argument must be used for the other cases. By some computations, one can prove the uniqueness of the finite renormalizations (233). □

Now we can address the question of the Wick property. We have the finite renormalizations from theorem 12 and the preceding theorem. If the Wick property is preserved, the finite renormalizations should verify identities of the type (40). This is true according to the following.

Theorem 15. *The finite renormalizations (198) and (233) preserve the Wick expansion property. Explicitly, we have*

$$v_a^\mu \cdot N(T, T) = 2N(C_a^\mu, T) \\ u_a \cdot N(T^\mu, T) = -N(C_a^\mu, T) - N(T^\mu, D_a) \\ v_a^\nu \cdot N(T^\mu, T) = N(C_a^{\nu\mu}, T) + N(T^\mu, C_a^\nu) \\ u_a \cdot N(T^\mu, T^\nu) = N(C_a^\mu, T^\nu) - N(T^\mu, C_a^\nu) \\ v_a^\rho \cdot N(T^\mu, T^\nu) = N(C_a^{\rho\mu}, T^\nu) + N(T^\mu, C_a^{\rho\nu}) \\ u_a \cdot N(T^{\mu\nu}, T) = -N(C_a^{\mu\nu}, T) + N(T^{\mu\nu}, D_a) \\ v_a^\rho \cdot N(T^{\mu\nu}, T) = N(T^{\mu\nu}, C_a^\rho) \tag{238}$$

and

$$N(B_a^\mu, T^I) = N(D_a, T^I) = N(E_a^{\mu\nu}, T^I) = 0, \quad \forall I. \tag{239}$$

Proof. We start with the relation

$$[v_a^\mu(y), T(T(x_1), T(x_2))] = i [D_0(y - x_1) T(C_a^\mu(x_1), T(x_2)) - \partial_\nu D_0(y - x_1) T(E_a^{\mu\nu}(x_1), T(x_2))] + (1 \leftrightarrow 2) \tag{240}$$

following from Wick expansion property (22). If we consider finite renormalizations of the type (198) and (233), we will obtain new terms in the left- and right-hand sides of the preceding identity. The identity is preserved *if and only if* we have the first relation from (238) and one of the relation (239). In the same way, we obtain

$$[\tilde{u}_a(y), T(T^\mu(x_1), T(x_2))] = -i D_0(y - x_1) T(C_a^\mu(x_1), T(x_2)) - i D_0(y - x_2) T(T^\mu(x_1), D_a(x_2)) \tag{241}$$

and the preservation of this relation leads to the second relation from (238). From

$$\begin{aligned}
 [v_a^\nu(y), T(T^\mu(x_1), T(x_2))] &= i D_0(y - x_1) T(C_a^{\nu\mu}(x_1), T(x_2)) \\
 -i\partial^\mu D_0(y - x_1) T(B_a^\nu(x_1), T(x_2)) &+ i \eta^{\mu\nu} \partial^\rho D_0(y - x_1) T(B_a^\rho(x_1), T(x_2)) \\
 +i D_0(y - x_2) T(T^\mu(x_1), C_a^\nu(x_2)) &- i \partial_\rho D_0(y - x_2) T(T^\mu(x_1), E_a^{\nu\rho}(x_2))
 \end{aligned} \tag{242}$$

and we obtain the third relation from (238). From

$$\begin{aligned}
 [\tilde{u}_a(y), T(T^\mu(x_1), T^\nu(x_2))] &= \\
 i D_0(y - x_1) T(C_a^\mu(x_1), T^\nu(x_2)) &- i D_0(y - x_2) T(T^\mu(x_1), C_a^\nu(x_2))
 \end{aligned} \tag{243}$$

we obtain the fourth relation from (238). From

$$\begin{aligned}
 [v_a^\rho(y), T(T^\mu(x_1), T^\nu(x_2))] &= i D_0(y - x_1) T(C_a^{\rho\mu}(x_1), T^\nu(x_2)) \\
 -i\partial^\mu D_0(y - x_1) T(B_a^\rho(x_1), T^\nu(x_2)) &+ i \eta^{\mu\rho} \partial_\sigma D_0(y - x_1) T(B_a^\sigma(x_1), T(x_2)) \\
 +i D_0(y - x_2) T(T^\mu(x_1), C_a^{\rho\nu}(x_2)) & \\
 -i \partial^\nu D_0(y - x_2) T(T^\mu(x_1), B_a^\rho(x_2)) &+ i \eta^{\nu\rho} \partial_\sigma D_0(y - x_2) T(T^\mu(x_1), B_a^\sigma(x_2))
 \end{aligned} \tag{244}$$

we obtain the fifth relation from (238). From

$$\begin{aligned}
 [\tilde{u}_a(y), T(T^{\mu\nu}(x_1), T(x_2))] &= \\
 -i D_0(y - x_1) T(C_a^{\mu\nu}(x_1), T(x_2)) &+ i D_0(y - x_2) T(T^{\mu\nu}(x_1), D_a(x_2))
 \end{aligned} \tag{245}$$

we obtain the sixth relation from (238). Finally, from

$$\begin{aligned}
 [v_a^\rho(y), T(T^{\mu\nu}(x_1), T(x_2))] &= i (\eta^{\mu\rho} \partial^\nu - \eta^{\nu\rho} \partial^\mu) D_0(y - x_1) T(B_a(x_1), T(x_2)) \\
 +i D_0(y - x_2) T(T^{\mu\nu}(x_1), C_a^\rho(x_2)) & \\
 -i \partial_\sigma D_0(y - x_2) T(T^{\mu\nu}(x_1), E_a^{\rho\sigma}(x_2)) &
 \end{aligned} \tag{246}$$

we obtain the last relation from (238).

It remains to consider the explicit form of the finite renormalizations (198) and (233) and prove that the identities (238) are true. \square

The anomalies cannot be always removed. We provide an example.

Theorem 16. *The following formulas are true*

$$\begin{aligned}
 sT(v_a^\mu(x_1), T(x_2)) &= \delta(x_1 - x_2) B_a^\mu(x_2) \\
 sT(v_a^\mu(x_1), T^\nu(x_2)) &= -\delta(x_1 - x_2) \eta^{\mu\nu} B_a(x_2) \\
 sT(v_a^\mu(x_1), T^{\rho\sigma}(x_2)) &= 0.
 \end{aligned} \tag{247}$$

$$\begin{aligned}
 sT(F_a^{\mu\nu}(x_1), T(x_2)) &= \partial^\mu \delta(x_1 - x_2) B_a^\nu(x_2) - (\mu \leftrightarrow \nu) \\
 sT(F_a^{\mu\nu}(x_1), T^\rho(x_2)) &= (\eta^{\mu\rho} \partial^\nu - \eta^{\nu\rho} \partial^\mu) \delta(x_1 - x_2) B_a(x_2) \\
 sT(F_a^{\mu\nu}(x_1), T^{\rho\sigma}(x_2)) &= 0.
 \end{aligned} \tag{248}$$

$$\begin{aligned}
 sT(u_a(x_1), T(x_2)) &= -\delta(x_1 - x_2) B_a(x_2) \\
 sT(u_a(x_1), T^\nu(x_2)) &= 0 \\
 sT(u_a(x_1), T^{\rho\sigma}(x_2)) &= 0.
 \end{aligned} \tag{249}$$

$$sT(\tilde{u}_a(x_1), T^I(x_2)) = 0, \quad \forall I. \tag{250}$$

Proof. We illustrate the first identity. We have by definition

$$sT(v_a^\mu(x_1), T(x_2)) = d_Q T(v_a^\mu(x_1), T(x_2)) - i \partial_1^\mu T(u_a(x_1), T(x_2)) - i \partial_v^2 T(v_a^\mu(x_1), T^\nu(x_2)) \tag{251}$$

so, using the formulas from (211), we obtain the first formula from the statement. \square

Theorem 17. From (199), we obtain also the finite renormalizations

$$\begin{aligned} N(F_a^{\mu\nu}, T) &= -i E_a^{\mu\nu} \\ N(F_a^{\mu\nu}, T^\rho) &= i (\eta^{\mu\rho} B_a^\nu - \eta^{\nu\rho} B_a^\mu) \\ N(F_a^{\mu\nu}, T^{\rho\sigma}) &= i (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma}) B_a \end{aligned} \tag{252}$$

Using these finite renormalizations, we obtain

$$sT^{\text{ren}}T(A_1(x_1), A_2(x_2)) = \delta(x_1 - x_2) \mathcal{A}(A_1, A_2)(x_2) \tag{253}$$

for $A_1 = \tilde{\zeta}_a = v_a^\mu, u_a, \tilde{u}_a$ etc., and $A_2 = T^I$ with the non-zero anomalies:

$$\begin{aligned} \mathcal{A}(v_a^\mu, T) &= B_a^\mu \\ \mathcal{A}(v_a^\mu, T^\nu) &= -\eta^{\mu\nu} B_a \\ \mathcal{A}(F_a^{\mu\nu}, T) &= C_a^{\mu\nu} \\ \mathcal{A}(u_a, T) &= -B_a. \end{aligned} \tag{254}$$

These anomalies cannot be removed by other finite renormalizations.

Proof. We consider

$$\begin{aligned} sT^{\text{ren}}(F_a^{\mu\nu}(x_1), T(x_2)) &= sT(F_a^{\mu\nu}(x_1), T(x_2)) \\ + d_Q [\delta(x_1 - x_2) N(F_a^{\mu\nu}, T)(x_2)] &- i \partial_\rho^2 [\delta(x_1 - x_2) N(F_a^{\mu\nu}, T^\rho)(x_2)] \end{aligned} \tag{255}$$

and use the finite renormalizations (252). We obtain the third relation from the statement.

It is rather simple to prove that the anomalies from the statement cannot be removed by other finite renormalizations. \square

We continue the analysis of the tree contributions of chronological products involving Wick submonomials. We first have

$$\begin{aligned} T(v_a^\mu(x_1)^{(0)}, B_b^\nu(x_2)) &= -f_{bcd} : u_c(x_2) T(v_a^\mu(x_1)^{(0)}, v_d^\nu(x_2)^{(0)}) : \\ T(v_a^\mu(x_1)^{(0)}, C_b^\nu(x_2)) &= f_{bcd} [: F_d^{\rho\nu}(x_2) T(v_a^\mu(x_1)^{(0)}, v_{c\rho}(x_2)^{(0)}) : \\ &+ : v_{c\rho}(x_2) T(v_a^\mu(x_1)^{(0)}, F_d^{\rho\nu}(x_2)^{(0)}) : \\ T(v_a^\mu(x_1)^{(0)}, D_b(x_2)) &= f_{bcd} : \partial_\nu \tilde{u}_c(x_2) T(v_a^\mu(x_1)^{(0)}, v_c^\nu(x_2)^{(0)}) : \\ T(v_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) &= f_{bcd} [: v_d^\sigma(x_2) T(v_a^\mu(x_1)^{(0)}, v_c^\rho(x_2)^{(0)}) : \\ &+ : v_c^\rho(x_2) T(v_a^\mu(x_1)^{(0)}, v_d^\sigma(x_2)^{(0)}) :] \\ T(v_a^\mu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) &= -f_{bcd} : u_c(x_2) T(v_a^\mu(x_1)^{(0)}, F_d^{\rho\sigma}(x_2)^{(0)}) : \\ T(F_a^{\mu\nu}(x_1)^{(0)}, B_b^\rho(x_2)) &= -f_{bcd} : u_c(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_d^\rho(x_2)^{(0)}) : \end{aligned}$$

$$\begin{aligned}
 T(F_a^{\mu\nu}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) &= f_{bcd} [: F_d^{\sigma\rho}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_{c\sigma}(x_2)^{(0)}) : \\
 &\quad + : v_{c\sigma}(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, F_d^{\sigma\rho}(x_2)^{(0)}) : \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, D_b(x_2)) &= f_{bcd} : \partial_\rho \tilde{u}_c(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_c^\rho(x_2)^{(0)}) : \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) &= f_{bcd} [: v_d^\sigma(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_c^\rho(x_2)^{(0)}) : \\
 &\quad + : v_c^\rho(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, v_d^\sigma(x_2)^{(0)}) :] \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) &= -f_{bcd} : u_c(x_2) T(F_a^{\mu\nu}(x_1)^{(0)}, F_d^{\rho\sigma}(x_2)^{(0)}) : \\
 T(u_a(x_1)^{(0)}, C_{b\mu}(x_2)) &= f_{bcd} : u_c(x_2) T(u_a(x_1)^{(0)}, \tilde{u}_{d,\mu}(x_2)^{(0)}) : \\
 T(u_a(x_1)^{(0)}, D_b(x_2)) &= f_{bcd} : v_c^\mu(x_2) T(u_a(x_1)^{(0)}, \tilde{u}_{d,\mu}(x_2)^{(0)}) : \\
 T(\tilde{u}_a(x_1)^{(0)}, B_b^\mu(x_2)) &= -f_{bcd} : v_d^\mu(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_c(x_2)^{(0)}) : \\
 T(\tilde{u}_a(x_1)^{(0)}, C_b^\mu(x_2)) &= -f_{bcd} : \partial^\mu \tilde{u}_d(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_c(x_2)^{(0)}) : \\
 T(\tilde{u}_a(x_1)^{(0)}, B_b(x_2)) &= f_{bcd} : u_d(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_c(x_2)^{(0)}) : \\
 T(\tilde{u}_a(x_1)^{(0)}, C_b^{\mu\nu}(x_2)) &= -f_{bcd} : F_d^{\mu\nu}(x_2) T(\tilde{u}_a(x_1)^{(0)}, u_c(x_2)^{(0)}) : \tag{256}
 \end{aligned}$$

If we use canonical splitting, we arrive at the more precise forms:

$$\begin{aligned}
 T(v_a^\mu(x_1)^{(0)}, B_b^\nu(x_2)) &= -i D_0^F(x_1 - x_2) \eta^{\mu\nu} f_{abc} u_c(x_2) \\
 T(v_a^\mu(x_1)^{(0)}, C_b^\nu(x_2)) &= -i f_{abc} [D_0^F(x_1 - x_2) F_c^{\mu\nu}(x_2) \\
 &\quad + \eta^{\mu\nu} \partial_\rho D_0^F(x_1 - x_2) v_c^\rho(x_2) - \partial^\nu D_0^F(x_1 - x_2) v_c^\mu(x_2)] \\
 T(v_a^\mu(x_1)^{(0)}, D_b(x_2)) &= -i D_0^F(x_1 - x_2) f_{abc} \partial^\mu \tilde{u}_c(x_2) \\
 T(v_a^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) &= -i f_{abc} D_0^F(x_1 - x_2) (\eta^{\mu\rho} v_c^\sigma - \eta^{\mu\sigma} v_c^\rho)(x_2) \\
 T(v_a^\mu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) &= i f_{abc} (\eta^{\mu\sigma} \partial^\rho - \eta^{\mu\rho} \partial^\sigma) D_0^F(x_1 - x_2) u_c(x_2) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, B_b^\rho(x_2)) &= i f_{abc} (\eta^{\mu\rho} \partial^\nu - \eta^{\nu\rho} \partial^\mu) D_0^F(x_1 - x_2) u_c(x_2) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, C_b^\rho(x_2)) &= i f_{abc} [\partial^\nu D_0^F(x_1 - x_2) F_c^{\mu\rho}(x_2) \\
 &\quad + \partial^\mu \partial^\rho D_0^F(x_1 - x_2) v_c^\nu(x_2) + \eta^{\mu\rho} \partial^\nu \partial_\sigma D_0^F(x_1 - x_2) v_c^\sigma(x_2)] - (\mu \leftrightarrow \nu) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, D_b(x_2)) &= i f_{abc} \partial^\nu D_0^F(x_1 - x_2) \partial^\mu \tilde{u}_c(x_2) - (\mu \leftrightarrow \nu) \\
 T(F_a^{\mu\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) &= i f_{abc} [\eta^{\mu\rho} \partial^\nu D_0^F(x_1 - x_2) v_c^\sigma(x_2) - (\mu \leftrightarrow \nu)] - (\rho \leftrightarrow \sigma)
 \end{aligned}$$

$$\begin{aligned}
 & T(F_a^{\mu\nu}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) = \\
 & i f_{abc} (\eta^{\nu\sigma} \partial^\mu \partial_\rho - \eta^{\mu\sigma} \partial^\nu \partial_\rho + \eta^{\mu\rho} \partial^\nu \partial_\sigma - \eta^{\nu\rho} \partial^\mu \partial_\sigma) D_0^F(x_1 - x_2) u_c(x_2) \\
 & T(u_a(x_1)^{(0)}, C_b^\mu(x_2)) = i f_{abc} \partial^\mu D_0^F(x_1 - x_2) u_c(x_2) \\
 & T(u_a^\mu(x_1)^{(0)}, D_b(x_2)) = i f_{abc} \partial_\mu D_0^F(x_1 - x_2) v_c^\mu(x_2) \\
 & T(\tilde{u}_a(x_1)^{(0)}, B_b^\mu(x_2)) = i f_{abc} D_0^F(x_1 - x_2) v_c^\mu(x_2) \\
 & T(\tilde{u}_a(x_1)^{(0)}, C_b^\mu(x_2)) = i f_{abc} D_0^F(x_1 - x_2) \partial^\mu \tilde{u}_c(x_2) \\
 & T(\tilde{u}_a(x_1)^{(0)}, B_b(x_2)) = -i f_{abc} D_0^F(x_1 - x_2) u_c(x_2) \\
 & T(\tilde{u}_a(x_1)^{(0)}, C_b^{\mu\nu}(x_2)) = i f_{abc} D_0^F(x_1 - x_2) F_c^{\mu\nu}(x_2) \tag{257}
 \end{aligned}$$

Now, we consider expressions of the type $T(\xi_a \cdot T^I, \xi_b \cdot T^J)$ and with the Wick theorem, we have

$$\begin{aligned}
 & T(B_a^\mu(x_1)^{(1)}, B_b^\nu(x_2)) = -f_{adc} : u_d(x_1) T(v_c^\mu(x_1)^{(0)}, B_b^\nu(x_2)) : \\
 & T(B_a^\mu(x_1)^{(1)}, C_b^\nu(x_2)) = -f_{adc} [: u_d(x_1) T(v_c^\mu(x_1)^{(0)}, C_b^\nu(x_2)) : \\
 & \quad - : v_d^\mu(x_1) T(u_c(x_1)^{(0)}, C_b^\nu(x_2)) :] \\
 & T(B_a^\mu(x_1)^{(1)}, D_b(x_2)) = -f_{adc} [: u_d(x_1) T(v_c^\mu(x_1)^{(0)}, D_b(x_2)) : \\
 & \quad - : v_d^\mu(x_1) T(u_c(x_1)^{(0)}, D_b(x_2)) :] \\
 & T(B_a^\mu(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)) = -f_{adc} : u_d(x_1) T(v_c^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) : \\
 & \quad T(B_a^\mu(x_1)^{(1)}, B_b(x_2)) = 0 \\
 & T(B_a^\mu(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)) = -f_{adc} : u_d(x_1) T(v_c^\mu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) : \\
 & T(C_a^\mu(x_1)^{(1)}, C_b^\nu(x_2)) = f_{adc} [: v_{d\rho}(x_1) T(F_c^{\rho\mu}(x_1)^{(0)}, C_b^\nu(x_2)) : \\
 & - : F_d^{\rho\mu}(x_1) T(v_{c\rho}(x_1)^{(0)}, C_b^\nu(x_2)) : - : u_d(x_1) T(\tilde{u}_c^\mu(x_1)^{(0)}, C_b^\nu(x_2)) : \\
 & \quad - : \partial^\mu \tilde{u}_c(x_1) T(u_c(x_1)^{(0)}, C_b^\nu(x_2)) :] \\
 & T(C_a^\mu(x_1)^{(1)}, D_b(x_2)) = f_{adc} [: v_{d\nu}(x_1) T(F_c^{\nu\mu}(x_1)^{(0)}, D_b(x_2)) : \\
 & - : F_d^{\nu\mu}(x_1) T(v_{c\nu}(x_1)^{(0)}, D_b(x_2)) : - : \partial^\mu \tilde{u}_d(x_1) T(u_c(x_1)^{(0)}, D_b(x_2)) :] \\
 & T(C_a^\mu(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)) = f_{adc} [: v_{d\nu}(x_1) T(F_c^{\nu\mu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) : \\
 & \quad - : F_d^{\nu\mu}(x_1) T(v_{c\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) :]
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, B_b(x_2)) &= -f_{adc} : u_d(x_1) T(\tilde{u}_c,^\mu(x_1)^{(0)}, B_b(x_2)) : \\
 T(C_a^\mu(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)) &= f_{adc} [: v_{dv}(x_1) T(F_c^{\nu\mu}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) : \\
 &- : F_d^{\nu\mu}(x_1) T(v_{cv}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) : - : u_d(x_1) T(\tilde{u}_c,^\mu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) :] \\
 T(D_a(x_1)^{(1)}, D_b(x_2)) &= -f_{adc} : \partial_\mu \tilde{u}_d(x_1) T(v_c^\mu(x_1)^{(0)}, D_b(x_2)) : \\
 T(D_a(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)) &= -f_{adc} : \partial_\mu \tilde{u}_d(x_1) T(v_c^\mu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) : \\
 T(D_a(x_1)^{(1)}, B_b(x_2)) &= f_{adc} : v_d^\mu(x_1) T(\tilde{u}_c,^\mu(x_1)^{(0)}, B_b(x_2)) : \\
 T(D_a(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)) &= f_{adc} [: v_d^\mu(x_1) T(\tilde{u}_{c,\mu}(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) : \\
 &- : \partial_\mu \tilde{u}_d(x_1) T(v_c^\mu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) :] \\
 T(E_a^{\mu\nu}(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)) &= f_{adc} : v_d^\mu(x_1) T(v_c^\nu(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) : -(\mu \leftrightarrow \nu) \\
 T(E_a^{\mu\nu}(x_1)^{(1)}, B_b(x_2)) &= 0 \\
 T(E_a^{\mu\nu}(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)) &= f_{adc} : v_d^\mu(x_1) T(v_c^\nu(x_1)^{(0)}, C_b^{\rho\sigma}(x_2)) : -(\mu \leftrightarrow \nu) \\
 T(B_a(x_1)^{(1)}, B_b(x_2)) &= 0 \\
 T(B_a(x_1)^{(1)}, C_b^{\mu\nu}(x_2)) &= 0 \\
 T(C_a^{\mu\nu}(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)) &= -f_{adc} : u_d(x_1) T(F_c^{\mu\nu}(x_1)^{(0)}, E_b^{\rho\sigma}(x_2)) : \tag{258}
 \end{aligned}$$

Using the canonical splitting, we have

$$\begin{aligned}
 T(B_a^\mu(x_1)^{(1)}, B_b^\nu(x_2)^{(1)}) &= i \eta^{\mu\nu} f_{ace} f_{bde} D_0^F(x_1 - x_2) : u_c(x_1) u_d(x_2) : \\
 T(B_a^\mu(x_1)^{(1)}, C_b^\nu(x_2)^{(1)}) &= i f_{ace} f_{bde} \{ D_0^F(x_1 - x_2) : u_c(x_1) F_d^{\mu\nu}(x_2) : \\
 + \eta^{\mu\nu} \partial_\rho D_0^F(x_1 - x_2) : u_c(x_1) v_d^\rho(x_2) : + \partial^\nu D_0^F(x_1 - x_2) [: v_c^\mu(x_1) u_d(x_2) : - : u_c(x_1) v_d^\mu(x_2) :] \} \\
 T(B_a^\mu(x_1)^{(1)}, D_b(x_2)^{(1)}) &= i f_{ace} f_{bde} [D_0^F(x_1 - x_2) : u_c(x_1) \partial^\mu \tilde{u}_d(x_2) : \\
 + \partial_\nu D_0^F(x_1 - x_2) : v_c^\mu(x_1) v_d^\nu(x_2) : \\
 T(B_a^\mu(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} D_0^F(x_1 - x_2) [\eta^{\mu\rho} : u_c(x_1) v_d^\sigma(x_2) : -(\rho \leftrightarrow \sigma)] \\
 T(B_a^\mu(x_1)^{(1)}, B_b(x_2)^{(1)}) &= 0 \\
 T(B_a^\mu(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} (\eta^{\mu\rho} \partial^\sigma - \eta^{\mu\sigma} \partial^\rho) D_0^F(x_1 - x_2) : u_c(x_1) u_d(x_2) :
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, C_b^\nu(x_2)^{(1)}) &= i f_{ace} f_{bde} \{ D_0^F(x_1 - x_2) : F_c^{\mu\rho}(x_1) F_d^\nu(x_2) : \\
 &\quad + \partial^\mu D_0^F(x_1 - x_2) [: v_{c\rho}(x_1) F_d^{\rho\nu}(x_2) : - : u_c(x_1) \partial^\nu \tilde{u}_d(x_2) :] \\
 &\quad - \partial^\nu D_0^F(x_1 - x_2) [: F_c^{\rho\nu}(x_1) v_{d\rho}(x_2) : + : \partial^\nu \tilde{u}_c(x_1) u_d(x_2) :] \\
 &\quad - \partial_\rho D_0^F(x_1 - x_2) [: F_c^{\mu\nu}(x_1) v_d^\rho(x_2) : + : v_c^\rho(x_1) F_d^{\mu\nu}(x_2) :] \\
 - \partial^\mu \partial^\nu D_0^F(x_1 - x_2) : v_{c\rho}(x_1) v_d^\rho(x_2) : &- \eta^{\mu\nu} \partial_\rho \partial_\sigma D_0^F(x_1 - x_2) : v_c^\rho(x_1) v_d^\sigma(x_2) : \\
 + \partial^\nu \partial_\rho D_0^F(x_1 - x_2) : v_c^\rho(x_1) v_d^\mu(x_2) : &+ \partial^\mu \partial_\rho D_0^F(x_1 - x_2) : v_c^\nu(x_1) v_d^\rho(x_2) : \}
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, D_b(x_2)^{(1)}) &= i f_{ace} f_{bde} \{ -D_0^F(x_1 - x_2) : F_c^{\mu\nu}(x_1) \partial_\nu \tilde{u}_d(x_2) : \\
 &\quad + \partial^\mu D_0^F(x_1 - x_2) : v_c^\nu(x_1) \partial_\nu \tilde{u}_d(x_2) : \\
 &\quad - \partial_\nu D_0^F(x_1 - x_2) [: v_c^\nu(x_1) \partial^\mu \tilde{u}_d(x_2) : + : \partial^\mu \tilde{u}_c(x_1) v_d^\nu(x_2) :]
 \end{aligned}$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} \{ D_0^F(x_1 - x_2) [: F_c^{\rho\mu}(x_1) v_d^\sigma(x_2) : - : F_c^{\sigma\mu}(x_1) v_d^\rho(x_2) :] \\
 &\quad + \partial^\mu D_0^F(x_1 - x_2) [: v_c^\rho(x_1) v_d^\sigma(x_2) : - : v_c^\sigma(x_1) v_d^\rho(x_2) :] \\
 &\quad - \partial_\nu D_0^F(x_1 - x_2) [\eta^{\mu\sigma} : v_c^\nu(x_1) v_d^\rho(x_2) : - \eta^{\mu\rho} : v_c^\nu(x_1) v_d^\sigma(x_2) :] \}
 \end{aligned}$$

$$T(C_a^\mu(x_1)^{(1)}, B_b(x_2)^{(1)}) = i f_{ace} f_{bde} \partial^\mu D_0^F(x_1 - x_2) : u_c(x_1) u_d(x_2) :$$

$$\begin{aligned}
 T(C_a^\mu(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} [-\partial^\mu D_0^F(x_1 - x_2) : u_c(x_1) F_d^{\rho\sigma}(x_2) : \\
 &\quad + \partial^\rho D_0^F(x_1 - x_2) : F_c^{\mu\sigma}(x_1) u_d(x_2) : - \partial^\sigma D_0^F(x_1 - x_2) : F_c^{\mu\rho}(x_1) u_d(x_2) : \\
 &\quad + (\eta^{\mu\sigma} \partial^\rho \partial_\nu - \eta^{\mu\rho} \partial^\sigma \partial_\nu) D_0^F(x_1 - x_2) : v_c^\nu(x_1) u_d(x_2) : \\
 &\quad + \partial^\mu \partial^\sigma D_0^F(x_1 - x_2) : v_c^\rho(x_1) u_d(x_2) : - \partial^\mu \partial^\rho D_0^F(x_1 - x_2) : v_c^\sigma(x_1) u_d(x_2) :]
 \end{aligned}$$

$$T(D_a(x_1)^{(1)}, D_b(x_2)^{(1)}) = i f_{ace} f_{bde} D_0^F(x_1 - x_2) : \partial_\mu \tilde{u}_c(x_1) \partial^\mu \tilde{u}_d(x_2) :$$

$$T(D_a(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)^{(1)}) = i f_{ace} f_{bde} D_0^F(x_1 - x_2) [: \partial^\rho \tilde{u}_c(x_1) v_d^\sigma(x_2) : - : \partial^\sigma \tilde{u}_c(x_1) v_d^\rho(x_2) :]$$

$$T(D_a(x_1)^{(1)}, B_b(x_2)^{(1)}) = -i f_{ace} f_{bde} \partial_\mu D_0^F(x_1 - x_2) : v_c^\mu(x_1) u_d(x_2) :$$

$$\begin{aligned}
 T(D_a(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} [\partial^\mu D_0^F(x_1 - x_2) : v_c^\mu(x_1) F_d^{\rho\sigma}(x_2) : \\
 &\quad + \partial^\sigma D_0^F(x_1 - x_2) : \partial^\rho \tilde{u}_c(x_1) u_d(x_2) : - \partial^\rho D_0^F(x_1 - x_2) : \partial^\sigma \tilde{u}_c(x_1) u_d(x_2) :]
 \end{aligned}$$

$$\begin{aligned}
 T(E_a^{\mu\nu}(x_1)^{(1)}, E_b^{\rho\sigma}(x_2)^{(1)}) &= -i f_{ace} f_{bde} D_0^F(x_1 - x_2) \\
 [\eta^{\nu\rho} : v_c^\mu(x_1) v_d^\sigma(x_2) : - \eta^{\nu\sigma} : v_c^\mu(x_1) v_d^\rho(x_2) : &+ \eta^{\mu\sigma} : v_c^\nu(x_1) v_d^\rho(x_2) : - \eta^{\mu\rho} : v_c^\nu(x_1) v_d^\sigma(x_2) :]
 \end{aligned}$$

$$T(E_a^{\mu\nu}(x_1)^{(1)}, B_b(x_2)^{(1)}) = 0$$

$$\begin{aligned}
 T(E_a^{\mu\nu}(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)^{(1)}) &= i f_{ace} f_{bde} \\
 [(\eta^{\nu\sigma} \partial^\rho - \eta^{\nu\rho} \partial^\sigma) D_0^F(x_1 - x_2) : v_c^\mu(x_1) u_d(x_2) : & \\
 - (\eta^{\mu\sigma} \partial^\rho - \eta^{\mu\rho} \partial^\sigma) D_0^F(x_1 - x_2) : v_c^\nu(x_1) u_d(x_2) : &]
 \end{aligned}$$

$$T(B_a(x_1)^{(1)}, B_b(x_2)^{(1)}) = 0$$

$$T(B_a(x_1)^{(1)}, C_b^{\mu\nu}(x_2)^{(1)}) = 0$$

$$T(C_a^{\mu\nu}(x_1)^{(1)}, C_b^{\rho\sigma}(x_2)^{(1)}) = -i f_{ace} f_{bde} (\eta^{\nu\sigma}\partial^\mu\partial^\rho - \eta^{\mu\sigma}\partial^\nu\partial^\rho + \eta^{\mu\rho}\partial^\nu\partial^\sigma - \eta^{\nu\rho}\partial^\mu\partial^\sigma) D_0^F(x_1 - x_2) : u_c(x_1)u_d(x_2) : \tag{259}$$

We can use the previous expressions to compute the corresponding anomalies. We have the following.

Theorem 18. *The following relations are true:*

$$s'T(B_a^\mu(x_1)^{(1)}, C_b^\mu(x_2)^{(1)}) = \delta(x_1 - x_2) \eta^{\mu\nu} f_{ace} f_{bde} (: u_c u_d :) (x_2) \tag{260}$$

$$s'T(B_a^\mu(x_1)^{(2)}, D_b(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (: v_c^\mu u_d : - : u_c v_d^\mu :) (x_2) \tag{261}$$

$$s'T(C_a^\mu(x_1)^{(2)}, C_b^\nu(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (: F_c^{\mu\nu} u_d : - : u_c F_d^{\mu\nu} :) (x_2) + \eta^{\mu\nu} f_{ace} f_{bde} \partial_\rho \delta(x_1 - x_2) [: v_c^\rho(x_1) u_d(x_2) : - : u_c(x_1) v_d^\rho(x_2) :] - f_{ace} f_{bde} [\partial^\mu \delta(x_1 - x_2) : v_c^\nu(x_1) u_d(x_2) : - \partial^\nu \delta(x_1 - x_2) : u_c(x_1) v_d^\mu(x_2) : \tag{262}$$

$$s'T(C_a^\mu(x_1)^{(2)}, D_b(x_2)^{(1)}) = -\delta(x_1 - x_2) f_{ace} f_{bde} (: u_c \partial^\mu \tilde{u}_d : + : F_c^{\nu\mu} v_{d\nu} : + : \partial^\mu \tilde{u}_c u_d :) (x_2) - f_{ace} f_{bde} \partial^\mu \delta(x_1 - x_2) : v_{c\rho}(x_1) v_d^\rho(x_2) : + f_{ace} f_{bde} \partial^\nu \delta(x_1 - x_2) : v_c^\nu(x_1) v_d^\mu(x_2) : \tag{263}$$

$$s'T(C^\mu(x_1)^{(2)}, E_b^{\rho\sigma}(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (\eta^{\mu\sigma} : u_c v_d^\rho : - \eta^{\mu\rho} : u_c v_d^\sigma :) (x_2) \tag{264}$$

$$s'T(C_a^\mu(x_1)^{(2)}, C_b^{\rho\sigma}(x_2)^{(1)}) = f_{ace} f_{bde} (\eta^{\mu\sigma}\partial^\rho - \eta^{\mu\rho}\partial^\sigma)\delta(x_1 - x_2) : u_c(x_1)u_d(x_2) : \tag{265}$$

$$s'T(D(x_1)^{(2)}, D_b(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (: \partial^\mu \tilde{u}_c v_d^\mu : - : v_c^\mu \partial^\mu \tilde{u}_d :) (x_2) \tag{266}$$

$$s'T(D_a(x_1)^{(2)}, E_b^{\rho\sigma}(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (- : v_c^\rho v_d^\sigma : + : v_c^\sigma v_d^\rho :) (x_2) \tag{267}$$

$$s'T(D_a(x_1)^{(2)}, B_b(x_2)^{(1)}) = -\delta(x_1 - x_2) f_{ace} f_{bde} (: u_c u_d :) (x_2) \tag{268}$$

$$s'T(D_a(x_1)^{(2)}, C_b^{\rho\sigma}(x_2)^{(1)}) = \delta(x_1 - x_2) f_{ace} f_{bde} (: u_c F_d^{\rho\sigma} :) (x_2) - f_{ace} f_{bde} [\partial^\rho \delta(x_1 - x_2) : v_c^\sigma(x_1) u_d(x_2) : - (\rho \leftrightarrow \sigma)] \tag{269}$$

We also have

$$s'T(v_a^\mu(x_1), C_b^\nu(x_2)) = \delta(x_1 - x_2) \eta^{\mu\nu} f_{abc} u_c(x_2) \\ s'T(v_a^\mu(x_1), D_b(x_2)) = \delta(x_1 - x_2) f_{abc} v_c^\mu(x_2) \\ s'T(F_a^{\mu\nu}(x_1), C_b^\rho(x_2)) = (\eta^{\nu\rho}\partial^\mu - \eta^{\mu\rho}\partial^\nu)\delta(x_1 - x_2) f_{abc} u_c(x_2) \\ s'T(F_a^{\mu\nu}(x_1), D_b^\rho(x_2)) = \partial^\mu \delta(x_1 - x_2) f_{abc} v_c^\nu(x_2) - (\mu \leftrightarrow \nu) \\ s'T(u_a(x_1), D_b(x_2)) = -\delta(x_1 - x_2) \eta^{\mu\nu} f_{abc} u_c(x_2) \tag{270}$$

Now we investigate if we can impose gauge invariance for expressions of the type $T(\xi_a \cdot T^I, \xi_b \cdot T^J)$ and $T(\xi_a, \xi_b \cdot T^J)$; the answer is negative.

Theorem 19. *The finite renormalization (199) induces also*

$$\begin{aligned} N(F_a^{\mu\nu}, C_b^\rho) &= i f_{abc} (\eta^{\nu\rho} v_c^\mu - \eta^{\mu\rho} v_c^\nu) \\ N(F_a^{\mu\nu}, C_b^{\rho\sigma}) &= -i f_{abc} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma}) u_c \end{aligned} \tag{271}$$

and

$$\begin{aligned} N(C_a^\mu, C_b^\nu) &= i f_{ace} f_{bde} (\eta^{\mu\nu} : v_{c\rho} v_d^\rho : - : v_c^\nu v_d^\mu :) \\ N(C_a^\mu, C_b^{\rho\sigma}) &= i f_{ace} f_{bde} (\eta^{\mu\rho} : v_c^\sigma u_d : - \eta^{\mu\sigma} : v_c^\rho u_d :) \\ N(C_a^{\mu\nu}, C_b^{\rho\sigma}) &= i f_{ace} f_{bde} (\eta^{\mu\rho} \eta^{\nu\sigma} - \eta^{\nu\rho} \eta^{\mu\sigma}) : u_c u_d : \end{aligned} \tag{272}$$

If we perform finite renormalizations, the expressions from the previous theorem go into

$$s' T^{\text{ren}} T(A_1(x_1), A_2(x_2)) = \delta(x_1 - x_2) \mathcal{A}(A_1, A_2)(x_2) \tag{273}$$

where A_1, A_2 are Wick submonomials of the type $\zeta_a \cdot T^I$. The non-trivial expressions $\mathcal{A}(A_1, A_2)$ are

$$\begin{aligned} \mathcal{A}(C_a^\mu, C_b^\nu) &= f_{abc} C_c^{\mu\nu} \\ \mathcal{A}(C_a^\mu, D_b) &= -f_{abc} C_c^\mu \\ \mathcal{A}(C_a^\mu, E_b^{\rho\sigma}) &= f_{abc} (\eta^{\mu\rho} B_c^\sigma - \eta^{\mu\sigma} B_c^\rho) \\ \mathcal{A}(D_a, D_b) &= -f_{abc} D_c \\ \mathcal{A}(D_a, E_b^{\rho\sigma}) &= -f_{abc} E_c^{\rho\sigma} \\ \mathcal{A}(D_a, B_b) &= -f_{abc} B_c \\ \mathcal{A}(D_a, C_b^{\rho\sigma}) &= -f_{abc} C_c^{\rho\sigma} \\ \mathcal{A}(B_a^\mu, C_b^\nu) &= \eta^{\mu\nu} f_{abc} B_c \\ \mathcal{A}(B_a^\mu, D_b) &= f_{abc} B_c^\mu. \end{aligned} \tag{274}$$

In the case when $A_1 = v_a^\mu, u_a, \tilde{u}_a$ and $A_2 = B_b^\mu, C_b^\mu, D_b, E_b^{\rho\sigma}, B_b, C_b^{\mu\nu}$ the non-trivial expressions $\mathcal{A}(A_1, A_2)$ are

$$\begin{aligned} \mathcal{A}(v_a^\mu, C_b^\nu) &= f_{abc} \eta^{\mu\nu} u_c \\ \mathcal{A}(v_a^\mu, D_b) &= f_{abc} v_c^\mu \\ \mathcal{A}(F_a^{\mu\nu}, D_b) &= f_{abc} F_c^{\mu\nu} \\ \mathcal{A}(u_a, D_b) &= -f_{abc} u_c. \end{aligned} \tag{275}$$

These anomalies cannot be removed by other finite renormalizations.

Proof. We illustrate the proof considering the first case from the statement. We have

$$\begin{aligned} s' T^{\text{ren}}(C_a^\mu(x_1), C_b^\nu(x_2)) &= s' T(C_a^\mu(x_1), T(x_2)) \\ &+ \delta(x_1 - x_2) R(C_a^\mu, C_b^\nu)(x_2) + i\partial_\nu \delta(x_1 - x_2) R^\rho(C_a^\mu, C_b^\nu)(x_2) \end{aligned} \tag{276}$$

where

$$\begin{aligned} R(C_a^\mu, C_b^\nu) &\equiv d_Q N(C_a^\mu, C_b^\nu) - i\partial_\rho N(C_a^\mu, C_b^{\nu\rho}) \\ R^\rho(C_a^\mu, C_b^\nu) &\equiv i [-N(C_a^{\mu\rho}, C_b^\nu) + N(C_a^\mu, C_b^{\nu\rho})] \end{aligned} \tag{277}$$

Using the expression $s' T(C_a^\mu(x_1), T(x_2))$ from the previous theorem and the expressions N from the statement, we obtain after some computations the first relation from (274). The rest of the relations can be obtained in the same way.

Next we prove that the anomaly from the first relation from (274) cannot be removed. The most general finite renormalizations would be

$$\begin{aligned} N(C_a^\mu(x_1), C_b^\nu(x_2)) &= i \delta(x_1 - x_2) R_{ab}^{\mu\nu}(x_2) \\ N(C_a^{\mu\nu}(x_1), C_b^\nu(x_2)) &= i \delta(x_1 - x_2) R_{ab}^{\mu\nu,\rho}(x_2) \end{aligned} \tag{278}$$

where the generic forms are

$$\begin{aligned} R_{ab}^{\mu\nu} &= f_{abcd}^{(1)} : v_c^\mu v_d^\nu : + f_{abcd}^{(2)} \eta^{\mu\nu} : v_{c\rho} v_d^\rho : + f_{abcd}^{(3)} \eta^{\mu\nu} : u_c \tilde{u}_d : \\ R_{ab}^{\mu\nu,\rho} &= f_{abcd}^{(4)} (\eta^{\mu\rho} : u_c v_d^\nu : - \eta^{\nu\rho} : u_c v_d^\mu :) \end{aligned} \tag{279}$$

with

$$f_{abcd}^{(1)} = f_{badc}^{(1)}, \quad f_{abcd}^{(2)} = f_{bacd}^{(2)} = f_{abdc}^{(2)}, \quad f_{abcd}^{(3)} = f_{bacd}^{(3)} \tag{280}$$

We insert everything in the equation

$$\delta(x_1 - x_2) \mathcal{A}(C_a^\mu, C_b^\nu)(x_2) = d_Q N C_a^\mu(x_1), C_b^\nu(x_2) - i \partial_\rho^1 N(C_a^{\mu\rho}(x_1), C_b^\nu(x_2)) - i \partial_\rho^2 N(C_a^\mu(x_1), C_b^{\nu\rho}(x_2)) \tag{281}$$

and after some computations, we obtain $f_{abc} = 0$ which is not possible. \square

However, the Wick property is preserved.

Theorem 20. *The following relations are true: the finite renormalizations (198) and (233) preserve the Wick expansion property. Explicitly, we have*

$$\begin{aligned} v_b^\nu \cdot N(C_a^\mu, T) &= f_{abc} N(F_c^{\nu\mu}, T) + N(C_a^\mu, C_b^\nu) \\ v_b^\rho \cdot N(C_a^\mu, T^\nu) &= f_{abc} N(F_c^{\rho\mu}, T^\nu) + N(C_a^\mu, C_b^{\rho\nu}) \\ u_b \cdot N(C_a^\mu, T^\nu) &= N(C_a^\mu, C_b^\nu) \\ v_b^\rho \cdot N(C_a^{\mu\nu}, T) &= N(C_a^{\nu\mu}, C_b^\rho) \\ u_b \cdot N(C_a^{\mu\nu}, T) &= f_{abc} N(F_c^{\mu\nu}, T) + N(C_a^{\mu\nu}, D_b) \\ v_b^\sigma \cdot N(C_a^{\mu\nu}, T^\rho) &= N(C_a^{\mu\nu}, C_b^{\sigma\rho}) \\ u_b \cdot N(C_a^{\mu\nu}, T^\rho) &= -f_{abc} N(F_c^{\mu\nu}, T^\rho) - N(C_a^{\mu\nu}, C_b^\rho) \\ v_b^\rho \cdot N(C_a^\mu, T^{\rho\sigma}) &= f_{abc} N(F_c^{\nu\mu}, T^{\rho\sigma}) \\ u_b \cdot N(C_a^\mu, T^{\rho\sigma}) &= -N(C_a^\mu, C_b^{\rho\sigma}) \end{aligned} \tag{282}$$

and

$$N(v_a^\mu, T^I) = 0, \quad N(u_a, T^I) = 0, \quad N(\tilde{u}_a, T^I) = 0. \tag{283}$$

We also have

$$\begin{aligned} v_c^\rho \cdot N(C_a^\mu, C_b^\nu) &= f_{acd} N(F_d^{\rho\mu}, C_b^\nu) + f_{bcd} N(C_a^\mu, F_d^{\rho\nu}) \\ v_c^\lambda \cdot N(C_a^{\mu\nu}, C_b^\rho) &= f_{bcd} N(C_a^{\mu\nu}, F_d^{\lambda\rho}) \\ u_c \cdot N(C_a^{\mu\nu}, C_{b\rho}) &= f_{acd} N(F_d^{\mu\nu}, C_{b\rho}) - f_{bcd} N(C_a^{\mu\nu}, \tilde{u}_{d,\rho}) \\ u_c \cdot N(C_a^{\mu\nu}, C_b^{\rho\sigma}) &= -f_{acd} N(F_d^{\mu\nu}, C_b^{\rho\sigma}) + f_{bcd} N(C_a^{\mu\nu}, F_d^{\rho\sigma}) \end{aligned} \tag{284}$$

$$\begin{aligned} v_b^\rho \cdot N(F_a^{\mu\nu}, T) &= N(F_a^{\mu\nu}, C_b^\rho) \\ v_b^\sigma \cdot N(F_a^{\mu\nu}, T^\rho) &= -N(F_a^{\mu\nu}, C_b^{\sigma\rho}) \\ u_b \cdot N(F_a^{\mu\nu}, T^\rho) &= -N(F_a^{\mu\nu}, C_b^\rho) \\ u_b \cdot N(F_a^{\mu\nu}, T^{\rho\sigma}) &= 0. \end{aligned} \tag{285}$$

and

$$\begin{aligned}
 v_c^\sigma \cdot N(F_a^{\mu\nu}, C_b^\rho) &= -f_{bcd} N(F_a^{\mu\nu}, F_d^{\rho\sigma}) \\
 u_c \cdot N(F_a^{\mu\nu}, C_b^{\rho\sigma}) &= f_{bcd} N(F_a^{\mu\nu}, F_b^{\rho\sigma})
 \end{aligned}
 \tag{286}$$

Proof: We start with the relation

$$\begin{aligned}
 [v_b^\mu(y), T(C_a^\mu(x_1), T(x_2))] &= \\
 &+ i f_{abc} D_0(y - x_1) T(F_c^{\nu\mu}(x_1), T(x_2)) \\
 &- i f_{abc} \partial_\mu D_0(y - x_1) [\eta^{\mu\rho} T(v_c^\nu(x_1), T(x_2)) - \eta^{\mu\nu} T(v_c^\rho(x_1), T(x_2))] \\
 &+ i D_0(y - x_2) T(C_a^\mu(x_1), C_b^\nu(x_2)) - i \partial_\rho D_0(y - x_1) T(C_a^\mu(x_1), E_b^{\nu\rho}(x_2))
 \end{aligned}
 \tag{287}$$

following from Wick expansion property (22). If we consider finite renormalizations of the type (233), (252) and (272), we will obtain new terms in the left- and right-hand sides of the preceding identity. The identity is preserved *if and only if* we have the first relation from (282) and some relations from (239) and (283).

The rest of the consistency relations go in the same way. \square

7. Finite Renormalizations

We proved in Theorem 12 that the anomalies can be eliminated by adding to the chronological products some quasi-local operators. However, the chronological products have still some arbitrariness: in principle, we can add other quasi-local operators in such a way that gauge invariance in the second order is preserved. This arbitrariness is described by the following.

Theorem 21. *The finite renormalizations are of the type*

$$R(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) N(T^I, T^J)(x_2)
 \tag{288}$$

where the polynomials $N(T^I, T^J)$ verify the symmetry property:

$$N(T^I, T^J) = (-1)^{|I||J|} N(T^J, T^I)
 \tag{289}$$

and

$$gh(N(T^I, T^J)) = |I| + |J|, \quad \omega(N(T^I, T^J)) \leq 4.
 \tag{290}$$

These finite renormalizations do not produce anomalies *if and only if* there exist expressions N^I such that

$$N(T^I, T^J) = N^{IJ}
 \tag{291}$$

and

$$d_Q N^I = i d_\mu N^{I\mu}.
 \tag{292}$$

Proof. The general form of the finite anomaly (288) follows from Bogoliubov axioms (power counting and ghost number assignment). We now want this redefinition of the chronological products not to create new anomalies, i.e., the expression

$$\begin{aligned}
 sR(T^I(x_1), T^J(x_2)) &\equiv d_Q R(T^I(x_1), T^J(x_2)) \\
 &- i \partial_\mu^1 R(T^{I\mu}(x_1), T^J(x_2)) - i (-1)^{|I|} \partial_\mu^2 R(T^I(x_1), T^{J\mu}(x_2))
 \end{aligned}
 \tag{293}$$

is null. We insert (288) and by direct computation, we have

$$sR(T^I(x_1), T^J(x_2)) = \delta(x_1 - x_2) [d_Q N(T^I, T^J) - i (-1)^{|I|} \partial_\mu N(T^I, T^{J^\mu})](x_2) - i \partial_\mu \delta(x_1 - x_2) [N(T^{I^\mu}, T^J) - (-1)^{|I|} N(T^I, T^{J^\mu})](x_2) \tag{294}$$

and this expression is null if and only if

$$d_Q N(T^I, T^J) - i (-1)^{|I|} \partial_\mu N(T^I, T^{J^\mu}) = 0 \tag{295}$$

$$N(T^{I^\mu}, T^J) - (-1)^{|I|} N(T^I, T^{J^\mu}) = 0$$

We define

$$N^I \equiv N(T^\emptyset, T^I) \tag{296}$$

and we can use the second relation (295) to prove (291) by induction. The relation (292) follows immediately from the first relation (295). □

It remains to investigate the solution of Equation (292) in the pure Yang–Mills case, with the restrictions

$$gh(N^I) = |I|, \quad \omega(N^I) \leq 4. \tag{297}$$

In canonical dimension $\omega = 4$ we have the tri-linear solution (45) and one can easily prove that there are no bilinear and quadri-linear solutions. In canonical dimension $\omega = 3$ there is no solution and, finally, in canonical dimension $\omega = 2$ we find the solution

$$N = f_{ab} \left(\frac{1}{2} v_a^\mu v_{b\mu} + u_a \tilde{u}_b \right) \tag{298}$$

$$N^\mu = f_{ab} u_a v_b^\mu$$

with f_{ab} symmetric in $a \leftrightarrow b$. However, we cannot find $N^{\mu\nu}$ such that

$$d_Q N^\mu = i d_\nu N^{\mu\nu} \tag{299}$$

so we remain only with the tri-linear solution (45). It follows that we have only one constant arbitration in the second order of the perturbation theory. In fact, this result remains true in all orders of the perturbation theory.

8. Conclusions

The analysis from this paper can be extended to gravity [31]. In further papers, we will extend the Hopf structure and the Wick expansion property to the case of the general Yang–Mills model (including massive vector Bosons and Dirac fields). An interesting point would be to see if our Hopf version of the Wick expansion property is connected with the Feynman graph version.

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