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Article

Spinor–Vector Duality and Mirror Symmetry

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Abstract: Mirror symmetry was first observed in worldsheet string constructions, and was shown to have profound implications in the Effective Field Theory (EFT) limit of string compactifications, and for the properties of Calabi–Yau manifolds. It opened up a new field in pure mathematics, and was utilised in the area of enumerative geometry. Spinor–Vector Duality (SVD) is an extension of mirror symmetry. This can be readily understood in terms of the moduli of toroidal compactification of the Heterotic String, which includes the metric the antisymmetric tensor field and the Wilson line moduli. In terms of the toroidal moduli, mirror symmetry corresponds to mappings of the internal space moduli, whereas Spinor–Vector Duality corresponds to maps of the Wilson line moduli. In the past few of years, we demonstrated the existence of Spinor–Vector Duality in the effective field theory compactifications of string theories. This was achieved by starting with a worldsheet orbifold construction that exhibited Spinor–Vector Duality and resolving the orbifold singularities, hence generating a smooth, effective field theory limit with an imprint of the Spinor–Vector Duality. Just like mirror symmetry, the Spinor–Vector Duality can be used to study the properties of complex manifolds with vector bundles. Spinor–Vector Duality offers a top-down approach to the "Swampland" program, by exploring the imprint of the symmetries of the ultra-violet complete worldsheet string constructions in the effective field theory limit. The SVD suggests a demarcation line between (2,0) EFTs that possess an ultra-violet complete embedding versus those that do not.

Keywords: string compactifications; Calabi–Yau manifolds; mirror symmetry; spinor–vector duality

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1. Introduction

Physics is, first and foremost, an experimental science. Be that as it may, the language which is used to encode the experimental data is mathematics. We build mathematical models that aim to describe the experimental data. A successful mathematical model is one that is able to account for the wider range of observational data.

On the other hand, continued advances in the experimental probes of matter require new mathematical tools to account for the experimental observations. The objective of describing wider ranges of experimental results using common mathematical frameworks leads to new insight into the required mathematical structures.

Experimental data in the sub-atomic domain since the end of the 19th century, cumulated in the Standard Model of particle physics. This model utilises the framework of relativistic point Quantum Field Theories (QFTs) to account for a wide range of experimental observations in the sub-atomic realm. The Standard Model consists of three sectors, and is underlined by invariance of its basic equations under spacetime and internal transformations. The gauge interactions in the Standard Model are mediated by spacetime vector bosons that transform as spin 1 representations of the Poincare group. The gauge sector contains three group factors that account for the strong, weak, and electromagnetic interactions. The matter sector of the Standard Model consists of three generations of chiral spin 1/2 states that carry identical charges under the three gauge sectors of the Standard Model. Finally, the Higgs sector of the Standard Model consists of a single spin 0 electroweak doublet that facilitates the synthesis of the short-range weak interactions with the long-range electromagnetic interactions. Ellis, Nanopoulos, and Gaillard were among the first to advocate the experimental searches for the Standard Model Higgs boson [1].

The success of the Standard Model opened the door to Grand Unified Theories (GUTs) [2–5] in which the three gauge groups of the Standard Model are unified into one simple GUT group and the matter and scalar sectors appear in representations of the GUT group. The development of GUTs is a watershed in the progression of physics because the perceived unification can only be manifest at a scale which is far removed from energy scales that are currently probed by experiments. Grand Unified Theories, and this vast separation, are supported by several experimental observations. In particular, the multiplet structure of the Standard Model states that are embedded in representations of the GUT group. In the context of $SO(10)$ GUTs [6], embedding of the Standard Model matter states entails that the number of free parameters that are required to account for the matter states gauge charges are reduced from 54 to 1. A remarkable coincidence indeed. Additionally, the vast separation between the GUT scale and the electroweak scale is supported by the longevity of the proton; the suppression of left-handed neutrino masses; and the logarithmic evolution of the Standard Model parameters, which is compatible with the observational data in the gauge sector and the heavy generation matter sector.

The high GUT scale, however, introduces a problem. While the lightness of the gauge and matter sectors, as compared to the GUT scale, can be explained by the existence of symmetries that protect them from being drawn to the GUT scale, nothing protects the scalar sector from this fate. To explain the lightness of the scalar sector, we can use a new spacetime symmetry, supersymmetry, or assume that the scalar states transform under a new gauge sector, which becomes strongly interacting near the electroweak scale. These proposals will be tested in future collider experiments.

Grand Unified Theories and their supersymmetric extentions introduce a new twist in the tale. If supersymmetry is localised, it requires the inclusion of a spin 2 state, which is the mediator of the gravitational interactions in the theory. Localised supersymmetry forces the merger of the gauge and gravitational interactions. The road, however, does not end there. Gravity is inconsistent as a point quantum field theory. It is plugged with infinities. It is then extremely rewarding that a small departure from point quantum field theories provides a consistent framework for the synthesis of the gauge and gravitational interactions. The local supersymmetric extensions of the Standard Model are the effective field theory limits of string theory, in which the point-like idealisation of elementary particles is replaced by a string representation.

String theory provides an elaborate mathematical structure that unifies the gauge, matter, and scalar sectors of the Standard Model with gravity. Its self-consistency conditions imply that additional degrees of freedom, beyond those that are observed in the Standard Model, should exist in nature. Some of these degrees of freedom may be interpreted as additional spacetime dimensions that are made small enough to evade detection by contemporary experiments, whereas others may be interpreted as additional gauge symmetries, beyond those in the Standard Model. The extra spacetime degrees of freedom can be compactified on an internal six-dimensional real manifold or a three-dimensional complex manifold. The compactified spaces determine many of the properties of the observed Standard Model matter spectrum, like the number of chiral generations and their masses. In this manner, string theory gives rise to models that reproduce the main phenomenological properties of the Standard Model. In particular, string theory provides a framework in which the Yukawa couplings of the Standard Model fermionic states to the scalar Higgs can be calculated in terms of the gauge coupling. This is a remarkable feature of string theory that provides a framework to calculate the Standard Model fermion masses.

The self-consistency conditions of string theory require the introduction of new worldsheet degrees of freedom that, in some guise, may be interpreted as extra spacetime dimensions. In ten dimensions, we have five supersymmetric theories that, together with 11-dimensional supergravity, are believed to be the effective perturbative limits of a more fundamental theory that is traditionally dubbed M-theory. Additionally, string theory in ten dimension gives rise to a tachyon-free non-supersymmetric vacuum and seven non-supersymmetric vacua that are tachyonic and unstable. The heterotic $E_8 \times E_8$ string is the effective stable string theory limit that reproduces the GUT picture hinted at by the Standard Model data, as it is the only string theory that gives rise to spinorial representations in the perturbative spectrum.

Phenomenological string models that reproduce the main phenomenological properties of the Standard Model (i.e., three chiral generations and the correct charges under the Standard Model gauge group) have been constructed since the mid-1980s. The free fermionic formulation of the Heterotic String in four dimensions [7–9] led to a particular class of quasi-realistic worldsheet constructions. These models correspond to toroidal $Z_2 \times Z_2$ orbifolds of six-dimensional compactified tori at special points in the moduli space [10,11]. The fermionic $Z_2 \times Z_2$ orbifolds provide a laboratory information on how to explore how the detailed phenomenology of the Standard Model and unification emergence from string theory. Among these are:

- Construction of the first Minimal Standard Heterotic String Model (MSHSM) that contains solely the states of the Minimal Supersymmetric Standard Model (MSSM) in the effective low energy field theory below the string scale [12,13].
- The prediction of the top quark mass at $\sim$175–180 GeV [14].
- Fermion masses and CKM mixing [15].
- Neutrino masses [16].
- Gauge coupling unification [17].
- Proton stability [18].
- Supersymmetry breaking [19].
- Moduli fixing [20].

The quasi-realistic free fermionic models motivate a deeper investigation of this class of string compactifications. Specifically, they highlight the potential relevance of the structure of the toroidal $Z_2 \times Z_2$ orbifolds. A few remarks are in order here. The first is with regard to the existence of a string landscape. The number of string vacua in ten dimensions is relatively small: five supersymmetric and eight non-supersymmetric. However, in four dimensions, the number of vacua is enormous, with some authors quoting the number 10^{500} or even more. The meaning of this expansive space is yet to be understood. One theme alluded to in this paper is that they may all, in fact, be connected. Our task is to unravel and to understand the symmetries that underlie this vast space of possibilities and their inner connections. However, even if this space is enormous, it is still believed to be finite. In each one of these possibilities, the parameters are supposedly determined in terms of the Vacuum Expectation Value (VEV) of a few fixed moduli, which determine the characteristics of the internally compactified space. This should be contrasted with the Standard Model (SM), which contains 19 continuous parameters (i.e., an infinite 19 dimensional space), and its Beyond the Standard Model (BSM) QFT extensions, which contain numerous more continuous parameters. String constructions are constrained by the straitjacket of quantum gravity. Although the space of possibilities is vast, it is finite rather than infinite, in contrast to point particle QFT constructions, which are not constrained by the consistency conditions imposed by quantum gravity. Furthermore, despite the fact that the space of string vacua is vast, the majority of string compactifications are not directly relevant to the observable world, i.e., they contain too many moduli fields and too many chiral generations to allow for a viable connection with the observable parameters of the Standard Model. To date, the fermionic $Z_2 \times Z_2$ orbifolds have been studied in the most detail, and provide concrete quasi-realistic examples to study how the parameters of the Standard Model can be determined in a theory of quantum gravity. Since their early days, they provide a framework to explore the phenomenology of the Standard Model on the one hand, and the mathematical properties that underlie string theory on the other hand. Spinor–Vector Duality (SVD) and its relation to mirror symmetry are among these mathematical properties.

2. Spinor–Vector Duality

While the early free fermionic models consisted of isolated examples [12,21–24], since 2003, systematic computerised methods were developed that enable scans of large number of free fermionic Heterotic String vacua with different unbroken $SO(10)$ subgroups [25–30]. Similar computerised tools for the classification of type II superstrings were developed in [31]. A recent comprehensive review of this subject is provided in [32]. The computerised classifications tools facilitated the discovery of a remarkable symmetry that underlies the space of (2,0) Heterotic String compactifications, which is akin to mirror symmetry. Spinor–Vector Duality operates in vacua in which the $N = 4$ spacetime supersymmetry is broken from $N = 4$ to $N = 2$ or $N = 1$ by Z_2 or $Z_2 \times Z_2$, respectively, of the internal compactified coordinates. The gauge symmetry from the ten-dimensional $E_8 \times E_8$ symmetry then depends on the action of a Wilson line. In the absence of a Wilson line, the gauge symmetries are $E_8 \times E_8$ in the $N = 4$ case, $E_7 \times SU(2) \times E_8$ in the $N = 2$ case, and $E_6 \times U(1)^2 \times E_8$ in the $N = 1$ case. The twists of the internal coordinates produce twisted sectors that give rise to massless states in the 56 representation of E_7, 27, and $\overline{27}$ representation of E_6. The inclusion of a specific Wilson line breaks the gauge symmetries in the three cases to $SO(16) \times SO(16)$, $SO(12) \times SU(2) \times SU(2) \times SO(16)$, and $SO(10) \times U(1)^3 \times SO(16)$, respectively. In terms of the unbroken subgroups, the 56, 27, and $\overline{27}$ are decomposed into spinorial and vectorial representations of the unbroken subgroup, which are $(32, 1)$, $(32', 1)$ and $(12, 2)$ in the case of $SO(12) \times SU(2)$, and 16, $\overline{16}$, and 10 in the case of $SO(10) \times U(1)$. The Spinor–Vector Duality (SVD) operates under the exchange of the spinor and vector representations. Focusing on the $N = 1$ case, the 27 and $\overline{27}$ representations of E_6 decompose as $27 = 16_{+1/2} + 10_{-1} + 1_{+2}$ and $\overline{27} = \overline{16}_{-1/2} + 10_{+1} + 1_{-2}$ under $SO(10) \times U(1)$. It is seen that, in the case of E_6 for every spinorial 16 state, there is a vectorial 10 state, and for every $\overline{16}$, there is a vectorial 10 state. Vacua with E_6 symmetry are, therefore, self-dual if we exchange the total number of $16 \oplus \overline{16}$ representations of $SO(10)$ with the total number of vectorial 10 representations. The remarkable property is that there is a remnant of this symmetry when the E_6 symmetry is broken to $SO(10) \times U(1)$. The statement is that, for a string vacuum with a number $\#_1(16 + \overline{16})$ of spinorial and anti-spinorial $SO(10)$ representations, and a $\#_2(10)$ of vectorial representations, there is a dual vacuum in which the two numbers are interchanged, i.e., $\#_1 \leftrightarrow \#_2$ [33–36].

The Spinor–Vector Duality resembles T–duality in the sense that the enhanced symmetry point with E_6 symmetry is self-dual under the SVD. The duality can also be seen to operate in terms of a spectral flow operator on the bosonic side of the Heterotic String [34,37]. The vacua that possess E_6 symmetry have $(2, 2)$ worldsheet supersymmetry. On the supersymmetric side of the Heterotic String, a vector in the basis that defines the free fermionic models operates as a spectral flow operator that mixes between sectors that produce spacetime fermions and bosons. Similarly, on the bosonic side of the Heterotic String, in the case with enhanced E_6 symmetry and $(2, 2)$ worldsheet supersymmetry, the models can be constructed such that the operation of a spectral flow operator on the bosonic side is manifest [34,37]. In this case, the spectral flow operator on the bosonic side mixes between the spinorial and vectorial representations in the breaking of E_6 under $SO(10) \times U(1)$. In the vacua in which the E_6 symmetry is broken to $SO(10) \times U(1)$ and the $N = 2$ worldsheet supersymmetry on the bosonic side is broken, the spectral flow operator induces the map between the dual vacua.

The SVD was discovered in free fermionic constructions of the Heterotic String in four dimensions by using the computerised classification tools that were developed for the analysis of the spectrum of free fermionic vacua [25,26,31,33]. The use of systematic computerised tools has been a hot pursuit over the past years (for a review and references, see, e.g., [38]). The SVD was observed initially by simple counting [33], which is illustrated in Table 1, where s, $\bar{s}$ and v are the total number of 16, $\overline{16}$, and 10 representations of $SO(10)$, respectively. It is easily seen, by adding the corresponding numbers of models, that the total number of vacua with two 16, two $\overline{16}$, and one $16 \oplus \overline{16}$ representations, is the same as the total number of models with two 10 representations. A more comprehensive depiction is illustrated in Figure 1.

Table 1. Number of models with a total number of 2 representations in the first twisted sector.

First, Plane			Second, Plane			Third Plane			
s	$\bar{s}$	v	s	$\bar{s}$	v	s	$\bar{s}$	v	**# of Models**
2	0	0	0	0	0	0	0	0	1325963712
0	2	0	0	0	0	0	0	0	1340075584
1	1	0	0	0	0	0	0	0	3718991872
0	0	2	0	0	0	0	0	0	6385031168

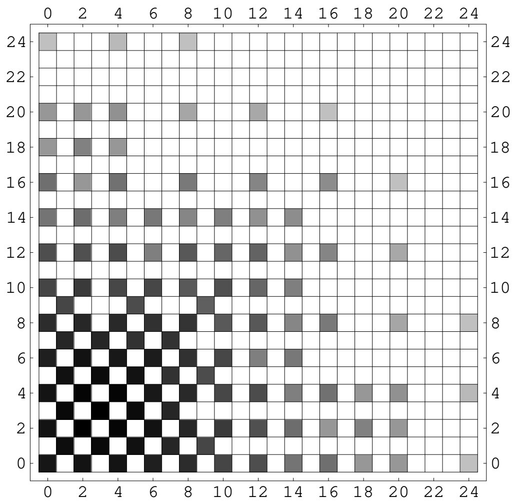

Figure 1. Density plot depicting the Spinor–Vector Duality in the space of fermionic $Z_2 \times Z_2$ Heterotic String orbifolds. The figure shows the number of models with a given number of $(16 + \overline{16})$ and 10 representations of $SO(10)$. It is symmetric under the exchange of rows and columns, reflecting the SVD that underlies the entire space of vacua.

So far, the SVD had been presented in terms of the free fermion constructions. In the free fermionic classification method, the set of boundary conditions is fixed, and the variation in the models is generated in terms of the one-loop Generalised-GSO (GGSO) phases. It is therefore apparent that, in these constructions, the SVD arises due to the exchange of the GGSO projection coefficients, which can be proven rigorously [33]. Further insight into the SVD can be gained by using a bosonic representation of the $Z_2 \times Z_2$ orbifolds [36,37]. Since the SVD operates plane by plane, it is sufficient to examine the case with a single Z_2 twist.

Using the level one $SO(2n)$ characters,

$$O_{2n} = \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n} + \frac{\theta_4^n}{\eta^n}\right) \quad , \qquad V_{2n} = \frac{1}{2}\left(\frac{\theta_3^n}{\eta^n} - \frac{\theta_4^n}{\eta^n}\right), \tag{1}$$

$$S_{2n} = \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n} + i^{-n}\frac{\theta_1^n}{\eta^n}\right) \quad , \qquad C_{2n} = \frac{1}{2}\left(\frac{\theta_2^n}{\eta^n} - i^{-n}\frac{\theta_1^n}{\eta^n}\right), \tag{2}$$

where

$$\theta_3 \equiv Z_f\binom{0}{0} \quad , \quad \theta_4 \equiv Z_f\binom{0}{1} \quad , \quad \theta_2 \equiv Z_f\binom{1}{0} \quad , \quad \theta_1 \equiv Z_f\binom{1}{1} \, ,$$

and Z_f is the partition function of a complex worldsheet fermion. The partition function of the $E_8 \times E_8$ Heterotic String in four dimensions is given by

$$Z_+ = (V_8 - S_8)\left(\sum_{m,n} \Lambda_{m,n}\right)^{\otimes 6} (\overline{O}_{16} + \overline{S}_{16})(\overline{O}_{16} + \overline{S}_{16}), \tag{3}$$

where, for each S_1,

$$p^i_{\mathrm{L,R}} = \frac{m_i}{R_i} \pm \frac{n_i R_i}{\alpha'} \qquad \text{and} \qquad \Lambda_{m,n} = \frac{q^{\frac{\alpha'}{4} p_{\mathrm{L}}^2} \bar{q}^{\frac{\alpha'}{4} p_{\mathrm{R}}^2}}{|\eta|^2}.$$

A $Z_2 \times Z_2' : g \times g'$ action on Z_+ is performed. The first Z_2 couples a fermion number in the observable and hidden sectors with a Z_2-shift in a compactified coordinate, and is given by $g : (-1)^{(F_1+F_2)}\delta$. Here, the fermion numbers $F_{1,2}$ operate on the spinorial representations of the observable and hidden $SO(16)$ groups as

$$F_{1,2} : (\overline{O}^{1,2}_{16}, \overline{V}^{1,2}_{16}, \overline{S}^{1,2}_{16}, \overline{C}^{1,2}_{16}) \longrightarrow (\overline{O}^{1,2}_{16}, \overline{V}^{1,2}_{16}, -\overline{S}^{1,2}_{16}, -\overline{C}^{1,2}_{16})$$

and δ identifies points shifted by a Z_2 shift in the X_9 direction, i.e., $\delta X_9 = X_9 + \pi R_9$. The result of the shift is to insert a factor of $(-1)^m$ into the lattice sum in Equation (3), i.e., $\delta : \Lambda^9_{m,n} \longrightarrow (-1)^m \Lambda^9_{m,n}$. The second Z_2 is a twist of the internal coordinates, given by

$$g' : (x_4, x_5, x_6, x_7, x_8, x_9) \longrightarrow (-x_4, -x_5, -x_6, -x_7, +x_8, +x_9). \tag{4}$$

Alternatively, the first Z_2 action can be interpreted as a Wilson line in X_9 [37],

$$g \; : (0^7, 1|1, 0^7) \;\to\; E_8 \times E_8 \to SO(16) \times SO(16).$$

The Z_2 twist in the internal space breaks $N = 4 \to N = 2$ spacetime supersymmetry and $E_8 \to E_7 \times SU(2)$, or with the inclusion of the Wilson line $SO(16) \to SO(12) \times SO(4)$. The orbifold partition function is

$$Z \;=\; \left(\frac{Z_+}{Z_g \times Z_{g'}}\right) \;=\; \left[\frac{(1+g)}{2}\frac{(1+g')}{2}\right] Z_+.$$

The partition function contains an untwisted sector and three twisted sectors. Its schematic form is shown in Figure 2.

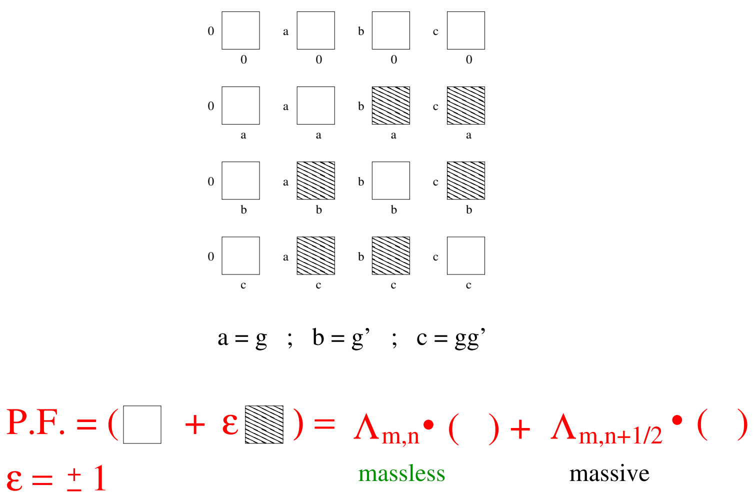

Figure 2. The $Z_2 \times Z_2'$ partition function of the g'-twist and g Wilson line with discrete torsion $\epsilon = \pm 1$.

The winding states in the sectors twisted by g and gg' are shifted by 1/2. Consequently, these sectors contain only massive states. The g' twisted sector produces massless matter states. The partition function has one discrete torsion $\epsilon = \pm 1$ between the two modular orbits, and produces massless states for zero lattice modes. The terms in the g' twisted sector contributing to the massless spectrum have the form

$$\Lambda_{p,q}\left\{\frac{1}{2}\left(\left|\frac{2\eta}{\theta_4}\right|^4+\left|\frac{2\eta}{\theta_3}\right|^4\right)\left[P_\epsilon^+ Q_s\overline{V}_{12}\overline{C}_4\overline{O}_{16}+P_\epsilon^- Q_s\overline{S}_{12}\overline{O}_4\overline{O}_{16}\right]+\right.$$
$$\left.\frac{1}{2}\left(\left|\frac{2\eta}{\theta_4}\right|^4-\left|\frac{2\eta}{\theta_3}\right|^4\right)\left[P_\epsilon^+ Q_s\overline{O}_{12}\overline{S}_4\overline{O}_{16}\right]\right\} \quad + \quad \text{massive} \tag{5}$$

where

$$P_\epsilon^+ \;=\; \left(\frac{1+\epsilon(-1)^m}{2}\right)\Lambda_{m,n} \quad ; \quad P_\epsilon^- \;=\; \left(\frac{1-\epsilon(-1)^m}{2}\right)\Lambda_{m,n} \tag{6}$$

From the sign of the discrete torsion $\epsilon = \pm$, it is noted from Equation (6) that either the vectorial states or the spinorial states are massless. It is readily seen from Equations (7) and (8) that the choice $\epsilon = +1$ gives rise to massless momentum modes from the shifted lattice in P_ϵ^+, whereas P_ϵ^- produces massive modes. Therefore, the vectorial character $\overline{V}_{12}$ in Equation (6) gives rise to massless states, whereas the spinorial character $\overline{S}_{12}$ produces massive states. Equation (8) shows that the choice $\epsilon = -1$ produces exactly the opposite result.

$$\epsilon \;=\; +1 \quad \Rightarrow \quad P_\epsilon^+ \;=\; \Lambda_{2m,n} \qquad P_\epsilon^- \;=\; \Lambda_{2m+1,n} \tag{7}$$
$$\epsilon \;=\; -1 \quad \Rightarrow \quad P_\epsilon^+ \;=\; \Lambda_{2m+1,n} \qquad P_\epsilon^- \;=\; \Lambda_{2m,n} \tag{8}$$

The Spinor–Vector Duality is generated due to the exchange of the discrete torsion $\epsilon = +1 \to \epsilon = -1$ in the $Z_2 \times Z_2'$ partition function. This is similar to the case of mirror symmetry in the $Z_2 \times Z_2$ orbifold of ref. [39], where the mirror symmetry transformation results from the exchange of the discrete torsion between the two Z_2 orbifold twists.

This particular example provides insight into the inner working of the SVD map. As shown in Figure 1 and Table 1, the SVD is manifested in the wider space of string vacua with $N = 2$ and $N = 1$ spacetime supersymmetry [26,33,34]. The analysis using the free fermionic formulation obscures the role of the geometrical moduli fields. In [33,34], it is shown, in terms of the GGSO projection coefficients of the one-loop partition function, that the SVD always exists in this space of vacua. The bosonic analysis in [37] reveals the role of the moduli fields, and demonstrates that the SVD arises due to an exchange of two Wilson lines. The SVD can then be interpreted to arise from the breaking of the $N = 2$ worldsheet supersymmetry on the bosonic side of the Heterotic String. It was further shown that the map between the dual vacua is induced in terms of a spectral flow operator. At the enhanced self-dual point, the spectral flow operator exchanges between the spinorial and vectorial components of the representations of the enhanced symmetry group. In the vacua with broken symmetry, the spectral flow operator induces the map between the dual Wilson lines and the dual vacua [34,37]. In ref. [40], this picture was generalised to string vacua with interacting internal CFTs [40] that utilise the Gepner construction [41]. The bosonic representation of the SVD is instrumental for studying the imprint of the SVD in the effective field theory limit.

The details of the relation between the discrete torsion and the Wilson line realisations of the SVD are discussed in ref. [37]. It is sufficient here to realise that there are choices of the background moduli fields that give rise to the spectra of the dual models. The Z_2 twist action of the internal coordinates is given by Equation (4), whereas the dual Wilson lines are given by

$$g = (0,0,0,0,0,1|0,0|1,0,0,0,0,0,0,0). \tag{9}$$

and

$$g = (0,0,0,0,0,0|1,0|1,0,0,0,0,0,0,0), \tag{10}$$

and the map between the two is induced by the spectral flow operator [37]. Relating the worldsheet symmetries to the properties of the effective field theory limit of the string compactifications is facilitated by using the bosonic data in the form of Equations (9) and (10). The interpretation of the worldsheet data in the effective field theory limit is often obscured, as, for example, in the case of mirror symmetry. For this purpose, the representation of the Spinor–Vector Duality in terms of the Wilson lines is particularly instrumental.

3. Mirror Symmetry

Mirror symmetry was observed initially in worldsheet constructions of string compactifications. Subsequently, the profound implications for complex geometrical manifolds that are used in the effective field theory limit of the string compactifications was understood. Mirror symmetry facilitates the counting of intersections between sub-surfaces of the complex manifolds, which are otherwise notoriously difficult to calculate.

The calculation is facilitated by the relation of the intersection curves to the calculation of the Yukawa couplings between the string states. Thus, the worldsheet constructions provide a useful tool to study the properties of the string vacua in the effective field theory limit. For brevity, we can consider the mirror models in the free fermionic formulation, though the mirror symmetry phenomena apply to the whole space of string configurations. The vacua with the unbroken $SO(10)$ group are produced by a set of twelve basis vectors,

$$\begin{aligned}
v_1 = \mathbf{1} &= \{\psi^\mu,\ \chi^{1,\dots,6}, y^{1,\dots,6}, \omega^{1,\dots,6} \ \ | \ \ \bar{y}^{1,\dots,6}, \bar{\omega}^{1,\dots,6}, \bar{\eta}^{1,2,3}, \bar{\psi}^{1,\dots,5}, \bar{\phi}^{1,\dots,8}\},\\
v_2 = S &= \{\psi^\mu, \chi^{1,\dots,6}\},\\
v_3 = z_1 &= \{\bar{\phi}^{1,\dots,4}\},\\
v_4 = z_2 &= \{\bar{\phi}^{5,\dots,8}\},\\
v_{4+i} = e_i &= \{y^i, \omega^i | \bar{y}^i, \bar{\omega}^i\},\ i = 1,\dots,6, \qquad\qquad N = 4 \ \text{Vacua}\\
\\
v_{11} = b_1 &= \{\chi^{34}, \chi^{56}, y^{34}, y^{56} | \bar{y}^{34}, \bar{y}^{56}, \bar{\eta}^1, \bar{\psi}^{1,\dots,5}\}, \qquad N = 4 \to N = 2\\
v_{12} = b_2 &= \{\chi^{12}, \chi^{56}, y^{12}, y^{56} | \bar{y}^{12}, \bar{y}^{56}, \bar{\eta}^2, \bar{\psi}^{1,\dots,5}\}, \qquad N = 2 \to N = 1.
\end{aligned} \tag{11}$$

The first ten vectors preserve $N = 4$ spacetime supersymmetry, and the last two are the $Z_2 \times Z_2$ orbifold twists. The e_i basis vectors correspond to shifts in the internal compactified coordinates, whereas the z_i basis vectors reduce the untwisted hidden sector gauge group to $SO(8) \times SO(8)$. The third twisted sector of the $Z_2 \times Z_2$ orbifold is obtained as the combination $b_3 = b_1 + b_2 + x$, where the x-sector is obtained from the combination

$$x = \mathbf{1} + S + \sum_{i=1}^{6} e_i + \sum_{k=1}^{2} z_k = \{\bar{\psi}^{1,\cdots,5}, \bar{\eta}^{1,2,3}\}. \tag{12}$$

The x-sector can give rise to additional massless spacetime vector bosons in the observable sector. If these are not projected by the GGSO projections, they enhance the $SO(10)$ gauge symmetry to E_6, and the matter representations are in the 27 and $\overline{27}$ of E_6. Mirror symmetry in the large space of free fermionic vacua corresponds to the exchange of the GGSO phase,

$$c\binom{b_1}{b_2} = +1 \to c\binom{b_1}{b_2} = -1 \tag{13}$$

which corresponds to the discrete torsion exchange of [39]. The effect in Heterotic String vacua with E_6 symmetry is to flip the net chirality of the chiral representations, which is counted by the Euler characteristic of the internal manifolds,

$$\frac{\chi}{2} \quad = \#(27 - \overline{27}) \quad \longrightarrow \quad -\frac{\chi}{2} \tag{14}$$

In string compactifications with (2,2) worldsheet supersymmetry, there is a one-to-one correspondence between the chiral and anti-chiral representations, and between the complex structure and Kähler moduli of the internal manifolds. In terms of the moduli fields of Narain toroidal compactfications, the metric G, the antisymmetric tensor field B, and the Wilson line moduli W, the mirror map (Equations (13) and (14)) corresponds to an exchange of the internal moduli, i.e., the metric field G and the antisymmetric tensor field B, which relate to the complex structure and Kähler moduli of the complex Calabi–Yau manifolds. The mirror symmetry map exchanges the complex structure and Kähler moduli of the internal compactified manifold.

Mirror symmetry was first observed [42,43] in Gepner constructions [41] of Heterotic String compactifications. It was not foreseen by mathematicians, and was a complete surprise from their point of view [44,45]. Moreover, it was shown to be instrumental in the field of enumerative geometry, in which the intersections between sub-surfaces of the complex manifolds are counted [46]. The observation of mirror symmetry in the space of complex manifolds is a profound observation from the purely mathematical point of view, and led to important developments in pure mathematics. In this respect, we should note that the string compactifications on Calabi–Yau entails the analysis of the string vacua in their effective field theory limit. This is an example where the symmetries of the ultra-violet complete string theory have fundamental imprints on the effective string theory limit of the string compactifications. The Yukawa couplings between massless states in the string spectrum of the worldsheet vacua are given in terms of correlators among vertex operators,

$$\langle V_1^f V_2^f V_3^b \cdots \cdot V_N^b \rangle,$$

where the vertex operators are given by [47]

$$\begin{aligned} V^f_{(-\frac{1}{2})} \quad = \quad & e^{(-\frac{c}{2})} \, \mathcal{L}^\ell \, e^{(i\alpha\chi_{12})} \, e^{(i\beta\chi_{34})} \, e^{(i\gamma\chi_{56})} \\ & \left(\prod_j e^{(iq_i\zeta_j)} \ \{\sigma' s\} \prod_j e^{(i\bar{q}_i\bar{\zeta}_j)} \right) \\ & e^{(i\bar{\alpha}\bar{\eta}_1)} \, e^{(i\bar{\beta}\bar{\eta}_2)} \, e^{(i\bar{\gamma}\bar{\eta}_3)} \, e^{(iW_R\cdot\bar{J})} e^{(i\frac{1}{2}KX)} \, e^{(i\frac{1}{2}K\cdot\bar{X})}, \end{aligned} \tag{15}$$

and the different components entering Equation (15) are detailed in [47]. The non-vanishing correlators have to be invariant under all the string symmetries. In the vacua with enhanced E_6 symmetry, the couplings are between three 27 chiral representations of E_6, and the mirror map implies that

$$27 \cdot 27 \cdot 27 \ \longleftrightarrow \ \overline{27} \cdot \overline{27} \cdot \overline{27}$$

On the Calabi–Yau manifolds that describe the string vacua in their effective field theory limit, the Yukawa couplings correspond to intersection of curves. Thus, one finds imprints of the worldsheet correlators in the geometrical data, and uses them to analyse the properties of the corresponding manifolds. Mirror symmetry proved its power in this domain by providing a tool to analyse the geometry. I should emphasise that it is not possible to describe the field of mirror symmetry here. Interested readers are referred to Sheldon Katz's book [44], which provides a very lucid introduction to the subject, and the more in-depth monograph [45]. Superficially, the analysis of the intersection of the rational curves on Calabi–Yau manifolds is related to the Yukawa couplings and, therefore, the calculation of the Yukawa couplings, which are related to the Gromov–Witten invariants, provides a tool to analyse the geometrical data of the manifolds. The message in the current paper is that the relation of Spinor–Vector Duality to mirror symmetry (i.e., both represents mappings under transformations of the moduli parameters of the Narain toroidal spaces) suggests that the Spinor–Vector Duality may have similar interesting mathematical implications in the Effective Field Theory (EFT) limit. In this context, it is noted that, similarly to mirror symmetry, the likely tool to be of use is the calculation of Yukawa couplings among the string states, albeit in the case of SVD, the picture is complicated, because it involves not only the internal space, but also the vector bundles that correspond to the gauged degrees of freedom on them.

4. Spinor–Vector Duality in the EFT Limit

I will not delve into technical details in the discussion here; that can be found in the original literature [48–51]. Rather, I will discuss the Spinor–Vector Duality in relation to mirror symmetry and articulate future directions for research in light of this relation. Just as in the case of mirror symmetry, the SVD, which was first observed in worldsheet constructions, may have profound implications for the mathematical properties of the geometrical manifolds with vector bundles, corresponding to the gauged degrees of freedom of the Heterotic String. In refs. [48,49], the SVD was analysed in the effective field theory limit of the string compactifications in six and five dimensions, respectively. The analysis was performed by starting with an orbifold model that exhibits SVD, and analysing the EFT limit on a smooth Calabi–Yau manifold with vector bundle by smoothing the orbifold singularities, using a well-established technique in this context [52,53]. Ref. [48] analyses the SVD on $T^4/Z_2 \times S^1$ in five dimensions by including a twist in the form of Equation (4), which acts on four internal coordinates, and a Wilson line in the form of Equation (9) or (10) on the additional circle. The subsequent step is to analyse the resolution of this orbifold to a smooth $K3 \times S^1$ by using some massless states in the orbifold model to blow up the singularities. We incorporate a discrete torsion in the analysis of the orbifold model between the twist and the Wilson line, as well as its effect on the resulting massless states. In the model that we analyse, the states used for the resolution transform under the $SO(10)$ GUT symmetry. This entails that the GUT symmetry is broken by the resolutions. As the available states for the resolution differ in the dual configurations and transform under the observable gauge symmetry, the gauged degrees of freedom are also different in the two cases. This is unlike the situation in some of the free fermionic SVD models [33,34] that contain twisted hidden sector states that may be used to resolve the singularities without affecting the gauge symmetry. Since the role of the discrete torsion in the effective field theory smooth limit is obscured, we make an educated guess on the resolved manifold for the orbifold that includes the discrete torsion. The case without torsion is well-defined on the resolved manifold, but the case with torsion introduces some subtleties that are discussed in detail in ref. [48]. The short summary is that the smooth geometries do exhibit a Spinor–Vector Duality-like phenomenon but, due to the different spectra available for the resolutions on the dual configurations, the gauge symmetries differ on the resolved manifolds. This phenomenon is expected to be generic in the resolved limit because of the different states available for the resolution (e.g., the spinorials in one case and the vectorials in the other) in the example discussed in Section 2. In ref. [49], the Spinor–Vector Duality was studied in six dimensions. It was found that the Spinor–Vector Duality also operates in these cases, though the vacua are self-dual under the Spinor–Vector Duality map, and satisfy the general anomaly consistence condition on the number of vector and spinor representations of any $SO(2N)$ unbroken subgroup in the string vacuum

$$N_V = 2^{N-5} N_S + 2N - 8. \quad (16)$$

The analysis of Spinor–Vector Duality in smooth $Z_2 \times Z_2$ orbifolds in four dimensions is complicated due to the large number of possible resolutions. The $T^6/Z_2 \times Z_2$ orbifold has 64 $C^3/Z_2 \times Z_2$ singularities, where Z_2-fixed tori intersect. All of the singularities have to be resolved to produce a smooth manifold. Each singularity can be resolved in four topologically distinct ways [50], resulting in 4^{64} distinct a priori possibilities. The symmetry structure of the $Z_2 \times Z_2$ orbifold can be used to reduce this number, but still leaves a large number, in the order of 10^{33} distinct configurations. Many of the physical properties of the effective field theory limits of the resolved geometries, like the spectra of massless states and the interactions between them, depend on the chosen resolution, and hinder the extraction of generic properties of the resolved $Z_2 \times Z_2$ orbifolds. In ref. [50], a formalism was developed that allows computations of any choice of the resolution, which opens the way to extract some properties of the resolved $T^6/Z_2 \times Z_2$ that are independent of the choice of the resolution and, therefore, hold for any such choice. The analysis of the Spinor–Vector Duality in four dimensions is still outstanding to this date.

Another tool in the analysis of the effective field theory limit of worldsheet string models is Gauged Linear Sigma Models (GLSM) [54], which provide a tool to interpolate between the singular orbifold constructions and their resolved smooth geometries. Some of the properties of the worldsheet string constructions that do not have a direct analogue in the smooth geometries can, therefore, be studied by using the GLSMs. An example of this is the discrete torsion that appears in the worldsheet string vacua between the different modular orbits in the string partition function, and has no direct analogue in the smooth geometries that underlie the effective field theory limit. In ref. [51], we used the GLSM to shed light on what becomes of the discrete torsion in the resolution of non-compact $C^3/Z_2 \times Z_2$ and the compact $T^3/Z_2 \times Z_2$ orbifolds. The GLSMs associated with the non-compact orbifold with or without torsion are to a large degree equivalent: only when expressed in the same superfield basis, a field redefinition anomaly arises among them, which in the orbifold limit reproduces the discrete torsion phases. The GLSMs associated with the torsional compact orbifold suffers from mixed gauge anomalies, which need to be cancelled by appropriate logarithmic superfield dependent Fayet–Iliopoulos terms on the worldsheet, signalling H-flux due to NS5-branes supported at the exceptional cycles.

5. Questions for Future Explorations

As discussed above, mirror symmetry is the key example of the relation between worldsheet string constructions and their Effective Field Theory (EFT) limit on smooth geometries. Mirror symmetry, which was discovered in worldsheet string constructions, relates couplings in the dual string vacua which, in the EFT limit on complex geometries, correspond to intersections of rational curves on Calabi–Yau manifolds. Mirror symmetry proved to be instrumental in counting the number of such intersections, i.e., it proved to be a useful tool in the purely mathematical field of enumerative geometry. Dedicated tools, such as the Gromov–Witten invariants, were developed for that purpose.

The Spinor–Vector Duality (SVD) is an extension of mirror symmetry in the sense described in Section 2. As such, it is natural to ask whether the SVD can be instrumental as a tool to explore the properties of algebraic complex curves with vector bundles on them and, in particular, to explore the effective field theory limits of worldsheet string constructions. We can pursue that in the first instance by analysing the correlators between the string states in the Spinor–Vector dual vacua, and seek to define the analogues of the Gromov–Witten invariants. The SVD provides a tool to study the complex Calabi–Yau manifolds with vector bundles, which correspond to the gauged degrees of freedom of the Heterotic String. The SVD, thus, provides a tool to study the moduli spaces of (2,0) string compactifications. In this respect, we can ask whether it is complete, i.e., does the SVD constrain the viable effective field theory limits of quantum gravity models that are compatible with the ultra-violet complete Heterotic String theory? We can pose a "Swampland" conjecture [55]: "Every EFT (2,0) Heterotic String compactification which has an ultra-violet complete embedding in string theory is connected to a (2,2) Heterotic String compactification by an orbifold or by continuous interpolation". If it is not, then it is necessarily in the "Swampland". The motivation to pose this conjecture stems from the question on whether the symmetries of the string worldsheet formalism are complete. We can view this in analogy with the celebrated T-dualities and mirror symmetry where, similarly, we may question whether a mirror manifold should always exist, and whether T-duality represent a complete symmetry of string theory, i.e., any string compactification must admit a symmetry that can be interpreted as T-duality, and can be connected to the self-dual point. That is: does T-duality provide a complete characterisation of string theories of quantum gravity, or is it merely a property of string compactification on tori? In this respect, we can note that the interpretation of T-duality as phase-space duality (see e.g., [56] and the references therein) may provide a generalisation that extends its realisation beyond the toroidal geometry. The Spinor–Vector Duality extends T-duality in the sense discussed in Section 2, i.e., by including transformations induced due to exchange of Wilson line moduli, rather than the moduli of the internal compactified torus. The self-dual point under the SVD is the enhanced E_6 symmetry point with (2,2) worldsheet supersymmetry, or enhanced E_7 symmetry in the case of a single Z_2 twist with $N = 2$ spacetime supersymmetry. The transformation between the dual Wilson lines is continuous in the later case and discrete in the former. In the discrete case, the moduli that enable the continuous interpolation in the E_7 case are simply projected out from the spectrum by the second Z_2 twist, and the map between the dual vacua is discrete. In both cases, the models are connected to the self-dual enhanced symmetry point by either a continuous interpolation or by an orbifold. This is similar to the case of T-duality in which the continuous interpolation can be nullified by asymmetric boundary condition assignments that project some or all of the internal torus moduli [20,57]. We can conjecture that:

SVD conjecture: Every EFT (2,2) Heterotic String compactification has to be connected to a (2,2) Heterotic String compactification by orbifold or continuous interpolation. Otherwise, it is in the swampland, i.e., it does not have an ultra-violet completion in string theory.

The approach articulated here, therefore, presents a top-down approach to the Swampland program [58]. The aim is to explore how the symmetries of the ultra-violet complete string theories, which are defined in terms of the worldsheet constructions, constrain the effective field theory limits of these theories. The "Swampland" program approach aims to explore which effective field theories of quantum gravity have an embedding in an ultra-violet complete string theory of quantum gravity (for a review and references, see, e.g., [59]), and can be viewed as a bottom-up approach to the construction of consistent theories of quantum gravity.

The SVD conjecture, therefore, provides a demarcation line between (2,0) effective field theories that do, and do not, possess an ultra-violet complete embedding in string theory. We can envision that there exist many (2,0) EFTs that do not satisfy the SVD conjecture, which will be in the "Swampland", whereas those that do satisfy the conjecture have an ultra-violet complete embedding in string theory. It should be noted that the SVD and T-duality are merely two examples of the symmetry structure of (2,0) string vacua, and it is anticipated that a much larger symmetry structure underlies them [60]. The proposition of the SVD conjecture as posed above is a physicist's proposition, and making it a proper mathematical statement is warranted. Likewise, we can ask what the tools that can substantiate this statement are, and it seems that the GLSMs might provide such a tool.

6. Conclusions

Our understanding of fundamental physics reached a juncture in which the mathematical description of all sub-atomic data are well accounted for by the Standard Model (SM) of particle physics, whereas observations at the celestial, galactic, and cosmological scales are well accounted for by Einstein's general relativity. Yet, the two theories are fundamentally incompatible. The Standard Model gives rise to a large vacuum energy, whereas observations using Einstein's general relativity are compatible with a much smaller vacuum energy. Furthermore, using the QFT framework that is used in the Standard Model to calculate quantum gravitational effects is plagued with infinities and, therefore, is inconsistent. Yet, the synthesis of the Standard Model with gravity is inevitable. The sub-atomic observational data indicate that further basic insights into the Standard Model parameters, e.g., in its flavour sector, can only be gained by synthesising it with gravity.

String theory is a self-consistent theoretical framework that accommodates perturbative quantum gravity with all the ingredients that make up the Standard Model. Detailed phenomenological models can be constructed that reproduce the structure of the SM, and enable the development of a phenomenological approach to quantum gravity. The characterisation of string theory in the literature is often misleading. It is often called the Theory of Everything, and is therefore poorly portrayed as a final step in our understanding of fundamental physics. First of all, we do not know what everything is. String theory is not a final step, but rather the relevant question is what and whether any of the ingredients of string theory are relevant in the real physical world. We can relate to the QFTs that underlie the Standard Model as point particle theories, whereas string theory is a string particle theory. The concept of a particle with specified properties is well defined in both. They are facets of the same object. One is compatible with quantum gravity, and one is not. Likewise, in the thriving field of amplitudes, the calculation methods are interchangeable.

String theory is also used as a tool to explore the fundamental mathematical structures that underlie the theory and their properties. The most celebrated example among those is that of mirror symmetry, which was first observed in worldsheet constructions of string compactifications, and its profound implications for complex manifolds and enumerative geometry were subsequently understood. Spinor–Vector Duality (SVD) is an extension of mirror symmetry that extends the duality map to include the transformation of Wilson-line moduli. The most general set of symmetries in toroidal orbifolds, hence, act on the internal and Wilson line moduli as $(G, B, W) \rightarrow (\tilde{G}, \tilde{B}, \tilde{W})$, where G, B, and W are the metric, anti-symmetric tensor, and Wilson-line moduli fields of the Narain moduli space. Similarly to mirror symmetry, the SVD was first noted in worldsheet formulations of Heterotic String theories, and may have profound implications for the complex algebraic curves with vector bundles that correspond to the EFT limits of the worldsheet constructions. The SVD may provide a demarcation line between the (2,0) quantum gravity EFTs that have an ultra-violet complete embedding in string theory and those that do not.

The SVD is, therefore, of pure mathematical interest. We should note, however, the use of SVD in the construction of string derived Z'-model that may remain light down to low scales [61], with implications for physics at the LHC [62].

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