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Article

New Classes of Solutions for Euclidean Scalar Field Theories

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Abstract: This paper presents new classes of exact radial solutions to the nonlinear ordinary differential equation that arises as a saddle-point condition for a Euclidean scalar field theory in D-dimensional spacetime. These solutions are found by exploiting the dimensional consistency of the radial differential equation for a single *massless* scalar field, which allows it to transform into an autonomous equation. For massive theories, the radial equation is not exactly solvable, but the massless solutions provide useful approximations to the results for the massive case. The solutions presented here depend on the power of the interaction and on the spatial dimension, both of which may be noninteger. Scalar equations arising in the study of conformal invariance fit into this framework, and classes of new solutions are found. These solutions exhibit two distinct behaviors as $D \to 2$ from above.

Keywords: quantum field theory; path-integral; vacuum decay; bounce; instantons; scale invariance; conformal invariance

1. Introduction

Tunneling is a process allowed in quantum physics but not in classical physics. Tunneling amplitudes in quantum mechanics can be calculated approximately by using WKB techniques [1–5]. In quantum field theory, the transition between a metastable (false) vacuum state and the true ground state (vacuum) is a tunneling process. This process can be studied in field theory in the spacetime dimension D by using the path-integral representation of the Euclidean partition function $\mathcal{Z}$. Here, we consider a field theory for a scalar field ϕ for which $\mathcal{Z}$ is given by the path-integral

$$\mathcal{Z} = N \int \mathcal{D}\phi \, \exp(-S[\phi]/\hbar). \tag{1}$$

In the semiclassical limit [6–9], $\mathcal{Z}$ is approximated by calculating fluctuations around the classical solution ϕ_c of the equation of motion $\frac{\delta S[\phi]}{\delta \phi(x)} = 0$ for the scalar field ϕ [10]. The usual form of $S[\phi]$ is

$$S[\phi] = \int d^4x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \tfrac{g}{4}\phi^4\right] \tag{2}$$

for which the equation of motion is

$$\partial^2\phi - m^2\phi - g\phi^3 = 0.$$

For the massless case $m^2 = 0$, there is a well-known explicit solution to this equation known as the *bounce* [6,11–15], but, for nonzero m^2, there are no known explicit solutions (for $D \neq 1$). This paper presents new classes of solutions for the case in which the Euclidean spacetime dimension D is noninteger and the power of the scalar interaction may differ from four. Generalizing the power of the interaction is natural in the context of the δ-expansion [16–18] approach to quantum field theory and in the field of

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$\mathcal{PT}$-symmetric quantum field theory [19–26]. Thus, instead of action (2), we consider the more general action

$$\mathcal{S}[\phi] = \int d^D x \left[\tfrac{1}{2}(\partial_\mu \phi)^2 + \tfrac{1}{2}m^2\phi^2 + \lambda\phi^{2+\epsilon}\right],$$

where ϵ is a real parameter. The equation of motion for this more general action is

$$\partial^2\phi - m^2\phi - (2+\epsilon)\lambda\phi^{1+\epsilon} = 0. \tag{3}$$

This nonlinear *partial* differential equation cannot be solved in closed form. However, in [27], the general partial differential equation

$$\partial^2\phi - U'(\phi) = 0 \tag{4}$$

is considered in D-dimensional Euclidean space. For the class of real admissible functions, U is assumed to have the following four properties:

1. $U(\phi)$ is continuously differentiable;
2. $U(0) = U'(0) = 0$;
3. $U(\phi)$ is not positive definite;
4. $U(\phi) - a|\phi|^\alpha + b|\phi|^\beta \geq 0$, where a, α, b, and β are positive and $\alpha < \beta < 2D/(D-2)$.

Apart from the trivial solution $\phi = 0$, (4) has at least one monotone spherically symmetric solution that vanishes at infinity. If the conditions above are satisfied, the action for such a solution is *less than that for any solution that is not spherically symmetric and monotone*.

Consequently, we replace the differential Equation (3) with the spherically symmetric *ordinary* differential equation

$$\phi''(r) + \tfrac{D-1}{r}\phi'(r) - m^2\phi(r) - (2+\epsilon)\lambda\phi^{1+\epsilon}(r) = 0. \tag{5}$$

In this paper, we apply techniques taken from the theory of ordinary differential equations to solve the massless version of the radial equation of motion (5) and discover infinite numbers of new solutions. We discuss the limiting behavior of two classes of these solutions as D approaches two from above. We clarify the role of scale invariance in restricting the class of solutions with finite action. We also show how the notions of boundary layer theory can be applied to justify the use of these massless solutions to approximate the contributions of massive solutions.

2. The Role of Scale Invariance

Consider the Euclidean action $\mathcal{S}_m[\phi] = \int d^D x\, \mathcal{L}_m[\phi(x)]$ derived from the Lagrangian

$$\mathcal{L}_m(\phi) = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \tfrac{1}{2}m^2\phi^2 - \lambda m^{2+\epsilon-\epsilon D/2}\phi^{2+\epsilon}, \tag{6}$$

where λ is a dimensionless coupling constant. (In $\mathcal{PT}$-symmetric models λ may be complex [20].) Scale invariance requires that a solution $\phi_c(x)$ of the equations of motion satisfies

$$\begin{aligned} 0 &= \frac{d}{d\eta} S[\phi_c/\eta]\Big|_{\eta=1} \\ &= -\int d^D x \left([\partial_\mu\phi_c(x)]^2 + m^2\phi_c^2(x)\right) + (2+\epsilon)\lambda m^{2+\epsilon-\epsilon D/2}\int d^D x\, \phi_c^{2+\epsilon}(x). \end{aligned} \tag{7}$$

Hence,

$$(2+\epsilon)S[\phi_c] = \tfrac{1}{2}\epsilon \int d^D x\, \left[\partial_\mu\phi_c(x)\right]^2 + \tfrac{1}{2}\epsilon\, m^2 \int d^D x\, [\phi_c(x)]^2\,. \tag{8}$$

Similarly, $\frac{d}{d\eta}S[\phi_c(\eta x)]\big|_{\eta=1} = 0$, which implies that

$$S[\phi_c] = \frac{1}{D}\int d^D x\,\left[\partial_\mu \phi_c(x)\right]^2 > 0. \tag{9}$$

Thus, assuming that $S[\phi_c]$ is finite, we deduce from (8) and (9) that

$$[(-\tfrac{1}{2}D+1)\epsilon + 2]S[\phi_c] = \tfrac{1}{2}\epsilon m^2 \int d^D x\,[\phi_c(x)]^2.$$

We conclude that, for $\epsilon = \frac{4}{D-2}$, there are only solutions with $m = 0$.

3. The Role of Mass

Explicit massive solutions cannot be found for $D \neq 1$. We redefine the field variable as $\phi_c(x) = \alpha\,\varphi(mx)$, where $\alpha = m^{D/2-1}(2+\epsilon)^{-1/\epsilon}\lambda^{-1/\epsilon}$. In terms of this new field, the classical equation of motion is

$$-\varphi''(mx) + \varphi(mx) - \varphi^{1+\epsilon}(mx) = 0.$$

As noted earlier, the smallest-action solution is radially symmetric in the variable $r = |mx - mx_0|$, where x_0 is an arbitrary point in Euclidean spacetime. The equation of motion now reads

$$-\varphi''(r) - \tfrac{D-1}{r}\varphi'(r) + \varphi(r) - \varphi^{1+\epsilon}(r) = 0.$$

This equation does not have solutions that obey the boundary conditions for a bounce [6,28], but a solution may have a bounce-like core. Indeed, for an $\epsilon > 0$ and a small ϕ, and using *dominant balance*, we can solve the equation

$$-\varphi''(r) - \tfrac{D-1}{r}\varphi'(r) + \varphi(r) = 0 \tag{10}$$

for a large r, which has the independent solutions $r^{(2-D)/2}I_{(D-2)/2}(r)$ and $r^{(2-D)/2}K_{(D-2)/2}(r)$, where $I_{(D-2)/2}(r)$ and $K_{(D-2)/2}(r)$ are modified Bessel functions of the first and second kind [1]. We are interested in finite actions, so we choose $K_{(D-2)/2}(r)$ since it falls off exponentially with r; the function $I_{(D-2)/2}(r)$ increases exponentially with r and will not be considered. This large-r *outer* solution can be matched [1] with the small-r *inner* solution, which we obtain in the massless approximation used in the next section. The scale of the core of this solution in x is of the order $1/m$. For a small r, the $\frac{D-1}{r}\frac{d}{dr}$ term in (10) is large compared to the terms of the order one. Ignoring these smaller terms, we obtain the massless equation considered in the remainder of this paper.[1]

The concepts used above originate in boundary layer analysis [1]. We treat the central core of a soliton as the *outer regions* and the edges of the soliton as the boundary layer or *inner regions*. In the inner regions, the soliton varies rapidly and decays exponentially to zero on a scale determined by the mass term. In the core regions, the soliton is slowly varying, and this regions makes the principal contribution to the action; in this core regions, the mass term in the Lagrangian is neglected. For massless classical solutions, there are quantum fluctuations that radiatively generate masses [6,29–31]. Consequently, massive solutions may not have a dominant role in the physics of tunneling.

4. Solutions of the Massless Equation

The massless field theory for a scalar field $\phi(x)$, where $x \in \mathbb{R}^D$, is given by the Lagrangian

$$\mathcal{L}(\phi) = \tfrac{1}{2}\partial_\mu\phi\partial^\mu\phi + \lambda\phi^{2+\epsilon}. \tag{11}$$

The semiclassical evaluation of the path-integral

$$Z = \int D\phi \, \exp\left[-\int d^D x \, \mathcal{L}(\phi)\right] \tag{12}$$

requires the solution of the saddle-point equation

$$\partial_\mu \partial^\mu \phi - (2+\epsilon)\lambda\phi^{1+\epsilon} = 0. \tag{13}$$

(The cases $\epsilon = -1, -2$ are trivial and will not be considered.)

This equation is invariant under a translation of x by x_0; in terms of $r = |x - x_0|$, the radially symmetric solution of (13) is found by solving

$$\phi''(r) + \tfrac{d}{r}\phi'(r) - \lambda(2+\epsilon)\phi^{1+\epsilon} = 0, \tag{14}$$

where $d = D - 1$. Ordinarily, the boundary conditions used in studies of vacuum decay [6] are $\frac{d\phi}{dr}\big|_{r=0} = 0$ and $\lim_{r\to\infty} \phi(r) \to \phi_0$.

Note that (14) is *dimensionally consistent*; that is, it is invariant under a scale change in r [1]. For dimensional consistency, we make a change of variable from $\phi(r)$ to $f(r)$: $\phi(r) = f(r)r^{-2/\epsilon}$. This leads to an *equidimensional* equation (an equation that is invariant under $r \to \alpha r$) [1]:

$$r^2 f''(r) + (d - \tfrac{4}{\epsilon}) r f'(r) + \tfrac{2}{\epsilon^2}[2 - (d-1)\epsilon] f(r) - \lambda(2+\epsilon) f^{1+\epsilon}(r) = 0. \tag{15}$$

We make the further change of variable $r = e^t$, which is equivalent to substituting $r\frac{d}{dr} = \frac{d}{dt}$. This gives a second-order *autonomous* equation in which the independent variable does not appear explicitly:

$$\epsilon^2(\tfrac{d^2}{dt^2} - \tfrac{d}{dt})u(t) + \epsilon(d\epsilon - 4)\tfrac{d}{dt}u(t) + 2[2 - (d-1)\epsilon]u(t) - \lambda\epsilon^2(2+\epsilon)u^{1+\epsilon}(t) = 0, \tag{16}$$

where $u(t) \equiv f(r)$. Equation (16) is an autonomous equation in t. This suggests substituting a new function $F(u)$ such that

$$u'(t) = F(u), \tag{17}$$

which reduces the order of the differential equation by one [1].

In terms of $F(u)$, we rewrite (16) as

$$F(u)F'(u) + (d - \tfrac{4}{\epsilon} - 1)F(u) + \tfrac{2}{\epsilon^2}[2 - (d-1)\epsilon]u - \lambda(2+\epsilon)u^{1+\epsilon} = 0, \tag{18}$$

which is a *first-order* equation.

5. Classes of Instanton Solutions

5.1. Solutions with Nonpolynomial $F(u)$

If $d = \frac{4}{\epsilon} + 1$, (18) is easy to solve:

$$F(u) = \pm\sqrt{2}(2\epsilon^{-2}u^2 + \lambda u^{\epsilon+2} + c)^{1/2}, \tag{19}$$

where c is a constant. This leads to

$$(2\epsilon^{-2}u^2 + \lambda u^{\epsilon+2} + c)^{-1/2} du = \pm 2^{1/2} dt. \tag{20}$$

It is easiest to integrate (20) for $c = 0$; if $c \neq 0$, one can represent the solution in terms of elliptic functions. For $c = 0$, we find the class of solutions

$$\phi(r) = a(r^2 + \rho^2)^{-2/\epsilon}, \tag{21}$$

where ρ is a scale that arises from a constant of integration and $a \equiv \left[(2\rho^2)/(\lambda\epsilon^2)\right]^{1/\epsilon}$. For $\epsilon = 2$, we recover the Fubini–Lipatov instanton [11,12], which has a finite action. For the general case, we also have finite action, which is proportional to $B = B_1 + B_2$, where

$$B_1 = \frac{1}{2}\int_0^\infty dr\, r^d\, (\partial_r\phi)^2, \qquad B_2 = \lambda\int_0^\infty dr\, r^d\, \phi^{2+\epsilon}. \tag{22}$$

The indefinite integrals $\mathcal{I}_1(r)$ and $\mathcal{I}_2(r)$ for B_1 and B_2 are given in terms of hypergeometric functions:

$$\mathcal{I}_1(r) = \frac{\epsilon}{4(1+\epsilon)\rho^{4+8/\epsilon}} r^{4(1+1/\epsilon)}\, {}_2F_1\left(2+\frac{2}{\epsilon}, 2+\frac{4}{\epsilon}, 3+\frac{2}{\epsilon}, -\frac{r^2}{\rho^2}\right)$$

and

$$\mathcal{I}_2(r) = \frac{\epsilon a^{2+\epsilon}}{(4+2\epsilon)\rho^{4+8/\epsilon}} r^{2+\frac{4}{\epsilon}}\, {}_2F_1\left(2+\frac{4}{\epsilon}, \frac{2+\epsilon}{\epsilon}, 2+\frac{2}{\epsilon}, -\frac{r^2}{\rho^2}\right).$$

Both $\mathcal{I}_1(r)$ and $\mathcal{I}_2(r)$ are finite at $r = 0$ and $r = \infty$, so these generalized instantons have finite action. In a $\mathcal{PT}$-symmetric theory $\lambda = gi^\epsilon$, where $g > 0$, ϕ is complex in general. For $\epsilon = 4$, $d = 2$ ($D = 3$). For $\epsilon = 2$, $d = 3$ ($D = 4$); the solution is real for a positive λ, and it is the well-known bounce solution studied in false vacuum decay [6]. The bounce solution for $\epsilon = 4$, $d = 2$ is not typically discussed and, to the best of our knowledge, is new. The general solution for $\epsilon = \frac{4}{d-1}$ is new. Here, d may be noninteger.

5.2. Solutions with Polynomial $F(u)$

If the linear term in F in (18) is present, the method above does not apply. However, a different method of solution can be devised for a special family of d. This is an entirely new family of solutions that can be considered as inner solutions for solitons.

The case $\epsilon = 2$ in (18) leads to

$$F(u)F'(u) + (d-3)F(u) + (2-d)u - 4\lambda u^3 = 0. \tag{23}$$

We substitute $F(u) = a_0 + a_1 u + a_2 u^2$ into (23).[2] Matching powers of u, we obtain four simultaneous equations:

$$\begin{aligned} a_0 a_1 + (d-3)a_0 &= 0, \\ a_1^2 + 2a_0 a_2 + (d-3)a_1 + 2 - d &= 0, \\ 3a_1 a_2 + (d-3)a_2 &= 0, \\ a_2^2 - 2\lambda &= 0. \end{aligned} \tag{24}$$

The key feature of these equations is that the number of equations exceeds the number of variables. (Note that $d = 3$ is a special case discussed in Appendix A, so we consider $d \neq 3$.) Upon solving (24) in sequence, we obtain two solutions:

$$d = 0 \text{ with } a_0 = 0,\ a_1 = 1,\ a_2 = \pm\sqrt{2\lambda};$$

$$d = 3/2 \text{ with } a_0 = 0,\ a_1 = 1/2,\ a_2 = \pm\sqrt{2\lambda}.$$

The case $\epsilon = 4$ in (18) leads to

$$F(u)F'(u) + (d-2)F(u) - 6\lambda u^5 + \tfrac{1}{4}(3-2d)u = 0. \tag{25}$$

We substitute $F(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3$ into (25). Then, matching powers of u, we obtain six equations in four unknowns:

$$\begin{aligned} (d-2+a_1)a_0 &= 0, \\ \tfrac{1}{4}(3-2d) - (2-d)a_1 + a_1^2 + 2a_0 a_2 &= 0, \\ (d+3a_1-2)a_2 + 3a_0 a_3 &= 0, \\ (d+4a_1-2)a_3 + 2a_2^2 &= 0, \\ 5a_2 a_3 &= 0, \\ a_3^2 - 2\lambda &= 0. \end{aligned} \tag{26}$$

The solution to these equations is

$$a_0 = 0, \quad a_1 = \tfrac{1}{2} - \tfrac{d}{4}, \quad a_2 = 0, \quad a_3 = \pm\sqrt{2\lambda}.$$

From the second of these equations, we deduce that either $d = 0$ or $d = \frac{4}{3}$; that is, $a_1 = \frac{1}{2}$ or $a_1 = \frac{1}{6}$. (We consider the cases $\epsilon = 6, 8$ in the Appendix A.) In these examples, there are always $\epsilon + 2$ equations and $\frac{\epsilon}{2} + 2$ unknowns, so there are always *more equations than unknowns*. It is remarkable that solutions to these coupled quadratic equations exist. This is because these systems of equations are actually *not independent*. Perhaps this is because these equations are related to the existence of an increasing set of conservation laws in the theory of solitons as $d \to 1$ or $d = 0$ [32].[3] Validating this conjecture would be difficult and is beyond the scope of this work [34,35].

The following pattern emerges and has been verified: For $\epsilon = 2n$ $(n = 1, 2, 3, \ldots)$, the solutions to (18) require that either $d = 0$ or $d = \frac{n+2}{n+1}$. For $d = \frac{n+2}{n+1} = \frac{\epsilon+4}{\epsilon+2}$, we have

$$F(u) = a_1 u + a_{n+1} u^{n+1}, \tag{27}$$

where $a_1 = \frac{1}{n(n+1)}$ and $a_{n+1} = \pm\sqrt{2\lambda}$. For $d = 0$, we have $a_1 = \frac{1}{n}$ and $a_{n+1} = \pm\sqrt{2\lambda}$. From (17), for $\epsilon = 2n$, we have

$$\frac{du}{dt} = \begin{cases} \frac{1}{n} u \pm \sqrt{2\lambda} u^{n+1} & (d = 0), \\ \frac{1}{n(n+1)} u \pm \sqrt{2\lambda} u^{n+1} & [d = (n+2)/(n+1)]. \end{cases} \tag{28}$$

To summarize, for $d = 0$, a family of solutions is

$$u = \left(\frac{\alpha \exp(t)}{1 + \sqrt{2\lambda}\,\alpha n \exp(t)} \right)^{1/n}, \tag{29}$$

where the constant $\alpha > 0$; the corresponding ϕ is given by

$$\phi = \left(\frac{\alpha}{1 + \sqrt{2\lambda}\alpha n \exp(t)} \right)^{1/n}. \tag{30}$$

This has a finite action, as we see by evaluating B_1 and B_2 in (22). A corresponding set of solutions arises if the negative sign is taken in (28). For $d = \frac{n+2}{n+1}$, a family of solutions is

$$\phi = \left(\frac{\alpha \exp\left(-\frac{nt}{n+1}\right)}{1 + \sqrt{2\lambda}\, n(n+1)\alpha \exp\left(\frac{t}{n+1}\right)} \right)^{1/n}, \tag{31}$$

whose action is not finite. It is possible that such solutions could play the role of an outer regions in a boundary layer analysis, as mentioned earlier. Note that $d = 0$ corresponds to $D = 1$ [32] and (11) resembles quantum mechanics with a power potential. If $n \to \infty$, this potential becomes a square well. The solutions above may have implications for strong coupling behavior in quantum mechanics in unstable or $\mathcal{PT}$-symmetric potentials.

Since $d = \frac{n+2}{n+1}$ corresponds to $D = 2 + \frac{1}{n+1}$, D approaches two as $n \to \infty$. As discussed below, such solutions for a large n are distinct from those leading to the singular limit $D = 2$ in the context of conformal field theory.

5.3. Conformally Invariant Equations

The dilatation operator involved in scale invariance is part of the conformal group, which is infinitely dimensional at $D = 2$. It is of interest to examine the approach to $D = 2$ from above in terms of our solutions where D is noninteger. We first consider the formulation of the action in terms of conformal invariance [36,37] and show that there are two different families of solutions as we approach $D = 2$ from above. The conformally invariant action [38] is

$$S = \int d^D x \left(\tfrac{1}{2} \partial_\mu \phi \partial^\mu \phi - \lambda \phi^{2+\frac{4}{D-2}} \right) \tag{32}$$

for $D \neq 2$. For $D = 4$, we have a ϕ^4 interaction, which is classically conformally invariant [36,39]. We are interested in solutions to (14) as $D \to 2$ from above. The classical Lagrangian is conformally invariant in the dimension $D \neq 2$. The scaling dimension [40,41] of ϕ is $\frac{1}{2}(D-2)$ and S in (32) is invariant under the infinitesimal transformation $\phi \to \phi + \delta\phi$, where

$$\delta\phi = v^\alpha \partial_\alpha \phi + \tfrac{D-2}{2D} \phi \partial_\alpha v^\alpha; \tag{33}$$

v^α is the conformal Killing vector in flat space [39]. We use our earlier methods for solving the radial form of the equation of motion for the action (32).

In our formalism, (32) represents a theory with $\epsilon = \frac{4}{D-2}$, so, for the *conformally invariant* equation

$$F(u)F'(u) - \tfrac{1}{4}(D-2)^2 u + \tfrac{2\lambda D}{D-2} u^{\frac{D+2}{D-2}} = 0. \tag{34}$$

This fits into our framework of nonpoynomial solutions. However, we find additional solutions not covered by this case as D approaches two. For $\epsilon = 2n$, we have $D = 2 + \frac{2}{n}$ and, using the methods outlined above, we find that

$$u(t) = \left[2\lambda n^2 \sinh^2(t+\beta)\right]^{-1/(2n)}, \tag{35}$$

where β is a constant of integration and $\lambda > 0$.

We have thus found two new families of solutions as D approaches two from above. One is through conformally invariant solutions, and the other is through solutions that can be real or complex depending on the sign of λ and that are not conformally invariant. A discussion of existence theorems of solutions for the Euler–Lagrange equation for (32) is given in [42] and related works [43,44].

6. Conclusions

We have found many novel solutions to the saddle-point equations for scalar field theories. The possibilities for fractional interactions allow for new deformations of Hermitian field theories that could be relevant for nonpolynomial field theory [16] and also for $\mathcal{PT}$-symmetric field theory [19,20,22–25]. Furthermore, we have produced two different limiting procedures toward $D = 2$. In principle, our solutions allow for semiclassical evaluation of path-integrals for unconventional field theories where canonical methods are cumbersome or impossible.

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Appendix A

In this Appendix, we briefly describe two more examples of exact instanton solutions. These examples are an additional illustration that the result of (27) holds.[4] The case $\epsilon = 6$ in (18) leads to

$$F(u)F'(u) + \left(d - \tfrac{5}{3}\right)F(u) - 8\lambda u^7 - \tfrac{1}{9}(3d-4)u = 0. \tag{A1}$$

We substitute $F(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4$ into (A1), match powers of u, and obtain eight equations in five unknowns:

$$\begin{aligned} a_0 a_1 + \left(d - \tfrac{5}{3}\right)a_0 &= 0, \\ a_1^2 + 2a_0 a_2 + \left(d - \tfrac{5}{3}\right)a_1 + \tfrac{1}{9}(4-3d) &= 0, \\ 3a_0 a_3 + 3a_1 a_2 + \left(d - \tfrac{5}{3}\right)a_2 &= 0, \\ 4a_0 a_4 + \left(d - \tfrac{5}{3}\right)a_3 + 4a_1 a_3 + 2a_2^2 &= 0, \\ \left(d - \tfrac{5}{3}\right)a_4 + 5a_1 a_4 + 5a_2 a_3 &= 0, \\ 2a_2 a_4 + a_3^2 &= 0, \\ a_3 a_4 &= 0, \\ a_4^2 - 2\lambda &= 0. \end{aligned} \tag{A2}$$

Solving (A2) in *reverse* order, we obtain $a_4 = \pm\sqrt{2\lambda}$, $a_2 = a_3 = a_0 = 0$, $a_1 = \left(\frac{1}{3} - \frac{d}{5}\right)$. From the second equation of (A2), we deduce that either $d = 0$ or $d = 5/4$.

The case $\epsilon = 8$ in (18) leads to

$$F(u)F'(u) + \left(d - \tfrac{3}{2}\right)F(u) - 10\lambda u^9 + \tfrac{1}{16}(5-4d)u = 0. \tag{A3}$$

We substitute $F(u) = a_0 + a_1 u + a_2 u^2 + a_3 u^3 + a_4 u^4 + a_5 u^5$ into (A3) to obtain ten equations in six unknowns:

$$\begin{aligned} \left(a_1 + d - \tfrac{3}{2}\right)a_0 &= 0, \\ 5 - 4d - 24a_1 + 16a_1^2 + 32a_0 a_2 + 16d a_1 &= 0, \\ a_1 a_2 + a_0 a_3 + \tfrac{1}{6}(2d-3)a_2 &= 0, \\ a_2^2 + 2a_1 a_3 + 2a_0 a_4 + \tfrac{1}{4}(2d-3)a_3 &= 0, \\ a_2 a_3 + a_1 a_4 + a_0 a_5 + \tfrac{1}{10}(2d-3)a_4 &= 0, \\ a_3^2 + 2a_2 a_4 + 2a_1 a_5 + \tfrac{1}{6}(2d-3)a_5 &= 0, \\ a_3 a_4 + a_2 a_5 &= 0, \\ a_4^2 + 2a_3 a_5 &= 0, \\ a_4 a_5 &= 0, \\ a_5^2 - 2\lambda &= 0. \end{aligned}$$

The solution to these equations follows the pattern of the solutions for $\epsilon = 2, 4, 6$ above:

$$a_0 = a_2 = a_3 = a_4 = 0, \quad a_5 = \pm\sqrt{2\lambda}, \quad a_1 = \frac{1}{4} - \frac{d}{6}.$$

Hence, either $d = 0$ or $d = 6/5$.

Notes

1 It was shown in [27] that massive solutions exist in our case for $D < D_c = (4 + 2\epsilon)/\epsilon$.

2 No new solutions are obtained by generalizing to Laurent polynomials. If F is a polynomial of degree p, the term $F(u)F'(u)$, which is of order $2p - 1$, needs to be $\epsilon + 1$ for F to satisfy (18).

3 A conformal field theory has an infinite number of conservation laws at $d = 1$ [33].

4 One should be aware of the 1-2-3-infinity fallacy in which one makes a general conclusion based on a few examples. A nice example of small-n behavior giving an incorrect answer for a general n is provided by the integral $\int_0^\infty dx\, f_n(x)$, where $f_n(x) = \prod_{k=1}^n \sin\left(\frac{x}{2k-1}\right) \Big/ \frac{x}{2k-1}$. The integrals $\int_0^\infty dx\, f_n(x) = \frac{\pi}{2}$ for $n \leq 7$, but this relation fails at $n = 8$. However, we have carefully checked that there are no such problems with the solutions presented in this paper.

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