SLAC-PUB-2287
March 1979
(T)

INSTABILITY OF ABELIAN FIELD CONFIGURATIONS IN YANG-MILLS THEORY*

P. Sikivie
Stanford Linear Accelerator Center
Stanford University, Stanford, California 94305

Abstract

It is shown that static purely electric or purely magnetic abelian field configurations in Yang-Mills theory are unstable when the electric or magnetic field strength is too large over too wide a region. The critical parameter is gEL^2 (gBL^2) where E (B) is a measure of the field strength, and L is a measure of the distance over which such field strengths extend.

Submitted to Physical Review D

* Work supported by the Department of Energy under contract number EY-76-C-03-0515.

I. Introduction and Summary

It was shown some time ago by Mandula[1] that the Coulomb solution to the Yang-Mills field equations in the presence of a static external source is unstable when $\frac{gQ}{4\pi}$ exceeds some critical value. Here g is the gauge coupling constant and Q is the total external charge. The method used in such a stability analysis is simply to linearize the Yang-Mills equations with respect to the small oscillations around the Coulomb solution and to see whether these small oscillations do or do not have eigenmodes which grow exponentially in time.

In this paper, it is our purpose to show that the instability of abelian field configurations in Yang-Mills theory is quite a general phenomenon. We call a field configuration abelian if it is lined up everywhere in the same direction of isospin, or if it can be lined up in this way by a gauge transformation. In Sections II and III, we will consider respectively purely electric and purely magnetic static abelian field configurations and show that they are unstable under small perturbations when gEL^2 (or gBL^2 in the magnetic case) exceeds some critical value which depends on the shape of the configuration. Here E (B) is a measure of the electric (magnetic) field and L is a length characterizing the volume over which such a electric (magnetic) field extends and in which the small perturbations are applied. The instability of these field configurations arises because gluons can be "pulled out of the vacuum" which screen the initial electric or magnetic field by creating opposing color charge and color current densities.

The instability of the Coulomb solution is a particular case of this general phenomenon. One should note in this regard that the critical parameter gEL^2, in the case of the Coulomb solution, is equivalent to the critical parameter $\frac{gQ}{4\pi}$, since $E = \frac{Q}{4\pi}\frac{1}{r^2}$ for the Coulomb solution. At any rate, the critical parameter is always the one dimensionless parameter available. Let us recall that g has dimension of $(\text{action})^{-\frac{1}{2}}$, Q has dimension of $(\text{action})^{\frac{1}{2}}$ and E has dimension of $(\text{action})^{\frac{1}{2}}\,(\text{length})^{-2}$.

We study the stability of Yang-Mills field configurations independently of the sources that create these configurations, by choosing to consider small oscillations only in the region of space from which the sources are absent. In particular, we make it clear that the instability Mandula found for the Coulomb solution, exists independently of the way the external source is treated.

II. Stability Analysis of Abelian Electric Field Configurations

Let us consider an abelian static electric field configuration which satisfies the Yang-Mills equations in a region R free of external sources:

$$\vec{E}^a = -\delta^{a3}\vec{\nabla} A_o(\vec{x}) \qquad , \qquad \nabla^2 A_o = 0 \quad \text{in R} \tag{2.1}$$

(Here, as in the rest of the paper, we have taken for simplicity the gauge group to be SU(2). The generalization to larger gauge groups is a straightforward.) We will show that if such an electric field is too strong over too wide a region, it becomes unstable under small perturbations. Let us thus consider small time-dependent deviations from our background field configuration:

$$A^a_o = \delta^{a3} A_o(\vec{x}) \qquad , \qquad A^a_i(x) = a^a_i(\vec{x},t) \tag{2.2}$$

where the $a^a_i(\vec{x},t)$ vanish on the boundary of the space region R and are small compared to A_o and to $\frac{1}{gA_o}\partial_i A_o$. We have chosen to work in the gauge where the oscillations in the A_o field vanish. Neglecting terms which are quadratic or higher in the a_i^a fields, we obtain for the field strengths:

$$E_i^a = F_{oi}^a = -\delta^{a3}\partial_i A_o + \dot{a}_i^a - g\varepsilon^{3ab} A_o a_i^b \qquad (2.3)$$

$$B_{ij}^a = F_{ij}^a = \partial_i a_j^a - \partial_j a_i^a + o(a)^2 \qquad (2.4)$$

We use the dot over a field to indicate that the time derivative has been taken. The linearized Yang-Mills equations are:

$$(D_\mu F^{\mu o})^a = (D_i E_i)^a$$

$$= \vec{\nabla}\cdot\dot{\vec{a}}^a - g\varepsilon^{3ab}[A_o(\vec{\nabla}\cdot\dot{\vec{a}}^b) + 2\vec{a}^b\cdot\vec{\nabla}A_o] = 0 \qquad (2.5a)$$

$$(D_\mu F^{\mu j})^a = -(D_o E_j)^a + (D_i B_{ij})^a$$

$$= -\ddot{a}_j^a + \partial_i\partial_i a_j^a - \partial_j\partial_i a_i^a$$

$$+ 2g\varepsilon^{3ab}A_o\dot{a}_j^b - g^2(\delta_3^a\delta_b^3 - \delta_b^a)(A_o)^2 a_j^b = 0 \qquad (2.5b)$$

The oscillations along the 3d direction of isospin just obey the free Maxwell equations:

$$\vec{\nabla}\cdot\dot{\vec{a}}^3 = 0$$

$$-\vec{\nabla}\times(\vec{\nabla}\times\vec{a}^3) - \ddot{\vec{a}}^3 = 0 \qquad (2.6)$$

These oscillations do not see the background field and do not produce any instabilities. This is of course identical to what happens in pure electromagnetism. On the other hand, the oscillations along the 1st and

2d directions of isospin do see the background field. It is useful to introduce the complex field $\vec{a} = \vec{a}^1 + i\vec{a}^2$, in terms of which Eqs. (2.5) for $a = 1$ and 2, become:

$$\vec{\nabla} \cdot \dot{\vec{a}} + ig[A_o(\vec{\nabla} \cdot \vec{a}) + 2\vec{a} \cdot \vec{\nabla} A_o] = 0 \tag{2.7a}$$

$$-\ddot{\vec{a}} - \vec{\nabla} \times (\vec{\nabla} \times a) - 2ig\, A_o \dot{\vec{a}} + (gA_o)^2 \vec{a} = 0 \tag{2.7b}$$

or, more compactly:

$$2\vec{\nabla} \cdot (D_o \vec{a}) - D_o(\vec{\nabla} \cdot \vec{a}) = 0 \tag{2.8a}$$

$$D_o D_o \vec{a} + \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) = 0 \tag{2.8b}$$

with $D_o \vec{a} = \dot{\vec{a}} + igA_o \vec{a}$. For an eigenmode of frequency ω, $\vec{a}(\vec{x},t) = \vec{a}(\vec{x})\, e^{i\omega t}$, we have:

$$(\omega + gA_o)\, \vec{\nabla} \cdot \vec{a} + 2g\vec{a} \cdot \vec{\nabla} A_o = 0 \tag{2.9a}$$

$$\omega^2 \vec{a} = \vec{\nabla} \times (\vec{\nabla} \times \vec{a}) - 2gA_o \omega \vec{a} - (gA_o)^2 \vec{a} \tag{2.9b}$$

Gauss' law, Eq. (2.9a), is now superfluous since it follows from taking the divergence of the equation of motion (2.9b).

In Coulomb gauge, $\vec{\nabla} \cdot \vec{a} = 0$, we have:

$$\omega^2 \vec{a} = -\nabla^2 \vec{a} - 2gA_o \omega \vec{a} - (gA_o)^2 \vec{a} \tag{2.10}$$

The first term on the RHS of Eq. (2.10) is positive definite but becomes relatively unimportant when the size of the region R, in which the small oscillations are allowed, is large. The last term on the RHS of Eq. (2.10) is negative definite and proportional to the square of the background E field. Consequently, Eq. (2.10) will in general have runaway solutions -- i.e., eigenmodes with imaginary ω -- when the background E field is large enough over wide enough regions.

As an illustrative example, we will consider a constant electric field in a cylindrical box of radius R and length 2L:

$$A_o(\rho,\phi,z) = -Ez \quad \text{for} \quad \rho < R \quad \text{and} \quad -L < z < +L$$

where ρ, ϕ and z are the usual cylindrical coordinates. Equation (2.10) becomes in this case:

$$\omega^2 \vec{a} = -\nabla^2 \vec{a} + 2g\, Ez\omega\, \vec{a} - (gEz)^2 \vec{a} \tag{2.11}$$

Let us use the ansatz:

$$\vec{a} = \hat{\phi}\, \chi(\rho)\, \psi(z) \tag{2.12}$$

$\chi(\rho)$ must then satisfy the Bessel equation of index one, and vanish at $\rho = R$. Thus

$$\chi_n(\rho) = J_1(k_n\rho) \qquad n = 1,2,3,\ldots \tag{2.13}$$

where $k_n = \frac{\beta_n}{R}$ and the β_n are the zeros of J_1 ($\beta_1 = 3.831$, $\beta_2 = 7.002$, $\beta_3 = 10.174,\ldots$). For a given n, $\psi(z)$ must be solution of:

$$[\omega^2 - (k_n)^2]\, \psi(z) = \left[-\frac{\partial^2}{\partial z^2} + 2gE\omega z - (gEz)^2 \right] \psi(z) \tag{2.14}$$

with the boundary condition: $\psi(z = \pm L) = 0$. It is useful to redefine: $u = \frac{z}{L}$, $\nu = \omega L$ and $\kappa_n = k_n L$, in terms of which Eq. (2.14) becomes:

$$[\nu^2 - (\kappa_n)^2]\, \psi(u) = \left[-\frac{\partial^2}{\partial u^2} + 2p\nu u - p^2u^2 \right] \psi(u) \tag{2.15}$$

where $p = gEL^2$ and with the boundary condition: $\psi(u = \pm 1) = 0$. When the parameter p increases beyond a certain critical value P_{crit}, solutions to Eq. (2.15) with negative ν^2 appear; these are unstable modes. Of course, the $n=1$ modes will become unstable first. P_{crit} is the value of p for which Eq. (2.15) with $n=1$, has its first eigenmode with $\nu=0$. Thus P_{crit} is determined by the condition that the solution of:

$$-(\kappa_1)^2 \psi(u) = \left(-\frac{\partial^2}{\partial u^2} - P^2_{crit}\, u^2\right)\psi(u) \qquad (2.16)$$

has its first zero (node) at $u=\pm 1$. The solutions of Eq. (2.16) are the parabolic cylinder functions. In Fig. 1, P_{crit} is plotted as a function of $\frac{L}{R} = \frac{\kappa_1}{3.831}$. We find, of course, that P_{crit} increases when R decreases. When $R\to\infty$, P_{crit} approaches ~8.0.

III. Stability Analysis of Abelian Magnetic Field Configurations

Let us now consider an abelian static magnetic field configuration which satisfies the Yang-Mills equations in a region R free of external sources:

$$B^a_{ij}(\vec{x}) = \delta^{a3}(\partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x})) \qquad (3.1)$$

with

$$\partial_i(\partial_i A_j(\vec{x}) - \partial_j A_i(\vec{x})) = 0 \quad \text{in R} \qquad (3.2)$$

We will show that such a B field configuration becomes unstable under small perturbations, when gBL^2 exceeds a certain critical value. Here B is a measure of the magnitude of the magnetic field, and L is a measure of the spatial size of the region over which such a magnetic field extends, and in which the small perturbations are applied. The distance over which the magnetic field extends in the transverse direction becomes unimportant when it is larger than a few times $1/\sqrt{gB}$. The critical value of $gB\ell^2$, where ℓ is the distance over which the magnetic field extends in the longitudinal direction, is then of order π^2.

Let us thus consider small oscillations around our background field configuration in $A_o = 0$ gauge:

$$A_o = 0$$
$$A_i^b(\vec{x},t) = \delta^{b3} A_i(\vec{x}) + a_i^b(\vec{x},t) \quad (3.3)$$

where the a_i^b are small and vanish outside and on the boundary of R. To first order in a_i^b, the electric and magnetic fields are:

$$E_i^b = \dot{a}_i^b$$
$$B_{ij}^b = \delta^{b3}(\partial_i A_j - \partial_j A_i) + \partial_i a_j^b - \partial_j a_i^b$$
$$- g\epsilon^{3bc}(A_i a_j^c - A_j a_i^c) \quad (3.4)$$

The linearized Yang-Mills equations are:

$$(\vec{D}\cdot\vec{E})^b = \vec{\nabla}\cdot\dot{\vec{a}}^b - g\epsilon^{3bc}\,\vec{A}\cdot\dot{\vec{a}}^c = 0 \quad (3.5a)$$

$$(-D_o E_j + D_i B_{ij})^b = -\ddot{a}_j^b + \partial_i\partial_i a_j^b - \partial_j\partial_i a_i^b$$
$$- g^2(\delta_c^b - \delta_3^b\delta_c^3)(A_i A_i a_j^c - A_j A_i a_i^c)$$
$$+ g\epsilon^{3bc}[-(\partial_i A_i)a_j^c + (\partial_i a_i^c)A_j - 2A_i\partial_i a_j^c$$
$$+ 2a_i^c\partial_i A_j - a_i^c\partial_j A_i + A_i\partial_j a_i^c] \quad (2.5b)$$

For an eigenmode of frequency $\omega \neq 0$, $\vec{a}^b(\vec{x},t) = \vec{a}^b(\vec{x})e^{i\omega t}$, Gauss' law

Eq. (3.5a) implies:

$$\vec{\nabla}\cdot\vec{a}^{b} - g\varepsilon^{3bc}\,\vec{A}\cdot\vec{a}^{c} = 0 \tag{3.6}$$

which can be used to simplify Eq. (3.5b) to:

$$-\ddot{a}_j^b + \partial_i\partial_i a_j^b - g^2(\delta_c^b - \delta_3^b\delta_c^3)\,A_i A_i a_j^c$$

$$+ g\varepsilon^{3bc}[-(\partial_i A_i)a_j^c - 2A_i\,\partial_i\,a_j^c + 2a_i^c(\partial_i A_j - \partial_j A_i)] = 0 \tag{3.7}$$

The modes along the 3d direction of isospin simply satisfy Maxwell's equations:

$$\vec{\nabla}\cdot\dot{\vec{a}}^3 = 0$$

$$-\ddot{\vec{a}}^3 + \nabla^2\vec{a}^3 = 0 \tag{3.8}$$

They do not see the background B field and do not produce any instabilities. On the other hand, the modes along the 1st and 2d directions of isospin do see the background field. It is useful to define the complex field:

$$\vec{a} = \vec{a}^1 + i\,\vec{a}^2 \tag{3.9}$$

in terms of which the linearized Yang-Mills equations become:

$$\vec{\nabla}\cdot\vec{a} + ig\,\vec{A}\cdot\vec{a} = 0 \tag{3.10a}$$

$$-\ddot{\vec{a}} + \nabla^2\vec{a} - g^2(\vec{A})^2\vec{a}$$

$$+ ig[(\vec{\nabla}\cdot\vec{A})\,\vec{a} + 2A_i\,\partial_i\,\vec{a} - 2\vec{B}\times\vec{a}] = 0 \tag{3.10b}$$

or more succintly:

$$\vec{D}\cdot\vec{a} = 0 \tag{3.11a}$$

$$\omega^2\vec{a} = -D_k D_k\,\vec{a} + 2ig\,\vec{B}\times\vec{a} \tag{3.11b}$$

with $D_k\vec{a} = \partial_k\vec{a} + igA_k\vec{a}$. Equation (3.11b) is the equation for a massless charged particle with spin 1 in a magnetic field. The kinetic energy term is positive definite, but the interaction energy of the spin with the magnetic field can have either sign. We will show, that because of the latter, Eq. (3.11b) has in general solutions with negative ω^2, i.e., unstable modes.

To illustrate this effect, we will use the example of a constant magnetic field oriented in the $\hat{z}$-direction of space:

$$\vec{B} = \hat{z}B \qquad \text{and} \qquad \vec{A} = \frac{1}{2}\hat{\phi}\rho B \tag{3.12}$$

where z, ρ and ϕ are the usual cylindrical coordinates. One obtains:

$$\begin{aligned} \omega^2 a_\varepsilon &= -D_k D_k a_\varepsilon - 2g\varepsilon B a_\varepsilon \\ &= \left[-\nabla^2 - igB\frac{\partial}{\partial\phi} + \frac{1}{4}g^2B^2\rho^2 - 2\varepsilon gB\right]a_\varepsilon \end{aligned} \tag{3.13}$$

where $\varepsilon = -1, 0, +1$ and:

$$a_o = a_z \qquad , \qquad a_{\pm1} = a_x \pm i a_y \tag{3.14}$$

Because of the presence of a harmonic oscillator potential in Eq. (3.13), it is useful to introduce the creation and annihilation operators:

$$\begin{aligned} C_x &= \frac{\partial}{\partial x} + \frac{1}{2}gBx \qquad , \qquad C_x^+ = -\frac{\partial}{\partial x} + \frac{1}{2}gBx \\ C_y &= \frac{\partial}{\partial y} + \frac{1}{2}gBy \qquad , \qquad C_y^+ = -\frac{\partial}{\partial y} + \frac{1}{2}gBy \end{aligned} \tag{3.15}$$

which satisfy:

$$[C_x, C_x^+] = [C_y, C_y^+] = gB \qquad , \qquad [C_x, C_y] = [C_x^+, C_y] = 0 \tag{3.16}$$

and

$$C_u = \frac{1}{\sqrt{2}} (C_x - iC_y) \quad , \quad C_u^+ = \frac{1}{\sqrt{2}} (C_x + iC_y^+)$$

$$C_d = \frac{1}{\sqrt{2}} (C_x + iC_y) \quad , \quad C_d^+ = \frac{1}{\sqrt{2}} (C_x^+ - iC_y^+) \qquad (3.17)$$

which satisfy similar commutation relations. We have:

$$- \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) + \frac{1}{4} g^2 B^2 \;\; {}^2 = C_x^+ C_x + C_y^+ C_y + gB$$

$$= C_u^+ C_u + C_d^+ C_d + gB \qquad (3.18)$$

and

$$L_z = (-i) \frac{\partial}{\partial \phi} = \frac{1}{gB} (C_u^+ C_u - C_d^+ C_d) \qquad (3.19)$$

Thus, Eq. (3.13) becomes:

$$\omega^2 a_\varepsilon = \left[- \frac{\partial^2}{\partial z^2} + 2C_u^+ C_u + (1 - 2\varepsilon)\, gB \right] a_\varepsilon \qquad (3.20)$$

whose solutions are:

$$a_{\varepsilon\, ;\, k,n,m} = e^{ikz} \, (C_u^+)^n \, (C_d^+)^{n-m} \, e^{-\frac{1}{4} gB\rho^2} \qquad (3.21)$$

for $n = 0,1,2,3,\ldots$ and $m = n,\ n-1,\ n-2,\ldots$ with eigenfrequencies:[2]

$$\omega^2 = k^2 + gB\,(1 - 2\varepsilon + 2n) \qquad (3.22)$$

and eigenvalues of the z-component of angular momentum:

$$L_z \, a_{\varepsilon\, ;\, k,n,m} = m \, a_{\varepsilon\, ;\, k,n,m} \qquad (3.23)$$

We thus find that the $\varepsilon = +1$, $n = 0$ eigenmodes are unstable for all values of m and small values of k. However so far, we have not yet made use of Gauss' law Eq. (3.10a), which for our configuration Eq. (3.12) takes the form:

$$\vec{D}\cdot\vec{a} = \frac{\partial a_x}{\partial x} + \frac{\partial a_y}{\partial y} + \frac{\partial a_z}{\partial z} + ig\,\frac{\rho B}{2}\,(-\sin\phi\, a_x + \cos\phi\, a_y)$$

$$= \frac{\partial a_o}{\partial z} + \left(\frac{\partial}{\partial x} - \frac{i}{2}\,gBy\right)\frac{a_{+1} + a_{-1}}{2} + \left(\frac{\partial}{\partial y} + \frac{i}{2}\,gBx\right)\frac{a_{+1} - a_{-1}}{2}$$

$$= \frac{\partial a_o}{\partial z} + \frac{1}{2}\,(C_x - iC_y)\,a_{+1} + \frac{1}{2}\,(-C_x^+ - iC_y^+)\,a_{-1}$$

$$= \frac{\partial a_o}{\partial z} + \frac{1}{\sqrt{2}}\,C_u\,a_{+1} - \frac{1}{\sqrt{2}}\,C_u^+\,a_{-1} = 0 \qquad (3.24)$$

It is clear that all the unstable modes (Eq. (3.21) with $\varepsilon = +1$ and $n = 0$) do satisfy Gauss' law.

If we only allow small oscillations inside a box of length ℓ along the direction of the magnetic field, we have $k = \frac{p\pi}{\ell}$ where p is a non-zero integer, and:

$$\omega^2 = \frac{p^2\pi^2}{\ell^2} + gB\,(1 - 2\varepsilon + 2n) \qquad (3.25)$$

The unstable modes appear as soon as

$$gB\ell^2 > \pi^2 \qquad (3.26)$$

Note added

While this work was being written up, we received a preprint form S.-J. Chang and N. Weiss[3] in which similar calculations are performed.

Acknowledgements

We would like to thank R. Blankenbecler, E. Farhi, L. McLerran and J. Richardson for useful discussions.

References and Footnotes

1. J. Mandula, Phys. Lett. 67B, 175 (1977); M. Magg, Phys. Lett. 74B, 246 (1978).

2. This result has already been stated in: N. K. Nielsen and P. Olesen, Nucl. Phys. B144, 376, 1978. We thank J. Sapirstein for pointing out the existence of this paper to us.

3. S. J. Chang and N. Weiss, University of Illinois preprint.

Figure Caption

Fig. 1. Plot of the critical value of gEL^2 versus R/L, for a constant electric field E in a cylindrical box of radius R and length 2L.

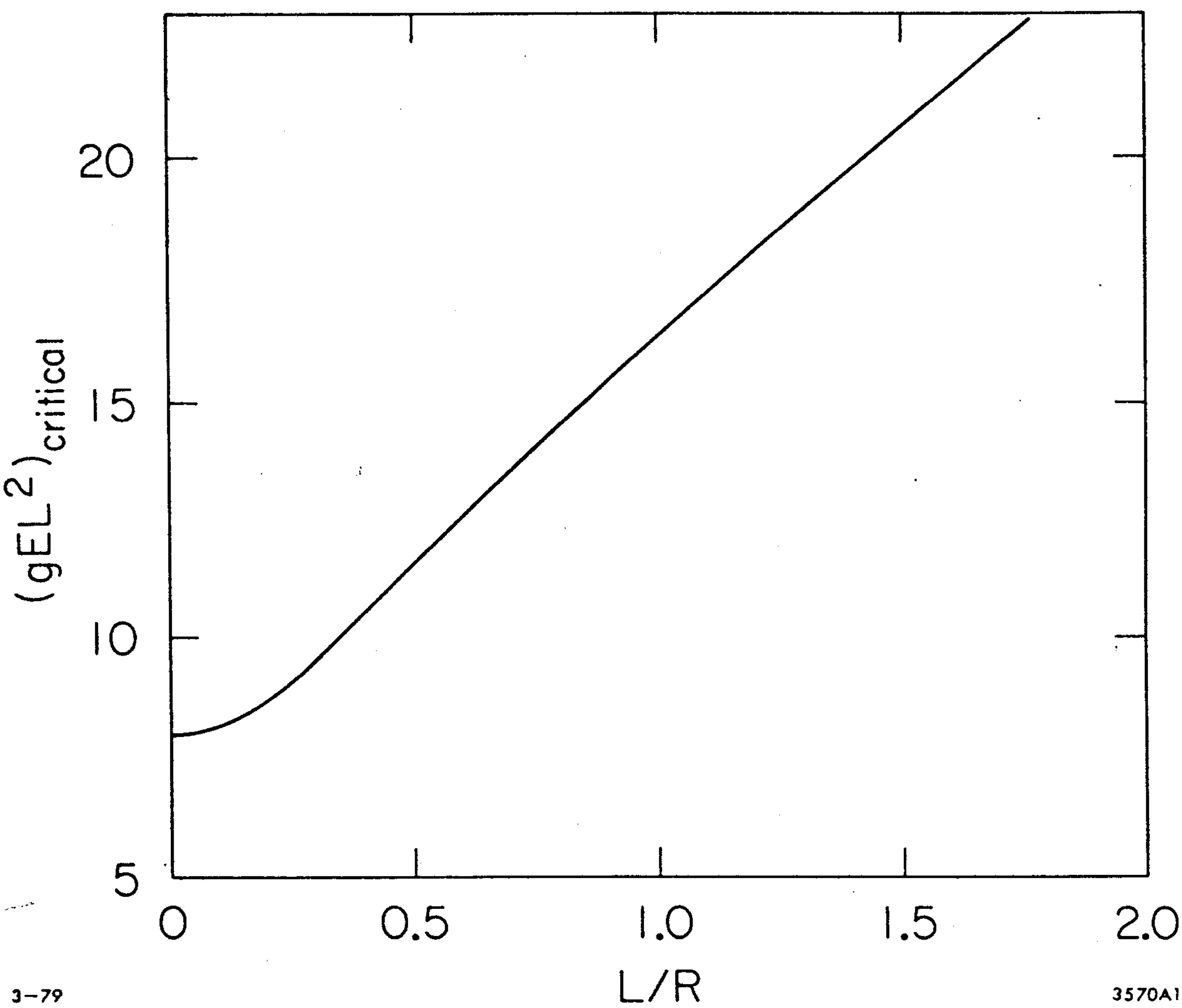

Fig. 1